Real Linear Spaces of Matrices

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ROSINA – E la pratica?
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Carlo Goldoni
Le avventure della villeggiatura

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CONTENTS

Chapter 1

INTRODUCTION

The object of this thesis is the study of a classical problem of linear algebra, that is:

given a set $\mathcal{X}$ of real or complex matrices, what is the maximal dimension $d(\mathcal{X})$ of a real vector space V such that $V \setminus \{0\} \subset \mathcal{X}$?

This question, apparently simple in its formulation, arises in many different fields of research.

In the case of real invertible matrices, it is deeply related to the existence of independent vector fields on a sphere.

The existence of a set $\{A_1, \dots, A_k\} \subset GL_n(\mathbb{R})$ such that any nonzero linear combination is still an invertible matrix, leads, indeed, to the existence of $k-1$ independent vector fields on $S^{n-1} \subset \mathbb{R}^n$. The basic fact is that the matrices $A_1A_k^{-1}, \dots, A_{k-1}A_k^{-1}$ have not real eigenvalues hence, if $x \in S^{n-1}$, the tangential projection of $A_iA_k^{-1}x$, over the unitary sphere, defines a nonvanishing vector field V_i. Since any nonzero linear combination of the matrices $A_iA_k^{-1}$ is still nonsingular, the V_i's are everywhere independent. Adams, in [1], determines the maximal number of independent vector fields on S^{n-1}. His work essentially consists in the translation of the problem to an homotopy-theoretic setting and its solution is given by means of a deep development of the K-theory for projective spaces. This is the starting point for many subsequent theories, that we briefly recall. In [3] (with Lax and Phillips), the solution of the initial problem is determined for real invertible symmetric matrices, as well as in the invertible complex and hermitian cases, using a recursive argument.

In the case of real rectangular matrices of maximal rank (and in their natural generalization to constant rank matrices), the above problem is related to the existence of bilinear maps $\mu : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^m$ with the nonsingularity condition: $\mu(x, y) = 0 \Rightarrow x = 0$ or $y = 0$. These maps have been largely studied (see Hurwitz [27], Radon [35] and also Lam *et al.* [30], [31] and Friedland *et al.*[11]) since they generalize the multiplication map of the classical division algebras over $\mathbb{R}$. Also, they provide estimates on the geometric dimension of real vector bundles over the projective space and, hence, are instrumental in the study of the immersion of such spaces into $\mathbb{R}^n$ (cf. [2]).

When $\mathcal{X}$ denotes the set of real invertible symmetric matrices, the problem is related to spectral theory and its solution provides an estimate on maximal dimension of a system of strictly hyperbolic PDEs (see [23]).

For what concerns this work, we are interested to analyze sets $\mathcal{X}$ defined by some restrictions on the rank of the involved matrices.

The content of Chapter 2 is an overview of K-theory and of some classical results concerning the initial problem. These are provided in order to introduce the theorems and technical tools used in our arguments. In particular, it is important to describe the structure of the ring $\widetilde{K}(\mathbb{RP}^d)$ of reduced K-theory, that we compute in details.

The original work of the thesis begins with Chapter 3.

Here, we analyze the case $\mathcal{X} = \mathcal{S}_n^{n-k}$ of $n \times n$ symmetric matrices with constant rank $n-k$, using the theory of stable vector bundles over real projective spaces. Then, we generalize our argument to the corresponding case of complex hermitian matrices, $\mathcal{X} = \mathcal{H}_n^{n-k}$.

Define the (generalized) Radon-Hurwitz numbers:

Definition *For real r, set: $\rho(r) = 2^a + 8b$ and $\rho_{\mathbb{C}}(r) = 2(a+4b)+2$, if $r = 2^{a+4b}(2c+1)$, with a, b, c integers, and $0 \le a \le 3$; $\rho_{\mathbb{C}}(r) = \rho(r) = 0$ otherwise. Moreover, for $k \le n$, set $\sigma(n, n-k) = \max\{\rho(\frac{n-k}{2} + i)$ with $0 \le i \le k\}$ and define $\sigma_{\mathbb{C}}$ in a similar fashion using $\rho_{\mathbb{C}}$.*

We prove the following estimates:

Theorem A *If $0 \le k \le 4$, then*

$$\sigma(n, n-k) \le \; d(\mathcal{S}_n^{n-k}) \; \le \max\{c_k, \sigma(n, n-k)\} + 1$$

where c_k depends only on k and has values: $c_0 = c_1 = 0$, $c_2 = 3$ and $c_3 = c_4 = 13$.

If $\sigma(n, n-k) = \rho\left(\frac{n-k}{2}\right) \ge c_k$, the upper bound is attained. Moreover, if $k = 1$ and $(n+1)/2 = 2^{a+4b}c$ with $a = 2, 3$, arbitrary b and c odd, then the lower bound is optimal, that is $d(S_n^{n-1}) = \sigma(n, n-1)$.

In particular, for $k = 0$, this theorem provides a new proof of the corresponding result of Adams in [3]. For $k = 1$, we improve the estimates given in [21]. For $2 \le k \le 4$, to the knowledge of the author, there does not seem to be analogous results in literature.

Theorem B *If $k = 0, 1$, then*

$$\sigma_{\mathbb{C}}(n, n-k) \le \; d(\mathcal{H}_n^{n-k}) \; \le \sigma_{\mathbb{C}}(n, n-k) + 1.$$

If $\sigma_{\mathbb{C}}(n, n-k) = \rho_{\mathbb{C}}\left(\frac{n-k}{2}\right)$, the upper bound is attained, otherwise, the lower bound is optimal, that is $d(\mathcal{H}_n^{n-1}) = \sigma_{\mathbb{C}}(n, n-1)$.

Again, for $k = 0$, the corresponding result in [3] is proven here by using different arguments. The case $k = 1$ seems new in literature.

The proof of the theorems above is based on the following arguments.

Any vector space V of $m \times n$ matrices of constant rank (0 excepted) induces an evaluation map between the trivial n-bundle $\underline{\mathbb{R}}^n$ and the sum of m copies of the tautological bundle $m\xi_{\mathbb{R}}$ over $\mathbb{P}(V)$: the map is $\Phi([A], v) = ([A], Av)$. The assumption that the matrices have constant rank implies that the kernel of Φ is a well-defined vector bundle K fitting in a long exact sequence

$$0 \to K \to \underline{\mathbb{R}}^n \xrightarrow{\Phi} m\xi_{\mathbb{R}} \to C \to 0 \quad \text{over} \quad \mathbb{P}(V).$$

Then, the exactness of this sequence can be interpreted in terms of stable bundles, that is, in (reduced) K-theory. The peculiar structure of the finite ring $\widetilde{K}(\mathbb{P}(V))$, together with a stable classification of the possible kernels K, yields a sharp estimate of the dimension of V, provided that the rank of K is little enough. This "natural" approach to the problem has been exploited in many work: see, among others, [31] (for the case of full rank rectangular matrices and nonsingular bilinear maps) and [36] and [41] (for square real and, respectively, complex constant rank matrices).

When working with symmetric matrices, we take into account an additional feature (which has been shown in [23]): the assumption of constant rank forces the eigenvalues of the matrices in V to be in pairs. This induces a decomposition of the pull-back to $S^{d-1} \subset V$ of the trivial bundle $\underline{\mathbb{R}}^n \to \mathbb{P}(V)$ into the sum of the pull-back of K with two isomorphic bundles E^+ and E^-. The fibers of E^+ and E^- over a matrix A represent the subspaces where the quadratic form ${}^t xAx$ has a defined sign. Under the further (and, as we will show, not so restrictive) assumption that E^+ is trivial, we can decompose the trivial bundle $\underline{\mathbb{R}}^n \to \mathbb{P}(V)$ as the sum of K with an equal number of trivial and tautological line bundles. Interpreting this decomposition in the ring of reduced K-theory, we provide a very sharp bound on the dimension of V (Theorem A). Moreover, by constructing explicit examples and by some considerations on the stable homotopy of classical groups, we determine the cases for which the bounds that we find are the best possible.

Furthermore, we extend our results to complex hermitian matrices of constant rank, since in this context they appear as a natural generalization of the symmetric case. The main results of Chapter 3 form the content of [16].

Using a slightly different argument, we also provide some estimates for skew-symmetric matrices of constant rank $\mathcal{A}_n^{n-k}$, generalizing a result of Yeung ([7]):

Theorem C *Let $\rho(n, n-k) = \max_{-k\leq i\leq k} \rho(n+i)$ and c_k as in Theorem A. If $0 \leq k \leq 3$, then*

$$\max\{\rho(n-k), \rho(n-k+2)\} \leq\ d(\mathcal{A}_n^{n-k}) + 1\ \leq \max\{c_k + 1, \rho(n, n-k)\}.$$

Chapter 4 is devoted to the study of matrices of rank bounded from below.

Denoting by $\mathcal{M}_n^{n-1,+}$ and $\mathcal{H}_n^{n-1,+}$ the sets of real and, respectively, hermitian $n \times n$ matrices of rank greater or equal than $n-1$, we show:

Theorem D *If n is even, then*

$$\max\{\, d(\mathcal{M}_n^{n-1,+}),\ d(\mathcal{H}_n^{n-1,+})\,\}\ \leq \rho_{\mathbb{C}}(n).$$

For the real case, this result improves the corresponding estimate in [21].

In this context, the fundamental tool is the following nonlinear characterization (due to Friedland *et al.* in [23], [21]): the existence of a d-dimensional vector space of complex $n \times n$ invertible matrices is equivalent to the existence of an odd map $\phi : S^{d-1} \to GL_n(\mathbb{C})$, that is a map verifying $\phi(-x) = -\phi(x)$.

Then, we remark the following two facts:

1. if n is even, then the map associating A to its adjoint matrix A^* is equivariant with respect to the multiplication by -1;

2. if the rank of A greater or equal than $n-1$, and $\det A$ is not of the form ir, with $i^2 = -1$ and $r < 0$, then ${}^t\bar{A} + iA^*$ is a complex invertible matrix (the bar stands for complex conjugation).

Thus, if n is even, any vector space $V \setminus \{0\}$ included in $\mathcal{M}_n^{n-1,+}$ or in $\mathcal{H}_n^{n-1,+}$, induces an odd map $\phi : V \supset S^{d-1} \to GL_n(\mathbb{C})$ by $\phi(A) = {}^t\bar{A} + iA^*$ and consequently, there exists a d-dimensional vector space of complex invertible matrices. The bound on d is a consequence of the estimate of Adams *et al.* in [3].

Moreover, we generalize to the hermitian case some results concerning real symmetric matrices obtained in [22] by Friedland and Libgober. In this work, they introduce a completely different method whose key point relies in an elegant application of the Lefschetz fixed-point theorem; this is used to show, by an Euler characteristic computation, the existence of real points of a complex variety of matrices.

The basic general idea of our argument is to consider the degeneracy locus $\{A \text{ hermitian}, \operatorname{rank} A\ \leq\ n-2\}$ as the real part of the complex variety $D_{\mathbb{C}} = \{A \text{ complex}, \operatorname{rank} A\ \leq\ n-2\}$ with respect to the involution $A \mapsto {}^t\bar{A}$. Then, we look for the minimal dimension of a real linear space intersecting D. We note that the calculations are particularly long. Our precise statement is:

Theorem E *The following estimate holds:* $7 \leq d(\mathcal{H}_5^{4,+}) \leq 8$.

The content of Chapter 5 is devoted to show the relations between hermitian matrices with rank bounded from below and the topology of Kähler varieties that do not admit Albanese fibrations. This was the original motivation for studying the dimension of linear spaces of matrices (see [15]).

Recall that the notion of *no Albanese fibration* (see [14]) for an n-dimensional compact Kähler variety X can be given by requiring that, for any $k < n$ and any independent $\beta_1, \ldots, \beta_k \in H^0(\Omega^1(X))$, the product $\beta_1 \wedge \cdots \wedge \beta_k \in H^0(\Omega^k(X)) = H^{k.0}(X)$ is not zero.

For X as above, we are interested in giving an estimate of the dimension of the kernel of cup-product map $\varphi : H^1(X, \mathbb{C}) \otimes H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$.

Thanks to the Hodge decomposition $H^m(X, \mathbb{C}) = \bigoplus_{i+j=m} H^{i.j}$, the cup product map φ defines maps:

1. $\varphi^{2.0} : \bigwedge^2 H^{1.0} \longrightarrow H^{2.0}$ (and its dual $\varphi^{0.2}$);
2. $\varphi^{1.1} : H^{1.0} \otimes H^{0.1} \longrightarrow H^{1.1}$.

The role of the assumption of no Albanese fibration becomes clear when dealing with the estimate of $\dim \ker \varphi^{2.0}$ (note, for instance, that there are not decomposable elements in the kernel).

Then, we look for some upper bound for the dimension of $\ker \varphi^{1.1}$. Set $H^{1.1}(X)_{\mathbb{R}} = H^{1.1}(X) \cap H^2(X, \mathbb{R})$ and call $M \subset H^{1.0} \otimes H^{0.1}$ the subspace of forms invariant under complex conjugation. The space M is naturally identified with the space $\mathcal{H}_q$ of Hermitian $q \times q$ matrices ($q = \dim H^{1.0}$) and $\varphi^{1.1}$ restricts to $\varphi^{1.1}_{\mathbb{R}} : M \to H^{1.1}(X)_{\mathbb{R}}$; it follows that $\dim \ker \varphi^{1.1}_{\mathbb{R}} = \dim \ker \varphi^{1.1}$. Assuming that X has no Albanese fibration, and using a positivity argument, we find restrictions on the signature and hence on the rank of the involved matrices: if $A \in N \setminus \{0\}$, then the rank of A must be greater or equal than $2n$.

In particular, if $n = 2q$, then A is invertible, and from [3] we achieve a very good bound on the dimension $\ker \varphi^{1.1}$. However, when $2n < q$, no good bound seems to be known in general. The Theorem (E) gives a sharp estimate in the case $n = 2$, $q = 5$. Precisely, we prove:

Theorem F *Let X be a compact Kähler variety without Albanese fibrations, and let $\phi\colon \wedge^2 H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C})$ be the cup product.*

1. *If $q \leq 2n-1$, then ϕ is injective;*
2. *if $q = 2n$, then $\dim\ker\phi \leq \rho_{\mathbb{C}}(q) + 1$;*
3. *if $q = 5$ and $n = 2$, then $\dim\ker\phi \leq 14$.*

Chapter 2

A SHORT REVIEW OF K-THEORY (AND OTHER USEFUL TOOLS)

The purpose of this chapter is to recall briefly some particular topics that will be used throughout the thesis. In order to introduce them, it seems appropriate to introduce the general setting of the K-theory. The topics that we would like to emphasize are: Bott periodicity and its relations with the stable homotopy of the classical groups, the ring structure in the real K-theory of $\mathbb{RP}^d$, the theorem of Adams on the number of vector fields on a sphere. Moreover, at the end of the chapter, we provide a new proof of an important theorem of Friedland ([23], Thm. A) concerning the relations between these arguments and the problem of determining the maximal dimension of a linear space of matrices. In this proof there are some of the essential ideas exploited in Chapters 3 and 4.

Throughout this chapter, X will denote a compact Haussdorf topological space, except when otherwise specified; $\mathbb{F}$ is either the field of real or complex numbers and $\underline{\mathbb{F}}^n \to X$ stands for the trivial vector bundle over X of fiber $\mathbb{F}^n$. We assume as known the general theories of vector bundles and homotopy groups (for exhaustive treatment of these subjects we refer to *e.g.* [28], [25]).

2.1 The K functor

Classes of vector bundles

Let us denote by $\mathrm{Vec}_{\mathbb{F}}(X)$ the set of isomorphism classes of $\mathbb{F}$-vector bundles over X. The exterior sum and the tensor product of bundles define operations

$$\oplus, \otimes \colon \mathrm{Vec}_{\mathbb{F}}(X) \times \mathrm{Vec}_{\mathbb{F}}(X) \longrightarrow \mathrm{Vec}_{\mathbb{F}}(X)$$

for which the triple $(\mathrm{Vec}_{\mathbb{F}}(X), \oplus, \otimes)$ verifies all the axioms of a ring excepted the existence of opposite elements, 0 is the class of the bundle with fiber $\{0\}$ while 1 is the class of the trivial line bundle; then, $\mathrm{Vec}_{\mathbb{F}}(X)$ has the structure of a *semiring*. The choice of a base point $x \in X$ defines a *semiring homomorphism*

$$\mathrm{rank}_x \colon \mathrm{Vec}_{\mathbb{F}}(X) \longrightarrow \mathbb{N} \quad \mathrm{rank}_x(E) = \dim_{\mathbb{F}} E_x.$$

Fixed E, if the above map is a constant function in the variable x, we get the rank of E; for instance, this happens for any E when X is a path-connected space. In this case, we set $\mathrm{Vec}^k_{\mathbb{F}}(X) = \mathrm{rank}^{-1}(k)$.

There are two maps $r : \mathrm{Vec}_{\mathbb{C}}(X) \to \mathrm{Vec}_{\mathbb{R}}(X)$ and $c : \mathrm{Vec}_{\mathbb{R}}(X) \to \mathrm{Vec}_{\mathbb{C}}(X)$ called, respectively, *realization* and *complexification* that act in the following way: $r(E)$ is the bundle E where the complex structure on each fiber is forgotten, $c(F)$ is the complex bundle with fiber $F_x \otimes_{\mathbb{R}} \mathbb{C}$.

Let us state an important property of vector bundles over compact spaces, which will be useful in the sequel:

Proposition 2.1.1. *If X is compact, for any $E \in Vec_{\mathbb{F}}(X)$ there exists $E' \in Vec_{\mathbb{F}}(X)$ such that $E \oplus E'$ is trivial.*

Proof. We give an outline of the proof, for the details refer to [29] or [28]. The key point is the existence of a finite number of trivializing open subsets for E, say $U_i, \ldots, U_n$ and of a corresponding partition of unity $\{\omega_i\}_{1\leq i\leq n}$. Over any U_i, E admits k_i independent sections $\{s_{ij}\}_{1\leq j\leq k_i}$; the $N = \sum_{i=1}^n k_i$ functions $\sigma_{ij} = \omega_i s_{ij}$ are defined everywhere on X and they generate the vector space E_x at each point x. Using these functions, we can define a morphism of bundles surjective at each fiber:

$$\alpha : \underline{\mathbb{F}}^N \to E \quad \text{defined by} \quad \alpha(x, r_1, \ldots, r_{n,k_n}) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} r_{ij}\sigma_{ij}(x).$$

It can be shown that α admits a cross section $\beta : E \to \underline{\mathbb{F}}^N$. Hence we can take E' as the kernel bundle (which is well defined) of α obtaining the isomorphism $E \oplus E' = \underline{\mathbb{F}}^N$. ♢

Ring completion

For any semiring S, there is a universal ring that extends it that is, a ring $K(S)$ and a semiring homomorphism $s : S \to K(S)$ so that, for any semiring homomorphism $f : S \to R$, there exists a ring homomorphism $g : K(S) \to R$ verifying $gs = f$. Any construction of $K(S)$ and s with the above properties gives, up to isomorphism, the same result; $K(S)$ is called the ring completion of S.

A standard construction is the following: $K(S)$ is the quotient of $S \times S$ by the relation

$$(n, m) \sim (n', m') \quad \text{when there exists } k \in S \text{ such that } n + m' + k = n' + m + k;$$

the class of (n, m) is noted $n - m$. Operations in $K(S)$ are given by

$$\begin{aligned}(n-m)+(n'-m') &= (n+n')-(m+m') \quad \text{and} \\ (n-m)\cdot(n'-m') &= (nn'+mm')-(nm'+mn')\end{aligned}$$

with zero and unit, respectively, $n - n$ and $(n + 1) - n$.

The ring K(X)

Definition 2.1.2. *The ring completion of the semiring $Vec_{\mathbb{F}}(X)$, which we will denote by $K_{\mathbb{F}}(X)$, is called the ring of* K-theory *of X.*

Remark 2.1.3. From the definition, it follows that, when X is written as the disjoint union of its connected components $X_1 \cup \cdots \cup X_n$, $\mathrm{Vec}_{\mathbb{F}}(X)$ decomposes as the semiring $\mathrm{Vec}_{\mathbb{F}}(X_1) \oplus \cdots \oplus \mathrm{Vec}_{\mathbb{F}}(X_n)$ with addition and multiplication defined componentwise; this implies that

$$K_{\mathbb{F}}(X) = K_{\mathbb{F}}(X_1) \oplus \cdots \oplus K_{\mathbb{F}}(X_n).$$

From now on, let us note by $[E]$ the element in K-theory represented by E, and $n = [\underline{\mathbb{F}}^n]$. Then, the elements of $K_{\mathbb{F}}(X)$ are written as formal differences of (isomorphism classes of) vector bundles: $[E] - [F] \in K_{\mathbb{F}}(X)$.

Proposition 2.1.4. *Each element $x \in K_{\mathbb{F}}(X)$ can be written as $[E] - N$ for some $E \in Vec_{\mathbb{F}}(X)$ and integer N. Moreover, $[E]-n = [F]-m$ if and only if $E \oplus \underline{\mathbb{F}}^{m+N} = F \oplus \underline{\mathbb{F}}^{n+N}$ for some N.*

Proof. If $x = [F]-[G]$, we can take, by proposition 2.1.1, G' such that $G \oplus G' = \underline{\mathbb{F}}^N$; then, $[F]-[G] = [F \oplus G'] - [G \oplus G']$. Similarly, the equality $[E]-n = [F]-m$ means that there is a bundle G such that $E \oplus \underline{\mathbb{F}}^m \oplus G = F \oplus \underline{\mathbb{F}}^n \oplus G$; then, by taking G' as above and adding it on both sides of the equality, we get $E \oplus \underline{\mathbb{F}}^{m+N} = F \oplus \underline{\mathbb{F}}^{n+N}$. $\diamondsuit$

Corollary 2.1.5. *Two bundles E and F represent the same element in $K_{\mathbb{F}}(X)$, that is $[E] = [F]$, if and only if there exists a trivial bundle $\underline{\mathbb{F}}^N$ such that $E \oplus \underline{\mathbb{F}}^N = F \oplus \underline{\mathbb{F}}^N$.*

Remark 2.1.6. The hypothesis of compactness of X can be weaken by requiring simply that X has the property that for any bundle E there exists another bundle E' such that $E \oplus E'$ is trivial. Spaces with this property are, for instance, the CW-complexes with a countable number of cells or, more generally, paracompact spaces, see *e.g* [28].

Example 2.1.7. *(The "trivial" tangent bundle of a sphere)* Consider the sphere S^d standardly embedded in $\mathbb{R}^{d+1}$ as the set of vectors of unitary length and call TS^d and N respectively the tangent bundle of S^d and the normal bundle of the embedding. Remark that N is the trivial line bundle $\underline{\mathbb{R}}$. We have that $TS^d \oplus N = \underline{\mathbb{R}}^{d+1}$, hence $[TS^d] = d$ in $K_{\mathbb{R}}(S^d)$. However, it is well-known that TS^d is trivial only when $d = 1, 3$ or 7.

Example 2.1.8. When X is a point $*$, a vector bundle is just a vector space and its dimension, *i.e.* a natural number, determines the isomorphism class. Then, $\mathrm{Vec}_{\mathbb{F}}(*)$ is isomorphic to the semiring $\mathbb{N}$, so that $K_{\mathbb{F}}(*) = \mathbb{Z}$. As a consequence, if X is the disjoint union of n points, $K_{\mathbb{F}}(X) = \mathbb{Z}^n$ thus $K_{\mathbb{F}}(S^0) = \mathbb{Z} \oplus \mathbb{Z}$.

Functoriality

Any (continuous) function $f : X \to Y$ between compact spaces induces a map

$$K_{\mathbb{F}}(Y) \to K_{\mathbb{F}}(X) \quad \text{sending } [E] - n \text{ to } [f^*E] - n,$$

where f^* is the pullback of bundles. Remarking that the pullback of a trivial bundle is still a trivial bundle of the same rank, this map of rings can be interpreted as the extension of the composition $\mathrm{Vec}_{\mathbb{F}}(Y) \xrightarrow{f^*} \mathrm{Vec}_{\mathbb{F}}(X) \xrightarrow{s} K_{\mathbb{F}}(X)$ via the ring completion of $\mathrm{Vec}_{\mathbb{F}}(Y)$, therefore we still denote it by f^*.

Proposition 2.1.9. *The K-theory gives a functor from the category of compact spaces and homotopy classes of maps to the category of rings and homomorphisms.*

Proof. All the functorial properties of f^* descend from the corresponding ones of the pullback map: $f^*(E \oplus F) = f^*E \oplus f^*F$, $f^*(E \otimes F) = f^*E \otimes f^*F$, $(f \circ g)^* = g^* \circ f^*$, $Id^* = Id$, as well as the fact that f homotopic to g implies $f^* = g^*$. ♢

Example 2.1.10. The choice of a base point $\{x\} \to X$ induces a ring homomorphism $r_x : K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(x) = \mathbb{Z}$; this is, indeed, the ring homomorphism that extends $\mathrm{rank}_x : \mathrm{Vec}_{\mathbb{F}}(X) \to \mathbb{N} \subset \mathbb{Z}$, thus $r_x([E] - n) = \mathrm{rank}_x E - n$.

The ring $\widetilde{K}(X)$ and stable vector bundles

Proposition 2.1.11. *The choice of a base point $x \in X$ induces a ring decomposition $K(X) = \ker(r_x) \oplus \mathbb{Z}$. If X is path-connected, this decomposition is canonical.*

Proof. By functoriality, the composition $\{x\} \to X \xrightarrow{p} \{x\}$ gives

$$\mathbb{Z} \xrightarrow{p^*} K(X) \xrightarrow{r_x} \mathbb{Z} \quad \text{with } r_x \circ p^* = Id$$

thus, in particular, r_x is surjective and has a cross section p^*: the splitting $K(X) = \ker(r_x) \oplus \mathbb{Z}$ holds.

Moreover, if X has only one path-connected component, any two choices of a base point are homotopic, hence give the same map in K-theory. $\diamondsuit$

Remark that $\ker(r_x)$, being an ideal, has a ring structure on its own, though it does not necessarily have a unit.

Definition 2.1.12. *The ring* $\ker(r_x)$ *is called the ring of* reduced K-theory *of* X *and it is noted* $\widetilde{K}_{\mathbb{F}}(X)$.

Remark 2.1.13. The projection $p : X \to \{x\}$ realizes a canonical isomorphism $\widetilde{K}_{\mathbb{F}}(X) = co\mathrm{ker}\, p^*$ and this is why the notation of x in the reduced K-theory is omitted. What really depends in the choice of the base point is the splitting $K(X) = \widetilde{K}(X) \oplus \mathbb{Z}$, as stated at the beginning. However, it is useful to think of $\widetilde{K}_{\mathbb{F}}(X)$ as the subring of the objects of the form $[E] - \mathrm{rank}_x E$.

The reduced K-theory gives a functor from the category of compact spaces to the one of rings. $\widetilde{K}_{\mathbb{F}}(X)$, being smaller than $K_{\mathbb{F}}(X)$, is more suitable for calculations. Furthermore, there is a very useful characterization in terms of *stable isomorphism* classes of bundles, as we are going to show.

Definition 2.1.14. *Two bundles* E *and* F *over* X *are said to be stable isomorphic (or simply s-isomorphic) when there are integers* n *and* m *such that* $E \oplus \underline{\mathbb{F}}^m = F \oplus \underline{\mathbb{F}}^n$ *in* $Vec_{\mathbb{F}}(X)$. *We note this by* $E \sim F$.

The relation $\sim$ clearly is an equivalence relation. Note by $[E]_s$ the class of E and remark that $[\underline{\mathbb{F}}^n]_s = 0$ for any n.

Proposition 2.1.15. *The semiring quotient* $Vec_{\mathbb{F}}(X)/\sim$ *is a ring. Moreover, it is isomorphic to* $\widetilde{K}_{\mathbb{F}}(X)$.

Proof. The only thing to prove in order to show that $\mathrm{Vec}_{\mathbb{F}}(X)/\sim$ is a ring is the existence of its inverse elements. By proposition 2.1.1, for any bundle E we get another bundle E' such that their sum is trivial, *i.e.* $[E]_s + [E']_s = 0$.

Consider, now, the quotient map $\mathrm{Vec}_{\mathbb{F}}(X) \to \mathrm{Vec}_{\mathbb{F}}(X)/\sim$: by the universal property of the ring completion, it extends to a ring homomorphism $K_{\mathbb{F}}(X) \to \mathrm{Vec}_{\mathbb{F}}(X)/\sim$. This latter map sends $[E] - n$ to $[E]_s$, hence it is surjective; its kernel is the ideal of elements $[E] - n$ such that E is stably trivial, hence it is the image of $p^* : K_{\mathbb{F}}(x) \to K_{\mathbb{F}}(X)$. Then, remark 2.1.13 concludes. $\diamondsuit$

More on stable bundles

In this part we present some general results about fibre bundles of large rank over a finite dimensional CW-complex. All statements essentially rely in homotopy arguments regarding the CW-cells codimension, moreover they have an interpretation in terms of the relations between classifying maps of bundles and reduced K-theory (see [28]).

In the following, set $\delta = \dim_{\mathbb{R}}\mathbb{F}$ as vector space and assume that X is a n-dimensional CW-complex. The following propositions show in which extent the dimension of X determines the maximum rank of stale non-triviality of a vector bundle.

Proposition 2.1.16. *Let $E \in Vec^k_{\mathbb{F}}(X)$ with $k \geq (n+1)/\delta$; then E decomposes as $F \oplus \underline{\mathbb{F}}$, for some $k-1$ vector bundle F.*

By an inductive argument, we get:

Corollary 2.1.17. *Any $E \in Vec^k_{\mathbb{F}}(X)$ decomposes as a sum $F \oplus \underline{\mathbb{F}}^{k-m}$, where F has rank m and m is the smallest integer greater or equal than $(n+1)/\delta - 1$.*

Remark 2.1.18. The number m of the corollary above is n in the case of real bundles and about $n/2$ for complex ones.

Moreover, the corollary can be interpreted as a generalization of the fact that any finite dimensional vector space has a basis.

Proposition 2.1.19. *Any two injective morphisms of vector bundles over X, $f, g : \underline{\mathbb{R}} \to E$, induce isomorphic cokernel bundles, provided* $\operatorname{rank} E \geq (n+2)/\delta$.

Again, by induction:

Corollary 2.1.20. *If $E, F \in Vec^k_{\mathbb{F}}(X)$ verify $k \geq (n+2)/\delta - 1$ and are stable isomorphic, then $E = F$.*

Remark 2.1.21. This second corollary can be viewed as a generalization of the fact that two basis of a finite-dimensional vector space have the same number of elements.

Also, note that the lower bound $(n+2)/\delta - 1$ is the best possible as the example 2.1.7 shows.

Let us conclude this section with some words about the geometric dimension of bundles:

Definition 2.1.22. *We say that a bundle $E \in Vec_{\mathbb{F}}(X)$ has geometric dimension $\leq k$ whenever E is stably isomorphic to F, with $F \in Vec^k_{\mathbb{F}}(X)$.*

Remark 2.1.23. This notion is consistent for elements of $\widetilde{K}_{\mathbb{F}}(X)$. Moreover, we say that $gd(E) = n$ when n is the minimum integer k for which $gd(E) \leq k$.

Two obvious consequences of the definition are that trivial bundles have geometric dimension 0 and $gd(E) \leq \mathrm{rank}\,(E)$.

By corollary 2.1.17, we get that any $E \in \mathrm{Vec}_{\mathbb{F}}(X)$ verifies $gd(E) \leq (n+1)/\delta$.

2.2 Vector bundles on the sphere

There is a very nice and general way to relate K-theory to homotopy theory and this is basically done by relating the isomorphism classes of vector bundles to the homotopy classes of certain classifying maps from X to high dimensional Grassmann spaces, see *e.g.* [28], [29]. For our purposes, however, we do not need the whole of such theory, since we are mainly interested in spheres and projective spaces. For spheres, in particular, a much simpler homotopic description of K-theory can be done, still retaining some of the basic ideas of the general case. Here we present the principal statements and results, referring to [28] for a very detailed treatise.

In the following, denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$.

Homotopy classification of bundles

Proposition 2.2.1. *For any $d \geq 1$ there is a map $\phi : [S^{d-1}, GL_k(\mathbb{F})] \to Vect^k_{\mathbb{F}}(S^d)$.*

Proof. Starting from $f : S^{d-1} \to GL_k(\mathbb{F})$, construct the vector bundle $E_f \to S^d$ using f as "gluing" function: write S^d as the union of its upper and lower hemispheres D^d_+ and D^d_- , with $D^d_+ \cap D^d_- = S^{d-1}$. Let E_f be the quotient of the disjoint union $D^d_+ \times \mathbb{F}^k \cup D^d_- \times \mathbb{F}^k$ obtained by identifying $(x, v) \in D^d_+ \times \mathbb{F}^k$ with $(x, f(x)v) \in D^d_- \times \mathbb{F}^k$. There is a natural projection $E_f \to S^d$ and this is a vector bundle of rank k, as one can most easily see by taking an equivalent definition in which the two hemispheres of S^d are enlarged slightly to open balls and the identification occurs over their intersection, a product $S^d \times (-\varepsilon, \varepsilon)$, with the map f used in each slice $S^d \times \{t\}$. The map f is called *clutching function.*

Given two homotopic maps $f, g : S^{d-1} \to GL_k(\mathbb{F})$ we can repeat the clutching construction using the homotopy F: we obtain a vector bundle $E_F \to S^d \times [0, 1]$ that restricts to E_f and E_g respectively over $S^d \times \{0\}$ and $S^d \times \{1\}$. Being X compact, it is a standard result (see *e.g.* [29]) that E_f is isomorphic to E_g. So the map $\phi : f \to E_f$ is well defined. ♢

Let us consider separately the complex and the real case.

Proposition 2.2.2. *The map $\phi : [S^{d-1}, GL_k(\mathbb{C})] \to Vec^k_{\mathbb{C}}(S^d)$ is bijective.*

Proof. We sketch how to construct an inverse ψ. Any vector bundle $F \to S^d$ restricts to two trivial bundles $F_\pm \to D^d_\pm$, since the disk is contractible. Choose trivializations $h_\pm : F_\pm \to D^d_\pm \times \mathbb{C}^k$: they induce a map $f_h = h_+ \circ h_-^{-1} : S^{d-1} \to GL_k(\mathbb{C})$. To show that the map ψ sending F to f_h is well defined, we need to prove that, for any two other trivializations $h'_\pm$ of $F_\pm$, the corresponding map $f_{h'}$ is homotopic to f_h. This is shown by using the fact that $GL_k(\mathbb{C})$ is path-connected. $\diamondsuit$

Example 2.2.3. Any complex vector bundle over the sphere S^1 is trivial: indeed, $[S^0, GL_k(\mathbb{C})]$ has only one point for any k, being $GL_k(\mathbb{C})$ path connected.

Example 2.2.4. The canonical line bundle $\xi_\mathbb{C} \to \mathbb{CP}^1 = S^2$ verifies $\xi_\mathbb{C} \oplus \xi_\mathbb{C} = (\xi_\mathbb{C} \otimes \xi_\mathbb{C}) \oplus \underline{\mathbb{C}}$: indeed, the two bundles appearing in each side of the equation are, respectively, obtained using the clutching functions

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix},$$

which are homotopic.

In the real case, the argument used above will not work, since $GL_k(\mathbb{R})$ has two path-connected components: $\{A \in GL_k(\mathbb{R}) \mid \pm \det A > 0\}$. The closest analogy with the complex case is obtained by considering oriented real vector bundles[1]. Not all vector bundles can be given an orientation: the Möbius bundle is not orientable, this is because an oriented line bundle over a compact base is always trivial since it has a canonical section formed by the unit vectors having positive orientation. Bundles constructed from clutching functions with positive determinant will admit an orientation, while others won't.

Let $\mathrm{Vec}^k_\mathbb{R}(X)_+$ the set of isomorphism classes of oriented rank k vector bundles (isomorphisms are required to preserve orientation) and set $GL_k(\mathbb{R})_+$ the path-connected component of $GL_k(\mathbb{R})$ defined by positive determinant.

Proposition 2.2.5. *The map $\phi : [S^{d-1}, GL_k(\mathbb{R})_+] \to Vec^k_\mathbb{R}(S^d)_+$ is bijective.*

Then, which vector bundles, if any, the set $[S^{k-1}, GL_k(\mathbb{R})]$ describe? If we choose a basepont $x_0 \in S^{k-1}$, we have (see [28]):

Proposition 2.2.6. *The set of maps $f \in [S^{k-1}, GL_k(\mathbb{R})]$ such that $f(x_0) \in GL_k(\mathbb{R})_+$ is in bijection with the set of bundles $E \in Vec^k_\mathbb{R}(S^d)$ such that E_{x_0} has a specified orientation and the isomorphisms preserve it.*

[1] An orientation of a real vector bundle is a function assigning an orientation to each fiber and defined by ordered n-tuples of independent local sections.

Example 2.2.7. The sphere S^1 admits two non isomorphic real vector bundles bundles of rank 1: indeed there are two nonhomotopic maps $S^0 \to GL_1(\mathbb{R})$ taking x_0 to $GL_1(\mathbb{R})_+$.

Remark 2.2.8. For $d > 1$, S^{d-1} is connected, hence, provided x_0 is sent to $GL_k(\mathbb{R})_+$,

$$[S^{d-1}, GL_k(\mathbb{R})] = [S^{d-1}, GL_k(\mathbb{R})_+] \leftrightarrow \mathrm{Vec}\,^k_{\mathbb{R}}(S^d)_+.$$

This means that any vector bundle bundle over S^d is orientable. As a consequence, the natural map

$$\mathrm{Vec}\,^k_{\mathbb{R}}(S^d)_+ \to \mathrm{Vec}\,^k_{\mathbb{R}}(S^d)$$

is surjective and at most $2-1$; moreover, it is $1-1$ for bundles that do not admit orientation-reversing isomorphisms.

About the homotopy of some classical group

Consider the spaces $GL_k(\mathbb{C})$ and $GL_k(\mathbb{R})_+$. They are, in facts, topological groups (with respect to the multiplication of matrices), hence the sets $[S^{d-1}, GL_k(\mathbb{C})]$ and $[S^{d-1}, GL_k(\mathbb{R})_+]$ naturally inherit a structure of group. Moreover, they deformation retract, respectively, onto the groups $U(k)$ and $SO(k)$ which are considerably smaller and more suitable for calculations.

Proposition 2.2.9. *The following group isomorphisms hold:*

$$[S^{d-1}, GL_k(\mathbb{C})] \simeq [S^{d-1}, U(k)] \simeq \pi_{d-1}(U(k)) \text{ and}$$

$$[S^{d-1}, GL_k(\mathbb{R})_+] \simeq [S^{d-1}, SO(k)] \simeq \pi_{d-1}(SO(k)).$$

This fact, coupled with the results of he previous section, has many consequences.

The first (and most obvious) one is that, since higher homotopy groups have good computational features (such as the exact sequence of a fibration, or a weak form of excision), they give a well-known algebraic instrument way to classify vector bundles over spheres. However, the calculation of homotopy of the classical groups has proven to be not simple at all! It is, indeed, related to the problem of calculating the homotopy group of spheres.

A second consequence, whose detailed description goes beyond the scope of this work but which is still worth mentioning (see [28]), is that, as one can expect, there is a description of stable bundles in terms of homotopy.

Let us call U and SO respectively the inductive limits of $U(k)$ and $SO(k)$; we get:

Theorem 2.2.10. *For $d > 0$, there are isomorphisms $\widetilde{K}_{\mathbb{C}}(S^d) \to \pi_{d-1}(U)$ and $\widetilde{K}_{\mathbb{R}}(S^d) \to \pi_{d-1}(SO)$.*

Periodicity

Probably, the most notable feature in K-theory is the Bott periodicity theorem. Despite to its proof, which, in particular in the real case, is rather involved and requires the full development of the theory, this theorem has a very simple statement.

For a compact space X with base point p, denote by SX its suspension, that is the quotient space of $X \times [0,1]$ in which the subsets $X \times \{0\}$, $X \times \{1\}$ and $\{p\} \times [0,1]$ are contracted to a common point. Iterated suspension are written $S^i X = S(S^{i-1}X)$.

One of the formulations of Bott periodicity is the following:

Theorem 2.2.11. *If X is compact, the rings $\widetilde{K}_{\mathbb{C}}(X)$ and $\widetilde{K}_{\mathbb{C}}(S^2X)$ are isomorphic. The same holds for $\widetilde{K}_{\mathbb{R}}(X)$ and $K_{\mathbb{R}}(S^8X)$.*

Remark 2.2.12. What explicitly realizes the above isomorphisms is the cup-product map

$$\widetilde{K}_{\mathbb{F}}(X) \otimes \widetilde{K}_{\mathbb{F}}(S^i) \to \widetilde{K}_{\mathbb{F}}(S^iX).$$

In the complex case, this map i the multiplication by the class of the Hopf bundle over $S^2 = \mathbb{CP}^1$; this gives a element of

$$\widetilde{K}_{\mathbb{C}}(S^2X) = \widetilde{K}_{\mathbb{C}}(S^2 \times X / S^2 \vee X).$$

Similarly, in the real case, the product with the 8-dimensional real Hopf bundle is considered. A very detailed proof of the theorem in both real and complex case can be found in [28].

Example 2.2.13. Since the i-th suspension of S^d is the $d+i$-dimensional sphere, Bott periodicity implies that the ring of (reduced) complex K-theory of S^d depends only on the parity of d. In particular, from (2.1.8) and (2.2.3) one has, for any i:

$$\widetilde{K}_{\mathbb{C}}(S^{2i}) = \widetilde{K}_{\mathbb{C}}(S^0) = \mathbb{Z} \quad \text{and} \quad \widetilde{K}_{\mathbb{C}}(S^{2i+1}) = \widetilde{K}_{\mathbb{C}}(S^1) = 0$$

Example 2.2.14. Similarly to the complex case, the ring $\widetilde{K}_{\mathbb{R}}(S^d)$ depends only on the modulo 8 congruence class of d, thus, we only need to know the reduced rings of spheres of dimension up to 7. We have $\widetilde{K}_{\mathbb{R}}(S^0) = \mathbb{Z}$ by (2.1.8). For studying spheres from S^1 to S^7, we can make use of theorem 2.2.10, since the lower homotopy groups of SO are fairly well-known (cf. [25]). In the table below, we resume the values:

$\widetilde{K}_{\mathbb{R}}(S^d)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
d (modulo 8)	0	1	2	3	4	5	6	7

Bott periodicity also allows to compute the *stable* homotopy groups of $U(k)$ and $SO(k)$. The word “stable” comes from the fact that, if k is sufficiently large, these groups are independent of k:

Lemma 2.2.15. *If $k \geq d+2$, then $\pi_d(SO(k)) = \pi_d(SO(k+1))$ and $\pi_d(U(k)) = \pi_d(U(k+1))$.*

Proof. Let us show only the SO case, the other one being completely similar (excepted, perhaps, that one can make the assumption $k \geq d/2$). First, consider the fibration

$$SO(k) \to SO(k+1) \xrightarrow{p} S^k$$

where the map p can be thought as the evaluation on a fixed vector of $\mathbb{R}^{k+1}$ and the first map behaves as an inclusion. Any fibration induces a long exact sequence of homotopy groups, thus, in our case, we get:

$$\cdots \to \pi_{d+1}(S^k) \to \pi_d(SO(k)) \to \pi_d(SO(k+1)) \xrightarrow{p_*} \pi_d(S^k) \to \ldots .$$

Under the assumption $k \geq d+2$, we get that the central map (the one induced by $SO(k) \subset SO(k+1)$) is an isomorphism, since $\pi_{d+1}(S^k)$ and $\pi_d(S^k)$ are both zero. $\diamondsuit$

In the following, we list the values of $\pi_d(SO(k))$ and $\pi_d(U(k))$, when k is in the stable range.

$\pi_d(U(k))$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_d(SO(k))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
d (modulo 8)	0	1	2	3	4	5	6	7

2.3 Projective spaces

In this section we go a bit deeper in the theory of stable vector bundles, by describing the reduced K-theory rings of real and projective spaces. This is done, in the spirit of Adams [1], in order to prepare a background for discussing the links that exist between the existence of non vanishing vector fields on a sphere, the existence of linear spaces of real invertible matrices and the existence of orthogonal multiplications.

Spheres, vector fields and orthogonal multiplications

Definition 2.3.1. *A vector field V over S^{d-1} is a map $V : S^{d-1} \to \mathbb{R}^d$ such that, for any $X \in S^{d-1}$, the (standard) scalar product $V(X) \cdot X = 0$. Moreover, we say that two vector fields V and W are orthonormal when $\|V(X)\| = \|W(X)\| = 1$ and $V(X) \cdot W(X) = 0$ for any $X \in S^{d-1}$.*

First, let us consider the problem of determining when a sphere S^d admits orthonormal vector fields.

The following proposition gives a constructive method:

Proposition 2.3.2. *If S^{d-1} admits k orthonormal vector fields, then also S^{rd-1} does.*

Proof. Let us call $V_1, \dots, V_k$the vector fields on S^{d-1}. The sphere $S^{rd-1} \subset \mathbb{R}^{rd}$ can be considered as th join of r spheres S^{n-1} lying in r orthogonal subspaces of $\mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d$; that is, any $x \in S^{rd-1}$ can be written as $x = (a_1 X_1, \dots, a_r X_r)$ where $\sum_i a_i = 1$ and $X_i \in S^{d-1}$. Then construct k vector fields on S^{rd-1} by putting

$$W_j(x) = a_1 V_j(X_1) + \cdots + a_r V_j(X_r).$$

To prove that the W_j's are orthogonal it is enough to observe that $X_i \cdot X_j = 0$. $\diamondsuit$

Corollary 2.3.3. *Any odd-dimensional sphere admits a nonzero vector field.*

Proof. S^1 has a nonzero vector field: $V(x, y) = (-y, x)$. $\diamondsuit$

Theorem 2.3.4. *The following condition for S^{d-1} are equivalent:* (1) $d-1$ *is odd;* (2) *the antipodal involution has degree* 1*;* (3) *the antipodal involution is homotopicto the identity;* (4) S^{d-1} *has at least a nonzero vector field.*

Definition 2.3.5. *A bilinear function $\mu : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ is called an orthogonal multiplication if, for any $a \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$,*

$$\|\mu(a, x)\| = \|a\| \, \|x\|$$

Remark 2.3.6. Any orthogonal multiplication can be viewed as an injective linear map $M : \mathbb{R}^d \to \mathcal{M}_n(\mathbb{R})$; the orthogonality property means that, for $a \in S^{d-1}$, $M(a)$ is an orthogonal matrix and also that, for $x \in S^{n-1}$, $a \mapsto \mu(a, x)$ is an isometry. So, forcefully, $d \leq n$.

Let us call $e_i = (0, \dots, 1, \dots, 0)$ the i-th standard vector of $\mathbb{R}^d$.

Lemma 2.3.7. *There exists a orthogonal multiplication $\mu : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ if and only if there is another one whose induced map M verifies $M(e_d) = I$.*

Proof. Call $N : \mathbb{R}^d \to O(n)$ the map induced by μ. Then, construct $M : \mathbb{R}^d \to O(n)$ by setting $M(e_i) = N(e_k)^{-1} N(e_i)$ for any i. $\diamondsuit$

Orthogonal multiplications such that $M(e_d) = I$ are called *normalized*. By the previous lemma, we can always suppose that this condition is always verified.

Orthogonal multiplications give rise to orthonormal vector fields on spheres:

Theorem 2.3.8. *If there exists an orthogonal multiplication $\mu : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$, then S^{n-1} admits $d-1$ orthonormal vector fields.*

Proof. Suppose, by lemma 2.3.7, that μ is normalized: then, for any $x \in S^n$, the vectors $\mu(e_1, x), \dots, \mu(e_d, x)$ are orthonormal. Construct vector fields by putting $V_i = \mu(e_i, x)$ with $i \leq k-1$. ♢

The next theorem gives a very nice representation of orthogonal multiplications in terms of matrices. It will be very useful in the next chapter.

Theorem 2.3.9. *The sets of normalized orthogonal multiplications $\mu : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ are in bijective correspondence with the sets of orthogonal matrices $M_1, \dots, M_{d-1} \in O(n)$ verifying the (Clifford) relations :*

$$M_i M_j + M_j M_i = 0 \quad i \neq j, \qquad M_i^2 = -I.$$

These correspondences are obtained by setting $M_i = \mu(e_i, \cdot)$.

Proof. The matrices $M_1, \dots, M_{d-1}, M_d := I$ are orthogonal: this is just remark 2.3.6. Moreover, they span (over $\mathbb{R}$) a linear space of orthogonal matrices and this is a consequence of the bilinearity of μ. Granting this, we show that $M_1 \dots, M_{d-1}$ are also skew-symmetric, and so Clifford relations hold.

By the orthogonality of any linear combination of M_i's,we get

$$1 = (\sum_i a_i^t M_i)(\sum_j a_j M_j) = \sum_i a_i^2\, {}^tM_i M_i + \sum_{i<j} a_i a_j ({}^tM_i M_j + {}^tM_j M_i).$$

Since $v = (a_1, \dots, a_d)$ is generic and can be chosen such that $\sum_i a_i^2 = 1$, this implies

$${}^tM_i M_j +^t M_j M_i = 0 \quad \text{ for any } i < j.$$

When $j = d$, so that $M_d = I$, we get ${}^tM_i = -M_i$ for any $i < d-1$. ♢

Radon-Hurwitz numbers

The question of how many real orthogonal, anti-commuting skew-symmetric matrices there can be in order n is a question that was completely settled in the early part of the XX century by J. Radon [35] as a development of earlier works of A. Hurwitz [27].

In what follows, we present an answer to the problem, in the spirit of [24]. Our presentation has the advantage of giving a simple inductive method for constructing the matrices.

We start by the definition of the generalized Radon-Hurwitz numbers, which are widely used throughout our work

Definition 2.3.10. *The generalized Radon-Hurwitz numbers are the function $\rho, \rho_{\mathbb{C}}$: $\mathbb{R} \to \mathbb{N}$ defined by the conditions: $\rho(r) = \rho_{\mathbb{C}}(r) = 0$ if $r \notin \mathbb{N}$ and, if $r \in \mathbb{N}$,*

$$\rho(r) = 2^a + 8b \qquad \text{if } r = 2^{a+4b}(2c+1) \text{ with } a, b, c \in \mathbb{N} \text{ and } 0 \leq a \leq 3; \quad (2.1)$$
$$\rho_{\mathbb{C}}(r) = 2v_2(r) + 2 \qquad \text{if } r = 2^{v_2(r)}(2c+1) \text{ with } v_2(r), c \in \mathbb{N}. \quad (2.2)$$

Definition 2.3.11. *A Radon-Hurwitz system (briefly: R-H system) of order n is a finite set of real orthogonal matrices $\{M_1, \dots, M_s\}$ verifying the Clifford relations:*

$$M_iM_j + M_jM_i = 0 \text{ for } i \neq j, \quad M_i^2 = -I, \quad {}^tM_i = -M_i.$$

Remark 2.3.12. It seems somewhat helpful to rephrase some congruence properties of ρ. First of all, note that if r is integer and decomposed as in the definition, then $\rho(r) = \rho(2^{a+4b})$.

Moreover, setting $r_1 = 2^{3+4b}, r_2 = 2^{4(b+1)}, r_3 = 2^{4(b+1)+1}, r_4 = 2^{4(b+1)+2}$ and $r_5 = 2^{4(b+1)+3}$, we get

$$\rho(n_2) = \rho(n_1) + 1, \quad \rho(n_3) = \rho(n_1) + 2, \quad \rho(n_4) = \rho(n_1) + 4, \quad \rho(n_3) = \rho(n_1) + 8.$$

The main theorem of Radon states:

Theorem 2.3.13. *For any integer n, there is a R-H system of order n having exactly $\rho(n) - 1$ elements.*

Proof. (following [24]) We will use elementary properties of the tensor product of matrices. Set

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 2.3.14. *1. $\{A\}$ is a R-H system of order 2 with $\rho(2) - 1$ elements;*

2. $\{A \otimes I_2, P \otimes A, Q \otimes A\}$ is a R-H system of order 4 with $\rho(4) - 1$ elements;

3. $\{I_2 \otimes A \otimes I_2, I_2 \otimes P \otimes A, Q \otimes Q \otimes A, P \otimes Q \otimes A, A \otimes P \otimes Q, A \otimes P \otimes P, A \otimes Q \otimes I\}$ is a R-H system of order 8 with $\rho(8) - 1$ elements.

Proof. By endurance and symmetry properties of A, P and Q with respect to $\otimes$. ♢

By remark 2.3.12, the lemma above handles the cases $n = 2t, 4t$ and $8t$ wit t odd: It is sufficient to tensor everything with I_t. We have the following two facts:

(a) If $\{M_1, \dots, M_s\}$ is a R-H system of order n, then $\{A \otimes I_n\} \cup \{Q \otimes M_i\}$ is a R-H system of order $2n$ with $s + 1$ elements;

(b) if $\{L_1, \dots, L_m\}$ is a R-H system of order k, then $\{A \otimes I_{nk}\} \cup \{P \otimes I_k \otimes M_i\} \cup \{Q \otimes L_j \otimes I_n\}$ is a R-H system of order $2nk$ with $s + m + 1$ elements.

Let, now, $n_1, \dots, n_5$ as in remark 2.3.12 and we proceed by induction: the starting step is the third point in the lemma above, since it gives the case $n_1 = 2^3$.

Point (a) gives us the transition from n_1 to n_2. Using point (b) with $k = n_1, n = 2$ (and, hence, by point 1. in the lemma, $s = 1$) we get the transition from n_1 to n_3. Again, using point (b), now with the second ad the third point in the lemma we get the transitions from n_1 to n_4 and n_1 to n_5. This concludes the proof. ♢

Corollary 2.3.15. *The sphere S^{n-1} admits at least $\rho(n) - 1$ orthonormal vector fields.*

Proof. By theorem 2.3.9, a R-H system of order n counting $\rho(n) - 1$ elements produces an orthogonal multiplication $\mathbb{R}^{\rho(n)} \times \mathbb{R}^n \to \mathbb{R}^n$; by theorem 2.3.8, this defines $\rho(n) - 1$ orthonormal vector fields over S^{n-1}. ♢

Remark 2.3.16. Thanks to remark 2.3.6, a R-H system of order n with $\rho(n) - 1$ elements gives a real linear space of invertible matrices (excepted 0, of course) of dimension $\rho(n)$. This space is, indeed, generated by the R-H system and the identity matrix.

Such space of matrices is an example of Clifford algebra (see, among others, *e.g.* [28], [23], [29]). Usually, Clifford algebras are the preferred instrument for proving Radon theorem 2.3.13 due to their strong connection with stable bundles.

Stable bundles over projective spaces

In this section we show how to calculate the groups of the reduced K-theory for the complex and real projective space (see *e.g.* [1], [28]).

First, remark that one can construct the groups of a cohomology theory by setting

$$K_{\mathbb{F}}^{-q}(X, Y) = \widetilde{K}_{\mathbb{F}}(S^q(X/Y)), \quad q \geq 0,$$

where X/Y denotes the space obtained by identifying the subset $Y \subset X$ to a new disjoint point. Clearly, $K_{\mathbb{F}}^0(X) := K_{\mathbb{F}}^0(X, \emptyset) = \widetilde{K}_{\mathbb{F}}(X \cup *) = K_{\mathbb{F}}(X)$; moreover, using Bott periodicity, we can extend this definition to positive exponents.

For $Y \subset X$ CW-complexes, it can be shown (by using the Puppe sequence) that $K_{\mathbb{F}}^n(X, Y)$ verifies all axioms of a *generalized cohomology theory* in the sense of Eilenberg and Steenrod [38], lacking only the dimension axiom. As a consequence, K-theory can be realized as the limit of the spectral sequence (which is natural with respect to maps):

$$\begin{cases} E_2^{p,q} = & \widetilde{H}^p(X, K_{\mathbb{F}}^q(*)) \\ E_\infty^{p,q} \Rightarrow & \widetilde{K}_{\mathbb{F}}^{p+q}(X) \end{cases} . \tag{2.3}$$

The filtration, having $E_\infty^{p,q}$ as graded groups, is defined using the $(p-1)$-skeleton X_{p-1} by $Im(\widetilde{K}_{\mathbb{F}}^n(X/X_{p-1}) \to \widetilde{K}_{\mathbb{F}}^n(X))$. The coefficient groups $K_{\mathbb{F}}^{-q}(*) = \widetilde{K}_{\mathbb{F}}(S^q)$ are given by examples 2.2.13 and 2.2.14.

Let us denote by $\xi_{\mathbb{F}}$ the tautological line bundle over the projective space $\mathbb{FP}^{d-1}$. Observe that the actions of $\mathbb{R}^*$ and $\mathbb{C}^*$, given by scalar multiplication over $\mathbb{C}^n = \mathbb{R}^{2n}$, induce a map

$$\varphi : \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1} \quad \text{such that } \varphi^*(\xi_{\mathbb{C}}) = c(\xi_{\mathbb{R}}) \tag{2.4}$$

(c is the complexification defined at the beginning of sect. 2.1).

Proposition 2.3.17. *The ring $K_{\mathbb{C}}(\mathbb{CP}^{d-1})$ is the truncated polynomial ring $\mathbb{Z}[u]/u^d$, where $u = [\xi_C] - 1$.*

Proof. The differentials in the page $E_2^{*,*}$ of the spectral sequence 2.3 are all zero, since $\widetilde{H}^{odd}(\mathbb{CP}^{d-1}, \mathbb{Z}) = 0$. Thus $E_2 = E_\infty$ and, in particular, one gets that, as groups, $K_{\mathbb{C}}(\mathbb{CP}^{d-1}) = \mathbb{Z}^{d-1}$. The ring structure is recovered, from the rational cohomology, by the Chern character (see *e.g.* [29]):

$$ch : K_{\mathbb{C}}(\mathbb{CP}^{d-1}) \to H^{ev}(\mathbb{CP}^{d-1}, \mathbb{Q}), \quad ch[F] = \sum_{i=0}^{\infty} \frac{c_i(F)}{i!}.$$

Proposition 2.3.18. *The ring $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1})$ is the (truncated) polynomial ring $\mathbb{Z}[\nu]$, with relations*

$$\nu^{g(d)+1} = 0, \quad \nu^2 = -2\nu$$

(so that $2^{g(d)}\nu = 0$), where $\nu = c([\xi_{\mathbb{R}}] - 1)$ and $g(d)$ is the integer part of $(d-1)/2$.

Proof. We get, $H^{odd}(\mathbb{RP}^{d-1}, \mathbb{Z}) = 0$ except, at most, $H^{d-1}(\mathbb{RP}^{d-1}, \mathbb{Z}) = \mathbb{Z}$ when $(d-1)$ is odd. In any case, the spectral sequence collapses at $E_2^{*,*}$. As groups, $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1}) = \mathbb{Z}_2^{g(d)}$; the ring structure is recovered via the map φ and from the fact that the bundle $\xi_{\mathbb{R}}$ verifies the relation $\xi_{\mathbb{R}} \otimes \xi_{\mathbb{R}} = \underline{\mathbb{R}}$, so that $(\nu+1)^2 = 1$. ◇

Proposition 2.3.19. *The ring $\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{d-1})$ is the (truncated) polynomial ring $\mathbb{Z}[\mu]$ with the relations*

$$\mu^{f(d)+1} = 0, \quad \mu^2 = -2\mu$$

(so that $2^{f(d)}\mu = 0$), where $\mu = [\xi_{\mathbb{R}}] - 1$ and $f(d)$ is the number of integers $i \equiv 0, 1, 2$ or 4 modulo 8 such that $0 < i < d$.

Proof. In the page $E_2^{*,*}$, the groups of total degree 0 ($i.e$ such that $p+q=0$) are all copies of $\mathbb{Z}_2$ (some $\mathbb{Z}$'s may appear in the page, but this is only in degrees $p=d-1$ and $q \equiv 0 \mod 4$ with $d-1$ odd, so that $p+q \neq 0$). Their number is equal to the number of nonzero coefficient groups $K_{\mathbb{R}}^{-q}(*)$ for $0<q<d$, that is $f(d)$.

Moreover, the complexification map

$$c : \widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{d-1}) \to \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1})$$

is surjective for any d, and injective for $d-1 \equiv 6, 7$ or 8 modulo 8: surjectivity comes from the fact that $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1})$ is generated by $c(\mu)$, while injectivity is a consequence of the fact that, for d as above, $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1})$ has the same number of elements than $\bigoplus_{p+q=0} E_2^{p,q}$. This implies that, for $d \equiv 0, 1, 7 \mod 8$, μ generates $\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{d-1})$.

For studying the other values of d we can argument as follows. In the page E_2, we have seen that there are exactly $f(d)$ copies of $\mathbb{Z}_2$ in total degree 0, and, when $d \equiv 0, 1, 7 \mod 8$, they remain unchanged to $E_\infty^{*,*}$. This forces them to remain unchanged also for the other values of d. In conclusion, we get that $\bigoplus_{p+q=0} E_2^{p,q} = \mathbb{Z}^{f(d)}$ is the graded group associated to $\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{d-1})$. The ring structure is recovered by the complexification map c. $\diamondsuit$

Example 2.3.20. The pictures 2.3 and 2.3 show the second pages of the spectral sequences for $\widetilde{K}_{\mathbb{F}}(\mathbb{RP}^9)$. Grey squares represent the trivial group and white ones stand for $\mathbb{Z}_2$ (or $\mathbb{Z}$ if $p=9$ and the coefficient group is $\mathbb{Z}$). In the highlightened diagonal are the groups of total degree 0, that is, those from which one recovers $\widetilde{K}_{\mathbb{F}}(-)$. Remark that, if $q = 6, 7$ or 8, the number of white squares in the diagonal is the same for both pages, as expected.

The function $f(d)$ of proposition (2.3.19) is strictly related to the Radon-Hurwitz number ρ of the previous section. Indeed, one has the following relation:

Proposition 2.3.21. *For any strictly positive integer m:*

$$m\mu = 0 \text{ in } \widetilde{K}_{\mathbb{F}}(\mathbb{RP}^{d-1}) \iff d \leq \rho(m).$$

Proof. We can show this as follows: in order to be $m\mu = 0$, m must be a positive multiple of $2^{f(d)}$, say $m = 2^{f(d)}2^k odd$, thus $\rho(m) = \rho(2^{f(d)+k}) \geq \rho(2^{f(d)})$ since $\rho(2^n)$ is a nondecreasing sequence. Now, assume that $f(d)$ decomposes as $a + 4b$ with $0 \leq a \leq 3$, then, by the definition of $f(d)$, one gets $d \leq 8b + 2^a$. The last number, on the other hand, is exactly the expression of $\rho(2^{f(d)})$. $\diamondsuit$

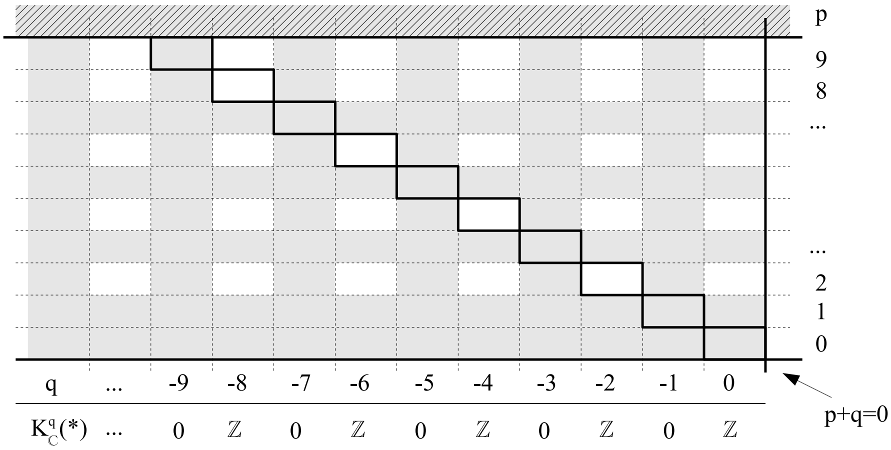

Figure 2.1. The page $E_2^{p,q} = \widetilde{H}^p(\mathbb{RP}^9, K^q_{\mathbb{C}}(*))$.

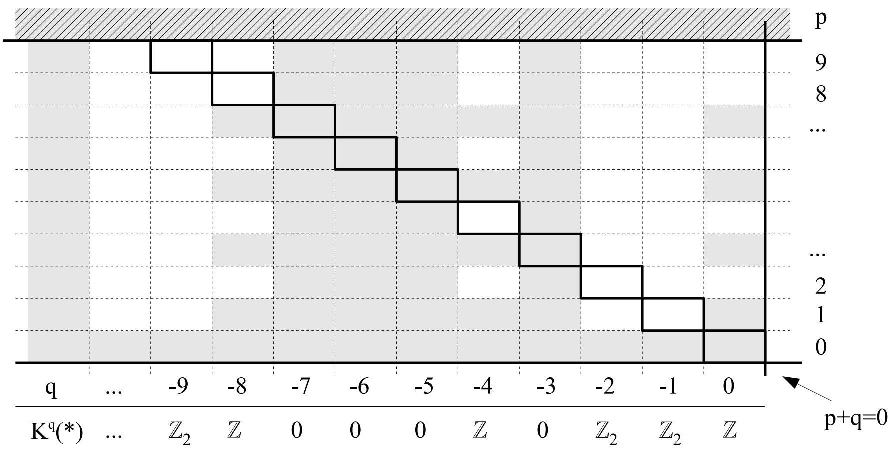

Figure 2.2. The page $E_2^{p,q} = \widetilde{H}^p(\mathbb{RP}^9, K^q_{\mathbb{R}}(*))$.

A summarizing theorem

In his famous paper [1], Adams shows that the existence of k linearly independent vector fields over S^{n-1} is equivalent to require the existence of a map $\mathbb{RP}^{m+k}/\mathbb{RP}^{m-1} \to S^m$, with $m =$ odd multiple of n, such that the composition

$$S^m = \mathbb{RP}^m/\mathbb{RP}^{m-1} \to \mathbb{RP}^{m+k}/\mathbb{RP}^{m-1} \to S^m$$

has degree $+1$. By a deep use of cohomological operations in K-theory, Adams also shows that this can happen only if $k < \rho(m) = \rho(n)$, that is:

Theorem 2.3.22. *The maximum number of linearly independent vector fields over S^{n-1} is $\rho(n) - 1$.*

This theorem settles the question about the relations between the maximum number of independent vector fields over a sphere S^{n-1} and the maximum number of linearly independent $n \times n$ matrices such that any their nonzero linear combination is invertible.

In conclusion, we can summarize all the discussions of the present chapter by the following Theorem (Friedland, Robbin and Sylvester: [23]):

Theorem 2.3.23. *For integers $k, n > 0$, the following propositions are equivalent:*

1. *$k \leq \rho(n)$;*
2. *there exist $k-1$ orthonormal vector fields over S^{n-1};*
3. *there exists a R-H system of order n counting $k-1$ elements;*
4. *there is a k-dimensional real linear space in $GL_n(\mathbb{R}) \cup \{0\}$;*
5. *there is a map $\phi : S^{k-1} \to GL_n(\mathbb{R})$ such that $\phi(-x) = -\phi(x)$;*
6. *the bundle $n\xi_{\mathbb{R}}$ over $\mathbb{RP}^{k-1}$ is trivial.*

Proof. the equivalence of points from 1 to 3 is just a synthesis of the theorems of Radon (2.3.11) and Adams (2.3.22), and of the arguments used to prove them.

Moreover, implications $3 \Rightarrow 4 \Rightarrow 5$ are trivial. Let us prove $5 \Rightarrow 6 \Rightarrow 1$.

First, remark that a map ϕ as in point 5, covariant with respect to the multiplication by -1, induces a map $\phi_{\mathbb{P}} : \mathbb{RP}^{k-1} \to \mathbb{P}(GL_n(\mathbb{R})) = GL_n(\mathbb{R})/\mathbb{R}^*$. Using ϕ, define a map of bundles over $\mathbb{RP}^{k-1}$:

$$\Phi : \underline{\mathbb{R}}^n \longrightarrow n\xi_{\mathbb{R}}$$

by setting, locally, $\Phi([x], v) = ([x], \phi(x)v)$. This is an isomorphism of vector bundles, since $\phi(x)$ is an invertible matrix for any x.

By looking at the equation $n\xi_{\mathbb{R}} = \underline{\mathbb{R}}^n$ in the ring $\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{k-1})$, we get that n times the ring generator is 0, so that $k \leq \rho(n)$ by proposition 2.3.21. $\diamondsuit$

Remark 2.3.24. Statement 5 of the theorem above can be interpreted as a *non-linear* equivalent of the existence of a vector pace of invertible matrices.

Furthermore, let us point out that that the proof of implications $5 \Rightarrow 6 \Rightarrow 1$ is different from the ones appearing in [23]: the argument that we use is more in the spirit of this presentation and has the advantage of being generalizable with ease.

Remark 2.3.25. Theorem 2.3.23 admits can be extended, for instance to the complex and the quaternionic case, as well as to their corresponding hermitian versions. Later on, we will use some of those generalization (*e.g.* Theorem 4.1.1).

Chapter 3

SPACES OF CONSTANT RANK MATRICES

In the present and in the next chapter, we deal with the following classical problem introduced in Chapter 1: *given a set $\mathcal{X}$ of real or complex matrices, determine the maximal dimension $d(\mathcal{X})$ of a real vector space V such that $V \setminus \{0\} \subset \mathcal{X}$.*

Using the the theory of stable vector bundles and of homotopy of the classical groups, in this chapter we give a very sharp estimate on $d(\mathcal{X})$ when $\mathcal{X}$ is the set of real symmetric matrices with some constant rank condition. Furthermore, we generalize our results to the case of complex hermitian matrices. Real skew-symmetric matrices are also taken into account, using a slightly different method. Some of the main results of this chapter can be found in [16].

3.1 Preliminaries and statement of the results

The assumption of constant rank allows to describe vector spaces of matrices with precision using vector bundles defined by kernels and cokernels of morphisms.

Odd maps and K-theory

In the proof of theorem 2.3.23, we sketched a way to relate vector bundles and, consequently, K-theory, to the dimension of a vector space of matrices. In the following, we would like to make this approach more explicit.

We recall that by *odd* (and, respectively, *complex odd*) map we denote a map $\phi : S^{d-1} \to \mathcal{M}_{m,n}(\mathbb{R})$ (resp. $\phi : S^{d-1} \to \mathcal{M}_{m,n}(\mathbb{C})$) such that $\phi(-x) = -\phi(x)$.

Proposition 3.1.1. *Any odd (complex odd) map ϕ induces a morphism of vector bundles*

$$\Phi : \underline{\mathbb{R}}^n \to m\xi_{\mathbb{R}} \quad (\Phi : \underline{\mathbb{C}}^n \to m\xi_{\mathbb{R}} \otimes \underline{\mathbb{C}}) \quad \textit{over} \quad \mathbb{RP}^{d-1}$$

where $\xi_{\mathbb{R}}$ is the canonical bundle, by defining locally $\Phi([x], v) = ([x], \phi(x)v)$.

Proof. We show the complex case, the real one being completely analogous. Consider the map $\Phi' : S^{d-1} \times \mathbb{C}^n \to S^{d-1} \times \mathbb{C}^m$ defined by $\Phi'(x, v) = (x, \phi(x)v)$. It is equivariant with respect to the actions of $\mathbb{R}^*$ given, for $\lambda \in \mathbb{R}^*$, by

$$f_\lambda(x,v) = \left(\frac{\lambda}{|\lambda|}x, v\right) \quad \text{and} \quad g_\lambda(x,v) = \left(\frac{\lambda}{|\lambda|}x, \lambda v\right),$$

thus defines a morphism of complex bundles over the real projective space $\mathbb{RP}^{d-1}$.

In particular, note that, under the isomorphism $\mathbb{C} = \mathbb{R}^2$, the the quotient space of $S^{d-1} \times \mathbb{R}^{2m}$ under the action g is $m(\xi_\mathbb{R} \oplus \xi_\mathbb{R}) = m\xi_\mathbb{R} \otimes \underline{\mathbb{C}}$. ♢

Remark 3.1.2. An interesting case is when ϕ comes from the inclusion of a real vector space: $V\setminus\{0\} \subset \mathcal{M}_{m,n}(\mathbb{R})$, $\dim V = d$. With the choice of a scalar product on V, we can construct an odd map $\phi : S^{d-1} \to \mathcal{M}_{m,n}(\mathbb{R})$ by restricting the inclusion to the vectors of norm 1; this map is $\phi(x) = x$. The same happens for complex matrices.

The aim is to extend the morphism Φ to an exact sequence of vector bundles in order to read it in K-theory, or in some other cohomology ring. This can certainly be done when $\phi(x)$ has constant rank.

Proposition 3.1.3. *Assume ϕ odd (complex odd); if $\phi(x)$ has rank k for any x, then there are exact sequences of vector bundles over $\mathbb{RP}^{d-1}$:*

$$0 \to K \to \underline{\mathbb{R}}^n \xrightarrow{\Phi} m\xi_\mathbb{R} \to C \to 0 \tag{3.1}$$
$$(0 \to K \to \underline{\mathbb{C}}^n \xrightarrow{\Phi} m\xi_\mathbb{R} \otimes \underline{\mathbb{C}} \to C \to 0).$$

Moreover, the following isomorphisms hold:

$$K \oplus m\xi_\mathbb{R} = \underline{\mathbb{R}}^n \oplus C \qquad (K \oplus m\xi_\mathbb{R} \otimes \underline{\mathbb{C}} = \underline{\mathbb{C}}^n \oplus C). \tag{3.2}$$

Proof. Since the rank of $\phi(x)$ is constant, Φ defines a kernel and a cokernel bundle that we call, respectively, K and C.

The isomorphism (3.2) is a consequence of the fact that any exact sequence of bundles splits. ♢

Let us show, as an example, how to recover, in this context, the result of Adams, Lax and Phillips [3] about complex invertible matrices:

Proposition 3.1.4. *If V is a real vector space such that $V \setminus \{0\} \subset GL_n(\mathbb{C})$, then* $\dim V \leq \rho_\mathbb{C}(n)$.

Proof. Set $d = \dim V$. As observed in example 3.1.2, the inclusion of V induces the "identity" map; in this case ($n = m$ and $K = C = \underline{0}$), isomorphism

(3.2) becomes $\mathbb{C}^n = n\xi_{\mathbb{R}} \otimes \underline{\mathbb{C}}$. This equation means that n times the generator ν of the ring $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1})$ is zero. Then, by proposition 2.3.18, $n\nu = 0$ implies that $n = 2^{g(d)}2^k(2c+1)$ for some $k \geq 0$. Applying v_2 to both sides of the last equation we get: $v_2(n) = g(d) + k \geq [(d-1)/2] \geq d/2 - 1$, that is $d \leq 2v_2(n) + 2$. ♢

From the proof of proposition 3.1.4 we get a relation involving complex K-theory and the Radon-Hurwitz number $\rho_{\mathbb{C}}$ (see definition 2.2), which is interesting on its own:

Proposition 3.1.5. *The following equivalence holds for any integer $m > 0$:*

$$m\nu = 0 \text{ in } \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1}) \quad \text{if and only if} \quad d \leq \rho_{\mathbb{C}}(m).$$

Compare this with proposition 2.3.21.

Remark 3.1.6. The idea of using stable bundles (or, more generally, the K-theory) to estimate the dimension of linear spaces of matrices has been used by many authors (see *e.g.* [36], [41], [31]).

Statement of the results

For integers n and $k \leq n$, in the following, let us denote by $\mathcal{S}_n^{n-k} \subset \mathcal{M}_{n,n}(\mathbb{R})$ the space of $n \times n$ real symmetric matrices with rank $n-k$. In a similar way, denote by $\mathcal{H}_n^{n-k}$ and $\mathcal{A}_n^{n-k}$ the corresponding spaces of complex hermitian and real skew-symmetric matrices.

Our aim is to estimate

$$d(\mathcal{S}_n^{n-k}) = \max\{\dim V \mid V \setminus \{0\} \subset \mathcal{S}_n^{n-k},\ V \text{ real vector space }\} \tag{3.3}$$

as well as the corresponding $d(\mathcal{H}_n^{n-k})$ and $d(\mathcal{A}_n^{n-k})$ defined for complex hermitian and real skew-symmetric matrices.

Set:

$$\sigma(n, n-k) = \max_{0 \leq r \leq k} \rho\left(\frac{n-k}{2} + r\right) \tag{3.4}$$

and define $\sigma_{\mathbb{C}}(n, n-k)$ in a completely analogous way, using $\rho_{\mathbb{C}}$ instead of ρ; moreover, define

$$\rho(n, n-k) = \max_{-k \leq i \leq k} \rho(n+i).$$

Theorem 3.1.7. *If $0 \leq k \leq 4$, then*

$$\sigma(n, n-k) \leq d(\mathcal{S}_n^{n-k}) \leq \max\{c_k, \sigma(n, n-k)\} + 1$$

where c_k depends only on k and has values: $c_0 = c_1 = 0$, $c_2 = 3$ and $c_3 = c_4 = 13$.

If $\sigma(n, n-k) = \rho\left(\frac{n-k}{2}\right) \geq c_k$*, the upper bound is attained. Moreover, if* $k = 1$ *and* $(n+1)/2 = 2^{a+4b}(2c+1)$ *with* $a = 2, 3$ *and arbitrary* b *and* c*, then the lower bound is optimal.*

Theorem 3.1.8. *For* $k = 0, 1$*, then*

$$\sigma_{\mathbb{C}}(n, n-k) \leq d(\mathcal{H}_n^{n-k}) \leq \sigma_{\mathbb{C}}(n, n-k) + 1.$$

If $\sigma_{\mathbb{C}}(n, n-k) = \rho_{\mathbb{C}}\left(\frac{n-k}{2}\right)$*, the upper bound is attained, otherwise, the lower bound is optimal.*

Theorem 3.1.9. *Let* c_k *be as in Theorem 3.1.7; if* $0 \leq k \leq 3$*, then*

$$\max\{\rho(n-k), \rho(n-k+2)\} \leq d(\mathcal{A}_n^{n-k}) + 1 \leq \max\{c_k + 1, \rho(n, n-k)\}.$$

Remark 3.1.10. The proof of Theorem 3.1.7 is the argument of the next section, which is organized in the following way.

The first part is devoted to determine the upper bound (proposition 3.2.6), and this is done by exploiting the symmetry of the matrices in terms of a peculiar decomposition of the sequence 3.1; the hint for finding such decomposition has been given by Friedland *et. al.* in [23]. It is worthy to remark that the number c_k, appearing in the statement of the theorem, is a quantity that arises when determining the stable class of the kernel bundle of sequence 3.1.

Then, we prove the lower bound (corollary 3.1) by exhibiting suitable vector spaces of matrices: in this case, we adapt to our purposes the constructions made by Adams *et al.* ([3], Radon [35] and Lam *et al.* [31]).

A third part aims to describe, at least partially, in which cases the bounds are reached (table 3.5 and proposition 3.2.15): in particular, for the case of rank $n-1$, a (perhaps generalizable) stable homotopy argument is used.

Remark 3.1.11. Theorems 3.1.8 and 3.1.9 are the topic of the third section. The hermitian case is studied as a direct generalization theorem 3.1.7; the reason is that hermitian matrices induce a decomposition of 3.1 extremely similar to the one induced by symmetric matrices.

Real skew symmetric matrices, on the other hand, are studied by explicit constructing and odd map taking values to the space of real invertible matrices.

3.2 Real symmetric matrices

In order to determine $d(\mathcal{S}_n^{n-k})$, first, let us argument as in the proof of proposition 3.1.4: suppose V is a d-dimensional real vector space such that $V \setminus \{0\} \subset \mathcal{S}_n^{n-k}$ and apply remark 3.1.2 and proposition 3.1.1 to construct

$$0 \to K \to \underline{\mathbb{R}}^n \xrightarrow{\Phi} n\xi_{\mathbb{R}} \to C \to 0 \quad \text{over } \mathbb{RP}^{d-1} \tag{3.1}$$

with $\Phi([A], v) = ([A], Av)$, $\operatorname{rank} K = \operatorname{rank} C = k$. The consequent relation in reduced K-theory is $n\mu = [C] - [K]$. We can exploit symmetry in this way: dualizing the above sequence and tensoring it with $\xi_{\mathbb{R}}$ gives:

$$0 \to C^* \otimes \xi_{\mathbb{R}} \to \underline{\mathbb{R}}^n \xrightarrow{\Psi} n\xi_{\mathbb{R}} \to K^* \otimes \xi_{\mathbb{R}} \to 0.$$

The bundle morphism Ψ is given by $\Psi([A], v) = ([A], {}^t Av)$ so that symmetry of matrices implies $\Psi = \Phi$ and, consequently, $C = K^* \otimes \xi_{\mathbb{R}}$. Moreover, the choice of a metric on K induces a perfect duality $K = K^*$, and we get the stable vector bundle equation: $n\mu = [K]\mu$.

Unfortunately, this last equation alone does not yields a good estimate for $d(\mathcal{S}_n^{n-k})$: for instance, when $k = 1$, this would imply $d \leq \max\{\rho(n-1), \rho(n+1)\}$ (depending on whether $K = \underline{\mathbb{R}}$ or $K = \xi_{\mathbb{R}}$), that is the same estimate for general matrices of rank $n - 1$ without taking into account symmetry (for general matrices of constant high rank, see *e.g.* [36]). However, in the following we will prove that a much sharper bound holds.

Symmetric decomposition

In this part we show how to describe the symmetry of matrices in terms of (stable) vector bundles in a more subtle way. As before, assume that $V \setminus \{0\} \subset \mathcal{S}_n^{n-k}$ and $\dim V = d$.

Let us begin with a simple fact:

Proposition 3.2.1. *For V as above, either* $\dim V = 1$ *or any nonzero matrix $A \in V$ has exactly $(n-k)/2$ positive (and negative) eigenvalues.*

Proof. Suppose $\dim V \geq 2$; then, $V \setminus \{0\}$ is path-connected. For any matrix A in $V \setminus \{0\}$, take a path γ joining $A = \gamma(0)$ with $-A = \gamma(1)$. Call $e_{\pm}(t)$ the number of positive (negative) eigenvalues of $\gamma(t)$. The signature $(e_+(t), e_-(t))$ is constant, since otherwise the rank would drop. This means that $e_+(0) = e_+(1)$ but, on the other hand, we know that $e_+(1) = e_-(0)$ so that $e_+(t) = e_-(t) = (n-k)/2$ for any t. ♢

Remark 3.2.2. Note that proposition 3.2.1 implies that, if $n - k$ is odd, then $\dim V = 1$. This V is the space generated by the scalar multiples of any matrix of rank $n - k$.

In the following, assume that $n-k$ is even, so that $d \geq 2$. The inclusion $V \setminus \{0\} \subset \mathcal{S}_n^{n-k}$ induces two interesting bundles over $S^{d-1} \subset \mathcal{S}_n^{n-k}$.

We say that an eigenvector is positive if it is relative to a positive eigenvalue and define negative eigenvectors in the corresponding way. Construct the vector bundle $E^+ \to S^{d-1}$ by requiring that the fibre of E^+ over each matrix A is the real vector space generated by the positive eigenvectors of A. Equivalently, E^+_A is the subspace of $\mathbb{R}^n$ where the quadratic form ${}^t vAv$ is positive definite. The vector bundle E^- is constructed as E^+, using negative eigenvectors. By proposition 3.2.1, $\operatorname{rank} E^+ = \operatorname{rank} E^- = (n-k)/2$.

Proposition 3.2.3. *The vector bundles E^+ and E^- are isomorphic.*

Proof. Since a positive eigenvector of A is a negative one for $-A$, we have that the multiplication by -1 on S^{d-1} lifts to an automorphism M of $E^+ \oplus E^-$ that interchanges the summands. By $M|_{E^-} \circ M|_{E^+} = Id : E^+ \to E^+$, we get the statement. $\diamondsuit$

Call $\pi : S^{d-1} \to \mathbb{RP}^{d-1}$ the quotient induced by the antipodal involution on the sphere and consider the pullback via π of the sequence 3.1. Since the pullback of Φ is the map $(A, v) \mapsto (A, Av)$, we immediately get

$$\pi^* \underline{\mathbb{R}}^n = \pi^* K \oplus E^+ \oplus E^-. \tag{3.2}$$

Proposition 3.2.4. *If E^+ is a trivial vector bundle, then*

$$E^+ \oplus E^- = s\,\pi^*(\underline{\mathbb{R}} \oplus \xi_{\mathbb{R}}) \quad \textit{with } s = \frac{n-k}{2}.$$

Proof. Since E^+ is trivial, it admits everywhere independent sections $v_1^+, \dots v_s^+$. Using these sections, construct a trivialization of E^- by setting $v_i^-(A) = v_i^+(-A)$, $i = 1, \dots, s$. Therefore, define new sections of $E^+ \oplus E^-$ by setting

$$r_i = v_i^+ + v_i^- \quad \text{and} \quad \tau_i = v_i^+ - v_i^- \qquad i = 1, \dots, s.$$

These new sections decompose $E^+ \oplus E^-$ as a sum of $2s$ trivial line bundles, say $R_1, \dots, R_s$ (the ones induced by r_i) and $T_1, \dots T_s$ (the ones induced by τ_i).

Since $r_i(A) = r_i(-A)$, each R_i is the pullback via π of a trivial line bundle $\underline{\mathbb{R}} \to \mathbb{RP}^{d-1}$. Moreover, $\tau_i(A) = -\tau_i(-A)$ implies that each T_i is the pullback of a copy of the canonical bundle $\xi_{\mathbb{R}}$. This concludes. $\diamondsuit$

Remark 3.2.5. Equation 3.2 together with proposition 3.2.4 imply, then, that

$$[\mathbb{R}^n] = \frac{n-k}{2}\,[\xi_{\mathbb{R}} \oplus \mathbb{R}] + [K] \quad \text{in } \widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{d-1}). \tag{3.3}$$

Of course, the same equation can be derived from the sequence (3.1), that becomes:

$$0 \to K \to s\xi_{\mathbb{R}} \oplus \underline{\mathbb{R}}^s \oplus K \xrightarrow{\Phi} n\xi_{\mathbb{R}} \to K \otimes \xi_{\mathbb{R}} \to 0, \qquad s = \frac{n-k}{2}.$$

Nonexistence of vector spaces

Let $d(\mathcal{S}_n^{n-k})$ and $\sigma(n, n-k)$ as defined, respectively, in (3.3) and (3.4). In this part we prove the following estimates:

Proposition 3.2.6. (Upper bounds for $d(\mathcal{S}_n^{n-k})$) *If k is integer and $0 \leq k \leq 4$, then*

$$d(\mathcal{S}_n^{n-k}) \leq \begin{cases} \sigma(n, n-k) + 1 & \text{if } k = 0, 1 \\ \max\{3, \sigma(n, n-k)\} + 1 & \text{if } k = 2 \\ \max\{13, \sigma(n, n-k)\} + 1 & \text{if } k = 3, 4 \end{cases}$$

Proof. By remark 3.1.2 we can suppose that $n-k$ is even, otherwise $d(\mathcal{S}_n^{n-k}) = 1$ and the statement is trivially true. Assume, then, that $d(\mathcal{S}_n^{n-k}) = d+1$ with $d \geq 1$.

Now consider the vector bundles E^+ and E^- over S^d which arise from the inclusion $S^d \subset \mathcal{S}_n^{n-k}$. We do not know if they are trivial or not, but, certainly, they become trivial when they are restricted to $S^{d-1} \subset S^d \setminus \{point\} \subset S^d$, since $S^d \setminus \{point\}$ is contractible.

Thus, we can apply proposition 3.2.4 and the subsequent remark 3.2.5 to the restriction $E^+ \to S^{d-1}$. In return, we obtain the equation:

$$\frac{n-k}{2}\,\mu = k - [K] \quad \text{in } \widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{d-1}) \tag{3.4}$$

where μ is the generator $[\xi_{\mathbb{R}}] - 1$. To obtain the estimates we need, it remains to determine the stable class of the kernel K in terms of μ, so that we can apply proposition 2.3.21. In order to do this, we use the classification of stable bundles of low geometric dimension provided by Adams in [2]. We distinguish some cases:

- If $k = 0$ or 1, then $[K] = r\mu$ for some $0 \leq r \leq k$. Consequently, equation 3.4 becomes:
$$\left(\frac{n-k}{2} + r\right)\mu = 0, \qquad \text{for some } 0 \leq r \leq k,$$
so that, certainly, $d \leq \sigma(n, n-k)$, thanks to proposition 2.3.21.

- When $k = 2$, we still have $[K] = r\mu$ for some $0 \leq 2 \leq k$, but provided $d \geq 4$. So, either $d \leq 3$ or $d \leq \sigma(n, n-2)$.

- Similarly to the previous case, when $k = 3$ and $d \geq 14$, any bundle of geometric dimension ≤ 3 has stable class of the form $r\mu$ for some $0 \leq r \leq 3$. This yields: $d \leq 13$ or $d \leq \sigma(n, n-3)$.

- For $k = 4$, again, according to Adams ([2], Thm. 7), the stable class of K is one of $0, \mu, \ldots, 4\mu$, provided that $d \geq 14$. This implies $d \leq \{13, \sigma(n, n-4)\}$.

$\diamondsuit$

Remark 3.2.7. The study of the geometric dimension of stable vector bundles over the projective space has been completely carried out in some recent works [18], [17]. From such classification, it results that the stable geometric dimension only depends on the order of the bundle, in the ring of reduced K-theory. This could probably be used for studying the dimension of spaces of matrices with constant rank and arbitrary large null-space.

Remark 3.2.8. The case $k = 0$ in the theorem above, is classical (see Adams [3]). Our proof, however, involves different arguments.

Examples of vector spaces

In this part we give an inductive method for constructing vector spaces of $n \times n$ real symmetric matrices with arbitrary rank $n - k$. In particular, for $0 \leq k \leq 4$, this will provide a lower bound for $d(\mathcal{S}_n^{n-k})$. Moreover, in some cases this bound will be optimal. As before, assume that $n - k$ is even, set $s = (n - k)/2$ and let $\sigma(n, n - k)$ be as in (3.4).

We begin by showing:

Proposition 3.2.9. *For any $k \leq n$ there exists a real vector space $V \setminus \{0\} \subset \mathcal{S}_n^{n-k}$ of dimension $\rho(s) + 1$.*

Proof. Recall, from theorem 2.3.13 and the subsequent remark, that there exists an $\rho(s)$-dimensional vector space of $s \times s$ invertible matrices; call $M_1, \dots, M_{\rho(s)}$ a R-H system spanning it (with $M_{\rho(s)} = I$). Now construct the matrices $S_1, \dots, S_{\rho(s)+1} \in \mathcal{S}_n^{n-k}$ by setting

$$S_i = \begin{pmatrix} O_s & M_i & 0 \\ {}^tM_i & O_s & 0 \\ 0 & 0 & O_k \end{pmatrix}, \quad 1 \leq i \leq \rho(s) \quad S_{\rho(s)+1} = \begin{pmatrix} I & O_s & 0 \\ O_s & -I & 0 \\ 0 & 0 & O_k \end{pmatrix}$$

where O_t is the $t \times t$ zero matrix.

These symmetric matrices verify the (Clifford-like) relations

$$S_i S_j + S_j S_i = 0, \qquad {}^tS_i S_i = S_i^2 = \begin{pmatrix} I_{2s} & 0 \\ 0 & O_k \end{pmatrix}$$

so any nonzero linear combination of the S_i has also rank $n - k = 2s$. $\diamondsuit$

To construct further examples, we need some preliminary considerations:

Lemma 3.2.10. *If A is a $i \times (i+j)$ matrix of full rank i, then the matrix* $B = \begin{pmatrix} 0 & A \\ {}^tA & 0 \end{pmatrix}$ *has rank $2i$.*

Proof. The first i columns of B are linearly independent and, among the last $i+j$ columns of B, there are i of them which are independent. Moreover, each column of the first group is independent from any one of the second group, since there are zeroes. ♢

Let us recall a (part of a) theorem of Lam and Yiu ([31],Thm. 1):

Theorem 3.2.11. *The maximal dimension of a real vector space of $i \times (i+j)$ real matrices of full rank i is greater or equal than $\rho = \max\{\rho(i), \rho(i+1), \ldots, \rho(i+j)\}$. Moreover, the inequality is an equality if $\rho \geq 9$.*

Proof. One obtains a vector space of dimension $\rho(i+r)$ for any $0 \leq r \leq j$, as follows. Let A be the generic element of a $\rho(i+r)$-dimensional space of $\rho(i+r) \times \rho(i+r)$ real matrices (given by theorem 2.3.13 and the subsequent remark). Cancel the last r rows of A and add $j-r$ zero columns at the end of what remains. The $i \times (i+j)$ matrix obtained in this way, still depends on $\rho(i+r)$ parameters and has full rank. For the details, and the proof of the second statement, we refer to the original paper. ♢

Now, we can construct the following spaces of symmetric matrices:

Proposition 3.2.12. *For any $k \leq n$ there exists a real vector space $V \setminus \{0\} \subset \mathcal{S}_n^{n-k}$ of dimension greater or equal than $\sigma(n, n-k)$.*

Proof. Consider the vector space of matrices of the form $B = \begin{pmatrix} 0 & A \\ {}^tA & 0 \end{pmatrix}$, where A is a $s \times (s+k)$ real matrix of rank s. Then, by theorem 3.2.11, we can find such A depending linearly on at least $\sigma(n, n-k) = \max\{\rho(s), \ldots, \rho(s+k)\}$ real parameters. Moreover, lemma 3.2.10 says that B has rank $2s = n-k$. ♢

Summarizing, we get:

Theorem 3.1. (Lower bounds for $dd(\mathcal{S}_n^{n-k})$) *For any n and $k \leq n$,*

$$d(\mathcal{S}_n^{n-k}) \geq \sigma(n, n-k).$$

Moreover, if $\sigma(n, n-k) = \rho\left(\frac{n-k}{2}\right)$, *then* $d(\mathcal{S}_n^{n-k}) \geq \rho\left(\frac{n-k}{2}\right) + 1$.

Remark 3.2.13. As we pointed out at the beginning of this section, there are cases for which the lower and the upper bound of $d(\mathcal{S}_n^{n-k})$ coincide. We make this explicit. Let $0 \le k \le 4$ and recall the critical values c_k in theorem 3.1.7; In order to make sure that the lower bound is equal to the upper bound, we must require that, for fixed k, both the following conditions hold:

$$\sigma(n, n-k) = \rho\left(\frac{n-k}{2}\right) \qquad \text{and } \sigma(n, n-k) \ge c_k.$$

This happens for the values of n listed below:

k	$\sigma(n,n-k) = \rho\left(\frac{n-k}{2}\right)$	$\sigma(n,n-k) = \rho\left(\frac{n-k}{2}\right) \ge c_k$
0	any n	any n
1	$n-1 \equiv 0 \mod 4$	any n
2	$n-2 \equiv 0 \mod 8$	$n \ge 10$
3	$n-3 \equiv 0 \mod 8$	$n \ge 2^8+3$
4	$n-4 \equiv 0 \mod 16$	$n \ge 2^8+4$

(3.5)

Proposition 3.2.14. *If k and n are as in the table (3.5), then*

$$d(\mathcal{S}_n^{n-k}) = \rho\left(\frac{n-k}{2}\right) + 1.$$

The case of constant rank $n-1$

We would like to refine the estimates we proved, on the case of constant rank $n-1$.

Remark that, from proposition 3.2.14 and remark 3.1.2, we know the exact value of $d(\mathcal{S}_n^{n-1})$ for all n, except the case $n \equiv 3 \mod 4$. We would like to fill in this missing case.

Assuming $n \equiv 3 \mod 4$ is equivalent to require $\sigma(n, n-1) = \rho\left(\frac{n+1}{2}\right)$ and $(n+1)/2 = 2^a(2c+1)$ with $a \ge 1$. We prove the following:

Proposition 3.2.15. *Assume that $a = 2+4d$ or $a = 3+4d$ (that is, $\rho(\frac{n+1}{2}) = 4+8d$ or $8+8d$) for some integer d; then, $d(\mathcal{S}_n^{n-1}) = \rho(\frac{n+1}{2})$.*

Thanks to the lower bound inequality 3.1, it is sufficient to prove that the case, given by theorem 3.2.6, $d(\mathcal{S}_n^{n-1}) = \sigma(n, n-1) + 1 = \rho(\frac{n+1}{2}) + 1$ does not hold. We show this by contradiction; assume $d(\mathcal{S}_n^{n-1}) = \sigma(n, n-1) + 1 = \rho(\frac{n+1}{2}) + 1$ and denote this number by $r+1$. The Lemma 3.2.3 gives isomorphic bundles $E^{\pm}$ of rank $\frac{n-1}{2}$ over S^r. If we can show that these bundles are trivial, then Proposition 3.2.4 and remark 3.2.5 would imply the relation

$$\frac{n-1}{2}(\underline{\mathbb{R}} \oplus \xi_{\mathbb{R}}) \oplus K = \underline{\mathbb{R}}^n \qquad \text{over } \mathbb{P}^r.$$

Reading such equation in K-theory, gives the contradiction $r+1 \leq \sigma(n,n-1) = r$.

It remains to prove that, under the hypothesis $d(\mathcal{S}_n^{n-1}) = \sigma(n,n-1)+1 = \rho(\frac{n+1}{2})+1$, the bundles $E^{\pm}$ are trivial.

Recall (see Chapter 2) that real bundles of rank k over S^r, with $r \geq 2$, are classified up to isomorphism by the homotopy groups $\pi_{r-1}(SO(k))$. If A and B are maps $S^{r-1} \to SO(k)$ representing bundles F and G, the map representing $F \oplus G$ is $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = A + B \in \pi_{r-1}(SO(rkF + rkG))$. This is a consequence of the fact that $SO(k)$ is a topological group.

Now, we can show the following two lemmas, corresponding to the cases we are dealing with.

Lemma 3.2.16. *If $r = 4+8d$, then E^+ and E^- are trivial vector bundles.*

Proof. We show that the map $i : \pi_{r-1}(SO(\frac{n-1}{2})) \to \pi_{r-1}(SO(n-1))$ induced by the inclusion of spaces is injective and the target group is isomorphic to $\mathbb{Z}$. This will conclude the proof, since $E^+ \oplus E^- = E^+ \oplus E^+ = \underline{\mathbb{R}}^{n-1}$ and if e represents E^+ we will get $2i(e) = 0$.

Observe that $r = 4+8d$ is equivalent to $\frac{n+1}{2} = 2^{2+4d}\gamma$ with γ odd, hence the above map is

$$\pi_{3+8d}(SO(16^d 4\gamma - 1)) \longrightarrow \pi_{3+8d}(SO(n-1))$$

and is a composition of isomorphisms provided $16^d 4\gamma - 1 > 3+8d+1$ that is $\gamma \neq 1$ and $d \neq 0$; moreover, all those groups are isomorphic to $\mathbb{Z}$ thanks to Bott periodicity and the fact that $\pi_3(SO(k)) = \mathbb{Z}$ stably.

Then, take $d = 0$ and $\gamma = 1$. The corresponding map is the composition

$$\pi_3(SO(3)) \xrightarrow{i} \pi_3(SO(4)) \xrightarrow{j} \pi_3(SO(5)) \to \pi_3(SO(6));$$

the last arrow is a stable isomorphism $\mathbb{Z} \to \mathbb{Z}$, thus we only need to show that ji is not zero. Computing the exact homotopy sequence of $SO(3) \to SO(4) \xrightarrow{p} S^3$ shows that $\pi_3(SO(3)) = \mathbb{Z}$, $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ and i is injective. Moreover, $Imi = \ker p_*$ and $\ker j = Im\partial$ where ∂ is the injective boundary in the sequence $\pi_4(S^4) \xrightarrow{\partial} \pi_3(SO(4)) \xrightarrow{j} \pi_3(SO(5)) \to \pi_3(S^4) = 0$. In ([28]) it is shown that $Im\partial$ is generated by the characteristic map $c : S^3 \to SO(4)$ of the principal bundle associated to the tangent bundle of S^4. It is also shown that the composition $pc : S^3 \to S^3$ has degree 2. This forces $\pi_4(S^4) \xrightarrow{\partial} \pi_3(SO(4)) \xrightarrow{p_*} \pi_3(S^3)$ to be the multiplication by 2, indeed $p_*\partial([id]) = p_*([c]) = [pc]$. Hence $Im\partial \cap \ker p_* = \{0\}$, so ji is not zero. $\diamondsuit$

Lemma 3.2.17. *If $r = 8+8d$, then E^+ and E^- are trivial vector bundles.*

Proof. We argue as in the previous lemma. Now we deal with maps

$$\pi_{7+8d}(SO(16^d 8\gamma - 1)) \longrightarrow \pi_{7+8d}(SO(n-1))$$

which are in the range of stable inclusion of homotopy groups when $d \neq 0$ and $\gamma \neq 1$, hence they all are isomorphisms $\mathbb{Z} \to \mathbb{Z}$. The only case left is $\pi_7(SO(7)) \to \pi_7(SO(14))$ which reduces to determine $\pi_7(SO(7)) \to \pi_7(SO(9))$, but this is done exactly as before (cf. [25], [28]). $\diamondsuit$

Remark 3.2.18. The methods exploited in this section can not be used to decide the optimality of the lower bound of $d(\mathcal{S}_n^{n-1})$ when $(n+1)/2 = 2^a(2c+1)$ with $a = 0$ or 1; indeed, in those cases, the stable homotopy groups $\pi_{r-1}(SO(k))$ are cyclic of order 2.

3.3 Generalizations in the constant rank case

In this section we show how to adapt the methods used for the symmetric case, to the complex hermitian and the real skew-symmetric matrices.

Complex hermitian matrices

In what follows, we sketch how to recover the upper bound, the lower bound and the optimality condition, in order to prove Theorem 3.1.8. The scheme of the proof is completely analogous to that of the corresponding theorem for symmetric matrices, hence we will provide only the necessary details. Recall that $\mathcal{H}_n^{n-k}$ denotes the space of hermitian matrices of rank $n-k$.

First, we prove the upper bound:

Proposition 3.3.1. *If $k = 0$ or 1, then $d(\mathcal{H}_n^{n-k}) \leq \sigma_{\mathbb{C}}(n, n-k) + 1$.*

Proof. Assume that $V \setminus \{0\} \subset \mathcal{H}_n^{n-k}$ has dimension $d+1$. In this case, the evaluation sequence of Proposition 3.1.3 becomes

$$0 \to K \to \mathbb{C}^n \xrightarrow{\Phi} m(\xi_{\mathbb{R}} \otimes \mathbb{C}) \to C \to 0 \quad \text{over } \mathbb{RP}^d$$

The key fact, providing the analogy with the symmetric case, is that hermitian matrices have real eigenvalues. Then,

Lemma 3.3.2. *Either $d = 1$ or any $A \in V \setminus \{0\}$ has exactly $(n-k)/2$ positive (and negative) eigenvalues.*

This implies that the pull-back of $\underline{\mathbb{C}}^n \to \mathbb{RP}^d$ to S^d, via the usual projection π, decomposes as the sum of $\pi^*K \oplus E^+ \oplus E^-$, where $E^\pm$ are the bundles whose fiber $E_A^\pm$ are the subspaces of $\mathbb{C}^n$ defined by $\pm\, {}^t\overline{x}Ax > 0$.

Lemma 3.3.3. *The bundles E^+ and E^- are isomorphic. Moreover, if they are trivial, then $E^+ \oplus E^-$ is the pull-back of $s(\xi_{\mathbb{R}} \otimes \mathbb{C}) \oplus \underline{\mathbb{C}}^s$, with $s = (n-k)/2$.*

Proof of Lemma 3.3.3. The argument is exactly the same as the one for Propositions 3.2.3 and 3.2.4. The only thing that we need to prove is that any nonvanishing section $\tau : S^d \to E^+ \oplus E^- = \underline{\mathbb{C}}^{n-k}$, such that $\tau(-x) = -\tau(x)$, induces a bundle inclusion $T \subset E^+ \oplus E^-$ with $T = \pi^*(\xi_{\mathbb{R}} \otimes \mathbb{C})$. But this is clear, since $\tau(x) = Re\tau(x) + iIm\tau(x)$ and $\xi_{\mathbb{R}} \otimes \mathbb{C} = \xi_{\mathbb{R}} \oplus (i\xi_{\mathbb{R}})$. $\diamondsuit$

Then, in order to make $E^\pm$ trivial, we restrict all bundles to an equatorial S^{d-1}, obtaining, in return, the stable bundle equation

$$[\underline{\mathbb{C}}^n] = \frac{n-k}{2}[(\xi_{\mathbb{R}} \otimes \mathbb{C}) \oplus \underline{\mathbb{C}}] + [K] \quad \text{in } \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1}),$$

that is $\frac{n-k}{2}\nu + [K] = 0$.

The upper bound is now a consequence of Proposition 3.1.5, together with the fact that $\mathbb{RP}^{d-1}$ has only one (stably) nontrivial complex line bundle. $\diamondsuit$

Remark 3.3.4. As in the symmetric case, the argument used here is general. The only point in which the hypothesis $k = 0, 1$ is needed is at the end of the proof, when determining the stable class of K. A further classification of stable complex bundles over $\mathbb{RP}^{d-1}$ would, clearly, lead to similar estimates for hermitian matrices of bigger rank.

Then, we show the lower bound:

Proposition 3.3.5. *If $k = 0$ or 1, then $\sigma_{\mathbb{C}}(n, n-k) \leq d(\mathcal{H}, n, n-k)$.*

Proof. Set $s = (n-k)/2$. The bound is a consequence of the two following lemmata:

Lemma 3.3.6. *For any $k \leq n$, there exists a vector space $V \setminus \{0\} \subset \mathcal{H}_n^{n-k}$ of dimension $\rho_{\mathbb{C}}(s) + 1$.*

Proof of Lemma 3.3.6. By the results of Adams, Lax and Phillips [3], there exists a linear space of $2s \times 2s$ invertible hermitian matrices of dimension $\rho_{\mathbb{C}}(s) + 1$. Let $\{B_i\}$ a basis of such space and construct the $n \times n$ matrices,

$$A_i = \begin{pmatrix} B_i & 0 \\ 0 & O_{n-2s} \end{pmatrix}.$$

Then, the span of the A_i's is our required V. ♢

Lemma 3.3.7. *For $k = 0, 1$ there exists a vector space $W \setminus \{0\} \subset \mathcal{H}_n^{n-k}$ of dimension $\rho_{\mathbb{C}}(s+1)$.*

Proof of Lemma 3.3.7. Let $\{D_i\}$ a basis of a $\rho_{\mathbb{C}}(s+1)$-dimensional space of $(s+1)\times(s+1)$ invertible matrices, whose existence is granted by [3]. Then, a basis for the required W is given by the matrices C_i obtained from

$$\begin{pmatrix} 0 & D_i \\ {}^t\bar{D}_i & 0 \end{pmatrix}$$

by eliminating the last row and column. ♢

The lower bound is proved, since $d(\mathcal{H}, n, n-k) \geq \max\{\rho_{\mathbb{C}}(s)+1, \rho_{\mathbb{C}}(s+1)\} \geq \sigma_{\mathbb{C}}(n, n-k)$. ♢

Finally, we have the optimality of our estimates:

Proposition 3.3.8. *With regard to Theorem 3.1.8, the upper bound is attained when $\sigma_{\mathbb{C}}(n, n-k) = \rho_{\mathbb{C}}(\frac{n-k}{2})$. Otherwise, the lower bound is optimal.*

Proof. Clearly, the only case we have to consider is $k = 1$: if n is even, there is nothing to prove. If $n \equiv 1 \mod 4$, then $\sigma_{\mathbb{C}}(n, n-1) = \rho_{\mathbb{C}}(\frac{n-1}{2})$ and the bound is reached by explicit examples. When $n \equiv 3 \mod 4$, then $\sigma_{\mathbb{C}}(n, n-1) = \rho_{\mathbb{C}}(\frac{n+1}{2})$; we denote this number by r, and we show that the upper bound can never be attained. Suppose by contradiction that it is attained; then, with the same argument of Proposition 3.2.15, we need to prove that the bundles $E^{\pm}$ are trivial. We have to study the homotopy maps $\pi_{r-1}(SU(k)) \to \pi_{r-1}(SU(2k))$. These maps are isomorphisms for $n \neq 3$, as can be seen by computing the homotopy sequences of $SU(m) \to SU(m+1) \to S^{2m-1}$. Since r is always even, complex Bott periodicity ensures that these groups are isomorphic to $\mathbb{Z}$. Finally, if $n = 3$, line bundles on the 4-sphere are trivial since $\pi_3(SU(1)) = 0$. ♢

Real skew-symmetric matrices

Recall that $\mathcal{A}_n^{n-k}$ denotes the set of n-square skew-symmetric matrices of constant rank $n-k$. In this section we prove Theorem 3.1.9. Our main tool is an explicit way of constructing odd maps with values in a set of invertible skew-symmetric matrices. Indeed, we have the following estimate (cf. also [7]):

Proposition 3.3.9. *If $\phi : S^{d-1} \to \mathcal{A}_n^n$ is an odd map, then $d \leq \rho(n) - 1$.*

Proof. Give $\mathbb{R}^d$ spherical coordinates $(x, r) \in S^{d-1} \times [0, +\infty)$, such that any nonzero point can be uniquely written as rx, and consider the sphere $S^d \subset \mathbb{R} \times \mathbb{R}^d$ as the set of points with coordinates (s, rx) such that $s^2 + r^2 = 1$. Define the map

$$\phi' : S^d \to \mathcal{M}_n(\mathbb{R}) \quad \text{by} \quad \phi'(s, rx) = sI + r\phi(x),$$

where I is the $n \times n$ identity. This is an odd map and $\phi'(S^d) \subset GL_n(\mathbb{R})$, since $\phi(x)$, being an invertible skew-symmetric matrix, has only imaginary eigenvalues. By Theorem 2.3.23, the inequality $d \leq \rho(n) - 1$ holds. ♢

Assume, now, that V is a d-dimensional vector space such that $V \setminus \{0\} \subset \mathcal{A}_n^{n-k}$ and consider the induced odd map $\phi : S^{d-1} \to A_n^{n-k}$. We would like to estimate the maximal value $d(\mathcal{A}_n^{n-k})$ that d can attain. Observe that there is no restriction in supposing $n - k$ even, since, if it is odd we can immediately deduce that $d = 0$ (skew-symmetric matrices have imaginary conjugate eigenvalues, hence are of even rank).

As pointed out in proposition 3.1.3, ϕ induces an an exact sequence of bundles over $\mathbb{RP}^{d-1}$:

$$0 \to K \to \mathbb{R}^n \xrightarrow{\Phi} n\xi_{\mathbb{R}} \to C \to 0.$$

The following statement (whose proof will be postponed to the end of this section) is the key for showing the upper bound of Theorem 3.1.9:

Proposition 3.3.10. *Suppose that $K = K' \oplus \Lambda$, with Λ line bundle. Then:*

1. *if Λ is not trivial, there exists an odd map $\phi' : S^{d-1} \to \mathcal{A}_{n+1}^{n-k+2}$;*

2. *if Λ is trivial, there exists an odd map $\phi' : S^{d-1} \to \mathcal{A}_{n-1}^{n-k}$.*

In both cases, the kernel bundle over $\mathbb{RP}^{d-1}$ induced by ϕ' is isomorphic to K'.

Corollary 3.3.11. *Let a, b be nonnegative integers. If $K = a\xi_{\mathbb{R}} \oplus \underline{\mathbb{R}}^b \oplus K'$, then there exists an odd map*

$$\phi' : S^{d-1} \to \mathcal{A}_{n+a-b}^{n-k+2a}.$$

Proof. Argument by induction on $a + b$. The initial step ($a + b = 1$) is the content of the first couple of statements in proposition 3.3.10, while the inductive step is shown thanks to the last statement of the same proposition. ♢

For n and k integers, define $\rho(n, n-k) = \max_{-k \leq i \leq k} \rho(n + i)$. A consequence of corollary 3.3.11 and proposition 3.3.9 is the following estimate:

Theorem 3.3.12. (Upper bounds for $d(\mathcal{A}_n^{n-k})$) *If k is an integer, $0 \le k \le 3$ and $n-k$ is even, then:*

$$d(A_n^{n-k}) \le \begin{cases} \rho(n, n-k) - 1 & \text{if } k = 0, 1 \\ \max\{3, \rho(n, n-k) - 1\} & \text{if } k = 2 \\ \max\{13, \rho(n, n-k) - 1\} & \text{if } k = 3 \end{cases}$$

If $n-k$ is odd, then $d(\mathcal{A}_n^{n-k}) = 0$.

Proof. Assume, as above, that $n-k$ is even and that V is a d-dimensional vector space such that $V \setminus \{0\} \subset \mathcal{A}_n^{n-k}$. Call ϕ the induced odd map.

- If $k = 0$, there is nothing to prove, since our statement is exactly proposition 3.3.9. If $k = 1$, then the kernel bundle K is either $\xi_{\mathbb{R}}$ or $\underline{\mathbb{R}}$; in the first case, proposition 3.3.10 provides an odd map $\phi' : S^{d-1} \to \mathcal{A}_{n+1}^{n+1}$ and prop. 3.3.9 consequently gives $d \le \rho(n+1) - 1$; in the second case, we get an od map $\phi' : S^{d-1} \to \mathcal{A}_{n-1}^{n-1}$ and the inequality $d \le \rho(n-1) - 1$.

- When $k = 2$, provided that $d \ge 4$, K is one of the following three bundles: $2\xi_{\mathbb{R}}$, $\xi_{\mathbb{R}} \oplus \underline{\mathbb{R}}$ or $\underline{\mathbb{R}}^2$. Said otherwise, $K = a\xi_{\mathbb{R}} \oplus \underline{\mathbb{R}}^{2-a}$, with a integer and $0 \le a \le 2$. Then, corollary 3.3.11 states the existence of an odd map

  $$\phi' : S^{d-1} \to \mathcal{A}_{n+a-(2-a)}^{n-2+2a} = \mathcal{A}_{n-2+2a}^{n-2+2a},$$

  thus $d \le \max_{0 \le a \le 2} \rho(n - 2 + 2a) - 1 = \rho(n, n-2) - 1$.

- For $k = 3$ we can not apply anymore the arguments above, since there is not a classification of rank-3 bundles over $\mathbb{P}^{d-1}$. Nevertheless, when $d \ge 14$, there is a stable classification (see [2]) that can be interpreted in terms of odd maps as follows.

  For α integer, call ϕ_α the map $S^{d-1} \to \mathcal{A}_{n+\alpha}^{n-3}$ defined as $\begin{pmatrix} \phi & 0 \\ 0 & \mathbf{0} \end{pmatrix}$, with $\mathbf{0}$ the $\alpha \times \alpha$ zero matrix, and remark that its induced kernel bundle over $\mathbb{RP}^{d-1}$ is $K \oplus \underline{\mathbb{R}}^\alpha$. If α is big enough, then $K \oplus \underline{\mathbb{R}}^\alpha = a\xi_{\mathbb{R}} \oplus \underline{\mathbb{R}}^{3-a} \oplus \underline{\mathbb{R}}^\alpha$, $0 \le a \le 3$. Then, corollary 3.3.11 produces an odd map

  $$(\phi_\alpha)' : S^{d-1} \to \mathcal{A}_{n+\alpha+a-(3-a)-\alpha}^{n-3+2a} = \mathcal{A}_{n-3+2a}^{n-3+2a}.$$

  Proposition 3.3.9 concludes.

$\diamondsuit$

In order to show the lower bound of Theorem 3.1.9 let us begin the classical Radon-Hurwitz construction in the invertible case:

Proposition 3.3.13. *For any n even, there exists a vector space $V_n \subset \mathcal{A}_n^n \cup \{0\}$ of dimension $\rho(n) - 1$.*

Proof. By Theorem 2.3.13, we can take V_n as the space generated over $\mathbb{R}$ by a R-H system of order n counting $\rho(n) - 1$ elements. ♢

Consider now the following two linear maps between spaces of matrices

$$I : \mathcal{M}_{n-1}(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R}) \quad \text{and} \quad R : \mathcal{M}_{n+1}(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$$

defined in this way: $R(A)$ is the matrix obtained by deleting the last row and column of A while

$$I(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 3.3.14. *The maps I and R respectively restrict to linear maps*

$$I : \mathcal{A}_{n-1}^{n-k} \to \mathcal{A}_n^{n-k} \quad \text{and} \quad R : \mathcal{A}_{n+1}^{n+1} \to \mathcal{A}_n^{n-1}.$$

Proof. We only prove that $\operatorname{rank} R(A) = \operatorname{rank} A - 2 = n - 1$: since R deletes a row (and a column) from an invertible matrix, we get $\operatorname{rank} R(A) < \operatorname{rank} A = n + 1$; $R(A)$ is skew-symmetric, hence its rank must be even and $\leq \operatorname{rank} A - 2 = n - 1$. On the other hand, by applying R, the rank can drop at most by 2. ♢

Lemma 3.3.15. *If V is a linear space in $\mathcal{A}_n^{n-k} \cup \{0\}$, then $\dim I(V) = \dim V$. If W is a linear space in $\mathcal{A}_{n+1}^{n+1} \cup \{0\}$ and $n > 1$, then $\dim R(W) = \dim W$.*

Proof. Take $A \in W$ such that $R(A) = 0$; then A has rank at most 2 hence, *a fortiori*, 0. This means that R is injective. ♢

We can now prove the second inequality in Theorem 3.1.9:

Proposition 3.3.16. (Lower bounds for $d(\mathcal{A}_n^{n-k})$) *For any integer k, $0 \leq k \leq 3$ such that $n - k$ is even, $d(\mathcal{A}_n^{n-k}) \geq \max\{\rho(n-k), \rho(n-k+2)\} - 1$*

Proof. The sequences of compositions

$$\mathcal{A}_{n-k}^{n-k} \xrightarrow{I} \mathcal{A}_{n-k+1}^{n-k} \xrightarrow{I} \cdots \xrightarrow{I} \mathcal{A}_n^{n-k};$$

and

$$\mathcal{A}_{n-k+2}^{n-k+2} \xrightarrow{R} \mathcal{A}_{n-k+1}^{n-k} \xrightarrow{I} \mathcal{A}_{n-k+2}^{n-k} \xrightarrow{I} \cdots \xrightarrow{I} \mathcal{A}_n^{n-k}$$

respectively map the $(\rho(n-k)-1)$-dimensional space V_{n-k} and the $(\rho(n-k+2)-1)$-dimensional space V_{k+2}, whose existence is granted by proposition 3.3.13, to a space in $\mathcal{A}_n^{n-k} \cup \{0\}$ of the appropriate dimension. $\diamondsuit$

We conclude the section with the proof of proposition 3.3.10:

Proof. Consider the antipodal projection map $\pi : S^{d-1} \to \mathbb{RP}^{d-1}$; the pull-back $\pi^*\Lambda$ is a trivial bundle, thus we can pick a nonvanishing section s such that the composition

$$S^{d-1} \xrightarrow{s} \pi^*\Lambda \to \pi^*K \to \underline{\mathbb{R}}^n \qquad \text{defined by} \qquad x \mapsto (x, v(x))$$

verifies $|v(x)| = 1$. Observe that both matrices $\phi(x)$ and $\phi(-x)$ have the same null-space spanned by $v(x)$; this implies that for any $x \in S^{d-1}$ either $v(-x) = v(x)$ or $v(-x) = -v(x)$. The parity of the map v characterizes Λ:

Lemma 3.3.17. *The map $v : S^{d-1} \to \mathbb{R}^n$ is odd if and only if Λ is not trivial. Otherwise, v is even, that is, $v(-x) = v(x)$ for any $x \in S^{d-1}$.*

Proof. If $d = 1$ there is nothing to prove, so we can assume that S^{d-1} is connected. First, remark that v is either odd or even. Indeed, the map $w(x) := v(x) - v(-x)$ takes values on the disconnected set $\{0\} \cup \{z \in \mathbb{R}^n \ : \ |z| = 2\}$ and, by continuity, either $w(S^{d-1}) = \{0\}$ (v is even) or $w(x) \neq 0$ for any x (which means that v is odd).

Suppose that v it is even, then the map $\mathbb{RP}^{d-1} \to \underline{\mathbb{R}}^n$ defined by $[x] \mapsto ([x], v(x))$ is a trivializing section of Λ. On the converse, if Λ is trivial, any section of $\pi^*\Lambda$ comes from a section of Λ hence must be invariant for the action of the antipodal map on S^{d-1}. $\diamondsuit$

We are now ready to show the statements of the proposition 3.3.10.

Case I: Λ not trivial. Take v odd as above. Thinking of $v(x)$ as a column vector, construct the odd map

$$\phi' : S^{d-1} \to \mathcal{A}_{n+1}^{n-k+2} \qquad \text{by posing} \qquad \phi'(x) = \begin{pmatrix} \phi(x) & v(x) \\ -{}^t v(x) & 0 \end{pmatrix}.$$

To see that $\phi'(x)$ has indeed the required rank, take a unitary matrix $W(x)$ which diagonalizes $\phi(x)$ over $\mathbb{C}$, define $U(x) := \begin{pmatrix} W(x) & 0 \\ 0 & 1 \end{pmatrix}$ and compute ${}^t\bar{U}(x)\phi'(x)U(x)$. After a suitable reordering of rows and columns, it results that

$${}^t\bar{U}\phi'U = \left(\begin{array}{c|c} {}^t\overline{W}\phi\, W & {}^t\overline{W}v \\ \hline -{}^t v W & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 0 & z \\ 0 & \Delta & 0 \\ \hline -{}^t\overline{z} & 0 & 0 \end{array}\right)$$

where Δ is a nonsingular diagonal $(n-k)\times(n-k)$ matrix and $|z|=1$. This complex matrix has rank $n-k+2$, so the same holds for ϕ'.

Case II: Λ trivial. Construct the complementary bundle N of $K=\underline{\mathbb{R}}\oplus K'\subset\underline{\mathbb{R}}^n$ by requiring that its fibres are orthogonal to those of K with respect to the standard scalar product on $\underline{\mathbb{R}}^n$. The relation $\underline{\mathbb{R}}\oplus K'\oplus N=\underline{\mathbb{R}}^n$ means that $K'\oplus N$ is stably trivial. We can now construct an odd map $\phi':S^{d-1}\to\mathcal{A}^{n-k}_{n-1}$.

Step 1: suppose $d-1<n-1$. By corollary 2.1.20, $E\oplus N$ is trivial, hence admits everywhere independent sections $y_2,\dots,y_n$ which can be thought as sections of $\underline{\mathbb{R}}^n$. As in the proof of 3.3.17, the y_i's provide even maps $w_i:S^{d-1}\to\mathbb{R}^n$ $i=2,\dots,n$ that we can also suppose to be orthonormal. Then, for $x\in S^{d-1}$, construct the orthogonal matrix $O(x)$ by putting in its first column the even map $v(x)$ defined by the trivial Λ and, in the other columns, the maps $w_i(x)$. This matrix is a continuous function of x and skew-symmetry of $\phi(x)$ yields:

$$ {}^t O(x)\phi(x)O(x)=\left(\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \phi'(x) & \\ 0 & & & \end{array}\right). $$

The rank of $\phi'(x)$ equals the rank of $\phi(x)$ while its order is one less; moreover, ϕ' is odd, so it is our required map.

Step 2: suppose $d-1\geq n-1$. We show that this case can not happen: take an equatorial $S^{n-2}\subset S^d$ and consider the bundles restricted to the base $\pi(S^{n-2})=\mathbb{RP}^{n-2}\subset\mathbb{RP}^{d-1}$. We rephrase step 1 above, using S^{n-2} in the place of S^{d-1}.

Since S^{n-2} is contained in a contractible subset of S^{d-1}, π^*K' and π^*N become trivial and we can construct, using their sections, orthonormal maps $v_2,\dots,v_{n-k}$ and $w_{n-k+1},\dots,w_n$ from S^{n-2} to $\mathbb{R}^n$ with the condition that $\phi(x)v_i(x)=0$. Define $O(x)$ orthogonal as in the step before using v, the v_i's and the w_j's as columns. Computation shows that ${}^tO(x)\phi(x)O(x)=\begin{pmatrix}\mathbf{0} & 0\\ 0 & \phi'(x)\end{pmatrix}$ where $\mathbf{0}$ is a k-square zero matrix. We get the odd map

$$ \phi':S^{n-2}\longrightarrow\mathcal{A}^{n-k}_{n-k}. $$

Then, proposition 3.3.9 implies the inequality

$$ n-1\leq\rho(n-k)-1 $$

which is always false since $n-k<n$.

$\diamondsuit$

Chapter 4

SPACES OF BOUNDED RANK MATRICES

In this chapter we deal with the problem of determining the maximal dimension of a linear space V of matrices with rank bounded from below. The methods of the previous chapter, for most part, are not directly applicable. This is essentially due to the fact that the bundle map induced by the inclusion of V (cf. proposition 3.1.1 and what follows it) has not a constant rank kernel and thus can not be completed to an exact sequence. The results that we prove in this chapter are obtained by finding a way to overcome this *empasse*.

4.1 Statement of the results

Denote by $\mathcal{M}_n^{n-1,+}$ the set of real $n \times n$ matrices of rank greater or equal than $n-1$ and set

$$d(\mathcal{M}_n^{n-1,+}) = \max\{\dim_{\mathbb{R}} V, V \setminus \{0\} \subset \mathcal{M}_n^{n-1,+}\};$$

similarly, define $\mathcal{H}_n^{n-1,+}$ and $d(\mathcal{H}_n^{n-1,+})$ for hermitian matrices.

The following result is obtained by replacing the inclusion of the vector spaces with another odd map defined using adjoint matrices.

Theorem 4.1.1. *If n is even, then*

$$\max\{\, d(\mathcal{M}_n^{n-1,+}),\, d(\mathcal{H}_n^{n-1,+})\,\} \le \rho_{\mathbb{C}}(n)$$

Another fruitful attempt to study spaces of matrices of rank bounded from below, comes from the consideration that, in the space of all matrices of the appropriate dimension, their complementary set is an algebraic variety defined by the vanishing of the minors. Thus, the algebraic degree and the intersection theory provide useful tools for determining whether a linear space of matrices has or not elements of a given rank. Following this line of arguments, we prove:

Theorem 4.1.2. *The following estimate holds:* $7 \le \, d(\mathcal{H}_5^{4,+}) \, \le 8$

.

4.2 Odd maps and adjoint matrices

This section is devoted to the proof of Theorem 4.1.1.

Denote by $\mathcal{M}_n(\mathbb{C})$ the space of $n \times n$ complex matrices and, for any $A \in \mathcal{M}_n(\mathbb{C})$, denote by A^c the transpose of its adjoint matrix. We define a map

$$\psi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \quad \text{by } \psi(A) = A + i\bar{A}^c.$$

Clearly, if $\operatorname{rank} A \leq n-2$, then $\psi(A) = A$ since all minors of order $n-1$ vanish. Now set:

$$\mathcal{Z} = \{A \in \mathcal{M}_n(\mathbb{C}) \,|\, \operatorname{rank} A \geq n-1,\, \det A \neq ir, r \in]-\infty, 0[\}$$

and remark that both $\mathcal{M}_n^{n-1,+}$ and $\mathcal{H}_n^{n-1,+}$ are included in $\mathcal{Z}$. We prove:

Proposition 4.2.1. *For ψ as above, $\psi(\mathcal{Z}) \subset GL_n(\mathbb{C})$.*

Proof. Argument by contradiction: take $A \in \mathcal{Z}$ and $v \in \mathbb{C}^n$, $v \neq 0$ such that

$$\psi(A)v = Av + i\bar{A}^c v = 0. \tag{4.1}$$

Multiplying this equation on the left by ${}^t\bar{A}$, gives

$${}^t\bar{A}Av + i\det(\bar{A})v. \tag{4.2}$$

Then, since $v \neq 0$, $-i\det\bar{A}$ is an eigenvalue of the hermitian matrix ${}^t\bar{A}A$, so it must be real and not negative. This means that $\det\bar{A} = ir$ with $r \leq 0$ but, since $A \in \mathcal{Z}$, we conclude that $\det\bar{A} = \det A = 0$. From equation 4.2 we deduce ${}^t\bar{A}Av = 0$ which implies

$${}^t\bar{v}\,{}^t\bar{A}Av = \|Av\| = 0.$$

But then, equation 4.1 entails

$$Av = \bar{A}^c v = 0. \tag{4.3}$$

From the relation $0 = \det(A)I = A \cdot {}^tA^c$ and the consequent fact that v generates the null-space of A, we get that v generates the image of ${}^tA^c$. Thus, equation 4.3 implies

$$\bar{A}^c \cdot {}^tA^c = 0$$

that is $A^c = 0$ or, said otherwise, all minors of order $n-1$ of A vanish. This is clearly impossible. $\diamondsuit$

Proposition 4.2.2. *If n is even, then $\psi : \mathcal{Z} \to GL_n(\mathbb{C})$ verifies $\psi(-A) = -\psi(A)$.*

Proof. This is almost obvious, since by the properties of the determinant, when $n-1$ is odd, $(-A)^c = -A^c$. ♢

Proposition 4.2.3. *If n is even and V is a real vector space such that $V \setminus \{0\} \subset \mathcal{Z}$, then $\dim V \leq \rho_{\mathbb{C}}(n)$.*

Proof. The restriction of ψ to the elements of V of length 1 is an odd map

$$\psi : S^{d-1} \to GL_n(\mathbb{C}) \qquad d = \dim V;$$

by proposition 3.1.3, ψ defines a vector bundle isomorphism $\underline{\mathbb{C}}^n = n(\xi_{\mathbb{R}} \otimes \mathbb{C})$ over $\mathbb{RP}^{d-1}$. In the ring $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{d-1})$, this means that n times the generator ν is zero hence proposition 3.1.5 implies $d \leq \rho_{\mathbb{C}}(n)$. ♢

As a corollary, we obtain:

Theorem 4.2.4. *If n is even, then*

$$\max\{\, d(\mathcal{M}_n^{n-1,+}),\ d(\mathcal{H}_n^{n-1,+})\,\} \leq \rho_{\mathbb{C}}(n)$$

Remark 4.2.5. Call X the set of matrices $A \in \mathcal{M}_n(\mathbb{C})$ of rank $\leq n-2$ and consider the map

$$\psi_s(A) = A + s \cdot i\bar{A}^c \quad s \in \mathbb{R}.$$

For $0 \leq s \leq 1$, this map realizes an homotopy, relatively to X, from $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ and the identity. Moreover, for any $s > 0$, it sends $\mathcal{Z}$ to $GL_n(\mathbb{C})$.

4.3 Real varieties of hermitian matrices

The present section is entirely dedicated to the proof of theorem 4.1.2.

First, we show how to obtain the lower bound:

Remark 4.3.1. The following matrix, depending on 7 real parameters, has $rank A \geq 4$ unless $\alpha = z = u = w = 0$, thus $d(\mathcal{H}_5^{4,+}) \geq 7$.

$$A(\alpha, z, u, w) = \begin{pmatrix} \alpha & z & u & w & 0 \\ \bar{z} & \alpha & \bar{w} & -\bar{u} & 0 \\ \bar{u} & w & -\alpha & z & 0 \\ \bar{w} & -u & \bar{z} & -\alpha & z \\ 0 & 0 & 0 & \bar{z} & 0 \end{pmatrix} \qquad \alpha \in \mathbb{R}, \quad z, u, w \in \mathbb{C}.$$

To see this, let A_k be the submatrix obtained from A by elimination of the k-th row and column; A_3 is invertible unless $z = 0$ or $|\alpha| = |z|$. When $z = 0$, $A_5^2 = (\alpha^2 + |u|^2 + |w|^2)I$ (see also [3]); when $|\alpha| = |z|$, $\det A_1 = |z|^2(|z|^2 + |w|^2)$.

In order to show the upper bound, we adapt to hermitian matrices the work developed in [22] in the case of real symmetric matrices, by thinking to the degeneracy locus $\{A \text{ hermitian}, rankA \leq n-2\}$ as a real variety. The main theoretic tool that we use is a theorem, proved by Friedland and Libgober in [22], which is a generalization, as a consequence of the Hodge decomposition (see Theorem 5.2.2 or [40]) and the Lefschetz fixed-point theorem, of the classical odd degree theorem for algebraic varieties. We recall it in the following part.

A generalization of the odd degree theorem

For any compact CW-complex X, and any map $f : X \to X$, denote by $Tr^k f$ the trace of the map $f^* : H^k(X, \mathbb{R}) \to H^k(X, \mathbb{R})$. Recall that the Lefschetz number $\lambda(f)$ is defined as $\sum_{k=0}^{\infty}(-1)^k Tr^k f$ and that the Lefschetz fixed point Theorem states that if $\lambda(f) \neq 0$, then $f(x) = x$ for some $x \in X$. In this part, we follow the set-up of [22].

Let $W \subset \mathbb{RP}^n$ be a real algebraic variety such that its complexification $W_{\mathbb{C}} \subset \mathbb{CP}^n$ is a smooth irreducible variety of (complex) dimension $m \geq 1$. Then, denoting with B the restriction to $W_{\mathbb{C}}$ of the antiholomorphic involution $\beta : \mathbb{CP}^n \to \mathbb{CP}^n$ fixing $\mathbb{RP}^n$ pointwise, we get:

Lemma 4.3.2. *For any nonnegative integer k, $Tr^{2k+1}B = 0$ and*

$$Tr^{2k}B = \text{ trace } B^* : H^{k.k}(W_{\mathbb{C}}) \to H^{k.k}(W_{\mathbb{C}}). \tag{4.1}$$

Proof. Since $B^*(H^{p.q}(W_{\mathbb{C}})) = H^{q.p}(W_{\mathbb{C}})$ we have for $p \neq q$, that the trace of B^* restricted to $H^{p,q}(W_{\mathbb{C}}) \oplus H^{q,p}(W_{\mathbb{C}})$ is zero.

The Hodge decomposition of $H^*(W_{\mathbb{C}}, \mathbb{C})$ yields the claim, since B reverses the orientation of $W_{\mathbb{C}}$ if m is odd and preserves the orientation of $W_{\mathbb{C}}$ if m is even. $\diamondsuit$

Corollary 4.3.3. *For W as above, the Lefschetz number $\lambda(W_{\mathbb{C}})$ of B is given by $\lambda(W_{\mathbb{C}}) = 0$ if m is odd and by*

$$\lambda(W_{\mathbb{C}}) = Tr^m B + 2\sum_{k=0}^{\frac{m-2}{2}} Tr^{2k}B \in \mathbb{Z} \quad \text{ if } m \text{ is even.} \tag{4.2}$$

If $\lambda(W_{\mathbb{C}}) \neq 0$ then $W \cap \mathbb{RP}^n \neq \emptyset$.

Corollary 4.3.4. *Let W be as above. Suppose that m is even and the Betti number $b_m(W_{\mathbb{C}})$ (equivalently the Euler characteristic $\chi(W_{\mathbb{C}})$) is odd. Then $W \cap \mathbb{RP}^n \neq \emptyset$.*

Proof. Since the eigenvalues of $B^* : H^m(W_{\mathbb{C}}) \to H^m(W_{\mathbb{C}})$ are ± 1, we have that $b_m(W_{\mathbb{C}}) = \lambda(W_{\mathbb{C}}) \bmod 2$. ♢

Theorem 4.3.5. *Let $U \subset \mathbb{RP}^n$ be a real algebraic variety such that its complexification $U_{\mathbb{C}} \subset \mathbb{CP}^n$ is an irreducible variety of codimension m. Suppose that the codimension of the variety of the singular points of $U_{\mathbb{C}}$ in $U_{\mathbb{C}}$ is at least k.*
If, for a generic vector space $L \in Gr(m+k, \mathbb{R}^{n+1})$, the Euler characteristic $\chi(U_{\mathbb{C}} \cap \mathbb{P}(L_{\mathbb{C}}))$ is odd, then $U \cap \mathbb{P}(N) \neq \emptyset$ for any $N \in Gr(m+k, \mathbb{R}^{n+1})$.

Proof. For $k = 1$ $U_{\mathbb{C}} \cap L_{\mathbb{C}}$ consists of $\deg U_{\mathbb{C}}$ distinct points if L is generic and the theorem follows. Assume that $k > 1$. Let $W = U \cap L$ and $W_{\mathbb{C}} = U_{\mathbb{C}} \cap L_{\mathbb{C}}$. The assumptions of the theorem yield that, for a generic L, $W_{\mathbb{C}}$ is a smooth irreducible variety. Hence $\lambda(W_{\mathbb{C}})$ is given by Corollary 4.3.3. Then, the claim of the theorem follows from Corollaries 4.3.3 and 4.3.4. ♢

Real degeneracy loci

In order to prove the upper bound of Theorem 4.1.2, call d the minimum integer number such that any d-dimensional vector space $P \subset \mathcal{H}_5$ of 5×5 hermitian matrices intersects the degeneracy locus

$$D = \{A \in \mathcal{H}_5 \,|\, \operatorname{rank} A \leq 3\}.$$

Indeed, we have:

Lemma 4.3.6. *The following holds: $d(\mathcal{H}_5^{4,+}) + 1 = d$.*

Proof. By the definition: 1) there exists a vector space (of hermitian matrices) of dimension $d(\mathcal{H}_5^{4,+})$ containing only elements of rank 4 or 5 and, thus, not intersecting D; 2) any vector space of dimension $d(\mathcal{H}_5^{4,+})+1$ has at least one element of rank 3 or less, hence intersects D. ♢

Now consider the map

$$\sigma : \mathcal{M}_5(\mathbb{C}) \to \mathcal{M}_5(\mathbb{C}) \quad \text{defined by } \sigma(A) = {}^t\bar{A};$$

σ is antiholomorphic and $\sigma^2 = id$ hence it defines a real structure on $\mathcal{M}_5(\mathbb{C})$ for which $\mathcal{H}_5 =$ fix (σ) is the real part.

The map σ realizes the affine variety

$$D_{\mathbb{C}} = \{A \in \mathcal{M}_5(\mathbb{C}) \,|\, \operatorname{rank} A \leq 3\}$$

as the complexification of D.

Moreover, the projectivization

$$\pi : \mathcal{M}_5(\mathbb{C}) \to \mathcal{M}_5(\mathbb{C})/\mathbb{C}^* = \mathbb{CP}^{24}$$

sends $D_{\mathbb{C}}$ to an irreducible real algebraic variety $U_{\mathbb{C}}$ whose real part U is $\pi(D)$.

We would like to show that:

Proposition 4.3.7. *For a generic $L \in Gr(9, \mathbb{R}^2 5)$, the Euler characteristic $\chi(U_{\mathbb{C}} \cap \pi(L_{\mathbb{C}}))$ is odd.*

Indeed, granting proposition 4.3.7, we can apply Theorem 4.3.5 to $U_{\mathbb{C}}$ and L since the hypothesis hold: $U_{\mathbb{C}} \subset \mathbb{CP}^{24}$ has codimension 4 and degree 50; its singular locus is

$$\Sigma = \pi(\{A \in \mathcal{M}_5(\mathbb{C}), \operatorname{rank} A \leq 2\})$$

with codimension 5 in $U_{\mathbb{C}}$ (see e.g. [6]). As a consequence, we get that U intersect any 9-dimensional real vector space. Thus, by lemma 4.3.6:

Theorem 4.1. *The estimate $d(\mathcal{H}_5^{4,+}) \leq 8$ holds.*

The rest of this section is devoted to prove proposition 4.3.7.

Euler characteristic: preparatory arguments

As in the previous part, let $U_{\mathbb{C}}$ the projectivization of the degeneracy locus D and $\mathbb{P}^8 := \pi(L_{\mathbb{C}})$ generic. Moreover, remark that, the statement of proposition 4.3.7 is equivalent to show that

$$\chi(U_{\mathbb{C}} \cap \mathbb{P}^8) \equiv 1 \mod (2)$$

Consider the following resolution of $U_{\mathbb{C}}$ with a smooth variety Y:

$$\begin{tikzcd}
O(1) \arrow[d] & \supset & O(1)|_Y \arrow[d] & \simeq & \mathcal{O}_{\mathbb{P}}(1) \arrow[d] \\
\mathbb{P}(M_5(\mathbb{C})) \times Gr(3, \mathbb{C}^5) & \supset & Y = \{([A], W) \mid ImA \subset W\} \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & \simeq & \mathbb{P}(S^{\otimes 5}) \arrow[d] \\
U_{\mathbb{C}} & & & & Gr(3, \mathbb{C}^5).
\end{tikzcd}$$

Here, $S \to Gr(3, \mathbb{C}^5)$ is the tautological bundle, $\mathbb{P}(S^5)$ the projective bundle associated to $S^{\otimes 5}$ and $\mathcal{O}_{\mathbb{P}}(1)$ and $O(1)$ are the duals to the tautological bundles over their respective projective bundles. Hyperplane sections of $O(1)|_Y$ are exactly the same of those of $\mathcal{O}_{\mathbb{P}}(1)$. Remark that, outside $\pi_1^{-1}(\Sigma)$, Y is formed by couples of type $([A], ImA)$ hence π_1 is generically $1-1$; remark also that the diffeomorphism $Y \xrightarrow{\sim} \mathbb{P}(S^5)$ is given by the map $([A], W) \mapsto A \in Hom(\mathbb{C}^5, W) \simeq W \otimes \mathbb{C}^{*5} \simeq W^{\otimes 5}$.

For a vector bundle $F \to B$, we denote its Chern classes by $c_i(F) \in H^{2i}(B, \mathbb{Z})$ and its Chern polynomial $\sum_{k=0}^{\infty} c_k(F)t^k$ by $c_t(F)$ (see, e.g. [28], for the definition of Chern classes)

Computing $\chi(U_{\mathbb{C}} \cap \mathbb{P}^8)$ is the same as computing $\chi(Y \cap \bigcap_{i=1}^{16} H_i)$ with H_i generic hyperplanes defined by sections of $O(1)$.

Since $Z = Y \cap \bigcap_{i=1}^{16} H_i$ is a smooth complex 4-dimensional manifold, denoting by T_Z its tangent bundle and by $[Z] \in H_8(Z, \mathbb{Z})$ its fundamental class, we get

$$\chi(Z) = c_4(T_Z)[Z].$$

Following [22] and [26] we have:

Proposition 4.3.8. *Let $i\colon Z \to Y$ be the embedding and $h = c_1(O(1)|_Y)$; then*

$$c_t(T_Y) = i^* c_t(T_Y|_Z)(1+ht)^{-16}$$

and, denoting by e_4 the coefficient of t^4 in $c_t(T_Y)(1+ht)^{-16}$:

$$\chi(Z) = h^{16} e_4[Y]. \tag{4.3}$$

Proof. The first equality is a consequence of the exact bundle sequence over Z:

$$0 \to T_Z \to i^* T_Y|_Z \to \bigoplus_{i=1}^{16} N_{H_i}|_Z \to 0$$

in which N_{H_i} is the normal bundle of $H_i \subset Y$. The second one follows as in [26, Ch. 9.2] since (the restriction of) h represents a submanifold in $Y \cap H_1 \cap \cdots \cap H_k$ for every k. $\diamondsuit$

The diffeomorphism $Y \simeq \mathbb{P}(S^5)$ allows us to compute $c_t(T_Y)$ by means of the exact sequences of vector bundles

$$0 \to T_{\mathbb{P}(S^5)/G} \to T_{\mathbb{P}(S^5)} \to p^* T_G \to 0$$
$$0 \to \mathbb{C} \to S^5 \otimes \mathcal{O}_{\mathbb{P}}(1) \to Q \otimes \mathcal{O}_{\mathbb{P}}(1) = T_{\mathbb{P}(S^5)/G} \to 0$$

where the first one comes from the projection $p : \mathbb{P}(S^5) \to G = G(3, \mathbb{C}^5)$ and the second one is the tautological sequence over $\mathbb{P}(S^5)$ tensorised with $\mathcal{O}_{\mathbb{P}}(1)$.

Working out calculations we find (see [22]):

$$c_t(T_{\mathbb{P}(S^5)}) = c_t(T_G) c_t(T_{\mathbb{P}(S^5)/G}) = c_t(T_G) \left(\sum_{j=0}^{15} c_j(S^5) t^j (1+ht)^{15-j} \right). \tag{4.4}$$

Euler characteristic: computation modulo 2

We now dispose of all the needed tools to complete the proof of the statement 4.3.7.

Remark that the universal coefficient theorem for cohomology implies that

$$H^*(B,\mathbb{Z})\otimes\mathbb{Z}_2 = H^*(B,\mathbb{Z}_2)$$

, where B is either $G(3,\mathbb{C}^5)$ or $\mathbb{P}(S^5)$. Thanks to this, since we are only interested to the parity of our objects, we will perform all computations in $H^*(\,\cdot\,,\mathbb{Z}_2)$; moreover all polynomials in the Chern classes will be truncated according to our real necessities. For the sake of bervity, denote by c_i the i-th Chern class of S.

Proposition 4.3.9. *Modulo* 2, *we have:*

1. $c_t(S^5) = 1 + c_1t + c_2t^2 + c_3t^3 + c_1^4t^4 + c_1^5t^5 + c_1^4c_2t^6$;
2. $c_t(T_G) = 1 + c_1t + c_2t^2 + c_3t^3 + c_2^2t^4 + c_1c_2^2t^5 + (c_1^4c_2 + c_3^2)t^6$;
3. *the ring* $H^*(\mathbb{P}(S^5),\mathbb{Z}_2)$ *is* $\mathbb{Z}_2[c_1,c_2,h]$ *together with the relations*

$$h^{15} = \sum_{j=1}^{15} c_j(S^5)h^{15-j} \qquad c_1c_2^2 = 0 \qquad c_1^4 + c_1^2c_2 + c_2^2 = 0 \tag{4.5}$$

Proof.

1. Follows directly from $c_t(S^5) = c_t(S)^5$.
2. Follows from the sequence $0 \to S\otimes\tilde{S} \to \tilde{S}^5 \to Q\otimes\tilde{S} = T_G \to 0$ where $\tilde{S}$ is the dual of S and $c_t(\tilde{S}) = c_{-t}(S) = c_t(S)$ since we are working in $\mathbb{Z}_2$. Hence: $c_t(T_G) = c_t(S\otimes\tilde{S})c_t(S)^5$.
3. (For a more detailed treatment, see[12]).

   The structure of the cohomology ring of a projective bundle $\mathbb{P}(E)\to B$ is well known: $H^*(\mathbb{P}(E)) \simeq H^*(B)[h]/(h^r + \sum_1^r c_i(E)h^{r-i})$ with h the first Chern class of the canonical bundle over $\mathbb{P}(E)$ and $r = rank(E)$.

   We have that $H^*(Gr(k,\mathbb{C}^n),\mathbb{Z}_2) = \mathbb{Z}_2[c_1,\dots,c_k]/(s_{n-k+1},\dots,s_n)$ where c_i are the Chern classes of the tautological bundle $S\to Gr(k,\mathbb{C}^n)$ and s_i those of the quotient bundle $\underline{\mathbb{C}}^n/S$, expressed as polynomials in the c_i's.

   In our context $s_3 = c_1^3 + c_3$, $s_4 = c_1^4 + c_1^2c_2 + c_2^2$ and $s_5 = c_1^5 + c_1^2c_3 + c_1c_2^2$; the statement follows from the elimination of c_3 and simplification of the expression of s_5.

♢

Let e_4 be the coefficient of t^4 in $c_t(T_{\mathbb{P}(S^5)})(1+ht)^{-16}$;

Lemma 4.3.10. *$e_4 = h^4 + c_1h^3 + c_1^2h^2 + c_2h^2 + c_1^3h + c_1c_2h + c_1^4$ modulo 2.*

Proof. From equation (4.4) we have

$$c_t(T_{\mathbb{P}(S^5)})(1+ht)^{-16} = c_t(T_G)(1+ht)^{-1}\left(\sum_{j=0}^{15} c_j(S^5)\left(\frac{t}{1+ht}\right)^j\right);$$

applying 4.3.9 and truncating polynomials to the 4th degree, this becomes

$$(1+c_1t+c_2t^2+c_1^3t^3+c_2^2t^4)(1+ht+h^2t^2+h^3t^3+h^4t^4)\sum_{j=0}^{4} c_j(S^5)(t+ht^2+h^2t^3+h^3t^4)^j.$$

The statement follows simply by working out calculations and by deleting couples of equal terms. ♢

Proposition 4.3.11. *The class $h^{16}e_4$ generates the top cohomology of $H^*(\mathbb{P}(S^5),\mathbb{Z}_2)$.*

Proof. The proof of this statement is almost entirely based on the third point of 4.3.9. The second and third relation of (4.5) imply that $H^{12}(Gr(3,\mathbb{C}^5),\mathbb{Z}_2)$ is generated by any degree 6 monomial in c_1, c_2, excepted $c_1^2c_2^2$.

We claim that $H^{40}(\mathbb{P}(S^5),\mathbb{Z}_2)$ is generated by gh^{14} where g is any generator of $H^{12}(Gr(3,\mathbb{C}^5),\mathbb{Z}_2)$. This can be seen directly by reducing the degree of h in any monomial of type $c_1^ac_2^bh^{14+k}$, $a+b+k=20$, with the aid of the first relation of (4.5). As an example, consider $c_1^2c_2h^{16}$:

$$\begin{aligned} c_1^2c_2h^{16} &= c_1^2c_2h\ h^{15} = c_1^2c_2h\ (c_1h^{14} + c_2h^{13} + \text{ higher order terms in } c_1, c_2) = \\ &= c_1^3c_2h^{15} + c_1^2c_2^2h^{14} = c_1^3c_2(c_1h^{14} + \dots) + 0 = c_1^4c_2h^{14}. \end{aligned}$$

Granting this, one finds (by analogous calculations) that the first and the three last terms in the expression of $h^{16}e_4$ given by 4.3.10 vanish, while the three others do not. Hence, $h^{16}e_4$ generates $H^{40}(\mathbb{P}(S^5),\mathbb{Z}_2)$. ♢

Corollary 4.3.12. *If L is a generic 8-dimensional linear subspace of $\mathbb{P}^{24}$, the Euler characteristic $\chi(L\cap U_{\mathbb{C}})$ is odd.*

Proof. By equation (4.3):

$$\chi(L \cap U_{\mathbb{C}}) = \chi(Z) = h^{16} e_4[\mathbb{P}(S^5)]$$

and the latter term is not zero modulo 2, thanks to 4.3.11 and Poincaré duality between $H^{40}(\mathbb{P}(S^5), \mathbb{Z})$ and $H_{40}(\mathbb{P}(S^5), \mathbb{Z})$. $\diamondsuit$

Chapter 5

HERMITIAN MATRICES AND KÄHLER VARIETIES

5.1 Statement of the results

One of the special features of the Kähler geometry is the interplay between topology and linear algebra. The goal of this chapter is to give some applications of our results about spaces of hermitian matrices to obtain bounds on the dimension of the kernel of the cup product mapping

$$\phi\colon \bigwedge^2 H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C}),$$

in the case where X is a compact Kähler variety admitting no Albanese fibration (for a precise definition, see 5.2.4).

The main result we prove here is:

Theorem 5.1.1. *Let X be a compact Kähler variety without Albanese fibrations, and let $\phi\colon \wedge^2 H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C})$ be the cup product.*

1. *If $q \leq 2n-1$, then ϕ is injective;*
2. *if $q = 2n$, then $\dim\ker\phi \leq \rho_{\mathbb{C}}(q)+1$;*
3. *if $q = 5$ and $n = 2$, then $\dim\ker\phi \leq 14$.*

5.2 Variety of Albanese type

Let us begin by recalling the notion of Kähler variety and Hodge decomposition (for a complete reference on the subject, see e.g. [40]).

For a complex variety X, denote by I the complex structure on the tangent bundle TX. We say that ω is a form of type $(p.q)$ if it is a complex $(p+q)$-differential form that locally can be written as a combination of forms $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$.

Definition 5.2.1. *We say that a complex variety X is Kähler if there exists a real closed* (1.1) *form ω (the Kähler form) such that $g(u,v)=\omega(u,Iv)$ is a Remannian metric.*

Denote by Ω_X^p the sheaf of holomorphic differential p-forms on X and consider the Hodge spaces $H^q(X,\Omega^p)$. Moreover, denote by $H^{p.q}(X)$ the subspace of $H^{p+q}(X,\mathbb{C})$ generated by the closed $(p.q)$ forms. We have (see e.g. [40]):

Theorem 5.2.2. (Hodge decomposition) *If X is a compact Kähler variety, then $H^{p.q}\simeq H^q(X,\Omega^p)$ and*

$$H^k(X,\mathbb{C})=\bigoplus_{p+q=k} H^{p,q}.$$

Albanese variety

Let X be a compact Kähler variety of dimension n; we set $V=H^{1,0}$ and $H^{0,1}=\overline{V}$ its conjugate and $q=\dim V$.

Then, integration defines

$$j: H_1(X,\mathbb{Z})\longrightarrow V^*,$$

where $*$ stands for dual; the Albanese variety is

$$Alb(X)=V^*/j(H_1(X,\mathbb{Z})).$$

The irregularity of X is denoted by $q_X=\dim V=\dim A$. The choice of a base point $p\in X$ defines the Albanese map

$$\alpha: X\longrightarrow Alb(X).$$

Definition 5.2.3. *We say that X is of Albanese type if α is generically finite. We say that X is of Albanese strict type if, moreover, α is not surjective. That is:*

$$\dim(\alpha(X))=\dim(X)<q_X.$$

In the sequel we will assume X of Albanese strict type. For our purposes, thanks to the result of Campana, we could also assume that α is generically one-to-one (see [13] and [5, Ch. 2, Sect. 4]).

Decomposable form and Albanese fibrations

We describe some Castelnuovo-de Franchis-type theorems. With the previous notations, the ma pin cohomology induced by the Albanese morphism $\alpha^*\colon H^k(A,\mathbb{C})\to H^k(X,\mathbb{C})$ is a Hodge structure map. We can moreover make the identifications: $H^k(A,\mathbb{C})\equiv\bigwedge^k H^1(X,\mathbb{C})$, $V=H^{1.0}(A)\equiv H^{1.0}(X)$ and $H^{p.q}(A)=\bigwedge^p V\otimes\bigwedge^q\overline{V}$. We have maps: $\alpha^{p,q}\colon \bigwedge^p V\otimes\bigwedge^q\overline{V}\to H^{p.q}(X)$.

Definition 5.2.4. *Given $k < n$, we say that a (rational map) $f : X \to Y$ is an $n-k$-Albanese fibration if*

1. *Y is of Albanese strict type;*
2. $\dim X - \dim Y = n - k$.

When $k = 1$, Y is a curve of genus $g > 1$ and f is usually called an irregular pencil.

We have (see [14]):

Proposition 5.2.5. *The following conditions are equivalent*

1. *X has no s-Albanese fibration for $s \leq n-k$*
2. *$\alpha^{k+1.0}$ is injective on the decomposable forms:*

$$0 \neq \beta_1 \wedge \cdots \wedge \beta_{k+1} \in H^{k+1,0}(X), \qquad \beta_i \text{ independent.}$$

Remark 5.2.6. More precisely, Fabrizio Catanese in [14] gave a one-to-one correspondence between fibrations of Albanese type and maximal isotropic subspaces of the first cohomology group of X.

5.3 Forms and matrices

In this section we show how the hermitian matrices are related to the real forms in $H^{1.1}(X)$ and how the assumprion that X does not have Albanese fibrations imposes conditions on the rank. This will lead to the proof of Theorem 5.1.1.

Real 1.1 forms

As before, X is of strict Albanese type. We will consider in details the map:

$$\alpha^{1.1} \colon H^{1.1}(A) \equiv V \otimes \overline{V} \to H^{1.1}(X). \tag{5.1}$$

This is the (1.1) part of the cup product map $\phi \equiv \alpha^2 : H^2(A, \mathbb{C}) \equiv \bigwedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$. We set

$$\kappa = \dim_{\mathbb{C}} \ker(\phi^{1.1}). \tag{5.2}$$

Since α^2 is a piece of a Hodge structure map, the kernel of α^2 is defined over the rational numbers and a fortiori over the real numbers.

Set $H^{1.1}_{\mathbb{R}}(\cdot) = H^{1.1} \cap H^2(\cdot, \mathbb{R})$; then we have a map:

$$\alpha^{1.1}_{\mathbb{R}} \colon H^{1.1}_{\mathbb{R}}(A) \to H^{1.1}_{\mathbb{R}}(X)$$

and in particular $\kappa = \dim_{\mathbb{R}} \ker \alpha^{1.1}_{\mathbb{R}}$.

Denote by $\mathcal{H}_q$ the space of complex hermitian matrices and remark that it is a real vector space.

Proposition 5.3.1. *The space $H^{1.1}_{\mathbb{R}}(A)$ can be identified with the space of sesquilinear forms on V^* or, equivalently, to $\mathcal{H}_q$.*

Proof. Fix a basis $\beta_j \;\; j = 1, ..., q$ of V. Write any element $\Omega \in H^{1.1}_{\mathbb{R}}(A)$ in the form

$$\Omega = i \sum_{l,m} a_{l,m} \beta_l \wedge \overline{\beta_m}, \quad i^2 = -1,\, a_{l,m} \in \mathbb{C};$$

since $\overline{\Omega} = \Omega$, the matrix $A_\Omega = (a_{l,m})$ is hermitian. $\diamondsuit$

We may define $\sigma \colon \mathcal{H}_q \to H^{1.1}_{\mathbb{R}}(X)$ as:

$$\sigma(A) = i \sum_{l,m} a_{l,m} \; \beta_l \wedge \overline{\beta_m}, \tag{5.3}$$

where $A = (a_{l,m})$. We have $\kappa = \dim \ker \sigma$. In particular, the rank and the signature of a form $\Omega \in H^{1.1}_{\mathbb{R}}(A)$ are well defined.

Definition 5.3.2. *If $\Omega \in H^{1.1}(A)_{\mathbb{R}}$ (respectively $A \in \mathcal{H}_q$) has signature (r, s) (and rank $r + s \leq q$), we set $m(\Omega) = \min\{r, s\}$ (respectively $m(A) = \min(r, s)$).*

The following proposition shows how assuming that X does not have Albanese fibrations imposes a condition on the rank of its (1.1) forms (hence on the matrices representing them).

Proposition 5.3.3. *Let $k < n$ be an integer, and assume that X has no $n-k$-Albanese fibration. Let $\Omega \in H^{1.1}_{\mathbb{R}}(A), \Omega \neq 0$. If $m(\Omega) \leq k - 1$ then $\alpha^{1.1}(\Omega) \neq 0$.*

Proof. Up to a change between Ω and $-\Omega$, we may assume $s = m(\Omega)$ where (r, s) is the signature of Ω, $s \leq k - 1 < n - 1$. We may find a basis β_i of V such that

$$\Omega = i \sum_{j=1}^{r} \beta_j \wedge \overline{\beta_j} - i \sum_{j=r+1}^{r+s} \beta_j \wedge \overline{\beta_j} = \Omega^+ - \Omega^-.$$

Setting $\varphi = \beta_{r+1} \wedge \cdots \wedge \beta_{r+s} \in H^{s,0}(X)$ and $\Theta = \varphi \wedge \overline{\varphi}$, we compute:

$$\Omega \wedge \Theta = i \sum_{j=1}^{r} \beta_j \wedge \overline{\beta_j} \wedge \Theta = (-1)^s i \sum_{j=1}^{r} \beta_j \wedge \varphi \wedge \overline{\beta_j \wedge \varphi}.$$

Now, posing $\varphi_j = \beta_j \wedge \varphi \in H^{s+1,0}(X)$ and $\Theta_j = \varphi_j \wedge \overline{\varphi_j}$, we have:

$$\Omega \wedge \Theta = i(-1)^s \sum_j \Theta_j.$$

Assume by contradiction $\Omega \in \ker \alpha^{1.1}$. It follows that $\Omega \wedge \Theta = 0$ in $H^{s+1,s+1}(X)$. Fix $\omega = i\sum_{j=1}^{q} \beta_j \wedge \overline{\beta_j}$; this is the pull-back of a Kähler form on A and is positive on a Zariski open set of X, since the Albanese map of X is generically finite. We get

$$0 = \int_X \Omega \wedge \Theta \wedge \omega^{n-s-1} = \sum_j \int_X i(-1)^s \Theta_j \wedge \omega^{n-s-1}.$$

All summands have the same sign. It follows that

$$\Theta_j \wedge \omega^{n-s-1} = 0;$$

this forces $\Theta_j = 0$ and finally $\varphi_j = \beta_j \wedge \beta_{r+1} \wedge \ldots \wedge \beta_{r+s} = 0$. Since $s+1 \leq k$, we get a contradiction with Prop. 5.2.5. ♢

Then, we have the following:

Corollary 5.3.4. *Assume that X has no Albanese fibration and $\Omega \in \ker \alpha^{1.1}_{\mathbb{R}}$, $\Omega \neq 0$; then, $m(\Omega) \geq n-1$ and $rank(\Omega) \geq 2n$.*

Hermitian matrices

Set $\mathcal{H}_q^{p,+} = \{A \in \mathcal{H}_q, \operatorname{rank} A \geq p\}$ and denote, as usual, $d(\mathcal{H}_q^{p,+})$ the maximal dimension of a real linear space included in $\mathcal{H}_q^{p,+} \cup \{0\}$.

Recalling that $\kappa = \dim \ker \phi^{1.1} = \dim \ker \sigma$ (see 5.3) we have the following:

Proposition 5.3.5. *Let $n = \dim X$, $q = \dim H^{1.0}(X)$ and κ as before. Assume that X has no Albanese fibration; then $\kappa \leq d_{\mathcal{H}_q^{2n,+}}$.*

Proof. It follows from 5.3.4 that $\ker \sigma$ is a real linear subspace of $\mathcal{H}_q^{p,+}$. ♢

Now we can prove Theorem 5.1.1:

Proposition 5.3.6. *If $q < 2n$, then $\alpha^2 \colon H^2(A, \mathbb{C}) \to H^2(X, \mathbb{C})$ is injective.*

Proof. First, one has that $\alpha^{1.1} \equiv \phi^{1.1}$ is injective, since $\mathcal{H}_q^{2n,+} = \emptyset$.

Then, consider $\omega \in H^{2.0}(A)$ such that $\alpha^*(\omega) = 0$; we show that $\omega = 0$. Indeed, if it is not, we can find a basis β_i of V for which $\omega = \sum_{i=1}^{k} \beta_i \wedge \beta_{i+k}$, with $k < n$; taking the $k-1$ form $\phi = \wedge_{i=2}^{k} \beta_i$, we get $\alpha^*(\omega \wedge \phi) = 0$ and consequently $\wedge_{i=1}^{k+1} \beta_i = 0$ on X. By 5.2.5, this would give an $n-k$-Albanese fibration, contradicting our assumptions. ♢

The previous results are essentially standard linear algebra. The first part of the following proposition is a consequence result of Adams [1], [3] (see, also, our Theorem 3.1.8).

Proposition 5.3.7. *Assume than X has no Albanese fibration and $q = 2n$; then we have*

1. $\dim(\ker(\alpha^{1.1})) \leq \rho_{\mathbb{C}}(\frac{q}{2}) + 1 = \rho_{\mathbb{C}}(q) - 1$ *and*
2. $\dim(\ker(\alpha^{2.0})) \leq 1$.

The two inequalities above can be unified by saying that $\dim\ker(\alpha^2) \leq \rho_{\mathbb{C}}(q) + 1$. *Consequently it holds:* $b_2(X) \geq \dim Im\phi \geq q(2q-1) - \rho_{\mathbb{C}}(q) - 1$.

Proof.

1. From proposition 5.3.5 and Theorem 3.1.8 we have $\dim\ker\sigma \leq \rho_{\mathbb{C}}(q) - 1$.
2. Arguing as in 5.3.4, the nontrivial forms in $\ker\alpha^{2.0}$ must be of maximal rank n. Representing them as skew-symmetric matrices, the elements in $\ker\alpha^{2.0} \setminus \{0\}$ are invertible. This is a complex space, and it follows that $\dim(\ker\alpha^{2.0}) \leq 1$.

$\diamondsuit$

In the following, let us show some consequences of Theorem 5.1.1.

Proposition 5.3.8. *Let X be a compact algebraic variety without irregular pencils whose fundamental group admits a presentation with γ generators and ρ relations.*

1. *If* $q = 2n$, *then* $\rho - \gamma \geq q(2q-3) - 2c - 3$; *moreover, if X is a surface i.e.* $n = 2$, $c_2(X) \geq 7$.
2. *If* $q = 5$ *and X is a surface, then* $b_2(X) \geq 31$, *hence* $\rho - \gamma \geq 31$ *and* $c_2(X) \geq 13$.

Proof. First, we recall that (see [4, Th. 1.1] and [5, Ch. 3]) the interplay between the cup product map and the fundamental group gives rise to the following estimate: $\rho - \gamma \geq \dim Im\phi - 2q$.

The estimate on $\rho - \gamma$ follows directly from the previous remark and 5.3.7. When $q = 2n = 4$, we have $b_1(X) = b_3(X) = 8$ and $b_2(X) \geq 21$.

To deduce $b_2(X) \geq 31$ one uses Theorem 4.1.2 together with the fact that $\dim Im\phi^{2.0} \geq 2q - 3 = 7$. The rest is as in the previous point. $\diamondsuit$

Remark 5.3.9. In [4] the following estimate for a compact Kähler variety of any dimension with no irrational pencil is given: $\rho - \gamma \geq 4q - 7$.

Our historical motivation was the following result:

Corollary 5.3.10. *Let X be a minimal projective algebraic surface with* $q = 4$ *and* $p_g = \dim H^{2.0}(X) = 5$. *Let K be the canonical bundle of X; then* $16 \leq K^2 \leq 17$.

Proof. In [8] it was proved that $K^2 \geq 16$ and that if X has an irregular pencil then $K^2 = 16$. When X has no irregular pencil Noether formula (see [9], [26]) $K^2 + c_2(X) = 12\chi_{\text{hol}} = 12(p_g - q + 1) = 24$ and the inequality $c_2(X) \geq 7$ forces then $K^2 \leq 17$. ♢

Remark 5.3.11. *In the previous example the Miyaoka-Bogomolov-Yau inequality (see for instance [9]) gives only $K^2 \leq 9\chi_{hol} = 18$. No examples of surfaces with $K^2 = 17$ are actually known.*

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