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TIME EVOLUTION OF INFINITELY EXTENDED CLASSICAL SYSTEMS IN STATISTICAL MECHANICS

Tesi di Dottorato

di

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Chapter 1

Introduction

This thesis deals with infinitely extended classical systems, and in this wide context we study two particular problems, both aimed to give a microscopic description of macroscopic phenomena which pertain to Statistical Physics.

The first problem, treated in Chapter 2, arises in the rigorous study of Non Equilibrium Statistical Mechanics, and it consists in giving a precise sense to the time evolution of states of infinitely many particles. We consider a physical system composed by infinitely many particles mutually interacting in three dimensions via a bounded superstable long-range potential. We want to establish existence and uniqueness of the time evolution of the system governed by the Newton Equations (1.1), which means essentially to show that a quasi-local observable evolves remaining quasi-local. This is not trivial because we can exhibit situations with an initial bounded density that after a finite time produce infinitely many particles in a bounded region. Consider in fact a one-dimensional system of point-like, non-interacting particles with initial positions and velocities $q_i = i$, $v_i = -i$, $i \in \mathbb{Z}$. At time $t = 1$ we have a configuration with all particles at the origin. So we need a careful choice of the initial conditions in order to exclude these bad data, but at the same time to take into account all the relevant states in Non Equilibrium Statistical Mechanics. The results depend in a very sensitive way on the dimension of the space in which the particles move and on the nature of the mutual interaction. The first pioneer results have been obtained by Lanford ([23], [24]) many years ago, in one dimension for bounded and finite range interactions, then the cases in one dimension with singular interactions ([18], [28]) and Coulomb force ([29]) have been solved. In two dimensions Fritz and Dobrushin solved the problem for finite range potentials ([20]), whereas Fritz ([19]) extended the previous results in two dimensions for superstable, singular, finite range potentials. The extension of this result for long-range potentials is due to Bahn *et al.* ([2]). The three dimensional case has been recently solved by Caglioti, Marchioro, and Pulvirenti for positive, bounded interactions ([10]). In this thesis we extend the results obtained in [10] to bounded superstable long-range potentials. As it was claimed in [10], the extension to superstable potentials seems quite natural and the problems of such a generalization are essentially of technical nature. With the generalizations introduced in the present work the more important potentials which are not yet included in this kind of analysis are those singular at the origin; they, although interesting from a physical point of view, seem to be out of a possible approach with the present techniques in three dimensions. An interaction which is singular at the origin in fact could produce a too fast growth of the maximal velocity assumed by the particles, which could diverge in a finite time. To confirm the difficulties that appear in three dimensions, J. Fritz and R. L. Dobrushin ([20]) have exhibited an example of a system of infinite particles with a hard-core potential, which preserves energy in the collisions but it's not hamiltonian, and it produces a collapse in three dimensions but not in two.

The above quoted papers exhibit explicits sets of initial conditions. Indeed there are other works regarding the Equilibrium (or Stationary) Dynamics, in which the existence of the dynamics can be proved in any dimension, without considering the existence for any single particle trajectory, but using the invariance properties of the equilibrium measure. The initial data have full-measure with respect to the Gibbs (or Stationary Non Equilibrium) measure, but they are not costructively specified ([1], [25], [27], [31], [35], [36], [37]). We recall also some papers dealing with the time evolution of special states ([7], [11], [32]).

Let us briefly describe the contents of Chapter 2, which are also contained in [16]. We are going to consider the motion of a countable collection of identical particles of unit mass in the 3-dimensional Euclidean space $\mathbb{R}^3$. A configuration of the system is represented as an infinite sequence $\{q_i, v_i\}_{i\in\mathbb{N}}$ of the positions and velocities of the particles, and its time evolution is characterized by the solutions of the Newton equations:

$$\ddot{q}_i(t) = \sum_{j\in\mathbb{N},\, j\neq i} F(q_i(t) - q_j(t)), \qquad i \in \mathbb{N}, \tag{1.1}$$

where $F(x) = -\nabla\phi(x)$. We assume that ϕ is a symmetric pair potential, superstable, bounded, and with infinite range, with a power-like decreasing rate (see Chapter 2 for the details). We establish existence and uniqueness of the solutions of Equations (1.1), complemented by the initial conditions $\{q_i(0), v_i(0)\}_{i\in\mathbb{N}}$, chosen in order to exclude data giving rise to a collapse of the system (i.e. infinitely many particles in a bounded region), but taking into account all the relevant states from a thermodynamical point of view.

A natural step forward would be to investigate in more detail the long time behavior of the dynamics. Unfortunately, in this kind of approaches the bounds one gets on the local density and energy are generally bad-behaving in time, so that it is difficult to say something about the time asymptotics of the system. On the other hand it is in this regime that many physical laws can be reproduced. Recently some results in this direction have been obtained in [4], [5], [6], [7], and [9], for some particular one-dimensional systems. In particular, in [5] and [6] authors aim at a microscopic rigorous derivation of Ohm's law. In these papers it is considered, in the framework of fully Hamiltonian models, a charged particle moving in a constant electric field and interacting with a medium composed by infinitely many neutral particles. In [5], for particles moving in an unbounded tube and for large electric field, and in [6], for a strict one-dimensional system and for electric field of any intensity, it has been rigorously proved that if the particle/medium interaction is bounded, positive, and short-range, the particle escapes in the direction of the electric field with a quasi-uniformly accelerated motion (*runaway particle* effect). Ohm's law, predicting a proportionality between the electric field and the mean velocity of the charge carriers, is in this case violated. This effect has been widely studied in kinetic theory to explain the so-called "runaway electrons" observed in plasma physics ([Landau]). The conditions under which this effect takes place are related to a fast decrease in the scattering cross-section of the particle/medium interaction.
Heuristic arguments suggest that the results obtained in [5] and [6] can be extended to singular interactions provided the singularity is integrable (the most important example being Coulomb interaction), see Section 5 of [5]. However, in general, a rigorous analysis on what conditions on the particle/medium interaction assure this *runaway particle* effect, in case of systems of infinitely many particles, seems too difficult, but a conjecture can be formulated (see [15]). If the charged particle interacts with the particles of the medium via a potential behaving as $g\,r^{-\alpha}$ ($g > 0$) for r small, and if the initial velocity of the charged particle is large enough, then:

Conjecture 1.1. *If $\alpha < 2$ the runaway particle effect happens;*
if $\alpha > 2$ the effect does not happen;

if $\alpha = 2$ there exists a positive constant g^*, depending on the intensity of the electric field and the state of the medium, such that for $g < g^*$ the effect happens, while for $g > g^*$ the effect does not happen.

In [8] the conjecture has been proved for $\alpha < 2$, when the medium is composed by infinitely many particles in the mean field approximation (i.e. moving via the Vlasov equation), the interaction between two particles of the medium is bounded, the motion of the charged particle does not affect the motion of the background, and finally the system has initially a one-dimensional symmetry. In [15] the conjecture is proved in case of a schematic model which keeps however the main features of the physical problem. It has been investigated the existence of a stationary state for the system [charged particle]+[medium] moving in a three-dimensional space, in the reference frame in which the charged particle is at rest and the background is composed by infinitely many free particles in the mean field approximation, coming from infinity with a velocity parallel to the electric field and constant flux. The problem is then reduced to a scattering one. It has been studied in detail the threshold case $\alpha = 2$, the solutions of the other cases following by the observation that they correspond to an increment ($\alpha > 2$) or a decrement ($\alpha < 2$) of the repulsive interaction.

Conjecture 1.1 has an immediate consequence on Ohm's law. In fact, in order to have a finite asymptotic velocity for the charged particle for any intensity of the electric field, the runaway particle effect has not to happen, hence a necessary condition for the validity of Ohm's law is that $\alpha > 2$.

When the charged particle interacts with the background via a hard-core interaction (i.e. the potential is infinite for $r \leq r_0$, otherwise it is zero) it has been proved the existence of a stationary motion for any intensity of the electric field. Moreover, for the initial velocity sufficiently close to the stationary one, the approach to the stationary velocity satisfies a power (in time) law (see [13], [12], [14]). We treat in Chapter 3 a problem of this kind, also studied in [12]. Let us introduce the model, describing the physical scenery and motivation of our analysis.

We consider a solid body moving along the x-axis under the action of an external horizontal force E, immersed in a homogeneous fluid. The macroscopic evolution equation is the following:

$$\ddot{X}(t) = -G(\dot{X}(t)) + E(X(t)), \tag{1.2}$$

where $X(t)$ is the position of the body, whose mass is assumed to be equal to one and G, the friction term, is the resultant of all interactions between the body and the medium. We assume the fluid to be a gas of free particles elastically interacting with the body, in the mean-field limit, that is, we let the mass of any particle go to zero, as the number of particles per unit volume goes to infinity, in such a way that the mass density remains finite. The reason for this assumption is related to possible difficulties which one meets when dealing with not averaged quantities, due to velocity fluctuations. Such a limit is well known for interacting particle systems in case of finite total mass ([3], [17], [30], [38]) and for one-dimensional particle systems with unbounded mass ([4]). We remark that the physical model here presented has been previously introduced in connection with the so called piston problem (see [21] and [26] with references quoted therein).

The friction term G, usually determined from phenomenological considerations, is mostly assumed to be positive and linear in $V(t) := \dot{X}(t)$. In particular, in the simple case of E positive and constant, if $G(V)$ has non-vanishing derivative, the solution $V(t)$ of Eq. (1.1) converges exponentially to the limiting velocity V_∞ which satisfies

$$G(V_\infty) = E. \tag{1.3}$$

In [13] it is shown that this trend to equilibrium is not the right one, since it is proved that the asymptotic time behavior of the body velocity to V_∞ is power like. More precisely, if $E > 0$ is constant, assuming the initial velocity V_0 such that $V_\infty - V_0$ is positive and small, it is proved that for large t

$$|V(t) - V_\infty| \approx \frac{C}{t^{d+2}} \tag{1.4}$$

where $d = 1, 2, 3$ is the dimension of the physical space and C is a constant, depending on the medium and on the shape of the obstacle. This trend, surprising for not being exponential, is due to the recollisions between gas particles and body (in fact, neglecting recollisions, we obtain the exponential decay). As it is already stressed in [13], the problem we are concerned with is a long memory one, since effects of very early collisions are retained for long time.

In the present analysis we go forward with respect to [13]. In Sections 3.1 and 3.2 we consider the more complex problem in which $E = 0$ and prove the same asymptotic behavior as in [13]. In this case additional difficulties come from the fact that the body changes its velocity sign, from positive to negative (if $V_0 > 0$), and this complicates the estimates on the effect of recollisions. The same techniques can be employed to study the case $E > 0$ with $V_0 > V_\infty$, which is the completion to the case treated in [13], but not its symmetric. A comment on this is made in Section 3.4, where some generalizations are discussed.

In Section 3.3 we approach a model in which a non constant force is acting on the body. It is evident that in general its motion could be very complicated, nevertheless we believe that, if we perturb slightly the body from its equilibrium position, then the power law approach to the equilibrium should still be valid, a part some exceptional cases. We are not able to prove this statement in general, thus we restrict ourselves to study a particular but significant problem, in which the external force is assumed to be harmonic, the initial data are chosen in a suitable way, and the friction force is large with respect to the external one. Even in this simple case, the problem is pretty hard to be handled. We are no more able, as in the previous cases, to predict the sign of the velocity, which could in principle go to zero oscillating from positive to negative values. Nevertheless we prove the asymptotic time behavior $X(t) \approx -C\, t^{-d-2}$, for t large, for sufficiently small X_0 and V_0.

This result can be immediately applied to a physical pendulum, that is a stick with a fixed point moving in a vertical plane under the action of its own weight and immersed in a viscous medium. Actually, it is usually expected that it reaches its rest point exponentially fast in time, while our results show that, at least for a suitable choice of the medium, the approach takes place with a power law.

We remark that our results are given for a simple shaped body, that is a disk, but this is not essential, since it can be changed into a more general one, as we discuss briefly in Section 3.4 (see also [14]).

Some comments on the result. As we already pointed out, the time behavior we prove is due to the recollisions between the gas particles and the obstacle, which create a long memory effect. On the other hand, it is reasonable to argue that this feature becomes negligible if the background is not constituted by a free gas but by a real gas with ergodic (mixing) properties. In this case we can say that our result remains valid, not as a strict asymptotic behavior, but as a transient long time behavior.

We can ask whether it is probable that a gas particle hits twice the obstacle: it depends on the data of the physical system, taking into account that a typical particle of the gas in thermodynamical equilibrium has a speed of the order of the sound velocity.

We conclude by recalling the works performed during these years of Ph.D., [7], [12], [14], [15], and [16], two of which contained in the present thesis ([12] and [16]).

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Chapter 2

Non Equilibrium Dynamics of infinite particle systems

2.1 Notations, Definitions and Main Results

In this chapter we show existence and uniqueness for the solutions to the Newton equations relative to a system of infinitely many particles moving in the three-dimensional space and mutually interacting via a bounded superstable long-range potential.

Let $X = \{q_i, v_i\}_{i\in\mathbb{N}}$ be the infinite sequence of positions and velocities of the particles. We assume that X is a locally finite configuration, that is in any compact set $\Lambda \subset \mathbb{R}^3$ the number of the particles in the region Λ:

$$n_\Lambda = \sum_{i\in\mathbb{N}} \chi(q_i \in \Lambda) \tag{2.1}$$

is finite. We denote by $\chi(A)$ the characteristic function of the set A, and by $B(\mu, R)$ the open ball centered in μ and of radius R. The integer part of the real number x is here denoted by $[x]$.
For simplicity in the sequel we will denote by $D_i, E_i, L_i, \widetilde{D}_i, \widetilde{E}_i, \widetilde{L}_i$ any positive constant, possibly depending on the interaction ϕ and on the initial configuration X of the system.
Let us now define the class of superstable interactions, which we are going to consider.
Given a symmetric pair potential $\phi(x) \equiv \phi(|x|)$, $x \in \mathbb{R}^3$, continuous with its first and second derivatives, we give the following definition:

Definition 2.1 (Superstability). *Let us divide the space* $\mathbb{R}^3$ *into cubes* Δ_α *of side* 1 *and centered in* $\alpha \in \mathbb{Z}^3$. *Let* n_{Δ_α} *be the number of particles in* Δ_α.
We say that the potential ϕ *is superstable if there exist constants* $A > 0$, $B \geq 0$ *for which* $\forall n$ *and* $\forall q_1, \ldots, q_n$ *we have:*

$$U(q_1, \ldots, q_n) \geq -Bn + A \sum_\alpha n^2_{\Delta_\alpha}, \tag{2.2}$$

with

$$U(q_1, \ldots, q_n) = \frac{1}{2} \sum_{i \neq j} \phi(|q_i - q_j|).$$

A superstable potential can be decomposed into the sum of a stable potential plus a potential not negative, strictly positive at the origin ([33], [34]). In spite of the presence of an attractive part, superstability avoids large concentrations of particles in small regions of space.
Here we consider the interaction due to a superstable, bounded, long-range potential, with a power-like decreasing rate, for which there exist positive constants γ, G_1, G_2, G_3, r_0, such that, for $|x| > r_0$:

$$|\, \phi(x) \,| \leq \frac{G_1}{|x|^\gamma}, \tag{2.3}$$

$$|\nabla \phi(x)| \leq \frac{G_2}{|x|^{\gamma+1}}, \tag{2.4}$$

and

$$|\nabla \phi(x) - \nabla \phi(y)| \leq \frac{G_3}{(1 + \min(|x|, |y|))^{\gamma+2}} \, |x - y|. \tag{2.5}$$

In the sequel we assume $\gamma > 7$. This technical assumption will be discussed at the end of this Section.

In order to consider configurations which are typical from a thermodynamical point of view, we must allow initial data with logarithmic divergences in the velocities and in the local densities.
More precisely, we define, using the short-hand notation $\phi_{i,j} = \phi(|q_i - q_j|)$,

$$Q(X; \mu, R) = \sum_{i \in \mathbb{N}} \chi(|q_i - \mu| \leq R) \left(\frac{v_i^2}{2} + \frac{1}{2} \sum_{\substack{j: j \neq i, \\ q_j \in B(\mu, R)}} \phi_{i,j} + b \right), \quad b > B, \tag{2.6}$$

and

$$Q_\xi(X) = \sup_\mu \sup_{R: R > \psi_\xi(|\mu|)} \frac{Q(X;\mu,R)}{R^3}, \tag{2.7}$$

where

$$\psi_\xi(x) = \{\log(\max(x,e))\}^\xi, \quad x \in \mathbb{R}^+. \tag{2.8}$$

For each $\xi \geq 1/3$, the set of all configurations for which $Q_\xi(X) < \infty$ constitutes a full measure set for all Gibbs states associated to the particle system (see [18], [20]).

If the initial configuration $X = \{q_i(0), v_i(0)\} \in \mathcal{X}_\xi$, with $\mathcal{X}_\xi = \{X : Q_\xi(X) < \infty\}$, we will make sense of the infinite set of Newton equations:

$$\ddot{q}_i(t) = F_i(X(t)) = \sum_{j \neq i} F_{i,j}(t), \tag{2.9}$$

where $F_{i,j} = -\nabla\phi(|q_i - q_j|)$ is the force exerted by the particle j on the particle i. The solutions to the Newton equations will be constructed by means of a limiting procedure. Neglecting all the particles outside $B(0,n)$, we consider, for an integer n:

$$\begin{aligned} \ddot{q}_i^n(t) &= F_i^n(t), \\ q_i^n(0) = q_i \quad &, \quad v_i^n(0) = v_i \quad i \in I_n, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} I_n &= \{i \in \mathbb{N} : q_i \in B(0,n)\}, \\ F_i^n(t) &= \sum_{\substack{j : j \neq i, \\ j \in I_n}} F(q_i^n(t) - q_j^n(t)), \end{aligned}$$

and

$$X^n(t) = \{q_i^n(t), v_i^n(t)\}_{i \in I_n}$$

is the time evolved finite configuration.

Even if we consider here the more general case of long-range potentials, it is useful to underline the differences that occur considering short-range and long-range potentials (in both cases of a superstable, bounded type). For short-range potentials the following Theorem holds:

Theorem 2.1. *If $X \in \mathcal{X}_\xi$, there exists a unique flow $t \to X(t)$, with*

$$X(t) = \{q_i(t), v_i(t)\}_{i\in\mathbb{N}} \in \mathcal{X}_{\frac{3}{2}\xi},$$

satisfying:

$$\ddot{q}_i(t) = F_i(X(t)) \qquad X(0) = X. \tag{2.11}$$

Moreover, $\forall t > 0$ and $\forall i \in \mathbb{N}$,

$$\lim_{n\to+\infty} q_i^n(t) = q_i(t), \quad \lim_{n\to+\infty} v_i^n(t) = v_i(t). \tag{2.12}$$

For long-range potentials the existence of the dynamics is defined starting from conditions for which ξ is not too large: $\xi < 4/9$ (this restriction for ξ will be clear in Section 5 where it will be used to make the iterative method work). In order to include states of physical interest we then take $\xi \in [1/3, 4/9)$. The Theorem in this case is the following:

Theorem 2.2. *If $X \in \mathcal{X}_\xi$, there exists a unique flow $t \to X(t)$, with*

$$X(t) = \{q_i(t), v_i(t)\}_{i\in\mathbb{N}} \in \tilde{\mathcal{X}}_\xi,$$

satisfying:

$$\ddot{q}_i(t) = F_i(X(t)) \qquad X(0) = X, \tag{2.13}$$

where

$$\tilde{\mathcal{X}}_\xi = \mathcal{X}_{\frac{3}{2}\xi} \cap \bar{\mathcal{X}}_\xi, \tag{2.14}$$

and

$$\bar{\mathcal{X}}_\xi = \{q_i, v_i : \forall i \in \mathbb{N} \ \ |v_i| \leq C\psi_\xi^{3/2}(|q_i|)\}, \tag{2.15}$$

with $C > 0$.
Moreover, $\forall t > 0$ and $\forall i \in \mathbb{N}$,

$$\lim_{n\to+\infty} q_i^n(t) = q_i(t), \quad \lim_{n\to+\infty} v_i^n(t) = v_i(t). \tag{2.16}$$

Theorems 2.1 and 2.2 are the main results and their proofs occupy the rest of the chapter. The proofs are based on several steps: we introduce a mollified version of the local energy and we study its evolution in time under the partial dynamics. The energy conservation allows to prove that the local energy grows in time at most as the cube of the maximal velocity of the particles. On the other hand a suitable time average allows to control the maximal velocity via the local energy in a good way. The result is achieved by letting $n \to \infty$. The philosophy of the proof is similar to that of reference [10]. Actually in that paper the authors use many times the positivity and the finite range of the interaction, while here the interaction can be negative and with long-range behavior. This fact requires a new mollifier and other cumbersome technical tools.

In the sequel we will need to split the potential into two terms: a short-range one, $\phi^{(1)}$, and a long-range one, $\phi^{(2)}$.
To do so, let us take, for $r > \max(r_0, \sqrt{3})$:

$$\begin{cases} \phi(x) &= \phi^{(1)}(x) + \phi^{(2)}(x) \\ \phi^{(1)}(x) &= \phi(x)\,\chi(|x| \le r) \\ \phi^{(2)}(x) &= \phi(x) - \phi^{(1)}(x) \\ |\phi^{(2)}(x)| &\le \dfrac{G_1}{|x|^\gamma} \end{cases} \tag{2.17}$$

The following Proposition holds:

Proposition 2.1. *Let ϕ as in (2.17). Then $\exists \bar{r} > 0$ such that, $\forall r \ge \bar{r}$, $\phi^{(1)}$ is superstable.*

Proof
From the superstability of ϕ we have

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \phi_{i,j} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} (\phi_{i,j}^{(1)} + \phi_{i,j}^{(2)}) \ge -B\,n + A\sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2 \Rightarrow$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \phi_{i,j}^{(1)} \ge -\frac{1}{2}\sum_{i,j}^{\star} \frac{G_1}{|q_i - q_j|^\gamma} - B\,n + A\sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2, \tag{2.18}$$

where $\sum_{i,j}^{\star}$ is the sum restricted to particles at distance greater than r.
Let us consider the first term on the right:

$$\begin{aligned}
&\sum_{i,j}^{\star} \frac{G_1}{|q_i - q_j|^\gamma} = G_1 \sum_{k=1}^{\infty} \sum_{i \neq j} \chi(kr < |q_i - q_j| \leq (k+1)r) \frac{1}{|q_i - q_j|^\gamma} \\
\leq\ & G_1 \sum_{k=1}^{\infty} \frac{1}{(rk)^\gamma} \sum_{i \neq j} \chi(kr < |q_i - q_j| \leq (k+1)r) \\
\leq\ & \sum_{k=1}^{\infty} \sum_{\substack{l \in \mathbb{Z}^3, \\ m \in \mathbb{Z}^3}} \chi(kr - \sqrt{3} \leq |l - m| < (k+1)r + \sqrt{3}) \frac{G_1}{(kr)^\gamma} n_{\Delta_l} n_{\Delta_m} \\
\leq\ & \sum_{i \in \mathbb{Z}^3} n_{\Delta_i}^2 \sum_{k=1}^{\infty} \frac{G_1}{(rk)^\gamma} \\
& \times \mathbf{Card}\{\mathbb{Z}^3 \cap (B(0,(k+1)r + \sqrt{3}) \setminus B(0, kr - \sqrt{3}))\} \\
\leq\ & \frac{D_1}{r^{\gamma-3}} \sum_{k=1}^{\infty} \frac{1}{k^{\gamma-2}} \sum_{i \in \mathbb{Z}^3} n_{\Delta_i}^2 \leq \frac{D_2}{r^{\gamma-3}} \sum_{i \in \mathbb{Z}^3} n_{\Delta_i}^2,
\end{aligned} \tag{2.19}$$

when $\gamma > 3$.
Inserting (2.19) in (2.18), we obtain:

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \phi_{i,j}^{(1)} \geq \left(-\frac{D_2}{2\, r^{\gamma-3}} + A \right) \sum_{i \in \mathbb{Z}^3} n_{\Delta_i}^2 - B\, n, \tag{2.20}$$

then for

$$r \geq \bar{r} = \max \left(\left(\frac{2D_2}{A} \right)^{\frac{1}{\gamma-3}}, r_0,\, \sqrt{3} \right) \tag{2.21}$$

we obtain the thesis:

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \phi_{i,j}^{(1)} \geq\ \frac{3}{4} A \sum_{i \in \mathbb{Z}^3} n_{\Delta_i}^2 - B\, n. \qquad \blacksquare \tag{2.22}$$

For a configuration X with finite cardinality, let us define a mollified version of the energy (plus b times the number of particles, with $b > B$) for the particles contained into the ball $B(\mu, R)$, by means of a suitable weight-function:

$$W(X;\mu,R) = \sum_{i\in\mathbb{N}} f_i^{\mu,R}\Big(\frac{v_i^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi_{i,j} + b\Big), \tag{2.23}$$

with a weight-function

$$f_i^{\mu,R} \equiv f(q_i-\mu,R) \equiv \int_{\mathbb{R}^3} \theta\left(\frac{|q_i-\mu-y|}{R}\right)\left(\frac{1}{1+\alpha|y|}\right)^{\lambda} dy, \tag{2.24}$$

where $\theta : \mathbb{R}^+ \to (0,1]$, is continuously differentiable and it is such that

1. $\theta(x) = (1+\alpha x)^{-\lambda}$, for $x \geq 2$,
2. $\theta(x)$ is concave for $x \leq 2$,
3. $\theta(x) = \theta(2) - \frac{1}{2}\theta'(2)$, for $x \leq 1$.

Notice that

$$\theta(x) \leq (1+\alpha x)^{-\lambda} \tag{2.25}$$

and

$$|\theta'(x)| \leq \lambda\,\alpha(1+\alpha x)^{-(\lambda+1)} \tag{2.26}$$

with $\lambda > 3$ and $\alpha \in (0,1]$. In the sequel we shall assume $\lambda \in (4, \gamma-3)$ and α small enough (for details see Appendix A).
Following [2], let us show the main properties of the weight-function:

Proposition 2.2. *There exist positive constants C_1, C_2, depending only on α and λ, such that, for any $R > 1$, the following properties hold*

1. $f(x,R) \leq C_1(1+\alpha|x|/R)^{-\lambda}$,
2. $f(x,R) \geq C_2(1+\alpha\,|x|/R)^{-\lambda}$,
3. $f(x,R) \leq (1+\alpha|x-y|)^{\lambda} f(y,R)$.

Proof

1. Let us prove the first property. Multiplying $f(x,R)$ by $(1+\alpha|x|/R)^\lambda$, using the triangular inequality we obtain:

$$\begin{aligned}\left(1+\alpha\frac{|x|}{R}\right)^\lambda f(x,R) \leq & \int_{\mathbb{R}^3} dy \left(\frac{R+\alpha|y|+\alpha|x-y|}{R+\alpha|x-y|}\right)^\lambda \frac{1}{(1+\alpha|y|)^\lambda} \\ \leq & 2^\lambda \int_{\mathbb{R}^3} dy \, \frac{(1+\alpha|y|)^\lambda + (R+\alpha|x-y|)^\lambda}{(R+\alpha|x-y|)^\lambda} \frac{1}{(1+\alpha|y|)^\lambda}\end{aligned}$$

considering that $\forall a,b \in \mathbb{R}^+$ it holds $(a+b)^\lambda \leq 2^\lambda\,(a^\lambda + b^\lambda)$. Then

$$\begin{aligned}\left(1+\alpha\frac{|x|}{R}\right)^\lambda f(x,R) \leq & 2^\lambda \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|y|)^\lambda} + 2^\lambda \int_{\mathbb{R}^3} dy \frac{1}{(R+\alpha|y-x|)^\lambda} \\ \leq & \;\; 2^{\lambda+1} \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|y|)^\lambda} \leq C_1\end{aligned}$$

for $R>1$.

2. Notice first that:

$$\theta\left(\frac{|x-y|}{R}\right) \geq \theta(2)\,\frac{1}{(1+\alpha|x-y|/R)^\lambda}\,, \tag{2.27}$$

being $\theta(2) = \min_{|x|\leq 2} \theta(x)$. So for the weight-function we have

$$f(x,R) \geq \theta(2) \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|x-y|/R)^\lambda} \frac{1}{(1+\alpha|y|)^\lambda}\;.$$

Multiplying $f(x,R)$ by $(1+\alpha|x|/R)^\lambda$, we obtain:

$$\begin{aligned}\left(1+\alpha\frac{|x|}{R}\right)^\lambda f(x,R) \geq & \frac{\theta(2)}{2^\lambda}\int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|y|)^\lambda} \frac{(1+\alpha|x|/R)^\lambda}{(1+\alpha|y|/R)^\lambda + (1+\alpha|x|/R)^\lambda} \\ \geq & \;\; \frac{\theta(2)}{2^\lambda} \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|y|)^\lambda} \frac{1}{1+\left(\frac{1+\alpha|y|/R}{1+\alpha|x|/R}\right)^\lambda} \\ \geq & \;\; \frac{\theta(2)}{2^\lambda} \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|y|)^\lambda} \frac{1}{1+(1+\alpha|y|/R)^\lambda} \\ \geq & \;\; \frac{\theta(2)}{2^\lambda} \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|y|)^\lambda} \frac{1}{1+(1+\alpha|y|)^\lambda} \geq C_2\,,\end{aligned}$$

for $R > 1$.
3. For the third relation let us write the function f in the following way, putting $x - y = z$:

$$f(x, R) = \int_{\mathbb{R}^3} \theta\left(\frac{|z|}{R}\right) \left(\frac{1}{1 + \alpha|x - z|}\right)^{\lambda} dz.$$

Since

$$\frac{1}{1 + \alpha|x - z|} \leq \frac{1 + \alpha|x - y|}{1 + \alpha|y - z|},$$

the thesis follows (last inequality becomes evident multiplying both sides by $(1 + \alpha|x - z|)(1 + \alpha|y - z|)$ and using the triangular inequality). ■

The choice of such a weight-function will be evident later, in the proof of Lemma 2.1. This function, unlike the mollifier function used in [10], allows also to give some superstability estimates for the energy of a bounded region of the space, essential in the proof of Lemma 2.2.
Notice that, if the interaction has finite range, we could use an explicit weight-function, i.e. $f(x) = 1/\cosh(x)$. In general an exponential decay for the weight-function is too fast for taking into account potentials with a power-law decay.

We give now a short explanation for the technical assumption on the power-law decay ($\gamma > 7$) of the interaction. The weight-function must decay slower than the interaction ($\gamma > 3+\lambda$) to handle the border terms of the mollified energy (see (A.12)); moreover the weight-function must decay fast enough ($\lambda > 4$, see (C.9)) to obtain the boundedness of the mollified density energy $W_\xi(X)$ defined in (2.31).

2.2 Properties of the mollified energy

We present here a lemma, whose proof is shown in Appendix A, that gives a superstability property of the mollified energy.

Lemma 2.1. *There exist $C_3 > 0$ and $\bar{\alpha} \in (0, 1)$, not depending on R, such that $\forall \alpha \in (0, \bar{\alpha})$:*

$$W(X; \mu, R) \geq C_3 \sum_{k \in \mathbb{Z}^3} f(|k - \mu|, R)\, n_{\Delta_k}^2. \tag{2.28}$$

We actually prove a stronger condition:

$$\begin{aligned} W(X;\mu,R) &\geq \sum_{i\in\mathbb{N}} f_i^{\mu,R}\Big(\frac{1}{2}\sum_{j:j\neq i}\phi_{i,j}+b\Big) \\ &\geq C_3\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 \geq 0, \end{aligned} \tag{2.29}$$

which implies that the interaction energy is non-negative. In the sequel the parameter α appearing in Lemma 2.1 will be considered fixed.
From the previous lemma we can derive the following corollaries:

Corollary 2.1. *There exist $C_3, C_4 > 0$, not depending on R, such that:*

$$\begin{aligned} C_3\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 \leq W(X;\mu,R) &\leq C_4\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 \\ &+ \sum_{i\in\mathbb{N}} f_i^{\mu,R}\,\frac{v_i^2}{2}. \end{aligned} \tag{2.30}$$

The first inequality reproduces Lemma 2.1, while the proof of the second inequality will be given in Appendix B.
The function W is a technical tool. The following Corollary (whose proof is given in Appendix C) shows the relation with the initial data. Defining

$$W_\xi(X) \equiv \sup_{\mu}\ \sup_{R>\psi_\xi(|\mu|)} \frac{W(X;\mu,R)}{R^3}, \tag{2.31}$$

then it holds:

Corollary 2.2. $\exists\, C_5,\, C_6 > 0$*, not depending on R, such that:*

$$C_5\, Q_\xi(X) \leq W_\xi(X) \leq C_6\, Q_\xi(X). \tag{2.32}$$

We can give now an estimate for the mollified energy, useful for the proof of the existence of the dynamics.

Lemma 2.2. *For $X \in \mathcal{X}_\xi$, there exists a positive constant C_7 such that*

$$\sup_{\mu} W(X^n(t); \mu, R(n,t)) \leq C_7 R^3(n,t)\,, \tag{2.33}$$

where

$$R(n,t) = \varphi(n) + \int_0^t ds\, V^n(s), \tag{2.34}$$

with

$$\varphi(n) = \psi_\xi^{3/2}(n)$$

and

$$V^n(s) = \max_{i \in I_n} \left\{ \sup_{0 \leq \tau \leq s} |v_i^n(\tau)| \right\}.$$

Proof
For $0 \leq s \leq t \leq T$ let us define

$$R(n,t,s) = R(n,t) + \int_s^t V^n(\tau)\, d\tau\,. \tag{2.35}$$

Notice that

$$\dot{R}(n,t,s) \equiv \frac{\partial R}{\partial s}(n,t,s) = -V^n(s) \leq 0,$$

moreover

$$R(n,t,t) = R(n,t), \quad R(n,t,0) < 2R(n,t).$$

Let us derive with respect to s the quantity:

$$\begin{aligned} W(X^n(s); \mu, R(n,t,s)) &= \sum_{i \in \mathbb{N}} f_i^{\mu, R(n,t,s)} \Big(\frac{v_i^2}{2} + \frac{1}{2} \sum_{\substack{j: j \neq i, \\ j \in \mathbb{N}}} \phi_{i,j} + b \Big) \\ &= \sum_{i \in \mathbb{N}} f_i^{\mu, R(n,t,s)}\, w_i, \end{aligned} \tag{2.36}$$

with

$$w_i \equiv \frac{v_i^2}{2} + \frac{1}{2} \sum_{j: j \neq i} \phi_{i,j} + b. \tag{2.37}$$

We have:

$$\frac{\partial W}{\partial s} = \dot{W}_1 + \dot{W}_2, \tag{2.38}$$

where

$$\begin{aligned}
\dot{W}_1 &\equiv \sum_i w_i \int_{\mathbb{R}^3} dy\, \theta' \left(\frac{|q_i - \mu - y|}{R} \right) \\
&\times \frac{1}{(1+\alpha|y|)^\lambda} \left(\frac{\mathrm{Vers}(q_i - y - \mu)\ \cdot v_i}{R(n,t,s)} - \frac{\dot{R}(n,t,s)}{R^2(n,t,s)} |q_i - y - \mu| \right), \\
\dot{W}_2 &\equiv \sum_{i \neq j} f_i^{\mu, R(n,t,s)} \Big(v_i \cdot F_{i,j} - \frac{1}{2} F_{i,j} \cdot (v_i - v_j) \Big).
\end{aligned} \tag{2.39}$$

We have denoted by Vers (x) the versor of the vector $x \in \mathbb{R}^3$. Let us consider now the first term $\dot{W}_1$. Thanks to (2.26) and to the definition of V^n, we have:

$$\begin{aligned}
|\dot{W}_1| &\leq \lambda |\frac{\dot{R}}{R}|\, \alpha \sum_i |w_i| \int_{\mathbb{R}^3} dy \frac{1}{(1 + \alpha|q_i - \mu - y|/R)^{\lambda+1}} \\
&\times \frac{1}{(1+\alpha|y|)^\lambda} \left(1 + \frac{|q_i - y - \mu|}{R(n,t,s)} \right) \\
&\leq D_3\, |\frac{\dot{R}}{R}| \sum_i |w_i| \int_{\mathbb{R}^3} dy \frac{1}{(1 + \alpha|q_i - \mu - y|/R)^{\lambda}} \frac{1}{(1+\alpha|y|)^\lambda} \\
&\leq D_4\, |\frac{\dot{R}}{R}| \sum_i f_i^{\mu, R(n,t,s)} |w_i|,
\end{aligned} \tag{2.40}$$

where in the last inequality we have applied (2.27). From the positivity of the mollified energy and from estimates analogous to those used to obtain (B.1) we have:

$$|\dot{W}_1| \leq D_5 \Big|\frac{\dot{R}}{R}\Big| \left(W(X;\mu,R) + \sum_{i \in \mathbb{Z}^3} n^2_{\Delta_i} f(|i-\mu|, R) \right), \tag{2.41}$$

and from Lemma 2.1 we obtain:

$$|\dot{W}_1| \leq D_6 \Big|\frac{\dot{R}}{R}\Big| W(x;\mu,R). \tag{2.42}$$

For the second term $\dot{W}_2$ we are going to give also an estimate of the form:

$$\dot{W}_2 \leq D_{11} \Big|\frac{\dot{R}}{R}\Big| W(X;\mu,R). \tag{2.43}$$

Let us evaluate

$$\dot{W}_2 = \sum_{i\neq j} f_i^{\mu,R(n,t,s)}\Big(v_i\cdot F_{i,j} - \frac{1}{2}F_{i,j}\cdot(v_i - v_j)\Big)$$
$$= \frac{1}{2}\sum_{i\neq j} f_i^{\mu,R(n,t,s)} F_{i,j}(v_i + v_j). \tag{2.44}$$

Since $F_{i,j} = -F_{j,i}$, it results:

$$\dot{W}_2 = \frac{1}{2}\sum_{i\neq j} f_i^{\mu,R(n,t,s)} F_{i,j}\cdot(v_i + v_j)$$
$$= \frac{1}{2}\sum_{i\neq j} f_i^{\mu,R(n,t,s)} F_{i,j}\cdot v_i - \frac{1}{2}\sum_{i\neq j} f_j^{\mu,R(n,t,s)} F_{i,j}\cdot v_i$$
$$= -\frac{1}{2}\sum_{i\neq j}(f_i^{\mu,R(n,t,s)} - f_j^{\mu,R(n,t,s)})\nabla\phi_{i,j}\cdot v_i(s). \tag{2.45}$$

Let us estimate now the addends of the sum one by one. From the properties of $\theta(x)$ (2.26), (2.27) and of the potential we have:

$$|f_i^{\mu,R(n,t,s)} - f_j^{\mu,R(n,t,s)}| \leq D_7 \frac{|q_i - q_j|}{R(n,t,s)}\,(f_i^{\mu,R(n,t,s)} + f_j^{\mu,R(n,t,s)}). \tag{2.46}$$

Being

$$|\nabla\phi(|q_i - q_j|)| \leq D_8(1 + |q_i - q_j|)^{-\gamma-1}, \tag{2.47}$$

then, using an estimate analogous to (B.1):

$$|\dot{W}_2| \leq D_9\Big|\frac{\dot{R}}{R}\Big| \sum_{i\in\mathbb{N}} \sum_{\substack{j\in\mathbb{N}:\\ i\neq j}} (f_i^{\mu,R(n,t,s)} + f_j^{\mu,R(n,t,s)})\,\frac{1}{(1 + |q_i - q_j|)^{\gamma}}$$
$$\leq D_{10}\Big|\frac{\dot{R}}{R}\Big| \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i}^2. \tag{2.48}$$

Using Lemma 2.1 we close the estimate with the function W:

$$|\dot{W}_2| \leq D_{11}\left|\frac{\dot{R}(n,t,s)}{R(n,t,s)}\right| W(X^n(s);\mu, R(n,t,s)). \tag{2.49}$$

We have so proved that

$$\left|\frac{\partial W(X^n(s);\mu,R(n,t,s))}{\partial s}\right| \le D_{12}\left|\frac{\dot R(n,t,s)}{R(n,t,s)}\right| W(X^n(s);\mu,R(n,t,s)). \tag{2.50}$$

Integrating we have:

$$\begin{aligned} W(X^n(s);\mu,R(n,t,s)) \le\, & W(X^n(0);\mu,R(n,t,0)) \\ & + D_{12}\int_0^s d\tau \left|\frac{\dot R(n,t,\tau)}{R(n,t,\tau)}\right| W(X^n(\tau);\mu,R(n,t,\tau)). \end{aligned}$$

Let us use now the Gronwall's lemma to handle the previous inequality:

$$W(X^n(s);\mu,R(n,t,s)) \le W(X^n(0);\mu,R(n,t,0)) \left(\frac{R(n,t,0)}{R(n,t,s)}\right)^{D_{12}}, \tag{2.51}$$

from which, being $\frac{R(n,t,0)}{R(n,t,s)} \le 2$, we obtain

$$W(X^n(s);\mu,R(n,t,s)) \le 2^{D_{12}}\, W(X^n(0);\mu,R(n,t,0)), \tag{2.52}$$

and since $R(n,t,t) = R(n,t)$, taking the supremum over μ, we have

$$\sup_\mu W(X^n(t);\mu,R(n,t)) \le D_{13} \sup_\mu W(X^n(0);\mu,R(n,t,0)). \tag{2.53}$$

From Corollary 2.2 and by the hypothesis on the initial data, being $R(n,t,0) > \psi_\xi(n)$, we get:

$$\sup_\mu W(X^n(0);\mu,R(n,t,0)) \le C_6 Q_\xi(X) R^3(n,t,0), \tag{2.54}$$

thus

$$\sup_\mu W(X^n(t);\mu,R(n,t)) \le D_{14} R^3(n,t). \quad \blacksquare$$

In the next lemma we present some relations that will be used in the sequel. The proof is in Appendix D.

Lemma 2.3. *Let X be a configuration with finite cardinality. Then, for any $R > 1$ there exist positive constants C_8, C_9, C_{10}, C_{11} such that* *i) if* $n \in \mathbb{N}, \ n > 1$

$$W(X;\mu,nR) \le C_8\, n^\lambda\, W(X;\mu,R); \tag{2.55}$$

ii) if $n \in \mathbb{N}, \ n > 1$

$$W(X;\mu,R) \le C_9 W(X;\mu,nR); \tag{2.56}$$

iii)

$$N(X,\mu,R) \equiv \sum_i \chi(|q_i-\mu|<R) \le C_{10} R^{3/2} W(X;\mu,R)^{1/2}; \tag{2.57}$$

iiii) for $0<\rho<R$

$$\sum_{i\neq j} \chi(|q_i-q_j|<\rho)\chi(|q_i-\mu|<R)\chi(|q_j-\mu|<R) \le C_{11}\, \rho^3\, W(X;\mu,R). \tag{2.58}$$

We will need an estimate for the force F_i that at time t acts on the particle i:

$$F_i(X^n(t)) \equiv -\sum_{j\in I_n} \nabla\phi(|q_i(t)-q_j(t)|). \tag{2.59}$$

We can make the following decomposition:

$$|F_i(X^n(t))| \le F_i^{(1)} + F_i^{(2)},$$

where $F_i^{(1)}$ represents a bound for the absolute value of the force acting on the particle i, due to the particles j contained in $B(q_i(t),r)$, with r not less than $\bar r$, defined in Proposition 2.1, and $F_i^{(2)}$ is a bound for the absolute value of the force acting on the particle i, due to the particles j contained in $B^c(q_i(t),r)$.
Using the third property of Lemma 2.3, the first term is bounded by:

$$\begin{aligned} F_i^{(1)} &\le \|F\|_\infty\, N(X^n(t),q_i(t),r) \le \|F\|_\infty\, C_{10}\, r^{3/2} W(X^n(t);q_i(t),r)^{1/2} \\ &\le \|F\|_\infty\, D_{15}\, r^{3/2} \sup_\mu W(X^n(t);\mu,R(n,t))^{1/2} \le D_{16}\ R^{3/2}(n,t), \end{aligned}$$

where, for sufficiently large n, we have used Lemma 2.2.
Let us give now a bound for the second term; for $R=R(n,t)\gg r$ we have

$$F_i^{(2)} \le G_2 \sum_{j:|q_i-q_j|>r} \frac{1}{|q_i-q_j|^{\gamma+1}}$$

$$
\begin{aligned}
&\leq G_2 \sum_{k=1}^{[R/r]+1} \sum_j \chi(kr < |q_i - q_j| \leq (k+1)r) \frac{1}{(kr)^{\gamma+1}} \\
&+ G_2 \sum_{k=1}^{+\infty} \sum_j \chi(kR < |q_i - q_j| \leq (k+1)R) \frac{1}{(kR)^{\gamma+1}} \\
&\leq G_2 \sum_{k=1}^{[R/r]+1} N(X, q_i, (k+1)r)) \frac{1}{(rk)^{\gamma+1}} \\
&+ G_2 \sum_{k=1}^{\infty} N(X, q_i, (k+1)R)) \frac{1}{(kR)^{\gamma+1}} \\
&\leq D_{17} \sum_{k=1}^{[R/r]+1} ((k+1)r)^{3/2}\, W^{1/2}(X, q_i, (k+1)r) \frac{1}{(rk)^{\gamma+1}} \\
&+ D_{17} \sum_{k=1}^{\infty} ((k+1)R)^{3/2}\, W^{1/2}(X, q_i, (k+1)R) \frac{1}{(Rk)^{\gamma+1}} \\
&\leq D_{18} \left(R^{3/2} + R^{3/2-\gamma-1} \sup_\mu W^{1/2}(X, \mu, R) \sum_{k=1}^{\infty} k^{\lambda/2} \frac{1}{k^{\gamma+1-3/2}} \right) \\
&\leq D_{19} \left(R^{3/2} + R^{3-\gamma-1} \sum_{k=1}^{+\infty} \frac{1}{k^{\gamma-\lambda/2-1/2}} \right), \qquad (2.60)
\end{aligned}
$$

where in the penultimate line we have used the first property of Lemma 2.3. Since $\gamma > 3 + \lambda$ we obtain:

$$
F_i^{(2)} \leq D_{20} R^{3/2}(n, t). \qquad (2.61)
$$

Then

$$
|F_i(X^n(t))| \leq D_{21}\, R^{3/2}(n, t). \qquad (2.62)
$$

In the proof of Proposition 2.3 we will need an estimate for the force, $|\bar{F}_i|$, due to the particles j at distance larger than $R(n,t)^{1/4}$ from the particle i:

$$
\begin{aligned}
|\bar{F}_i| &\leq G_2 \sum_{j:|q_i-q_j|>R^{1/4}} \frac{1}{|q_i - q_j|^{\gamma+1}} \\
&\leq G_2 \sum_{k=1}^{+\infty} \sum_j \chi(kR^{1/4} \leq |q_i - q_j| < (k+1)R^{1/4}) \frac{1}{(kR^{1/4})^{\gamma+1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le G_2 \sum_{k=1}^{\infty} N(X, q_i, (k+1)R^{1/4})) \frac{1}{(kR^{1/4})^{\gamma+1}} \\
&\le D_{22} \sum_{k=1}^{\infty} ((k+1)R^{1/4})^{3/2}\, W^{1/2}(X, q_i, (k+1)R) \frac{1}{(kR^{1/4})^{\gamma+1}} \\
&\le D_{23}\, R^{3/8-(\gamma+1)/4} \sup_{\mu} W^{1/2}(X, \mu, R) \sum_{k=1}^{\infty} k^{\lambda/2} \frac{1}{k^{\gamma+1-3/2}} \\
&\le D_{24} R^{3/8+3/2-(\gamma+1)/4} \sum_{k=1}^{+\infty} \frac{1}{k^{\gamma-\lambda/2-1/2}}.
\end{aligned} \tag{2.63}
$$

2.3 Dynamical estimates

The following two propositions give bounds on the maximal velocity of a particle and on the work done by the system over a single particle.
In this section we shall omit any explicit notational dependence on n for $R(n,t)$ and $\{q_i^n(t), v_i^n(t)\}$ for simplicity, since, from now on, n will be fixed.

Proposition 2.3. *For any positive $T < +\infty$, there exists a positive constant C_{12} such that, for $t \le T$,*

$$V^n(t) \le C_{12}\, R(t), \tag{2.64}$$

where

$$R(t) = \varphi(n) + \int_0^t V^n(s)\, ds, \tag{2.65}$$

and

$$\varphi(n) = \psi_\xi^{3/2}(n). \tag{2.66}$$

Proposition 2.4. *For $0 \le s \le t \le T$ and any $\zeta \in [1/2, 1]$, we set:*

$$\Delta = \zeta\, R(t)^{-4/6}. \tag{2.67}$$

Suppose that, for some $i \in I_n$ and some suitable constant $\bar{A} > 1$:

$$\inf_{\tau \in [s-\Delta, s]} |\, v_i(\tau)\,| = \bar{A}\, R(t). \tag{2.68}$$

Then there exists a constant C_{13} independent of $\bar{A}$ such that:

$$\left| \int_{s-\Delta}^{s} d\tau \sum_j v_j \cdot F_{i,j} \right| \leq C_{13} \Delta R(t)^2 \tag{2.69}$$

Using Proposition 2.4 we are able to prove Proposition 2.3.

Proof of Proposition 2.3

The proof will be achieved by contradiction. We first notice that, by the initial conditions, $V^n(0) \leq Q_\xi(X)^{1/2}\varphi(n) = Q_\xi(X)^{1/2}R(0)$ and then (2.64) is verified for $t = 0$. Suppose that, for some $t^\star \in [0, t]$ and $i \in I_n$ we have:

$$V^n(t^\star) = |v_i(t^\star)| = \tilde{A}\, R(t) \tag{2.70}$$

for a suitable constant $\tilde{A}$ to be fixed later and satisfying $\tilde{A} > 2(Q_\xi(X)^{1/2} + 1)$. We also fix $t_1 \in [0, t^\star)$, such that

$$|v_i(t_1)| = (Q_\xi(X)^{1/2} + 1)R(t); \tag{2.71}$$

$$\inf_{\tau \in (t_1, t^\star)} |v_i(\tau)| \geq (Q_\xi(X)^{1/2} + 1)R(t) \tag{2.72}$$

and $|t^\star - t_1| = H\Delta$ for some integer $H \geq 1$ and a suitable choice of ζ. This can be done because by

$$v_1(t^\star) = v_i(t_1) + \int_{t_1}^{t^\star} F_i(X^n(\tau))\, d\tau \tag{2.73}$$

and by (2.62), we find

$$\tilde{A}R(t) \leq (Q_\xi(X)^{1/2} + 1)R(t) + D_{21}(t^\star - t_1)R(t)^{3/2} \tag{2.74}$$

and hence

$$(t^\star - t_1) \geq E_1\, R(t)^{-1/2} \gg R(t)^{-4/6}, \tag{2.75}$$

therefore, for a suitable choice of $\zeta \in [1/2, 1]$, $\frac{R(t)^{4/6}|t^\star - t_1|}{\zeta}$ is integer.
Furthermore, defining the set

$$\bar{Y}_n = \{j \in I_n : |q_i(\tau) - q_j(\tau)| \leq R(t)^{1/4} \text{ for some } \tau \in [t_1, t^*]\}, \tag{2.76}$$

we have

$$\frac{1}{2}v_i^2(t^\star) - \frac{1}{2}v_i^2(t_1) = \int_{t_1}^{t^\star} ds \sum_j v_i \cdot F_{i,j}$$
$$= \mathcal{L}_1 + \mathcal{L}_2, \tag{2.77}$$

where

$$\mathcal{L}_1 \equiv \int_{t_1}^{t^\star} ds \sum_{j\in \bar{Y}_n^c} v_i \cdot F_{i,j} \quad \text{and} \quad \mathcal{L}_2 \equiv \int_{t_1}^{t^\star} ds \sum_{j\in \bar{Y}_n} v_i \cdot F_{i,j}. \tag{2.78}$$

For $\mathcal{L}_1$ we have:

$$|\mathcal{L}_1| \le \max_{s\in[t_1,t^*]} \Big(\sum_{j\in \bar{Y}_n^c} |F_{i,j}(s)| \Big) \int_{t_1}^{t^\star} ds\, |v_i(s)| \le E_2 R(t)^{\frac{23}{8}-\frac{\gamma+1}{4}} \tag{2.79}$$

where the time integral is bounded by $R(t)$ (see (2.34)), and for the sum of the force we have used (2.63). Eq. (2.79) clearly gives $|\mathcal{L}_1| \le E_3\, R(t)^2$.
Let us consider the second term $\mathcal{L}_2$:

$$\begin{aligned}\mathcal{L}_2 &= \int_{t_1}^{t^\star} ds \sum_{j\in\bar{Y}_n} (v_i - v_j)\cdot F_{i,j} + \sum_{h=1}^{H} \int_{t_1+(h-1)\Delta}^{t_1+h\Delta} ds \sum_{j\in\bar{Y}_n} v_j \cdot F_{i,j} \\ &= -\sum_{j\in\ \bar{Y}_n} \phi(q_i(t^\star) - q_j(t^\star)) + \sum_{j\in\bar{Y}_n} \phi(q_i(t_1) - q_j(t_1)) \\ &\quad + \sum_{h=1}^{H} \int_{t_1+(h-1)\Delta}^{t_1+h\Delta} ds \sum_{j\in\bar{Y}_n} v_j \cdot F_{i,j}\end{aligned} \tag{2.80}$$

and, following a similar method to that used to obtain (2.62), we get

$$\left| \sum_{j\in\bar{Y}_n} \phi(q_i(t^\star) - q_j(t^\star)) \right| \le E_4 R(t)^{3/2}. \tag{2.81}$$

The same bound holds for $\sum_j \phi((q_i(t_1) - q_j(t_1))$. Thus, using Proposition 2.4 to control the last term of (2.80), we have:

$$\frac{1}{2}v_i^2(t^\star) \le (Q_\xi(X) + 1 + E_3)R(t)^2 + 2E_4 R(t)^{3/2} + C_{13}R(t)^2\, |t^\star - t_1|, \tag{2.82}$$

hence

$$\tilde{A}^2\, R(t)^2 \leq 2\,(Q_\xi(X) + 1 + E_3 + 2E_4 + C_{13}T)\, R(t)^2. \tag{2.83}$$

The above inequality can't be satisfied for any $\tilde{A}^2$ larger than $2(Q_\xi(X)+1+E_3+2E_4+C_{13}T)$. This clearly contradicts (2.70) (for this choice of $\tilde{A}$), therefore the proposition is proved. ■

Proof of Proposition 2.4
Let us set

$$J = [s-\Delta, s], \tag{2.84}$$

$$Y_n = \{j \in I_n : |q_i(\tau) - q_j(\tau)| \leq R(t)^{1/4} \text{ for some } \tau \in J\}. \tag{2.85}$$

The particles belonging to ${Y_n}^c$ can be easily handled: as shown in (2.79) we have

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in Y_n^c} v_j \cdot F_{i,j} \right| \leq E_2 R(t)^{\frac{23}{8}-\frac{\gamma+1}{4}} \leq E_5 R(t)^2 \Delta, \tag{2.86}$$

being $\gamma > 7$. Hence from now on we consider only the particles $j \in Y_n$. Let us split the set Y_n according to the following partition:

$$a_k = \{j \in Y_n : 2^{k-1}\, R(t)^{4/6} \leq \sup_{\tau\in J} |v_j(\tau)| < 2^k\, R(t)^{4/6} \quad k = 1, \dots k_{max}\}, \tag{2.87}$$

where k_{max} is the maximum integer for which

$$2^{k_{max}} \leq \frac{1}{2} R(t)^{2/6}, \tag{2.88}$$

$$a_0 = \{j \in Y_n : \sup_{\tau\in J} |v_j(\tau)| < R(t)^{4/6}\}, \tag{2.89}$$

$$\tilde{a} = \bigcup_{k=1}^{k_{max}} a_k, \tag{2.90}$$

$$\bar{a} = Y_n \setminus (a_0 \cup \tilde{a}). \tag{2.91}$$

Therefore

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in Y_n} v_j \cdot F_{i,j} \right| = \left| \int_{s-\Delta}^{s} d\tau \,\{ \sum_{j\in\bar{a}} + \sum_{j\in\tilde{a}} + \sum_{j\in a_0} \}\, v_j \cdot F_{i,j} \right| \tag{2.92}$$

and we give below a bound for each term of the previous equality.
First of all we give an upper bound for the cardinality of $\bar{a}$. If $j \in \bar{a}$

$$|v_j(t^*)| = \max_{\tau \in J} |v_j(\tau)| \geq \frac{1}{4} R(t), \tag{2.93}$$

then by (2.62),

$$|v_j(\tau)| \geq \frac{1}{4} R(t) - D_{21}\Delta\, R(t)^{3/2} \geq \frac{1}{4} R(t) - D_{21}\, R(t)^{5/6} \geq \frac{1}{8} R(t), \tag{2.94}$$

for n (and so for $R(t)$) large enough.
By definition $R(t)$ is larger than the maximal displacement that a particle can undergo during the time interval $[0,t]$, then all the particles with indices in Y_n must be contained into the ball $B(q_i(0), 3R(t))$. Thus it follows from (C.1), (2.55) and (2.33) that

$$\begin{aligned}\sum_{j \in \bar{a}} v_j^2(\tau) &\leq 2\, Q(X^n(\tau); q_i(0), 3R(t)) \leq 2\widetilde{L}\, W(X^n(\tau); q_i(0), 3R(t)) \\ &\leq 2\, C_8 3^\lambda\, \widetilde{L} W(X^n(\tau); q_i(0), R(t)) \leq 2\, C_8\, 3^\lambda \widetilde{L}\, C_7 R(t)^3,\end{aligned} \tag{2.95}$$

then, by (2.94):

$$\frac{1}{64} |\bar{a}| R(t)^2 \leq E_6 R(t)^3, \tag{2.96}$$

which implies

$$|\bar{a}| \leq 64\, E_6\, R(t). \tag{2.97}$$

As a consequence, we have

$$\begin{aligned}\left| \int_{s-\Delta}^{s} d\tau \sum_{j \in \bar{a}} v_j \cdot F_{i,j} \right| &\leq \|F\|_\infty \int_{s-\Delta}^{s} d\tau \left(\sum_{j \in \bar{a}} |v_j|^2 \right)^{1/2} |\bar{a}|^{1/2} \\ &\leq E_7 R(t)^{3/2}\, R(t)^{1/2} \Delta = E_7 R(t)^2\, \Delta.\end{aligned} \tag{2.98}$$

Let us consider now the contribution of the set $\tilde{a}$. Let $l \in \mathbb{N}$ with $1 \leq l \leq l_{max}$ and $l_{max} = [R(t)^{1/4}]$. In this way, using the decreasing property (2.4), we get:

$$\begin{aligned}\left| \int_{s-\Delta}^{s} d\tau \sum_{j \in a_k} v_j \cdot F_{i,j} \right| \leq & E_8\, R(t)^{4/6}\, 2^k \sum_{j \in a_k} \Big\{ \sum_{l=1}^{l_{max}} \frac{1}{l^{\gamma+1}} \int_{s-\Delta}^{s} d\tau\, \chi_{i,j}^{(l)}(\tau) \\ & + \frac{1}{[R(t)^{1/4}]^{(\gamma+1)}} \int_{s-\Delta}^{s} d\tau\, \chi(|q_i(\tau) - q_j(\tau)| > [R(t)^{1/4}]) \Big\}\end{aligned}$$

$$
\leq E_8 \, R(t)^{4/6} \, 2^k \sum_{j \in a_k} \Big\{ \sum_{l=1}^{l_{max}} \frac{1}{l^\gamma} \int_{s-\Delta}^{s} d\tau \, \chi_{i,j}^{(l)}(\tau)
$$

$$
+ \frac{1}{[R(t)^{1/4}]^\gamma} \int_{s-\Delta}^{s} d\tau \, \chi(|q_i(\tau) - q_j(\tau)| > [R(t)^{1/4}]) \Big\}, \tag{2.99}
$$

where

$$
\chi_{i,j}^{(l)}(\tau) = \chi(|q_i(\tau) - q_j(\tau)| \leq l).
$$

Now we want to study the time integral $\int_{s-\Delta}^{s} d\tau \, \chi_{i,j}^{(l)}(\tau)$, with $1 \leq l \leq l_{max}$.
In order to estimate this integral, we notice that for n sufficiently large:

$$
\begin{aligned}
|v_i(\tau) - v_j(\tau)| &\geq \inf_{\tau \in J} |v_i(\tau)| - \sup_{\tau \in J} |v_j(\tau)| \\
&\geq R(t) - 2^{k_{max}} \, R(t)^{4/6} \geq \frac{1}{2} \, R(t).
\end{aligned} \tag{2.100}
$$

Suppose that $|q_i(t_0) - q_j(t_0)| = l$ at time $t_0 \in [s - \Delta, s]$, with outgoing velocities (i.e. $(v_i(t_0) - v_j(t_0)) \cdot (q_i(t_0) - q_j(t_0)) \geq 0$). Then we are going to prove that the pair (i, j), once reached a relative distance larger than l, it will never reach a distance smaller than l. Let $t_1 \in (s - \Delta, s)$ denote the time in which $(q_i(\tau) - q_j(\tau))^2$ reaches its maximum value, say r_1^2 (for this reason $(v_i(t_1) - v_j(t_1)) \cdot (q_i(t_1) - q_j(t_1)) = 0$).
By the identity

$$
\begin{aligned}
\frac{1}{2} \frac{d^2}{d\tau^2} (q_i(\tau) - q_j(\tau))^2 &= (v_i(\tau) - v_j(\tau))^2 \\
&\quad + (q_i(\tau) - q_j(\tau)) \cdot (F_i(\tau) - F_j(\tau)),
\end{aligned}
$$

and using (2.100), (2.62) we get:

$$
(q_i(\tau) - q_j(\tau))^2 \geq r_1^2 + \frac{(\tau - t_1)^2}{2} \left(\frac{R(t)^2}{4} - D_{21} \, r_1 R(t)^{3/2} \right), \tag{2.101}
$$

for $\tau > t_1$. By the definition of r_1 it follows that $r_1 \geq R(t)^{1/2} / (4 \, D_{21})$, otherwise $(q_i(\tau) - q_j(\tau))^2 > r_1^2$. In this case

$$\frac{(\tau - t_1)^2}{2} r_1\, R(t)^{3/2} \le \frac{\Delta^2}{2} r_1\, R(t)^{3/2} \le \frac{\zeta^2}{2} r_1\, R(t)^{1/6} \le E_9 {r_1}^{4/3}. \tag{2.102}$$

Therefore

$$(q_i(\tau) - q_j(\tau))^2 \ge r_1^2 - E_9 {r_1}^{4/3} \gg l^2, \tag{2.103}$$

then the pair (i, j) will keep a relative distance larger than l in the time interval (t_0, s) (note that the last inequality clearly holds because $l_{max} = [R(t)^{1/4}] \ll R(t)^{1/2}$).

Now we repeat this argument when r_1 is the minimum distance between particles i and j; we again denote by t_1 the time in which this distance is reached. Supposing $r_1 < l$, we want to establish the exit time of the particle j from the ball $B(q_i(\tau), l)$; this time can be derived from the equation $(q_i(\tau) - q_j(\tau))^2 = l^2$, hence (2.101) implies

$$l^2 \ge r_1^2 + E_{10} \frac{(\tau - t_1)^2}{2} \frac{R(t)^2}{4},$$

$$(\tau - t_1)^2 \le \frac{8(l^2 - r_1^2)}{E_{10} R(t)^2} \le \frac{8l^2}{E_{10} R(t)^2} \;\Rightarrow\; |\tau - t_1| \le \frac{E_{11}\, l}{R(t)}.$$

Thus

$$\int_{s-\Delta}^{s} \chi^{(l)}_{i,j}(\tau) \le \frac{E_{12}\, l}{R(t)}. \tag{2.104}$$

In order to estimate the cardinality of a_k, we use again an upper bound of the energy as we have done for the set $\bar{a}$. Let be $\tau_j \in J$ such that $|v_j(\tau_j)| = \max_{\tau \in J} |v_j(\tau)|$. Thus

$$\begin{aligned} |a_k|\, 2^{2(k-1)}\, R^{8/6} &\le \sum_{j \in a_k} |v_j(\tau_j)|^2 \le \sum_{j \in a_k} |v_j(s - \Delta)|^2 \\ &+ \int_{s-\Delta}^{s} d\tau \sum_{j \in a_k} |v_j(\tau)| \sum_p |F_{p,j}(\tau)|. \end{aligned} \tag{2.105}$$

Multiplying (2.105) by 2^{-k} and summing over k, we have

$$\begin{aligned} \sum_k \frac{1}{2} |a_k|\, 2^{(k-1)}\, R^{8/6} &\le \sum_k 2^{-k} \sum_{j \in a_k} |v_j(s - \Delta)|^2 \\ &+ E_{13}\, R^{4/6} \int_{s-\Delta}^{s} d\tau \sum_{j \in \tilde{a}} \sum_p |F_{p,j}(\tau)|. \end{aligned} \tag{2.106}$$

The latter term can be bounded as follows:

$$\sum_p |F_{p,j}(\tau)| \le E_{14} \sum_p \sum_{l=1}^{\infty} \frac{1}{l^\gamma}\, \chi_{p,j}^{(l)}(\tau). \tag{2.107}$$

By means of (2.58) and (2.33), taking the supremum over μ, we can state:

$$\sum_{j\in\tilde{a}} \sum_p \chi_{j,p}^{(l)}(\tau) \le E_{15}\, l^3\, R(t)^3, \tag{2.108}$$

and by (2.95) it follows that

$$\sum_k 2^{-k} \sum_{j\in a_k} |v_j(s-\Delta)|^2 \le E_{16}\, R(t)^3, \tag{2.109}$$

hence, combining these two relations and using the definition (2.67) of Δ, we get

$$\sum_k |a_k| 2^k \le E_{17}\, R(t)^{14/6}\, \Delta. \tag{2.110}$$

It follows from (2.99), (2.104) and (2.110) that

$$\sum_k \left| \int_{s-\Delta}^{s} d\tau \sum_{j\in a_k} v_j \cdot F_{i,j} \right| \le E_{18} R(t)^2\, \Delta \tag{2.111}$$

since the sum over k of the second term of (2.99) can be easily bounded by:

$$\begin{aligned} R(t)^{4/6} \sum_k 2^k \sum_{j\in a_k} \frac{1}{[R(t)^{1/4}]^\gamma} \int_{s-\Delta}^{s} d\tau \;&\le E_{19}\, R(t)^{4/6}\, R(t)^{14/6} \Delta^2\, R(t)^{-\gamma/4} \\ &\le E_{19} \Delta\, R(t)^2. \end{aligned} \tag{2.112}$$

It remains to estimate the last contribution, namely that associated to the set of indices a_0. We have

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in a_0} v_j \cdot F_{i,j} \right| \le E_{20} \sum_{h=0}^{H-1} R(t)^{4/6} \int_{s_h}^{s_{h+1}} d\tau \sum_{k=1}^{k_{max}} \frac{1}{(kr)^\gamma} N^{(k)}(\tau), \tag{2.113}$$

with

$$N^{(k)}(\tau) = \sum_{j\in a_0} \chi(\, |q_i(\tau) - q_j(\tau)| < kr\,)$$

and where $J = [s - \Delta, s]$ has been decomposed into H identical intervals:

$$J = \bigcup_{h=0}^{H-1} [s_h, s_{h+1}], \tag{2.114}$$

with $s_H = s$, $s_0 = s - \Delta$, and $|s_{h+1} - s_h| = \delta \in [\frac{1}{2\bar{A}\,R(t)}, \frac{1}{\bar{A}\,R(t)}]$.
Moreover k_{max} is such that

$$k_{max} = \left[R(t)^{1/4}/r \right] + 1, \tag{2.115}$$

(such a choice for the maximum value of k will be clear later).
Since $|v_j(\tau)| \leq R(t)^{4/6}$, the maximal displacement of a particle belonging to the set a_0 is less than 1, in the time interval J. Moreover, defining

$$N_h^{(k)} = \sum_{j \in a_0} \chi(\inf_{\tau \in (s_h, s_{h+1})} |q_i(\tau) - q_j(s_0)| < kr + 1), \tag{2.116}$$

for $\tau \in (s_h, s_{h+1})$, we get $N^{(k)}(\tau) \leq N_h^{(k)}$.
Then for (2.113) we have:

$$\begin{aligned} \left| \int_{s-\Delta}^{s} d\tau \sum_{j \in a_0} v_j \cdot F_{i,j} \right| &\leq E_{20}\, R(t)^{4/6}\, \delta \sum_{k=1}^{k_{max}} \frac{1}{(kr)^{\gamma}} \sum_{h=0}^{H-1} N_h^{(k)} \\ &\leq E_{20}\, R(t)^{4/6} \sqrt{H}\, \delta \sum_{k=1}^{k_{max}} \frac{1}{(kr)^{\gamma}} \left(\sum_{h=0}^{H-1} \left(N_h^{(k)} \right)^2 \right)^{1/2}. \end{aligned} \tag{2.117}$$

Let us define

$$\mathcal{T}_h^k = \{ y \in \mathbb{R}^3 : \inf_{\tau \in (s_h, s_{h+1})} |q_i(\tau) - y| < kr + 1 \} \tag{2.118}$$

and

$$E(\mathcal{T}_h^k) = \sum_{l<j} \phi(q_l(s_0) - q_j(s_0)) + b\, N(X^n(s_0), \mathcal{T}_h^k), \tag{2.119}$$

where the sum is restricted to the pairs of particles in $\mathcal{T}_h^k$ and E is a positive quantity because $b > B$. Let us note that $N(X^n(s_0), \mathcal{T}_h^k) \leq N_h^{(k)}$.
We want to estimate now the sum in (2.117):

$$\sum_{h=0}^{H-1} \left(N_h^{(k)} \right)^2. \tag{2.120}$$

If the sets $\mathcal{T}_h^k$ were all disjoint, then, defining

$$\mathcal{T}^k = \bigcup_h \mathcal{T}_h^k, \tag{2.121}$$

by superstability we would simply have

$$E(\mathcal{T}^k) \geq A \sum_{i\in\mathbb{Z}^3\cap\mathcal{T}^k} n_{\Delta_i}^2 \geq A \sum_h \sum_{i\in\mathbb{Z}^3\cap\mathcal{T}_h^k} n_{\Delta_i}^2 \geq E_{21} \sum_{h=0}^{H-1} \frac{\left(N_h^{(k)}\right)^2}{|\mathcal{T}_h^k|}. \tag{2.122}$$

We anticipate two results that will be proved afterwards: the first regarding $|\mathcal{T}_h^k|$:

$$|\mathcal{T}_h^k| \leq E_{22}\, k^3, \tag{2.123}$$

the other dealing with the fact that a set $\mathcal{T}_h^k$ has a non empty intersection with no more than $(8+4rk)$ other sets (we consider k fixed).
In this way (2.122) becomes

$$E(\mathcal{T}^k) \geq \frac{E_{23}}{k^4} \sum_{h=0}^{H-1} \left(N_h^{(k)}\right)^2. \tag{2.124}$$

Putting the previous relation into (2.117), we can write:

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in a_0} v_j \cdot F_{i,j} \right| \leq E_{24} R(t)^{4/6} \sqrt{H}\, \delta \sum_{k=1}^{k_{max}} \frac{1}{(k)^{\gamma-2}}\, E(\mathcal{T}^k)^{1/2}. \tag{2.125}$$

By the bound on the maximal velocity of the i-th particle

$$|v_i(\tau)| \leq \bar{A}\, R(t) + D_{21} R(t)^{3/2}\, \Delta \leq \frac{3}{2} \bar{A}\, R(t), \tag{2.126}$$

we get

$$\mathcal{T}^k \subset B(q_i(\tau), 2 + kr + R(t)^{1/3}), \tag{2.127}$$

with τ belonging to the interval $[s-\Delta, s]$ (for the proof see below).
Therefore

$$\begin{aligned} E(\mathcal{T}^k) &\le \sum_{\substack{l<j: \\ q_l,q_j \in \mathcal{T}^k}} |\phi(q_l(s_0) - q_j(s_0))| + b\, N(X^n(s_0), \mathcal{T}^k) \\ &\le \sum_{\substack{l<j: \\ q_l,q_j \in B}} |\phi_{l,j}| + b\, N(B) \le E_{25} \sup_\mu \sum_l f_l^{\mu,R} \big(\sum_j |\phi_{l,j}| + b \big) \\ &\le E_{26} \sup_\mu W(X^n(s_0); \mu, R(s_0)) \le E_{27} R(t)^3, \end{aligned} \tag{2.128}$$

where in the fourth inequality we have used an estimate like the one given in (B.1) and Lemma 2.1, while in the last inequality we have used Lemma 2.2. Putting the last relation into (2.125) we get:

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j \in a_0} v_j \cdot F_{i,j} \right| \le E_{28} \Delta\, R(t)^2, \tag{2.129}$$

since $\sqrt{H} = (\Delta/\delta)^{1/2} \le \sqrt{2\zeta \bar{A}} R^{1/6}$.

It remains to prove that a fixed set $\mathcal{T}_h^k$ has a non empty intersection with no more than $(8 + 4rk)$ other sets, that $|\mathcal{T}_h^k| \le E_{22}\, k^3$, and (2.127) (we will see that these three statements are consequences of the inclusion (2.132)).

For a given h, let $e = \frac{v_i(s_{h+1})}{|v_i(s_{h+1})|}$ and $\xi(\tau) = (q_i(\tau) - q_i(s_{h+1})) \cdot e$.

Then:

$$\xi(\tau) = |v_i(s_{h+1})|\, (\tau - s_{h+1}) + \int_{s_{h+1}}^{\tau} d\sigma\, (\tau - \sigma)\, F_i(\sigma) \cdot e, \tag{2.130}$$

hence

$$\begin{aligned} |\xi(\tau)| &\ge |v_i(s_{h+1})|\, (\tau - s_{h+1}) - \frac{|\tau - s_{h+1}|^2}{2} D_{21} R(t)^{3/2} \\ &\ge |\tau - s_{h+1}|\, (\bar{A}R(t) - D_{21} R(t)^{3/2} R(t)^{-4/6}) \\ &\ge |\tau - s_{h+1}| \frac{\bar{A}R(t)}{2}, \end{aligned} \tag{2.131}$$

for n large enough. On the other hand from (2.126) it follows that

$$\mathcal{T}_h^k \subset B(q_i(s_{h+1}); \frac{3}{2} \bar{A} R(t) \delta + kr) \subset B(q_i(s_{h+1}); 2 + kr). \tag{2.132}$$

Let us choose $|\tau - s_{h+1}| > (8+4rk)\delta$, with $(8+4rk)\delta \ll \Delta$ (such a condition guarantees us to remain in $[s-\Delta, s]$), that is $k \le k_{max} \ll R(t)^{1/3}$; from this last condition, the choice (2.115) previously done of taking $k_{max} \sim R(t)^{1/4}$ is clear. Now, from (2.131), we have that $|\xi(\tau)| > 2 + kr$, and for this reason, after the time τ, q_i will not enter anymore into the ball $B(q_i(s_{h+1}); 2+kr)$, in such a way that $\mathcal{T}_h^k$ will have a non empty intersection with no more than $(8+4rk)$ other different $\mathcal{T}_y^k$'s.
The bound on $|\mathcal{T}_h^k|$ and the inclusion (2.127) are straightforward consequences of (2.132).
■

We have now all the results necessary to prove the main theorem of this work.

2.4 Proof of Theorem 2.2

Let us define the quantity

$$\delta_i(n,t) = \left| q_i^n(t) - q_i^{n-1}(t) \right|. \tag{2.133}$$

From the equations of motion in integral form we have:

$$q_i^n(t) = q_i(0) + v_i(0)\, t + \int_0^t ds(t-s) \sum_{j:j\neq i} F\left(q_i^n(s) - q_j^n(s)\right). \tag{2.134}$$

From (2.133) and (2.134) it follows that, for any $i \in I_{n-1}$,

$$\delta_i(n,t) \le \int_0^t ds(t-s) \Big| \sum_{j:j\neq i} \left\{ \nabla\phi(q_i^n(s) - q_j^n(s)) - \nabla\phi(q_i^{n-1}(s) - q_j^{n-1}(s)) \right\} \Big| \tag{2.135}$$

and, because of the long-range of the interaction, it is useful to split up the last sum in the following way. Let

$$\min\left\{ \left| q_i^{n-1}(s) - q_j^{n-1}(s) \right|, \left| q_i^n(s) - q_j^n(s) \right| \right\} = m_{ij}^n(s) \tag{2.136}$$

and, fixing a particle i, consider the following sets of indices:

$$\begin{aligned} \mathcal{A}_i^n(s,k) &= \left\{ j \neq i : (k-1)\varphi(n) \le m_{ij}^n(s) \le k\varphi(n) \right\}, \\ \tilde{\mathcal{A}}_i^n(s) &= \left\{ j \neq i : m_{ij}^n(s) \ge k_{\max}\varphi(n) \right\} \end{aligned}$$

where $\varphi(n) = \psi_\xi(n)^{3/2}$ (ψ_ξ has been defined in (2.8)), $k = 1, 2, \ldots, k_{\max}$ and $k_{\max} = \left[n^{3/4}/\varphi(n)\right]$. We can write, using the property (2.5) of the interaction:

$$\begin{aligned}
&\left|\sum_{j:j\neq i}\left\{\nabla\phi(q_i^n(s)-q_j^n(s))-\nabla\phi(q_i^{n-1}(s)-q_j^{n-1}(s))\right\}\right| \\
&\leq L_1\sum_{j\in\mathcal{A}_i^n(s,1)}(\delta_i(n,s)+\delta_j(n,s)) \\
&+L_1\sum_{k=2}^{k_{\max}}\frac{1}{((k-1)\varphi(n))^{\gamma+2}}\sum_{j\in\mathcal{A}_i^n(s,k)}(\delta_i(n,s)+\delta_j(n,s)) \\
&+L_1\frac{1}{(k_{\max}\varphi(n))^{\gamma+2}}\sum_{j\in\tilde{\mathcal{A}}_i^n(s)}\left|q_i^n(s)-q_j^n(s)-q_i^{n-1}(s)+q_j^{n-1}(s)\right|. \qquad (2.137)
\end{aligned}$$

Defining

$$d_n(t) = \sup_{s\in[0,t]}\sup_{i\in I_n}\left|q_i^n(s)-q_i(0)\right|, \qquad (2.138)$$

from the bound

$$V^n(t) \leq L_2\varphi(n) \qquad (2.139)$$

(it is a consequence of (2.64), (2.65) and of Gronwall's lemma) we get, for $t \leq T$: $d_n(t) \leq L_3\varphi(n)$, where $L_3 = L_2 T$.
Hence, putting

$$p^{(k)}(n,t) = k\varphi(n) + L_3\varphi(n), \qquad (2.140)$$

the number of particles contained in $\mathcal{A}_i^n(s,k)$ is bounded by the number of particles that, at the initial time, were in a ball of radius $p^{(k)}(n,t)$, and therefore, according to the definition (2.7), it is bounded by the quantity:

$$g^{(k)}(n,t) = Q_\xi(X)\left(p^{(k)}(n,t)\right)^3 \leq L_4 k^3\varphi(n)^3. \qquad (2.141)$$

For the same reason, the number of particles belonging to $\tilde{\mathcal{A}}_i^n(s)$ is bounded by $L_5 Q_\xi(X) n^3$, so the last term in (2.137) is bounded by

$$L_5\frac{Q_\xi(X)\,n^4}{\left(n^{3/4}\right)^{\gamma+2}}. \qquad (2.142)$$

We define

$$u_k(n,t) = \sup_{i \in I_k} \delta_i(n,t) \tag{2.143}$$

and we fix an integer $k_0 \ll n$. Putting

$$k_1 = \left[k_0 + p^{(k_{\max})}(n,t)\right], \tag{2.144}$$

we can bound the r.h.s. of (2.137) in the following way (using (2.140), (2.141), (2.142)):

$$(2.137) \le L_1 \Big(L_4 \varphi(n)^3 + \sum_{k \ge 2} \frac{L_4\, k^3 \varphi(n)^3}{((k-1)\varphi(n))^{\gamma+2}} \Big)\, u_{k_1}(n,s) + \frac{L_5\, Q_\xi(X)\, n^4}{(n^{3/4})^{\gamma+2}}. \tag{2.145}$$

Hence by (2.135), (2.145), we get:

$$u_{k_0}(n,t) \le L_6 \varphi(n)^3 \int_0^t ds\,(t-s)\, u_{k_1}(n,s) + \frac{L_7}{n^{(3/4)\gamma - 5/2}}. \tag{2.146}$$

We iterate now (2.146) m times, where m is

$$m = \left[\frac{n - k_0}{p^{(k_{\max})}(n,t)} \right]. \tag{2.147}$$

Since $u_m(n,t) \le L_3 \varphi(n)$, we have

$$\begin{aligned} u_{k_0}(n,t) &\le (L_8 \varphi(n))^{3m+1} \frac{t^{2m}}{(2m)!} + \frac{L_7}{n^{(3/4)\gamma-5/2}} \sum_{h=1}^{m} \frac{(\varphi(n)^3)^h\, t^{2h}}{(2h)!} \\ &\le (L_8 \varphi(n))^{3m+1} \frac{t^{2m}}{(2m)!} + \frac{L_7}{n^{(3/4)\gamma-5/2}} \exp(\varphi(n)^{3/2} t). \end{aligned} \tag{2.148}$$

By the choice (2.147), using Stirling formula, since $\varphi(n)^{3/2} < L_9 (\log n)^{\frac{9}{4}\xi}$, where $\xi < 4/9$, and since $\gamma > 7$, it follows that $u_{k_0}(n,t)$ converges summably to zero as $n \to \infty$.

For what concernes the velocities we have:

$$\left| v_i^n(t) - v_i^{n-1}(t) \right| \le \int_0^t ds \left| \sum_{j:j \ne i} F\left(q_i^n(s) - q_j^n(s)\right) - F\left(q_i^{n-1}(s) - q_j^{n-1}(s)\right) \right| \tag{2.149}$$

and we can bound the r.h.s. of (2.149) by the same estimates used to bound (2.137). In this way recalling (2.146) we obtain, for any $i \in I_{k_0}$:

$$\left|v_i^n(t) - v_i^{n-1}(t)\right| \leq L_6 \varphi(n)^3 \int_0^t ds\ u_{k_1}(n,s) + \frac{L_7}{n^{(3/4)\gamma - 5/2}}, \tag{2.150}$$

where for $u_{k_1}(n,s)$ it holds (2.148) replacing m with $m-1$:

$$u_{k_1}(n,t) \leq (L_8 \varphi(n))^{3(m-1)+1} \frac{t^{2(m-1)}}{(2(m-1))!} + \frac{L_7}{n^{(3/4)\gamma-5/2}} \exp(\varphi(n)^{3/2} t). \tag{2.151}$$

Substituting (2.151) into (2.150) we have

$$\begin{aligned} &\left|v_i^n(t) - v_i^{n-1}(t)\right| \leq L_6 \varphi(n)^3 \\ &\times \int_0^t ds\ \left((L_8 \varphi(n))^{3(m-1)+1} \frac{s^{2(m-1)}}{(2(m-1))!} + \frac{L_7}{n^{(3/4)\gamma-5/2}} \exp(\varphi(n)^{3/2} s) \right) \\ &+ \frac{L_7}{n^{(3/4)\gamma-5/2}} \end{aligned} \tag{2.152}$$

from which it follows that $\left|v_i^n(t) - v_i^{n-1}(t)\right|$ converges summably to zero as $n \to \infty$.

To prove that the limit solution belongs to (2.15) for any time $0 \leq t \leq T$, with T arbitrary but a priori fixed, let us fix $i \in \mathbb{N}$ and choose k_0 such that $k_0 - 1 \leq |q_i| \leq k_0$. We choose n^* of the form

$$n^* = [k_0^2 + L_{10}], \tag{2.153}$$

in such a way that we have a uniform convergence of $\sum_{n \geq n^*} u_{k_0}(n,t)$ with respect to k_0 (as it appears evident from (2.147)). Now we have:

$$|v_i(t) - v_i^{n^*}(t)| \leq \sum_{n \geq n^*} |v_i^n(t) - v_i^{n-1}(t)|, \tag{2.154}$$

hence by (2.152) and by the choice made for n^*, the r.h.s. of (2.154) is bounded by a constant indipendent from k_0:

$$|v_i(t)| \leq |v_i^{n^*}(t)| + L_{11}. \tag{2.155}$$

Thus from (2.139) it follows

$$|v_i^{n^*}(t)| \leq L_{12}(\log(e+n^*))^{\frac{3}{2}\xi} \leq L_{13}(\log(e+k_0))^{\frac{3}{2}\xi} \\ \leq L_{14}(\log(e+|q_i|))^{\frac{3}{2}\xi} = L_{14}\psi_\xi^{3/2}(|q_i|), \tag{2.156}$$

so that, from (2.156) and (2.155), it follows

$$|v_i(t)| \leq L_{15}\psi_\xi^{3/2}(|q_i|). \tag{2.157}$$

We want to prove now that, if $X \in \mathcal{X}_\xi$, then $X(t) \in \mathcal{X}_{\frac{3}{2}\xi}$.
Given $\mu \in \mathbb{R}^3$ and $R > (\log(e+|\mu|))^{\frac{3}{2}\xi}$ let

$$n_0 = \left[L_{16}\exp\left(2\,R^{\frac{2}{3\xi}}\right)\right]. \tag{2.158}$$

Clearly $(\log(e+n_0))^{\frac{3}{2}\xi} \geq R$ so that, by Lemma 2.2 and from the relation

$$Q(X;\mu,R) \leq \widetilde{L}\,W(X;\mu,R),$$

(see (C.1)), we have

$$Q(X^{n_0}(t);\mu,R) \leq \widetilde{L}\,W(X^{n_0}(t);\mu,2R(n_0,t)) \leq L_{17}R^3(n_0,t) \\ \leq L_{18}(\log(e+n_0))^{\frac{9}{2}\xi} \leq L_{19}\left(R^{\frac{2}{3\xi}}\right)^{\frac{9\xi}{2}} \leq L_{20}R^3. \tag{2.159}$$

On the other hand

$$Q(X(t);\mu,R) \leq Q(X^{n_0}(t);\mu,R) \\ + \sum_{n>n_0}\left|Q(X^n(t);\mu,R) - Q(X^{n-1}(t);\mu,R)\right| \tag{2.160}$$

and the sum on the r.h.s. of (2.160), by the choice (2.158) of n_0 (which in particular implies that $n_0 > |\mu|$), converges uniformly with respect to $\mu \in \mathbb{R}^3$ and $R > (\log(e+|\mu|))^{\frac{3}{2}\xi}$, so it is bounded by a constant independent from μ and R.
Notice that the following inequalities hold:

$$(\log(e+n_0))^{\frac{3}{2}\xi} \geq R \geq (\log(e+|\mu|))^{\frac{3}{2}\xi} \tag{2.161}$$

in order that, combining (2.159) and (2.160), taking the supremum over $\mu \in \mathbb{R}^3$ and over $R > (\log(e+|\mu|))^{\frac{3}{2}\xi}$, we obtain that $X(t) \in \mathcal{X}_{\frac{3}{2}\xi}$.
We want to underline that we cannot say that the solution surely exits from $\mathcal{X}_\xi$, we have only proved that the maximal set of existence for $X(t)$ is $\mathcal{X}_{\frac{3}{2}\xi} \supset \mathcal{X}_\xi$.

For what concerns the uniqueness of the solution, let us assume that there is a solution $\{q_i^\star, v_i^\star\}$ different from the one obtained as the limit of the partial dynamics and deduce a contradiction. In the space defined by (2.14) and (2.15) it can be easily proved that the difference $|q_i^n - q_i^\star|$ converges to zero as $n \to \infty$ by an iterative method identical to the one just used, in particular we need the restriction over the velocities provided by (2.15) in order to make the iterative method work. This last condition on the velocities is imposed by the long-range character of the interaction, which gives origin to a term like the last present in (2.148).
We want to point out that the restriction (2.15) is a requirement imposed to prove the uniqueness of the solution. In particular we need a velocity bound (better than the one following by energy conservation) for the non-limit (hypothetical) solution $\{q_i^\star, v_i^\star\}$, necessary to make the iterative method work. Nevertheless we remark that we have proved that the limit solution, $\lim_{n\to\infty}\{q_i^n, v_i^n\}$, belongs directly to (2.15). ■

The proof of Theorem 2.1, dealing with the short-range interaction, is analogous to Theorem 2.2's, with obvious simplifications.

Chapter 3

A Microscopic Model of Viscous Friction

In this chapter we discuss some features of a simple (but not trivial) microscopic model of viscous friction. In particular we want to show that a careful mathematical analysis of the long time asymptotics allows to outline some unexpected behaviors. We consider a body moving along the x-axis under the action of an external force E and immersed in an infinitely extended perfect gas. We assume the gas to be described by the mean-field approximation and interacting elastically with the body. In this set up, we discuss the following statement: "Let V_0 be the initial velocity of the body and V_∞ its asymptotic velocity, then for $|V_0 - V_\infty|$ small enough it results $|V(t) - V_\infty| \approx C\, t^{-d-2}$ for t large, where $V(t)$ is the velocity of the body at time t, d the dimension of the space and C is a positive constant depending on the medium and on the shape of the body".
The reason for the power law approach to the stationary state instead of the exponential one (usually assumed in viscous friction problems), is due to the long memory of the dynamical system.
In a recent paper by Caprino, Marchioro and Pulvirenti ([13]), it has been discussed the case of E constant and positive, with $0 < V_0 < V_\infty$, for a disk orthogonal to the x-axis. Here we complete the analysis in the cases $E > 0$ with $V_0 > V_\infty$ and $E = 0$. We also approach the problem of an x-dependent external force, by choosing E of harmonic type. In this case we obtain the power-like asymptotic time behavior for the body position $X(t)$. The investigation is done in detail for a disk orthogonal to the x-axis and then, by a sketched proof, extended to a body with a general convex shape.

3.1 Model and Results for $E = 0$

Here we consider the case with null external force. The body is a disk of radius R in dimension $d = 3$, a stick of lenght $2R$ for $d = 2$ and a point particle on the line for $d = 1$. We assume, for simplicity, its mass to be unitary. The disk has its center placed on the x-axis and it is assumed orthogonal to the same axis. We remark that the assumption that the center moves along the x-axis becomes redundant if we initially place the disk orthogonal to the x-axis, property which is conserved during the motion by symmetry. The thickness of the disk is assumed to be negligible, even if this is not essential and it is useful just for notational simplicity. Moreover the disk is immersed in a perfect gas in equilibrium at inverse temperature proportional to β and with constant density ρ.

We give the body an initial small velocity and we investigate how its velocity vanishes in time.

We assume the perfect gas in the mean-field approximation. The motion of the disk modifies the equilibrium of the gas, which evolves according to the free transport equation. Let $f(x, v; t)$, $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ be the mass density in the phase space of the gas particles, then its evolution equation is:

$$(\partial_t + v \cdot \nabla_x) f(x, v; t) = 0, \qquad \text{for } x \notin D(t), \tag{3.1}$$

where:

$$D(t) = \{y \in \Pi^{\perp}(X(t)) : |y - X(t)|^2 < R^2\}, \tag{3.2}$$

$X(t)$ is the position of the center of the disk at time t and $\Pi^{\perp}(X(t))$ the plane orthogonal to the x-axis at the point $X(t)$.

Now we give the boundary conditions, which express the continuity of f along the trajectories with elastic reflection on $D(t)$. According to the elastic reflection law, denoting by v' the outgoing velocity of a gas particle with incoming velocity v, after a collision with the body, we have:

$$v'_x = 2V(t) - v_x, \qquad v'_{\perp} = v_{\perp}, \tag{3.3}$$

where $V(t) = \dot{X}(t)$ is the velocity of the disk and v_x, $v_{\perp}$, the velocity components of the gas particles on the x-axis and the orthogonal plane respectively. We set

$$f_+(x, v'; t) = f_-(x, v; t), \qquad \text{for } x \in D(t) \tag{3.4}$$

where

$$f_{\pm}(x,v;t) = \lim_{\varepsilon\to 0^+} f(x \pm \varepsilon v, v; t \pm \varepsilon), \qquad \text{for } x \in D(t). \tag{3.5}$$

Eq. (3.4) gives the boundary conditions and it describes both the continuity along the collisions from the right ($V(t) > v_x$) and from the left ($V(t) < v_x$).

Coupled to Eq. (3.1) we consider the evolution equation for the disk:

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -F(t), \tag{3.6}$$

$$X(0) = 0, \qquad V(0) = V_0,$$

where

$$F(t) = 2\int_{D(t)} dx \int_{v_x < V(t)} dv\,(V(t)-v_x)^2 f_-(x,v;t)$$
$$-2\int_{D(t)} dx \int_{v_x \geq V(t)} dv\,(V(t)-v_x)^2 f_-(x,v;t) \tag{3.7}$$

is the action of the gas on the disk.

As initial state for the gas distribution we assume the thermal equilibrium, namely

$$f_+(x,v;0) = \rho \left(\frac{\beta}{\pi}\right)^{d/2} e^{-\beta v^2}, \tag{3.8}$$

for $\beta > 0$.

The choice of such initial datum is not binding, since the results of the present paper hold for any initial function of the form $\rho g(v^2)$, with g integrably decreasing.

A solution to the viscous friction problem is a pair (f, V) such that $V{=}V(t)$ solves, for almost all $t \in \mathbb{R}^+$, Eq.ns (3.6), (3.7) and f satisfies

$$\frac{d}{dt} f(x+vt, v; t) = 0 \qquad \text{a.e. } (x,v), \tag{3.9}$$

together with boundary conditions (3.4) and initial condition (3.8).

Eq. (3.6) can be derived in a heuristic way from the balance of momentum. In fact, formally, the quantity

$$\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dv\, v_x f_-(x,v;t) + V(t) \tag{3.10}$$

is conserved along the motion and its time derivative yields the equation of motion (for a short proof of this fact, see Appendix E). An equivalent heuristic derivation is given in [13].

Notice that, were the time evolution of the disk given, we could solve Eq. (3.1) by means of characteristics, that is by following back in time the trajectory of a gas particle which has position x and velocity v at time t. Any such particle has a free backward evolution up to the last time $\tau < t$ in which it hits the disk. At this instant we use condition (2.3) and then we continue on the backward trajectory in this way, up to time $t = 0$. Setting $x(s,t;x,v)$, $v(s,t;x,v)$ to denote position and velocity at time $s \leq t$ of a gas particle whose position and velocity at time t are x, v, at the end we obtain:

$$F(t) = 2\rho \left(\frac{\beta}{\pi}\right)^{d/2} \Big[\int_{D(t)} dx \int_{v_x < V(t)} dv\, (V(t) - v_x)^2 e^{-\beta v^2(0,t;x,v)} - \int_{D(t)} dx \int_{v_x \geq V(t)} dv\, (V(t) - v_x)^2 e^{-\beta v^2(0,t;x,v)} \Big]. \tag{3.11}$$

Note that, if $x \in D(t)$, then v has to be interpreted as a precollisional velocity, that is $v = \lim_{s \to t} v(s,t;x,v)$.

Unfortunately, to compute $F(t)$ we need to know $v(0,t;x,v)$ and hence to know all the previous history $\{X(s), V(s),\ s < t\}$. However, if a light particle goes back without undergoing any collision, then

$$v(0,t;x,v) = v.$$

If this is the case, where no recollisions take place, the friction term becomes:

$$F_0(V) = 2\rho \left(\frac{\beta}{\pi}\right)^{d/2} \sigma_d \Big[\int_{v_x < V} dv\, (V - v_x)^2 e^{-\beta v^2} - \int_{v_x \geq V} dv\, (V - v_x)^2 e^{-\beta v^2} \Big], \tag{3.12}$$

being σ_d the area of the disk.

It can be seen (see Lemma 3.1) that F_0 is an increasing odd function, positive and convex in the interval $(0, +\infty)$. This properties enable us to solve our problem, with F_0 in place of F, in a quite easy way.

Indeed Eq. (3.6) becomes:

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -F_0(V(t)) = -K(t)V(t), \tag{3.13}$$

$$X(0) = 0, \qquad V(0) = V_0,$$

where

$$K(t) = \frac{F_0(V(t))}{V(t)}. \tag{3.14}$$

We take, without loss of generality, $V_0 > 0$ (the case $V_0 < 0$ is the symmetrical one). Since F_0 is an odd function, it results, for $V \in [0, V_0]$,

$$C_2 \leq K(t) \leq C_1 \tag{3.15}$$

where

$$0 < F_0'(0) = C_2 < C_1 = F_0'(V_0), . \tag{3.16}$$

By estimate (2.15) we can deduce that V is decreasing in time and moreover it satisfies:

$$V_0 \, e^{-C_1 t} \leq V(t) \leq V_0 \, e^{-C_2 t}. \tag{3.17}$$

Equation ((2.9) can finally be solved by characteristics.

In the full problem, where we include recollisions, the long memory effect makes the problem much more difficult.

Let us rewrite the full friction term F as:

$$F(t) = F_0(V(t)) + r^+(t) + r^-(t) \tag{3.18}$$

where $r^+(t)$ and $r^-(t)$ are:

$$r^+(t) = 2\rho \left(\frac{\beta}{\pi}\right)^{d/2} \int_{D(t)} dx \int_{v_x < V(t)} dv \, (v_x - V(t))^2 (e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}) \tag{3.19}$$

and

$$r^-(t) = 2\rho \left(\frac{\beta}{\pi}\right)^{d/2} \int_{D(t)} dx \int_{v_x \geq V(t)} dv \, (v_x - V(t))^2 (e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}). \tag{3.20}$$

The quantities ρ, β, R and $\gamma = V_0$ are the data of the problem.
We are now in the position to state the following

Theorem 3.1. *There exists $\gamma_0 = \gamma_0(\rho, \beta, R) > 0$ sufficiently small such that, for any initial velocity $V_0 = \gamma \in (0, \gamma_0)$ there exists at least one solution $(V(t), f(t))$ to problem (3.1)-(3.9). Moreover there exist two positive constants A_1, A_2 independent of γ, such that any solution $(V(t), f(t))$ satisfies the following properties:*
(i) for any $t \geq 0$ it is:

$$V(t) \geq \gamma\, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^{d+2}}, \tag{3.21}$$

(ii) there exists a sufficiently large $\bar{t}$, depending on γ, such that for any $t \geq 0$:

$$V(t) \leq \gamma\, e^{-C_2 t} - \gamma^5 \frac{A_2}{t^{d+2}} \chi(\{t \geq \bar{t}\}) \tag{3.22}$$

where $\chi(\{\dots\})$ is the characteristic function of the set $\{\dots\}$.

Note that (3.22) establishes the power law approach to the equilibrium state.

Estimates (2.21) and (2.22) show that in this model the disk slows down its velocity in an unexpected way, in spite of what intuition suggests. The velocity $V(t)$ goes from positive to negative values, crossing the zero value in a finite time. Then, it tends asymptotically to zero from negative values. The fact that $V(t)$ changes sign is due to the memory of the recollisions, whose effect is contained in the terms $r^+(t)$ and $r^-(t)$.

For the sake of concreteness we will prove Theorem 3.1 for the three-dimensional case. The remaining cases $d = 1, 2$ follow by the same arguments with obvious modifications.

We discuss now some properties of F_0.

Lemma 3.1. *F_0 is an increasing odd function, positive and convex in $(0, +\infty)$.*

Proof
By (3.12) it is, for a constant $C > 0$:

$$F_0(V) = C \int dv_\perp\, e^{-\beta v_\perp^2} \Big[\int_{-\infty}^{V} dv_x\, (V - v_x)^2 e^{-\beta v_x^2} - \int_{V}^{+\infty} dv_x\, (V - v_x)^2 e^{-\beta v_x^2} \Big]. \tag{3.23}$$

By the simple change of variables $v_x \to -v_x$ we obtain, for $V > 0$,

$$F_0(V) = C \Big[\int_{-\infty}^{V} dv_x\, (V - v_x)^2 e^{-\beta v_x^2} - \int_{-\infty}^{-V} dv_x (V + v_x)^2 e^{-\beta v_x^2} \Big]$$

$$\geq C \int_{-\infty}^{-V} dv_x [(V - v_x)^2 - (V + v_x)^2] \, e^{-\beta v_x^2}$$
$$= -4CV \int_{-\infty}^{-V} dv_x \, v_x e^{-\beta v_x^2} > 0, \tag{3.24}$$

moreover $F_0(0) = 0$. From the previous expression we can also see that $F_0(-V) = -F_0(V)$.
Furthermore, for $V \geq 0$,

$$F_0'(V) = 2C \Big[\int_{-\infty}^{V} dv_x \, (V - v_x) e^{-\beta v_x^2} - \int_{-\infty}^{-V} dv_x \, (V + v_x) e^{-\beta v_x^2} \Big]$$
$$\geq -4C \int_{-\infty}^{-V} dv_x \, v_x e^{-\beta v_x^2} > 0. \tag{3.25}$$

Finally, for $V > 0$,

$$F_0''(V) = 2C \Big[\int_{-\infty}^{V} dv_x \, e^{-\beta v_x^2} - \int_{-\infty}^{-V} dv_x \, e^{-\beta v_x^2} \Big] > 0. \quad \blacksquare \tag{3.26}$$

3.2 Proof of Theorem 3.1

In the sequel of the paper C will denote positive constants, possibly depending on β, ρ, R, but not on γ, which is our small parameter.

For any $\gamma \in (0, \gamma_0)$ with γ_0 suitably small, we introduce an a.e. differentiable function with bounded derivative, $t \to W(t)$, such that $W(0) = V_0$, $\lim_{t \to \infty} W(t) = 0$, and satisfying the following properties:

(i) W is decreasing in any time interval in which $W(t) > 0$.
(ii) There exist two positive constants A_1, A_2 such that, for any $t \geq 0$, it is:

$$W(t) \geq \gamma \, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^5} \equiv f_1(t) \tag{3.27}$$

and

$$W(t) \leq \gamma \, e^{-C_2 t} - \gamma^5 \frac{A_2}{t^5} \chi(\{t \geq \bar{t}\}) \equiv f_2(t). \tag{3.28}$$

where

$$\bar{t} = \bar{K} \log \frac{1}{\gamma} \tag{3.29}$$

with the constant $\bar{K}$ satisfying $\bar{K} > 4/C_2$. The two constants A_1 and A_2, independent of each other and also of γ and γ_0, will be fixed later on.

The strategy of the proof of Theorem 3.1 is the following. We assign the disk a velocity $W(t)$ with the properties just stated and we consider the free transport equation (3.9) outside the disk, with boundary conditions (3.4). We can compute the terms r_W^+ and r_W^- defined below in (3.31) and (3.32), since the light particles velocities $v(s,t;x,v)$ for $s < t$ are known, once the motion of the body is given. At this point, we solve Eq. (3.6) for the disk with assigned r_W^+ and r_W^-, finding a new velocity, call it V_W. The solution to our problem is the fixed point of the map $W \to V_W$. To this aim, we have to prove that V_W enjoys the same properties established above for W, with the same constants.

Let $X(t) = \int_0^t W(\tau)d\tau$ be the position of the disk at time t. Consider the modified problem:

$$\dot{V}_W(t) = -K(t)\, V_W(t) - r_W^+(t) - r_W^-(t), \tag{3.30}$$

where $K(t)$ is the function introduced in (3.14) with $W(t)$ in place of $V(t)$.
We define

$$r_W^+(t) = 2\rho\Big(\frac{\beta}{\pi}\Big)^{\frac{3}{2}} \int_{D(t)} dx \int_{v_x \le W(t)} dv\, (v_x - W(t))^2 (e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}) \tag{3.31}$$

and

$$r_W^-(t) = 2\rho\Big(\frac{\beta}{\pi}\Big)^{\frac{3}{2}} \int_{D(t)} dx \int_{v_x \ge W(t)} dv\, (v_x - W(t))^2 (e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}). \tag{3.32}$$

We notice that as long as W is decreasing $r_W^+(t) = 0$, so that this term appears only for negative velocities and moreover, by the collision law (3.3), it is negative. The analysis of the sign of $r_W^-(t)$ is more involved, since there are positive and negative contributions. We will show, in any way, that the sum $r_W^+ + r_W^-$ is positive.

We may ask whether Eq. (3.30) is well posed. The following Proposition, proved in Ref. [13], shows that this dynamical system is well defined for almost all initial data and almost all $t \in \mathbb{R}^+$. More precisely we can neglect in the sequel all the initial configurations giving rise to infinitely many or tangential collisions, namely those for which there exists a time $s < t$ such that $x \in D(t)$, $x(s,t;x,v) \in D(s)$ and $v_x(s,t;x,v) = W(s)$.

Proposition 3.1. *Consider the dynamics of the disk with given velocity $W = W(t)$ and the fluid trajectories $x(s,t;x,v)$, $v(s,t;x,v)$ computed according to the evolution of the disk and the law of the elastic reflection (3.3). Assume W differentiable for almost all t and such that*

$$\operatorname{ess\,sup}_{t\in\mathbb{R}^+}(|W(t)| + |\dot{W}(t)|) = L < +\infty. \tag{3.33}$$

Then the set of all $t \in \mathbb{R}^+$, $x \in D(t)$, $v \in \mathbb{R}^3$ for which $x(s,t;x,v)$, $v(s,t;x,v)$, $0 \le s < t$, delivers infinitely many backward collisions, or has a tangential collision, has vanishing Lebesgue measure.

Now we start the analysis of the two terms $r_W^+(t)$ and $r_W^-(t)$. Let us set, for $0 \le s < t$

$$\langle W\rangle_{s,t} = \frac{1}{t-s}\int_s^t W(\tau)d\tau \tag{3.34}$$

and

$$\langle W\rangle_{0,t} = \langle W\rangle_t. \tag{3.35}$$

We have a recollision (from the right or from the left) if $x \in D(t)$, $v(0,t;x,v) \neq v$ and it exists a time $s < t$ such that

$$v_x(t-s) = X(t) - X(s) = \int_s^t W(\tau)\,d\tau \tag{3.36}$$

that is $v_x = \langle W\rangle_{s,t}$ for some $s \in (0,t)$ and

$$|x_\perp - v_\perp(t-s)| \le 2R. \tag{3.37}$$

Thus, necessary condition for a recollision to happen is that:

$$v_x = \langle W\rangle_{s,t} \quad \text{for some } s \in (0,t) \quad \text{and} \quad |v_\perp| \le \frac{2R}{t-s}. \tag{3.38}$$

Let us start by estimating $r_W^+(t)$, proving the following Lemma:

Lemma 3.2. *For any $t \ge 0$ and γ sufficiently small,*

$$0 \le -r_W^+(t) \le C\frac{\gamma^9 A_1^3}{(1+t)^5}\,\chi(\{t > t_0\}) \tag{3.39}$$

where

$$t_0 = K_0 \log\frac{1}{\gamma} \tag{3.40}$$

and K_0 is a constant satisfying $1/C_1 \le K_0 < 2/C_1$.

Proof

As already pointed out, $r_W^+(t) = 0$ as far as W is decreasing (i.e. as far as $W(t) > 0$). Now we give an upper and lower bound for the first time t^* for which $W(t^*) = 0$. For $t = t_0$ it results

$$f_1(t_0) = \gamma^{1+K_0 C_1} - \gamma^3 \frac{A_1}{(1 + K_0 \log \frac{1}{\gamma})^5} > 0, \tag{3.41}$$

the last inequality being satisfied by taking γ sufficiently small. By the properties (i) and (ii) of the function W this implies that, for $0 \le t \le t_0$, $W(t) \ge W(t_0) \ge f_1(t_0) > 0$.

Moreover by (3.29):

$$f_2(\bar{t}) = \gamma^{1+\bar{K} C_2} - \gamma^5 \frac{A_2}{(\bar{K} \log \frac{1}{\gamma})^5} < 0 \tag{3.42}$$

for γ sufficiently small. Then $W(t) < 0$ for $t \ge \bar{t}$, and so the first time t^* for which $W(t^*) = 0$ satisfies $t^* \in (t_0, \bar{t})$; moreover $W(t) \le 0$ for $t^* \le t \le \bar{t}$.

It is also evident from the law of elastic reflection (3.3) that $r_W^+(t) \le 0$, since it appears for negative velocities. Let us establish an upper bound for $|r_W^+(t)|$. Recalling the necessary condition (3.36) to have a recollision, we have by (3.27):

$$\begin{aligned} v_x = \langle W \rangle_{s,t} &\ge \frac{1}{t-s} \int_s^t \Big(\gamma e^{-C_1 \tau} - \gamma^3 \frac{A_1}{(1+\tau)^5} \Big) d\tau \\ &\ge \frac{1}{t-s} \int_s^t \Big(-\gamma^3 \frac{A_1}{(1+\tau)^5} \Big) d\tau \ge -C A_1 \frac{\gamma^3}{1+t} \end{aligned} \tag{3.43}$$

if $s < t/2$. Since by (3.3) $v_\perp(0, t; x, v) = v_\perp$, from (3.31), (3.38) and (3.43) it follows, for v_x such that $s < t/2$ and $t > t^*$, that a first contribution to the estimate of $|r_W^+(t)|$ is:

$$\begin{aligned} & C \int dv_x \, (v_x - W(t))^2 \, \chi(\{ -\frac{C A_1 \gamma^3}{1+t} \le v_x \le W(t) \}) \\ & \quad \times \int dv_\perp \, e^{-\beta v_\perp^2} \, \chi(\{ |v_\perp| < \frac{2R}{t-s} \}) \\ & \le \frac{C}{(1+t)^2} \int dv_x \, (v_x - W(t))^2 \, \chi(\{ -\frac{C A_1 \gamma^3}{1+t} \le v_x \le W(t) \}) \\ & \le \frac{C}{(1+t)^2} \Big(W(t) + \frac{C A_1 \gamma^3}{1+t} \Big)^3 \le C \frac{A_1^3 \gamma^9}{(1+t)^5} \end{aligned} \tag{3.44}$$

since $W(t) < 0$.

If $s \geq t/2$ and $t > t^*$ we have by (3.27):

$$v_x = \langle W \rangle_{s,t} \geq \frac{1}{t-s} \int_s^t \Big(-\gamma^3 \frac{A_1}{(1+\tau)^5} \Big) d\tau$$
$$\geq -C \frac{A_1 \gamma^3}{(1+t)^5}, \tag{3.45}$$

hence the second contribution to the estimate of $|r_W^+(t)|$ is:

$$C \int dv_x \, (v_x - W(t))^2 \, \chi(\{-C \frac{A_1\gamma^3}{(1+t)^5} \leq v_x \leq W(t)\}) \int dv_\perp \, e^{-\beta v_\perp^2}$$
$$\leq C \Big(\frac{A_1 \gamma^3}{(1+t)^5} \Big)^3. \tag{3.46}$$

Collecting estimates (3.44) and (3.46) we finally achieve the proof of Lemma 3.2. ■

For $r_W^-(t)$ we have an upper bound expressed by the following Lemma:

Lemma 3.3. *For any $t \geq 0$ and γ sufficiently small,*

$$r_W^-(t) \leq C \frac{(\gamma + A_1\gamma^3)^3}{(1+t)^5}. \tag{3.47}$$

Proof
By (3.36) and (3.28) we obtain:

$$v_x = \langle W \rangle_{s,t} \leq \frac{1}{t-s} \int_s^t \Big[\gamma \, e^{-C_2 \tau} - \gamma^5 \frac{A_2}{\tau^5} \chi(\{\tau \geq \bar{t}\}) \Big] d\tau$$
$$\leq \frac{1}{t-s} \int_s^t \gamma \, e^{-C_2 \tau} d\tau, \tag{3.48}$$

then we have

$$v_x \leq C \frac{\gamma}{1+t} \qquad \text{if } \; s < \frac{t}{2} \tag{3.49}$$
$$v_x \leq C \, \gamma \, e^{-C_2 t/2} \qquad \text{if } \; s \geq \frac{t}{2}. \tag{3.50}$$

Hence by (3.32) we get a first contribution to the estimate of $r_W^-(t)$, in case $s < t/2$, recalling (3.27) and condition (3.38) on $v_\perp$:

$$\frac{C}{(1+t)^2}\int_{W(t)}^{\frac{C\gamma}{1+t}}(v_x - W(t))^2\, dv_x \le \frac{C}{(1+t)^2}\Big(\frac{C\gamma}{1+t} - W(t)\Big)^3$$
$$\le \frac{C}{(1+t)^2}\Big(\frac{C\gamma}{1+t} - \gamma e^{-C_1 t} + \gamma^3\frac{A_1}{(1+t)^5}\Big)^3$$
$$\le C\frac{(\gamma + A_1\gamma^3)^3}{(1+t)^5}. \tag{3.51}$$

If $s \ge t/2$, using (3.50), we have by (3.27) the second contribution to the estimate of $r_W^-(t)$:

$$C\int dv_x\,(v_x - W(t))^2\,\chi(\{W(t) \le v_x \le C\gamma e^{-C_2\frac{t}{2}}\})$$
$$\le C(C\gamma e^{-C_2\frac{t}{2}} - W(t))^3 \le C\Big(\frac{A_1\gamma^3 + \gamma}{(1+t)^5}\Big)^3, \tag{3.52}$$

therefore, collecting (3.51) and (3.52), we obtain the thesis. ■

By the collision law (3.3) it follows that $r_W^-(t) \ge 0$ for any $t \le t^*$ (as long as $W(t) \ge 0$). Actually the positivity of $r_W^-(t)$ for any $t \ge 0$ is not obvious, since for negative velocities of the disk, $r_W^-(t)$ could change sign. We can prove that this is not the case. Moreover we will show that the sum $(r_W^+(t) + r_W^-(t))$ is not negative for any $t \ge 0$, which is a key ingredient in the proof of Theorem 3.1.

Lemma 3.4. *Suppose γ sufficiently small. Then, for $t \ge t_0$ we have:*

$$r_W^-(t) \ge C\frac{\gamma^5}{t^5}. \tag{3.53}$$

Proof

The only reason for $r_W^-(t)$ to become negative at some instant t is that the absolute value of the particles' velocities has been increased by the collisions with the body. By the law (2.3), this can happen only for those particles who are to collide at time t with $W(t) < 0$, while their last but one collision was at some time $s < t$ with $W(s) < 0$.

Moreover, any such particle has negative x-component of its pre-collisional velocity at time t, being

$$v_x = 2W(s) - v_x' \leq 2W(s) - W(s) < 0, \tag{3.54}$$

for some $v_x' > W(s)$. Denoting by $\tilde{r}_W^-(t)$ the restriction of $r_W^-(t)$ to such configurations, we give an upper bound for $|\tilde{r}_W^-(t)|$.

From Definition (3.32) of $r_W^-(t)$ and using (3.27) we obtain, by (3.28):

$$\begin{aligned} |\tilde{r}_W^-(t)| &\leq C\int_{W(t)}^0 (v_x - W(t))^2 dv_x \leq C(-W(t))^3 \\ &\leq C\Big(-\gamma e^{-C_1 t} + \gamma^3 \frac{A_1}{(1+t)^5}\Big)^3 \\ &\leq C\frac{\gamma^9 A_1^3}{(1+t)^{15}}, \end{aligned} \tag{3.55}$$

where $W(t) \leq 0$, since $t > s$ and $W(s) < 0$ (see the comment on the sign of W in the proof of Lemma 3.2). Up to now we could obviously write

$$r_W^-(t) \geq -|\tilde{r}_W^-(t)|. \tag{3.56}$$

To improve this lower bound, let us denote by $\hat{r}_W^-(t)$ a term which contains some "good" contribution to $r_W^-(t)$, namely the one due to particles colliding at time t and coming from a single collision in the past, at some instant $s < t$ for which $W(s) > 0$. Hence we have

$$r_W^-(t) \geq \hat{r}_W^-(t) - |\tilde{r}_W^-(t)| \tag{3.57}$$

and the difficulty now shifts to get a lower bound for $\hat{r}_W^-(t)$. To this end we restrict our analysis to a subset of the set of "good" contributions. Therefore let us introduce $s_0 > 0$ defined as:

$$s_0 = \min\left\{ s \in (0,t) : W(s) \leq \frac{V_0 + \langle W\rangle_{s,t}}{2} \right\}. \tag{3.58}$$

Such s_0 does exist by continuity, since we have at time 0:

$$W(0) = V_0 > \frac{V_0 + \langle W\rangle_{0,t}}{2} \Longleftrightarrow \frac{V_0}{2} > \frac{\langle W\rangle_t}{2} \tag{3.59}$$

which holds because it is $W(t) < V_0$ $\forall t > 0$; at time $s = t$ we have

$$W(t) < \frac{V_0 + \langle W\rangle_{t,t}}{2} = \frac{V_0 + W(t)}{2} \Longleftrightarrow \frac{W(t)}{2} < \frac{V_0}{2}. \tag{3.60}$$

Define the set $\Gamma = \{(x,v) : x \in D(t),\ \langle W\rangle_{s_0,t} \le v_x \le \langle W\rangle_{0,t} = \langle W\rangle_t\}$.
First of all, we show that Γ is a non-empty set. Indeed for $s < s_0$, by definition (3.58), it follows:

$$\frac{d}{ds}\langle W\rangle_{s,t} = \frac{1}{t-s}[\langle W\rangle_{s,t} - W(s)] < \frac{1}{t-s}(W(s) - V_0) \le 0. \tag{3.61}$$

Hence, for $s < s_0$, $\langle W\rangle_{s_0,t} < \langle W\rangle_{s,t} < \langle W\rangle_t$.
Moreover, any particle belonging to Γ had at most one collision with the disk in the past. In fact, consider a light particle which is to collide at x with velocity $v_x = \langle W\rangle_{s,t}$, being $s \in (0, s_0]$ the time of its last collision with the disk. Then, denoting by $v_x(s^-)$ the $x-$ component of the precollisional velocity, for $s < s_0$ by (3.58) we have:

$$v_x(s^-) = -v_x + 2W(s) = -\langle W\rangle_{s,t} + 2W(s) \ge V_0. \tag{3.62}$$

Before s the light particle cannot undergo another recollision, since it should be:

$$V_0 \le v_x(s^-) = \langle W\rangle_{\tau,s} \tag{3.63}$$

for some $\tau \in (0,s)$, and $\langle W\rangle_{\tau,s} \ge V_0$ is absurd.

Hence, for $s < s_0$,

$$v_x(0,t;x,v) = 2W(s) - v_x. \tag{3.64}$$

The time s_0 can be bounded from above and from below, independently of γ, in the following way:

$$\frac{1}{C_1}\log\frac{3}{2} \le s_0 \le \frac{1}{C_2}\log 4 \tag{3.65}$$

provided that t is sufficiently large independently of γ.

We have that s_0 is the minimal solution of the equation:

$$W(s_0) = \frac{V_0 + \langle W\rangle_{s_0,t}}{2}. \tag{3.66}$$

Moreover, by (3.28),

$$\frac{\gamma}{2} + \frac{\langle W\rangle_{s_0,t}}{2} = W(s_0) \le \gamma e^{-C_2 s_0}, \tag{3.67}$$

and since, by (3.27),

$$\langle W\rangle_{s_0,t} \ge -\frac{1}{t-s_0}\int_{s_0}^{t}\frac{A_1\gamma^3}{(1+\tau)^5}d\tau \ge -A_1\gamma^3, \tag{3.68}$$

we have, for $A_1\gamma^2 \le 1/2$,

$$\gamma e^{-C_2 s_0} \ge \frac{\gamma}{2} - \frac{A_1\gamma^3}{2} \ge \frac{\gamma}{4} \tag{3.69}$$

so we have proved the right bound in (3.65).

To prove the left bound, we have, by (3.27):

$$\frac{\gamma}{2} + \frac{\langle W\rangle_{s_0,t}}{2} = W(s_0) \ge \gamma e^{-C_1 s_0} - \gamma^3 \frac{A_1}{(1+s_0)^5}. \tag{3.70}$$

By (3.28) and the upper bound just proved for s_0, we have for large t

$$\langle W\rangle_{s_0,t} \le \frac{1}{t-s_0}\int_{s_0}^{t} \gamma e^{-C_2\tau} d\tau \le \frac{1}{t}\gamma C. \tag{3.71}$$

Hence, for t large independently of γ and for $A_1\gamma^2 < 1/12$,

$$\gamma e^{-C_1 s_0} \le \frac{\gamma}{2} + \frac{C\gamma}{t} + \frac{A_1\gamma^3}{(1+s_0)^5} \le \frac{2}{3}\gamma \tag{3.72}$$

proving also the left bound in (3.65).

Now we consider the restriction of $r_W^-(t)$ to the set Γ, setting

$$I(t) = \int_{D(t)} dx \int_{|v_\perp|<\frac{C}{t}} dv_\perp \int_{\langle W\rangle_{s_0,t}}^{\langle W\rangle_t} dv_x\, (v_x - W(t))^2\, [e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}]. \tag{3.73}$$

By definition of $\hat{r}_W^-(t)$ and (3.64), it results that $\hat{r}_W^-(t)$ is non-negative, since $[e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}] \ge 0$. Moreover it is

$$\hat{r}_W^-(t) \ge CI(t). \tag{3.74}$$

This is due to the fact, which now we prove, that for $t > s_0$,

$$W(t) < \langle W\rangle_{s_0,t}. \tag{3.75}$$

We have, by (3.27) and (3.28), for t large enough,

$$\begin{aligned} W(t) - \langle W\rangle_{s_0,t} &= W(t) - \frac{1}{t-s_0}\int_{s_0}^{t} ds\, W(s) \\ &\le \gamma e^{-C_2 t} + \frac{1}{t-s_0}\int_{s_0}^{t} ds \Big[-\gamma e^{-C_1 s} + \gamma^3 \frac{A_1}{(1+s)^5}\Big] \\ &\le \gamma\Big[e^{-C_2 t} - \frac{e^{-C_1 s_0} - e^{-C_1 t}}{C_1(t-s_0)} + \frac{1}{t} C A_1 \gamma^2\Big]. \end{aligned} \tag{3.76}$$

For γ small enough, so that t_0 is sufficiently large and $A_1\gamma^2$ is small, we obtain that the r.h.s. of (3.76) is negative for $t > t_0$. For $s_0 < t \le t_0$ $W(t)$ is decreasing, so that (3.75) holds for any $t > s_0$.

Let us go back to the investigation of $I(t)$. For $s \le s_0$ by (3.64) and (3.58) we get:

$$\begin{aligned} v_x^2(0,t;x,v) - v_x^2 &= (2W(s) - v_x)^2 - v_x^2 = 4W(s)(W(s) - \langle W\rangle_{s,t}) \\ &\ge 2W(s)(V_0 - \langle W\rangle_{s,t}), \end{aligned} \tag{3.77}$$

and for $s \le s_0 \le (\log 4)/C_2$ and for $A_1\gamma^2 < C$ it is

$$W(s) \ge W\left(\frac{\log 4}{C_2}\right) \ge C\gamma. \tag{3.78}$$

We want to show also that

$$V_0 - \langle W\rangle_{s,t} \ge C\gamma. \tag{3.79}$$

In fact $\langle W\rangle_{s,t}$ is decreasing with respect to s, for $s \le s_0$, then by (3.28)

$$\begin{aligned} V_0 - \langle W\rangle_{s,t} &\ge V_0 - \langle W\rangle_{0,t} = V_0 - \frac{1}{t}\int_0^t W(\tau)\,d\tau \\ &\ge \gamma + \frac{1}{t}\int_0^t \left(-\gamma e^{-C_2\tau} + \gamma^5\frac{A_2}{\tau^5}\chi(\{\tau \ge \bar{t}\})\right)d\tau \\ &\ge \gamma - \gamma\frac{1 - e^{-C_2 t}}{C_2 t} \ge \frac{\gamma}{2} \end{aligned} \tag{3.80}$$

for t sufficiently large independently of γ. Therefore, for t sufficiently large:

$$2W(s)(V_0 - \langle W\rangle_{s,t}) \ge C\gamma^2. \tag{3.81}$$

By these considerations, we have:

$$\begin{aligned} I(t) = &\int_{D(t)} dx \int_{|v_\perp| < \frac{C}{t}} dv_\perp \\ &\times \int_{\langle W\rangle_{s_0,t}}^{\langle W\rangle_t} dv_x\,(v_x - W(t))^2\, e^{-\beta v^2}[1 - e^{\beta(v_x^2 - v_x^2(0,t;x,v))}] \end{aligned}$$

and since, by (3.77),

$$v_x^2(0,t;x,v) - v_x^2 < 4V_0(V_0 + V_0) = 8\gamma^2, \tag{3.82}$$

it results, using also (3.81),

$$-8\gamma^2 < v_x^2 - v_x^2(0,t;x,v) < 0, \tag{3.83}$$

and in this interval we can write, since γ is small,

$$1 - e^{\beta(v_x^2 - v_x^2(0,t;x,v))} \geq -C\beta(v_x^2 - v_x^2(0,t;x,v)). \tag{3.84}$$

Therefore, by (3.77), (3.81) and t large independently of γ,

$$\begin{aligned}
I(t) \geq\ & C \int_{D(t)} dx \int_{|v_\perp| < \frac{C}{t}} dv_\perp \\
& \times \int_{\langle W \rangle_{s_0,t}}^{\langle W \rangle_t} dv_x\, (v_x - W(t))^2\, e^{-\beta v^2} [v_x^2(0,t;x,v) - v_x^2] \\
\geq\ & C\gamma^2 \int_{\langle W \rangle_{s_0,t}}^{\langle W \rangle_t} dv_x\, (v_x - W(t))^2\, e^{-\beta v_x^2} \int_{|v_\perp| < \frac{C}{t}} dv_\perp\, e^{-\beta v_\perp^2} \\
\geq\ & \frac{C\gamma^2}{t^2} \Big[(\langle W \rangle_t - W(t))^3 - (\langle W \rangle_{s_0,t} - W(t))^3 \Big] \\
\geq\ & \frac{C\gamma^2}{t^2} \Big[(\langle W \rangle_t - \langle W \rangle_{s_0,t})\, (\langle W \rangle_t - W(t))^2 \Big].
\end{aligned} \tag{3.85}$$

We now estimate both differences appearing in (3.85), showing that they are both $O(\frac{1}{t})$.

$$\begin{aligned}
\langle W \rangle_t - \langle W \rangle_{s_0,t} &= \frac{1}{t} \int_0^t W(\tau)\, d\tau - \frac{1}{t - s_0} \int_{s_0}^t W(\tau)\, d\tau \\
&= \left(\frac{1}{t} - \frac{1}{t - s_0} \right) \int_0^t W(\tau)\, d\tau + \frac{1}{t - s_0} \int_0^{s_0} W(\tau)\, d\tau \\
&= \frac{s_0}{t - s_0} \left[\frac{1}{s_0} \int_0^{s_0} W(\tau)\, d\tau - \frac{1}{t} \int_0^t W(\tau)\, d\tau \right] \\
&= \frac{s_0}{t - s_0} \left(\langle W \rangle_{s_0} - \langle W \rangle_t \right),
\end{aligned} \tag{3.86}$$

and it is, by (3.78) and (3.28),

$$\begin{aligned}
\langle W \rangle_{s_0} - \langle W \rangle_t &\geq C\gamma + \frac{1}{t} \int_0^t \left(-\gamma e^{-C_2 \tau} \right) d\tau \\
&\geq C\gamma - \gamma \frac{1 - e^{-C_2 t}}{C_2 t} \geq C\gamma,
\end{aligned} \tag{3.87}$$

for γ sufficiently small and t large independently of γ. Thus by (3.86) and (3.65) we arrive at:

$$\langle W\rangle_t - \langle W\rangle_{s_0,t} \geq C\frac{\gamma}{t}. \tag{3.88}$$

Let us now estimate the remaining term in (3.85). Proceeding as in (3.76) we have:

$$\langle W\rangle_t - W(t) \geq -\gamma\Big[e^{-C_2 t} - \frac{1-e^{-C_1 t}}{C_1 t} + \frac{1}{t}CA_1\gamma^2\Big], \tag{3.89}$$

therefore, for γ sufficiently small (so that $A_1\gamma^2$ is small enough) and t large independently of γ,

$$\langle W\rangle_t - W(t) \geq C\frac{\gamma}{t}. \tag{3.90}$$

Inserting estimates (3.88) and (3.90) in (3.85), by (3.74) we conclude that, for γ sufficiently small and t sufficiently large, independently of γ,

$$\hat{r}_W^-(t) \geq C\frac{\gamma^5}{t^5}. \tag{3.91}$$

Recalling (3.57), choosing $t \geq t_0$ (see (3.40)) and γ sufficiently small, by (3.91) and (3.55),

$$r_W^-(t) \geq \hat{r}_W^-(t) - |\tilde{r}_W^-(t)| \geq C\frac{\gamma^5}{t^5} - C\frac{\gamma^9 A_1^3}{(1+t)^{15}}, \tag{3.92}$$

so that, for $\gamma^4 A_1^3$ small enough,

$$r_W^-(t) \geq C\frac{\gamma^5}{t^5}. \quad \blacksquare \tag{3.93}$$

We remark that, from (3.39) and (3.53) it follows immediatly, for γ small and any t,

$$r_W^+(t) + r_W^-(t) \geq 0. \tag{3.94}$$

Now we prove that the function $V_W(t)$ satisfying Eq. (3.30) enjoys, for γ suitably small, the same properties as the function W, with the same constants A_1, A_2. After this the proof of Theorem 3.1 will follow easily.

Proposition 3.2. *Suppose $\gamma > 0$ sufficiently small. Then:*
(i) $t \to V_W(t)$ is an a.e. differentiable function with bounded derivative, decreasing in any time interval in which $V_W(t) > 0$.
(ii) For any $t \geq 0$:

$$V_W(t) > \gamma\, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^5}. \tag{3.95}$$

(iii) For any $t > 0$:

$$V_W(t) < \gamma\, e^{-C_2 t} - \gamma^5 \frac{A_2}{t^5} \chi(\{t \geq \bar{t}\}). \tag{3.96}$$

Proof
(i) From Eq. (3.30) and the Duhamel formula we have:

$$V_W(t) = \gamma e^{-\int_0^t K(\tau)d\tau} - \int_0^t ds\, e^{-\int_s^t K(\tau)d\tau} (r_W^+(s) + r_W^-(s)), \tag{3.97}$$

and since $r_W^+(t)$ and $r_W^-(t)$ are bounded, by (3.97) and (3.30) V_W is a.e. differentiable with bounded derivative. The fact that $V_W(t)$ is decreasing in any time interval in which $V_W(t) > 0$ is obvious by Eq. (3.30) and (3.94).

(ii) By (3.97), (3.39), and (3.47) it follows:

$$V_W(t) \geq \gamma e^{-C_1 t} - C(\gamma + A_1\gamma^3)^3 \int_0^t ds\, e^{-C_2(t-s)} \frac{1}{(1+s)^5}. \tag{3.98}$$

Let us evaluate the integral:

$$\begin{aligned} \int_0^t ds\, \frac{e^{C_2 s}}{(1+s)^5} &= \int_0^{\frac{t}{2}} (\cdot)\, ds + \int_{\frac{t}{2}}^t (\cdot)\, ds \\ &\leq \frac{e^{C_2 \frac{t}{2}} - 1}{C_2} + \frac{2^5}{(2+t)^5} \frac{e^{C_2 t} - e^{C_2 \frac{t}{2}}}{C_2}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^t ds\, \frac{e^{-C_2(t-s)}}{(1+s)^5} &\leq \frac{e^{-C_2 \frac{t}{2}} - e^{-C_2 t}}{C_2} + \frac{2^5}{(2+t)^5} \frac{1 - e^{-C_2 \frac{t}{2}}}{C_2} \\ &\leq \frac{1}{C_2} \left(e^{-C_2 \frac{t}{2}} + \frac{2^5}{(2+t)^5} \right) \leq \frac{C}{(1+t)^5}. \end{aligned} \tag{3.99}$$

To conclude, there exists a constant $\bar{C}$ such that:

$$V_W(t) \geq \gamma e^{-C_1 t} - \bar{C}(\gamma + A_1\gamma^3)^3 \frac{1}{(1+t)^5} \tag{3.100}$$

and hence, to achieve the thesis, it is sufficient that

$$\bar{C}(\gamma + A_1\gamma^3)^3 < A_1\gamma^3, \tag{3.101}$$

which is satisfied, for instance, by choosing $A_1 = 2\bar{C}$ (this fixes A_1) and γ consequently small (also to satisfy the previous constraints on A_1).

(iii) First, by (3.94) and (3.97), we have that, for any $t > 0$,

$$V_W(t) < \gamma e^{-C_2 t}. \tag{3.102}$$

By (3.97), (3.39), and (3.53), for γ suitably small and $t \geq \bar{t} > 2t_0$, where $\bar{t}$ is defined in (3.29) and t_0 in (3.40), it follows:

$$\begin{aligned} V_W(t) &< \gamma e^{-C_2 t} + \int_0^t ds\, e^{-\int_s^t K(\tau)d\tau} \left(C\frac{\gamma^9}{(1+s)^5} - C\frac{\gamma^5}{s^5}\right) \chi(\{s > t_0\}) \\ &< \gamma e^{-C_2 t} - C\gamma^5 \int_0^t ds\, e^{-\int_s^t K(\tau)d\tau} \frac{1}{s^5}\chi(\{s > t_0\}). \end{aligned} \tag{3.103}$$

We have that

$$\begin{aligned} \int_0^t ds\, e^{-\int_s^t K(\tau)d\tau} \frac{1}{s^5}\chi(\{s > t_0\}) &\geq \int_{t_0}^t ds\, e^{-C_1(t-s)} \frac{1}{s^5} \\ &\geq \frac{1 - e^{-C_1(t-t_0)}}{C_1 t^5} \\ &\geq \frac{1 - e^{-C_1 t_0}}{C_1 t^5} \geq \frac{1}{2C_1 t^5}, \end{aligned} \tag{3.104}$$

since $t \geq \bar{t} > 2t_0$.

Then, by (3.103) and (3.104),

$$V_W(t) < \gamma e^{-C_2 t} - C\frac{\gamma^5}{t^5}. \tag{3.105}$$

Last inequality enables us to choose A_2, in such a way that (3.96) is satisfied for any $t > 0$. This can be done in a consistent manner, since the constant C appearing in

(3.105) does not depend of A_2. Actually it depends on A_1, nevertheless A_2 can be chosen independently of A_1 for γ sufficiently small. ■

Using Proposition 3.2 we easily prove Theorem 3.1. We only give a sketch of the proof and, for a better explanation of it, we address to [13]. We construct a sequence $\{V_n\}_{n=1}^{\infty}$ defined by

$$V_n = V_{V_{n-1}}, \qquad n \geq 2 \tag{3.106}$$

setting $V_1 = W$, being W any function with the properties established at the beginning of this Section. By Proposition 3.2 such properties hold for the whole sequence (for suitable values of A_1, A_2, $\bar{t}$ independent of n). By compactness (the sequence is equibounded and equicontinuous), we can extract a subsequence $V_{n'}$ converging to a limit point $V = V(t)$. Moreover, for any $n \geq 1$ we can solve the free transport equation, with reflecting boundary conditions on the disk moving according to the velocity $V_{n'}(t)$, by means of the characteristics which are a.e. defined. The convergence of $V_{n'}$ implies the convergence of almost all characteristics to a family of characteristics satisfying the reflecting boundary conditions on the disk moving with velocity $V(t)$. This yields a solution to the Vlasov equation (3.1) producing the friction term (3.7). Therefore we have obtained a solution to the problem (3.1)-(3.9).

Moreover, any solution to this problem satisfies bounds (3.21) and (3.22). Consider in fact any solution (V, f) of the problem. By continuity of V there exists a time interval in which inequalities (3.21)-(3.22) hold strictly. Let T be the first time for which our strict inequalities are violated. The same arguments used in Proposition 3.2 (replacing W by V) show that (3.21)-(3.22) hold strictly in the interval $(0, T]$, since in this interval V enjoys the same properties as W. Then T must be infinite. This concludes the proof of Theorem 3.1.

3.3 The case with elastic force

We consider here the case in which an elastic force acts on the disk. We consider the same model introduced in Section 3.1, the only difference consisting in the evolution equation for the disk (see 3.6), which becomes:

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -F(t) - S\,X(t), \tag{3.107}$$

$$X(0) = X_0 < 0, \qquad V(0) = V_0 > 0,$$

where $F(t)$ is defined in (3.7) and $S > 0$ is the elastic constant.

We analyse first a simplified case in which recollisions are neglected and only the first term of the Taylor expansion, centered in $V = 0$, of $F_0(V(t))$ is retained. We obtain:

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -F_0'(0)\, V(t) - S\, X(t), \tag{3.108}$$

$$X(0) = X_0 < 0, \qquad V(0) = V_0 > 0,$$

It is useful to put (3.108) into the form (recalling that $C_2 = F_0'(0)$):

$$\begin{pmatrix} \dot{X}(t) \\ \dot{V}(t) \end{pmatrix} = \mathbf{A} \begin{pmatrix} X(t) \\ V(t) \end{pmatrix}, \tag{3.109}$$

where

$$\mathbf{A} \equiv \begin{pmatrix} 0 & 1 \\ -S & -C_2 \end{pmatrix}. \tag{3.110}$$

Therefore we have

$$\begin{pmatrix} X(t) \\ V(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} X_0 \\ V_0 \end{pmatrix}. \tag{3.111}$$

The eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = \frac{-C_2 + \sqrt{C_2^2 - 4S}}{2}, \tag{3.112}$$

$$\lambda_2 = \frac{-C_2 - \sqrt{C_2^2 - 4S}}{2}. \tag{3.113}$$

We want to restrict ourselves to study the case in which the body position goes to zero without oscillations, hence we have to choose the friction force large with respect to the elastic one, and the initial data in a suitable way. We assume

$$C_2^2 - 4S > 0, \tag{3.114}$$

so that it results $\lambda_2 < \lambda_1 < 0$. Decomposing the initial datum into the eigenvectors of $\mathbf{A}$, we have:

$$\begin{pmatrix} X_0 \\ V_0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}, \tag{3.115}$$

finding

$$a_1 = \frac{V_0 - \lambda_2 X_0}{\lambda_1 - \lambda_2}, \qquad a_2 = \frac{\lambda_1 X_0 - V_0}{\lambda_1 - \lambda_2}. \tag{3.116}$$

Hence from (3.111) we obtain:

$$X(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}, \tag{3.117}$$

$$V(t) = a_1 \lambda_1 e^{\lambda_1 t} + a_2 \lambda_2 e^{\lambda_2 t}. \tag{3.118}$$

Now we assume

$$\lambda_1 X_0 < V_0 < \lambda_2 X_0. \tag{3.119}$$

This condition, together with (3.114), ensures, by (3.117) and (3.118), that both $X(t)$ and $V(t)$ decay exponentially to zero, being $X(t) < 0$ and $V(t) > 0$ for all $t \geq 0$.

Let us examine now the full problem (3.107), including recollisions. In this case the sign of $V(t)$ is no more evident, nevertheless we can state for $X(t)$ a result analogous to the one in Theorem 3.1.

We make assumption (3.114) on the quantities ρ, β, R, S, which are the data of the problem. Moreover, we put

$$\gamma = V_0, \quad |X_0| = \bar{C}\gamma \tag{3.120}$$

with the constant $\bar{C}$, independent of γ, chosen in such a way that (3.119) is satisfied. Let us define:

$$C_3 = \frac{2}{\gamma} \max\{|a_1|;\ |a_2|\}, \tag{3.121}$$

$$C_4 = \frac{1}{\gamma} \min\{|a_1|;\ |a_2|\}, \tag{3.122}$$

$$C_5 = \frac{2}{\gamma} \max\{|a_1\lambda_1|;\ |a_2\lambda_2|\}. \tag{3.123}$$

Note that, with the choice (3.120), the constants C_3, C_4, and C_5 are independent of γ.

Theorem 3.2. *There exists $\gamma_0 = \gamma_0(\rho, \beta, R, S) > 0$ sufficiently small such that, for any $\gamma \in (0, \gamma_0)$ and for any choice of initial conditions X_0, V_0 satisfying (3.120), there exists at least one pair (X, f), where $X = X(t)$ solves, for almost all $t \in \mathbb{R}^+$, Eq. (3.107) and f solves Eq.s (3.9), (3.4), and (3.8). Moreover there exist three positive constants B_1, B_2, and B_3, independent of γ, such that any solution (X, f) satisfies the following properties:*
(i) for any $t \geq 0$, we have:

$$X(t) < 0, \qquad |X(t)| \leq C_3\, \gamma\, e^{\lambda_1 t} + \gamma^3 \frac{B_1}{(1+t)^{d+2}}, \tag{3.124}$$

(ii) there exists a sufficiently large $\bar{t}$, depending on γ, such that for any $t \geq 0$:

$$|X(t)| \geq C_4 \, \gamma \, e^{\lambda_2 t} + \gamma^5 \frac{B_2}{t^{d+2}} \chi(\{t \geq \bar{t}\}), \tag{3.125}$$

(iii) for any $t \geq 0$:

$$|\dot{X}(t)| \leq C_5 \, \gamma \, e^{\lambda_1 t} + \gamma^3 \frac{B_3}{(1+t)^{d+2}}. \tag{3.126}$$

We notice that in this case the power-law decay is stated on the position of the disk, while in Theorem 3.1 it was stated on its velocity. This is due to the fact that in Theorem 3.1 the equation governing the evolution of the disk was a first order differential equation involving only $V(t)$, while in the present case we have a second order differential equation involving $X(t)$.

Also in this case, as we did in the previous Section, we give the proof of the Theorem for the three-dimensional case.

Proof of Theorem 3.2

In the same way as in Section 3.2, for any $\gamma \in (0, \gamma_0)$ (with γ_0 sufficiently small) we introduce an a.e. differentiable function with bounded derivative, $t \to W(t)$, such that $W(0) = V_0$, $\lim_{t \to \infty} W(t) = 0$, and satisfying the following properties:

(i) W is positive and decreasing for $t \in [0, \tilde{t}]$, where

$$\tilde{t} = \frac{4}{C_1} \log \frac{1}{\gamma}. \tag{3.127}$$

(ii) There exists a positive constant B_3, such that, for any $t \geq 0$, it is:

$$|W(t)| \leq C_8 \, \gamma \, e^{\lambda_1 t} + \gamma^3 \frac{B_3}{(1+t)^5}. \tag{3.128}$$

The constant B_3, independent of γ and γ_0, will be fixed later on.

Let us consider the modified problem:

$$\begin{cases} \dot{X}_W(t) = V_W(t) \\ \dot{V}_W(t) = -F_0'(0) \, V_W(t) - S \, X_W(t) - [F_0(W(t)) - F_0'(0) \, W(t)] \\ \qquad\qquad -r_W^+(t) - r_W^-(t) \end{cases} \tag{3.129}$$

with initial conditions X_0, V_0, satisfying (3.107), (3.119), and r^+_W, r^-_W, defined in (3.31), (3.32). The fixed point of the map $W \to V_W$ (if any), defined by (3.129), solves Eq. (3.107).
The solution, by the Duhamel formula, is:

$$\begin{pmatrix} X_W(t) \\ V_W(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t ds\, e^{(t-s)\mathbf{A}} \begin{pmatrix} 0 \\ -r^+_W(s) - r^-_W(s) \end{pmatrix}$$
$$+ \int_0^t ds\, e^{(t-s)\mathbf{A}} \begin{pmatrix} 0 \\ -F_0(W(s)) + F_0'(0)\, W(s) \end{pmatrix} \tag{3.130}$$

where $\mathbf{A}$ is defined in (3.110). We decompose the last two vectors in (3.130) into the eigenvectors of $\mathbf{A}$:

$$\begin{pmatrix} 0 \\ -r^+_W(s) - r^-_W(s) \end{pmatrix} + \begin{pmatrix} 0 \\ -F_0(W(s)) + F_0'(0)\, W(s) \end{pmatrix}$$
$$= b^{(1)}_W(s) \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + b^{(2)}_W(s) \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}, \tag{3.131}$$

finding

$$b^{(1)}_W(s) = \frac{-r^+_W(s) - r^-_W(s) - F_0(W(s)) + F_0'(0)\, W(s)}{\lambda_1 - \lambda_2}, \quad b^{(2)}_W(s) = -b^{(1)}_W(s). \tag{3.132}$$

The r.h.s. of (3.130) can be written then in the form (recalling (3.115)):

$$a_1\, e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + a_2\, e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{3.133}$$
$$+ \int_0^t ds \left[b^{(1)}_W(s)\, e^{\lambda_1 (t-s)} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - b^{(1)}_W(s)\, e^{\lambda_2 (t-s)} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \right]$$

and the solution in (3.130) is, recalling (3.111), (3.117), and (3.118):

$$X_W(t) = a_1\, e^{\lambda_1 t} + a_2\, e^{\lambda_2 t}$$
$$+ \int_0^t ds\, b^{(1)}_W(s) \left[e^{\lambda_1 (t-s)} - e^{\lambda_2 (t-s)} \right], \tag{3.134}$$

$$V_W(t) = a_1 \lambda_1\, e^{\lambda_1 t} + a_2 \lambda_2\, e^{\lambda_2 t}$$
$$+ \int_0^t ds\, b^{(1)}_W(s) \left[\lambda_1 e^{\lambda_1 (t-s)} - \lambda_2 e^{\lambda_2 (t-s)} \right]. \tag{3.135}$$

In order to establish the properties of $X_W(t)$, $V_W(t)$, and to infer that V_W satisfies the same properties of W, we proceed to analyse the recollision terms, $r^+_W(t)$ and $r^-_W(t)$, and the function $b^{(1)}_W(t)$.

Let us start by estimating $r^+_W(t)$.

Lemma 3.5. *For any $t \geq 0$ and γ sufficiently small,*

$$|r^+_W(t)| \leq C\frac{\gamma^9 B_3^3}{(1+t)^5}\,\chi(\{t \geq \tilde{t}\}). \tag{3.136}$$

Proof

First of all let us notice that $r^+_W(t) = 0$ as far as W is decreasing (i.e. for $t \in [0, \tilde{t}]$). For $t > \tilde{t}$ we have, by (3.127) and (3.128),

$$\begin{aligned}|W(t)| &\leq C_8\gamma\, e^{\lambda_1 \frac{t}{2}} e^{\lambda_1 \frac{t}{2}} + \gamma^3\frac{B_3}{(1+t)^5} \leq C_8\gamma^3\, e^{\lambda_1 \frac{t}{2}} + \gamma^3\frac{B_3}{(1+t)^5} \\ &\leq \gamma^3\frac{CB_3}{(1+t)^5},\end{aligned} \tag{3.137}$$

therefore, recalling (3.38), we have for $s < t/2$:

$$v_x = \langle W\rangle_{s,t} \geq \frac{1}{t-s}\int_s^t \left(-\gamma^3\frac{CB_3}{(1+\tau)^5}\right) d\tau \geq -CB_3\frac{\gamma^3}{1+t}. \tag{3.138}$$

Then recalling that, by (3.3), $v_\perp(0,t;x,v) = v_\perp$, from (3.31), (3.38), (3.137) and (3.138) it follows, for v_x such that $s < t/2$ and $t > \tilde{t}$, that a first contribution to the estimate of $|r^+_W(t)|$ is:

$$\begin{aligned}&C\int dv_x\,(v_x - W(t))^2\,\chi(\{-\frac{CB_3\gamma^3}{1+t} \leq v_x \leq W(t)\}) \\ &\quad\times \int dv_\perp\, e^{-\beta v_\perp^2}\,\chi(\{|v_\perp| < \frac{2R}{t-s}\}) \\ &\leq \frac{C}{(1+t)^2}\int dv_x\,(v_x - W(t))^2\,\chi(\{-\frac{CB_3\gamma^3}{1+t} \leq v_x \leq W(t)\}) \\ &\leq \frac{C}{(1+t)^2}\Big(W(t) + \frac{CB_3\gamma^3}{1+t}\Big)^3 \leq C\frac{B_3^3\gamma^9}{(1+t)^5}.\end{aligned} \tag{3.139}$$

If $s \geq t/2$ and $t > \tilde{t}$, we have by (3.137):

$$v_x = \langle W \rangle_{s,t} \geq \frac{1}{t-s} \int_s^t \Big(-\gamma^3 \frac{CB_3}{(1+\tau)^5} \Big) d\tau$$
$$\geq -C \frac{B_3 \gamma^3}{(1+t)^5}, \tag{3.140}$$

hence the second contribution to the estimate of $|r_W^+(t)|$ is:

$$C \int dv_x \, (v_x - W(t))^2 \; \chi(\{-C \frac{B_3\gamma^3}{(1+t)^5} \leq v_x \leq W(t)\})$$
$$\leq C \Big(\frac{B_3 \gamma^3}{(1+t)^5} \Big)^3. \tag{3.141}$$

Collecting estimates (3.139) and (3.141) we obtain (3.136). ■

For $r_W^-(t)$ we have an upper bound expressed by the following Lemma:

Lemma 3.6. *For any $t \geq 0$ and γ sufficiently small,*

$$r_W^-(t) \leq C \frac{(\gamma + B_3 \gamma^3)^3}{(1+t)^5}. \tag{3.142}$$

Proof
By (3.36) and (3.128) we obtain:

$$v_x = \langle W \rangle_{s,t} \leq \frac{1}{t-s} \int_s^t \Big[C_8 \gamma \, e^{\lambda_1 \tau} + \gamma^3 \frac{B_3}{(1+\tau)^5} \Big] d\tau, \tag{3.143}$$

then we have

$$v_x \leq C \frac{\gamma + B_3 \gamma^3}{1+t} \qquad \text{if } s < \frac{t}{2} \tag{3.144}$$
$$v_x \leq C \, \frac{\gamma + B_3 \gamma^3}{(1+t)^5} \qquad \text{if } s \geq \frac{t}{2}. \tag{3.145}$$

Hence, again by (3.38) and (3.128), a first contribution to the estimate of $r_W^-(t)$ is, in the case $s < t/2$:

$$\frac{C}{(1+t)^2} \int dv_x \, (v_x - W(t))^2 \, \chi(\{W(t) \leq v_x \leq C \frac{\gamma + B_3 \gamma^3}{1+t}\})$$

$$\leq \frac{C}{(1+t)^2}\Big(C\frac{\gamma + B_3\gamma^3}{1+t} - W(t)\Big)^3$$

$$\leq \frac{C}{(1+t)^2}\Big(C\frac{\gamma + B_3\gamma^3}{1+t} + C_8\gamma e^{\lambda_1 t} + \gamma^3\frac{B_3}{(1+t)^5}\Big)^3$$

$$\leq C\frac{(\gamma + B_3\gamma^3)^3}{(1+t)^5}. \tag{3.146}$$

If $s \geq t/2$, by (3.145) we analogously get the second contribution to the estimate of $r^-_W(t)$:

$$C\int dv_x\,(v_x - W(t))^2\,\chi(\{W(t) \leq v_x \leq C\frac{\gamma + B_3\gamma^3}{(1+t)^5}\})$$

$$\leq C\left(\frac{\gamma + B_3\gamma^3}{(1+t)^5}\right)^3, \tag{3.147}$$

therefore, collecting (3.147) and (3.146), we obtain the thesis. ■

The lower bound for $r^-_W(t)$ is given in the following Lemma:

Lemma 3.7. *Suppose γ sufficiently small. Then, for $t \geq \tilde{t}$ we have:*

$$r^-_W(t) \geq C\frac{\gamma^5}{t^5}. \tag{3.148}$$

Proof

Following the same argument as in Lemma 3.4, we notice that the "bad" contributions to $r^-_W(t)$, which tend to turn it into a negative quantity, are uniquely due to particles whose last collision time $s < t$ is such that $W(s) < 0$. By the definition of W, it has to be necessarily $s > \tilde{t}$. We denote by $\tilde{r}^-_W(t)$ such "bad" contributions, and we give an upper bound for $|\tilde{r}^-_W(t)|$. It is for $s \in (\tilde{t}, t)$, by (3.38) and (3.137),

$$v_x = \langle W\rangle_{s,t} \leq \frac{1}{t-s}\int_s^t \gamma^3\frac{CB_3}{(1+\tau)^5}\,d\tau, \tag{3.149}$$

and it results

$$v_x \leq C\frac{B_3\gamma^3}{1+t} \qquad \text{if } s < \frac{t}{2}, \tag{3.150}$$

$$v_x \leq C\frac{B_3\gamma^3}{(1+t)^5} \qquad \text{if } s \geq \frac{t}{2}. \tag{3.151}$$

Hence a first contribution to the estimate of $|\tilde{r}_W^-(t)|$ is, in the case $s < t/2$, by (3.137) and (3.38):

$$\frac{C}{(1+t)^2}\int dv_x\,(v_x - W(t))^2\,\chi(\{W(t)\le v_x \le C\frac{B_3\gamma^3}{1+t}\})$$
$$\le \frac{C}{(1+t)^2}\Big(C\frac{B_3\gamma^3}{1+t} - W(t)\Big)^3 \le \frac{C}{(1+t)^2}\Big(C\frac{B_3\gamma^3}{1+t} + C\frac{B_3\gamma^3}{(1+t)^5}\Big)^3$$
$$\le C\frac{\gamma^9 B_3^3}{(1+t)^5}. \tag{3.152}$$

If $s \ge t/2$, using (3.151) and (3.137), we have the second contribution to the estimate of $|\tilde{r}_W^-(t)|$:

$$C\int dv_x\,(v_x - W(t))^2\,\chi(\{W(t)\le v_x \le C\frac{B_3\gamma^3}{(1+t)^5}\})$$
$$\le C\frac{\gamma^9 B_3^3}{(1+t)^{15}}, \tag{3.153}$$

therefore, collecting (3.152) and (3.153), we obtain that $\forall\, t > \tilde{t}$

$$|\tilde{r}_W^-(t)| \le C\frac{\gamma^9 B_3^3}{(1+t)^5}. \tag{3.154}$$

Let us now look for some positive contribution to r_W^-, call it $\hat{r}_W^-$, which turns out to contrast $\tilde{r}_W^-$. As in Lemma 3.4, we consider a set of particles coming from a single collision in the past, happening for some $s < t$ such that $W(s) > 0$. The analysis is now pretty much the same as the one made before (the only change being the small difference in the definition of W), and the lower bound obtained for $\hat{r}_W^-(t)$ is the same:

$$\hat{r}_W^-(t) \ge C\frac{\gamma^5}{t^5}, \tag{3.155}$$

for γ small enough and t large. Then we have, by (3.154) and (3.155), for $t \ge \tilde{t}$ and γ sufficiently small,

$$r_W^-(t) \ge \hat{r}_W^-(t) - |\tilde{r}_W^-(t)| \ge C\frac{\gamma^5}{t^5} - C\frac{\gamma^9 B_3^3}{(1+t)^5}, \tag{3.156}$$

so that, for $\gamma^4 B_3^3$ small enough, we obtain (3.148). ■

As an immediate consequence of Lemmas 3.5 and 3.7 we have, for γ sufficiently small and $\forall\, t \geq 0$,

$$r_W^+(t) + r_W^-(t) \geq 0. \tag{3.157}$$

The following Lemma concerns the function $b_W^{(1)}(t)$, introduced in (3.132):

Lemma 3.8. *Suppose γ sufficiently small. Then for any $t \geq 0$ it is $b_W^{(1)}(t) < 0$. Moreover, for any $t \geq 0$, it results:*

$$C\frac{\gamma^5}{t^5}\chi(\{t \geq \tilde{t}\}) \leq |b_W^{(1)}(t)| \leq C\frac{(\gamma + B_3\gamma^3)^3}{(1+t)^5}. \tag{3.158}$$

Proof

For $t \leq \tilde{t}$ it is $W(t) > 0$, then the quantity $[F_0(W(t)) - F_0'(0)\,W(t)]$ is positive, by the properties of the function F_0 (see Lemma 3.1). Hence, by the definition of $b_W^{(1)}(t)$ (3.132), and by (3.157), we have that $b_W^{(1)}(t) < 0$ for $t \leq \tilde{t}$. For $t > \tilde{t}$ and γ sufficiently small, by (3.137) we have that

$$|F_0(W(t)) - F_0'(0)\,W(t)| \leq C\frac{\gamma^9 B_3^3}{(1+t)^{15}}, \tag{3.159}$$

since, by the properties of F_0, the l.h.s. of (3.159) is $O(W^3)$ for W small. Hence by Lemma 3.5, Lemma 3.7, (3.159), and (3.132), we have that

$$\begin{aligned} b_W^{(1)}(t) &\leq C\frac{\gamma^9 B_3^3}{(1+t)^5} - C\frac{\gamma^5}{t^5} + C\frac{\gamma^9 B_3^3}{(1+t)^{15}} \\ &\leq -C\frac{\gamma^5}{t^5}, \end{aligned} \tag{3.160}$$

for $\gamma^4 B_3^3$ small enough, so that the negativity of $b_W^{(1)}(t)$ is proved for any time.

In the same way it can be proved the lower bound for $|b_W^{(1)}(t)|$, while the upper bound is achieved by using Lemma 3.6. ∎

We show now that the function $X_W(t)$ (satisfying Eq. (3.129)) enjoys, for γ sufficiently small, properties (3.124) and (3.125) stated for $X(t)$ in Theorem 3.2, while the function $V_W(t)$ satisfies the same properties as $W(t)$ (3.127)-(3.128).

Proposition 3.3. *Suppose $\gamma > 0$ sufficiently small. Then, for any $t \geq 0$, the following properties hold:*
(i)

$$X_W(t) < 0, \qquad |X_W(t)| < C_6\, \gamma\, e^{\lambda_1 t} + \gamma^3 \frac{B_1}{(1+t)^5}. \tag{3.161}$$

(ii) There exists a sufficiently large $\bar{t}$, depending on γ, such that:

$$|X_W(t)| > C_7\, \gamma\, e^{\lambda_2 t} + \gamma^5 \frac{B_2}{t^5} \chi(\{t \geq \bar{t}\}). \tag{3.162}$$

(iii) $t \to V_W(t)$ is an a.e. differentiable function with bounded derivative, positive and decreasing for $t \in [0, \tilde{t}]$.
(iv)

$$|V_W(t)| < C_8\, \gamma\, e^{\lambda_1 t} + \gamma^3 \frac{B_3}{(1+t)^5}. \tag{3.163}$$

Proof
(i) The negativity of $X_W(t)$ is obvious from (3.134), as it follows from assumptions (3.114), (3.119), and by the negativity of $b_W^{(1)}(s)$, proved in Lemma 3.8.

Moreover, by (3.134) and Lemma 3.8,

$$\begin{aligned} |X_W(t)| &\leq 2 \max\{|a_1|; |a_2|\}\, e^{\lambda_1 t} + \int_0^t ds\, |b_W^{(1)}(s)|\, e^{\lambda_1 (t-s)} \\ &\leq C_6 \gamma\, e^{\lambda_1 t} + \int_0^t ds\, C \frac{(\gamma + B_3 \gamma^3)^3}{(1+s)^5}\, e^{\lambda_1 (t-s)}. \end{aligned} \tag{3.164}$$

By the same calculations as those made in (3.99) we have

$$\int_0^t ds\, \frac{e^{\lambda_1 (t-s)}}{(1+s)^5} \leq \frac{C}{(1+t)^5}, \tag{3.165}$$

so that (3.161) is obtained, by choosing the constant B_1 independently of B_3, for γ suitably small.

(ii) We have, for any $t \geq 0$:

$$\begin{aligned} |X_W(t)| &= |a_1|\, e^{\lambda_1 t} + |a_2|\, e^{\lambda_2 t} + \int_0^t ds\, |b_W^{(1)}(s)|\, \left[e^{\lambda_1 (t-s)} - e^{\lambda_2 (t-s)} \right] \\ &> C_7 \gamma\, e^{\lambda_2 t}. \end{aligned} \tag{3.166}$$

Moreover, choosing $\bar{t} = 2\tilde{t}$ (where $\tilde{t}$ is defined in (3.127)), by Lemma 3.8 we have, for $t \geq \bar{t}$:

$$\int_0^t ds\, |b_W^{(1)}(s)|\, \left[e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right] \geq C \int_{\tilde{t}}^t ds\, \frac{\gamma^5}{s^5} \left[e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right]. \tag{3.167}$$

Let us evaluate the integral:

$$\begin{aligned} \int_{\tilde{t}}^t ds\, \frac{e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}}{s^5} &\geq \frac{1}{t^5} \left[\frac{1 - e^{\lambda_1(t-\tilde{t})}}{-\lambda_1} - \frac{1 - e^{\lambda_2(t-\tilde{t})}}{-\lambda_2} \right] \\ &\geq \frac{1}{t^5} \left[\frac{1 - e^{\lambda_1 \tilde{t}}}{-\lambda_1} + \frac{1}{\lambda_2} \right] \geq \frac{C}{t^5}. \end{aligned} \tag{3.168}$$

This implies that (3.162) is verified, for some constant B_2, with $\bar{t} = 2\tilde{t}$.

(iii) Since $b_W^{(1)}(t)$ is bounded, by (3.134), (3.135), and (3.129) V_W is a.e. differentiable with bounded derivative. We have also

$$V_W(t) \geq a_1\lambda_1\, e^{\lambda_1 t} + a_2\lambda_2\, e^{\lambda_2 t} - \int_0^t ds\, |b_W^{(1)}(s)|\, \left|\lambda_1 e^{\lambda_1(t-s)} + \lambda_2 e^{\lambda_2(t-s)}\right|. \tag{3.169}$$

By Lemma 3.8 and (3.165),

$$\begin{aligned} \int_0^t ds\, |b_W^{(1)}(s)|\, \left|\lambda_1 e^{\lambda_1(t-s)} + \lambda_2 e^{\lambda_2(t-s)}\right| &\leq C \int_0^t ds\, \frac{(\gamma + B_3\gamma^3)^3}{(1+s)^5}\, e^{\lambda_1(t-s)} \\ &\leq C\,\gamma^3, \end{aligned} \tag{3.170}$$

therefore in the time interval $[0, \tilde{t}]$,

$$V_W(t) \geq C\gamma\, e^{\lambda_1 \tilde{t}} - C\gamma^3 = C\gamma\, \gamma^{\frac{-4\lambda_1}{C_1}} - C\gamma^3 > 0, \tag{3.171}$$

for γ sufficiently small, since by (3.112) and (3.16)

$$0 < \frac{-4\lambda_1}{C_1} < \frac{2C_2}{C_1} < 2. \tag{3.172}$$

By taking the derivative with respect to time in (3.135), it can be seen similarly that $\dot{V}_W(t) < 0$ for $t \in [0, \tilde{t}]$. Indeed, from Lemma 3.8 it follows:

$$\begin{aligned} \dot{V}_W(t) &= a_1\lambda_1^2\, e^{\lambda_1 t} + a_2\lambda_2^2\, e^{\lambda_2 t} \\ &\quad + \int_0^t ds\, b_W^{(1)}(s)\, \left[\lambda_1^2\, e^{\lambda_1(t-s)} - \lambda_2^2\, e^{\lambda_2(t-s)}\right] + (\lambda_1 - \lambda_2) b_W^{(1)}(t) \\ &\leq a_1\lambda_1^2\, e^{\lambda_1 t} + C \int_0^t ds\, |b_W^{(1)}(s)| \leq -C\gamma\, e^{\lambda_1 \tilde{t}} + C\gamma^3 < 0, \end{aligned} \tag{3.173}$$

for γ sufficiently small.

(iv) The result is achieved following the same reasoning as in (i). ■

Theorem 3.2 can finally be proved, through the same steps as in the proof of Theorem 3.1.

3.4 Comments

We want to discuss briefly some possible generalizations of the present investigation.

Our techniques employed in Section 3.2 work as well in case of a constant field $E > 0$ directed along the x-direction, with the choice $V_0 > V_\infty$. Putting $V_0 - V_\infty = \gamma$ as small parameter, Theorem 3.1 can be slightly modified, to obtain the power-like time asymptotics. This case $E > 0$, $V_0 > V_\infty$, is even easier than the one we faced in Section 3.2, with $E = 0$ and $V_0 > 0$. Indeed we are no more troubled with the signes of r^+ and r^-: they are always positive since the velocity of the disk never changes sign. Hence we have the same power-law approach to the limiting velocity V_∞, and we can prove that $V(t)$ initially decreases monotonically, crosses its limiting value V_∞ in a finite time and then reaches it from below.

Another improvement of the model consists in the generalization of the shape of the obstacle. We have considered the simplified shape of a disk, but the same results remain valid in case of a general convex body. A detailed analysis of this case is performed in [14], and we give here a short sketch of such a generalization.

Let us consider the same physical problem of Section 3.1 in which the disk is replaced by a convex solid Ω in $\mathbb{R}^3$, taken for simplicity with unitary mass and constant density. Let R be the diameter of Ω (i.e. the maximum distance between two points on its boundary $\partial\Omega$) and $X(t)$ the position of its center of mass at time t. We impose that the center of mass is constrained to move along the x-axis, and that the solid cannot undergo any kind of rotation (if the solid has a rotational symmetry around the x-axis, these constraints are superfluous). The outward normal to $\partial\Omega$ is denoted by $\hat{n}$. Moreover we denote by $\partial\Omega^+$ the right face of the solid, on which $\hat{n} \cdot \hat{x} \geq 0$ (being $\hat{x}$ the unit vector of the x-axis) and by $\partial\Omega^-$ the left face of the solid, on which $\hat{n} \cdot \hat{x} < 0$. We require that $\hat{n}$ is continuously varying a.e. on $\partial\Omega$ and that there are two disjoint subsets of $\partial\Omega$, having positive measure, on which $\hat{n} \cdot \hat{x} > 0$ and $\hat{n} \cdot \hat{x} < 0$ respectively.

A necessary condition for which a gas particle with velocity v hits the solid at time t is that, at the collision point $P \in \partial\Omega$, it results

$$v_n \leq V_n(t), \tag{3.174}$$

denoting by

$$v_n = v \cdot \hat{n}, \qquad V_n(t) = V(t)\,\hat{n} \cdot \hat{x}, \tag{3.175}$$

where $V(t) = \dot{X}(t)$ and $\hat{n}$ is calculated at P.
The gas particles hitting the body are reflected according to the usual collision law:

$$v'_n = 2V_n(t) - v_n, \qquad v'_{n_\perp} = v_{n_\perp}, \tag{3.176}$$

denoting by $v_{n_\perp} = v - v_n\,\hat{n}$.

Ignoring recollisions, the viscous friction term is

$$\begin{aligned} F_0(V) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \leq V_n} dv\,(V_n - v_n)^2\,\hat{n} \cdot \hat{x}\, e^{-\beta v^2} \\ +k \int_{\partial\Omega^-} d\sigma \int_{v_n \leq V_n} dv\,(V_n - v_n)^2\,\hat{n} \cdot \hat{x}\, e^{-\beta v^2} \end{aligned} \tag{3.177}$$

where $k = 2\rho(\beta/\pi)^{3/2}$ and $d\sigma$ is the surface element on $\partial\Omega$. The scalar product $\hat{n} \cdot \hat{x}$ in the integral is due to the fact that we have to consider the projection of the force along the x-axis. It is also convenient to separate the whole integral on $\partial\Omega$ in the two integrals (on $\partial\Omega^+$ and $\partial\Omega^-$) appearing in (3.177), since the first one is positive ($\hat{n}\cdot\hat{x} \geq 0$ on $\partial\Omega^+$) and the second one is negative ($\hat{n}\cdot\hat{x} < 0$ on $\partial\Omega^-$). It can be easily seen that the function $F_0(V)$ defined in (3.177) is an increasing odd function (not necessarily convex for $V > 0$).

Taking into account recollisions the full friction term is

$$F(t) = F_0(V(t)) + r^+(t) + r^-(t)$$

where

$$r^+(t) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \leq V_n(t)} dv\,(V_n(t) - v_n)^2\,\hat{n} \cdot \hat{x}\,[e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}] \tag{3.178}$$

and

$$r^-(t) = -k \int_{\partial\Omega^-} d\sigma \int_{v_n \leq V_n(t)} dv\,(V_n(t) - v_n)^2\,\hat{n} \cdot \hat{x}\,[e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}]. \tag{3.179}$$

Define a function $W(t)$ with the same properties stated in Section 3.2, and correspondingly compute the terms

$$r_W^+(t) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \le W_n(t)} dv\, (W_n(t) - v_n)^2\, \hat{n} \cdot \hat{x}\, [e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}], \tag{3.180}$$

$$r_W^-(t) = -k \int_{\partial\Omega^-} d\sigma \int_{v_n \le W_n(t)} dv\, (W_n(t) - v_n)^2\, \hat{n} \cdot \hat{x}\, [e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}], \tag{3.181}$$

where $W_n(t) = W(t)\, \hat{n} \cdot \hat{x}$.
Let $s < t$ be the first backward recollision time, and let us denote by $P \in \partial\Omega$ the collision point at time t, and by $Q \in \partial\Omega$ the collision point at time s. The condition to have two subsequent collisions is the following:

$$v_n(t-s) = \overrightarrow{QP} \cdot \hat{n} + \langle W_n \rangle_{s,t}(t-s) \ge \langle W_n \rangle_{s,t}(t-s) \tag{3.182}$$

where $\hat{n}$ is calculated at P and $\overrightarrow{QP}$ is the vector joining the points Q and P at time s. In fact $v_n(t-s)$ is the space along the $\hat{n}$ direction covered by the gas particle in the time interval $[s,t]$, $\langle W_n \rangle_{s,t}(t-s)$ is the space along the $\hat{n}$ direction covered by the body in the time interval $[s,t]$, and $\overrightarrow{QP} \cdot \hat{n}$ is the distance along the $\hat{n}$ direction between the two points Q and P at time s, which, by the convex shape of the body, is always non-negative.
Hence a first necessary condition to have a recollision is

$$v_n \ge \langle W_n \rangle_{s,t}. \tag{3.183}$$

Another necessary condition is the following:

$$|v_{n_\perp}|\,(t-s) \le 2R + |\langle W \rangle_{s,t}|\,(t-s). \tag{3.184}$$

In fact the r.h.s. of (3.184) represents the maximum displacement that a particle can undergo along the x-direction, and along any direction orthogonal to the x-axis, to have a recollision with the body. Then from (3.184) we obtain

$$|v_{n_\perp}| \le \frac{C}{t-s}. \tag{3.185}$$

Summarizing, conditions (3.183) and (3.185) replace conditions (3.38). In this way we can obtain estimates analogous to those of Lemma 3.2, Lemma 3.3 and Lemma 3.4, hence we are able to prove for $V(t)$ the same time-decay as that contained in Theorem 3.1, in case of an obstacle with a general convex shape.

As a final comment, we remark that, even if we have explicitly treated the case of elastic force, the same techniques work also in case that anharmonic terms are added.

Appendix A

Proof of Lemma 2.1

We make a partition of the physical space with large cubes of side mr and we divide the interaction into a short-range and a long-range one. The last one can be handled using Proposition 2.1. Concerning the short-range interaction we choose the parameter α in the definition of the weight-function f (see (2.24)) so small in such a way that, in a cube, f is constant, and then, neglecting the interaction with the other cubes, W is superstable. Of course the interaction between different cubes exists, but it gives a surface effect, and it becomes negligible with respect to a volume effect, as m is very large.
Let us define the set $\Gamma_u^l(r)$ in the following way:

$$\Gamma_u^l(r) \equiv \{x \in \mathbb{R}^3 : u^{(i)} + l^{(i)}mr \leq x^{(i)} < u^{(i)} + (l^{(i)}+1)mr, \\ u \in \mathbb{R}^3;\, m \in \mathbb{N},\, l \in \mathbb{Z}^3\},$$

where r is the parameter appearing in Proposition 2.1.
From this definition, it follows that $|x-y| \leq \sqrt{3}mr, \quad \forall x, y \in \Gamma_u^l(r)$, then, by the properties of the weight-function:

$$f(|y-\mu|, R) \leq f(|x-\mu|, R)\left(1 + \alpha\sqrt{3}\,mr\right)^{\lambda}. \tag{A.1}$$

We define also the following quantities:

$$\widehat{f}_{u,l}^{\mu,R} \equiv \inf_{i\in\mathbb{N}, q_i \in \Gamma_u^l} f_i^{\mu,R}, \tag{A.2}$$

$$P_u^l(r) \equiv \{(i,j) \in \mathbb{N} \otimes \mathbb{N} : i > j,\ q_i \in \Gamma_u^l,\ q_j \in \Gamma_u^l, |q_i - q_j| < r\}, \tag{A.3}$$

$$T_u(r) \equiv \bigcup_{l \in \mathbb{Z}^3} P_u^l(r), \tag{A.4}$$

$$M \equiv \sup_{x \in \mathbb{R}^3} |\phi(x)|, \tag{A.5}$$

$$V(r) \equiv \{(i,j) \in \mathbb{N} \otimes \mathbb{N} : i > j,\ |q_i - q_j| < r\}. \tag{A.6}$$

As it follows from its definition, $V(r)$ is the set of all the pairs of particles with relative distance smaller than r, while in T_u there are no pairs with particles in two adjoining Γ_u^l.

Let ϵ be a real positive number such that:

$$\epsilon > \sqrt{3}\,\alpha m r, \tag{A.7}$$

then for each x and y in Γ_u^l we have:

$$f(|y-\mu|, R) \leq (1+\epsilon)^\lambda f(|x-\mu|, R). \tag{A.8}$$

Since the potential can be decomposed into $\phi = \phi^{(1)} + \phi^{(2)}$ (see (2.17)), the mollified energy becomes

$$W(X;\mu,R) \equiv W^{(1)}(X;\mu,R) + W^{(2)}(X;\mu,R), \tag{A.9}$$

where

$$W^{(1)}(X;\mu,R) \equiv \sum_{i \in \mathbb{N}} f_i^{\mu,R} \Big(\frac{v_i^2}{2} + \frac{1}{2} \sum_{j:j \neq i} \phi_{i,j}^{(1)} + b \Big) \tag{A.10}$$

$$W^{(2)}(X;\mu,R) \equiv \sum_{i \in \mathbb{N}} f_i^{\mu,R} \frac{1}{2} \sum_{j:j \neq i} \phi_{i,j}^{(2)}. \tag{A.11}$$

Let us estimate now the second term $W^{(2)}$. For r large enough we have:

$$|W^{(2)}| \leq \widetilde{D}_1 \,(1+\sqrt{3}\,\alpha)^\lambda \sum_{i \in \mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i} \times \sum_{j \in \mathbb{Z}^3} n_{\Delta_j} \frac{\chi(|i-j| > r-2)}{(|i-j| - \sqrt{3})^\gamma}$$

$$
\begin{aligned}
&\leq \widetilde{D}_2 \sum_{i\in\mathbb{Z}^3} \sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|i-\mu|,R)\, n_{\Delta_i} n_{\Delta_j}\, \frac{1}{|i-j|^{\gamma}} \\
&\leq \widetilde{D}_3 \sum_{i\in\mathbb{Z}^3} \sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|i-\mu|,R)\, (n^2_{\Delta_i}+n^2_{\Delta_j})\, \frac{1}{|i-j|^{\gamma}} \\
&\leq \widetilde{D}_4 \Big\{ \sum_{i\in\mathbb{Z}^3} \sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|i-\mu|,R)\, n^2_{\Delta_i}\, \frac{1}{|i-j|^{\gamma}} \\
&+ \sum_{\substack{i,j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|j-\mu|,R)\, n^2_{\Delta_j}\, \frac{1}{|i-j|^{\gamma}}(1+|i-j|)^{\lambda}) \Big\} \\
&\leq \widetilde{D}_5 \sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\, n^2_{\Delta_i} \\
&\times \sum_{k=r}^{\infty} \sum_{j\in\mathbb{Z}^3} \chi(k \leq |j| < (k+1))\, \frac{(1+k+1)^{\lambda}}{k^{\gamma}} \\
&\leq \widetilde{D}_6 \sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\, n^2_{\Delta_i} \sum_{k=r}^{\infty} \frac{1}{k^{\gamma-2-\lambda}} \\
&\leq \widetilde{D}_7(r) \sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\, n^2_{\Delta_i}\,, \qquad \text{(A.12)}
\end{aligned}
$$

with $\widetilde{D}_7(r)$ such that:

$$\lim_{r\to+\infty} \widetilde{D}_7(r) = 0, \qquad \text{(A.13)}$$

as

$$\gamma > 3 + \lambda. \qquad \text{(A.14)}$$

Therefore $\exists\, r_1 > 0 : \forall r > r_1 \Rightarrow \widetilde{D}_7(r) \leq \frac{1}{4}A$, hence

$$W^{(2)} \geq -\frac{1}{4} A \sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\, n^2_{\Delta_i}, \qquad \text{(A.15)}$$

for any $r > r_1$.

Now it remains to examine the first term $W^{(1)}$.

If we define the quantity:

$$E(X;\mu,\Gamma^l_u) = \sum_{(i,j)\in P^l_u} f^{\mu,R}_i \phi^{(1)}_{i,j}\,, \qquad \text{(A.16)}$$

by the superstability of $\phi^{(1)}$ we have:

$$
\begin{aligned}
E(X;\mu,\Gamma_u^l) &= \sum_{(i,j)\in P_u^l} (f_i^{\mu,R} - \widehat{f}_{u,l}^{\mu,R})\phi_{i,j}^{(1)} + \widehat{f}_{u,l}^{\mu,R} \sum_{(i,j)\in P_u^l} \phi_{i,j}^{(1)} \\
&\geq -M((1+\epsilon)^\lambda - 1) \sum_{(i,j)\in P_u^l} \widehat{f}_{u,l}^{\mu,R} - B\widehat{f}_{u,l}^{\mu,R} \sum_{k\in\mathbb{Z}_u^3} n_{\Delta_k} + \frac{3}{4} A \widehat{f}_{u,l}^{\mu,R} \sum_{k\in\mathbb{Z}_u^3} n_{\Delta_k}^2,
\end{aligned}
$$

and from the following definition

$$
\mathbb{Z}_u^3 \equiv \mathbb{Z}^3 \cap \Gamma_u^l \tag{A.17}
$$

we get:

$$
\begin{aligned}
E(X;\mu,\Gamma_u^l) \geq& -B\,\widehat{f}_{u,l}^{\mu,R} \sum_{k\in\mathbb{Z}_u^3} n_{\Delta_k} + \frac{3A}{4(1+\epsilon)^\lambda} \sum_{k\in\mathbb{Z}_u^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 \\
&- \frac{M}{2}((1+\epsilon)^\lambda - 1) \sum_{(i,j)\in P_u^l} (f_i^{\mu,R} + f_j^{\mu,R}).
\end{aligned} \tag{A.18}
$$

Choosing $z = u$, where $z \in \Gamma_0^0 \cap r\mathbb{Z}^3$, we have $\bigcup_{l\in\mathbb{Z}^3} \Gamma_z^l = \mathbb{R}^3$, and to each z it is associated a partition $\mathcal{P}_z$ of the space.
For a fixed partition, considering the definition (2.23), summing (A.16) over the sets $\Gamma_z^l \in \mathcal{P}_z$ and taking into account all the contributions of the pairs not belonging to a set of the partition, we finally obtain a lower bound for the mollified energy. Indeed choosing $b > B$, we have:

$$
\begin{aligned}
W^{(1)}(X;\mu,R) \geq& \sum_{l\in\mathbb{Z}^3} E(X;\mu,\Gamma_z^l) + b\sum_{l\in\mathbb{Z}^3} \widehat{f}_{z,l}^{\mu,R}\, n_{\Delta_l} \\
&- \frac{M}{2} \sum_{(i,j)\notin T_z} (f_i^{\mu,R} + f_j^{\mu,R}) \\
\geq& \frac{3A}{4(1+\epsilon)^\lambda} \sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 - M \sum_{(i,j)\notin T_z} (f_i^{\mu,R} + f_j^{\mu,R}) \\
&- \frac{M}{2}((1+\epsilon)^\lambda - 1) \sum_{(i,j)\in T_z} (f_i^{\mu,R} + f_j^{\mu,R})
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{3A}{4(1+\epsilon)^\lambda}\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n^2_{\Delta_k} - M\sum_{(i,j)\notin T_z}(f_i^{\mu,R}+f_j^{\mu,R})\\
&- \frac{M}{2}((1+\epsilon)^\lambda-1)\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R}), \qquad \text{(A.19)}
\end{aligned}
$$

If we sum over z, the term in the left hand side is clearly independent of z. On the contrary, given a pair of particles (i,j), the number of z such that $(i,j)\in T_z$ is larger than $(m-2)^3$, thus the number of pairs of particles with a relative distance smaller than r, but such that they do not belong to T_z, is less than $m^3-(m-2)^3\leq 14m^2$.
In this way we obtain

$$
\begin{aligned}
m^3 W^{(1)}(X;\mu,R) \geq\ & \frac{3A\, m^3}{4(1+\epsilon)^\lambda}\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n^2_{\Delta_k}\\
&- 14M\, m^2\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R})\\
&- \frac{M\, m^3}{2}((1+\epsilon)^\lambda-1)\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R}). \qquad \text{(A.20)}
\end{aligned}
$$

Let us estimate now the last two terms of the sum:

$$
\begin{aligned}
\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R}) &\leq \sum_{i<j}(f_i^{\mu,R}+f_j^{\mu,R})\chi(|q_i-q_j|<r)\\
&\leq \widetilde{D}_8\sum_{i\in\mathbb{Z}^3}\sum_{\substack{l\in\mathbb{N}:\\ q_l\in\Delta_i}}\sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|<r+2}}\sum_{g\in\mathbb{N}:q_g\in\Delta_j} f(|i-\mu|,R)\\
&= \widetilde{D}_8\sum_{i\in\mathbb{Z}^3}\sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|<r+2}} f(|i-\mu|,R)\, n_{\Delta_i}n_{\Delta_j}.
\end{aligned}
$$

Obviously for a fixed j

$$
\textbf{Card}\{i\in\mathbb{Z}^3 : |i-j|<r+2\}\leq \widetilde{D}_9\, r^3,
$$

then, for r large enough:

$$
\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R})\leq \widetilde{D}_{10}\, r^{3+\lambda}\sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\, n^2_{\Delta_i}. \qquad \text{(A.21)}
$$

In conclusion the term $W^{(1)}$ is bounded by

$$W^{(1)}(X;\mu,R) \geq D(\epsilon,m,r) \sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\, n^2_{\Delta_i}, \tag{A.22}$$

where

$$D(\epsilon,m,r) \equiv \left(\frac{3A}{4(1+\epsilon)^\lambda} - \frac{14\widetilde{D}_{10}M}{m} r^{3+\lambda} - \frac{M}{2}\widetilde{D}_{10} r^{3+\lambda}((1+\epsilon)^\lambda - 1) \right). \tag{A.23}$$

Let r be such that $r > \max\{\bar{r}, r_1\}$, and m such that:

$$m \geq \bar{m} \equiv \frac{112\widetilde{D}_{10}Mr^{3+\lambda}}{A}, \tag{A.24}$$

and let ϵ satisfy the following bound:

$$\epsilon \leq \min\left\{ (3/2)^{\frac{1}{\lambda}} - 1, (\frac{A}{4M\widetilde{D}_{10}r^{3+\lambda}} + 1)^{\frac{1}{\lambda}} - 1 \right\}, \tag{A.25}$$

so that we have

$$D(\epsilon,m,r) \geq \frac{1}{4}A. \tag{A.26}$$

Finally we fix α in such a way that $\alpha m r\sqrt{3} < \epsilon$, so the thesis immediately follows with $C_3 = 1/4A$.
Summing up, first we choose r so large that the tail term W_2 is small enough. Then, for a fixed r, m can be chosen in such a way that (A.24) holds, and ϵ small enough to satisfy (A.25). Finally, as we have fixed r, m, ϵ, from (A.7) the bound on α follows. ■

Appendix B

Proof of Corollary 2.1

The only part which remains to prove is:

$$\sum_{i\in\mathbb{N}} f_i^{\mu,R}\,(\frac{1}{2}\sum_{\substack{j\in\mathbb{N}:\\ j\neq i}}\phi_{i,j}+b)\le C_4\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n_{\Delta_k}^2. \tag{B.1}$$

Let us define

$$\sum_{i\in\mathbb{N}} f_i^{\mu,R}\,(\frac{1}{2}\sum_{\substack{j\in\mathbb{N}:\\ j\neq i}}\phi_{i,j}+b)\equiv W^{(a)}+W^{(b)}, \tag{B.2}$$

where

$$W^{(a)}\equiv\sum_{i\in\mathbb{N}} f_i^{\mu,R}\,(\frac{1}{2}\sum_{\substack{j\in\mathbb{N}:\\ j\neq i}}\phi_{i,j}^{(1)}+b), \tag{B.3}$$

$$W^{(b)}\equiv\frac{1}{2}\sum_{i\in\mathbb{N}} f_i^{\mu,R}\sum_{\substack{j\in\mathbb{N}:\\ j\neq i}}\phi_{i,j}^{(2)}. \tag{B.4}$$

Using the third property of Proposition 2.2, the first term can be easily bounded by

$$\begin{aligned} W^{(a)} &\le (1+\alpha\sqrt{3})^{\lambda}\,\widetilde{E}_1\sum_{l\in\mathbb{Z}^2} f(|l-\mu|,R)\,n_{\Delta_l} \\ &\quad+\frac{\|\phi^{(1)}\|_\infty}{2}\sum_{i\neq j} f_i^{\mu,R}\,\chi(|q_i-q_j|\le r). \end{aligned} \tag{B.5}$$

Thus

$$W^{(a)} \leq \widetilde{E}_2 \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l} + \widetilde{E}_3 \sum_{i \neq j} f_i^{\mu,R} \chi(|q_j - q_j| \leq r). \tag{B.6}$$

Let us give an upper bound for the second term that we denote with $\widetilde{W}$:

$$\begin{aligned}
\widetilde{W} \equiv & \sum_{i \neq j} f_i^{\mu,R} \chi(|q_j - q_j| \leq r) \\
\leq & \sum_{l,m \in \mathbb{Z}^3} \sum_{i \neq j} \chi_i(\Delta_l)\chi_j(\Delta_m)(1+\alpha\sqrt{3})^{\lambda} f(|l-\mu|, R)\, \chi(|l-m| \leq r+\sqrt{3}) \\
\leq & \ \widetilde{E}_4 \sum_{l \in \mathbb{Z}^3} \sum_{m \in \mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l} n_{\Delta_m} \chi(|l-m| \leq r+\sqrt{3}) \\
\leq & \ \frac{\widetilde{E}_4}{2} \sum_{l \in \mathbb{Z}^3} \sum_{m \in \mathbb{Z}^3} f(|l-\mu|, R)\, (n^2_{\Delta_l} + n^2_{\Delta_m}) \chi(|l-m| \leq r+\sqrt{3}) \\
\leq & \ \frac{\widetilde{E}_4}{2} \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l} \sum_{m \in \mathbb{Z}^3} \chi(|l-m| \leq r+\sqrt{3}) \\
+ & \frac{\widetilde{E}_4}{2} \sum_{m \in \mathbb{Z}^3} f(|m-\mu|, R)\, n^2_{\Delta_m} \sum_{l \in \mathbb{Z}^3} (1+\alpha(r+\sqrt{3}))^{\lambda} \chi(|l-m| \leq r+\sqrt{3}) \\
\leq & \ \widetilde{E}_5\, r^{3+\lambda} \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l} \leq \widetilde{E}_6 \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l},
\end{aligned}$$

where we denote with $\chi_i(\Delta_l)$ the characteristic function of the set $\{i \in \mathbb{N} : q_i \in \Delta_l\}$ and with n_{Δ_l} the number of particles in the unit cube Δ_l with its center in l. Moreover we have used the fact that, for a fixed l, $\sum_{m \in \mathbb{Z}^3} \chi(|l-m| \leq r+\sqrt{3})$ is bounded by the cardinality of the set $\mathbb{Z}^3 \cap B(0, r+\sqrt{3})$.
Thus for $W^{(a)}$ we have:

$$W^{(a)} \leq \widetilde{E}_7 \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l}. \tag{B.7}$$

Let us give a similar estimate for $W^{(b)}$.
From the forth property of (2.17) we have:

$$
\begin{aligned}
W^{(b)} &\leq \widetilde{E}_8 \sum_{i\in\mathbb{N}} f_i^{\mu,R} \sum_{j\in\mathbb{N}} \chi(|q_j - q_j| \geq r) \frac{1}{|q_i - q_j|^\gamma} \\
&= \widetilde{E}_8 \sum_{k=1}^{\infty} \sum_{i,j} f_i^{\mu,R} \chi(kr \leq |q_i - q_j| < (k+1)r) \frac{1}{|q_i - q_j|^\gamma},
\end{aligned}
$$

thus

$$
\begin{aligned}
W^{(b)} &\leq \widetilde{E}_8 \sum_{k=1}^{\infty} \frac{1}{(kr)^\gamma} \sum_{\substack{l,m\in \\ \mathbb{Z}^3}} \sum_{\substack{i,j\in \\ \mathbb{N}}} \chi_i(\Delta_l)\chi_j(\Delta_m) \\
&\quad \times (1+\alpha\sqrt{3})^\lambda f(|l-\mu|, R)\ \chi(kr - \sqrt{3} \leq |l-m| \leq (k+1)r + \sqrt{3}) \\
&\leq \widetilde{E}_9 \sum_{k=1}^{\infty} \frac{1}{(kr)^\gamma} \\
&\quad \times \sum_{\substack{l,m\in \\ \mathbb{Z}^3}} f(|l-\mu|, R)\, n_{\Delta_l} n_{\Delta_m} \chi(kr - \sqrt{3} \leq |l-m| \leq (k+1)r + \sqrt{3}) \\
&\leq \frac{\widetilde{E}_9}{2} \sum_{k=1}^{\infty} \frac{1}{(kr)^\gamma} \sum_{\substack{l,m\in \\ \mathbb{Z}^3}} \Big(f(|l-\mu|, R) n_{\Delta_l}^2 \\
&\quad + f(|m-\mu|, R)(1+\alpha|l-m|)^\lambda n_{\Delta_m}^2 \Big) \\
&\quad \times \chi(kr - \sqrt{3} \leq |l-m| \leq (k+1)r + \sqrt{3}) \\
&\leq \widetilde{E}_{10} \sum_{k=1}^{\infty} \frac{k^2 r^3}{(kr)^\gamma} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 \\
&\quad + \widetilde{E}_{11} \sum_{k=1}^{\infty} \frac{k^2 (1 + \alpha((k+1)r + \sqrt{3}))^\lambda}{(kr)^\gamma} \sum_{m\in\mathbb{Z}^3} f(|m-\mu|, R)\, n_{\Delta_m}^2 \\
&\leq \widetilde{E}_{12} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2
\end{aligned}
\tag{B.8}
$$

where in the last inequality the convergence of the series follows from the bound on γ. Thus we have:

$$
\sum_{i\in\mathbb{N}} f_i^{\mu,R} \Big(\frac{1}{2} \sum_{\substack{j\in\mathbb{N}: \\ j\neq i}} \phi_{i,j} + b\Big) \leq C_4 \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2,
$$

and then the proof easily follows. ∎

Appendix C

Proof of Corollary 2.2

For the first inequality we prove a stronger bound: there exists a positive constant $\widetilde{L}$ such that:

$$Q(X;\mu,R) \leq \widetilde{L}\, W(X;\mu,R). \tag{C.1}$$

From definition (2.6) we can write:

$$\begin{aligned}
Q(X;\mu,R) &= \sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leq R)\frac{v_i^2}{2} \\
&+ \sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leq R)\Big(b+\frac{1}{2}\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}}\phi^{(1)}_{i,j}\Big) \\
&+ \frac{1}{2}\sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leq R)\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}}\phi^{(2)}_{i,j} \\
&\equiv T+U^{(1)}+U^{(2)},
\end{aligned} \tag{C.2}$$

where

$$\begin{aligned}
T\ &\equiv \sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leq R)\frac{v_i^2}{2} \\
U^{(1)} &\equiv \sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leq R)\Big(b+\frac{1}{2}\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}}\phi^{(1)}_{i,j}\Big)
\end{aligned}$$

$$U^{(2)} \equiv \frac{1}{2} \sum_{i \in \mathbb{N}} \chi(|q_i - \mu| \le R) \sum_{\substack{j \neq i: \\ q_j \in B(\mu,R)}} \phi^{(2)}_{i,j}.$$

Because of the boundness of the weight-function $f_i^{\mu,R}$ and from the positivity of the interaction energy (2.29), we have

$$T \le \widetilde{L}_1 \sum_{i \in \mathbb{N}} f_i^{\mu,R} \frac{v_i^2}{2} \le \widetilde{L}_1 W(X;\mu,R), \tag{C.3}$$

The second term can be easily bounded by

$$\begin{aligned} U^{(1)} &\le \widetilde{L}_1 \sum_{i \in \mathbb{N}} f_i^{\mu,R} b + \frac{1}{2} \sum_{i \neq j} \chi(|q_i - \mu| \le R) \chi(|q_j - \mu| \le R) \; |\phi^{(1)}_{i,j}| \\ &\le (1 + \alpha\sqrt{3})^{\lambda} \, \widetilde{L}_1 \sum_{l \in \mathbb{Z}^3} f(|l - \mu|, R) \, n_{\Delta_l} \\ &+ \frac{\|\phi^{(1)}\|_\infty}{2} \sum_{i \neq j} \chi(|q_i - \mu| \le R) \chi(|q_i - q_j| \le r), \end{aligned} \tag{C.4}$$

where for the first addendum we have used the third property of Proposition 2.2.
Thus

$$U^{(1)} \le \widetilde{L}_2 \sum_{l \in \mathbb{Z}^3} f(|l - \mu|, R) \, n^2_{\Delta_l} + \widetilde{L}_3 \sum_{i \neq j} f_i^{\mu,R} \chi(|q_j - q_j| \le r). \tag{C.5}$$

Let us give an upper bound for the second term that we denote with $\widetilde{U}$:

$$\begin{aligned} \widetilde{U} &= \sum_{i \neq j} f_i^{\mu,R} \chi(|q_j - q_j| \le r) \\ &\le \sum_{l,m \in \mathbb{Z}^3} \sum_{i \neq j} \chi_i(\Delta_l) \chi_j(\Delta_m) (1 + \alpha\sqrt{3})^{\lambda} f(|l - \mu|, R) \, \chi(|l - m| \le r + \sqrt{3}) \\ &\le \widetilde{L}_4 \sum_{l \in \mathbb{Z}^3} \sum_{m \in \mathbb{Z}^3} f(|l - \mu|, R) \, n_{\Delta_l} n_{\Delta_m} \chi(|l - m| \le r + \sqrt{3}) \\ &\le \frac{\widetilde{L}_4}{2} \sum_{l \in \mathbb{Z}^3} \sum_{m \in \mathbb{Z}^3} f(|l - \mu|, R) \, (n^2_{\Delta_l} + n^2_{\Delta_m}) \chi(|l - m| \le r + \sqrt{3}) \end{aligned}$$

$$\leq \frac{\widetilde{L}_4}{2} \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 \sum_{m \in \mathbb{Z}^3} \chi(|l-m| \leq r+\sqrt{3})$$
$$+ \frac{\widetilde{L}_4}{2} \sum_{m \in \mathbb{Z}^3} f(|m-\mu|, R)\, n_{\Delta_m}^2 \sum_{l \in \mathbb{Z}^3} (1+\alpha(r+\sqrt{3}))^\lambda \chi(|l-m| \leq r+\sqrt{3})$$
$$\leq \widetilde{L}_5\, r^{3+\lambda} \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 \leq \widetilde{L}_6 \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2,$$

where we denote with $\chi_i(\Delta_l)$ the characteristic function of the set $\{i \in \mathbb{N} : q_i \in \Delta_l\}$ and with n_{Δ_l} the number of particles in the unit cube Δ_l with its center in l. Moreover we have used the fact that, for a fixed l, $\sum_{m \in \mathbb{Z}^3} \chi(|l-m| \leq r+\sqrt{3})$ is bounded by the cardinality of the set $\mathbb{Z}^3 \cap B(0, r+\sqrt{3})$.
Thus for $U^{(1)}$ we have:

$$U^{(1)} \leq \widetilde{L}_7 \sum_{l \in \mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2. \tag{C.6}$$

Let us give a similar estimate for $U^{(2)}$.
From the forth property of (2.17) we have:

$$U^{(2)} \leq G_1 \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \chi(|q_j-\mu| \leq R) \chi(|q_j-q_j| \geq r) \frac{1}{|q_i-q_j|^\gamma}$$
$$\leq \widetilde{L}_8 \sum_{i \in \mathbb{N}} f_i^{\mu,R} \sum_{j \in \mathbb{N}} \chi(|q_j-q_j| \geq r) \frac{1}{|q_i-q_j|^\gamma}$$
$$\leq \widetilde{L}_8 \sum_{k=1}^{k_{max}} \sum_{i,j} f_i^{\mu,R} \chi(kr \leq |q_i-q_j| \leq (k+1)r) \frac{1}{|q_i-q_j|^\gamma},$$

where $k_{max} = [4/3\, \pi\, R^3/r] + 1$, then:

$$U^{(2)} \leq \widetilde{L}_8 \sum_{k=1}^{k_{max}} \frac{1}{(kr)^\gamma} \sum_{\substack{l,m \in \\ \mathbb{Z}^3}} \sum_{\substack{i,j \in \\ \mathbb{N}}} \chi_i(\Delta_l) \chi_j(\Delta_m)$$
$$\times\ (1+\alpha\sqrt{3})^\lambda f(|l-\mu|, R)\ \chi(kr-\sqrt{3} \leq |l-m| \leq (k+1)r+\sqrt{3})$$
$$\leq \widetilde{L}_9 \sum_{k=1}^{k_{max}} \frac{1}{(kr)^\gamma}$$

$$\times \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} f(|l-\mu|,R)\, n_{\Delta_l} n_{\Delta_m} \chi(kr-\sqrt{3} \le |l-m| \le (k+1)r+\sqrt{3})$$

$$\le \frac{\widetilde{L}_9}{2}\sum_{k=1}^{k_{max}} \frac{1}{(kr)^\gamma} \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} (f(|l-\mu|,R) n^2_{\Delta_l} + f(|m-\mu|,R)(1+\alpha|l-m|)^\lambda n^2_{\Delta_m})$$

$$\times\, \chi(kr-\sqrt{3} \le |l-m| \le (k+1)r+\sqrt{3})$$

$$\le \widetilde{L}_{10} \sum_{k=1}^{\infty} \frac{k^2 r^3}{(kr)^\gamma} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\, n^2_{\Delta_l}$$

$$+ \widetilde{L}_{11} \sum_{k=1}^{\infty} \frac{k^2(1+\alpha((k+1)r+\sqrt{3}))^\lambda}{(kr)^\gamma} \sum_{m\in\mathbb{Z}^3} f(|m-\mu|,R)\, n^2_{\Delta_m}$$

$$\le \widetilde{L}_{12} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\, n^2_{\Delta_l} \tag{C.7}$$

where in the last inequality the convergence of the series follows from the bound on γ. Thus, from Lemma 2.1 we have:

$$Q(X;\mu,R) \le T + U^{(1)} + U^{(2)} \le \widetilde{L}_{13} W(X;\mu,R),$$

and then the proof of the first inequality of the Corollary follows.
Let us consider the second one.
From the definition of $Q(X;\mu,R)$ and from the superstability of the potential we have:

$$Q(X;\mu,R) \ge A \sum_{\substack{k\in\mathbb{Z}^3:\\ |k-\mu|<R}} n^2_{\Delta_k} \ge \widetilde{L}_{14} \sum_{\substack{k\in\mathbb{Z}^3:\\ |k-\mu|<R}} f(|k-\mu|,R)\, n^2_{\Delta_k}, \tag{C.8}$$

and from Corollary 2.1:

$$W(X;\mu,R) \le C_4 \sum_{i\in\mathbb{Z}^3:} f(|i-\mu|,R)\, n^2_{\Delta_i} + \sum_{i\in\mathbb{N}} f_i^{\mu,R} \frac{v_i^2}{2}$$

$$\le C_4 \sum_{k\ge 0} \sum_{\substack{i\in\mathbb{Z}^3:\\ i\in B(\mu,(k+1)R)\setminus B(\mu,kR)}} f(|i-\mu|,R)\, n^2_{\Delta_i}$$

$$+ \sum_{k\ge 0}\sum_{i\in\mathbb{N}} \chi(kR \le |q_i-\mu| < (k+1)R)\; f_i^{\mu,R} \frac{v_i^2}{2}$$

$$
\begin{aligned}
&\leq \widetilde{L}_{15} \sum_{k\geq 1} \frac{1}{k^{\lambda}} \sum_{\substack{i\in\mathbb{Z}^3:\\ i\in B(\mu,(k+1)R)}} n_{\Delta_i}^2 \\
&+ \widetilde{L}_{16} \sum_{k\geq 0} \frac{1}{(1+k)^{\lambda}}\, Q(X;\mu,(k+1)R) \\
&\leq \widetilde{L}_{17}\, R^3 \sum_{k\geq 1} \frac{1}{k^{\lambda-3}} \frac{Q(X;\mu,(k+1)R)}{((k+1)R)^3}.
\end{aligned}
$$

Dividing by R^3

$$
\frac{W(X;\mu,R)}{R^3} \leq \widetilde{L}_{17} \sum_{k\geq 1} \frac{1}{k^{\lambda-3}} \frac{Q(X;\mu,(k+1)R)}{((k+1)R)^3},
$$

from which, taking the supremum over $\mu \in \mathbb{R}^3$ and over $R > \psi_\xi(|\mu|)$:

$$
\sup_{\mu} \sup_{R>\psi_\xi(|\mu|)} \frac{W(X;\mu,R)}{R^3} \leq \widetilde{L}_{18}\, Q_\xi \sum_{k\geq 1} \frac{1}{k^{\lambda-3}} \leq \widetilde{L}_{19}\, Q_\xi, \tag{C.9}
$$

being $\lambda > 4$. ■

Appendix D

Proof of Lemma 2.3

i) Since

$$\frac{1}{(1+\alpha\,\frac{|y|}{nR})^{\lambda}} \leq \frac{n^{\lambda}}{(1+\alpha\,\frac{|y|}{R})^{\lambda}}$$

then, from the first two properties of Proposition 2.2, $\exists\,\widetilde{L}_{20} > 0$ such that

$$f_i^{\mu,nR} \leq \widetilde{L}_{20}\, n^{\lambda}\, f_i^{\mu,R}.$$

By Corollary 2.1 it follows that

$$\begin{aligned} W(X;\mu,nR) &\leq \sum_{i\in\mathbb{N}} f_i^{\mu,nR}\,\frac{v_i^2}{2} + C_4 \sum_{k\in\mathbb{Z}^3} f(|k-\mu|,nR)\, n_{\Delta_k}^2 \\ &\leq \widetilde{L}_{20}\, n^{\lambda}\, W(X;\mu,R) \\ &+ C_4\,\widetilde{L}_{20} n^{\lambda} \sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 \leq C_8\, n^{\lambda}\, W(X;\mu,R). \end{aligned}$$

ii) From the definition of the weight-function we have $f(x,R_1) < f(x,R_2)$, if $R_1 < R_2$. Using again Corollary 2.1 we get

$$\begin{aligned} W(X;\mu,R) &\leq C_4 \sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\, n_{\Delta_k}^2 + \sum_{i\in\mathbb{N}} f_i^{\mu,R}\frac{v_i^2}{2} \\ &\leq C_4 \sum_{k\in\mathbb{Z}^3} f(|k-\mu|,nR)\, n_{\Delta_k}^2 + W(X;\mu,nR) \\ &\leq C_9\, W(X;\mu,nR). \end{aligned}$$

iii) We use the superstability of the interaction and the bound (C.1):

$$\begin{aligned} W(X;\mu,R) &\geq \frac{1}{\widetilde{L}}\,Q(X;\mu,R) \\ &\geq \frac{1}{2\widetilde{L}}\sum_{i,j}\chi(|q_i-\mu|\leq R)\,\chi(|q_j-\mu|\leq R)\,\phi_{i,j} \\ &\geq \frac{\widetilde{L}_{21}}{R^3}\,N^2(X,\mu,R) - B\,\frac{1}{2\widetilde{L}}\,N(X,\mu,R). \end{aligned}$$

Since the interaction energy is positive:

$$N^2(X,\mu,R) \leq \widetilde{L}_{22}\,R^3\,\left(\,N(X,\mu,R)+W(X;\mu,R)\,\right) \leq \widetilde{L}_{23}\,R^3\,W(X;\mu,R).$$

iiii) Let us cover the ball $B(\mu,R)$ by a collection of disjoint cubes $\{\Delta_\alpha\}_{\alpha\in\mathbb{Z}^3}$ of side one. Therefore

$$\sum_{i\neq j}\chi(|q_i-q_j|<\rho)\chi(|q_i-\mu|<R)\chi(|q_j-\mu|<R) \leq \sum_{(\alpha,\beta)} n_{\Delta_\alpha}\,n_{\Delta_\beta} + \sum_{\alpha} n^2_{\Delta_\alpha}, \tag{D.1}$$

where (α,β) means the sum restricted to all pairs of different cubes at distance not larger than ρ. Thus we have the bound:

$$\begin{aligned} &\sum_{i\neq j}\chi(|q_i-q_j|<\rho)\chi(|q_i-\mu|<R)\chi(|q_j-\mu|<R) \\ &\leq \sum_{\alpha} n^2_{\Delta_\alpha} + \frac{1}{2}\sum_{(\alpha,\beta)}\left(n^2_{\Delta_\alpha}+n^2_{\Delta_\beta}\right) \leq \widetilde{L}_{24}\,\rho^3\sum_{\alpha} n^2_{\Delta_\alpha} \\ &\leq \widetilde{L}_{25}\,\rho^3\sum_{i\in\mathbb{Z}^3} f(|i-\mu|,R)\,n^2_{\Delta_i} \leq \widetilde{L}_{26}\,\rho^3\,W(X;\mu,R). \quad \blacksquare \end{aligned} \tag{D.2}$$

Appendix E

Derivation of the viscous friction force

We give a derivation of the equation of motion (3.6)-(3.7) in the one-dimensional case, the d-dimensional case following by using theorems on multiple integrals. We will obtain the equation of motion from the time derivative of the total momentum of the system gas+disk, which is conserved along the motion:

$$\frac{d}{dt}\int_{\mathbb{R}} dx \int_{\mathbb{R}} dv\, v f(x,v;t) + \dot{V}(t) = 0. \tag{E.1}$$

Let us denote by

$$f^L(X(t),v;t) = \lim_{x\to X(t)^-} f(x,v;t), \tag{E.2}$$

$$f^R(X(t),v;t) = \lim_{x\to X(t)^+} f(x,v;t). \tag{E.3}$$

Let us calculate the first term in (E.1). Using (3.1) and the fact that

$$\lim_{x\to-\infty} f(x,v;t) = \lim_{x\to+\infty} f(x,v;t) = \rho\left(\frac{\beta}{\pi}\right)^{1/2} e^{-\beta v^2}, \tag{E.4}$$

we have:

$$\begin{aligned}
&\frac{d}{dt}\left[\int_{-\infty}^{X(t)} dx \int_{\mathbb{R}} dv\, v f(x,v;t) + \int_{X(t)}^{+\infty} dx \int_{\mathbb{R}} dv\, v f(x,v;t)\right] \\
&= \int_{\mathbb{R}} dv\, v f^L(X(t),v;t) V(t) - \int_{\mathbb{R}} dv\, v f^R(X(t),v;t) V(t)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}} dv \int_{-\infty}^{X(t)} dx\, v(-v\partial_x f(x,v;t)) + \int_{\mathbb{R}} dv \int_{X(t)}^{+\infty} dx\, v(-v\partial_x f(x,v;t)) \\
&= \int_{\mathbb{R}} dv\, vV(t) f^L(X(t),v;t) - \int_{\mathbb{R}} dv\, vV(t) f^R(X(t),v;t) \\
&- \int_{\mathbb{R}} dv\, v^2 f^L(X(t),v;t) + \int_{\mathbb{R}} dv\, v^2 f^R(X(t),v;t) \\
&= \int_{\mathbb{R}} dv\, v(V(t)-v) f^L(X(t),v;t) - \int_{\mathbb{R}} dv\, v(V(t)-v) f^R(X(t),v;t).
\end{aligned} \tag{E.5}
$$

We consider first the integral involving f^L, taking into account the fact that

$$
\begin{aligned}
f^L(X(t),v;t) = \;& f_-^L(X(t),v;t)\,\chi(\{v \geq V(t)\}) \\
&+ f_+^L(X(t),v;t)\,\chi(\{v < V(t)\}),
\end{aligned} \tag{E.6}
$$

with the definition of $f_\pm$ given in (3.5), since for $v \geq V(t)$ the velocity v is necessarily a pre-collisional velocity (we are on the left side of the obstacle), while for $v < V(t)$ the velocity v is a post-collisional velocity.

Hence

$$
\begin{aligned}
&\int_{\mathbb{R}} dv\, v(V(t)-v) f^L(X(t),v;t) \\
&= \int_{-\infty}^{V(t)} dv'\, v'(V(t)-v') f_+^L(X(t),v';t) + \int_{V(t)}^{+\infty} dv\, v(V(t)-v) f_-^L(X(t),v;t).
\end{aligned} \tag{E.7}
$$

Performing the change of variable $v' = 2V(t) - v$ in the first integral in the r.h.s. of (E.7), we have

$$
\begin{aligned}
(E.7) = &- \int_{+\infty}^{V(t)} dv\, (2V(t)-v)(-V(t)+v) f_+^L(X(t), 2V(t)-v;t) \\
&+ \int_{V(t)}^{+\infty} dv\, v(V(t)-v) f_-^L(X(t),v;t),
\end{aligned} \tag{E.8}
$$

and for the continuity of f^L along the collisions, by (3.4) it is

$$
f_+^L(X(t), 2V(t)-v;t) = f_-^L(X(t),v;t), \tag{E.9}
$$

therefore

$$
\begin{aligned}
(\text{E.8}) &= \int_{V(t)}^{+\infty} dv\, (V(t)-v)(v-2V(t)+v) f_-^L(X(t),v;t) \\
&= -2\int_{V(t)}^{+\infty} dv\, (V(t)-v)^2 f_-^L(X(t),v;t).
\end{aligned} \tag{E.10}
$$

In the same way it can be handled the integral with f^R in (E.5), arriving at:

$$
\begin{aligned}
(\text{E.5}) = 2&\int_{-\infty}^{V(t)} dv\, (V(t)-v)^2 f_-^R(X(t),v;t) \\
&-2\int_{V(t)}^{+\infty} dv\, (V(t)-v)^2 f_-^L(X(t),v;t),
\end{aligned} \tag{E.11}
$$

which is the friction term (3.7) in the one-dimensional case.

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