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A Model Hamiltonian for Condensed Matter Physics

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Introduction

In this thesis we give a rigorous construction of a (Rayleigh) model for the dynamics of a quantum mechanical test particle interacting with condensed matter via point interactions.
From the very beginning point interactions have been widely used in solid state physics: the model of Kronig and Penney, see [1], described the motion of an electron in a one dimensional crystal; the crystal was represented by a periodic array of point interactions.
In this work we have considered a different situation: condensed matter have been represented by N quantum harmonic oscillators with the same mass and proper frequency. It's clear that this model is much more difficult since we are dealing with a many-body problem.
At a formal level, the model was introduced in physics by Fermi in his seminal paper [F] for the analysis of the scattering of slow neutrons from the nuclei of a target of condensed matter. Since the interaction between the neutron and the nuclei of the target is strong and very short range with respect to the wavelength of the neutron, Fermi introduced a formal hamiltonian where the interaction is modelized by the singular delta-potential, also called point interaction. Then he analyzed the three cases of fixed, free and harmonically bounded nuclei, computing in each case the scattering cross section in the Born approximation.
Later, these kind of models have been extensively used in nuclear physics to describe the low-energy dynamics of neutrons and then to investigate the structure of crystals and in general of condensed matter ([LA],[D],[LO], [DO]).
Indeed point interactions provide non trivial models indexed by a minimal set of parameters: the strengths and the positions of the interaction centers. It is then natural, in these cases, to find very simple representations for studied objects, such as the resolvent and the unitary group.
This remains true even for more complicate models, that is if one considers time dependent parameters or non linear forces, see [Fi] for a survey on these generalizations.
It should be noted that in all the applications one can find in the physical literature the computation of the relevant quantities, are only given in the Born approximation in dimension two and three. This is due to the fact that the second order term diverges if one formally applies perturbation theory to these cases.
From the mathematical point of view this means that point interactions cannot be considered as small perturbation of the free laplacian, even in the sense of quadratic forms.
Then it naturally arises the problem to give (if possible) a rigorous meaning to the formal

hamiltonian as a self adjoint and bounded below operator in the appropriate Hilbert space. In particular this would open the possibility to have a well defined perturbation theory and then to compute the relevant quantities at any order of approximation.
This program started in the sixties when Berezin and Faddeev gave the first rigorous definition ([BF]) for the hamiltonian of a particle undergoing the influence of a single delta interaction and it has been successfully completed (see e.g. [AGH-KH]) for the case of one particle subject to point interactions placed at fixed positions.
The case of a system of N particles interacting via a two-body local point interaction is more difficult. In the case of three particles it was shown in ([MF1],[MF2]) that the most natural self adjoint realization of the hamiltonian is in fact unbounded from below and then physically unreasonable.
Surprisely enough, the same collapse phenomenon occurs in the case of one particle interacting via a point interaction with a system of N non interacting particles, if N is sufficiently large ([DFT], [Mi]).
This means that a satisfactory model for the dynamics of a neutron interacting with free nuclei is still lacking.
Here we approach the last case treated in [F], i.e. the case of a test particle (neutron) interacting via point interaction with a system of N particles (nuclei) harmonically bounded around their equilibrium positions. The corresponding formal hamiltonian can be written as

$$\begin{aligned} H_\gamma &= -\frac{\hbar^2}{2M}\Delta_x + \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\Delta_{y_i} + \frac{1}{2}m\omega^2(y_i - y_i^0)^2\right) + \alpha\sum_{i=1}^{N}\delta(x - y_i) \\ &\equiv H_0 + \alpha\sum_{i=1}^{N}\delta(x - y_i) \end{aligned} \tag{1}$$

In (1) we have denoted by x the coordinate of the test particle and by M its mass, by y_i the coordinates of the oscillators, by m their mass, by ω their frequency and by y_i^0 their equilibrium positions.
The hamiltonian H_0 refers to the free system, i.e. the system of the test particle and the oscillators without point interactions.
The parameter $\alpha \in \mathbb{R}$ plays the role of the strength of the point interaction but in (1) it should be understood only at a formal level since, as we have pointed out, point interactions cannot be treated as an additive interaction.
In chapter one we have studied hamiltonian (1) in dimension one using the theory of quadratic forms and we have constructed integral representations for the resolvent and for the unitary group. Since we are dealing with point interactions with codimension one, the definition of the hamiltonian is straightforward.
In chapter two we have studied hamiltonian (1) in dimension three; the model is much more singular due to the higher codimension of the support of the interaction and a renormalization

of the quadratic form is required in order to define a meaningful quadratic form. The operator and a representation for the resolvent have been constructed when $N = 1$.
In chapter three we have studied hamiltonian (1) in dimension two; a renormalization is still required but the model is less singular than the three dimensional case, indeed the hamiltonian (1) has been constructed for every N and a representation for the resolvent has been given.
In chapter four we have considered an application of this model. We have considered in one dimension, a light test particle interacting with one heavy harmonic oscillator placed at the origin and we have derived an asymptotic formula for the evolution in the limit of small mass ratio.
In such a scaling of physical parameters, an adiabatic decoupling of the degrees of freedom of the test particle and of the harmonic oscillator is expected. We have considered a superposition of two spatially separated wave packets as the initial state for the heavy particle and we have studied how the interaction with the light particle produces (partial) decoherence on the heavy one.
Decoherence induced by interaction with the environment is a key argument for understanding the classical behaviour of macro objects, see [KJ] and ref.cit. In our simple model the light particle represents the environment, while the heavy one is the "macroscopic" system. It worths noticing that in this model decoherence emerges as purely quantum effect, due to a scattering process and that no assumptions on the environment are required. This model should be the first step tosards a more realistic model where the environment is represented by N light particles.
A similar model has been considered in [DuFiT], where the "macroscopic" system has been represented by a similar state of a free particle. Notice that in [DuFiT] the decohence process can happen only once because of the free motion of the unpertubed system; in our model the periodic motion of the system can lead, in principle, to the iteration of the decoherence phenomenon.

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Contents

Chapter 1

The One Dimensional Case

In this chapter we give a rigorous definition of the hamiltonian (1) as a self-adjoint and semibounded operator in dimension one and we give representations for the resolvent and for the unitary group.
By means of the theory of quadratic forms, the construction of the operator is quite straightforward: by Sobolev trace theorems we are concerned with an infinitesimal perturbation in Kato sense of the free quadratic form. Using standard perturbative arguments we prove closure and semiboundedness of the quadratic form in theorem (1.1.1).
Next, in theorems (1.2.1) and (1.2.2), we construct the operator and the resolvent, proving the decomposition of the elements of the domain in a regular and a singular part which must fulfill a suitable boundary condition, see (1.15).
In theorem (1.3.1) we give an integral representation of the unitary group by means of a system of N coupled integral equations.
It is easy to see that if one puts $\alpha = 0$ in (1.2.1), (1.2.2), (1.3.1) one recovers the unperturbed operator, resolvent and propagator: as already pointed out, even if the interaction is singular, i.e. it is supported by a set of zero measure, we could construct perturbative expansions in α which is the true coupling constant of the system. As we shall see in the next chapters, this is no longer true in two and three dimensional case where the role of α is different.

1.1 The Quadratic Form

In this section we introduce the quadratic form F_α associated to (1) in dimension one and we prove that it's closed and bounded from below.
Let's consider the quadratic form F_0 associated to H_0:

$$F_0[u] = \int dx dy_1 \dots dy_N \left\{ \frac{\hbar^2}{2M} |\partial_x u|^2 + \sum_{i=1}^{N} \left(\frac{\hbar^2}{2m} |\partial_{y_i} u|^2 + \frac{1}{2} m\omega^2 (y_i - y_i^0)^2 |u|^2 \right) \right\} \tag{1.1}$$

$$\mathscr{D}(F_0) = \{u(x, y_1, \ldots, y_N) \in L^2(\mathbb{R}^{N+1})\ s.t.\ F_0[u] < +\infty\}$$

It is well known that F_0 is a closed positive quadratic form and that:

$$\mathscr{D}(F_0) = \{u \in L^2(\mathbb{R}^{N+1}) s.t. u \in H^1(\mathbb{R}^{N+1}),\ y_i u \in L^2(\mathbb{R}^{N+1})\ i = 1 \ldots N\}$$

We shall often use y without under script to indicate $(y_1, \ldots, y_N)$, and we shall omit summations over the index i to make formulas less heavy. We shall deserve the square modulus only to complex quantities; y^2 stands for $y_1^2 + \ldots + y_N^2$. For instance, with these conventions, (1.1) is written as:

$$F_0[u] = \int dxdy \left\{ \frac{\hbar^2}{2M} |\partial_x u|^2 + \frac{\hbar^2}{2m} |\partial_y u|^2 + \frac{1}{2} m\omega^2 (y - y^0)^2 |u|^2 \right\} \tag{1.2}$$

We can now introduce the quadratic form F_α:

$$F_\alpha[u] = \int dxdy \left\{ \frac{\hbar^2}{2M} |\partial_x u|^2 + \frac{\hbar^2}{2m} |\partial_y u|^2 + \frac{1}{2} m\omega^2 (y - y^0)^2 |u|^2 \right\} + \alpha \sum_{i=1}^{N} \int dy\, |u(y_i, y_1, \ldots, y_N)|^2 \tag{1.3}$$

$$\equiv F_0[u] + \alpha F_{int}[u]$$

$$\mathscr{D}(F_\alpha) = \{u(x, y_1, \ldots, y_N) \in L^2(\mathbb{R}^{N+1}) s.t. F_\alpha[u] < +\infty\}$$

The main properties of F_α are proved in the following theorem:

Theorem 1.1.1 *The quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ is closed and bounded from below on $\mathscr{D}(F_\alpha) = \mathscr{D}(F_0)$.*

Proof
By KLMN theorem, see [RSII], it is sufficient to show that F_{int} is infinitesimal with respect to F_0 in Kato sense. By Sobolev trace theorems we have:

$$F_{int}[u] \leq c\|u\|^2_{H^{\frac{1}{2}}} \leq c\varepsilon \|u\|^2_{H^1} + c\frac{1}{\varepsilon}\|u\|^2_{L^2}$$

Since $\|u\|^2_{H^1} \leq cF_0[u]$ the theorem holds. □

1.2 The Operator and the Resolvent

In this section we construct the operator H_α corresponding to the quadratic formula of the resolvent $(H_\alpha + \lambda)^{-1}$.
Let's introduce some more notation. Let $G^\lambda(x, y; x', y')$ be the integral kernel of the Green's function of the free system; an integral representation for it have been derived in appendix (A).

For given functions $q = (q_1 \dots q_N)$, $q_i : \pi_i \to \mathbb{C}$ $i = 1 \dots N$, which we call charges, we shall consider the potential produced by q_i in $\mathbb{R}^{N+1}$

$$\left(G^\lambda q_i\right)(x,y) = \int_{\pi_i} dy' G^\lambda(x,y;y'_i,y') q_i(y')$$

and also its restriction on π_j, $i \neq j$

$$\left(G^\lambda_j q_i\right)(y) = \left(G^\lambda q_i\right)(y_j,y) = \int_{\pi_i} dy' G^\lambda(y_j,y;y'_i,y') q_i(y')$$

Now we can state the main theorem of this section.

Theorem 1.2.1 *The resolvent $(H_\alpha+\lambda)^{-1}$ has the following representation: there exists $\lambda_0 > 0$ s.t.for $\lambda > \lambda_0$ and for any $f \in L^2(\mathbb{R}^{N+1})$ there are N charges $(q_1,\dots,q_N)$ solutions of the following system*

$$q_i(y) + \alpha\left(\sum_{j=1}^N G^\lambda_i q_j(y) + G^\lambda f(y_i,y)\right) = 0 \qquad i = 1 \dots N \tag{1.4}$$

and the following representation holds

$$(H_\alpha + \lambda)^{-1} f = G^\lambda f + G^\lambda q \tag{1.5}$$

Proof
If $u = (H_\alpha+\lambda)^{-1} f$ and $v \in C_0^\infty(\mathbb{R}^{N+1})$ by the first representation theorem of quadratic forms, one has

$$F_\alpha[u,v] + \lambda(u,v) = (f,v)$$

It's natural to start from the ansatz

$$Rf = G^\lambda f + G^\lambda q$$

which is the typical structure of the resolvent when point interactions are involved, and to look for conditions on q in order to satisfy

$$F_\alpha[Rf,v] + \lambda(Rf,v) = (f,v) \tag{1.6}$$

Equation (1.6) can be explicitly written as

$$\int dxdy \left[\frac{\hbar^2}{2M}\overline{\partial_x G^\lambda f}\,\partial_x v + \frac{\hbar^2}{2m}\overline{\partial_y G^\lambda f}\,\partial_y v + \frac{1}{2}m\omega^2(y-y^0)^2\overline{G^\lambda f}\,v + \lambda\overline{G^\lambda f}\,v\right] +$$
$$+ \int dxdy \left[\frac{\hbar^2}{2M}\overline{\partial_x G^\lambda q}\,\partial_x v + \frac{\hbar^2}{2m}\overline{\partial_y G^\lambda q}\,\partial_y v + \frac{1}{2}m\omega^2(y-y^0)^2\overline{G^\lambda q}\,v + \lambda\overline{G^\lambda q}\,v\right] +$$
$$+ \alpha\sum_{i=1}^N \int dy \left(\overline{G^\lambda f(y_i,y) + G^\lambda q(y_i,y)}\right) v(y_i,y) = \int dxdy \overline{f} v \tag{1.7}$$

Since $G^\lambda f \in \mathscr{D}(H_0)$, we can integrate by parts the first piece of (1.7) and we obtain

$$\int dxdy \left[\frac{\hbar^2}{2M}\overline{\partial_x G^\lambda q}\,\partial_x v + \frac{\hbar^2}{2m}\overline{\partial_y G^\lambda q}\,\partial_y v + \frac{1}{2}m\omega^2(y-y^0)^2\overline{G^\lambda q}\,v + \lambda\overline{G^\lambda q}\,v\right] + \\ + \alpha\sum_{i=1}^{N}\int dy \left(\overline{G^\lambda f(y_i,y) + G^\lambda q(y_i,y)}\right) v(y_i,y) = 0$$

Using the regularity of v we can integrate by parts again obtaining:

$$\sum_{i=1}^{N}\int dy \left(\overline{q_i(y) + \alpha G^\lambda f(y_i,y) + \alpha G^\lambda q(y_i,y)}\right) v(y_i,y) = 0 \tag{1.8}$$

In order to satisfy (1.8) for any v, each term of the sum must be zero, then we get (1.4).
Now we prove that the domain of R is $L^2(\mathbb{R}^{N+1})$, that is for every f, exists λ_0 such that for $\lambda > \lambda_0$ (1.4) admits a solution q such that $G^\lambda q_i \in L^2(\mathbb{R}^{N+1})\,\forall i$.
If we prove

$$\sup_{x,y}\int dy\, G^\lambda(x,y;y_i',y') \le c_1 \qquad\qquad \sup_{y'}\int dxdy\, G^\lambda(x,y;y_i',y') \le c_2 \tag{1.9}$$

Schur's test, see [HS], implies

$$\left\|G^\lambda q\right\|_{L^2(\mathbb{R}^{N+1})} \le \sqrt{c_1 c_2}\,\|q\|_{L^2(\mathbb{R}^N)} \tag{1.10}$$

It is straightforward to see that (1.9) holds if we take:

$$c_1 = \frac{1}{\hbar\omega}\int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}-\frac{1}{2}} \qquad\qquad c_2 = \frac{1}{\sqrt{2\pi}}\frac{1}{\hbar\omega}\left(\frac{M\omega}{\hbar}\right)^{\frac{1}{2}}\int_0^1 d\nu\,\frac{\nu^{\frac{\lambda}{\hbar\omega}-\frac{1}{2}}}{(\ln\frac{1}{\nu})^{\frac{1}{2}}} \tag{1.11}$$

By (1.10), it is then sufficient to show that (1.4) admits a square integrable solution.
Using again Schur's test it is easy to see that for $\forall i,j = 1,\dots N$ one has:

$$\int dy\left|G^\lambda q_i(y_j,y)\right|^2 \le \left(\frac{1}{\sqrt{2\pi}}\frac{1}{\hbar\omega}\left(\frac{M\omega}{\hbar}\right)^{\frac{1}{2}}\int_0^1 d\nu\,\frac{\nu^{\frac{\lambda}{\hbar\omega}-\frac{1}{2}}}{(\ln\frac{1}{\nu})^{\frac{1}{2}}}\right)\int dy\,|q_i(y)|^2 \tag{1.12}$$

System (1.4) can be written in a more compact form as a fixed point equation in the Hilbert space $\bigoplus_{i=1}^N L^2(\pi_i)$: if we put $\left(T_f^\lambda q\right)_i(y) = G^\lambda f(y_i,y) + \sum_j G^\lambda q_j(y_i,y)$ system (1.4) is equivalent to the equation

$$q + \alpha T_f^\lambda q = 0 \tag{1.13}$$

By estimate (1.12), T_f^λ becomes a contraction as $\lambda \to +\infty$, then exists $\lambda_0 > 0$ s.t. for $\lambda > \lambda_0$ (1.13) has a solution.

Now we prove that the operator R is symmetric: the condition $(Rf, g) = (f, Rg)$ is equivalent to $(G^\lambda q^f, g) = (f, G^\lambda q^g)$ which can be written as

$$\sum_{i=1}^{N} \int dy \overline{q_i^f(y)} G^\lambda g(y_i, y) = \sum_{i=1}^{N} \int dy \overline{G^\lambda f(y_i, y)} q_i^g(y) \tag{1.14}$$

If we solve (1.4) with respect to $G^\lambda f(y_i, y)$ and to $G^\lambda g(y_i, y)$ and we substitute into (1.14) we immediately obtain an identity.
Since we have proved that R is everywhere defined and symmetric, it is a bounded self adjoint operator operator by Hellinger-Toeplitz theorem, see [RSI].
The last step to conclude the proof of the theorem is showing that R has empty kernel, which is equivalent to saying that H_α is densely defined. If we suppose $Rf = 0$ then we have $Rf(y_i, y) = 0\ \forall i = 1 \ldots N$. Notice that the range of R is contained in $H^1(\mathbb{R})$ and then we are allowed to take the traces of Rf over π_i. System (1.4) immediately implies $q = 0$ and then $G^\lambda f = 0$ which means $f = 0$. □

The knowledge of the resolvent is equivalent to the knowledge of the operator itself; next theorem is just a different rephrasing of the previous one.

Theorem 1.2.2 *The domain and the action of H_α are the following:*

$$\mathscr{D}(H_\alpha) = \left\{ u \in L^2(\mathbb{R}^{N+1})\, s.t.\, u = \varphi^\lambda + G^\lambda q,\, \varphi^\lambda \in \mathscr{D}(H_0),\, q_i + \alpha Tr_{\pi_i} u = 0\, \forall i = 1 \ldots N \right\} \tag{1.15}$$

$$(H_\alpha + \lambda)u = (H_0 + \lambda)\varphi^\lambda \tag{1.16}$$

Proof
The key observation is that $u \in \mathscr{D}(H_\alpha)$ if and only $u = (H_\alpha + \lambda)^{-1} f$. If we put $\varphi^\lambda = G^\lambda f$, (1.15) and (1.16) follow immediately from the previous theorem. □

1.3 The Unitary Group

In this section we give an integral representation for the unitary group $e^{-i\frac{t}{\hbar}H_\alpha}$.
The free propagator is simply the product between the free propagator and the Mehler kernel:

$$e^{-i\frac{t}{\hbar}H_0}(x, y; x', y') = \left(\frac{M}{2\pi i \hbar t}\right)^{\frac{1}{2}} e^{i\frac{M}{2\hbar t}(x-x')^2} \left(\frac{m\omega}{\pi\hbar(1 - e^{-2i\omega t})}\right)^{\frac{N}{2}} e^{-\frac{m\omega}{2i\hbar\tan(\omega t)}(y^2+y'^2)} e^{\frac{m\omega}{i\hbar\sin(\omega t)}yy'}$$

In order to formulate the main theorem of this section we introduce the operators W^t_{jk} which are the restrictions of $e^{-i\frac{t}{\hbar}H_0}$ over the planes π_j, π_k.

$$W^t_{j,k}(y;y') = \left(\frac{M}{2\pi i\hbar t}\right)^{\frac{1}{2}} e^{i\frac{M}{2\hbar t}(y_j-y_k)^2} \left(\frac{m\omega}{\pi\hbar(1-e^{-2i\omega t})}\right)^{\frac{N}{2}} e^{-\frac{m\omega}{2i\hbar\tan(\omega t)}(y^2+y'^2)} e^{\frac{m\omega}{i\hbar\sin(\omega t)}yy'}$$

Theorem 1.3.1 *Let $\psi_0 \in \mathscr{D}(H_\alpha)$ a regular initial datum, then $\psi(t)$ has the following representation: there are N charges $q_j(t)$ depending on time such that*

$$\psi(t;x,y) = \left(e^{-i\frac{t}{\hbar}H_0}\psi_0\right)(x,y) + \frac{i}{\hbar}\int_0^t ds \sum_{j=1}^N \int dy' e^{-i\frac{t-s}{\hbar}H_0}(x,y;y'_j,y')q_j(s;y') \tag{1.17}$$

The charges $q_i(t)$ are solution of the following system:

$$q_j(t) + \alpha Tr_{\pi_j}\left(e^{-i\frac{t}{\hbar}H_0}\psi_0\right) + \frac{i\alpha}{\hbar}\sum_{k=1}^N \int_0^t ds W^{t-s}_{jk} q_k(s) = 0 \quad \forall j = 1\dots N \tag{1.18}$$

Proof

We start from the ansatz (1.17); if we diagonalize the free evolution with a Fourier transform and a projection on the basis of the harmonic oscillator we obtain:

$$\psi(t;k,\underline{n}) = e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)t}\psi_0(k,\underline{n}) + \frac{i}{\hbar}\int_0^t ds\, e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)(t-s)} \sum_{j=1}^N \check{q}_j(s;k,\underline{n}) \tag{1.19}$$

$$\check{q}_j(t;k,\underline{n}) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int dy'\, e^{-i\frac{ky'_j}{\hbar}} e_{\underline{n}}(y')q_j(t;y') \tag{1.20}$$

In the previous formula we have used standard notation on multi-indexes: $\underline{n} = (n_1,\dots,n_N)$ and $|\underline{n}| = n_1+\dots+n_N$; we have denoted the n-th eigenvector of the one dimensional harmonic oscillator with e_n while $e_{\underline{n}}(y)$ stands for $e_{n_1}(y_1)e_{n_2}(y_2)\dots e_{n_N}(y_N)$.
With an integration by parts we obtain:

$$\begin{aligned}\psi(t;k,\underline{n}) = {} & e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)t}\varphi_0^\lambda(k,\underline{n}) - \frac{1}{\frac{k^2}{2M}+|\underline{n}|\hbar\omega+\lambda}\int_0^t ds\, e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)(t-s)}\sum_{i=j}^N \mathring{\check{q}}_j(s;k,\underline{n}) + \\ & + \frac{i}{\hbar}\frac{\lambda}{\frac{k^2}{2M}+|\underline{n}|\hbar\omega+\lambda}\int_0^t ds e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)(t-s)}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) + \frac{1}{\frac{k^2}{2M}+|\underline{n}|\hbar\omega+\lambda}\sum_{j=1}^N \check{q}_j(s;k,\underline{n})\end{aligned} \tag{1.21}$$

If q are regular functions the first three terms on the r.h.s. belong to $\mathscr{D}(H_0)$ while the last one is $G^\lambda q(t)$ which doesn't belong to $\mathscr{D}(H_0)$; we impose the boundary condition contained in (1.15) such that $\psi(t) \in \mathscr{D}(H_\alpha)$, then we obtain (1.20). It is now straightforward to verify that (1.17) satisfies the time dependent Schroedinger equation: using respectively (1.17) and (1.21) we have

$$i\hbar\frac{d\psi}{dt}(t) = \left(\frac{k^2}{2M} + |\underline{n}|\hbar\omega\right) e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)t}\psi_0(k,\underline{n}) - \sum_{j=1}^N \check{q}_j(s;k,\underline{n})+$$
$$+\frac{i}{\hbar}\left(\frac{k^2}{2M} + |\underline{n}|\hbar\omega\right)\int_0^t ds\, e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)(t-s)}\sum_{j=1}^N \check{q}_j(s;k,\underline{n})+ \tag{1.22}$$

$$H_\alpha\psi(t;k,\underline{n}) = \left(\frac{k^2}{2M} + |\underline{n}|\hbar\omega\right) e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)t}\varphi_0^\lambda(k,\underline{n}) - \frac{\frac{k^2}{2M}+|\underline{n}|\hbar\omega}{\frac{k^2}{2M}+|\underline{n}|\hbar\omega+\lambda}\cdot$$
$$\int_0^t ds\, e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)(t-s)}\sum_{i=1}^N \overset{\circ}{\check{q}_i}(s;k,\underline{n})+\frac{i}{\hbar}\lambda\frac{\left(\frac{k^2}{2M}+|\underline{n}|\hbar\omega\right)}{\frac{k^2}{2M}+|\underline{n}|\hbar\omega+\lambda}\int_0^t ds\, e^{-i\left(\frac{k^2}{2M\hbar}+|\underline{n}|\omega\right)(t-s)}\sum_{j=1}^N \check{q}_j(s;k,\underline{n})+$$
$$-\frac{\lambda}{\frac{k^2}{2M}+|\underline{n}|\hbar\omega+\lambda}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) \tag{1.23}$$

The r.h.s. of (1.22) and (1.23) are equal and then $\psi(t)$ is the solution of the time dependent Schroedinger equation. The last step to conclude the proof of the theorem is proving that (1.18) has a global solution. System (1.18) can be written in a more compact form as a linear equation in $C(0,T;\oplus_{i=1}^N L^2(\pi_i))$.

$$q(t) + \frac{i\alpha}{\hbar}\int_0^t ds W^{t-s}q(s) + \alpha\Psi(t) = 0 \tag{1.24}$$

We have put

$$\left(\Psi(t)\right)_j = Tr_{\pi_j}e^{-i\frac{t}{\hbar}H_0}\psi_0 \qquad \left(W^{t-s}q(s)\right)_j = \sum_{k=1}^N W_{jk}^{t-s}q_k(s) \tag{1.25}$$

In order to prove the local existence of $q(t)$ it sufficient to prove the following estimate:

$$\|W_{jk}^t\| \le C\frac{1}{\sqrt{t}} \tag{1.26}$$

If (1.26) holds then exists T_0 s.t. for $T < T_0$ $\int_0^t ds W^{t-s}q(s)$ becomes a contraction and (1.24) has a solution. The solution can be extended for any T by iteration. Estimate (1.26) is trivial if $j = k$, otherwise it follows from standard results on Fourier Integral Operators, see [Fu]. □

Chapter 2

The Three Dimensional Case

In this chapter we introduce the quadratic form F_α in dimension three. Even if we are still concerned by a perturbation of the free quadratic form F_0 supported by a null set, i.e. the planes π_i, the situation is very different from the one previous case. The perturbation has now support on a set of codimension three and this gives much stronger singularities.
It is then necessary a renormalization procedure to construct a well-posed quadratic form. We adapt to our case the techniques developed in [T] and [DFT] which lead to definition (2.1.1).
As a matter of fact, the domain of the quadratic form consists of functions in $\mathscr{D}(F_0)$ plus the potentials $G^\lambda q$. It is then clear that we are no longer concerned with a bounded perturbation of F_0; indeed one can see that if one puts $\alpha = 0$ the unperturbed quadratic form is not recovered, that is α is no longer the coupling constant of the system.
In section one we introduce the quadratic form; it is required some technical work, see propositions (2.1.2), (2.1.3), (2.1.4), (2.1.5), to prove that F_α is well defined.
Closure and semiboundedness of the quadratic form are proved in the case $N = 1$, see (2.1.7) and (2.1.8).
In section two, theorem (2.2.1) characterizes the domain and the action of the operator.

2.1 The Quadratic Form

In this section we introduce the quadratic form F_α, we show that it is well defined and we prove it is closed and positive if $N = 1$. The quadratic form F_α is defined as follows.

Definition 2.1.1

$$\mathscr{D}(F_\alpha) = \left\{ u \in L^2(\mathbb{R}^{3N+3})\, s.t.\, \exists q \in \mathscr{D}(\Phi_\alpha^\lambda),\, \varphi^\lambda \equiv u - G^\lambda q \in \mathscr{D}(F_0) \right\}$$

$$F_\alpha[u] = \mathcal{F}^\lambda[u] + \Phi_\alpha^\lambda[u]$$

$$\mathcal{F}^\lambda[u] = \int dxdy \left\{ \frac{\hbar^2}{2M} |\nabla_x \varphi^\lambda|^2 + \frac{\hbar^2}{2m} |\nabla_y \varphi^\lambda|^2 + \lambda |\varphi^\lambda|^2 - \lambda |u|^2 + \frac{1}{2} m\omega^2 (y - y^0)^2 |\varphi^\lambda|^2 \right\}$$

$$\mathscr{D}(\Phi^\lambda_\alpha) = \big\{(q_1,\dots,q_N),\ q_i \in L^2(\pi_i)\ s.t.\ \Phi^\lambda[q] < +\infty\big\}$$

$$\Phi^\lambda_\alpha[q] = \alpha \sum_{i=1}^N \int dy |q_i(y)|^2 - 2\Re \sum_{i<j} \int dy \overline{q_i(y)} (G^\lambda_i q_j)(y) + \\ + \sum_{i=1}^N \int dy\, a^\lambda_i(y) |q_i(y)|^2 + \frac{1}{2}\sum_{i=1}^N \int dy dy' G^\lambda(y_i, y; y'_i, y') |q_i(y) - q_i(y')|^2$$

In the previous definition we have introduced the functions $a^\lambda_i(y)$ given by:

$$a^\lambda_i(y) = \frac{1}{(2\pi)^{\frac{3}{2}}} \omega^{\frac{1}{2}} \frac{1}{\hbar^{\frac{5}{2}}} \left(\frac{mM}{m+M}\right)^{\frac{3}{2}} \Bigg\{ \frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \Bigg[1 - \nu^{\frac{\lambda}{\hbar\omega} + \frac{3N}{2} - 1} \left(\frac{2}{1+\nu^2}\right)^{\frac{3(N-1)}{2}} \\ \left(\frac{m+M}{\frac{1+\nu^2}{2}\frac{\ln\frac{1}{\nu}}{1-\nu} m + \frac{1+\nu}{2} M}\right)^{\frac{3}{2}} \exp\left(-\frac{1}{2}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}\sum_{\substack{j=1\\ j\neq i}}^N |y_j - y^0_j|^2\right) \\ \exp\left(-\frac{\omega}{\hbar}\frac{\frac{1}{4}m^2 + \frac{1-\nu}{1+\nu}\frac{1}{2\ln\frac{1}{\nu}} mM}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2} m + \frac{1}{2\ln\frac{1}{\nu}} M} |y_i - y^0_i|^2\right) \Bigg] \Bigg\} \tag{2.1}$$

In order to give a heuristic motivation for this definition one can proceed as follows. The naive quadratic form associated to H_γ is:

$$\tilde{F}_\gamma[u] = F_0[u] + \sum_j \gamma \int dx dy |u|^2 \delta(x - y_j)$$

which can be written as

$$\tilde{F}_\gamma[u] = F_0[u - G^\lambda q] + \lambda \int dx dy |u - G^\lambda q|^2 - \lambda \int dx dy |u|^2 + \big((H_0+\lambda)G^\lambda q, u\big) + \\ + \big(u, (H_0+\lambda)G^\lambda q\big) - \big((H_0+\lambda)G^\lambda q, G^\lambda q\big) + \sum_j \gamma \int dx dy |u|^2 \delta(x - y_j) \tag{2.2}$$

for any $\lambda > 0$ and any charge q. Now we fix $q_i = -\gamma\, u|_{\pi_i}$ and denote $\varphi^\lambda = u - G^\lambda q$. Then we use the fact the $(H_0 + \lambda)G^\lambda q = q$ and we finally obtain

$$\tilde{F}_\gamma[u] + \lambda \int dx dy |u|^2 = F_0[\varphi^\lambda] + \lambda \int dx dy |\varphi^\lambda|^2 + \tilde{\Phi}_\gamma[q]$$

$$\tilde{\Phi}_\gamma[q] = \sum_i \int_{\pi_i} dy\, \overline{q_i}(y) \left[-\frac{q_i(y)}{\gamma} - G_i^\lambda q_i(y) \right] - 2\Re \sum_{i<j} \int dy \overline{q_i}(y) G_i^\lambda q_j(y) \tag{2.3}$$

It's clear that the form $\tilde{\Phi}_\gamma[q]$ on the charges is ill defined due to the presence of the first term of the r.h.s. of (2.3) (diagonal terms) since the potential $G^\lambda q_i$ is divergent on π_i. Its singularity around $y \in \pi_i$ is only due to the value of q_i in y, i.e. it has the same singularity of the potential generated by the uniform charge distribution $q_i(y)$ on U_ε, a small neighborhood of y. Then we can write

$$G^\lambda q_i(x,y) = q_i(y)\zeta_{1_\varepsilon}(x,y) + \tau_{q_i}(x,y)$$

where ζ_{1_ε} is the divergent part of $G^\lambda 1_\varepsilon$ and $\tau_{q_i}(x,y)$ remains finite in the limit $x \to y_i$; we have denoted the characteristic function of U_ε with 1_ε. Moreover we define $G^\lambda_{i,ren} q_i(y) = \lim_{x \to y_i} \tau_{q_i}(x,y)$.
The idea is to separate the divergent part and the finite part of the quadratic form and the to compensate the divergence through a renormalization of the coupling constant γ. We only outline here the procedure.
Define $\pi_i^\delta = \{(x,y)\, s.t.\, \frac{1}{\sqrt{2}}|x - y_i| \leq \delta\}$ and consider a charge distribution q_i^δ on π_i^δ such that $q_i^\delta \to q_i$ and $G^\lambda q_i^\delta \to G^\lambda q_i$ for $\delta \to 0$. Moreover $G^\lambda q_i^\delta(x,y) = q_i^\delta(y)\zeta^\delta(x,y) + \tau^\delta_{q_i}(x,y)$ where $\zeta^\delta \to \zeta_{1_\varepsilon}$ and $\tau^\delta_{q_i} \to \tau_{q_i}$ for $\delta \to 0$. Introducing a new coupling constant $\gamma = \gamma^\delta(x,y)$ the ill defined diagonal term can now be replaced by

$$\lim_{\delta\to 0} \int_{\pi_i^\delta} \overline{q_i^\delta}(x,y) \left[-\frac{q_i^\delta(x,y)}{\gamma^\delta(x,y)} - G^\lambda q_i^\delta(x,y) \right] =$$

$$\lim_{\delta\to 0} \int_{\pi_i^\delta} \overline{q_i^\delta}(x,y) \left[-\frac{q_i^\delta(x,y)}{\gamma^\delta(x,y)} - q_i^\delta(x,y)\zeta^\delta(x,y) \right] - \int_{\pi_i^\delta} \overline{q_i^\delta}(x,y)\tau^\delta_{q_i}(x,y) \tag{2.4}$$

In order to get a finite limit in (2.4) it is sufficient to define

$$-\frac{1}{\gamma^\delta(x,y)} = \zeta^\delta(x,y) + \alpha, \qquad \alpha \in \mathbb{R}$$

so that the diagonal term reduces to

$$\alpha \int_{\pi_i} |q_i|^2 - \int_{\pi_i} \overline{q_i}(y) G^\lambda_{i,ren} q_i(y) \tag{2.5}$$

We shall now rewrite the renormalized potential $G^\lambda_{i,ren} q_i$ in a more convenient form

$$G^\lambda_{i,ren} q_i(y) = \lim_{x\to y_i} \left[G^\lambda q_i(x,y) - q_i(y)\zeta_{1_\varepsilon}(x,y) \right] = \lim_{x \to y_i} \Bigg[\int_{|y'-y|>\varepsilon} dy' q_i(y') G^\lambda(x,y;y_i',y') +$$
$$+ \int_{|y'-y|<\varepsilon} dy' G^\lambda(x,y;y_i',y') \big(q_i(y') - q_i(y)\big) +$$
$$+ q_i(y) \left(\int_{|y'-y|<\varepsilon} dy' G^\lambda(x,y;y_i',y') - \zeta_{1_\varepsilon}(x,y) \right) \Bigg]$$

Then the second term of (2.5) becomes:

$$
\begin{aligned}
-\int dy\overline{q_i(y)}G^{\lambda}_{i,ren}q_i(y) =& -\int_{|y'-y|>\varepsilon} dydy'\overline{q_i(y)}q_i(y')G^{\lambda}(y_i,y;y'_i,y')+\\
&-\int_{|y'-y|<\varepsilon} dydy'\overline{q_i(y)}(q_i(y')-q_i(y))G^{\lambda}(y_i,y;y'_i,y')+\\
&-\int dy|q_i(y)|^2\tau_{1_\varepsilon}(y_i,y)\\
=&\int dydy'\overline{q_i(y)}(q_i(y)-q_i(y'))G^{\lambda}(y_i,y;y'_i,y')+\\
&+\int dy|q_i(y)|^2\left(-\tau_{1_\varepsilon}(y_i,y)-\int_{|y'-y|>\varepsilon} dy'G^{\lambda}(y_i,y;y'_i,y')\right)\\
=&\frac{1}{2}\int dydy'|q_i(y)-q_i(y')|^2G^{\lambda}(y_i,y;y'_i,y')+\int dy\, a^{\lambda}_i(y)|q_i(y)|^2
\end{aligned}
$$

Where we have denoted

$$
a^{\lambda}_i(y) = -\lim_{x\to y_i}\left(\tau_{1_\varepsilon}(x,y)+\int_{|y'-y|>\varepsilon} dy'G^{\lambda}(x,y;y'_i,y')\right)
$$

and τ_{1_ε} is the finite part of the potential generated by 1_ε.
It remains to analyze $a^{\lambda}_i(y)$. To be concrete we fix $i=1$, put $y^0=0$ and introduce the coordinates ξ and η defined in the following way.

$$
\begin{cases}
\xi = \dfrac{1}{\sqrt{2}}(x-y_1)\\
\eta = \dfrac{1}{\sqrt{2}}(x+y_1)
\end{cases}
$$

With this choice π_1 is defined by $\xi=0$. We have to study the behavior for small ξ of

$$
\left(G^{\lambda}1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) = \int_{U_\varepsilon} d\eta' dy'_2\dots dy'_N\; G^{\lambda}(\xi,\eta,y_2,\dots,y_N;0,\eta',y'_2,\dots,y'_N) \tag{2.6}
$$

The shape of U_ε doesn't matter, indeed a^{λ}_i doesn't depend on ε; we choose

$$
U_\varepsilon = \{|\eta'-\eta|<\varepsilon, |y'_2-y_2|<\varepsilon,\dots,|y'_N-y_N|<\varepsilon\}
$$

From (A.1) one has that, with simple manipulations, (2.6) can be written as:

$$\begin{aligned}
\left(G^\lambda 1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\int_0^1 d\nu \frac{\nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}}{(1+\nu)^{\frac{3N}{2}}}\frac{1}{(1-\nu)^{\frac{3N}{2}}(\log\frac{1}{\nu})^{\frac{3}{2}}} \\
&\prod_{i=2}^N \left\{ e^{-\frac{m\omega}{\hbar}\frac{1-\nu}{1+\nu}y_i^2}\left(\frac{m\omega}{\hbar}\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\right)^{-\frac{3}{2}}\int_{|y_i'|<\varepsilon\left(\frac{m\omega}{\hbar}\frac{(1+\nu^2)}{2(1-\nu^2)}\right)^{\frac{1}{2}}} dy_i'\, e^{-y_i'^2}e^{-y_i\cdot y_i'\left(\frac{m\omega}{\hbar}\frac{2(1-\nu^2)}{(1+\nu^2)}\right)^{\frac{1}{2}}\frac{1-\nu}{1+\nu}} \right\} \\
&e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}(\eta^2-\eta\cdot\xi)}e^{-\xi^2\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}+\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)}\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)^{-\frac{3}{2}} \\
&\int_{|\eta'|<\varepsilon\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}+\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)^{\frac{1}{2}}} d\eta' e^{-\eta'^2}\exp\left\{\frac{2\eta'\cdot\left[\left(\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{1}{2}\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)\xi+\frac{1}{4}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\eta\right]}{\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}+\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)^{\frac{1}{2}}}\right\} \qquad (2.7)
\end{aligned}$$

With the changes of variable $\nu = 1-\nu'$ and $\nu'\xi^2 = \nu''$ we recover a coulombian singularity for the potential:

$$\left(G^\lambda 1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) \simeq_{\xi\to 0} \frac{1}{\pi}\frac{1}{\hbar^2}\frac{mM}{m+M}\frac{1}{|\xi|}$$

This expression can be written in a more useful way as

$$\left(G^\lambda 1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) \simeq_{\xi\to 0} \frac{1}{\pi^{\frac{3}{2}}}\frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{mM}{m+M}\right)^{\frac{3}{2}}\int_0^\infty d\nu \frac{1}{\nu^{\frac{3}{2}}}e^{-\frac{\omega}{\hbar}\frac{1}{\nu}\frac{mM}{m+M}|\xi|^2}$$

We separate this integral into two pieces with ν going from 0 to 1 and from 1 to $+\infty$; this latter term is finite when $\xi \to 0$ and will give the one half in brace brackets in (2.1), while the former is subtracted from (2.7). Adding to it $\int_{|y'-y|>\varepsilon} dy' G^\lambda(y_i, y; y_i', y')$ we finally obtain (2.1).

Now we show that definition (2.1.1) is well posed. First we prove that the decomposition $u = \varphi^\lambda + G^\lambda q$ is well defined and unique; the proof is divided into propositions (2.1.2), (2.1.3) and (2.1.4).

Proposition 2.1.2 *If $q_i \in L^2(\mathbb{R}^{3N})$ then $G^\lambda q_i \in L^2(\mathbb{R}^{3N+3})$.*

Proof
Let's fix $i = 1$.

$$\int dx\, dy |G^\lambda q_1(x,y)|^2 = \frac{1}{(2\pi)^3}\frac{1}{(\hbar\omega)^2}\frac{1}{\pi^{3N}}\left(\frac{m\omega}{\hbar}\right)^{3N}\left(\frac{M\omega}{\hbar}\right)^3$$
$$\int dx dy \int dz \overline{q_1(z)} \int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}} e^{-\frac{m\omega}{\hbar}\frac{1-\nu}{2(1+\nu)}(y^2+z^2)} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-z_1|^2}$$
$$e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-z|^2}\int dw q_1(w)\int_0^1 d\mu\, \mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\mu^2)^{\frac{3}{2}N}}\frac{1}{\left(\ln\frac{1}{\mu}\right)^{\frac{3}{2}}} e^{-\frac{m\omega}{\hbar}\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}$$
$$e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}|x-w_1|^2} e^{-\frac{m\omega}{\hbar}\frac{\mu}{1-\mu^2}|y-w|^2}$$

$$= \frac{1}{(2\pi)^3}\frac{1}{(\hbar\omega)^2}\frac{1}{\pi^{3N}}\left(\frac{m\omega}{\hbar}\right)^{3N}\left(\frac{M\omega}{\hbar}\right)^3\int dz dw\, \overline{q_1}(z) q_1(w)$$
$$\int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\int_0^1 d\mu\, \mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\mu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\mu}\right)^{\frac{3}{2}}}$$
$$\int dx dy e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-z|^2} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-z_1|^2} e^{-\frac{m\omega}{\hbar}\frac{1-\nu}{2(1+\nu)}(y^2+z^2)}$$
$$e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}|x-w_1|^2} e^{-\frac{m\omega}{\hbar}\frac{\mu}{1-\mu^2}|y-w|^2} e^{-\frac{m\omega}{\hbar}\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}$$

It's then sufficient to show that

$$T(z,w) = \int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\int_0^1 d\mu\, \mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\mu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\mu}\right)^{\frac{3}{2}}}$$
$$\int dx dy e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-z|^2} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-z_1|^2} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}|x-w_1|^2} e^{-\frac{m\omega}{\hbar}\frac{\mu}{1-\mu^2}|y-w|^2}$$

is the integral kernel of a bounded operator in $L^2(\mathbb{R}^{3N})$.
This kernel can be pointwise estimated in the following way:

$$T(z,w) \le \left(\frac{M\omega}{\hbar}\right)^{-\frac{3}{2}}\left(\frac{m\omega}{\hbar}\right)^{-\frac{3N}{2}}\pi^{\frac{3N+3}{2}}\int_0^1 d\nu d\mu\left(\frac{\nu}{1-\nu^2}+\frac{\mu}{1-\mu^2}\right)^{-\frac{3N+3}{2}}\nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}$$
$$\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\mu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\mu}\right)^{\frac{3}{2}}} e^{-\frac{1}{4\left(\frac{1-\nu^2}{\nu}+\frac{1-\mu^2}{\mu}\right)}\frac{m\omega}{\hbar}|w-z|^2} \tag{2.8}$$

Using the Schur's test, see [HS], and (2.8) we have

$$\|T\| \le \sup_w \int dz\, T(z,w) \le \left(\frac{M\omega}{\hbar}\right)^{-\frac{3}{2}} \pi^{3N+\frac{3}{2}} \int_0^1 d\nu d\mu \nu^{\frac{\lambda}{\hbar\omega}-1} \mu^{\frac{\lambda}{\hbar\omega}-1} \frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}} \frac{1}{\left(\ln\frac{1}{\mu}\right)^{\frac{3}{2}}}$$

$$\left(\frac{\nu}{1-\nu^2}+\frac{\mu}{1-\mu^2}\right)^{-\frac{3}{2}} \le \left(\frac{M\omega}{\hbar}\right)^{-\frac{3}{2}} \pi^{3N+\frac{3}{2}} \int_0^1 d\nu d\mu \frac{\nu^{\frac{\lambda}{\hbar\omega}-1}\mu^{\frac{\lambda}{\hbar\omega}-1}}{\left[\frac{\nu}{1+\nu}\frac{\ln\frac{1}{\nu}}{1-\nu}\ln\frac{1}{\mu}+\frac{\mu}{1+\mu}\frac{\ln\frac{1}{\mu}}{1-\mu}\ln\frac{1}{\nu}\right]^{\frac{3}{2}}} < +\infty$$

and this concludes the proof. □

Proposition 2.1.3 *If $q_i \in \mathscr{D}(\Phi^\lambda_\alpha)$ then $G^\lambda q_i \notin \mathscr{D}(F_0)$.*

Proof
Fix $i=1$ and put $y^0=0$. Since $G^\lambda q_1$ is divergent near π_1, we shall consider a regularized version of F_0 restricting the integration to the whole space minus a small cylinder around π_1 and, with integrations by parts, we shall show that it diverges when the small cylinder shrinks to π_1.
Let's show that the following identity holds:

$$\int_{\mathbb{R}^{3N+3}\setminus\pi_1^\delta} dxdy \left\{\frac{\hbar^2}{2M}|\nabla_x G^\lambda q_1|^2 + \frac{\hbar^2}{2m}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}m\omega^2 y^2+\lambda\right)\left|G^\lambda q_1\right|^2\right\} =$$
$$= -\frac{1}{2}\int dydy'|q_1(y)-q_1(y')|^2 \int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'',y'';y_1,y)\frac{\partial G^\lambda}{\partial n}(x'',y'';y_1',y')+$$
$$+\int_{\partial\pi_1^\delta} dxdy\left(G^\lambda 1-\zeta_1\right)\frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right)+\int_{\partial\pi_1^\delta} dxdy\,\zeta_1\frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right) \quad (2.9)$$

Formula (2.9) is equivalent to:

$$\int_{\mathbb{R}^{3N+3}\setminus\pi_1^\delta} dxdy \left\{\frac{\hbar^2}{2M}|\nabla_x G^\lambda q_1|^2 + \frac{\hbar^2}{2m}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}m\omega^2 y^2+\lambda\right)\left|G^\lambda q_1\right|^2\right\} =$$
$$= -\frac{1}{4}\int dydy'|q_1(y)-q_1(y')|^2 \left(\int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'',y'';y_1,y)\frac{\partial G^\lambda}{\partial n}(x'',y'';y_1',y')+\right.$$
$$\left.+\int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'',y'';y_1',y')\frac{\partial G^\lambda}{\partial n}(x'',y'';y_1,y)\right)+\int_{\partial\pi_1^\delta} dxdy\,G^\lambda 1\frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right) \quad (2.10)$$

If we further simplify (2.10), we obtain

$$\int_{\mathbb{R}^{3N+3}\setminus\pi_1^\delta} dxdy \left\{ \frac{\hbar^2}{2M}|\nabla_x G^\lambda q_1|^2 + \frac{\hbar^2}{2m}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}m\omega^2 y^2 + \lambda\right) \left|G^\lambda q_1\right|^2 \right\} = \\ = \int_{\partial\pi_1^\delta} dxdy\, \overline{G^\lambda q_1}\, \frac{\partial G^\lambda}{\partial n}(q_1) \tag{2.11}$$

which is true by Stoke's theorem and because we have removed π_1^δ from the domain of integration.

In order to prove that $F_0[G^\lambda q_1] = +\infty$ we shall consider the r.h.s. of (2.9) and we shall prove the following steps: the sum of first and the second term converge, up to a multiplicative constant, to $\Phi_\alpha^\lambda[q]$ as $\delta \to 0$ while the third term diverges; since by proposition (2.1.2), the l.h.s. of (2.9) differs from $F_0[G^\lambda q_1]$ only by a bounded term it's clear that $F_0[G^\lambda q_1] = +\infty$.

Now we show that

$$\lim_{\delta\to 0} \int_{\partial\pi_1^\delta} dxdy \left(G^\lambda 1 - \zeta_1\right)(x,y) \frac{\partial G^\lambda}{\partial n}(x,y;y_1',y') = -\frac{4}{\hbar^2}\frac{mM}{m+M} a_1^\lambda(y')$$

$$\lim_{\delta\to 0} \int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'',y'';y_1,y) \frac{\partial G^\lambda}{\partial n}(x'',y'';y_1',y') = \frac{4}{\hbar^2}\frac{mM}{m+M} G^\lambda(y_1,y;y_1',y')$$

Notice that in this case $\frac{\partial}{\partial n} = -\frac{\partial}{\partial|\xi|}$, so the integral kernel of $\frac{\partial G^\lambda}{\partial n}$ can be easily obtained from (A.1).

$$\frac{\partial G^\lambda}{\partial n}(\xi,\eta,y_2,\ldots,y_n;y_1',y') = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\int_0^1 d\nu\, \frac{\nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}}{(1-\nu^2)^{\frac{3N}{2}}\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}} \\ e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\left[\frac{1}{2}(\eta-\xi)^2+y_2^2+\ldots y_N^2+y'^2\right]}\, e^{-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\left(|\frac{1}{\sqrt{2}}(\eta-\xi)-y_1'|^2+|y_2-y_2'|^2+\ldots+|y_N-y_N'|^2\right)}\, e^{-\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}|\frac{1}{\sqrt{2}}(\eta+\xi)-y_1'|^2} \\ \left\{\left[|\xi| + \sqrt{2}\frac{\xi}{|\xi|}\cdot\left(\frac{1}{\sqrt{2}}\eta + y_1'\right)\right]\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar} + \left[|\xi| + \sqrt{2}\frac{\xi}{|\xi|}\cdot\left(\frac{1}{\sqrt{2}}\eta - y_1'\right)\right]\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right\} \tag{2.12}$$

The potential $G^\lambda 1$ and its divergent part ζ_1 have the following expressions:

$$\left(G^\lambda 1\right)(\xi,\eta,y_2,\dots,y_n) = \frac{1}{\pi^{\frac{3}{2}}}\frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{mM}{m+M}\right)^{\frac{3}{2}}\int_0^1 d\mu\,\frac{1}{(1-\mu)^{\frac{3}{2}}}\mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\left(\frac{2}{1+\mu^2}\right)^{\frac{3(N-1)}{2}}$$
$$\left(\frac{1+\mu^2}{2}\frac{\ln\frac{1}{\mu}}{1-\mu}m+\frac{1+\mu}{2}M\right)^{-\frac{3}{2}}\exp\left(-\frac{1}{2}\frac{1-\mu^2}{1+\mu^2}\frac{m\omega}{\hbar}\sum_{j=2}^N y_j^2\right)\exp\left(-\frac{\omega}{\hbar}\frac{\frac{1}{4}m^2+\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}mM}{\frac{1}{2}\frac{1+\mu^2}{1-\mu^2}m+\frac{1}{2\ln\frac{1}{\mu}}M}\frac{1}{2}\eta^2\right)$$
$$\exp\left(-4\frac{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\xi^2\right)\exp\left(-\frac{\frac{1}{8}\left(\frac{m\omega}{\hbar}\right)^2}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\eta\cdot\xi\right)$$

$$\zeta_1 = \frac{1}{\pi^{\frac{3}{2}}}\frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{mM}{m+M}\right)^{\frac{3}{2}}\int_0^{+\infty} d\mu\frac{1}{\mu^{\frac{3}{2}}}e^{-\frac{1}{\mu}\frac{\omega}{\hbar}\frac{mM}{m+M}\xi^2}$$

It is useful to write $G^\lambda 1-\zeta_1$ in the following way:

$$\left(G^\lambda 1-\zeta_1\right)(\xi,\eta,y_2\dots y_N) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\omega^{\frac{1}{2}}}{\hbar^{\frac{5}{2}}}\left(\frac{mM}{m+M}\right)^{\frac{3}{2}}\left\{\int_1^{+\infty} d\mu\frac{1}{\mu^{\frac{3}{2}}}e^{-\frac{1}{\mu}\frac{\omega}{\hbar}\frac{mM}{m+M}\xi^2}+\int_0^1 d\mu\,\frac{1}{(1-\mu)^{\frac{3}{2}}}\right.$$
$$\left[\mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\left(\frac{2}{1+\mu^2}\right)^{\frac{3(N-1)}{2}}\frac{1}{\left(\frac{1+\mu^2}{2}\frac{\ln\frac{1}{\mu}}{1-\mu}m+\frac{1+\mu}{2}M\right)^{\frac{3}{2}}}\exp\left(-\frac{1}{2}\frac{1-\mu^2}{1+\mu^2}\frac{m\omega}{\hbar}\sum_{j=2}^N y_j^2\right)\right.$$
$$\exp\left(-\frac{\omega}{\hbar}\frac{\frac{1}{4}m^2+\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}mM}{\frac{1}{2}\frac{1+\mu^2}{1-\mu^2}m+\frac{1}{2\ln\frac{1}{\mu}}M}\frac{1}{2}\eta^2\right)\exp\left(-4\frac{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\xi^2\right)$$
$$\left.\left.\exp\left(-\frac{\frac{1}{2}\frac{1-\mu}{1+\mu}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\eta\cdot\xi\right)-\exp\left(\frac{1}{1-\mu}\frac{mM}{m+M}\frac{\omega}{\hbar}\xi^2\right)\right]\right\} \tag{2.13}$$

Equation (2.13) shows that $a_1^\lambda(y) = -2^{\frac{3}{2}}\left(G^\lambda 1-\zeta_1\right)(0,\eta,y_2\dots y_N)$; the numerical factor $2^{\frac{3}{2}}$ has been absorbed in (2.1) by a change of variable.
The integral kernel $\frac{\partial G^\lambda}{\partial n}$ has the following property:

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta} dx dy f(x,y)\frac{\partial G^\lambda}{\partial n}(x,y;y_1',y') = \frac{4}{\hbar^2}\frac{mM}{m+M}f(y_1',y') \tag{2.14}$$

If we prove (2.14) when f is $(G^\lambda-\zeta_1)(x,y)$ or $G^\lambda(x,y;y_1'',y'')$ we conclude the first step of the proof; if we interchange the integral over the parameter μ with the limit over δ and the integral over (x, y) we can restrict ourselves to prove (2.14) when f is a gaussian. Now it's only a matter of long calculations to prove that:

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta} d\xi d\eta dy_2\dots dy_N e^{-\frac{\omega}{\hbar}(a\xi^2+b\eta^2+c\eta\cdot\xi)}e^{-\frac{\omega}{\hbar}d(y_2^2+\ldots+y_N^2)}\frac{\partial G^\lambda}{\partial n}(\xi,\eta,y_2,\dots,y_n;y_1',y')$$
$$=\frac{4}{\hbar^2}\frac{mM}{m+M}e^{-\frac{\omega}{\hbar}b\eta'^2}e^{-\frac{\omega}{\hbar}d(y_2'^2+\ldots+y_N'^2)}$$

Using (2.12) and integrating over $\eta, y_2, \dots, y_N$, we have to study

$$\lim_{\delta\to 0}\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\int_0^1 d\nu\,\frac{\nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}}{(1-\nu^2)^{\frac{3N}{2}}\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\left[\frac{1}{\pi}\frac{\omega}{\hbar}\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}m+d\right)\right]^{-\frac{3(N-1)}{2}}$$
$$\exp\left(-\frac{\omega}{\hbar}(y_2'+\dots y_N'^2)\frac{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}md+\frac{1}{4}m^2}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}m+d}\right)\left(\frac{1}{4}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\frac{\omega}{\hbar}b\right)^{\frac{3}{2}}$$
$$\exp\left\{-\frac{1}{2}\delta^2\left[\frac{\frac{1+\nu^2}{1-\nu^2}\frac{1}{\ln\frac{1}{\nu}}+2\frac{\omega}{\hbar}b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)+2\frac{\omega}{\hbar}c\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}-\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)+\left(\frac{\omega}{\hbar}c\right)^2}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right]\right\}$$
$$\exp\left\{-y_1'^2\left[\frac{\frac{1}{4}\left(\frac{m\omega}{\hbar}\right)^2+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}+2\frac{\omega}{\hbar}b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{M\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right]\right\}$$
$$\int_{|\xi|=\delta}d\xi\exp\left\{\sqrt{2}\xi\cdot y_1'\left[\frac{\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)+\frac{\omega}{\hbar}b\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right]\right\}$$
$$\left\{|\xi|\frac{4\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)+\frac{\omega}{\hbar}c\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}+\right.$$
$$\left.-\sqrt{2}\frac{\xi}{|\xi|}\cdot y_1'\frac{\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+\frac{\omega}{\hbar}b\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right\} \qquad (2.15)$$

Making the change of variables $\nu = \nu'$ and $\nu' = \delta^2\nu''$ when $\delta \to 0$ only the most singular terms of (2.15) survives, in particular the surface integral becomes trivial; then we obtain

$$\frac{8}{\pi^{\frac{1}{2}}}\frac{\omega^{\frac{3}{2}}}{\hbar^{\frac{7}{2}}}\left(\frac{mM}{m+M}\right)^{\frac{5}{2}}\int_0^{+\infty}d\nu\frac{1}{\nu^{\frac{5}{2}}}e^{-\frac{1}{\nu}}e^{-2\frac{\omega}{\hbar}by_1'^2}e^{-\frac{\omega}{\hbar}d(y_2'^2+\ldots+y_N'^2)}=\frac{4}{\hbar^2}\frac{mM}{m+M}e^{-2\frac{\omega}{\hbar}by_1'^2}e^{-\frac{\omega}{\hbar}d(y_2'^2+\ldots+y_N'^2)}$$

Since on π_1 $\eta = \sqrt{2}y_1$ we have proved (2.14).

The third term on the r.h.s. of (2.9) is equal to:

$$\frac{1}{\pi}\frac{1}{\hbar^2}\frac{mM}{m+M}\frac{1}{\delta}\int_{\partial\pi_1^\delta}\frac{\partial G^\lambda}{\partial n}(|q_1|^2) \tag{2.16}$$

because ζ_1 is a function of ξ only. Equation (2.14), considering constant as a particular case of gaussian functions, also implies that

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta}\frac{\partial G^\lambda}{\partial n}(|q_1|^2) = \frac{1}{\hbar^2}\frac{mM}{m+M}\int dy'|q_1(y')|^2 < +\infty$$

which shows that (2.16) diverges when $\delta \to 0$. The proof of the proposition is concluded.

□

Notice that we have showed that $\Phi_\alpha^\lambda[q]$ can be looked as a renormalized energy of the potential $G^\lambda q$.

Proposition 2.1.4 *The decomposition $u = \varphi^\lambda + G^\lambda q$ is well defined and unique.*

Proof

By proposition (2.1.2) $G^\lambda q \in L^2(\mathbb{R}^{3N+3})$ so that the decomposition is meaningful in the Hilbert space; uniqueness easily follows from proposition (2.1.3). If by absurd $u = \varphi^\lambda + G^\lambda q = \tilde{\varphi}^\lambda + G^\lambda \tilde{q}$ then $0 = \varphi^\lambda - \tilde{\varphi}^\lambda = G^\lambda(\tilde{q} - q)$; but then $\tilde{q} - q = 0$ because the operator G^λ has empty kernel. □

In definition (2.1.1) it is understood that one must fix $\lambda > 0$; the role of this parameter is only to regularize the behavior of the potential at infinity in such a way that it belongs to $L^2(\mathbb{R}^{3N+3})$.

Proposition 2.1.5 *The quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ doesn't depend on λ.*

Proof

First we note that $\mathscr{D}(F_\alpha)$ doesn't depend on λ. If $u = \varphi^\lambda + G^\lambda q$ then also $u = \varphi^{\lambda'} + G^{\lambda'} q$ holds because $G^\lambda q - G^{\lambda'} q \in \mathscr{D}(F_0)$ and then we can put $\varphi^{\lambda'} \equiv \varphi^\lambda + G^\lambda q - G^{\lambda'} q$. Now we show that

$$\mathcal{F}^\lambda[u] - \mathcal{F}^{\lambda'}[u] = \Phi^{\lambda'}[u] - \Phi^\lambda[u] \tag{2.17}$$

$$\begin{aligned}\mathcal{F}^\lambda[u]-\mathcal{F}^{\lambda'}[u] = &\int dxdy\left(\frac{\hbar^2}{2M}\overline{\nabla_x\left(u-G^\lambda q\right)}\cdot\nabla_x\left(u-G^\lambda q\right)+\frac{\hbar^2}{2m}\overline{\nabla_y\left(u-G^\lambda q\right)}\cdot\nabla_y\left(u-G^\lambda q\right)\right)+\\ &-\int dxdy\left(\overline{\nabla_x\left(u-G^{\lambda'} q\right)}\cdot\nabla_x\left(u-G^{\lambda'} q\right)+\frac{\hbar^2}{2m}\overline{\nabla_y\left(u-G^{\lambda'} q\right)}\cdot\nabla_y\left(u-G^{\lambda'} q\right)\right)+\\ &+\lambda\int dxdy\overline{(u-G^\lambda q)}(u-G^\lambda q)-\lambda'\int dxdy\overline{(u-G^{\lambda'} q)}(u-G^{\lambda'} q)\\ &+\int dxdy\,\frac{1}{2}m\omega^2y^2\overline{(u-G^\lambda q)}(u-G^\lambda q)-\int dxdy\,\frac{1}{2}m\omega^2y^2\overline{(u-G^{\lambda'} q)}(u-G^{\lambda'} q)+(\lambda'-\lambda)\int dxdy|u|^2\end{aligned} \tag{2.18}$$

The first part of the r.h.s. of (2.18) can be written as:

$$\begin{aligned}
&\int dxdy \left(\frac{\hbar^2}{2M}\overline{\nabla_x(u-G^\lambda q)}\cdot\nabla_x(u-G^\lambda q)+\frac{\hbar^2}{2m}\overline{\nabla_y(u-G^\lambda q)}\cdot\nabla_y(u-G^\lambda q)\right)+\\
&\quad-\int dxdy\left(\frac{\hbar^2}{2M}\overline{\nabla_x(u-G^{\lambda'}q)}\cdot\nabla_x(u-G^{\lambda'}q)+\frac{\hbar^2}{2m}\overline{\nabla_y(u-G^{\lambda'}q)}\cdot\nabla_y(u-G^{\lambda'}q)\right)\\
&=\int dxdy\left(\frac{\hbar^2}{2M}\overline{\nabla_x(u-G^\lambda q)}\cdot\nabla_x(G^{\lambda'}q-G^\lambda q)+\frac{\hbar^2}{2m}\overline{\nabla_y(u-G^\lambda q)}\cdot\nabla_y(G^{\lambda'}q-G^\lambda q)\right)+\\
&\quad+\int dxdy\left(\frac{\hbar^2}{2M}\overline{\nabla_x(G^{\lambda'}q-G^\lambda q)}\cdot\nabla_x(u-G^{\lambda'}q)+\frac{\hbar^2}{2m}\overline{\nabla_y(G^{\lambda'}q-G^\lambda q)}\cdot\nabla_y(u-G^{\lambda'}q)\right)\\
&\quad=-\int dxdy\left(\frac{\hbar^2}{2M}\overline{(u-G^\lambda q)}\Delta_x(G^{\lambda'}q-G^\lambda q)+\frac{\hbar^2}{2m}\overline{(u-G^\lambda q)}\Delta_y(G^{\lambda'}q-G^\lambda q)\right)+\\
&\quad-\int dxdy\left(\frac{\hbar^2}{2M}\overline{\Delta_x(G^{\lambda'}q-G^\lambda q)}(u-G^{\lambda'}q)+\frac{\hbar^2}{2m}\overline{\Delta_y(G^{\lambda'}q-G^\lambda q)}(u-G^{\lambda'}q)\right)
\end{aligned}\tag{2.19}$$

The integration by parts in (2.19) is allowed since $G^\lambda q - G^{\lambda'}q \in \mathscr{D}(H_0)$.
Using the identity

$$-\frac{\hbar^2}{2M}\Delta_x(G^{\lambda'}q-G^\lambda q)-\frac{\hbar^2}{2m}\Delta_y(G^{\lambda'}q-G^\lambda q)=-\frac{1}{2}m\omega^2(y-y^0)^2(G^{\lambda'}q-G^\lambda q)+\lambda G^\lambda q-\lambda' G^{\lambda'}q$$

and (2.19), (2.18) becomes

$$\mathcal{F}^\lambda[u]-\mathcal{F}^{\lambda'}[u]=(\lambda'-\lambda)\int dxdy\,\overline{G^\lambda q}\,G^{\lambda'}q$$

Now we analyze the r.h.s. of (2.17). Let's first concentrate on diagonal terms.

$$\begin{aligned}
&\frac{1}{2}\int dydy'\left(G^{\lambda'}(y_i,y;y_i',y')-G^\lambda(y_i,y;y_i',y')\right)|q_i(y)-q_i(y')|^2+\int dy\,\left(a_i^{\lambda'}(y)-a_i^\lambda(y)\right)|q_i(y)|^2=\\
&\qquad-\int dydy'\left(G^{\lambda'}(y_i,y;y_i',y')-G^\lambda(y_i,y;y_i',y')\right)\overline{q_i(y)}q_i(y')+\\
&\quad+\int dy\left[\left(a_i^{\lambda'}(y)-a_i^\lambda\right)(y)+\int dy'\left(G^{\lambda'}(y_i,y;y_i',y')-G^\lambda(y_i,y;y_i',y')\right)\right]|q_i(y)|^2=\\
&\qquad\qquad-\int dydy'\left(G^{\lambda'}(y_i,y;y_i',y')-G^\lambda(y_i,y;y_i',y')\right)\overline{q_i(y)}q_i(y')
\end{aligned}\tag{2.20}$$

In (2.20) we have used $\big(a_i^{\lambda'}(y) - a_i^{\lambda}(y)\big) + \int dy' \big(G^{\lambda'}(y_i, y; y_i', y') - G^{\lambda}(y_i, y; y_i', y')\big) = 0$; this identity can be seen from (2.1) and (2.13). So we can write

$$\Phi^{\lambda}[u] - \Phi^{\lambda'}[u] = -\sum_{i=1}^{N} \int dy dy' \left(G^{\lambda'}(y_i, y; y_i', y') - G^{\lambda}(y_i, y; y_i', y')\right) \overline{q_i(y)} q_i(y') + \\ - \sum_{i,j=1}^{N} \int dy \overline{q_i(y)} \big((G_j^{\lambda'} - G_j^{\lambda}) q_i\big)(y) \quad (2.21)$$

The first resolvent identity applied to (2.1) and (2.21) proves (2.17). □

Proposition (2.1.3) shows that $G^{\lambda} q_i$ are the most singular part of u; the converse is also true: from the singularities of u around π_i it is possible to recover q_i. More precisely we have:

Proposition 2.1.6 *The following limit holds in $L^2(\pi_i)$.*

$$\lim_{\delta \to 0} \frac{\hbar^2}{2} \frac{m+M}{mM} \frac{1}{\delta} \int_{|x-y_i|=\delta} dx u = q_i \quad (2.22)$$

Proof
First, using Sobolev embedding theorems we prove that

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{|x-y_i|=\delta} dx u = \lim_{\delta \to 0} \frac{1}{\delta} \int_{|x-y_i|=\delta} dx G^{\lambda} q_i \quad (2.23)$$

and then it's enough to show that

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{|x-y_i|=\delta} dx G^{\lambda} q_i = \frac{2}{\hbar^2} \frac{mM}{m+M} q_i \quad (2.24)$$

Since $\varphi^{\lambda}(\cdot, y) \in H^1$ for a.e. y, its trace over $B_\delta = \{x \, s.t. \, |x - y_i| = \delta\}$ belongs to $H^{\frac{1}{2}}$; due to the embedding $H^{\frac{1}{2}}(B_\delta) \hookrightarrow L^4(B_\delta)$ we can write the following chain of inequalities

$$\int_{|x-y_i|=\delta} dx |\varphi^{\lambda}(\cdot, y)| \le c \|\varphi^{\lambda}(\cdot, y)\|_{L^4(B_\delta)} \|1\|_{L^{\frac{4}{3}}(B_\delta)} \le c \|\varphi^{\lambda}(\cdot, y)\|_{H^{\frac{1}{2}}(B_\delta)} \delta^{\frac{3}{2}} \le c \delta^{\frac{3}{2}} \|\varphi^{\lambda}(\cdot, y)\|_{H^1(\mathbb{R}^3)} \quad (2.25)$$

If we take the L^2 norm of the first and the last term in (2.25) we see that the regular part of u gives no contribute to the limit in (2.22). Since $G^{\lambda} q_j$ is a regular function on π_i apart from the intersection with π_j, which has zero measure, then (2.23) holds.

Now we introduce the following two integral operators:

$$T_\delta(y;y') = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\frac{1}{\delta}\int_{|x-y_i|=\delta}dx\int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\cdot$$
$$e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}(y^2+y'^2)}e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-y_1'|^2}e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-y'|^2} \qquad (2.26)$$

$$T'_\delta(y;y') = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\frac{1}{\delta}\int_{|x-y_i|=\delta}dx\int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\cdot$$
$$e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-y_1'|^2}e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-y'|^2} \qquad (2.27)$$

Notice that $T_\delta(y;y') \le T'_\delta(y;y')$ and that, with a straightforward integration, $\int dy' T'_\delta(y;y') \le \frac{2}{\hbar 2}\frac{mM}{m+M}$; then $\{T_\delta\}$ and $\{T'_\delta\}$ are two equibounded family of bounded operators.
In order to prove (2.24), since $\frac{1}{\delta}\int_{|x-y_i|=\delta} dx G^\lambda q_i = \int dy' T_\delta(y;y')q_1(y')$, it is sufficient to prove that T'_δ is an approximation of the unity, see [R], and that

$$\int dy'\, T_\delta(y;y') \to \frac{2}{\hbar^2}\frac{mM}{m+M} \qquad \text{uniformly on compact sets} \qquad (2.28)$$

Indeed, if these two statements holds, consider $q \in C_0^\infty$; (2.24) follow from

$$\lim_{\delta\to 0}\int dy \left|\int dy'\, T_\delta(y;y')\,(q(y)-q(y'))\right|^2 = 0 \qquad (2.29)$$

which is true since

$$\lim_{\delta\to 0}\int dy \left|\int dy'\, T_\delta(y;y')\,(q(y)-q(y'))\right|^2 \;\le\; \lim_{\delta\to 0}\int dy \left[\int dy'\, T'_\delta(y;y')\,|q(y)-q(y')|\right]^2 \;=\; 0 \qquad (2.30)$$

The second limit in (2.30) is 0 because we can interchange the limit and the integral over y by dominate convergence theorem, and because T'_δ is an approximation of the unity.
Now we prove that $T'_\delta(y;y')$ is an approximation of the identity; first with a change of variable we see that T'_δ is a convolution operator.

$$T'_\delta(y;y') = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\frac{1}{\delta}\int_{|x|=\delta}dx\int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\frac{1}{(1-\nu^2)^{\frac{3N}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{3}{2}}}\cdot$$
$$e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}(x+y_1-y_1')^2}e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}(y-y')^2} \qquad (2.31)$$

We have already noticed that the L^1 norm of the kernel has an upper bound uniform with respect to δ. With further changes of variable we see that $T'_\delta(y;y')$ can be written in the following way which resembles the usual expression for an approximation of the identity.

$$T'_\delta(y;y') = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\frac{1}{\delta^{3N}}\int_{|x|=1} dx \int_0^{\frac{1}{\delta^2}} d\nu\, \frac{(1-\delta^2\nu)^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}}{(\nu)^{\frac{3N}{2}}\,(2-\delta^2\nu)^{\frac{3N}{2}}}$$
$$\left(\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}\right)^{\frac{3}{2}} e^{-\frac{M\omega}{\hbar}\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}(x+y_1-y'_1)^2}\, e^{-\frac{m\omega}{\hbar}\frac{1-\delta^2\nu}{2-\delta^2\nu}\frac{1}{\nu}(y-y')^2} \quad (2.32)$$

The integral kernel $T'_\delta(y-y')$ is pointwise positive and its integral is a constant; using (2.31) it is straighfroward to verify that the integral outside a compact containing the origin goes to zero as delta goes to zero In order to conclude the proof, we have only to prove (2.28); using the same change of variable of (2.32) we obtain:

$$\int dyp\, T_\delta(y;y') = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\hbar\omega}\frac{1}{\pi^{\frac{3N}{2}}}\left(\frac{m\omega}{\hbar}\right)^{\frac{3N}{2}}\left(\frac{M\omega}{\hbar}\right)^{\frac{3}{2}}\int_{|x|=1} dx \int dy' \int_0^{\frac{1}{\delta^2}} d\nu\, \frac{(1-\delta^2\nu)^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}}{(\nu)^{\frac{3N}{2}}\,(2-\delta^2\nu)^{\frac{3N}{2}}}$$
$$\left(\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}\right)^{\frac{3}{2}} e^{-\frac{m\omega}{\hbar}\frac{1}{2}\frac{\delta^2\nu}{2-\delta^2\nu}\left(y^2+(\delta y+y')^2\right)} e^{-\frac{M\omega}{\hbar}\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}(x-y'_1)^2}\, e^{-\frac{m\omega}{\hbar}\frac{1-\delta^2\nu}{2-\delta^2\nu}\frac{1}{\nu}y'^2} \quad (2.33)$$

If we exchange the limit and the integrals by dominate convergence theorem and we compute the obtained integrals, we prove (2.28) □

Now we turn our attention to the most important issue of this section: the study of the closure and of the boundedness from below of the quadratic form F_α. The real object to study is Φ^λ_α, the part of F_α containing most of the information about the interaction. Once we have proved that Φ^λ_α is closed and positive it is quite straightforward proving that F_α is closed and bounded from below as we show in the next theorem.

Theorem 2.1.7 *Let Φ^λ_α be closed and positive, then F_α is closed and bounded from below.*

Proof
It is sufficient to prove that $F^\lambda_\alpha[u] = F_\alpha[u] + \lambda\|u\|^2$ is positive and closed. Positivity of $F^\lambda_\alpha[u]$ is trivial, let's discuss closure.
If $\{u_n = \varphi^\lambda_n + G^\lambda q_n\} \subset \mathscr{D}(F_\alpha)$ is a sequence converging to u in $L^2(\mathbb{R}^{3N+3})$ s.t. $\lim_{n,m} F^\lambda_\alpha[u_n - u_m] = 0$, one has

$$\lim_{n,m\to\infty}(F_0+\lambda)[\varphi^\lambda_n - \varphi^\lambda_m] = 0 \qquad \lim_{n,m\to\infty}\Phi^\lambda_\alpha[q_n - q_m] = 0$$

Since $(F_0+\lambda)[\varphi] \geq \lambda\|\varphi\|^2$ and $\|G^\lambda q\|^2 \geq c\|q\|^2$ these conditions, toegheter with $u_n \xrightarrow{L^2} u$, imply $\varphi_n^\lambda \xrightarrow{L^2} \varphi^\lambda$ and $q_n \xrightarrow{L^2} q$.
By the closure of F_0 and of Φ_α^λ, it follows that $\varphi^\lambda \in \mathscr{D}(F_0)$, $q \in \mathscr{D}(\Phi_\alpha^\lambda)$ and

$$\lim_{n\to\infty}(F_0+\lambda)[\varphi_n^\lambda - \varphi^\lambda] = 0 \qquad \lim_{n\to\infty}\Phi_\alpha^\lambda[q_n - q] = 0$$

It is clear then that $u \in \mathscr{D}(F_\alpha)$ and $F_\alpha^\lambda[u_n - u] \to 0$. □

Now we prove the positivity and the closure of Φ_α^λ when $N = 1$.

Lemma 2.1.8 *Let $N = 1$, if $\lambda \geq \lambda_0$ then Φ_α^λ is closed and positive.*

Proof
If $N = 1$, Φ_α^λ reduces to:

$$\Phi^\lambda[q] = \alpha\int dy|q(y)|^2 + \int dy\, a^\lambda(y)|q(y)|^2 + \frac{1}{2}\int dydy' G^\lambda(y,y;y',y')|q(y)-q(y')|^2$$

If α is non negative then Φ_α^λ is positive since $a^\lambda(y) > 0$; if α is negative it sufficient to observe that $\lim_{\lambda\to+\infty}\inf_y a^\lambda(y) = +\infty$: if we take λ such that $\inf_y a^\lambda + \alpha > c$ then $\Phi_\alpha^\lambda[q] > c\|q\|^2$.
In order to prove the closure of Φ_α^λ we can mimic the Fischer-Riesz proof of the completeness of L^2. Given a Cauchy sequence $\{q_n\} \subset \mathscr{D}(\Phi_\alpha^\lambda)$ we can pick a subsequence, still denoted by $\{q_n\}$, such that $\Phi_\alpha^\lambda[q_n - q_{n+1}] < 2^{-n}$ and prove the convergence of such a subsequence.
Let's consider $Q_N(y) = \sum_{i=1}^N |q_i(y) - q_{i+1}(y)|$ and $Q(y) = \sum_{i=1}^\infty |q_i(y) - q_{i+1}(y)|$; by the monotone convergence theorem we have $Q \in L^2(\mathbb{R}^3)$ and $Q < +\infty$ *a.e.*.
In order to prove that $Q \in \mathscr{D}(\Phi_\alpha^\lambda)$ we notice that $\Phi_\alpha^\lambda[Q_N]$ is uniformly bounded with respect to N, indeed one has:

$$\sqrt{\Phi_\alpha^\lambda[Q_N]} \leq \sum_{i=1}^N \sqrt{\Phi_\alpha^\lambda[|q_i - q_{i+1}|]} \leq c$$

By the Fatou lemma we get $Q \in \mathscr{D}(\Phi_\alpha^\lambda)$.
Then $q_1 + \sum_i (q_{i+1} - q_i)$ is absolutely convergent to a sum q with $|q - q_1| \leq Q$ so that $q \in \mathscr{D}(\Phi_\alpha^\lambda)$.
It is straightforward to show that $\Phi_\alpha^\lambda[q - q_n] \to 0$. □

We state the main result of this section which follows from (2.1.7) and (2.1.8) for completeness.

Theorem 2.1.9 *For $\lambda > \lambda_0$ and $N = 1$ the quadratic form F_α is closed and bounded from below.*

If $N \geq 2$ the positivity and the closure of Φ_α^λ are not obvious: we can divide Φ_α^λ into diagonal terms $\Phi_{\alpha,diag}^\lambda$ and non diagonal terms $\Phi_{\alpha,ndiag}^\lambda$

$$\Phi_{\alpha,diag}^\lambda[q] = \alpha\sum_{i=1}^N\int dy|q_i(y)|^2 + \sum_{i=1}^N\int dy\, a_i^\lambda(y)|q_i(y)|^2 + \frac{1}{2}\sum_{i=1}^N\int dydy' G^\lambda(y_i,y;y_i',y')|q_i(y)-q_i(y')|^2$$

$$\Phi^{\lambda}_{\alpha,ndiag}[q] = -2\Re \sum_{i<j} \int dy \overline{q_i(y)} (G^{\lambda}_j q_i)(y)$$

It's clear, by the same argument used in the previous lemma, that for $\lambda > \lambda_0$ diagonal terms are closed and positive; non diagonal terms don't have a definite sign.
In order to prove that Φ^{λ}_{α} is positive one should prove that the diagonal terms dominate, in some sense to be precised, the non diagonal terms.
There are physical models with zero range interactions, in which this vague condition is not satisfies. It is well known, see [Mi] [DFT] [MF1] [MF2], that the three body problem with delta potentials has an unbounded from below hamiltonian; that is if one consider the local and self-adjoint extension of $\tilde{H}_0$ defined by

$$\mathscr{D}(\tilde{H}_0) = \left\{ u \in C_0^{\infty}(\mathbb{R}^9 \setminus \Sigma),\, \Sigma = \cup_{i<j} \sigma_{ij},\, \sigma_{ij} = \{(x_1, x_2, x_3) \in \mathbb{R}^9 \; s.t. \; x_i = x_j\} \right\}$$

$$\tilde{H}_0 u = \left(-\frac{\hbar^2}{2m_1} \Delta_{x_1} - \frac{\hbar^2}{2m_2} \Delta_{x_2} - \frac{\hbar^2}{2m_3} \Delta_{x_3} \right) u$$

they are all unbounded from below.
In [DFT] [M], it has been proved that also a system composed of a test particle interacting with N free particles with delta potentials has an unbounded from below hamiltonian if N is large enough.
It's clear that in these models diagonal terms don't dominate the non diagonal terms otherwise we would obtain a semibounded operator.
Nevertheless we think that Φ^{λ}_{α} is positive for any N, at least for suitable values of the physical parameters. We give only some remarks in this direction: in [DFT], a sequence of states $\{u_n = G^{\lambda} q_n\}$ such that $\widehat{F_\alpha}[G^{\lambda} q_n] \to -\infty$ for $n \to +\infty$, has been constructed for the three body problem, $\widehat{F_\alpha}$ is the quadratic form corresponding to F_α. Such a sequence is not admissible for Φ^{λ}_{α}, i.e. $q_n \notin \mathscr{D}(\Phi^{\lambda}_{\alpha})$: this is due to the slow decay at infinity of q_n. The functions a^{λ}_i diverge at infinity with a power law:

$$a^{\lambda}_i(y) \simeq_{y \to \infty} \frac{1}{(2\pi)^{\frac{3}{2}} \hbar^3} m^{\frac{1}{2}} \left(\frac{mM}{m+M} \right)^{\frac{3}{2}} \left(\int_0^{+\infty} d\nu \, \frac{1}{\nu^{\frac{3}{2}}} (1 - e^{-\frac{\nu}{2}}) \right) \omega |y|$$

Then a necessary condition for $q \in \mathscr{D}(\Phi^{\lambda}_{\alpha})$ is $|y|^{\frac{1}{2}} q(y) \in L^2$. This condition is not satisfied for the sequence $\{q_n\}$.
The divergence at infinity of a^{λ}_i is due to presence of the harmonic potential: in the three body problem no such a condition on the behavior of the charges at infinity is required.
We expect that, in a regime where the harmonic potential gives a large contribute to the energy of the state, i.e. large frequency limit or large separation of the equilibrium position limit, the quadratic form Φ^{λ}_{α} should be positive: we expect the non diagonal terms to become small in Kato sense with respect to diagonal ones; the taking big enough λ, Φ^{λ}_{α} should become positive.

2.2 The Operator and the Resolvent

In this section we construct, in the case $N = 1$, the operator H_α and the resolvent $(H_\alpha + \lambda)^{-1}$. Let's call Γ_α^λ the positive self adjoint operator associated to Φ_α^λ.

Theorem 2.2.1 *The domain and the action of H_α are the following:*

$$\mathscr{D}(H_\alpha) = \left\{ u \in L^2(\mathbb{R}^6)\, s.t.\, u = \varphi^\lambda + G^\lambda q,\ \varphi^\lambda \in \mathscr{D}(H_0), q \in \mathscr{D}(\Gamma_\alpha^\lambda), \Gamma_\alpha^\lambda q = Tr_\pi \varphi^\lambda \right\}$$

$$(H_\alpha + \lambda)u = (H_0 + \lambda)\varphi^\lambda$$

The resolvent has the following representation:

$$(H_\alpha + \lambda)^{-1} f = G^\lambda f + G^\lambda q \qquad q \text{ solution of } \Gamma_\alpha^\lambda q = Tr_\pi G^\lambda f$$

Proof
A function u belongs to $\mathscr{D}(H_\alpha)$ if and only if it exists $w \in L^2$, which is by definition $H_\alpha u$, such that

$$F_\alpha[u, v] = (w, v) \qquad \forall v \in \mathscr{D}(F_\alpha) \tag{2.34}$$

In particular for $v \in C_0^\infty$, which means $q_v \equiv 0$, (2.34) reduces to:

$$\int dxdy \left[\frac{\hbar^2}{2M} \overline{\partial_x \varphi_u^\lambda} \partial_x v + \frac{\hbar^2}{2m} \overline{\partial_y \varphi_u^\lambda} \partial_y v + \frac{1}{2} m\omega^2 (y - y^0)^2 \overline{\varphi_u^\lambda} v + \lambda \overline{\varphi_u^\lambda} v \right] = \int dxdy \overline{(H_\alpha + \lambda)u}\, v \tag{2.35}$$

Equation (2.35) implies

$$\varphi_u^\lambda \in \mathscr{D}(H_0) \qquad (H_\alpha + \lambda)u = (H_0 + \lambda)\varphi_u^\lambda \tag{2.36}$$

If we now take a generic element of the form domain as test function we obtain:

$$\int dxdy \left[\frac{\hbar^2}{2M} \overline{\partial_x \varphi_u^\lambda} \partial_x \varphi_v^\lambda + \frac{\hbar^2}{2m} \overline{\partial_y \varphi_u^\lambda} \partial_y \varphi_v^\lambda + (\frac{1}{2} m\omega^2 (y - y^0)^2 + \lambda) \overline{\varphi_u^\lambda} \varphi_v^\lambda \right] + \\ + \Phi_\alpha^\lambda [q_u, q_v] = \int dxdy \overline{(H_\alpha + \lambda)u} v$$

Using (2.36), we obtain

$$\Phi_\alpha^\lambda [q_u, q_v] = \int dxdy \overline{(H_0 + \lambda)\varphi_u^\lambda}\, G^\lambda q_v$$

which is equivalent to

$$\Phi_\alpha^\lambda [q_u, q_v] = \int dxdy\, \overline{Tr_\pi \varphi_u^\lambda}\, q_v$$

The previous equation says

$$q_u \in \mathscr{D}(\Gamma_\alpha^\lambda) \qquad \Gamma_\alpha^\lambda q_u = Tr_\pi \varphi_u^\lambda \tag{2.37}$$

Equation (2.37) is a boundary condition, which must be satisfied by the elements of the domain. The boundary condition can be written also as $q_u = \left(\Gamma_\alpha^\lambda\right)^{-1} Tr_\pi \varphi_u^\lambda$; the advantage of this reformulation is that $\left(\Gamma_\alpha^\lambda\right)^{-1}$ is a bounded operator and then it's clear that for any φ_u^λ, q_u is well defined and continuously depends on φ_u^λ. This statement follows from lemma (2.1.8) when we observed that for $\lambda > \lambda_0$ one has $\Phi_\alpha^\lambda[q] > c\|q\|^2$

□

Chapter 3

The Two Dimensional Case

In this chapter we introduce the quadratic form in dimension two. The situation is similar to the three dimensional case: a renormalization of the quadratic form is still required in order to achieve a meaningful quadratic form since we are not concerned with a bounded perturbation of F_0.
In section we introduce F_α; propositions (3.1.2) (3.1.3) (3.1.4) and (3.1.5) proves that definition (3.1.1) is well put. Closure and semiboundednes of the quadratic form are proved for any N since in the two dimensional case we can give an L^2 estimate of non diagonal term of Φ_α^λ, see lemma (3.1.7). In section two, theorem (3.2.1) characterizes the domain and the action of the operator.

3.1 The Quadratic Form

In this section we introduce the quadratic form F_α in two dimension and we prove that it is closed and bounded from below for any N. The definition of F_α is similar to the three dimanional case.

Definition 3.1.1

$$\mathscr{D}(F_\alpha) = \big\{ u \in L^2(\mathbb{R}^{2N+2})\; s.t.\, \exists q \in \mathscr{D}(\Phi^\lambda)\,,\, \varphi^\lambda \equiv u - G^\lambda q \in \mathscr{D}(F_0) \big\}$$

$$F_\alpha[u] = \mathcal{F}^\lambda[u] + \Phi_\alpha^\lambda[u]$$

$$\mathcal{F}^\lambda[u] = \int dxdy \left\{ \frac{\hbar^2}{2M} |\nabla_x \varphi^\lambda|^2 + \frac{\hbar^2}{2m} |\nabla_y \varphi^\lambda|^2 + \lambda |\varphi^\lambda|^2 - \lambda |u|^2 + \frac{1}{2} m\omega^2 (y - y^0)^2 |\varphi^\lambda|^2 \right\}$$

$$\mathscr{D}(\Phi_\alpha^\lambda) = \big\{ (q_1, \ldots, q_N),\, q_i \in L^2(\pi_i)\, s.t.\, \Phi_\alpha^\lambda[q] < +\infty \big\}$$

$$\Phi^\lambda_\alpha[q] = \alpha \sum_{i=1}^{N} \int dy |q_i(y)|^2 - 2\Re \sum_{i<j} \int dy \overline{q_i(y)} (G^\lambda_i q_j)(y) +$$
$$+ \sum_{i=1}^{N} \int dy\, a^\lambda_i(y) |q_i(y)|^2 + \frac{1}{2} \sum_{i=1}^{N} \int dy dy' G^\lambda(y_i, y; y'_i, y') |q_i(y) - q_i(y')|^2$$

In the two dimensional case the functions $a^\lambda_i(y)$ have the following expression:

$$a^\lambda_i(y) = \frac{1}{(2\pi)} \frac{1}{\hbar^2} \frac{mM}{m+M} \Big\{ C + \int_0^1 d\nu \frac{1}{(1-\nu)^2} \Big[1 - \nu^{\frac{\lambda}{\hbar\omega}+N-1} \left(\frac{2}{1+\nu^2} \right)^{(N-1)}$$
$$\left(\frac{m+M}{\frac{1+\nu^2}{2} \frac{\ln\frac{1}{\nu}}{1-\nu} m + \frac{1+\nu}{2} M} \right) \exp\left(-\frac{1}{2} \frac{1-\nu^2}{1+\nu^2} \frac{m\omega}{\hbar} \sum_{\substack{j=1 \\ j\neq i}}^{N} |y_j - y^0_j|^2 \right)$$
$$\exp\left(-\frac{\omega}{\hbar} \frac{\frac{1}{4} m^2 + \frac{1-\nu}{1+\nu} \frac{1}{2\ln\frac{1}{\nu}} mM}{\frac{1}{2} \frac{1+\nu^2}{1-\nu^2} m + \frac{1}{2\ln\frac{1}{\nu}} M} |y_i - y^0_i|^2 \right) \Big] \Big\} \tag{3.1}$$

$$C = -\left(\int_0^1 d\nu\, \frac{1}{\nu} e^{-\frac{1}{\nu}} + \int_1^\infty d\nu\, \frac{1}{\nu^2} \ln\frac{1}{\nu} e^{-\frac{1}{\nu}} \right)$$

The heuristic argument at the basis of definition (3.1.1) is the same of the three dimensional case: the potential generated by a charge distribution is still divergent near the planes π_i and then a renormalization of the quadratic form is required. Nevertheless the singularity of the potential is of logarithmic type, much weaker than the coulombian type of the previous case; as we shall see later this fact makes easier the study of the closure of the quadratic form.
The analysis of $a^\lambda_i(y)$ follows the same line of the three dimensional case. To be concrete we fix $i = 1$, put $y^0 = 0$ and introduce the coordinates ξ and η.

$$\begin{cases} \xi = \frac{1}{\sqrt{2}}(x - y_1) \\ \eta = \frac{1}{\sqrt{2}}(x + y_1) \end{cases}$$

With this choice π_1 is defined by $\xi = 0$. We have to study the behavior for small ξ of

$$\left(G^\lambda 1_\varepsilon\right)(\xi, \eta, y_2, \dots, y_N) = \int_{U_\varepsilon} d\eta' dy'_2 \dots dy'_N\, G^\lambda(\xi, \eta, y_2, \dots, y_N; 0, \eta', y'_2, \dots, y'_N) \tag{3.2}$$

The shape of U_ε doesn't matter, indeed a^λ_i doesn't depend on ε; we choose

$$U_\varepsilon = \{ |\eta' - \eta| < \varepsilon, |y'_2 - y_2| < \varepsilon, \dots, |y'_N - y_N| < \varepsilon \}$$

From (A.1) one has that, with simple manipulations, (3.2) can be written as:

$$\left(G^\lambda 1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) = \frac{1}{2\pi}\frac{1}{\hbar\omega}\frac{1}{\pi^N}\left(\frac{m\omega}{\hbar}\right)^N\left(\frac{M\omega}{\hbar}\right)\int_0^1 d\nu \frac{\nu^{\frac{\lambda}{\hbar\omega}+N-1}}{(1+\nu)^N}\frac{1}{(1-\nu)^N(\log\frac{1}{\nu})}$$
$$\prod_{i=2}^N\left\{e^{-\frac{m\omega}{\hbar}\frac{1-\nu}{1+\nu}y_i^2}\left(\frac{m\omega}{\hbar}\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\right)^{-1}\int_{|y_i'|<\varepsilon\left(\frac{m\omega}{\hbar}\frac{(1+\nu^2)}{2(1-\nu^2)}\right)^{\frac{1}{2}}} dy_i'\, e^{-y_i'^2}e^{-y_i\cdot y_i'\left(\frac{m\omega}{\hbar}\frac{2(1-\nu^2)}{(1+\nu^2)}\right)^{\frac{1}{2}}\frac{1-\nu}{1+\nu}}\right\}$$
$$e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}(\eta^2-\eta\cdot\xi)}e^{-\xi^2\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)}\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}+\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)^{-1}$$
$$\int_{|\eta'|<\varepsilon\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)^{\frac{1}{2}}} d\eta' e^{-\eta'^2}\exp\left\{\frac{2\eta'\cdot\left[\left(\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}\right)\xi+\frac{1}{4}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\eta\right]}{\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{m\omega}{\hbar}\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)^{\frac{1}{2}}}\right\} \quad (3.3)$$

With the changes of variable $\nu = 1-\nu'$ and $\nu'\xi^2 = \nu''$ we recover a logarithmic singularity for the potential:

$$\left(G^\lambda 1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) \simeq_{\xi\to 0} -\frac{1}{\pi}\frac{1}{\hbar^2}\frac{mM}{m+M}\ln\left(\xi^2\frac{\omega}{\hbar}\frac{mM}{m+M}\right) \quad (3.4)$$

If we subtract (3.3) from (3.4) and we add $\int_{|y'-y|>\varepsilon} dy' G^\lambda(y_i,y;y_i',y')$, we obtain (3.1).
The proof that (2.1.1) is well defined is divided into several propositions; the proofs differ from the three dimensional case only by minor modifications.
First we prove that the decomposition $u = \varphi^\lambda + G^\lambda q$ is well defined and unique.

Proposition 3.1.2 *If $q_i \in L^2(\mathbb{R}^{2N})$ then $G^\lambda q_i \in L^2(\mathbb{R}^{2N+2})$.*

Proof
Let's fix $i = 1$.

$$\int dx\, dy|G^\lambda q_1(x,y)|^2 = \frac{1}{(2\pi)^2}\frac{1}{(\hbar\omega)^2}\frac{1}{\pi^{2N}}\left(\frac{m\omega}{\hbar}\right)^{2N}\left(\frac{M\omega}{\hbar}\right)^2$$
$$\int dx dy\int dz\,\overline{q_1(z)}\int_0^1 d\nu\,\nu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}e^{-\frac{m\omega}{\hbar}\frac{1-\nu}{2(1+\nu)}(y^2+z^2)}e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-z_1|^2}$$
$$e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-z|^2}\int dw\, q_1(w)\int_0^1 d\mu\,\mu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)}e^{-\frac{m\omega}{\hbar}\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}$$
$$e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}|x-w_1|^2}e^{-\frac{m\omega}{\hbar}\frac{\mu}{1-\mu^2}|y-w|^2}$$

$$= \frac{1}{(2\pi)^2}\frac{1}{(\hbar\omega)^2}\frac{1}{\pi^{2N}}\left(\frac{m\omega}{\hbar}\right)^{2N}\left(\frac{M\omega}{\hbar}\right)^2\int dz dw\, \overline{q_1}(z) q_1(w)$$
$$\int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)}\int_0^1 d\mu\, \mu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)}$$
$$\int dx dy e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-z|^2} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-z_1|^2} e^{-\frac{m\omega}{\hbar}\frac{1-\nu}{2(1+\nu)}(y^2+z^2)}$$
$$e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}|x-w_1|^2} e^{-\frac{m\omega}{\hbar}\frac{\mu}{1-\mu^2}|y-w|^2} e^{-\frac{m\omega}{\hbar}\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}$$

It's then sufficient to show that

$$T(z,w) = \int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}\int_0^1 d\mu\, \mu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)}$$
$$\int dx dy\, e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}|y-z|^2} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}|x-z_1|^2} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}|x-w_1|^2} e^{-\frac{m\omega}{\hbar}\frac{\mu}{1-\mu^2}|y-w|^2}$$

is the integral kernel of a bounded operator in $L^2(\mathbb{R}^{2N})$. This kernel can be pointwise estimated in the following way:

$$T(z,w) \le \left(\frac{M\omega}{\hbar}\right)^{-1}\left(\frac{m\omega}{\hbar}\right)^{-N}\pi^{N+1}\int_0^1 d\nu d\mu\left(\frac{\nu}{1-\nu^2}+\frac{\mu}{1-\mu^2}\right)^{-(N+1)}\nu^{\frac{\lambda}{\hbar\omega}+N-1}$$
$$\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}\mu^{\frac{\lambda}{\hbar\omega}+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)} e^{-\frac{1}{4\left(\frac{1-\nu^2}{\nu}+\frac{1-\mu^2}{\mu}\right)}\frac{m\omega}{\hbar}|w-z|^2} \tag{3.5}$$

Using the Schur's test, and (3.5) we have

$$\|T\| \le \sup_w \int dw\, T(z,w) \le \left(\frac{M\omega}{\hbar}\right)^{-1}\pi^{N+1}\int_0^1 d\nu d\mu\, \nu^{\frac{\lambda}{\hbar\omega}-1}\mu^{\frac{\lambda}{\hbar\omega}-1}\frac{1}{\left(\ln\frac{1}{\nu}\right)}\frac{1}{\left(\ln\frac{1}{\mu}\right)}$$
$$\left(\frac{\nu}{1-\nu^2}+\frac{\mu}{1-\mu^2}\right)^{-1} \le \left(\frac{M\omega}{\hbar}\right)^{-1}\pi^{N+1}\int_0^1 d\nu d\mu\, \nu^{\frac{\lambda}{\hbar\omega}-1}\mu^{\frac{\lambda}{\hbar\omega}-1}\frac{1}{\left[\frac{\nu}{1+\nu}\frac{\ln\frac{1}{\nu}}{1-\nu}\ln\frac{1}{\mu}+\frac{\mu}{1+\mu}\frac{\ln\frac{1}{\mu}}{1-\mu}\ln\frac{1}{\nu}\right]} < +\infty$$

and this concludes the proof. □

Proposition 3.1.3 *If $q_i \in \mathscr{D}(\Phi_\alpha^\lambda)$ then $G^\lambda q_i \notin \mathscr{D}(F_0)$.*

Proof

Fix $i = 1$ and put $y^0 = 0$. Since $G^\lambda q_1$ is divergent near π_1, we shall consider a regularized version of F_0 restricting the integration to the whole space minus a small cylinder around π_1 and, with integrations by parts, we shall show that it diverges when the small cylinder shrinks to π_1.

Let's show that the following identity holds:

$$\int_{\mathbb{R}^{2N+2}\setminus\pi_1^\delta} dxdy \left\{ \frac{\hbar^2}{2M}|\nabla_x G^\lambda q_1|^2 + \frac{\hbar^2}{2m}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}m\omega^2 y^2 + \lambda\right)\left|G^\lambda q_1\right|^2 \right\} =$$
$$= -\frac{1}{2}\int dydy'|q_1(y) - q_1(y')|^2 \int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'', y''; y_1, y)\frac{\partial G^\lambda}{\partial n}(x'', y''; y_1', y') +$$
$$+ \int_{\partial\pi_1^\delta} dxdy \left(G^\lambda 1 - \zeta_1\right)\frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right) + \int_{\partial\pi_1^\delta} dxdy\, \zeta_1 \frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right) \quad (3.6)$$

Formula (3.6) is equivalent to:

$$\int_{\mathbb{R}^{2N+2}\setminus\pi_1^\delta} dxdy \left\{ \frac{\hbar^2}{2M}|\nabla_x G^\lambda q_1|^2 + \frac{\hbar^2}{2m}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}m\omega^2 y^2 + \lambda\right)\left|G^\lambda q_1\right|^2 \right\} =$$
$$= -\frac{1}{4}\int dydy'|q_1(y) - q_1(y')|^2 \left(\int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'', y''; y_1, y)\frac{\partial G^\lambda}{\partial n}(x'', y''; y_1', y') +\right.$$
$$\left. + \int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'', y''; y_1', y')\frac{\partial G^\lambda}{\partial n}(x'', y''; y_1, y)\right) + \int_{\partial\pi_1^\delta} dxdy\, G^\lambda 1 \frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right) \quad (3.7)$$

If we further simplify (3.7), we obtain

$$\int_{\mathbb{R}^{2N+2}\setminus\pi_1^\delta} dxdy \left\{ \frac{\hbar^2}{2M}|\nabla_x G^\lambda q_1|^2 + \frac{\hbar^2}{2m}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}m\omega^2 y^2 + \lambda\right)\left|G^\lambda q_1\right|^2 \right\} =$$
$$= \int_{\partial\pi_1^\delta} dxdy\, \overline{G^\lambda q_1}\frac{\partial G^\lambda}{\partial n}(q_1) \quad (3.8)$$

which is true by Stoke's theorem and because we have removed π_1^δ from the domain of integration.

In order to prove that $F_0[G^\lambda q_1] = +\infty$ we shall consider the r.h.s. of (3.6) and we shall prove the following steps: the sum of first and the second term converge, up to multiplicative constant, to $-\frac{4}{\hbar 2}\frac{mM}{m+M}\Phi_\alpha^\lambda[q]$ as $\delta \to 0$ while the third term diverges; since by proposition (3.1.2), the l.h.s. of (3.6) differs from $F_0[G^\lambda q_1]$ only by a bounded term it's clear that $F_0[G^\lambda q_1] = +\infty$.

Now we show that

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta} dxdy \left(G^\lambda 1 - \zeta_1\right)(x, y)\frac{\partial G^\lambda}{\partial n}(x, y; y_1', y') = -\frac{4}{\hbar^2}\frac{mM}{m+M}a_1^\lambda(y')$$

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta} dx'' dy'' G^\lambda(x'',y'';y_1,y)\frac{\partial G^\lambda}{\partial n}(x'',y'';y_1',y') = \frac{4}{\hbar^2}\frac{mM}{m+M}G^\lambda(x'',y'';y_1',y')$$

In this case $\frac{\partial}{\partial n} = -\frac{\partial}{\partial|\xi|}$, then the integral kernel of $\frac{\partial G^\lambda}{\partial n}$ can be easily obtained from (A.1).

$$\frac{\partial G^\lambda}{\partial n}(\xi,\eta,y_2,\ldots,y_n;y_1',y') = \frac{1}{2\pi}\frac{1}{\hbar\omega}\frac{1}{\pi^N}\left(\frac{m\omega}{\hbar}\right)^N\left(\frac{M\omega}{\hbar}\right)\int_0^1 d\nu\,\frac{\nu^{\frac{\lambda}{\hbar\omega}+N-1}}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)}$$
$$e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\left[\frac{1}{2}(\eta-\xi)^2+y_2^2+\ldots y_N^2+y'^2\right]}\,e^{-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\left(|\frac{1}{\sqrt{2}}(\eta-\xi)-y_1'|^2+|y_2-y_2'|^2+\ldots+|y_N-y_N'|^2\right)}\,e^{-\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}|\frac{1}{\sqrt{2}}(\eta+\xi)-y_1'|^2}$$
$$\left\{\left[|\xi|+\sqrt{2}\frac{\xi}{|\xi|}\cdot\left(\frac{1}{\sqrt{2}}\eta+y_1'\right)\right]\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\left[|\xi|+\sqrt{2}\frac{\xi}{|\xi|}\cdot\left(\frac{1}{\sqrt{2}}\eta-y_1'\right)\right]\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right\} \quad (3.9)$$

The potential $G^\lambda 1$ and its divergent part ζ_1 have the following expressions:

$$\left(G^\lambda 1\right)(\xi,\eta,y_2,\ldots,y_n) = \frac{1}{2\pi}\frac{1}{\hbar\omega}\frac{m\omega}{\hbar}\frac{M\omega}{\hbar}\int_0^1 d\mu\,\frac{1}{(1-\mu^2)\left(\ln\frac{1}{\mu}\right)}\mu^{\frac{\lambda}{\hbar\omega}+\frac{3N}{2}-1}\left(\frac{2}{1+\mu^2}\right)^{N-1}$$
$$\left(\frac{1+\mu^2}{2}\frac{\ln\frac{1}{\mu}}{1-\mu}m+\frac{1+\mu}{2}M\right)^{-1}\exp\left(-\frac{1}{2}\frac{1-\mu^2}{1+\mu^2}\frac{m\omega}{\hbar}\sum_{j=2}^N y_j^2\right)\exp\left(-\frac{\omega}{\hbar}\frac{\frac{1}{4}m^2+\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}mM}{\frac{1}{2}\frac{1+\mu^2}{1-\mu^2}m+\frac{1}{2\ln\frac{1}{\mu}}}\frac{1}{2}\eta^2\right)$$
$$\exp\left(-4\frac{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\xi^2\right)\exp\left(-\frac{\frac{1}{8}\left(\frac{m\omega}{\hbar}\right)^2}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\eta\cdot\xi\right)$$
$$\zeta_1 = \frac{1}{\pi}\frac{1}{\hbar^2}\frac{mM}{m+M}\ln\left(\frac{1}{\xi^2}\frac{\hbar}{\omega}\frac{m+M}{mM}\right)$$

It is useful to write $G^\lambda 1-\zeta_1$ in the following way:

$$\left(G^\lambda 1-\zeta_1\right)(\xi,\eta,y_2\ldots y_N) = \frac{1}{\pi}\frac{1}{\hbar^2}\frac{mM}{m+M}\left\{C+\int_0^1 d\mu\,\frac{1}{(1-\mu)^{\frac{3}{2}}}\left[\mu^{\frac{\lambda}{\hbar\omega}+N-1}\left(\frac{2}{1+\mu^2}\right)^{N-1}\right.\right.$$
$$\frac{1}{\left(\frac{1+\mu^2}{2}\frac{\ln\frac{1}{\mu}}{1-\mu}m+\frac{1+\mu}{2}M\right)}\exp\left(-\frac{1}{2}\frac{1-\mu^2}{1+\mu^2}\frac{m\omega}{\hbar}\sum_{j=2}^N y_j^2\right)\exp\left(-\frac{\omega}{\hbar}\frac{\frac{1}{4}m^2+\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}mM}{\frac{1}{2}\frac{1+\mu^2}{1-\mu^2}m+\frac{1}{2\ln\frac{1}{\mu}}}\frac{1}{2}\eta^2\right)$$
$$\left.\left.\exp\left(-4\frac{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\xi^2\right)\exp\left(-\frac{\frac{1}{2}\frac{1-\mu}{1+\mu}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}}{\frac{1}{4\ln\frac{1}{\mu}}\frac{M\omega}{\hbar}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}\frac{m\omega}{\hbar}}\eta\cdot\xi\right)-\exp\left(\frac{1}{1-\mu}\frac{mM}{m+M}\frac{\omega}{\hbar}\xi^2\right)\right]\right\}$$
$$(3.10)$$

Equation (3.10) shows that $a_1^\lambda(y) = -2\left(G^\lambda 1 - \zeta_1\right)(0, \eta, y_2 \dots y_N)$, the numerical factor 2 is absorbed in (3.1) by a change of variable.
The integral kernel $\frac{\partial G^\lambda}{\partial n}$ has the following property:

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta} dx dy f(x,y)\frac{\partial G^\lambda}{\partial n}(x,y;y_1',y') = \frac{4}{\hbar^2}\frac{mM}{m+M}f(y_1',y') \tag{3.11}$$

If we prove (3.11) when f is $(G^\lambda - \zeta_1)(x,y)$ or $G^\lambda(x,y;y_1'',y'')$ we conclude the first step of the proof; if we interchange the integral over the parameter μ with the limit over δ and the integral over (x,y) we can restrict ourselves to prove (3.11) when f is a gaussian. Now it's only a matter of long calculations to prove that:

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta} d\xi d\eta dy_2\dots dy_N e^{-\frac{\omega}{\hbar}(a\xi^2+b\eta^2+c\eta\cdot\xi)}e^{-\frac{\omega}{\hbar}d(y_2^2+\dots+y_N^2)}\frac{\partial G^\lambda}{\partial n}(\xi,\eta,y_2,\dots,y_n;y_1',y')$$
$$= \frac{4}{\hbar^2}\frac{mM}{m+M}e^{-\frac{\omega}{\hbar}b\eta'^2}e^{-\frac{\omega}{\hbar}d(y_2'^2+\dots+y_N'^2)}$$

Using (3.9) and integrating over $\eta, y_2, \dots, y_N$, we have to study

$$\lim_{\delta\to 0}\frac{1}{\pi}\frac{1}{\hbar\omega}\frac{1}{\pi^N}\left(\frac{m\omega}{\hbar}\right)^N\left(\frac{M\omega}{\hbar}\right)\int_0^1 d\nu\,\frac{\nu^{\frac{\lambda}{\hbar\omega}+N-1}}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)}\left[\frac{1}{\pi}\frac{\omega}{\hbar}\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}m+d\right)\right]^{-(N-1)}$$
$$\exp\left(-\frac{\omega}{\hbar}(y_2'+\dots y_N'^2)\frac{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}md+\frac{1}{4}m^2}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}m+d}\right)\left(\frac{1}{4}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{4\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\frac{\omega}{\hbar}b\right)^{\frac{3}{2}}$$
$$\exp\left\{-\frac{1}{2}\delta^2\left[\frac{\frac{1+\nu^2}{1-\nu^2}\frac{1}{\ln\frac{1}{\nu}}+2\frac{\omega}{\hbar}b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)+2\frac{\omega}{\hbar}c\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}-\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)+\left(\frac{\omega}{\hbar}c\right)^2}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right]\right\}$$
$$\exp\left\{-y_1'^2\left[\frac{\frac{1}{4}\left(\frac{m\omega}{\hbar}\right)^2+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}+2\frac{\omega}{\hbar}b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{M\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right]\right\}$$
$$\int_{|\xi|=\delta}d\xi\exp\left\{\sqrt{2}\xi\cdot y_1'\left[\frac{\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)+\frac{\omega}{\hbar}b\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right]\right\}$$
$$\left\{|\xi|\frac{4\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}\right)+\frac{\omega}{\hbar}c\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}+\right.$$
$$\left.-\sqrt{2}\frac{\xi}{|\xi|}\cdot y_1'\frac{\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+\frac{\omega}{\hbar}b\left(\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{m\omega}{\hbar}+\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}+2\frac{\omega}{\hbar}b}\right\} \tag{3.12}$$

Making the change of variables $\nu = \nu'$ and $\nu' = \delta^2\nu''$ when $\delta \to 0$ only the most singular terms of (3.12) survives, in particular the surface integral becomes trivial; then we obtain

$$\frac{4}{\hbar\omega}\left(\frac{\omega}{\hbar}\right)^2\left(\frac{mM}{m+M}\right)^2\int_0^{+\infty} d\nu \frac{1}{\nu^2} e^{-\frac{1}{\nu}\frac{\omega}{\hbar}\frac{mM}{m+M}} e^{-2\frac{\omega}{\hbar}by_1'^2} e^{-\frac{\omega}{\hbar}d(y_2'^2+\ldots+y_N'^2)} = \frac{4}{\hbar^2}\frac{mM}{m+M} e^{-2\frac{\omega}{\hbar}by_1'^2} e^{-\frac{\omega}{\hbar}d(y_2'^2+\ldots+y_N'^2)}$$

Since on π_1 $\eta = \sqrt{2}y_1$ we have proved (3.11).
The third term on the r.h.s. of (2.9) is equal to:

$$\frac{1}{\pi}\frac{1}{\hbar^2}\frac{mM}{m+M}\ln\left(\frac{1}{\delta^2}\frac{\hbar}{\omega}\frac{m+M}{mM}\right)\int_{\partial\pi_1^\delta}\frac{\partial G^\lambda}{\partial n}(|q_1|^2) \tag{3.13}$$

because ζ_1 is a function of ξ only. Equation (3.11), considering constant as a particular case of gaussian functions, also implies that

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta}\frac{\partial G^\lambda}{\partial n}(|q_1|^2) = \frac{1}{\hbar^2}\frac{mM}{m+M}\int dy'|q_1|^2 < +\infty$$

which shows that (3.13) diverges when $\delta \to 0$. The proof of the proposition is concluded. □

Proposition 3.1.4 *The decomposition $u = \varphi^\lambda + G^\lambda q$ is well defined and unique.*

Proof
By proposition (3.1.2) $G^\lambda q \in L^2(\mathbb{R}^{2N+2})$ so that the decomposition is meaningful; uniqueness follows from proposition (3.1.3). If by absurd $u = \varphi^\lambda + G^\lambda q = \tilde{\varphi}^\lambda + G^\lambda \tilde{q}$ then $0 = \varphi^\lambda - \tilde{\varphi}^\lambda = G^\lambda(\tilde{q} - q)$; but then $\tilde{q} - q = 0$ because the operator G^λ has empty kernel. □

Proposition 3.1.5 *The quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ doesn't depend on λ.*

Proof
The proof is identical to the three dimensional case since in (2.1.5) we have used only identities independent from the dimension. □

Proposition 3.1.6 *The following limit holds in $L^2(\pi_i)$.*

$$\lim_{\delta\to 0} \hbar^2 \frac{m+M}{mM} \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx u = q_i \tag{3.14}$$

Proof
First, using Sobolev embedding theorems we prove that

$$\lim_{\delta\to 0} \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx u = \lim_{\delta\to 0} \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx q_i \tag{3.15}$$

and then it's enough to show that

$$\lim_{\delta\to 0} \hbar^2 \frac{m+M}{mM} \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx q_i = q_i \tag{3.16}$$

Since $\varphi^\lambda(\cdot, y) \in H^1$ for a.e. y, its trace over $B_\delta = \{x\, s.t.\, |x-y_i| = \delta\}$ belongs to $H^{\frac{1}{2}}$; due to the embedding $H^{\frac{1}{2}}(B_\delta) \hookrightarrow L^\infty(B_\delta)$ we can write the following chain of inequalities

$$\int_{|x-y_i|=\delta} dx |\varphi^\lambda(x,y)| \leq c\|\varphi^\lambda(\cdot,y)\|_{L^\infty(B_\delta)} \|1\|_{L^1(B_\delta)} \leq c\|\varphi^\lambda(\cdot,y)\|_{H^{\frac{1}{2}}(B_\delta)} \delta \leq c\delta \|\varphi^\lambda(\cdot,y)\|_{H^1(\mathbb{R}^2)} \tag{3.17}$$

If we take the L^2_y norm of the first and the last term in (3.17) Then the regular part of u gives no contribute to the limit in (3.14). Since $G^\lambda q_j$ is a regular function on π_i apart from the intersection with π_j, which has zero measure, then (3.15) holds.
Now we introduce the following two integral operators:

$$T_\delta(y;y') = \frac{1}{2\pi} \frac{1}{\hbar\omega} \frac{1}{\pi^N} \left(\frac{m\omega}{\hbar}\right)^N \left(\frac{M\omega}{\hbar}\right) \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx \int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+N-1} \frac{1}{(1-\nu^2)^N} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \cdot e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}(y^2+y'^2)} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}(x-y'_1)^2} e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}(y-y')^2} \tag{3.18}$$

$$T'_\delta(y;y') = \frac{1}{2\pi} \frac{1}{\hbar\omega} \frac{1}{\pi^N} \left(\frac{m\omega}{\hbar}\right)^N \left(\frac{M\omega}{\hbar}\right) \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx \int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+N-1} \frac{1}{(1-\nu^2)^N} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \cdot e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}(x-y'_1)^2} e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}(y-y')^2} \tag{3.19}$$

Notice that $T_\delta(y;y') \leq T'_\delta(y;y')$ and that, with a straightforward integration, $\int dy' T'_\delta(y;y') \leq \frac{1}{\hbar 2}\frac{mM}{m+M}$; then $\{T_\delta\}$ and $\{T'_\delta\}$ are two equibounded family of bounded operators.

In order to prove (2.24), since $\frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x-y_i|=\delta} dx G^\lambda q_i = \int dy' T_\delta(y;y') q_1(y')$, it is sufficient to prove that T'_δ is an approximation of the unity, see [R], and that

$$\int dy' T_\delta(y;y') \to \frac{1}{\hbar^2} \frac{mM}{m+M} \qquad \text{uniformly on compact sets} \tag{3.20}$$

Indeed, if these two statements holds, consider $q \in C_0^\infty$; (2.24) follows from

$$\lim_{\delta \to 0} \int dy \left| \int dy'\, T_\delta(y;y')\,(q(y) - q(y')) \right|^2 = 0 \tag{3.21}$$

which is true since

$$\lim_{\delta \to 0} \int dy \left| \int dy'\, T_\delta(y;y')\,(q(y) - q(y')) \right|^2 \le \lim_{\delta \to 0} \int dy \left[\int dy'\, T'_\delta(y;y')\,|q(y) - q(y')| \right]^2 = 0 \tag{3.22}$$

The second limit in (3.22) is 0 because we can interchange the limit and the integral over y by dominate convergence theorem, and because T'_δ is an approximation of the unity.
Now we prove that $T'_\delta(y;y')$ is an approximation of the identity; first with a change of variable we see that T'_δ is a convolution operator.

$$T'_\delta(y;y') = \frac{1}{2\pi} \frac{1}{\hbar\omega} \frac{1}{\pi^N} \left(\frac{m\omega}{\hbar}\right)^N \left(\frac{M\omega}{\hbar}\right) \frac{1}{\delta \ln \frac{1}{\delta^2}} \int_{|x|=\delta} dx \int_0^1 d\nu\, \nu^{\frac{\lambda}{\hbar\omega}+N-1} \frac{1}{(1-\nu^2)^N} \frac{1}{\left(\ln \frac{1}{\nu}\right)} \cdot e^{-\frac{M\omega}{\hbar} \frac{1}{2\ln \frac{1}{\nu}} (x+y_i-y'_1)^2} e^{-\frac{m\omega}{\hbar} \frac{\nu}{1-\nu^2} (y-y')^2} \tag{3.23}$$

We have already noticed that the L^1 norm of the kernel has an upper bound uniform with respect to δ. With futher change of variable we see that $T'_\delta(y;y')$ can be written in the following way which resembles the usual expression for an approximation of the identity.

$$T'_\delta(y;y') = \frac{1}{2\pi} \frac{1}{\hbar\omega} \frac{1}{\pi} \left(\frac{m\omega}{\hbar}\right) \left(\frac{M\omega}{\hbar}\right) \frac{1}{\delta^{2N} \ln \frac{1}{\delta^2}} \int_{|x|=1} dx \int_0^{\frac{1}{\delta^2}} d\nu\, \frac{(1-\delta^2\nu)^{\frac{\lambda}{\hbar\omega}+N-1}}{\nu^N (2-\delta^2\nu)^N} \left(\frac{\delta^2}{\ln \frac{1}{1-\delta^2\nu}}\right) e^{-\frac{M\omega}{\hbar} \frac{\delta^2}{\ln \frac{1}{1-\delta^2\nu}} (x+y_1-y'_1)^2} e^{-\frac{m\omega}{\hbar} \frac{1-\delta^2\nu}{2-\delta^2\nu} \frac{1}{\nu} (y-y')^2} \tag{3.24}$$

We have already noticed that $T'_\delta(y-y')$ is positive and its integral is a constant; using (3.23) it is easy to see that the integral of T'_δ outside every compact containing the origin goes to zero. In order to conclude the proof, we have only to prove (3.20); with the same change of variable of (3.24) we see that

$$\int dyp\, T_\delta(y;y') = \frac{1}{2\pi}\frac{1}{\hbar\omega}\frac{1}{\pi^N}\left(\frac{m\omega}{\hbar}\right)^N\left(\frac{M\omega}{\hbar}\right)\int_{|x|=1} dx \int dy'$$
$$\int_0^{\frac{1}{\delta^2}} d\nu\, \frac{(1-\delta^2\nu)^{\frac{\lambda}{\hbar\omega}+N-1}}{(\nu)^N(2-\delta^2\nu)^N}\left(\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}\right) e^{-\frac{m\omega}{\hbar}\frac{1}{2}\frac{\delta^2\nu}{2-\delta^2\nu}\left(y^2+(\delta y+y')\right)} e^{-\frac{M\omega}{\hbar}\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}(x-y_1')^2} e^{-\frac{m\omega}{\hbar}\frac{1-\delta^2\nu}{2-\delta^2\nu}\frac{1}{\nu}y'^2} \tag{3.25}$$

If we exchange the limit and the integrals by dominate convergence theorem and we compute the obtained integrals, we prove (2.28) □

Now we study the closure and semiboundedness of F_α. Theorem (2.1.7) holds also in dimension two: its proof is based only on the form of F_α since no information about the dimension has been used. Then we are still concerned with the study of Φ^λ_α.

Lemma 3.1.7 *If $\lambda > \lambda_0$ then Φ^λ_α is closed and positive on*

$$\mathscr{D}(\Phi^\lambda_{\alpha,diag}) = \{q \in L^2(\mathbb{R}^{2N})\; s.t.\; \Phi^\lambda_{\alpha,diag}[q] < +\infty\}$$

Proof
Let's define $C^\lambda_\alpha = \alpha + \inf_y a^\lambda_i(y)$; for $\lambda > \lambda_0$ $\Phi^\lambda_{\alpha,diag}$ is closed, see the argument used in (2.1.8) and positive since

$$\Phi^\lambda_{\alpha,diag} \geq C^\lambda_\alpha \|q\|^2$$

and $\lim_{\lambda\to+\infty} C^\lambda_\alpha = +\infty$
It is sufficient to prove the L^2 boundedness of non diagonal terms. Indeed if we prove

$$\left|\int dy\, \overline{q_i} G^\lambda q_j\right| \leq C\|q_i\|\|q_j\| \tag{3.26}$$

then we have

$$\left|\Phi^\lambda_{\alpha,ndiag}[q]\right| \leq \frac{C(N-1)}{C^\lambda_\alpha}\Phi^\lambda_{\alpha,diag}[q]$$

If we define λ_0 such that $\frac{CN}{C^{\lambda_0}_\alpha} < 1$, by KLMN theorem we have that Φ^λ_α is closed and positive on $\mathscr{D}(\Phi^\lambda_{\alpha,diag})$.
Now we prove (3.26); take $i = 1$ $j = 2$ to be concrete. Since

$$G^\lambda(y_1,y;y_2',y') \leq \frac{1}{2\pi}\frac{1}{\hbar\omega}\frac{1}{\pi^N}\left(\frac{m\omega}{\hbar}\right)^N\frac{M\omega}{\hbar}\int_0^1 d\nu\, \frac{\nu^{\frac{\lambda}{\hbar\omega}+N-1}}{(1-\nu^2)^N\ln\frac{1}{\nu}} e^{-\frac{\nu}{1-\nu^2}\frac{m\omega}{\hbar}(y-y')^2} e^{-\frac{1}{2\ln\frac{1}{\nu}}\frac{M\omega}{\hbar}(y_1-y_2')^2} \tag{3.27}$$

it is sufficient to prove that the r.h.s. of (3.27) is a bounded operator. Making a Fourier transform, we are concerned with the estimate of the following term.

$$\left| \frac{2^{N-5}}{\hbar\omega} \int dk_1 dk_2 dk_2' dk_3 \dots dk_N \overline{q_1(k_1,\dots,k_N)} q_2(k_1+k_2-k_2', k_2', k_3, \dots, k_N) \cdot \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} e^{-\frac{1}{2}\ln\frac{1}{\nu}\frac{\hbar}{M\omega}(k_2-k_2')^2} e^{-\frac{1-\nu^2}{4\nu}\frac{\hbar}{m\omega}(k_2^2+(k_1+k_2-k_2')^2)} \right|$$
$$\le \frac{2^{N-5}}{\hbar\omega} \int dk dk_2' |q_1(k_1, k_2+k_2', k_3, \dots, k_N)||q_2(k_1+k_2, k_2', k_3, \dots, k_N)| \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} \cdot e^{-\frac{1-\nu}{2}\frac{\hbar}{m^*\omega}\left[k_2^2+(k_2+k_2')^2+(k_1+k_2)^2\right]} \tag{3.28}$$

In the previous formula we have introduced $m^* = \max\{m, M\}$. With the following change of variables

$$\begin{cases} k_1 = -2p_1 + p_2 \\ k_2 = p_1 - p_2 - p_3 \\ k_2' = -2p_1 + p_3 \end{cases}$$

equation (3.28) becomes

$$\frac{9\,2^{N-5}}{\hbar\omega} \int dp_1 dp_2 dp_3 dk_3 \dots dk_N |q_1(-2p_1+p_2, -p_1-p_2, k_3, \dots, k_N)||q_2(-p_1-p_3, -2p_1+p_3, k_3, \dots, k_N)| \cdot \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} e^{-\frac{1-\nu}{2}\frac{\hbar}{m^*\omega}\left[3p_1^2+2p_2^2+2p_3^2+2p_2\cdot p_3\right]}$$
$$\le \frac{9\,2^{N-5}}{\hbar\omega} \int dp_1 dp_2 dp_3 dk_3 \dots dk_N |\tilde{q}_1(p_1, p_2, k_3, \dots, k_N)||\tilde{q}_2(p_1, p_3, k_3, \dots, k_N)| \cdot \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} e^{-(1-\nu)\frac{\hbar}{m^*\omega}\left[p_2^2+p_3^2+p_2\cdot p_3\right]}$$

We have put $q_1(-2p_1+p_2, -p_1-p_2, k_3, \dots, k_N) = \tilde{q}_1(p_1, p_2, k_3, \dots, k_N)$ and $q_2(-p_1-p_3, -2p_1+p_3, k_3, \dots, k_N = \tilde{q}_2(p_1, p_3, k_3, \dots, k_N)$. Now by the Cauchy-Schwartz inequality, in order to conclude the proof, it is sufficient to prove that

$$S(p_2; p_3) = \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} e^{-(1-\nu)\frac{\hbar}{m^*\omega}\left[p_2^2+p_3^2+p_2\cdot p_3\right]}$$

is a bounded operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Using Schur's test we have

$$\begin{aligned}\|S\| &\leq \sup_{p_2} |p_2|^{\frac{1}{2}} \int dp_3 \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} e^{-(1-\nu)\frac{\hbar}{m^*\omega}\left[p_2^2+p_3^2+p_2\cdot p_3\right]} \frac{1}{|p_3|^{\frac{1}{2}}} \\ &\leq \sup_{p_2} |p_2|^{\frac{1}{2}} \int dp_3 \int_0^1 d\nu \nu^{\frac{\lambda}{\hbar\omega}-1} e^{-\frac{1-\nu}{2}\frac{\hbar}{m^*\omega}\left[p_2^2+p_3^2\right]} \frac{1}{|p_3|^{\frac{1}{2}}} \\ &= \pi \left(2\frac{m^*\omega}{\hbar}\right)^{\frac{3}{4}} \Gamma(\frac{3}{4}) \sup_{p_2} \int_0^{p_2^2} d\nu \frac{1}{\nu^{\frac{3}{4}}}(1-\frac{\nu}{p_2^2})^{\frac{\lambda}{\hbar\omega}-1} e^{-\nu} \leq \pi \left(2\frac{m^*\omega}{\hbar}\right)^{\frac{3}{4}} \Gamma(\frac{3}{4})\Gamma(\frac{1}{4})\end{aligned}$$

and this concludes the proof. □

We state for completeness the theorem following from the previous lemma and (2.1.7).

Theorem 3.1.8 *The quadratic form F_α is closed and bounded from below for $\lambda > \lambda_0$.*

The main difference between the two dimensional case and the three dimensional case, as regards closure and semiboundedness of F_α, is the L^2 boundedness of $\Phi^\lambda_{\alpha,ndiag}$ which is due to the much weaker typical singularity of the potential.

3.2 The Operator and the Resolvent

In this section we construct, the operator H_α and the resolvent $(H_\alpha + \lambda)^{-1}$. Let's call Γ^λ_α the positive self adjoint operator associated to Φ^λ_α. Unless we specify more informations on Γ^λ_α and on its properties the construction is the same of the three dimensional case.

Theorem 3.2.1 *The domain and the action of H_α are the following:*

$$\mathscr{D}(H_\alpha) = \left\{u \in L^2(\mathbb{R}^{2N+2})\, s.t.\, u = \varphi^\lambda + G^\lambda q,\ \varphi^\lambda \in \mathscr{D}(H_0), q \in \mathscr{D}(\Gamma^\lambda_\alpha), \Gamma^\lambda_\alpha q = Tr_\pi \varphi^\lambda\right\}$$

$$(H_\alpha + \lambda)u = (H_0 + \lambda)\varphi^\lambda$$

The resolvent has the following representation:

$$(H_\alpha + \lambda)^{-1} f = G^\lambda f + G^\lambda q \qquad q \text{ solution of } \Gamma^\lambda_\alpha q = Tr_\pi G^\lambda f$$

Proof
A function u belongs to $\mathscr{D}(H_\alpha)$ if and only if it exists $w \in L^2$, which is by definition $H_\alpha u$, such that

$$F_\alpha[u, v] = (w, v) \qquad \forall v \in \mathscr{D}(F_\alpha) \tag{3.29}$$

In particular for $v \in C_0^\infty$, which means $q_v \equiv 0$, (3.29) reduces to:

$$\int dxdy\,\left[\frac{\hbar^2}{2M}\overline{\partial_x\varphi_u^\lambda}\partial_x v + \frac{\hbar^2}{2m}\overline{\partial_y\varphi_u^\lambda}\partial_y v + \frac{1}{2}m\omega^2(y-y^0)^2\overline{\varphi_u^\lambda}v + \lambda\overline{\varphi_u^\lambda}v\right] = \int dxdy\overline{(H_\alpha+\lambda)u}\,v \tag{3.30}$$

Equation (3.30) implies

$$\varphi_u^\lambda \in \mathscr{D}(H_0) \qquad\qquad (H_\alpha+\lambda)u = (H_0+\lambda)\varphi_u^\lambda \tag{3.31}$$

If we now take a generic element of the form domain as test function we obtain:

$$\int dxdy\,\left[\frac{\hbar^2}{2M}\overline{\partial_x\varphi_u^\lambda}\partial_x\varphi_v^\lambda + \frac{\hbar^2}{2m}\overline{\partial_y\varphi_u^\lambda}\partial_y\varphi_v^\lambda + (\frac{1}{2}m\omega^2(y-y^0)^2+\lambda)\overline{\varphi_u^\lambda}\varphi_v^\lambda\right] +$$
$$+\,\Phi_\alpha^\lambda[q_u,q_v] = \int dxdy\overline{(H_\alpha+\lambda)u}v$$

Using (3.31), we obtain

$$\Phi_\alpha^\lambda[q_u,q_v] = \int dxdy\overline{(H_0+\lambda)\varphi_u^\lambda}\,G^\lambda q_v$$

which is equivalent to

$$\Phi_\alpha^\lambda[q_u,q_v] = \int dxdy\,\overline{Tr_\pi\varphi_u^\lambda}\,q_v$$

The previous equation says

$$q_u \in \mathscr{D}(\Gamma_\alpha^\lambda) \qquad\qquad \Gamma_\alpha^\lambda q_u = Tr_\pi\varphi_u^\lambda \tag{3.32}$$

Equation (3.32) is a boundary condition, written as integral equation, which must be satisfied by the elements of the domain. Even in the two dimensional case the boundary condition can be written in a more convenient form as $q_u = \left(\Gamma_\alpha^\lambda\right)^{-1} Tr_\pi\varphi_u^\lambda$ since $\left(\Gamma_\alpha^\lambda\right)^{-1}$ is a bounded operator.
□

Chapter 4

Approximate Dynamics for Small Mass Ratio

In this chapter we study an application of our model: we consider a one dimensional light particle incident over an heavy harmonic oscillator and we show that the interaction produces a partial decoherence between different components of the wave function of the heavy particle.
In section one we derive an asymptotic formula for the evolution in the limit of small mass ratio, see theorem (4.1.1); as it is expected, we find an adiabatic decoupling between the light particle and the heavy particle. This formula has been used to obtain a representation form for the reduced density matrix of the harmonic oscillator.
In section two, with additional assumptions on the physical parameters of the system, we find a more precise form for the reduced density matrix, see theorem (4.1.1); formula (4.69) shows the decoherence effect on the heavy particle.

4.1 Approximated Evolution

In this section we study the one dimensional time dependent Schroedinger equation:

$$\begin{cases} \imath\hbar\frac{\partial\psi}{\partial t}(t) = H\psi(t) \\ H = -\frac{\hbar^2}{2M}\Delta_x - \frac{\hbar^2}{2m}\Delta_y + \frac{1}{2}m\omega^2 y^2 + \alpha\delta(x-y) \\ \psi(0;x,y) = \psi(x,y) = g(x)f(y) \end{cases} \tag{4.1}$$

We take as initial state the product of two functions belonging to Schwartz class and we derive an approximate solution of (4.1) in the limit of small mass ratio $\varepsilon = \frac{M}{m}$. In such an asymptotic regime it is expected an adiabatic decoupling between the fast degrees of freedom of the system (i.e. the incident particle) and the slow ones, a situation very similar to the Born-Oppenheimer approximation; the main difference lies on the absence of bound states for the light particle.

We introduce some notation in order to formulate the main result of this section. Let μ and ν be the reduced and the total mass of the system. It will be useful to use baricentral coordinates.

$$\begin{cases} x_1 = x - y \\ x_2 = \frac{Mx+my}{m+M} \end{cases}$$

Let $U_\gamma^{m,x_0}(t;x,x')$ be the integral kernel of the propagator of $-\frac{\hbar^2}{2m}\Delta + \gamma\delta(\cdot - x_0)$; in [S] it is proved that

$$\left(U_\gamma^{m,x_0}(t)\phi\right)(x) = \int dx' U^m(t;x-x')\phi(x'+x_0) - \frac{m\gamma}{\hbar^2}\int_0^{+\infty} e^{-\frac{m\gamma}{\hbar^2}u}\int dx' U^m(t;u+|x|+|x'|)\phi(x'+x_0) \tag{4.2}$$

In (4.2) we have denoted the kernel of the free propagator with $U^m(t;x)$.

$$U^m(t;x) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} e^{-\frac{m}{2i\hbar t}x^2}$$

We shall omit x_0 as superscript for $U_\gamma^{m,x_0}(t;x,x')$ whenever is 0. We define also the integral operator

$$\left(W_+^{\gamma,x_0}h\right)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int dx h(x)\left(e^{-ikx} + \mathscr{R}_\gamma(k)e^{-ix_0k}e^{i|k||x-x_0|}\right) \tag{4.3}$$

$$\mathscr{R}_\gamma(k) = -\frac{\gamma}{\gamma - i|k|}$$

The integral kernel of (4.3) is the generalized eigenfunction of $-\frac{1}{2}\Delta + \gamma\delta(\cdot - x_0)$ and $\mathscr{R}_\gamma(k)$ is the reflection coefficient; see [AGH-KH]. Let Ω_+^{γ,x_0} be the corresponding wave operator.

$$\left(\Omega_+^{\gamma,x_0}h\right)(x) = \left[\left(W_+^{\gamma,x_0}\right)^{-1}\mathcal{F}h\right](x)$$

Let $V^{m,\omega}(t;x,x')$ be the Mehler kernel for the unitary propagator of the harmonic oscillator:

$$V^{m,\omega}(t;x,x') = \left[\frac{m\omega}{\pi\hbar(1-e^{-2i\omega t})}\right]^{\frac{1}{2}} e^{-\frac{m\omega}{\hbar}\frac{x^2+x'^2}{2i\tan(\omega t)}} e^{\frac{m\omega}{\hbar}\frac{xx'}{i\sin(\omega t)}}$$

We put $\tilde{\omega} = \frac{\omega}{\sqrt{1+\varepsilon}}$. With the above notation we are ready to formulate the main theorem of this section.

Theorem 4.1.1 *Fix t such that $\sqrt{\varepsilon}t \ll 1$, then for every $0 < \delta < \frac{1}{2}$ the following estimate holds:*

$$\|\psi(t) - \psi_a(t)\| \leq C\varepsilon^{\frac{1}{2}-\delta}$$

$$\psi_s(t;x,y)\left(\frac{M}{i\hbar t}\right)^{\frac{1}{2}} e^{i\frac{M}{2\hbar t}x^2}\int dy'\, V^{m,\omega}(t;y,y')\, W_+^{\alpha_0,y}g\left(\frac{mx}{\hbar t}\right)$$

The proof of the theorem will be divided into three lemmas.

Lemma 4.1.2 *Fix t such that $\sqrt{\varepsilon}t \ll 1$, then for every $0 < \delta < \frac{1}{2}$ the following estimate holds:*

$$\|\psi(t) - \psi_1(t)\| \leq C\varepsilon^{\frac{1}{2}-\delta}$$
$$\psi_1(t;x,y) = \int dy'\, V^{\nu,\tilde{\omega}}(t; \frac{Mx+my}{m+M}, y')\, f(y') \int dx' U^{\mu}_{\alpha}(t; x-y, x') g(x'+y')$$

Where C is explicitly given as a function of t, M, α, g, f.

Proof
The proof of this lemma will be divided into two steps: let $\hat{\psi}_1$ and ψ_1 be the solution of the following time-dependent Schroedinger equations:

$$\begin{cases} i\hbar\frac{\partial\hat{\psi}_1}{\partial t}(t) = \hat{H}_1\hat{\psi}_1(t) \\ \hat{H}_1 = -\frac{\hbar^2}{2\mu}\Delta_{x_1} + \alpha\delta(x_1) + \frac{1}{2}\mu\varepsilon\tilde{\omega}^2\left(\frac{\varepsilon}{1+\varepsilon}\right)x_1^2 - \frac{\hbar^2}{2\nu}\Delta_{x_2} + \frac{1}{2}\nu\tilde{\omega}^2 x_2^2 \\ \hat{\psi}_1(0;x_1,x_2) = \hat{\psi}_1(x_1,x_2) = g(x_1+x_2)f(x_2) \end{cases}$$
$$\begin{cases} i\hbar\frac{\partial\psi_1}{\partial t}(t) = H_1\psi_1(t) \\ H_1 = -\frac{\hbar^2}{2\mu}\Delta_{x_1} + \alpha\delta(x_1) - \frac{\hbar^2}{2\nu}\Delta_{x_2} + \frac{1}{2}\nu\tilde{\omega}^2 x_2^2 \\ \psi_1(0;x_1,x_2) = \psi_1(x_1,x_2) = g(x_1+x_2)f(x_2) \end{cases}$$

In order to prove (4.1.2) it is sufficient to estimate $\|\psi(t) - \hat{\psi}_(t)\|$ and $\|\psi_1(t) - \hat{\psi}_1(t)\|$ separately. It is useful to write (4.1) using (x_1, x_2) as coordinates.

$$\begin{cases} i\hbar\frac{\partial\psi}{\partial t}(t) = H\psi(t) \\ H = -\frac{\hbar^2}{2\mu}\Delta_{x_1} + \alpha\delta(x_1) - \frac{\hbar^2}{2\nu}\Delta_{x_2} + \frac{1}{2}\nu\tilde{\omega}^2\left[x_2^2 - 2\frac{\varepsilon}{1+\varepsilon}x_1\,x_2 + \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 x_2^2\right] \\ \psi(0;x_1,x_2) = \psi(x_1,x_2) = g(\frac{1}{1+\varepsilon}x_1 + x_2)f(x_2 - \frac{\varepsilon}{1+\varepsilon}x_1) \end{cases}$$

We proceed as follows:

$$\begin{aligned} \|\psi(t) - \hat{\psi}_1(t)\| &= \|e^{-i\frac{t}{\hbar}H}\psi - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1\| \\ &\leq \|e^{-i\frac{t}{\hbar}H}\psi - e^{-i\frac{t}{\hbar}H}\hat{\psi}_1\| + \|e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1\| \\ &= \|\psi - \hat{\psi}_1\| + \|e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1\| \end{aligned}$$

The estimate of $\|\psi - \hat{\psi}_1\|$ can be obtained in the following way:

$$\begin{aligned}
\|\psi - \hat{\psi}_1\|^2 &= \int dx_1 dx_2 \left| g\left(\frac{1}{1+\varepsilon}x_1 + x_2\right) f\left(x_2 - \frac{\varepsilon}{1+\varepsilon}x_1\right) - g(x_1+x_2)f(x_2)\right|^2 \\
&\leq 2\int dx_1 dx_2 \left| g\left(\frac{1}{1+\varepsilon}x_1 + x_2\right)\right|^2 \left| f\left(x_2 - \frac{\varepsilon}{1+\varepsilon}x_1\right) - f(x_2)\right|^2 + \\
&\quad + 2\int dx_1 dx_2\, |f(x_2)|^2 \left| g\left(\frac{1}{1+\varepsilon}x_1 + x_2\right) - g(x_1+x_2)\right|^2 \\
&= \int dx_1\, |g(x_1)|^2 \int dx_2 \left| f\left(\left(1 + \frac{\varepsilon}{1+\varepsilon}\right)x_2 - \varepsilon x_1\right) - f(x_2)\right|^2 + \\
&\quad + 2\int dx_2\, |f(x_2)|^2 \int dx_1 \left| g\left(\frac{1}{1+\varepsilon}x_1 + \frac{\varepsilon}{1+\varepsilon}x_2\right) - g(x_1)\right|^2
\end{aligned} \tag{4.4}$$

The action of translations and dilatations on regular functions can be estimated as follows:

$$\int dx |h(ax+by) - h(x)|^2 \leq c(1-a)^2 \left(\|h\|^2 + \int dx\; x^2 |h'(x)|^2\right) + cb^2y^2\|h'\|^2 \tag{4.5}$$

Using (4.5) in (4.4) we prove that $\|\psi - \hat{\psi}_1\| \leq C\,\varepsilon$.
Let's turn our attention to $\|e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1\|$. Using Duhamel's formula for the representation of the unitary group

$$e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1 = -\frac{i}{\hbar}\int_0^t ds\, e^{-i\frac{(t-s)}{\hbar}H}(H - \hat{H}_1)e^{-i\frac{s}{\hbar}\hat{H}_1}\hat{\psi}_1$$

it is sufficient to show that the following inequality holds:

$$\int dx_1 dx_2 \left| x_1 x_2 \left(e^{-i\frac{s}{\hbar}\hat{H}_1}\hat{\psi}_1\right)(x_1,x_2)\right|^2 \leq c \tag{4.6}$$

Indeed, if (4.6) holds, we can immediately complete the proof of step one in the following way.

$$\begin{aligned}
\|e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1\| &\leq \frac{1}{\hbar}\int_0^t ds\, \|e^{-i\frac{(t-s)}{\hbar}H}(H - \hat{H}_1)e^{-i\frac{s}{\hbar}\hat{H}_1}\hat{\psi}_1\| \\
&= \frac{1}{\hbar}\int_0^t ds\, \|(H - \hat{H}_1)e^{-i\frac{s}{\hbar}\hat{H}_1}\hat{\psi}_1\| \\
&\leq c\frac{\varepsilon}{1+\varepsilon}\frac{t}{\hbar}\nu\tilde{\omega}^2
\end{aligned} \tag{4.7}$$

In order to prove (4.6) we introduce $V_\alpha^{m,\omega}(t)$, the unitary group generated by $-\frac{\hbar^2}{2m}\Delta_x + \frac{1}{2}m\omega^2x^2 + \alpha\delta(x)$. In appendix (B) it is proved that the following representation holds for $V_\alpha^{m,\omega}(t)$:

$$\begin{cases} \left(V_\alpha^{m,\omega}\phi\right)(t;x) = \left(V^{m,\omega}\phi\right)(t;x) + \frac{i}{\hbar}\int_0^t ds\, V^{m,\omega}(t-s;x,0)q(s) \\ q(t) + \alpha\left(V^{m,\omega}\phi\right)(t;0) + i\frac{\alpha}{\hbar}\int_0^t ds\, V^{m,\omega}(t-s;0,0)q(s) = 0 \end{cases} \tag{4.8}$$

Estimate (4.6) can be proved in this way:

$$\int dx_1 dx_2 \left| x_1 x_2 \left(e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1 \right)(x_1,x_2) \right|^2 \leq$$
$$\leq 2\int dx_1 dx_2 \left| x_1 x_2 \int dx_2'\, V^{\nu,\tilde{\omega}}(t;x_2,x_2') f(x_2') \int dx_1'\, V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t;x_1,x_1') g(x_1'+x_2') \right|^2 +$$
$$+ 2\int dx_1 dx_2 \left| \frac{\alpha}{\hbar} x_1 x_2 \int dx_2'\, V^{\nu,\tilde{\omega}}(t;x_2,x_2') f(x_2') \int_0^t ds\, V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t-s;x_1,0) q_\varepsilon(s;x_2') \right|^2 \quad (4.9)$$

We have added a dependence on x_2' to $q_\varepsilon(t)$ because it solves the following integral equation.

$$q_\varepsilon(t) + \alpha \left[\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{\pi\hbar(1-e^{-2i\sqrt{\varepsilon}\tilde{\omega}t})} \right]^{\frac{1}{2}} \int dx_1' e^{-\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{2i\hbar\tan(\sqrt{\varepsilon}\tilde{\omega})}x_1'^2} g(x_1'+x_2') +$$
$$+ \frac{i}{\hbar}\alpha \int_0^t ds \left[\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{\pi\hbar(1-e^{-2i\sqrt{\varepsilon}\tilde{\omega}(t-s)})} \right]^{\frac{1}{2}} q_\varepsilon(s) = 0 \quad (4.10)$$

The "initial datum" of (4.10) has a parametric dependence on x_2', and the same is true for the solution of the equation.
Now we introduce two identities which will be used several times in the following:

$$y\int dy'\, V^{m,\omega}(t;y,y') f(y') = \int dy'\, V^{m,\omega}(t;y,y') \left(\cos(\omega t) y' f(y') - \frac{i\hbar}{m\omega}\sin(\omega t) f'(y') \right) \quad (4.11)$$
$$\frac{d}{dy}\int dy'\, V^{m,\omega}(t;y,y') f(y') = \int dy'\, V^{m,\omega}(t;y,y') \left(\cos(\omega t) f'(y') - \frac{m\omega}{i\hbar}\sin(\omega t) y' f(y') \right) \quad (4.12)$$

In order to prove (4.11) and (4.12) it is sufficient to carefully integrate by parts.
We shall estimate the two terms on the r.h.s. of (4.9) separately: the first one can be handled using (4.11)

$$x_1 x_2 \int dx_2'\, V^{\nu,\tilde{\omega}}(t;x_2,x_2') f(x_2') \int dx_1'\, V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t;x_1,x_1') g(x_1'+x_2') =$$
$$\int dx_1' dx_2'\, V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t;x_1,x_1') V^{\nu,\tilde{\omega}}(t;x_2,x_2') \frac{i\hbar\sin(\mu\sqrt{\varepsilon}\tilde{\omega}t)}{\mu\sqrt{\varepsilon}\tilde{\omega}} \frac{i\hbar\sin(\nu\tilde{\omega}t)}{\nu\tilde{\omega}}$$
$$\left[\frac{\nu\tilde{\omega}}{2i\hbar\tan(\tilde{\omega}t)} x_2' \left(\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{2i\hbar\tan(\sqrt{\varepsilon}\tilde{\omega}t)} x_1' g(x_1'+x_2') f(x_2') - g'(x_1'+x_2') f(x_2') \right) + \right.$$
$$\left. - \frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{2i\hbar\tan(\sqrt{\varepsilon}\tilde{\omega}t)} x_1' (g'(x_1'+x_2') f(x_2') + g(x_1'+x_2') f'(x_2')) + g''(x_1'+x_2') f(x_2')) + g'(x_1'+x_2') f'(x_2')) \right]$$

Since f and g belong to Schwartz class each term in the square bracket has bounded L^2 norm then by unitarity we have proved the first half (4.6).

Now, we estimate the norm of

$$x_1 \int_0^t ds V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t-s;x_1,0)\left[x_2 \int dx_2' V^{\nu,\tilde{\omega}}(t;x_2,x_2')f(x_2')q_\varepsilon(s;x_2')\right] \tag{4.13}$$

In [AT] it is proved that:

$$q(t) \in L^\infty(0,T) \Rightarrow \int_0^t ds\, U^\mu(t-s;x)q(s) \in L^2(\mathbb{R}) \tag{4.14}$$

$$q(t) \in H^{\frac{1}{4}}(0,T) \Rightarrow x \int_0^t ds U^\mu(t-s;x,0)q(s) \in L^2(\mathbb{R}) \tag{4.15}$$

These two propositions can be extended in order to be used in our case, in particular, due to the the expression of (4.13), we need a vector valued version of (4.14) and (4.15). We shall use these two simple generalizations:

$$q(t) \in L^\infty(0,T;L^2(\mathbb{R})) \Rightarrow \int_0^t ds\, V^{\mu,\omega}(t-s;x)q(s) \in L^2(\mathbb{R}^2) \tag{4.16}$$

$$q(t) \in H^{\frac{1}{4}}(0,T;L^2(\mathbb{R})) \Rightarrow x \int_0^t ds V^{\mu,\varepsilon^{\frac{1}{2}}\tilde{\omega}}(t-s;x,0)q(s) \in L^2(\mathbb{R};L^2(\mathbb{R})) = L^2(\mathbb{R}^2)$$

Then we need a regularity condition on q_ε to ensure:

$$x_2 \int dx_2' V^{\nu,\tilde{\omega}}(t;x_2,x_2')f(x_2')q_\varepsilon(s;x_2') \in H^{\frac{1}{4}}(0,T;L^2(\mathbb{R})) \tag{4.17}$$

Using again (4.11), it is sufficient to show

$$q_\varepsilon(s;x_2') \in H^{\frac{1}{4}}(0,T;W^{1,\infty}(\mathbb{R})) \tag{4.18}$$

Indeed if (4.18) holds one has:

$$x_2 \int dx_2'\, V^{\nu,\tilde{\omega}}(t;x_2,x_2')f(x_2')q(s;x_2') = \int dx_2'\, V^{\nu,\tilde{\omega}}(t;x_2,x_2')\frac{i\hbar\sin(\nu\tilde{\omega}t)}{\nu\tilde{\omega}}\cdot$$
$$\left[\frac{\nu\tilde{\omega}}{2i\hbar\tan(\tilde{\omega}t)}x_2'f(x_2')q(s;x_2') - f'(x_2')q(s;x_2') - f(x_2')q'(s;x_2')\right]$$

All the terms in square bracket belong to $H^{\frac{1}{4}}(0,T;L^2(\mathbb{R}))$ and then by unitarity (4.17) holds. It remains to prove (4.18) to conclude step one of the lemma. The initial datum of (4.10) is a regular function of both time and space variables; in particular it belongs to $C(0,T;L^\infty(\mathbb{R}))$. Using a simple fixed point argument it is possible to show that $q_\varepsilon(t;x_2') \in L^2(0,T;L^\infty(\mathbb{R}))$ for small enough T; then the solution con be extended for any T by iteration.

Exploiting the Abel-like regularization properties of the kernel in (4.10), see [GV], we find $q_\varepsilon(t;x_2') \in H^{\frac{1}{4}}(0,T;L^\infty(\mathbb{R}))$. If we derive both sides of (4.10) with respect to x_2', we see that q_ε' satisfies the same equation with g' as new initial datum. Then we can repeat the previous argument to prove $q_\varepsilon'(t,x_2') \in H^{\frac{1}{4}}(0,T;L^\infty(\mathbb{R}))$.
Now we estimate $\|e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1\|$.
Using (4.8) and the corresponding representation for U^m_α

$$\begin{cases} \left(U^m_\alpha \phi\right)(t;x) = \left(U^m\phi\right)(t;x) + \frac{i}{\hbar}\int_0^t ds\, U^m(t-s;x,0)q(s) \\ q(t) + \alpha\left(U^m\phi\right)(t;0) + i\frac{\alpha}{\hbar}\int_0^t ds\, U^m(t-s;0,0)q(s) = 0 \end{cases}$$

we can write:

$$\begin{aligned} e^{-i\frac{t}{\hbar}H}\hat{\psi}_1 - e^{-i\frac{t}{\hbar}\hat{H}_1}\hat{\psi}_1 =& \int dx_1' dx_2' \Big(V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t;x_1-x_1') - U^\mu(t;x_1,x_1')\Big)\, V^{\nu,\tilde{\omega}}(t;x_2,x_2') g(x_1'+x_2') f(x_2') + \\ &+ \frac{1}{\hbar}\left[\int_0^t ds\, \Big(V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t-s;x_1,0) - U^\mu(t-s;x_1,0)\Big)\cdot \right. \\ &\left. \int dx_2' V^{\nu,\tilde{\omega}}(t;x_2,x_2') f(x_2') q_\varepsilon(s;x_2')\right] + \\ &+ \frac{i}{\hbar}\int_0^t ds U^\mu(t-s;x_1,0)\int dx_2' V^{\nu,\tilde{\omega}}(t;x_2,x_2') f(x_2')(q_\varepsilon(s;x_2') - q(s;x_2')) \end{aligned} \tag{4.19}$$

We will separately estimate the three terms on the r.h.s. of (4.19). Now we consider the first term on r.h.s. of (4.19): in order to prove that it is of order ε^2 it is sufficient to show that

$$\sup_{x_2'}\int dx_1 \left|\int dx_1' \Big(V^{\mu,\sqrt{\varepsilon}\tilde{\omega}}(t;x_1,x_1') - U^\mu(t;x_1,x_1')\Big)\, g(x_1'+x_2')\right|^2 \leq C\varepsilon^2$$

Using Duhamel's formula as in (4.7), the estimate reduces to proving $\int dx_1\, |x_1^2\, U^\mu g(t;x_1+x_2)|^2 \leq C$. Usin (4.11) twice we obtain:

$$\begin{aligned} x_1^2 U^\mu g(t;x_1+x_2) = \int dx_1' U^\mu(t;x_1,x_1')\, \Big(& x_1'^2 g(x_1'+x_2') + 2\frac{it}{m}x_1' g'(x_1'+x_2') + \\ & + \frac{it}{m}g(x_1'+x_2') + \frac{it}{m}g''(x_1'+x_2')\Big) \end{aligned}$$

Each term in brackets has finite norm bounded by a constant non depending on x_2'.
Now we consider the second term of (4.19). We denote $\int dx_2' V^{\nu,\tilde{\omega}}(t;x_2,x_2')f(x_2')q_\varepsilon(s;x_2')$ with $\hat{q}_\varepsilon(s)$ in order to make following formulas more compact. With a Fourier transform, we have to evaluate the norm of

$$\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^t ds\, e^{-i\frac{k^2}{2\mu\hbar}s}\hat{q}_\varepsilon(t-s) - \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^t ds\, \left(\frac{2\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{1-e^{-2i\sqrt{\varepsilon}\tilde{\omega}s}}\right)^{\frac{1}{2}} e^{-i\frac{k^2}{2\mu\hbar}\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}}\hat{q}_\varepsilon(t-s) \tag{4.20}$$

Since $\left(\frac{2\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{1-e^{-2i\sqrt{\varepsilon}\tilde{\omega}s}}\right)^{\frac{1}{2}} = \left(\frac{e^{i\sqrt{\varepsilon}\tilde{\omega}s}}{\cos(\sqrt{\varepsilon}\tilde{\omega}s)}\right)^{\frac{1}{2}}$, this quantity differs from one by a term of order $\sqrt{\varepsilon}$ uniformly in s. Then we can simply estimate

$$\int_0^t ds\, e^{-i\frac{k^2}{2\mu\hbar}s}\hat{q}_\varepsilon(t-s) - \int_0^t ds\, e^{-i\frac{k^2}{2\mu\hbar}\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}}\hat{q}_\varepsilon(t-s) \tag{4.21}$$

It is useful to make the following change of variable in the second term of (4.21)

$$\begin{aligned}\int_0^t ds\, e^{-i\frac{k^2}{2\mu\hbar}\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}}\hat{q}_\varepsilon(t-s) = \int_0^{\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} e^{-i\frac{k^2}{2\mu\hbar}s}\hat{q}_\varepsilon\left(t-\frac{\arctan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}\right)\frac{1}{1+\varepsilon s^2} = \\ \int_0^{\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} e^{-i\frac{k^2}{2\mu\hbar}s}\hat{q}_\varepsilon\left(t-\frac{\arctan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}\right)\left(\frac{1}{1+\varepsilon s^2}-1\right) + \\ + \int_0^{\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} e^{-i\frac{k^2}{2\mu\hbar}s}\left[\hat{q}_\varepsilon\left(t-\frac{\arctan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}\right) - \hat{q}_\varepsilon(t-s)\right] + \\ + \int_0^{\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} e^{-i\frac{k^2}{2\mu\hbar}s}\hat{q}_\varepsilon(t-s)\end{aligned} \tag{4.22}$$

The L^2 norm of the first term on the r.h.s. of (4.22) can be estimated with $c\varepsilon t^2\|\hat{q}_\varepsilon\|_\infty$ by (4.16). In order to estimate the second term notice that, fully exploiting the regularizing properties of (4.10), one has $\hat{q}_\varepsilon \in H^{1-\delta}(0,T)$ for every positive δ, and then $\hat{q}_\varepsilon \in C^{\frac{1}{2}-\delta}(0,T)$ due to Sobolev embeddings theorems, see [A]. Then the norm of this term can be estimated by $c\|\hat{q}_\varepsilon\left(t-\frac{\arctan(\sqrt{\varepsilon}\tilde{\omega}\cdot)}{\sqrt{\varepsilon}\tilde{\omega}}\right) - \hat{q}_\varepsilon(t-\cdot)\|_\infty \le c(\varepsilon t^2)^{\frac{1}{2}-\delta}$. Then we have only to estimate

$$\begin{aligned}\int dk \left|\int_t^{\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} e^{-i\frac{k^2}{2\mu\hbar}s}\hat{q}_\varepsilon(t-s)\right|^2 = 2\Re\int_{t<s'<s<\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} ds ds'\, \frac{\overline{\hat{q}}_\varepsilon(s)\hat{q}_\varepsilon(s')}{\sqrt{s-s'}} \le \\ \le 2\|\hat{q}_\varepsilon\|_\infty^2 \int_{t<s'<s<\frac{\tan(\sqrt{\varepsilon}\tilde{\omega}s)}{\sqrt{\varepsilon}\tilde{\omega}}} ds ds'\, \frac{1}{\sqrt{s-s'}} \le c\|\hat{q}_\varepsilon\|_\infty^2(\varepsilon t^2)^{\frac{3}{2}}\end{aligned}$$

Let's now estimate the last term of (4.19). Let's introduce $\tilde{q}(t,x_2') = q_\varepsilon(t,x_2') - q(t,x_2')$. If we prove

$$\sup_t \|\tilde{q}(t;\cdot)\|_\infty \le C\varepsilon$$

the proof of the lemma is completed by (4.16). Notice that the integral equation satisfied by q has the same properties of (4.10), any argument made about q_ε can be repeated for q; in particular both functions belong to $C(0,T;L^2(\mathbb{R}))$ and so $\tilde{q}$. The equation satisfied by $\tilde{q}$ is

$$\tilde{q}(t;x_2') + \alpha \int dx_1' \left\{ \left[\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{\pi\hbar(1-e^{-2i\sqrt{\varepsilon}\tilde{\omega}t})} \right]^{\frac{1}{2}} e^{-\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{2i\hbar\tan(\sqrt{\varepsilon}\tilde{\omega}t)}x_1'^2} - \left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{-\frac{\mu}{2i\hbar t}x_1'^2} \right\} g(x_1'+x_2') +$$
$$+ i\frac{\alpha}{\hbar}\int_0^t ds \left[\left(\frac{\mu\sqrt{\varepsilon}\tilde{\omega}}{\pi\hbar(1-e^{-2i\sqrt{\varepsilon}\tilde{\omega}(t-s)})} \right)^{\frac{1}{2}} - \left(\frac{\mu}{2\pi\hbar i(t-s)} \right)^{\frac{1}{2}} \right] \tilde{q}(s;x_2') = 0$$
$$= \alpha Tg(t;x_2') + i\frac{\alpha}{\hbar}\int_0^t ds K(t-s)\tilde{q}(s;x_2') = 0$$

Since $\|K(\cdot)\|_\infty \leq C\varepsilon$ and $\sup_t \|Tg(t,\cdot)\|_\infty \leq C\varepsilon$ we obtain $\sup_t \|\tilde{q}(t,\cdot)\|_\infty \leq C\varepsilon$. □

Since a small value of M in the interacting unitary group $U^\mu_\alpha(t)$ is equivalent to a large value of time, in the next lemma we use a typical scattering estimate to approximate $U^\mu_\alpha(t)$.

Lemma 4.1.3 *The following estimate holds:*

$$\|\psi_1(t) - \psi_2(t)\| \leq C\varepsilon$$

$$\psi_2(t;x_1,x_2) = \left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{\mu}{2\hbar t}x_1^2} \int dx_2'\, f(x_2') V^{\nu,\tilde{\omega}}(t;x_2,x_2') \int dx_1' g(x_1'+x_2') \left(e^{-i\frac{\mu}{\hbar t}x_1x_1'} - \frac{e^{-i\frac{\mu}{\hbar t}|x_1||x_1'|}}{1-i\frac{\hbar}{\alpha t}|x_1|} \right)$$

and C is explicitly given as a function of t, m, M, α, g.

Proof
Using the isometric character of $V^{\nu,\tilde{\omega}}(t)$ and the explicit expression of $U^\mu_\alpha(t)$, see (4.2), we have:

$$\|\psi_1(t) - \psi_2(t)\|^2 \leq 2\int dx_1 dx_2 \left| f(x_2) \int dx_1'\, g(x_1'+x_2) \left(U^\mu(t;x_1-x_1') - \left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{\mu}{2\hbar t}x_1^2 - i\frac{\mu}{\hbar t}x_1x_1'} \right) \right|^2$$
$$+ 2\int dx_1 dx_2 \left| f(x_2) \int dx_1'\, g(x_1'+x_2) \left(\frac{\mu\alpha}{\hbar^2} \int_0^\infty du\, e^{-\frac{\mu\alpha}{\hbar^2}u}\, U^\mu(t;u+|x_1|+|x_1'|) + \right.\right.$$
$$\left.\left. - \left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} \frac{1}{1-i\frac{\hbar}{\alpha t}|x_1|} e^{i\frac{\mu}{2\hbar t}x_1^2 - i\frac{\mu}{\hbar t}|x_1||x_1'|} \right) \right|^2 \qquad (4.23)$$

The first term of (4.23) can be easily estimated in the following way:

$$
\begin{aligned}
& 2\int dx_1 dx_2 \left| f(x_2) \int dx_1' g(x_1'+x_2) \left(U^\mu(t;x_1-x_1') - \left(\frac{\mu}{2\pi\hbar i t}\right)^{\frac{1}{2}} e^{i\frac{\mu}{2\hbar t}x_1^2 - i\frac{\mu}{\hbar t}x_1 x_1'} \right) \right|^2 \\
& = \frac{\mu}{\pi\hbar t}\int dx_2 |f(x_2)|^2 \int dx_1 \left| \int dx_1'\, g(x_1'+x_2') \left(e^{i\frac{\mu}{2\hbar t}x_1'^2} - 1 \right) e^{-i\frac{\mu}{\hbar t}x_1 x_1'} \right|^2 \\
& \leq \int dx_2 |f(x_2)|^2 \int dx_1' |g(x_1'+x_2)|^2 \left| e^{i\frac{\mu}{2\hbar t}x_1'^2} - 1 \right|^2 \\
& \leq \frac{\varepsilon^2 m^2}{2(1+\varepsilon)^2\hbar^2 t^2} \int dx_2 |f(x_2)|^2 \int dx_1' |g(x_1'+x_2)|^2 x_1'^4
\end{aligned}
$$

Concerning the second term, we introduce $\alpha_0 = \frac{\alpha m}{\hbar^2}$ and the change of variables

$$
v = \frac{\mu\alpha}{\hbar^2}u \qquad\qquad y_1 = \frac{\mu}{\hbar t}x_1
$$

and use the identity

$$
\int_0^\infty dv\, e^{-v + i\frac{\hbar|x_1|}{\alpha t}v} = \frac{1}{1 - i\frac{\hbar|x_1|}{\alpha t}}
$$

It can be estimated in the following way:

$$
\begin{aligned}
& 2\int dx_1 dx_2 \left| f(x_2) \int dx_1'\, g(x_1'+x_2) \left(\frac{\mu\alpha}{\hbar^2} \int_0^\infty du\, e^{-\frac{\mu\alpha}{\hbar^2}u}\, U^\mu(t; u + |x_1| + |x_1'|) \;+ \right.\right. \\
& \qquad \left.\left. -\; \left(\frac{\mu}{2\pi\hbar i t}\right)^{\frac{1}{2}} \frac{1}{1 - i\frac{\hbar}{\alpha t}|x_1|} e^{i\frac{\mu}{2\hbar t}x_1^2 - i\frac{\mu}{\hbar t}|x_1||x_1'|} \right) \right|^2 \\
& = \frac{1}{\pi}\int dx_2\, |f(x_2)|^2 \int dy_1 \left| \int dx_1'\, g(x_1'+x_2) e^{i|y_1||x_1'|} \left(e^{i\frac{\mu}{2\hbar t}x_1'^2} \int_0^\infty dv\, e^{-v+i\frac{\hbar|x_1|}{\alpha t}v} e^{i\frac{(1+\varepsilon)M}{2\hbar t\alpha_0^2}v^2 + i\frac{M}{\hbar t\alpha_0}|x_1'|v} + \right.\right. \\
& \qquad \left.\left. - \int_0^\infty dv\, e^{-v+i\frac{(1+\varepsilon)|y_1|}{\alpha_0}v} \right) \right|^2 \\
& \leq \frac{2}{\pi}\int dx_2\, |f(x_2)|^2 \int dy_1 \left| \int dx_1'\, g(x_1'+x_2) e^{i|y_1||x_1'|} e^{i\frac{\mu}{2\hbar t}x_1'^2} \cdot \cdot \right. \\
& \qquad \left. \int_0^\infty dv\, e^{-v+i\frac{(1+\varepsilon)|y_1|}{\alpha_0}v} \left(e^{i\frac{(1+\varepsilon)M}{2\hbar t\alpha_0^2}v^2 + i\frac{M}{\hbar t\alpha_0}|x_1'|v} - 1 \right) \right|^2 + \\
& + \frac{2}{\pi}\int dx_2\, |f(x_2)|^2 \int dy_1 \left| \int dx_1'\, g(x_1'+x_2) e^{i|y_1||x_1'|} \left(e^{i\frac{\mu}{2\hbar t}x_1'^2} - 1 \right) \frac{1}{1 - i\frac{1+\varepsilon}{\alpha_0}|y_1|} \right|^2
\end{aligned}
\tag{4.24}
$$

To estimate the first term of (4.24) we proceed as follows:

$$\frac{2}{\pi}\int dx_2\,|f(x_2)|^2 \int dy_1 \left| \int dx_1' g(x_1'+x_2) e^{i|y_1||x_1'|} e^{i\frac{\mu}{2\hbar t}x_1'^2} \int_0^\infty dv\, e^{-v+i\frac{(1+\varepsilon)|y_1|}{\alpha_0}v} \left(e^{i\frac{(1+\varepsilon)M}{2\hbar t\alpha_0^2}v^2 + i\frac{M}{\hbar t\alpha_0}|x_1'|v} - 1 \right) \right|^2$$
$$\leq \frac{1}{2\pi}\left(\frac{\mu}{\hbar t}\right)^2 \int dx_2\,|f(x_2)|^2 \int dy_1 \frac{1}{1+\left(\frac{1+\varepsilon}{\alpha_0}\right)^2 y_1^2} \left(\int dx_1'\, x_1'^2 |g(x_1'+x_2)| \right)^2$$
$$\leq \frac{\varepsilon^2 m^2 \alpha_0}{2(1+\varepsilon)^3\hbar^2 t^2} \int dx_2\,|f(x_2)|^2 \left(\int dx_1'\, x_1'^2 |g(x_1'+x_2)| \right)^2$$

It is convenient to integrate the second term of (4.24) by parts.

$$\frac{2}{\pi}\int dx_2\,|f(x_2)|^2 \int dy_1 \left| \int dx_1'\, g(x_1'+x_2) e^{i|y_1||x_1'|} \left(e^{i\frac{\mu}{2\hbar t}x_1'^2} - 1 \right) \frac{1}{1 - i\frac{1+\varepsilon}{\alpha_0}|y_1|} \right|^2$$
$$= \frac{2}{\pi}\int dx_2\,|f(x_2)|^2 \int dy_1 \frac{1}{1+\left(\frac{1+\varepsilon}{\alpha_0}\right)^2 |y_1|^2} \left| \int dx_1'\, g(x_1'+x_2) e^{i|y_1||x_1'|} e^{i\frac{\mu}{2\hbar t}x_1'^2} \cdot \right.$$
$$\left. \int_0^\infty dv\, e^{-v+i\frac{(1+\varepsilon)|y_1|}{\alpha_0}v} e^{i\frac{(1+\varepsilon)M}{2\hbar t\alpha_0^2}v^2 + i\frac{M}{\hbar t\alpha_0}|x_1'|v} \frac{(1+\varepsilon)\mu}{\hbar t\alpha_0} \left(\frac{1+\varepsilon}{\alpha_0}v + |x_1'| \right) \right|^2$$
$$\leq \frac{2(1+\varepsilon)\mu^2}{\pi\hbar^2 t^2\alpha_0} \int dx_2\,|f(x_2)|^2 \int dy \frac{1}{1+y^2} \left(\int dx_1'\, |g(x_1'+x_2)| \int_0^\infty dv e^{-v} \left(\frac{1+\varepsilon}{\alpha_0}v + |x_1'| \right) \right)^2$$
$$\leq \frac{2\varepsilon^2 m^2}{(1+\varepsilon)\hbar^2 t^2\alpha_0} \int dx_2\,|f(x_2)|^2 \left(\int dx_1'\, |g(x_1'+x_2)| \left(\frac{1+\varepsilon}{\alpha_0} + |x_1'| \right) \right)^2$$

□

Notice that one must assume that α is of order ε^{-1}, that is $\alpha_0 = \alpha m$, in order to have an effect of finite order on the wave function.

Lemma 4.1.4 *The following estimate holds:*

$$\|\psi_2(t) - \psi_a(t)\| \leq C\sqrt{\varepsilon} \tag{4.25}$$

Proof

The two wave functions have the following expressions:

$$\psi_2(t;x,y) = \left(\frac{\mu}{2\pi\hbar i t}\right)^{\frac{1}{2}} e^{i\frac{\mu}{2\hbar t}(x-y)^2} \int dy'\, f(y') V^{\nu,\tilde{\omega}}(t; \frac{my+Mx}{m+M}, y') \cdot$$
$$\int dx' g(x'+y') \left(e^{-i\frac{\mu}{\hbar t}x'(x-y)} - \frac{e^{i\frac{\mu}{\hbar t}|x'||x-y|}}{1 - i\frac{\hbar}{\alpha t}|x-y|} \right)$$

$$\psi_a(t;x,y)=\left(\frac{M}{2\pi\hbar it}\right)^{\frac{1}{2}}e^{i\frac{M}{2\hbar t}x^2}\int dy'\, f(y')V^{m,\omega}(t;y,y')e^{-i\frac{M}{\hbar t}xy'}\int dx' g(x'+y')\left(e^{-i\frac{M}{\hbar t}x'x}-\frac{e^{i\frac{M}{\hbar t}|x||x'|}}{1-i\frac{M\hbar}{\alpha_0 t}|x|}\right)$$

If we make the change of variable

$$\begin{cases}\sqrt{\mu}x=z\\ \sqrt{\nu}y=w\end{cases}$$

we arrive at

$$\psi_2(t;z,w)=\left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}}e^{i\frac{1}{2\hbar t}(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w)^2}\int dy'\, f(\frac{1}{\sqrt{\nu}}y')V^{1,\tilde{\omega}}(t;\frac{1}{1+\varepsilon}w+\sqrt{\varepsilon}z,y')$$
$$\int dx' g(x'+\frac{1}{\sqrt{\nu}}y')\left(e^{-i\frac{\sqrt{\mu}}{\hbar t}x'(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w)}-\frac{e^{i\frac{\mu}{\hbar t}|x'||z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w|}}{1-i\frac{\hbar}{\alpha t}|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w|}\right)$$

$$\psi_a(t;z,w)=\left(\frac{M}{2\pi\hbar it}\right)^{\frac{1}{2}}e^{i\frac{1}{2\hbar t}z^2}\int dy'\, f(\frac{1}{\sqrt{\nu}}y')V^{1,\tilde{\omega}}(t;w,y')e^{-i\frac{\sqrt{\varepsilon}}{(1+\varepsilon)^2}\frac{1}{\hbar t}z\frac{1}{\sqrt{\nu}}y'}$$
$$\int dx' g(x'+\frac{1}{\sqrt{\nu}}y')\left(e^{-i\frac{\sqrt{\mu}}{\hbar t}x'z}-\frac{e^{i\frac{\mu}{\hbar t}|x'||z|}}{1-i\frac{\hbar}{\alpha t}|z|}\right)$$

We shall estimate estimate separately the difference between the scattered and the free part of the wave functions. Let's consider the free parts. First we prove that the free part of ψ_2 is near to

$$\psi^\star(t;z,w)\quad=\quad\left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}}e^{i\frac{1}{2\hbar t}z^2}\int dy'\, f(\frac{1}{\sqrt{\nu}}y')V^{1,\tilde{\omega}}(t;w,y')\int dx' g(x'\quad+\quad\frac{1}{\sqrt{\nu}}y')e^{-i\frac{\sqrt{\mu}}{\hbar t}x'z}$$

In order to prove this statement it is sufficient to estimate these two norms:

$$\int dzdw\frac{1}{\sqrt{\mu\nu}}\Bigg|\int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right)V^{1,\tilde{\omega}}\left(t;\frac{1}{1+\varepsilon}w+\sqrt{\varepsilon}z,y'\right)$$
$$\left[e^{i\frac{1}{2\hbar t}(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w)^2}\int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right)e^{-i\frac{\sqrt{\mu}}{\hbar t}x'(z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w)}-e^{i\frac{1}{2\hbar t}z^2}\int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right)e^{-i\frac{\sqrt{\mu}}{\hbar t}x'z}\right]\Bigg|^2 \tag{4.26}$$

$$\int dzdw\frac{1}{\sqrt{\mu\nu}}\Bigg|\left[\int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right)V^{1,\tilde{\omega}}\left(t;\frac{1}{1+\varepsilon}w+\sqrt{\varepsilon}z,y'\right)+\right.$$
$$\left.-\int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right)V^{1,\tilde{\omega}}(t;w,y')\right]e^{i\frac{1}{2\hbar t}z^2}\int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right)e^{-i\frac{\sqrt{\mu}}{\hbar t}x'z}\Bigg|^2 \tag{4.27}$$

Using (4.5) with (4.11) and (4.12), it is straightforward to see that (4.26) and (4.27) are of order ε. In the same way it can be proved that the difference between $\psi^\star$ and $\psi^\sharp$ is order $\sqrt{\varepsilon}$ with

$$\psi^\sharp(t;z,w) = \left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{1}{2\hbar t}z^2} \int dy'\, f(\frac{1}{\sqrt{\nu}}y')V^{1,\tilde{\omega}}(t;w,y')e^{-i\frac{\sqrt{\varepsilon}}{(1+\varepsilon)^2}\frac{1}{\hbar t}z\frac{1}{\sqrt{\nu}}y'} \int dx' g(x'+\frac{1}{\sqrt{\nu}}y')e^{-i\frac{\sqrt{\mu}}{\hbar t}x'z}$$

Coming back to (x,y) variables we have

$$\psi^\sharp(t;x,y) = \left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{\mu}{2\hbar t}x^2} \int dy'\, f(y')V^{\nu,\omega}(t;y,y')e^{-i\frac{M}{\hbar t}xy'} \int dx' g(x'+y')e^{-i\frac{\mu}{\hbar t}x'x} =$$

$$= \left(\frac{1}{1+\varepsilon}\frac{M}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{1}{1+\varepsilon}\frac{M}{2\hbar t}x^2} \left(\frac{m\omega}{\pi\hbar(1-e^{-2i\tilde{\omega}t})}\sqrt{1+\varepsilon}\right)^{\frac{1}{2}} \int dy'\, f(y')e^{-\frac{m\omega}{2i\hbar\tan(\tilde{\omega}t)}(y^2+y'^2)\sqrt{1+\varepsilon}}$$
$$e^{\frac{m\omega}{i\hbar\sin(\tilde{\omega}t)}yy'\sqrt{1+\varepsilon}} \int dx' g(x')e^{-i\frac{1}{1+\varepsilon}\frac{M}{\hbar t}x'x}$$

Using (4.5) with (4.11) and (4.12), one proves that $\psi^\sharp$ is near to $\psi^\natural$.

$$\psi^\natural(t;x,y) = \left(\frac{M}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{M}{2\hbar t}x^2} \left(\frac{m\omega}{\pi\hbar(1-e^{-2i\tilde{\omega}t})}\right)^{\frac{1}{2}} \int dy'\, f(y')e^{-\frac{m\omega}{2i\hbar\tan(\tilde{\omega}t)}(y^2+y'^2)}e^{\frac{m\omega}{i\hbar\sin(\tilde{\omega}t)}yy'}.$$

$$\int dx' g(x')e^{-i\frac{M}{\hbar t}x'x} = \left(\frac{M}{2\pi\hbar it}\right)^{\frac{1}{2}} e^{i\frac{M}{2\hbar t}x^2} \int dy'\, f(y')V^{m,\omega}\left(\frac{t}{\sqrt{1+\varepsilon}};y,y'\right) \int dx' g(x')e^{-i\frac{M}{\hbar t}x'x}$$

This term is near order ε to the free part of ψ_a by Stone's theorem.
Now we turn our attention to the scattered part of ψ_2, first we prove that it is near order ε to

$$\left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} \int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right) V^{1,\tilde{\omega}}(t;w,y') \int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right) \frac{e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}}{1-i\frac{\hbar}{\alpha t\sqrt{\mu}}|z|}$$

In order to prove this statement it is sufficient to estimate these two differences:

$$\left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} \int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right) V^{1,\tilde{\omega}}(t;w,y') \left[\frac{1}{1-i\frac{\hbar}{\alpha\sqrt{\mu}t}|z|} \int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right) e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} + \right.$$
$$\left. - \frac{1}{1-i\frac{\hbar}{\alpha\sqrt{\mu}t}|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w|} \int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right) e^{i\frac{\sqrt{\mu}}{\hbar t}|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w||x'|}\right] \tag{4.28}$$

$$\left(\frac{\mu}{2\pi\hbar it}\right)^{\frac{1}{2}} \left[\int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right) V^{1,\tilde{\omega}}\left(t;\frac{1}{1+\varepsilon}w+\sqrt{\varepsilon}z,y'\right) - \int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right) V^{1,\tilde{\omega}}(t;w,y')\right]$$
$$\frac{1}{1-i\frac{\hbar}{\alpha\sqrt{\mu}t}|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w|} \int dx' g\left(x'+\frac{1}{\sqrt{\nu}}y'\right) e^{i\frac{\sqrt{\mu}}{\hbar t}|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w||x'|} \tag{4.29}$$

Now we introduce an auxiliary function $F_0(w, x')$.

$$F_0(w,x') = \int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right) g\left(x' + \frac{1}{\sqrt{\nu}}y'\right) V^{1,\tilde{\omega}}\left(t; \frac{1}{1+\varepsilon}w + \sqrt{\varepsilon}z, y'\right)$$

The following estimate holds:

$$\int \frac{dzdw}{\sqrt{\mu\nu}} \left| \left(\frac{\mu}{2\pi\hbar i t}\right)^{\frac{1}{2}} \left(\frac{1}{1 - i\frac{\hbar}{\alpha\sqrt{\mu}t}|z - \frac{\sqrt{\varepsilon}}{1+\varepsilon}w|} - \frac{1}{1 - i\frac{\hbar}{\alpha\sqrt{\mu}t}|z|} \right) \int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} F_0(w,x') \right|^2 \leq c\varepsilon \tag{4.30}$$

Using the estimate

$$\left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 \frac{\left(|z - \frac{\sqrt{\varepsilon}}{1+\varepsilon}w| - |z|\right)^2}{\left(1 + \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 z^2\right)\left(1 + \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 \left(z - \frac{\sqrt{\varepsilon}}{1+\varepsilon}w\right)^2\right)} \leq c \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 \frac{w^2}{1 + \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 z^2}$$

in order to prove (4.30) it is sufficient to prove

$$\left| w \int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} F_0(w,x') \right| \leq c_1 \tag{4.31}$$

$$\left| w^2 \int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} F_0(w,x') \right| \leq c_2 \tag{4.32}$$

Indeed, if (4.31) and (4.32) hold, then (4.30) is less or equal than

$$\varepsilon \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 \frac{1}{2\pi\hbar t} \int \frac{dzdw}{\sqrt{\mu\nu}} \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 \frac{1}{1 + \left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 z^2} \frac{c_1 + c_2}{(1+|w|)^2} = C\varepsilon^2$$

Now we prove (4.31): using (4.5) with (4.11) and (4.12) it is straightforward to see that

$$\left| w \int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} F_0(w,x') \right| \leq \nu \int dx |g(x)| \int dy |y\, f(y)| + \frac{\hbar}{\omega} \int dx |g'(x)| \int dy |f(y)| + \\ + \frac{\hbar}{\omega} \int dx |g(x)| \int dy |f'(y)|$$

Using (4.11) twice, it is possible to prove (4.32).
Now we prove that

$$\int \frac{dzdw}{\sqrt{\mu\nu}} \left| \left(\frac{\mu}{2\pi\hbar i t}\right)^{\frac{1}{2}} \frac{1}{1 - i\frac{\hbar}{\alpha\sqrt{\mu}t}|z - \frac{\sqrt{\varepsilon}}{1+\varepsilon}w|} \int dx' \left(e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} - e^{i\frac{\sqrt{\mu}}{\hbar t}|z - \frac{\sqrt{\varepsilon}}{1+\varepsilon}w||x'|} \right) F_0(w,x') \right|^2 \leq c\varepsilon^2 \tag{4.33}$$

Using the inequality

$$\left| e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|} - e^{i\frac{\sqrt{\mu}}{\hbar t}|z-\frac{\sqrt{\varepsilon}}{1+\varepsilon}w||x'|} \right| \le \frac{\sqrt{\varepsilon}}{1+\varepsilon}\frac{\sqrt{\mu}}{\hbar t}|w||x'|$$

in (4.33), we arrive at

$$\varepsilon\frac{\mu}{2\pi\hbar t}\frac{\mu}{(\hbar t)^2}\int\frac{dzdw}{\sqrt{\mu\nu}}\frac{1}{1+\left(\frac{\hbar}{\alpha t\sqrt{\mu}}\right)^2 z^2}\left(\int dx'\,|wx'F_0(w,x')|\right)^2 \tag{4.34}$$

In order to prove that (4.34) is order ε^2 it is then sufficient to prove

$$\left|wx'\int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}F_0(w,x')\right| \le c_3 \tag{4.35}$$

$$\left|w^2x'\int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}F_0(w,x')\right| \le c_4 \tag{4.36}$$

The proof of (4.35) and (4.36) is similar to the proof of (4.31) and (4.32). From (4.30) and (4.33) it follows that (4.28) is order ε.
In order to estimate (4.29) it is sufficient to estimate

$$\int\frac{dzdw}{\sqrt{\mu\nu}}\left(\frac{\mu}{2\pi\hbar t}\right)\frac{1}{1+\left(\frac{\hbar}{\alpha\sqrt{\mu}t}\right)^2 z^2}\left|\int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}\left(F_0(w+\sqrt{\varepsilon}z,x')-F_0(w,x')\right)\right|^2 \tag{4.37}$$

It is convenient to divide the domain of integration of the z variable in two regions and use

$$\int dw\left|\int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}\left(F_0(w+\sqrt{\varepsilon}z,x')-F_0(w,x')\right)\right|^2 \le$$
$$\varepsilon z^2\int dw\left(\int dx'\left|\frac{\partial}{\partial w}F_0(w+\sqrt{\varepsilon}z,x')-F_0(w,x')\right|\right)^2 \text{ if } \sqrt{\varepsilon}|z|\le 1 \tag{4.38}$$

$$\int dw\left|\int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}\left(F_0(w+\sqrt{\varepsilon}z,x')-F_0(w,x')\right)\right|^2 \le$$
$$2\int dw\left(\int dx'\left|F_0(w+\sqrt{\varepsilon}z,x')-F_0(w,x')\right|\right)^2 \text{ if } \sqrt{\varepsilon}|z|<1 \tag{4.39}$$

In order to prove that the quantities appearing at the r.h.s of (4.38) and (4.39) are finite, one must proceed as in the estimate of (4.30) It is now straightforward to prove that the integral over z is order ε and conclude the estimate of (4.29). Now we introduce a second auxiliary function

$$F_\varepsilon(w,z,x') = \int dy'\, f\left(\frac{1}{\sqrt{\nu}}y'\right) g\left(x' + \frac{1}{\sqrt{\nu}}y'\right) e^{-i\frac{\sqrt{\varepsilon}}{1+\varepsilon}\frac{1}{\hbar t}y'z} V^{1,\tilde{\omega}}\left(t; \frac{1}{1+\varepsilon}w + \sqrt{\varepsilon}z, y'\right)$$

Using (4.5) with (4.11) and (4.12), it is straightforward to prove that:

$$\int \frac{dz dw}{\sqrt{\mu\nu}} \left(\frac{\mu}{2\pi\hbar t}\right) \frac{1}{1+\left(\frac{\hbar}{\alpha\sqrt{\mu}t}\right)^2 z^2} \left|\int dx' e^{i\frac{\sqrt{\mu}}{\hbar t}|z||x'|}\left(F_\varepsilon(w,z,x') - F_0(w,x')\right)\right|^2 \leq c\varepsilon \tag{4.40}$$

Coming back to (x, y) variables, so far we have proved that the scattered part of ψ_2 is near to

$$\psi_a^\flat(t;x,y) = \left(\frac{\mu}{2\pi\hbar i t}\right)^{\frac{1}{2}} e^{i\frac{\mu}{2\hbar t}x^2} \int dy'\, f(y') V^{\nu,\tilde{\omega}}(t;y,y') e^{-i\frac{\mu}{\hbar t}xy'} \int dx' g(x'+y') \frac{e^{i\frac{\mu}{\hbar t}|x||x'|}}{1 - i\frac{\mu\hbar}{\alpha_0 t}|x|}$$

Repeating the arguments used at the end of the estimate of the free part, we prove that $\psi^\flat$ is near order ε to the free part of ψ_a and this concludes the proof. □

Proof of theorem 4.1.1
It is an obvious consequence of lemmas 4.1.2, 4.1.3, 4.1.4.

The result of theorem 4.1.1 can be more conveniently rephrased in terms of reduced density matrix for the heavy particle, which is defined by the integral operator $\hat{\rho}(t)$ in $L^2(\mathbb{R})$ given by the kernel

$$\hat{\rho}(t;y,y') = \int dx\, \psi(t;x,y)\overline{\psi}(t;x,y') \tag{4.41}$$

We also introduce the integral operator $\hat{\rho}_a(t)$ defined by

$$\begin{aligned}\hat{\rho}_a(t;y,y') &= \int dx\, \psi_a(t;x,y)\overline{\psi_a}(t;x,y') \\ &= \int dz f(z) V^{m,\omega}(t;y,z) \int dz' \overline{f}(z') \overline{V^{m,\omega}}(t;y',z') \mathcal{I}(z,z') \end{aligned}\tag{4.42}$$

where

$$\mathcal{I}(z,z') \equiv \int dk (W_+^{\alpha_0,z} g_\delta)(k) \overline{(W_+^{\alpha_0,z'} g_\delta)}(k) = \left((\Omega_+^{\alpha_0,z'})^{-1} g_\delta, (\Omega_+^{\alpha_0,z})^{-1} g_\delta\right) \tag{4.43}$$

Notice that, from (4.42),(4.43) one has

$$\hat{\rho}_a(t) = V^{m,\omega}(t)\hat{\rho}^a(0)V^{m,\omega}(-t) \tag{4.44}$$

where $\hat{\rho}^a_0$ is defined by the integral kernel

$$\hat{\rho}^a(0; z, z') = f(z)\overline{f}(z')\mathcal{I}(z, z') \tag{4.45}$$

It is easily seen that $\mathcal{I}(z, z') = \overline{\mathcal{I}}(z', z)$, $|\mathcal{I}(z, z')| \leq 1$ and the equality holds only if $z = z'$. Then $\hat{\rho}_a(0)$ is a self-adjoint and trace-class operator, with $Tr(\hat{\rho}_a(0)) = 1$; it is also positive since

$$\begin{aligned}(h, \hat{\rho}_a(0)h) &= \int dz\overline{h}(z) \int dz' h(z')f(z)\overline{f}(z') \int dk (W^{\alpha_0,z}g)(k)\overline{(W^{\alpha_0,z'}g)}(k) \\ &= \int dk \left| \int dz\overline{h}(z)f(y)(W^{\alpha_0,z}g)(k) \right|^2 \end{aligned} \tag{4.46}$$

Moreover we have

$$Tr((\hat{\rho}_a(0))^2) = \int dz dz' |f(z)|^2 |f_\sigma(z')|^2 |\mathcal{I}(z, z')|^2 < 1 \tag{4.47}$$

We conclude that $\hat{\rho}_a(0)$ and its harmonic evolution $\hat{\rho}_a(t)$ are density matrices describing mixture states and, by theorem (4.1.1),

$$Tr(|\hat{\rho}(t) - \hat{\rho}_a(t)|) < C\,\varepsilon^{\frac{1}{2}} \tag{4.48}$$

This means that in our asymptotic regime the motion of the heavy particle is a free evolution. On the other hand the presence of the light particle has a relevant effect, since it produces a transition of the initial state of the heavy particle from $\hat{\rho}_0(z, z') = f(z)\overline{f}(z')$ to $\hat{\rho}_a(0; z, z')$. We shall see in the next section that this is the origin of the decoherence effect on the heavy particle.

4.2 Decoherence Effect

Here we discuss an application of formula (4.42),(4.43) to a concrete example of quantum evolution and we give an explicit computation of the decoherence effect.

We shall make further assumptions on the initial state: take $h_1, h_2 \in C_0^\infty(-1,+1)$ such that $\|f\| = \|g\| = 1$; for later use it is convenient to choose h_2 even. We put

$$f(y) = \frac{1}{\sqrt{\sigma}}[f^+(y) + f^-(y)] \qquad f^\pm(y) = h_1\left(\frac{y \pm y_0}{\sigma}\right) e^{i\frac{p_0}{\hbar}y} \tag{4.49}$$

$$g(x) = \frac{1}{\sqrt{\delta}} h_2\left(\frac{x - x_0}{\delta}\right) e^{i\frac{q_0}{\hbar}x} \tag{4.50}$$

The heavy particle is assumed to be in a typical superposition state of two spatially separated wave packets, one localized in $y = y_0$ with mean value of the momentum p_0 and the other localized in $y = -y_0$ with mean value of the momentum $-p_0$.
The parameters σ, δ denote the spreading in position at time zero for the heavy and the light particle respectively.
We define $\beta = \alpha_0 \delta$ and $\delta^\pm = (\pm y_0 - \sigma, \pm y_0 + \sigma)$.
We shall make the following assumptions on the physical parameters of the system.

$$r_0 \in (-R_0, R_0), \qquad \frac{\delta}{R_0 - |r_0|} \ll 1, \qquad \frac{\sigma}{\delta} \ll 1 \tag{4.51}$$

i.e. we consider the light particle localized in the region between the two wave packets describing the heavy particle and well separated from each wave packet. Moreover the spreading in position of the wave packets is much smaller than the spreading of the light particle. Notice that (4.51) obviously imply $\frac{\sigma}{R_0} \ll 1$.
Using assumptions (4.51) we can give an estimate of the basic object $\mathcal{I}(z,z')$ for $z \in \Delta^\pm$ and $z' \in \Delta^\pm$ and then we can find a more suitable expression for the reduced density matrix of the heavy particle.

Lemma 4.2.1 *Assume (4.51). Then*

$$\sup_{z,z' \in \Delta^\pm} |\mathcal{I}(z,z') - 1| < C_4 \left(\frac{\sigma}{\delta} + \frac{\delta}{R_0 - |r_0|}\right) \tag{4.52}$$

$$\sup_{z \in \Delta^+, z' \in \Delta^-} |\mathcal{I}(z,z') - \Lambda| = \sup_{z \in \Delta^-, z' \in \Delta^+} \left|\mathcal{I}(z,z') - \overline{\Lambda}\right| < C_5 \, \frac{\delta}{R_0 - |r_0|} \tag{4.53}$$

where

$$\Lambda = \int dk |\tilde{g}_\delta(k)|^2 \mathcal{T}_{\alpha_0}(k) = \int dk |\tilde{g}(k - k_0\delta)|^2 \mathcal{T}_\beta(k) \tag{4.54}$$

$$k_0 = \frac{q_0}{\hbar} \tag{4.55}$$

$$\mathcal{T}_\gamma(k) = -\frac{ik}{\gamma - ik} \qquad \gamma > 0 \tag{4.56}$$

and $C_4 > 0$,$C_5 > 0$ are explicitly given as functions of β,g.

Proof
First we note that for $z \in \Delta^-$

$$\frac{1}{\sqrt{\delta}}\left(W_+^{\alpha_0,z} g_\delta\right)\left(\frac{k}{\delta}\right) = e^{i(k_0\delta - k)\frac{x_0}{\delta}}\tilde{g}(k - k_0\delta) + \mathcal{R}_\beta(k) e^{i(k_0\delta + |k|)\frac{x_0}{\delta} - i(k+|k|)\frac{z}{\delta}}\tilde{g}(|k| + k_0\delta) \tag{4.57}$$

and for $z \in \Delta^+$

$$\frac{1}{\sqrt{\delta}}\left(W_+^{\alpha_0,z} g_\delta\right)\left(\frac{k}{\delta}\right) = e^{i(k_0\delta - k)\frac{x_0}{\delta}}\tilde{g}(k - k_0\delta) + \mathcal{R}_\beta(k) e^{i(k_0\delta - |k|)\frac{x_0}{\delta} - i(k-|k|)\frac{z}{\delta}}\tilde{g}(|k| - k_0\delta) \tag{4.58}$$

where we have used the fact that

$$\tilde{g}_\delta(k) = \sqrt{\delta}\tilde{g}(k\delta - k_0\delta)e^{-i(k-k_0)r_0} \tag{4.59}$$

Then for $z, z' \in \Delta^-$ we have

$$\begin{aligned}
\mathcal{I}(z,z') &= 1 + \int dk |\tilde{g}(|k| + k_0\delta)|^2 |\mathcal{R}_\beta(k)|^2 e^{-i(k+|k|)\frac{z-z'}{\delta}} \\
&+ \int dk \overline{\tilde{g}}(k - k_0\delta)\tilde{g}(|k| + k_0\delta)\mathcal{R}_\beta(k) e^{i(k+|k|)\frac{r_0 - z}{\delta}} \\
&+ \int dk \overline{\tilde{g}}(|k| + k_0\delta)\tilde{g}(k - k_0\delta)\overline{\mathcal{R}_\beta}(k) e^{-i(k+|k|)\frac{r_0 - z'}{\delta}} \\
&= 1 + \int_0^\infty dk |\tilde{g}(k + k_0\delta)|^2 |\mathcal{R}_\beta(k)|^2 \left(e^{-2ik\frac{z-z'}{\delta}} - 1\right) \\
&+ \int_0^\infty dk \overline{\tilde{g}}(k - k_0\delta)\tilde{g}(k + k_0\delta)\mathcal{R}_\beta(k) e^{2ik\frac{x_0 - z}{\delta}} \\
&+ \int_0^\infty dk \overline{\tilde{g}}(k + k_0\delta)\tilde{g}(k - k_0\delta)\overline{\mathcal{R}_\beta}(k) e^{-2ik\frac{x_0 - z'}{\delta}} \\
&\equiv 1 + a_1 + a_2 + a_3
\end{aligned} \tag{4.60}$$

where we have used the identity $\mathcal{R}_\beta + \overline{\mathcal{R}_\beta} + 2|\mathcal{R}_\beta|^2 = 0$ and the fact that $\tilde{g}$ is even.
Using (4.51) we easily estimate a_1

$$|a_1| \leq 2\frac{|z-z'|}{\delta}\int_0^\infty dk|\tilde{g}(k+k_0\delta)|^2 k|\mathcal{R}_\beta(k)|^2 \leq 4\frac{\sigma}{\delta}\int_0^\infty dk|\tilde{g}(k+k_0\delta)|^2 k|\mathcal{R}_\beta(k)|^2$$
$$\leq 2\beta\|g\|^2\frac{\sigma}{\delta} \tag{4.61}$$

For the estimate of a_2 it is convenient to integrate by parts

$$\begin{aligned}
|a_2| &= \left|\frac{1}{2i}\frac{\delta}{r_0-z}\int_0^\infty dk\bar{\tilde{g}}(k-k_0\delta)\tilde{g}(k+k_0\delta)\mathcal{R}_\beta(k)\frac{d}{dk}e^{2ik\frac{r_0-z}{\delta}}\right| \\
&= \frac{\delta}{2|r_0-z|}\left|\int_0^\infty dk\frac{d}{dk}\left(\bar{\tilde{g}}(k-k_0\delta)\tilde{g}(k+k_0\delta)\mathcal{R}_\beta(k)\right)e^{2ik\frac{r_0-y}{\delta}} - |\tilde{g}(k_0\delta)|^2\right| \\
&\leq \frac{\delta}{R_0-|r_0|}\left[\frac{1}{2\pi}\left(\int dr|g(r)|\right)^2 + \frac{1}{\beta}\int_0^\infty dk\left|\bar{\tilde{g}}(k-k_0\delta)\tilde{g}(k+k_0\delta)\right|\right. \\
&\left.+\int_0^\infty dk\left|\bar{\tilde{g}}'(k-k_0\delta)\tilde{g}(k+k_0\delta)\right| + \int_0^\infty dk\left|\bar{\tilde{g}}(k-k_0\delta)\tilde{g}'(k+k_0\delta)\right|\right] \\
&\leq \frac{\delta}{R_0-|r_0|}\left[\frac{1}{2\pi}\left(\int dr|g(r)|\right)^2 + \frac{1}{\beta}\|g\|^2 + 2\|\tilde{g}'\|\|g\|\right]
\end{aligned} \tag{4.62}$$

The term a_3 is analyzed exactly in the same way and then we get the estimate (4.52) for $z, z' \in \Delta^-$. Since in the case $z, z' \in \Delta^+$ the computation is similar we conclude that (4.52) holds.

In order to prove (4.53) we consider the case $z \in \Delta^+$ and $z' \in \Delta^-$ (the case $z \in \Delta^-$ and $z' \in \Delta^+$ can be treated exactly in the same way) and we obtain

$$\begin{aligned}
\mathcal{I}(z,z') = 1 + \int dk\overline{\tilde{g}}(|k|+k_0\delta)\tilde{g}(|k|-k_0\delta)|\mathcal{R}_\beta(k)|^2 e^{-2i|k|\frac{r_0}{\delta}+i(|k|-k)\frac{z}{\delta}+i(|k|+k)\frac{z'}{\delta}} \\
+ \int dk\overline{\tilde{g}}(k-k_0\delta)\tilde{g}(|k|-k_0\delta)\mathcal{R}_\beta(k)e^{-i(|k|-k)\frac{r_0-z}{\delta}} \\
+ \int dk\overline{\tilde{g}}(|k|+k_0\delta)\tilde{g}(k-k_0\delta)\overline{\mathcal{R}_\beta}(k)e^{-i(|k|+k)\frac{r_0-z'}{\delta}} \\
= 1 + \int_0^\infty dk\left(|\tilde{g}(k-k_0\delta)|^2\mathcal{R}_\beta(k) + |\tilde{g}(k+k_0\delta)|^2\overline{\mathcal{R}_\beta}(k)\right) \\
+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)|\mathcal{R}_\beta(k)|^2 e^{-2ik\frac{r_0-z'}{\delta}} \\
+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)\overline{\mathcal{R}_\beta}(k)e^{-2ik\frac{r_0-z'}{\delta}} \\
+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)|\mathcal{R}_\beta(k)|^2 e^{-2ik\frac{r_0-z}{\delta}} \\
+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)\mathcal{R}_\beta(k)e^{-2ik\frac{r_0-z}{\delta}}
\end{aligned} \tag{4.63}$$

The estimate of the last four terms of (4.63) proceeds exactly as the estimate of a_2 in (4.62). On the other hand

$$\begin{aligned}
1 + \int_0^\infty dk\left(|\tilde{g}(k-k_0\delta)|^2\mathcal{R}_\beta(k) + |\tilde{g}(k+k_0\delta)|^2\overline{\mathcal{R}_\beta}(k)\right) \\
= \int dk|\tilde{g}(k-k_0\delta)|^2\left(\frac{-ik}{\beta-ik}\right)
\end{aligned} \tag{4.64}$$

and this concludes the proof of the proposition. □

Proposition (4.2.1) allows us to find a further approximate form for the reduced density matrix.

Theorem 4.2.2 *Under the assumptions (4.51) and for any $t \geq 0$ we have*

$$\left[Tr\left(\left(\hat{\rho}^a(t)-\hat{\rho}^f(t)\right)^2\right)\right]^{1/2} < C_6\left(\frac{\sigma}{\delta}+\frac{\delta}{R_0-|r_0|}\right) \tag{4.65}$$

where

$$\hat{\rho}^f(t) = V^{m,\omega}(t)\hat{\rho}_0^f V^{m,\omega}(-t) \tag{4.66}$$

$$\hat{\rho}_0^f(z,z') = \frac{1}{2}f^+(z)\overline{f^+}(z') + \frac{1}{2}f^-(z)\overline{f^-}(z') + \frac{\Lambda}{2}f^+(z)\overline{f^-}(z') + \frac{\overline{\Lambda}}{2}f^-(z)\overline{f^+}(z') \tag{4.67}$$

and $C_6 > 0$ is explicitely given as function of β,g.

Proof

$$\begin{aligned}
&Tr\left(\left(\hat{\rho}^a(t) - \hat{\rho}^f(t)\right)^2\right) = Tr\left(\left(\hat{\rho}_0^a - \hat{\rho}_0^f\right)^2\right)\\
&= \frac{1}{4}\int dydz' \left| f^+(z)\overline{f^+}(z')(\mathcal{I}(z,z')-1) + f^-(z)\overline{f^-}(z')(\mathcal{I}(z,z')-1)\right.\\
&\left. + f^+(z)\overline{f^-}(z')(\mathcal{I}(z,z')-\Lambda) + f^-(z)\overline{f^+}(z')(\mathcal{I}(z,z')-\overline{\Lambda})\right|^2\\
&\leq \sup_{z,z'\in\Delta^+}|\mathcal{I}(z,z')-1|^2 + \sup_{z,z'\in\Delta^-}|\mathcal{I}(z,z')-1|^2\\
&+ \sup_{z\in\Delta^+,z'\in\Delta^-}|\mathcal{I}(z,z')-\Lambda|^2 + \sup_{z\in\Delta^-,z'\in\Delta^+}|\mathcal{I}(z,z')-\overline{\Lambda}|^2
\end{aligned} \tag{4.68}$$

Using lemma (4.2.1) we conclude the proof. □

From theorems (4.2.2) and (4.1.1) we conclude that the reduced density matrix for the heavy particle in the position representation can be approximated by the density matrix

$$\begin{aligned}
\hat{\rho}^f(t,y,y') &= \frac{1}{2}(V^{m,\omega}(t)f^+)(y)(V^{m,\omega}(-t)\overline{f^+})(y') + \frac{1}{2}(V^{m,\omega}(t)f^-)(y)(V^{m,\omega}(-t)\overline{f^-})(R')\\
&+\frac{\Lambda}{2}(V^{m,\omega}(t)f^+)(y)(V^{m,\omega}(-t)\overline{f^-})(y') + \frac{\overline{\Lambda}}{2}(V^{m,\omega}(t)f^-)(y)(V^{m,\omega}(-t)\overline{f^+})(y')
\end{aligned} \tag{4.69}$$

with an explicit control of the error.
If the interaction with the light particle is switched off, i.e. for $\alpha_0 = 0$, we have $\Lambda = 1$ and then (4.69) reduces to the pure state corresponding to the coherent superposition of the free evolution of the two wave packets $f^{\pm}$.
On the other hand, if $\alpha_0 > 0$ one easily sees that $0 < |\Lambda| < 1$ and then (4.69) is a mixed state for which the interference terms are reduced by the factor Λ and this is the typical manifestation of the (partial) decoherence effect induced by the light particle on the heavy one.
The relevant parameter Λ (see (4.56)) is defined in terms of the probability distribution of the momentum of the light particle at time zero $|g_\delta(k)|^2$ and of the transmission coefficient $\mathcal{T}_{\alpha_0}(k)$

associated to the hamiltonian $H_{\alpha_0,0}$ defined by the laplacian perturbed by a point interaction of strength α_0 placed at the origin,(see e.g. [AGH-KH]).
Then the decoherence effect is emphasized if the fraction of transmitted wave for the light particle is small.
In fact for $k_0 \gg \alpha_0$ the light particle is completely transmitted and $\Lambda \to 1$, i.e. there is no decoherence effect.
On the other hand, for $k_0 \ll \alpha_0$ the decoherence effect is present since $\Lambda \to \int dk|\tilde{g}(k)|^2|\mathcal{T}_\beta(k)|^2$.
In such case one can say that the decoherence effect is proportional to the reflection probability for a particle described by the hamiltonian $H_{\beta,0}$ which at time zero is in the state g.

Appendix A

The Green's function of the free system

Here we derive the expression of the integral kernel of the resolvent of H_0 used in the paper; let $d = 1, 2, 3$ be the dimension of the space

$$G^{\lambda}(x, y; x', y') = (H_0 + \lambda)^{-1}(x, y; x'y') = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\hbar\omega} \frac{1}{\pi^{\frac{dN}{2}}} \left(\frac{m\omega}{\hbar}\right)^{\frac{dN}{2}} \left(\frac{M\omega}{\hbar}\right)^{\frac{d}{2}}$$
$$\int_0^1 d\nu \, \nu^{\frac{\lambda}{\hbar\omega} + \frac{dN}{2} - 1} \frac{1}{(1-\nu^2)^{\frac{dN}{2}}} \frac{1}{\left(\ln \frac{1}{\nu}\right)^{\frac{d}{2}}} e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\frac{m\omega}{\hbar}(y^2+y'^2)} e^{-\frac{M\omega}{\hbar}\frac{1}{2\ln\frac{1}{\nu}}(x-x')^2} e^{-\frac{m\omega}{\hbar}\frac{\nu}{1-\nu^2}(y-y')^2} \quad \text{(A.1)}$$

Starting from the Mehler kernel and performing the Laplace transform with respect to time one finds the Green's function for the harmonic oscillator in $\mathbb{R}^n$.

$$G^{\lambda}_{\text{osc}}(y, y') = (H_{\text{osc}} + \lambda)^{-1}(y, y') = \int_0^{+\infty} dt \, \frac{e^{-\frac{\lambda}{\hbar}t}}{\hbar} \left(\frac{m\omega}{\pi\hbar(1-e^{-2\omega t})}\right)^{\frac{n}{2}} e^{-\frac{m\omega}{\hbar}\frac{y^2+y'^2}{2\tanh(\omega t)} + \frac{m\omega}{\hbar}\frac{y\cdot y'}{\sinh(\omega t)}} e^{-t\frac{n\omega}{2}}$$

With the change of variable $\nu = e^{-\omega t}$ one obtains:

$$G^{\lambda}{}_{\text{osc}}(y, y') = (H_{\text{osc}} + \lambda)^{-1}(y, y') = \left(\frac{m\omega}{\hbar}\right)^{\frac{n}{2}} \frac{1}{\hbar\omega} \frac{1}{\pi^{\frac{n}{2}}} e^{-\frac{1}{2}\frac{m\omega}{\hbar}(y^2+y'^2)}$$
$$\int_0^1 d\nu \, \nu^{\frac{\lambda}{\hbar\omega} + \frac{n}{2} - 1} \frac{1}{(1-\nu^2)^{\frac{n}{2}}} e^{\frac{m\omega}{\hbar}\frac{2\nu y\cdot y' - \nu^2\left(y^2+y'^2\right)}{(1-\nu^2)}} \quad \text{(A.2)}$$

Note that formula A.2 can be also derived using Hermite functions (see e.g. [BC]). To compute $(H_0 + \lambda)^{-1}$, we solve the following equation with respect to f for a given g.

$$\left[\left(-\frac{\hbar^2}{2M}\Delta_x - \frac{\hbar^2}{2m}\Delta_y + \frac{1}{2}m\omega^2 y^2 + \lambda\right) f\right](x, y) = g(x, y)$$

We have put $y^0 = 0$ for simplicity. Taking the Fourier transform with respect to the x variable we find

$$\begin{aligned}\mathcal{F}g(k,y) &= \int dy' G_{\mathrm{osc}}^{\lambda+\frac{\hbar^2 k^2}{2M}}(y,y')\mathcal{F}f(k,y') \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int dx' dy' e^{-\imath k\cdot x'} G_{\mathrm{osc}}^{\lambda+\frac{\hbar^2 k^2}{2M}}(y,y') f(x',y')\end{aligned}$$

and then

$$(H_0+\lambda)^{-1}(x,y;x',y') = \frac{1}{(2\pi)^d}\int dk e^{\imath k\cdot(x-x')} G_{\mathrm{osc}}^{\lambda+\frac{\hbar^2 k^2}{2M}}(y,y')$$

Substituting (A.2) in this equation we finally get (A.1).

Appendix B

Unitary Group for the Harmonic Oscillator perturbed by a Point Interaction

In this appendix we give an integral representation for the unitary group of the one dimensional hamiltonian

$$H_\alpha = H_0 + \alpha\delta(y) = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2 + \alpha\delta(y)$$

In [FI] it was proved that

$$(H_\alpha + \lambda)^{-1}(y, y') = G^\lambda(y, y') - \frac{1}{\frac{1}{\alpha} + G^\lambda(0,0)} G^\lambda(y, 0) G^\lambda(y', 0) \tag{B.1}$$

In the previous formula $G^\lambda(y, y')$ is the integral kernel of $(H_0 + \lambda)^{-1}$. Equation (B.1) is equivalent to:

$$\mathscr{D}(H_\alpha) = \{u \in L^2(\mathbb{R})\, s.t.\, u = \varphi^\lambda + G^\lambda q, \varphi^\lambda \in \mathscr{D}(H_0), \alpha u(0) = -q\}$$

$$(H_\alpha + \lambda)u = (H_0 + \lambda)\varphi^\lambda$$

The domain of H_α can be written also in the equivalent way:

$$\mathscr{D}(H_\alpha) = \{u \in L^2(\mathbb{R})\, s.t.\, x^2 u \in L^2(\mathbb{R}),\, u \in H^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^-) \cap H^1(\mathbb{R}), \frac{\hbar^2}{2m}(u'(0^+) - u'(0^-)) = \alpha u(0)\} \tag{B.2}$$

Now we shall prove that, for a regular initial datum $\psi_0 \in \mathscr{D}(H_\alpha)$,

$$\psi(t; y) = \left(e^{-i\frac{t}{\hbar}H_0}\psi_0\right)(y) + \frac{i}{\hbar}\int_0^t ds\, e^{-i\frac{t-s}{\hbar}H_0}(y, 0) q(s) \tag{B.3}$$

is the solution of the time dependent Schrodinger equation.
The function $q(t)$ is solution of

$$q(t) + \alpha \frac{i}{\hbar} \int_0^t ds \left[\frac{m\omega}{\pi\hbar(1 - e^{-2i\omega(t-s)})} \right]^{\frac{1}{2}} q(s) + \alpha \left(e^{-i\frac{t}{\hbar}H_0} \psi_0 \right)(0) = 0 \tag{B.4}$$

If we project equation (B.3) on the harmonic oscillator eigenvalue basis we obtain

$$\psi(t;n) = e^{-in\omega t} \psi_0(n) \frac{i}{\hbar} \int_0^{t-s} ds\, e^{-in\omega(t-s)} q(s) \tag{B.5}$$

Integrating by parts we get

$$\psi(t;n) = e^{-in\omega t} \varphi_0^\lambda(n) + \frac{i}{\hbar} \frac{\lambda}{n\hbar\omega + \lambda} \int_0^t ds e^{-in\omega(t-s)} q(s) - \frac{1}{n\hbar\omega + \lambda} \int_0^t ds e^{-in\omega(t-s)} \overset{\circ}{q}(s) + \\ + \frac{1}{n\hbar\omega + \lambda} q(t) \tag{B.6}$$

If ψ_0 is a regular function then the first three terms on the r.h.s.of (B.6) belong to $\mathscr{D}(H_0)$ and then they have continuous first derivative; the only discontuities of the first derivative of $\psi(t;y)$ come from the last term of (B.6) which is proportional to $G^\lambda(y,0)$.
It is easy to show that $\frac{d}{dy} G^\lambda(y,0) \simeq -\frac{m}{\hbar^2} sgm(y)$ near the origin. Then using (B.2), $\psi(t)$ belongs to $\mathscr{D}(H_\alpha)$ if and only if (B.4) holds. Taking the time derivative of (B.3), and using (B.6) one shows that $i\hbar \frac{d\psi}{dt}(t) = H_\alpha \psi(t)$ holds.

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