On modular and cusp forms with respect to the congruence subgroup, over which the map given by the gradients of odd Theta functions in genus 2 factors, and related topics

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Introduction

The theory of automorphic functions rose around the second half of the nineteenth century as the study of functions on a space, which are invariant under the action of a group and consequently well defined on the quotient space.

The so-called elliptic functions, for which the first systematical exposition was given by Karl Weierstrass, are one of the simplest examples of automorphic functions. These are meromorphic functions of one variable with two independent periods and can be regarded, therefore, as functions on the complex torus $\mathbb{C}/L$, where L is the lattice generated by the two periods [1].

A first classical generalization of this notion is suggested by working with functions defined on isomorphism classes of complex tori. Since there exists a bijection between the isomorphic classes of complex tori and the points of the quotient space $\mathbb{H}/SL(2,\mathbb{Z})$ [2], where $\mathbb{H}$ is the complex upper half-plane, a remarkable example pertaining to the theory of automorphic functions is provided by holomorphic functions on $\mathbb{H}$, which are invariant under the action of $SL(2,\mathbb{Z})$; this is what is meant by a modular function [3].

More in general, the classical theory of the so-called elliptic modular forms pertained to the study of holomorphic functions on $\mathbb{H}$, which transform under the action of a discrete subgroup of $SL(2,\mathbb{R})$ as a multiplication by a factor satisfying the 1-cocycle condition. In the first half of the twentieth century Carl Ludwig Siegel was the first to generalize the elliptic modular theory to the case of more variable, by discovering some prominent examples of automorphic functions in several complex variables; these are named after him Siegel modular forms. In this theory ([F], [Kl], [VdG]), $\mathbb{H}$ is generalized by the upper half-plane $\mathfrak{S}_g := \{\tau \in Sym(g,\mathbb{C}) \mid Im\tau > 0\}$ and a transitive action of the symplectic group $Sp(2g,\mathbb{R})$ is defined on $\mathfrak{S}_g$, thus generalizing the action of $SL(2,\mathbb{R})$ on $\mathbb{H}$:

[1] A classical result completely describes these functions by means of the Weierstrass $\wp$-function and its derivative; the elementary theory is detailed, for instance, in [FB]

[2] See, for instance, [FB].

[3] Thanks to a classical result, the field of modular functions is known to be generated by the so-called absolute modular invariant, discovered by Felix Klein in 1879 (see Chapter 2, Example 2.3)

$$Sp(2g,\mathbb{R}) \times \mathfrak{S}_g \longrightarrow \mathfrak{S}_g$$

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tau\right) \to (A\tau + B)\cdot(C\tau + D)^{-1}$$

The arithmetic subgroup $\Gamma_g := Sp(2g,\mathbb{Z})$ plays, in particular, a meaningful role according to this action from a geometrical point of view, since the points of the quotient space $A_g := \mathfrak{S}_g/\Gamma_g$ describe the isomorphism classes of the principally polarized abelian varieties [4].

Modular forms can be used as coordinates in order to construct suitable projective immersions of level quotient spaces with respect to the action of remarkable subgroups of the group Γ_g. Riemann Theta functions with characteristics:

$$\theta_m(\tau,z) := \sum_{n\in\mathbb{Z}^g} e^{\pi i\left[{}^t\left(n+\frac{m'}{2}\right)\tau\left(n+\frac{m'}{2}\right)+2\,{}^t\left(n+\frac{m'}{2}\right)\left(z+\frac{m''}{2}\right)\right]}$$

are, in particular, are a good instrument at disposal of the theory, in order to construct suitable modular forms, because the so-called Theta constants:

$$\theta_m(\tau) := \theta_m(\tau,0)$$

and the Jacobian determinants:

$$D(n_1,\ldots,n_g)(\tau) := \begin{vmatrix} \frac{\partial}{\partial z_1}\theta_{n_1}|_{z=0}(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_1}|_{z=0}(\tau) \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_1}\theta_{n_g}|_{z=0}(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_g}|_{z=0}(\tau) \end{vmatrix}$$

turn out to be modular forms with respect to a technical level subgroup $\Gamma_g(4,8) \subset \Gamma_g$. A remarkable map can be defined in particular on the level moduli space $A_g^{4,8} := \mathfrak{S}_g/\Gamma_g(4,8)$, whose image lies in the Grassmannian of g-dimensional complex subspaces in $\mathbb{C}^{2^{g-1}(2^g-1)}$ and whose Plücker coordinates are the Jacobian determinants:

$$\mathbb{P}grTh : A_g^{4,8} \longrightarrow Gr_{\mathbb{C}}(g, 2^{g-1}(2^g-1))$$

In their paper [GSM] Samuel Grushevski and Riccardo Salvati Manni proved that the map $\mathbb{P}grTh$ is generically injective whenever $g \geq 3$ and injective on tangent spaces when $g \geq 2$.
The map is conjectured to be injective whenever $g \geq 3$, albeit it has not been proved yet. On the other hand, when $g = 2$, the map:

[4] See, for instance, [GH], [De] or [SU]

$$\mathbb{P}grTh : A^{4,8} \longrightarrow \mathbb{P}^{14}$$

is known not to be injective.

In this thesis, a description is provided for the congruence subgroup Γ such that the the map $\mathbb{P}grTh$ in genus 2 is still well defined on the correspondent level moduli space $A_\Gamma := \mathfrak{S}_g/\Gamma$ and also injective. It is also proved that Γ is a normal subgroup of Γ_2 with no fixed points; hence, the correspondent level moduli space A_Γ turns out to be smooth.
A structure theorem is also proved for the even part of the rings of the graded rings of modular forms $A(\Gamma)$ and for the even part of the ideal of cusp forms $S(\Gamma)$ (namely the modular forms which vanish on the boundary of Satake's compactification) with respect to Γ. These are, indeed, the only parts which counts in studying the geometry of the desingularization of the map on the boundary of Satake's compactification. Alas, a geometrical description still misses, due to the the not simple structure of $A(\Gamma)$ and $S(\Gamma)$.

Chapters 1 and 2 are designed to provide an outline of the basic theory; the reader is obviously referred to the bibliography for a deeper knowledge of the topics.

Chapter 3 is mainly focused on Theta constants and related topics. Section 3.6 is, in particular, devoted to a new description found for a known modular form of weight 30 in genus 2, which is characterized by transforming with a non-trivial character under the action of Γ_2.

Chapter 4 is, finally, centered around the above mentioned results concerning the group Γ.

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Notation

For each set A the symbol $|A|$ will stand for its cardinality. The symbol $A \subset B$ will mean that A is a subset of B, which is not necessarily proper. To state that the set A is a proper subset of B, the symbol $A \subsetneq B$ will be used.
For an associative ring with unity R, the symbol $M(g, R)$ will denote the $g \times g$ matrices, whose entries are elements of R, while the symbol $Sym_g(R)$ will stand for the symmetric $g \times g$ matrices.
For a field F, $GL(g, F)$ and $SL(g, F)$ will stand respectively for the general linear group of degree n and for the special linear group of degree n.
The symbol S_n will stand for the symmetric group of order $n!$.
The exponential function $e^{\pi i z}$ will be denoted by the symbol $\exp(z)$.

Chapter 1

The Siegel Upper Half-plane and the Simplectic Group

A more exhaustive discussion concerning these topics can be found in Siegel's classical work [Si], as well as in Freitag's book [F] and Klingen's book [Kl], where a focus on Minkowski's reduction theory is also available. Most of the topics are also outlined in Namikawa's Lecture notes [Na], where Satake's and Mumford's compactifications of the quotient space are discussed.

1.1 The Symplectic Group

Definition 1.1. *The* **symplectic group of degree** g *is the following subgroup of* $GL(2g, \mathbb{R})$*:*

$$Sp(g, \mathbb{R}) :== \left\{ \gamma \in M(2g, \mathbb{R}) \mid {}^t\gamma \cdot J_g \cdot \gamma = J_g \right\}$$

where:

$$J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

is the so called **symplectic standard form** *of degree* g*.*

The symplectic group naturally arises as the group of the automorphisms of the lattice $\mathbb{Z}^{2g}$, provided with the form J_g [1]; as it clearly turns out from the definition, it is an algebraic group.
A generic element of the symplectic group can be depicted in a standard block notation as:

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \qquad a_\gamma, b_\gamma, c_\gamma, d_\gamma \in M(g, \mathbb{R})$$

[1]More in general, a symplectic group $Sp(g, \Lambda)$ can be defined, whose elements are the linear transformations preserving a given non-degenerate skew-symmetric bilinear form Λ of degree 2; such transformations are in fact named symplectic transformations.

By expliciting the conditions describing the elements of the group, one has:

$$\begin{aligned} Sp(g,\mathbb{R}) &= \left\{ \gamma \in GL(2g,\mathbb{R} \mid \begin{array}{c} {}^t a_\gamma c_\gamma = {}^t c_\gamma a_\gamma \\ {}^t b_\gamma d_\gamma = {}^t d_\gamma b_\gamma \end{array} \; ; \; {}^t a_\gamma d_\gamma - {}^t c_\gamma b_\gamma = 1_g \right\} = \\ &= \left\{ \gamma \in Gl(2g,\mathbb{R}) \mid \gamma^{-1} = J_g^{-1}\,{}^t\gamma J_g = \begin{pmatrix} {}^t d_\gamma & -{}^t b_\gamma \\ -{}^t c_\gamma & {}^t a_\gamma \end{pmatrix} \right\} \end{aligned} \tag{1.1}$$

As it immediately turns out, the symplectic group is stable under the transposition $\gamma \to {}^t\gamma$; thus, an equivalent characterization follows, by expliciting the conditions for the transposed element:

$$Sp(g,\mathbb{R}) = \left\{ \gamma \in GL(2g,\mathbb{R} \mid \begin{array}{c} a_\gamma {}^t b_\gamma = b_\gamma {}^t a_\gamma \\ c_\gamma {}^t d_\gamma = d_\gamma {}^t c_\gamma \end{array} \; ; \; a_\gamma {}^t d_\gamma - b_\gamma {}^t c_\gamma = 1_g \right\} \tag{1.2}$$

The following definition introduces a remarkable subgroup of $Sp(g,\mathbb{R})$, on which the theory focuses:

Definition 1.2. *The subgroup:*

$$\Gamma_g := Sp(g,\mathbb{Z}) = \left\{ \gamma \in Sp(g,\mathbb{R}) \mid a_\gamma, b_\gamma, c_\gamma, d_\gamma \in M(g,\mathbb{Z}) \right\} = Sp(g,\mathbb{R}) \cap M(2g,\mathbb{Z})$$

is called the **Siegel modular group** *of degree* g [2]*. When* $g = 1$*, the group* $\Gamma_1 = SL(2,\mathbb{Z})$ *is called the* **elliptic modular group** [3]

Proposition 1.1. *The set:*

$$S := \left\{ \; J_g \quad , \quad \left\{ \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \right\}_{S \in Sym_g(\mathbb{Z})} \right\}$$

is a set of generators for the modular group Γ_g.

Proof. For each $\eta \in \Gamma_1$ and $h = 1, \dots g$, let $A^{(g)}_{\eta,h}$ be the $2g \times 2g$ matrix, whose entries are:

$$(A^{(g)}_{\eta,h})_{ij} = \begin{cases} a_\eta - 1 & \text{if} \quad i = j = h; \\ b_\eta & \text{if} \quad i = h, j = h + g; \\ c_\eta & \text{if} \quad i = h + g, j = h; \\ d_\eta - 1 & \text{if} \quad i = j = h + g; \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\gamma^{(g)}_{\eta,h} := 1_{2g} + A^{(g)}_{\eta,h} \in \Gamma_g \qquad \forall \eta \in \Gamma_1, \quad \forall h = 1, \dots g$$

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[2]One can observe that Γ_g is an example of arithmetic subgroup of $Sp(g,\mathbb{R})$

[3]A classical result states that $\Gamma_1/\{\pm 1\}$ is the group of the biholomorphic automorphisms of the Riemann sphere $\hat{\mathbb{C}}$.

By multiplying $\gamma \in \Gamma_g$ by suitable elements of the kind $\gamma^{(g)}_{\eta,h}$ from the left, it can be checked that the matrix:

$$N_\gamma = \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \prod_{\eta,h} \gamma^{(g)}_{\eta,h} \gamma$$

with a suitable $u \in GL_g(\mathbb{Z})$, has the unit vector e_{g+1} as $(g+1)$-th column. Since $\gamma \in \Gamma_g$, the first row of N_γ have to be e_1. Then, by induction, one concludes that the group Γ_g is generated by:

$$\left\{ \ \{\gamma^{(g)}_{\eta,h}\}_{\eta,h} \ , \left\{ \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \right\}_{u \in GL_g(\mathbb{Z})} \ , \ \left\{ \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \right\}_{S \in Sym_g(\mathbb{Z})} \ \right\}$$

However, it is easily checked that the elements $\gamma^{(g)}_{\eta,h}$ are generated by the other elements and J_g. Therefore, the modular group turns out to be generated by the set:

$$\left\{ \ J_g \ , \ \left\{ \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \right\}_{u \in GL_g(\mathbb{Z})} \ , \ \left\{ \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \right\}_{S \in Sym_g(\mathbb{Z})} \ \right\}$$

Moreover, for each $u \in GL_g(\mathbb{R})$ one has:

$$\begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} = \begin{pmatrix} 1_g & u \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \begin{pmatrix} 1_g & u \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

and, therefore, the thesis follows. □

Corollary 1.1. *The modular group Γ_g is also generated by matrices of the form:*

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{pmatrix} \ , \ {}^t\gamma^{-1} = \begin{pmatrix} d_\gamma & 0 \\ -b_\gamma & a_\gamma \end{pmatrix} \qquad {}^t b_\gamma d_\gamma = {}^t d_\gamma b_\gamma \ ; \ {}^t a_\gamma d_\gamma = 1_g$$

Proof. Thanks to Proposition 1.1, one has only to check that the set S is generated by such matrices. However, since:

$$J_g = \begin{pmatrix} 1_g & 0 \\ -1_g & 1_g \end{pmatrix} \begin{pmatrix} 1_g & 1_g \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} 1_g & 0 \\ -1_g & 1_g \end{pmatrix}$$

the thesis easily follows.

□

1.2 Congruence subgroups of the Siegel modular group

The aim of this section is to introduce some remarkable subgroups of the Siegel modular group Γ_g, which reveal themselves to be a natural tool to generalize the notion of modular functions.

Definition 1.3. *For each $n \in \mathbb{N}$ let $\Gamma_g(n)$ be the kernel of the natural homomorphism $\Gamma_g \to Sp(g, \mathbb{Z}/n\mathbb{Z})$, induced by the canonical projection. $\Gamma_g(n)$ is known as the* **principal congruence subgroup** *of level n.*

As a kernel of a group homomorphism, $\Gamma_g(n)$ is a normal subgroup of Γ_g. Moreover, since:

$$\Gamma_g(n) = \{\gamma \in \Gamma_g \mid \gamma \equiv 1_{2g} \operatorname{mod} n\}$$

an immediate characterization can be derived:

$$\Gamma_g(n) = \left\{\gamma \in M(2g, \mathbb{Z}) \mid \gamma \equiv 1_{2g} + nM_\gamma\right\}$$

$$\text{with} \quad M_\gamma = \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix} \quad \text{s.t.} \quad \begin{cases} {}^t b_M = b_M + n({}^t d_M b_M - {}^t b_M d_M) \\ {}^t c_M = c_M + n({}^t a_M c_M - {}^t c_M a_M) \\ d_M + {}^t a_M + n({}^t a_M d_M - {}^t c_M b_M) = 0 \end{cases} \tag{1.3}$$

It is a remarkable property of such subgroups that they are of finite index in the Siegel modular group; in particular, the following Lemma holds:

Lemma 1.1. *For each $n \in \mathbb{N}$:*

$$[\Gamma_g : \Gamma_g(n)] = n^{g(2g+1)} \prod_{p|n} \prod_{1 \le k \le g} \left(1 - \frac{1}{p^{2k}}\right)$$

Proof. A proof can be found in [Ko]. □

Pertaining principal congruence subgroup with even level, some elementary lemmas can be stated:

Lemma 1.2. *If $\gamma \in \Gamma_g(2n)$, $\gamma^2 \in \Gamma_g(4n)$.*

Proof. Let $\gamma_1 = 1_{2g} + 2nM_1, \gamma_2 = 1_{2g} + 2nM_2 \in \Gamma_g(2n)$ with reference to the notation introduced in (1.3). Then, one has:

$$\gamma_1\gamma_2 = 1_{2g} + 2n(M_1 + M_2 + 2nM_1M_2) \tag{1.4}$$

hence, the thesis follows. □

Lemma 1.3. *For each $\gamma \in \Gamma_g(2n)$:*

$$diag({}^t a_\gamma c_\gamma) \equiv diag(c_\gamma)\, mod 4n \qquad diag({}^t b_\gamma d_\gamma) \equiv diag(b_\gamma)\, mod 4n$$

Proof. Using again the notation introduced in (1.3), one has the following simple chain of congruences:

$$\begin{aligned} diag({}^t a_\gamma c_\gamma) &= diag[{}^t(1_g + 2na_M) \cdot 2nc_M] = diag[2nc_M + 4n^2({}^t a_M c_M)] = \\ &= 2n\, diag(c_M) + 4n^2 diag({}^t a_M c_M) \equiv 2n\, diag(c_M) \operatorname{mod} 4n \end{aligned}$$

which proves the first relation, since $c_\gamma = 2nc_M$. Likewise, the second relation is proved. □

Definition 1.4. *A subgroup* Γ *of* Γ_g*, such that* $\Gamma_g(n) \subset \Gamma$ *for some* $n \in \mathbb{N}$ *is called a* **congruence subgroup** *of level* n*.*

Remarkable examples of proper congruence subgroups (namely congruence subgroups which are not principal) of level n are clearly given by the following family:

$$\Gamma_{g,0}(n) := \left\{\gamma \in \Gamma_g \mid c_\gamma \equiv 0 \,\mathrm{mod} n\right\} \tag{1.5}$$

A notable family of congruence subgroups, on which this work mainly focuses, is the following one:

$$\begin{aligned}\Gamma_g(n, 2n) &:= \left\{\gamma \in \Gamma_g(n) \mid diag({}^t a_\gamma c_\gamma) \equiv diag({}^t b_\gamma d_\gamma) \equiv 0 \,\mathrm{mod} 2n\right\} = \\ &= \left\{\gamma \in \Gamma_g(n) \mid diag(a_\gamma {}^t b_\gamma) \equiv diag(c_\gamma {}^t d_\gamma) \equiv 0 \,\mathrm{mod} 2n\right\}\end{aligned}$$

It is immediately seen that $\Gamma_g(2n) \subset \Gamma_g(n, 2n)$; therefore, such subgroups are congruence subgroups of level $2n$. Moreover, Lemma 1.3 implies the characterization:

$$\Gamma_g(2n, 4n) = \left\{\gamma \in \Gamma_g(2n) \mid diag(b_\gamma) \equiv diag(c_\gamma) \equiv 0 \,\mathrm{mod} 4n\right\} \tag{1.6}$$

The congruence subgroups of the type (1.6) satisfy some notable properties:

Lemma 1.4. *If* $\gamma \in \Gamma_g(2n, 4n)$*,* $\gamma^2 \in \Gamma_g(4n, 8n)$*.*

Proof. Let $\gamma = 1_{2g} + 2nM$ be as in (1.3); then, the thesis follows from (1.4), since for hypothesis $diag(b_M) \equiv diag(c_M) \equiv 0 \,\mathrm{mod} 2$. □

Proposition 1.2. *The congruence subgroup* $\Gamma_g(2n, 4n)$ *is normal in* Γ_g *for each* $n \in \mathbb{N}$*. Moreover,* $[\Gamma_g(2n) : \Gamma_g(2n, 4n)] = 2^{2g}$*.*

Proof. Let $\gamma = 1_{2g} + 2nM$ be, as in (1.3), the generic element of $\Gamma_g(2n)$; then for each n one can define the map:

$$\begin{array}{cccc} D_n & : & \Gamma_g(2n) & \longrightarrow & \mathbb{Z}_2^g \times \mathbb{Z}_2^g \\ & & \gamma & \longrightarrow & (diag(b_M) \,\mathrm{mod} 2, \, diag(c_M) \,\mathrm{mod} 2) \end{array}$$

Due to (1.4), D_n is a group homomorphism. Moreover the condition $D_n(\gamma) = 0$ is equivalent to $diag(b_\gamma) \equiv diag(c_\gamma) \equiv 0 \,\mathrm{mod} 4n$; therefore $\Gamma_g(2n, 4n)$ is the kernel of D_n; in particular, $\Gamma_g(2n, 4n)$ is normal in $\Gamma_g(2n)$. In order to prove that $\Gamma_g(2n, 4n)$ is normal in Γ_g, one has to prove that $\eta\gamma\eta^{-1} \in KerD_n$ whenever $\gamma \in \Gamma_g(2n, 4n)$ and $\eta \in \Gamma_g$. One has:

$$\eta\gamma\eta^{-1} = \begin{pmatrix} a_\eta & b_\eta \\ c_\eta & d_\eta \end{pmatrix} \left[\begin{pmatrix} 1_g & 0 \\ 0 & 1_g \end{pmatrix} + 2n \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix}\right] \begin{pmatrix} {}^t d_\eta & -{}^t b_\eta \\ -{}^t c_\eta & {}^t a_\eta \end{pmatrix} = 1_{2g} + 2n \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

where, for instance, $b' = a_\eta b_M {}^t a_\eta - b_\eta c_M {}^t b_\eta + {}^t(a_\eta {}^t d_M {}^t b_\eta) - (a_\eta a_M {}^t b_\eta)$; by (1.3) $a_M + {}^t d_m \equiv 0 \,\mathrm{mod} 2n$, hence:

$$diag(b') \equiv a_\eta \cdot diag(b_M) + b_\eta \cdot diag(c_M) \,\mathrm{mod} 2$$

Since $\gamma \in KerD_n$, it follows that $diag(b') \equiv 0 \,\mathrm{mod} 2$; likewise one has $diag(c') \equiv 0 \,\mathrm{mod} 2$.
To prove the second part of the statement, one observes that D_n is also surjective. In fact, for $(\epsilon_1, \epsilon_2) \in \mathbb{Z}_2^g \times \mathbb{Z}_2^g$, by (1.3) one can choose $\gamma_1 \in \Gamma_g(2n)$ and $\gamma_2 \in \Gamma_g(2n)$ satisfying respectively $a_{M_1} = c_{M_1} = d_{M_1} = 0$ and $diag(b_{M_1}) \equiv \epsilon_1 \,\mathrm{mod} 2$, and $a_{M_2} = b_{M_2} = d_{M_2} = 0$ and $diag(c_{M_2}) \equiv \epsilon_2 \,\mathrm{mod} 2$. Then, $D(\gamma_1) = (\epsilon_1, 0)$ e $D(\gamma_2) = (0, \epsilon_2)$, and the surjectivity of D_n follows, because $D(\gamma_1\gamma_2) = (\epsilon_1, \epsilon_2)$. Therefore, the following isomorphism is given:

$$\Gamma_g(2n)/\Gamma_g(2n, 4n) \cong (\mathbb{Z}_2)^{2g}$$

and consequently $[\Gamma_g(2n) : \Gamma_g(2n, 4n)] = 2^{2g}$. □

Proposition 1.3. *$\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$ is a $g(2g+1)$-dimensional vector space on $\mathbb{Z}_2$.*

Proof. Lemma 1.4 implies that each element in $\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$, which differs from identity, has order 2. $\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$ is, in particular, an abelian group. By Lemma 1.1 and Proposition 1.2, one immediately has $[\Gamma_g(2n, 4n) : \Gamma_g(4n, 8n)] = 2^{g(2g+1)}$. Then:

$$\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n) \cong \mathbb{Z}_2^{g(2g+1)} \tag{1.7}$$

and the thesis follows. □

Proposition 1.4. *For each couple of indices $1 \leq i, j \leq g$ one can denote by $\tilde{O}_{ij}$ the matrix $g \times g$, whose coordinates are $\tilde{O}_{ij}^{(hk)} = \delta_{ih}\delta_{jk}$. Then, the following matrices:*

$$A_{ij} := \begin{pmatrix} a_{ij} & 0 \\ 0 & {}^t a_{ij}^{-1} \end{pmatrix} \qquad (1 \leq i, j \leq g) \qquad con \quad a_{ij} := \begin{cases} 1_g + 2\tilde{O}_{ij} & se\ i \neq j \\ 1_g - 2\tilde{O}_{ij} & se\ i = j \end{cases}$$

$$B_{ij} := \begin{pmatrix} 1_g & b_{ij} \\ 0 & 1_g \end{pmatrix} \qquad (1 \leq i \leq j \leq g) \qquad con \quad b_{ij} := \begin{cases} 2\tilde{O}_{ij} + 2\tilde{O}_{ji} & se\ i \neq j \\ 2\tilde{O}_{ij} & se\ i = j \end{cases}$$

$$C_{ij} := {}^t B_{ij} \qquad (1 \leq i \leq j \leq g)$$

are a set of generators for $\Gamma_g(2)$.

Proof. A proof can be found in [I3]. □

As a consequence of Proposition 1.4 one has the following Corollary:

Corollary 1.2. *The $g(2g+1)$ elements A_{ij} (for $i, j \neq g$), B_{ij}, C_{ij} (for $i < j$), $B_{ii}^2, C_{ii}^2, -1_{2g}$ are a basis for the $\mathbb{Z}^2$ - vector space $\Gamma_g(2,4)/\Gamma_g(4,8)$.*

Proof. Thanks to the characterization (1.6), the elements A_{ij} $(i, j \neq g)$, B_{ij}, C_{ij} $(i < j)$, B_{ii}^2, C_{ii}^2 are plainly checked to belong to $\Gamma_g(2,4)$. Moreover it can be straightly verified that such elements belong to distinct independent cosets of $\Gamma_g(4,8)$ in $\Gamma_g(2,4)$. Then, the thesis follows from Proposition 1.3. □

Another important family of remarkable congruence subgroups this work will focus on defined by:

$$\Gamma_g(n, 2n, 4n) := \left\{\gamma \in \Gamma_g(2n, 4n) \mid Tr(a_\gamma) \equiv g \bmod 4n\right\} \tag{1.8}$$

In the next section the main properties of the action of these subgroups on a meaningful tubular domain of the complex euclidean space will be discussed.

1.3 Action of the Symplectic Group on the Siegel Upper Half-Plane

Definition 1.5. *The* **Siegel Upper Half-Plane** *of degree g is the following subset:*

$$\mathfrak{S}_g := \left\{\tau \in Sym_g(\mathbb{C}) \mid Im\tau > 0\right\} \tag{1.9}$$

The Siegel upper half-plane clearly turns out to be a generalization of the usual complex upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, Im\tau > 0\} = \mathfrak{S}_1$.

An action of $Sp(g, \mathbb{R})$ on $\mathfrak{S}_g$, can be defined, by generalizing the known action of $SL(2, \mathbb{R})$ on $\mathbb{H}$:

$$\begin{aligned} &Sp(g, \mathbb{R}) \times \mathfrak{S}_g \longrightarrow \mathfrak{S}_g \\ &(\gamma, \tau) \to (a_\gamma\tau + b_\gamma) \cdot (c_\gamma\tau + d_\gamma)^{-1} \end{aligned} \tag{1.10}$$

One has the following statement:

Proposition 1.5. *The action (1.10) is well defined and transitive.*

Proof. First of all, for each $\gamma \in Sp(g, \mathbb{R})$ and $\tau \in Sym_g(\mathbb{C})$ the following identities are easily verified:

$$^t(a_\gamma\tau + b_\gamma)(c_\gamma\tau + d_\gamma) - {}^t(c_\gamma\tau + d_\gamma)(a_\gamma\tau + b_\gamma) = \tau - {}^t\tau = 0 \tag{1.11}$$

$$^t(a_\gamma\tau + b_\gamma)\overline{(c_\gamma\tau + d_\gamma)} - {}^t(c_\gamma\tau + d_\gamma)\overline{(a_\gamma\tau + b_\gamma)} = \tau - \overline{\tau} = 2i(Im\tau) \tag{1.12}$$

In order to be sure the expression in (1.10) is well defined, it will be needed first to prove that $c_\gamma\tau + d_\gamma$ is invertible for each $\tau \in \mathfrak{S}_g$ and for each $\gamma \in Sp(g, \mathbb{R})$.
If $c_\gamma\tau + d_\gamma$ were such an element which is not invertible, a nonzero vector $z \in \mathbb{C}^g$ would exist, such that $(c_\gamma\tau + d_\gamma)z = 0$, and consequently:

$$0 = {}^tz^t(a_\gamma\tau + b_\gamma)\overline{(c_\gamma\tau + d_\gamma)}\overline{z} = {}^tz^t(c_\gamma\tau + d_\gamma)\overline{(a_\gamma\tau + b_\gamma)}\overline{z}$$

Then, (1.12) would imply $2i^tz(Im\tau)\overline{z} = 0$, which contradicts the hypothesis $\tau \in \mathfrak{S}_g$.
Now, one has to prove that $\gamma\tau := (a_\gamma\tau + b_\gamma) \cdot (c_\gamma\tau + d_\gamma)^{-1} \in \mathfrak{S}_g$ for each $\tau \in \mathfrak{S}_g$

and for each $\gamma \in Sp(g,\mathbb{R})$. Since $c_\gamma\tau + d_\gamma$ is invertible under such hypothesis, (1.11) is equivalent to $\gamma\tau \in Sym_g(\mathbb{C})$. This assertion and (1.12) imply:

$$\begin{aligned} Im(\gamma\tau) &= \frac{1}{2i}[{}^t(\gamma\tau) - (\overline{\gamma\tau})] = \\ &= \frac{1}{2i}{}^t(c_\gamma\tau + d_\gamma)^{-1}[{}^t(a_\gamma\tau + b_\gamma)(\overline{c_\gamma\tau + d_\gamma}) - {}^t(c_\gamma\tau + d_\gamma)(\overline{a_\gamma\tau + b_\gamma})](\overline{c_\gamma\tau + d_\gamma})^{-1} = \\ &= {}^t(c_\gamma\tau + d_\gamma)^{-1} \cdot Im\tau \cdot (\overline{c_\gamma\tau + d_\gamma})^{-1} \end{aligned}$$

from which it follows that $Im(\gamma\tau) > 0$, because $\tau \in \mathfrak{S}_g$.
Then it is immediately checked that (1.10) satisfies the properties of an action. Finally, thanks to the Cholesky decomposition (Corollary A.2), for each $\tau \in \mathfrak{S}_g$ there exists a matrix $u \in GL(g,\mathbb{R})$ such that $Im\tau = u^t u$. Hence, one clearly has:

$$\tau = \begin{pmatrix} 1_g & Re\tau \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} (-i1_g)$$

and the transitivity of the action follows. □

The action of the symplectic group on $\mathfrak{S}_g$ provides a complete description for the group $Aut(\mathfrak{S}_g)$ of the biholomorphic automorphisms of $\mathfrak{S}_g$:

Proposition 1.6. *$Sp(g,\mathbb{R})/\{\pm 1_g\} \cong Aut(\mathfrak{S}_g)$.*

Proof. The action (1.10) allows to define for each $\gamma \in Sp(g,\mathbb{R})$ the holomorphic maps $T_\gamma : \tau \to \gamma\tau$ on $\mathfrak{S}_g$ to itself. Each map T_γ is clearly invertible with inverse $T_{\gamma^{-1}}$; a group homomorphism is therefore defined:

$$\begin{array}{cccc} T & : & Sp(g,\mathbb{R}) & \to & Aut(\mathfrak{S}_g) \\ & & \gamma & \to & T_\gamma \end{array} \tag{1.13}$$

whose kernel is precisely $\{\pm 1_g\}$. The proof of the surjectivity of T is provided in [Si], by applying a generalized version of Schwartz lemma for several complex variables. □

Some remarkable properties of the Siegel upper half-plane related to the action of the symplectic group can be stated here.

Proposition 1.7. *$\mathfrak{S}_g$ is a symmetric space.*

Proof. One needs to prove that each point of $\mathfrak{S}_g$ admits a symmetry.
For such a purpose, one can consider the generator:

$$\gamma = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \in Sp(g,\mathbb{R})$$

and the related holomorphic map T_γ as in (1.13).
T_γ is an involution of $\mathfrak{S}_g$, for $T_\gamma^2 = Id_{\mathfrak{S}_g}$. Moreover, $T_\gamma(i1_g) = i1_g$, hence T_γ is a symmetry for the point $i1_g \in \mathfrak{S}_g$. Since by Proposition 1.5 $Sp(g,\mathbb{R})$ acts transitively on $\mathfrak{S}_g$, for each $\tau \in \mathfrak{S}_g$ there exists $\eta \in Sp(g,\mathbb{R})$ such that $\tau = \eta i 1_g$. $T_{\eta\gamma\eta^{-1}}$ is thus a symmetry for τ. □

To prove other important properties of this action, a classical result concerning the group action on topological spaces have to be recalled:

Theorem 1.1. *Let G be a second-countable locally compact Hausdorff topological group which acts continuously and transitively on a locally compact Hausdorff topological space X. Then, for each $x \in X$ one has the homeomorphism between topological spaces $G/St_x \cong X$.*

Proof. The following application:

$$T: \quad G/St_x \to X$$

$$gSt_x \to gx$$

is clearly well defined and injective; moreover, it is also surjective for the action of G is transitive. In order to prove that T is indeed a homeomorphism, one has to show that the map $g \mapsto gx$, which is continuous by hypothesis, is an open map.
Then let $U \subset G$ be an open set; one has to show that $gU = \{gx | g \in U\}$ is an open set. Then, let $gx \in gU$ and let V a compact neighbourhood of $e \in G$ such that $V^{-1} = V$ and $gV^2 \subset U$. Since G is second-countable, there exists a collection of elements $\{g_n\}_{n \in \mathbb{N}} \subset G$ such that $G = \cup_{n=1}^{\infty} g_n V$ and, consequently, $X = \cup_{n=1}^{\infty} g_n Vx$. For each n, the set $g_n Vx$ is a closed set, for it is compact in X; since X is a locally compact Hausdorff space, it is a Baire space, and therefore the interiors of $g_n Vx$ can not be all empty. Then, there must exists $n_0 \in \mathbb{N}$ such that $g_{n_0} Vx$ has interior points. Therefore, the interior of Vx is not empty, since Vx is homeomorphic to $g_{n_0} Vx$. Then, let $x_0 \in X$ such that $x_0 x \in Int(g_{n_0} Vx)$; one has:

$$gx \in gx_0^{-1} Int(g_{n_0} Vx) \subset gV^2 x \subset Ux$$

hence gx is an interior point of gU; since each point of gU is interior, gU is an open set, and the thesis follows. □

Proposition 1.8. *$\mathfrak{S}_g$ is a homogeneous space.*

Proof. The subgroup:

$$U(g) = \left\{ \gamma \in Sp(g, \mathbb{R}) \,|\, d_\gamma = a_\gamma, \; c_\gamma = -b_\gamma, \; a_\gamma{}^t a_\gamma + b_\gamma{}^t b_\gamma = 1_g \right\}$$

is the stabilizer St_{i1_g} of the point $i1_g$. As a consequence of Theorem 1.1 one has the homeomorphism:

$$\mathfrak{S}_g \cong Sp(g, \mathbb{R})/U(g) \tag{1.14}$$

□

The following Proposition concerns the behaviour of the action of discrete subgroups on homogeneous spaces:

Proposition 1.9. *Let $X \cong G/K$ a homogeneous space. Then, each discrete subgroup of G acts properly discontinuously on X.*

Proof. It is a straightforward consequence of the fact that $\pi : G \to G/K$ is a proper map. □

The following Corollary for the action of the Siegel modular group is an immediate consequence:

Corollary 1.3. *The Siegel modular group Γ_g acts properly discontinuously on $\mathfrak{S}_g$ by (1.10).*

Proof. Since Γ_g is a discrete subgroup of $Sp(g, \mathbb{R})$, the statement follows by plainly applying Proposition 1.9. □

Siegel provided in [Si] an explicit description of a fundamental domain for the action of Γ_g on $\mathfrak{S}_g$:

$$\mathfrak{F}_g = \left\{ \tau \in \mathfrak{S}_g \;\middle|\; \begin{array}{l} n\, Im\tau\, {}^t n \geq Im\tau_{kk} \quad \forall\, n = (n_1, \cdots, n_g) \in \mathbb{Z}^g \;\; k-\text{admissible} \\ Im\tau_{k,k+1} \geq 0 \quad \forall\, k \\ |\det(c_\gamma \tau + d_\gamma)| \geq 1 \quad \forall\, \gamma \in \Gamma \\ |Re\tau| \leq 1/2 \end{array} \right\}$$

where $n = (n_1, \cdots, n_g) \in \mathbb{Z}^g$ is called k admissible for $1 \leq k \leq g$ whenever $n_k, \cdots, n_g$ are coprime. This domain is known as the **fundamental Siegel's domain** of degree g, and will be here denoted by the symbol $\mathfrak{F}_g$ [4].

Example 1.1. *The Siegel fundamental domain in the case $g = 1$ can be easily described as:*

$$\mathfrak{F}_1 = \{\tau \in \mathbb{H} \mid |Re\tau| \leq 1/2,\ |\tau| \geq 1\}$$

The following property is a remarkable consequence of Corollary 1.3:

Corollary 1.4. *The coset space $A_g := \mathfrak{S}_g/\Gamma_g$ admits a normal analytic space structure.*

Proof. It is a straightforward application of Cartan's Theorem on the existence of an analytic space structure for the quotients by the action of a finite group (cf. [Ca]). □

The coset space $A_g := \mathfrak{S}_g/\Gamma_g$ turns out to be remarkably meaningful in the theory of abelian varieties, since its points can be set in a one-to-one correspondence with the classes of isomorphic polarized abelian varieties (cf. [De]).

Some Lemmas will be stated now, in order to prove a useful property of Siegel's fundamental domain $\mathfrak{F}_g$, namely that it is contained in a so-called generalized vertical strip.

Lemma 1.5. *Whenever $\tau \in \mathfrak{F}_g$, one has:*

1. $Im\tau_{kk} \leq Im\tau_{k+1,k+1} \qquad \forall k \in \{1, \ldots, g-1\}$
2. $|2Im\tau_{kl}| \leq Im\tau_{kk} \qquad \forall k < l$

[4] The conditions $Im\tau_{k,k+1} \geq 0$ and $nIm\tau^t n \geq Im\tau_{kk}$ for each n which is k-admissible, are traditionally expressed in literature by stating the matrix $Im\tau$ is reduced in the sense of Minkowski (or, equivalently, it belongs to a Minkowski reduced domain)

3. $\exists\, c > 0$ *such that:*

$$\det Im\tau \leq \prod_{i=1}^{g} Im\tau_{ii} \leq c \det Im\tau$$

Proof. Let $1 \leq k \leq g-1$ be fixed. Since $n\, Im\tau^t n \geq Im\tau_{kk}$ for each k-admissible n, the condition 1. is obtained by choosing $n = e_{k+1}$. By setting $n = e_k \pm e_l$ for each l such that $k < l \leq g$, one has:

$$Im\tau_{kk} + Im\tau_{l,l} \pm (Im\tau_{kl} + Im\tau_{l,k}) = Im\tau_{kk} + Im\tau_{l,l} \pm 2Im\tau_{kl} \geq Im\tau_{kk}$$

and consequently the condition 2. is also verified.
Finally, the condition 3. can be derived as a consequence of Hermite inequality, which holds for real positive definite matrices $M \in Sym_n(\mathbb{R})$:

$$\min_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} {}^t kMk \leq c \det M^{\frac{1}{n}}$$

where $c > 0$ is a constant depending only on n (cf. [Kl]). □

Using Proposition A.4, the following technical statement can be derived by Lemma 1.5 (cf. [Kl]):

Lemma 1.6. *For each $\tau \in \mathfrak{F}_g$ let $Im\tau^D$ be the following diagonal matrix:*

$$Im\tau^D := \begin{pmatrix} Im\tau_{11} & 0 & \cdots & 0 \\ 0 & Im\tau_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Im\tau_{gg} \end{pmatrix} \tag{1.15}$$

Then, there exists $c > 0$ such that:

$$c\, Im\tau - Im\tau^D > 0$$

Thanks to this Lemma, Siegel's fundamental domain $\mathfrak{F}_g$ is proved to be contained in a generalized vertical strip:

Lemma 1.7. *There exists $\lambda > 0$ such that:*

$$\mathfrak{F}_g \subset \{\tau \in \mathfrak{S}_g \mid Im\tau - \lambda 1_g \geq 0\}$$

Proof. Let $\tau \in \mathfrak{F}_g$, $Im\tau^D$ as in (1.15) and η the element:

$$\eta := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1_{g-1} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{g-1} \end{pmatrix} \in \Gamma_g$$

Since $\tau \in \mathfrak{F}_g$, in particular, $|\det(c_\eta\tau + d_\eta)| = |\tau_{11}| > 1$; moreover, $|Re\tau_{11}| \leq 1/2$, hence $Im\tau_{11} \geq \sqrt{3}/2$. Now, let $\lambda_1 := \sqrt{3}/2$. Since for construction $Im\tau^D_{ii}$ is diagonal and $Im\tau^D_{ii} = Im\tau_{ii}$ for each i, (by condition 1. in Lemma 1.5), one has $(Im\tau^D - \lambda_1 1_g) \geq 0$. Then, let $c > 0$ be as in Lemma 1.6; by setting $\lambda = \lambda_1 c^{-1}$, one has $Im\tau - \lambda 1_g = (Im\tau - c^{-1}Im\tau^D) + (c^{-1}Im\tau^D - \lambda 1_g) \geq 0$, and the statement is proved.

□

Pertaining to level moduli spaces, which are quotients of Siegel's upper-half plane with respect to the action of congruence subgroups, the following Proposition, guarantees that an important family of such spaces admits a complex structure:

Proposition 1.10. *Let $n \geq 3$. The action $\Gamma_g(n)$ on the Siegel upper half-plane $\mathfrak{S}_g$ is free; $\mathfrak{S}_g/\Gamma_g(n)$ is, therefore, a $g(g+1)/2$-dimensional complex manifold.*

Proof. A proof can be found in [Se]. □

The following properties descend:

Proposition 1.11. *Let $\gamma \in \Gamma_g(4,8)$ an element which has fixed points on $\mathfrak{S}_g$; then $\gamma = 1_g$. In particular the so-called level moduli space $A_g^{4,8} := \mathfrak{S}_g/\Gamma_g(4,8)$ is smooth.*

Proposition 1.12. *An element $\gamma \in \Gamma_g$, which has fixed point on $\mathfrak{S}_g$, has finite order.*

In particular, the following statement holds:

Corollary 1.5. *An element $\gamma \in \Gamma_g(2,4)$, which has fixed points on $\mathfrak{S}_g$, has order 2.*

Proof. Let $\gamma \in \Gamma_g(2,4)$ such an element. By Proposition 1.12, γ has finite order; moreover, $\gamma^2 \in \Gamma_g(4,8)$ by Lemma 1.4; then, Proposition 1.11 implies $\gamma^2 = 1$. □

Chapter 2

Siegel Modular Forms

2.1 Definition and Examples

The aim of this section is to introduce a brief overview on the basic aspects of the theory of Siegel modular forms, which generalize the notion of modular functions, as already stated in the introduction.
For a detailed exposition on this interesting topic, Freitag's book [F] is the main reference. Klingen's introductory book [Kl] is also an important reference to quote. Van der Geer's lectures [VdG] provide an exhaustive overview of the theory, while Lang's book [L] Diamond and Shurman's book [DS], Miyake's book [Mi] and Zagier's lectures [Z] focus on a detailed description of the classical theory in the case $g = 1$.

Definition 2.1. *Let $k \in \mathbb{Z}$ and let Γ be a congruence subgroup. A classical* **Siegel modular form** *of weight k with respect to Γ is a function $f : \mathfrak{S}_g \to \mathbb{C}$, satisfying the following conditions:*

1. *f is holomorphic on $\mathfrak{S}_g$*
2. $f(\gamma\tau) = \det(c_\gamma\tau + d_\gamma)^k f(\tau) \qquad \forall\gamma \in \Gamma, \quad \forall\tau \in \mathfrak{S}_g$
3. *When $g = 1$ the function $\tau \mapsto \det(c_\gamma\tau + d_\gamma)^{-k} f(\gamma\tau)$ is bounded on $\mathfrak{F}_1$ for each $\gamma \in \Gamma_1$* [1]

It is useful to introduce for each $\gamma \in Sp(g, \mathbb{R})$, the function:

$$(\gamma|_k f)(\tau) := \det(c_{\gamma^{-1}}\tau + d_{\gamma^{-1}})^{-k} f(\gamma^{-1}\tau) \tag{2.1}$$

In fact, since for each $k \in \mathbb{Z}$:

$$det(c_{\gamma\gamma'}\tau + d_{\gamma\gamma'})^k = det(c_\gamma\gamma'\tau + d_\gamma)^k det(c_{\gamma'}\tau + d_{\gamma'})^k \qquad \forall\gamma, \gamma' \in Sp(g, \mathbb{R})$$

one therefore has:

[1]This condition, which is equivalent to the request for the function f to be holomorphic on ∞ when $\Gamma = \Gamma_1$, is indeed redundant when $g > 1$ (see Corollary 2.1)

$$\gamma|_k(\gamma'|_k f) = \gamma\gamma'|_k f \qquad 1_{2g}|_k f = f$$

and consequently (2.1) defines an action of $Sp(g, \mathbb{R})$ on the space of the holomorphic function on $\mathfrak{S}_g$ [2]
By using the notation introduced in (2.1), the conditions appearing in Definition 2.1 can be translated into the following ones:

1. f is holomorphic on $\mathfrak{S}_g$;
2. $\gamma^{-1}|_k f = f \qquad \forall \gamma \in \Gamma$
3. When $g = 1$, $\gamma^{-1}|_k f$ is bounded on $\mathfrak{F}_1$ for each $\gamma \in \Gamma_1$

Example 2.1. (Eisenstein series of weight $k \geq 3$) *For each $k \geq 3$ the series:*

$$E_k(\tau) := \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ MCD(c,d)=1}} \frac{1}{(c\tau + d)^k}$$

converges absolutely. Moreover it uniformly converges on the compacts of the complex upper half-plane $\mathbb{H}$; therefore, when $k \geq 3$ the series E_k defines a holomorphic function, which is easily seen to be a modular form of weight k with respect to the Siegel modular group (see, for instance, [DS]). This modular form is called the Eisenstein series.

Example 2.2. (Generalized Eisenstein Series) *The series*

$$E_k(\tau) := \sum_{\substack{c,d\in Sym_g(\mathbb{Z}) \\ c,d\ coprime}} \det(c\tau + d)^{-k}$$

is seen to be uniformly convergent on each compact and also a modular form with respect to the Siegel modular group Γ_g of weight k whenever $k > g+1$; hence, it defines a modular form which is the generalization of the one introduced in Example 2.1.

Besides Eisenstein series other remarkable Siegel modular forms can be constructed by means of the so-called Theta constants, as it will be shown in Chapter 3.

The additional condition in case $g = 1$ is not redundant, as the following counterexample shows:

Example 2.3. (The absolute modular invariant J). *By setting:*

$$e_4 := 60E_4 \qquad e_6 := 140E_6$$

[2]More in general, if M is a complex manifold on which a group G acts biholomorphically, a non vanishing function $R : G \times M \to \mathbb{C}$, which is holomorphic on M, is called a factor of automorphy if:

$$R(gg', p) = R(g, g'p)R(g', p) \qquad \forall g, g' \in G, \quad \forall p \in M$$

Then, for each function f holomorphic on M, one can set:

$$(g \cdot f)(p) := R(g^{-1}, p)^{-1} f(g^{-1}p)$$

and action of G turns out to be thus defined on the space of holomorphic functions on M.

where E_4 and E_6 are the Eisenstein series respectively of weight 4 and 6, as described in Example 2.1, one can define the so-called absolute modular invariant as:

$$J(\tau) := 1728 \frac{e_4^3(\tau)}{\Delta(\tau)}$$

where $\Delta(\tau) := (e_4^3(\tau) - 27e_6^2(\tau))$. The function J clearly verifies condition 1. and condition 2. for $k = 0$ and $\Gamma = \Gamma_g$ in Definition 2.1. However, J is not bounded on the Siegel's fundamental domain $\mathfrak{F}_1$ (cf. Example 1.1), because:

$$\lim_{t \to \infty} |J(it)| = \infty$$

Therefore, the condition 3. is not satisfied.

Henceforward, this work will refer to classical Siegel modular forms simply as modular forms.

The set of modular forms of weight k, with respect to a congruence subgroup Γ is naturally provided with a complex vector space structure; throughout this work this complex vector space will be denoted by $A(\Gamma)_k$.

Definition 2.2. *Let Γ be a congruence subgroup. The graded ring $A(\Gamma) = \bigoplus_{k \in \mathbb{Z}} A(\Gamma)_k$ is called the* **ring of the modular forms with respect to** Γ.

As it will be proved in the following Section, this ring is a positively graded ring.

Clearly by definition one has, in particular:

$$A(\Gamma') \subset A(\Gamma) \qquad \text{whenever} \quad \Gamma' \subset \Gamma$$

In Sections 3.5 and 3.7 some remarkable examples of rings of modular forms will be reviewed in more details. Concerning this Section, one can conclude by noting that for each weight l and for each character χ of Γ, a complex vector space is defined:

$$A_l(\Gamma, \chi) := \{f \in O(\mathfrak{S}_g) \mid f(\sigma\tau) = \chi(\sigma) det(c\tau + d)^l f(\tau) \ \ \forall \sigma \in \Gamma\}$$

where the symbol $O(\mathfrak{S}_g)$ stands for the space of holomorphic functions on $\mathfrak{S}_g$, also satisfying condition 3. in Definition 2.1 when $g = 1$.
Then, if Γ is a fixed subgroup of the Siegel modular group Γ_g, and $\Gamma_0 \subset \Gamma_g$ is such that Γ is a normal subgroup of Γ_0, the properties of transformation of the Siegel modular forms under the action of Γ_0 induce a decomposition of the homogeneous part $A_l(\Gamma)$ of the ring $A(\Gamma)$:

$$A_l(\Gamma) = \bigoplus_{\chi \in \hat{G}_0} A_l(\Gamma_0, \chi) \tag{2.2}$$

where $\hat{G}_0$ is the group of characters of Γ_0/Γ.

2.2 Fourier series of a modular form

First of all, one has to recall that a Laurent series exapansion can be obtained for holomorphic functions on Reinhardt's domains as a consequence of Cauchy's formula in several complex variables [3]; in particular, one has:

Theorem 2.1. *Let f be a holomorphic function on a Reinhardt domain R. Then, for each z in the product of annuli $A(r_1, a_1) \times \cdots \times A(r_n, a_n) \subset R$, one has:*

$$f(z) = \sum_{k_1,\dots k_n=-\infty}^{\infty} c_{k_1,\dots k_n}(z_1 - a_1)_1^k \cdots (z_n - a_n)_n^k$$

with:

$$c_{k_1,\dots k_n} = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\overline{D}_{r_n}} \cdots \int_{\partial\overline{D}_{r_1}} \frac{f(\xi_1, \dots \xi_n)}{(\xi_1 - a_1)^{k_1+1} \cdots (\xi_n - a_n)^{k_n+1}} d\xi_1 \cdots d\xi_n$$

where the convergence of the series is absolute and uniform on the compacts in $A(r_1, a_1) \times \cdots \times A(r_n, a_n)$ [4]

Definition 2.3. *A matrix $N \in Sym_g(\mathbb{Q})$ is defined half-integer whenever $2N \in Sym_g(\mathbb{Z})$ and $diag(2N) \equiv 0 \, mod 2$.*

Henceforth the symbol $Sym_g^s(\mathbb{Q}) \subset Sym_g(\mathbb{Q})$ will conventionally denote the set of half-integer matrices.

Proposition 2.1. *Let be $n \in \mathbb{N}$ and let $f : \mathfrak{S}_g \to \mathbb{C}$ be a holomorphic function such that $f(\tau + nN) = f(\tau)$ for each $N \in Sym_g(\mathbb{Z})$. Then the function f admits an expansion as a Fourier series:*

$$f(\tau) = \sum_{N \in Sym_g^s(\mathbb{Q})} a(S)\, e^{\frac{2\pi i}{n} Tr(N\tau)} \tag{2.3}$$

with coefficients:

$$a(N) = \int_{[0,n]^K} f(x + iy) e^{\frac{2\pi i}{n} Tr[N(x+iy)]} dx \qquad \forall y > 0 \tag{2.4}$$

where $K = \frac{g(g+1)}{2}$. In particular, the series (2.3) converges absolutely on $\mathfrak{S}_g$ and uniformly on each compact in $\mathfrak{S}_g$.

Proof. One can consider the holomorphic map:

$$\begin{aligned} e_n : \quad & \mathfrak{S}_g \to \mathbb{C}^N \\ & \tau \mapsto \{e^{\frac{2\pi}{n}\tau_{ij}}\}_{i \le j} \end{aligned} \tag{2.5}$$

[3] See, for instance,[GR], [O], [Ra] or [Sh]

[4] More in general, the convergence domain of the Laurent series expansion is a logaritmically convex and relatively complete Reinhardt's domain.

It turns out from the definition of e_n that the range $A := \operatorname{Ran} e_n$ is a Reinhardt's domain. Moreover, for a fixed $q \in A$ one clearly has $\tau_1, \tau_2 \in e^{-1}(q)$ of and only if $\tau_1^{(i,j)} - \tau_2^{(i,j)} \in \mathbb{Z}$ for each (i, j); since the periodicity condition on f implies, in particular, that f is periodic with period n in each variable $\tau^{(i,j)}$, a function $g : A \to \mathbb{C}$ is well defined by:

$$g(q) := \hat{f}(\hat{e}_n{}^{-1}(q)) \qquad \forall q \in A$$

where $\hat{f}$ e $\hat{e}_n$ are the functions respectively induced on the quotient by f and e_n:

$$\hat{f} : \mathfrak{S}_g/Sym_g(\mathbb{Z}) \to \mathbb{C} \qquad \hat{e}_n : \mathfrak{S}_g/Sym_g(\mathbb{Z}) \to A$$

By construction the function g verifies the property $g \cdot e_n = f$ and is, therefore, a holomorphic function. In particular, g admits a Laurent expansion on each product of annuli contained in A, by Theorem 2.1):

$$g(q) = \sum_{n_1 \ldots n_K = -\infty}^{\infty} c_{n_1 \ldots n_K} q_1^{n_1} \cdots q_K^{n_K} \tag{2.6}$$

with:

$$\begin{aligned} c_{n_1,\ldots n_K} &= \left(\frac{1}{2\pi i}\right)^K \int_{\partial \overline{D}_{r_K}} \cdots \int_{\partial \overline{D}_{r_1}} \frac{g(\xi_1, \ldots \xi_K)}{\xi_1^{n_1+1} \cdots \xi_K^{n_K+1}} d\xi_1 \cdots d\xi_K = \\ &= \int_{[0,n]^K} g(r_1 e^{\frac{2\pi i}{n} t_1}, \ldots, r_K e^{\frac{2\pi i}{n} t_K}) \left(\prod_{j=1}^K r_j^{-n_j} e^{\frac{2\pi i}{n} n_j t_j} \right) dt_1 \cdots dt_K \end{aligned}$$

Hence, by setting:

$$r_j = e^{\frac{2\pi}{n} y_j}; \qquad x_j = t_j; \qquad N = \begin{pmatrix} n_1 & \frac{n_2}{2} & \cdots & \frac{n_g}{2} \\ \frac{n_2}{2} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{n_g}{2} & \cdots & \cdots & n_K \end{pmatrix};$$

one has:

$$\begin{aligned} c_{n_1,\ldots n_K} &= \int_{[0,n]^K} g(e^{\frac{2\pi i}{n}(x_1 + i y_1)}, \ldots, e^{\frac{2\pi i}{n}(x_K + i y_K)})\, e^{\frac{2\pi i}{n} \sum_{j=1}^K n_j (x_j + i y_j)} dx_1 \cdots dx_K = \\ &= \int_{[0,n]^K} f(x + iy) e^{\frac{2\pi i}{n} Tr(S\tau)} dx_1 \cdots dx_K \end{aligned}$$

Since $g \cdot e_n = f$, (2.6) implies (2.3) with $a(N) = c_{n_1 \ldots n_K}$ and one is supplied with the desired properties of convergence of the (2.3) by Theorem 2.1. □

Theorem 2.2. (Götzky-Koecher Principle)[5] *Let $\Gamma \subset \Gamma_g$ be a congruence subgroup and let $n > 1$ be such that $\Gamma_g(n) \subset \Gamma$. If $f \in A_k(\Gamma)$, then f admits the following Fourier expansion:*

[5]The following basic property was discovered by Götzky in 1928 for particular Hilbert modular forms. In 1954 Koecher provided a general demonstration ([Ko]).

$$f(\tau) = \sum_{\substack{N \in Sym_g^s(\mathbb{Q}) \\ N \geq 0}} a(N)\, e^{\frac{2\pi i}{n} Tr(N\tau)} \tag{2.7}$$

with coefficients $a(N)$ as in (2.4).

Proof. Since:

$$\gamma_S^{(n)} := \begin{pmatrix} 1_g & nS \\ 0_g & 1_g \end{pmatrix} \in \Gamma \qquad \forall S \in Sym_g(\mathbb{Z})$$

in particular, one has $f(\tau + nS) = f(\gamma_S^{(n)}\tau) = f(\tau)$ for each $\tau \in \mathfrak{S}_g$ (namely f is periodic with period n in each variable $\tau^{(i,j)}$); therefore, the function f satisfies the hypothesis in Proposition 2.1 and consequently admits a Fourier expansion as in (2.3).

Moreover, since the series in (2.3) absolutely converges for each $\tau \in \mathfrak{S}_g$, by evaluating it on $\tau = i1_g$, one gets the following convergent numerical series:

$$\sum_{N \in Sym_g^s(\mathbb{Q})} |\, a(N)\,|\, e^{-\frac{2\pi}{n} Tr(N)}$$

Therefore, there exists a constant $C \geq 0$ such that:

$$|\, a(N)\,| \leq C e^{\frac{2\pi}{n} Tr(N)} \qquad \forall\, N \in Sym_g^s(\mathbb{Q}) \tag{2.8}$$

The Fourier coefficients $a(N)$ also satisfy a special transformation law; in order to show it, one has to consider for each $u \in Gl(g, \mathbb{Z})$ such that $u \equiv 1_g \bmod n$ the following elements:

$$\gamma_u := \begin{pmatrix} {}^t u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \Gamma_g(n)$$

Thanks to the modularity condition stated on f, for each $N \in Sym_g^s(\mathbb{Q})$, such a matrix u satisfies:

$$a({}^t uNu) = det(u)^k\, a(N) \tag{2.9}$$

In fact:

$$\begin{aligned} a({}^t uNu) &= \int_{[0,n]^K} f(x+iy) e^{\frac{2\pi i}{n} Tr[\,{}^t uNu(x+iy)]} dx = \\ &= det(u)^k \int_{[0,n]^K} f(\gamma_u(x+iy)) e^{\frac{2\pi i}{n} Tr[\,{}^t uNu(x+iy)]} dx = \\ &= det(u)^k \int_{[0,n]^K} f({}^t u(x+iy)u) e^{\frac{2\pi i}{n} Tr[Nu(x+iy)^t u]} dx = det(u)^k\, a(N) \end{aligned}$$

Now, the estimate (2.8) and the transformation law (2.9) can be used to show that $a(N) = 0$ for each $N \in Sym_g^s(\mathbb{Q})$ which is not positive semi-definite. Let, then, $N \in Sym_g^s(\mathbb{Q})$ be non positive semi-definite; then, there exists a primitive vector $v \in \mathbb{Z}^g$ such that ${}^t vNv < 0$. Thanks to Corollary A.1, the vector v can be completed to a matrix $u' \in GL(g, \mathbb{Z})$; in particular, by the choice of the vector v, the (1,1)-entry of the matrix ${}^t u' N u'$ is negative. By elementary operations on the columns of u', one can obtain a new matrix u with $\det u' = detu = 1$, in such a way that $u \equiv 1_g \bmod n$ and the matrix $N' := {}^t u N u$ still satisfies $N'_{11} < 0$. Then, for each $h \in \mathbb{Z}$ one can define the matrix:

$$M_h := \begin{pmatrix} 1 & nh & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & & \\ \vdots & \vdots & & 1_{g-2} & & \\ 0 & 0 & & & & \end{pmatrix} \in GL(g, \mathbb{Z})$$

which also satisfies $M_h \equiv 1_g \bmod n$ for each $h \in \mathbb{Z}$. Now, one has:

$$\lim_{h\to\infty} Tr({}^t M_h N' M_h) = \lim_{h\to\infty} [Tr(M_h) + N'_{11} n^2 h^2 + 2N'_{12} nh] = -\infty \tag{2.10}$$

Therefore, by reiterating (2.9) and applying the estimate (2.8), one obtains:

$$\begin{aligned} |\,a(N)\,| = |\,a(N')\,| = |\,\det({}^t M_h)|^{-1}\,|\,a({}^t M_h N' M_h)\,| = \\ = |\,a({}^t M_h N' M_h)\,| \leq C e^{\frac{2\pi}{n} Tr({}^t M_h N' M_h)} \qquad \forall h \in \mathbb{Z} \end{aligned}$$

from which $a(N) = 0$ follows, due to (2.10).

□

The following Proposition is an immediate consequence of the Götzky-Koecher Principle:

Proposition 2.2. *Let Γ be a congruence subgroup of the Siegel modular group Γ_g for $g > 1$ and let $f \in A_k(\Gamma)$. Then f is bounded on each set of the kind:*

$$\mathfrak{S}_g^\lambda := \{\tau \in \mathfrak{S}_g \,|\, Im\tau - \lambda 1_g > 0\}$$

with $\lambda \geq 0$.

Proof. Let $n_0 > 1$ be such that $\Gamma_g(n_0) \subset \Gamma$; by applying the Götzky-Koecher Principle, one has:

$$f(\tau) = \sum_{N\geq 0} a(N) e^{\frac{2\pi i}{n_0} Tr(N\tau)}$$

Moreover, for each $\tau \in \mathfrak{S}_g^\lambda$ and for each $N \in Sym_g^s(\mathbb{Q})$ such that $N \geq 0$, one has:

$$|\,a(N)\, e^{\frac{2\pi i}{n_0} Tr(N\tau)}| = |\,a(N)\,|\, e^{-\frac{2\pi}{n_0} Im[Tr(N\tau)]} \leq |\,a(N)\,|\, e^{-\frac{2\pi}{n_0} Tr(N\lambda)}$$

Hence:

$$|f(\tau)| \leq \sum_{N\geq 0} |a(N)|\, e^{-\frac{2\pi}{n_0} Tr(N\lambda)} \qquad \forall \tau \in \mathfrak{S}_g^\lambda$$

and the numerical series on the right is convergent, for it is the absolute Fourier expansion of f in $i1_g \in \mathfrak{S}_g$ □

Corollary 2.1. *Let Γ be a congruence subgroup of the Siegel modular group Γ_g for $g > 1$ and let $f \in A_k(\Gamma)$. Then $\gamma^{-1}|_k f$ is bounded on Siegel's fundamental domain $\mathfrak{F}_g$ for each $\gamma \in \Gamma_g$.*

Proof. It follows from Proposizton 2.2 and Lemma 1.7. □

Corollary 2.1 shows that condition 3. in Definition 2.1 is a consequence of both conditions 1. e 2. when $g > 1$.

Another important consequence of Götzky-Koecher Principle is the following:

Corollary 2.2. *Modular forms of negative weight vanish.*

Proof. Let Γ be a congruence subgroup of the Siegel modular group Γ_g and let $n_0 > 0$ be such that $\Gamma(n_0) \subset \Gamma$.
If $f \in A_k(\Gamma)$ the function f is bounded on Siegel's fundamental domain $\mathfrak{F}_g$ (by definition when $g = 1$ and by Corollary 2.1 when $g > 1$).
Moreover, if $k < 0$, by Lemma 1.7, the function $\tau \mapsto |\det Im\tau|^{\frac{k}{2}}$ is bounded on $\mathfrak{F}_g$. Then, there exists a constant $C > 0$ such that:

$$|\det Im\tau|^{\frac{k}{2}} f(\tau) \leq C \qquad \forall \tau \in \mathfrak{F}_g$$

The coefficients $a(N)$ of the Fourier expansion of f satisfy thus for each $y > 0$:

$$\begin{aligned} |a(N)|\, e^{-\frac{2\pi}{n} Tr(Ny)} &\leq \int_{[0,n]^K} |f(x+iy)|\,|e^{\frac{2\pi i}{n} Tr(Nx)}|\,dx \leq \\ &\leq \sup_{x\in[0,n]^K} f(x+iy) \leq C\,|\det Im\tau|^{-\frac{k}{2}} \end{aligned}$$

Then, letting y tend to zero, one obtains $a(N) = 0$ whenever $N \geq 0$. □

Corollary 2.3. *The ring of modular forms with respect to a congruence subgroup Γ is a positive graded ring $A(\Gamma) = \bigoplus_{k\geq 0} A(\Gamma)_k$.*

An important theorem generally guarantees the algebraic dependence of suitably many modular forms of given weight:

Theorem 2.3. *Let $\Gamma \subset \Gamma_g$ be a congruence subgroup. Then, for each k the complex vector space $A_k(\Gamma)$ is finite-dimensional.*

The existence of non-vanishing modular forms, however, is not a trivial question. Concerning the modularity with respect to the Siegel modular group Γ_g, Eisenstein series are examples of non trivial modular forms of even weight. Other fundamental examples of modular forms with respect to the remarkable congruence subgroups introduced in the Section 1.2 will be given in the following chapter by the Theta constants with characteristic. The next section will be devoted, instead, to the introduction of a particular kind of modular forms.

2.3 Cusp forms

Proposition 2.3. *Let be $1 \leq k \leq g$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathfrak{S}_k$ a sequence such that:*

$$\tau_n = \begin{pmatrix} \tau' & u_n \\ {}^t u_n & w_n \end{pmatrix} \qquad \forall n \in \mathbb{N} \tag{2.11}$$

where $\tau' \in \mathfrak{S}_k$ is a fixed point, $\{u_n\}_{n\in\mathbb{N}} \subset Sym_{k,g-k}(\mathbb{C})$ is bounded, and $\{w_n\}_{n\in\mathbb{N}}$ is such that all the eigenvalues of $Im(w_n)$ tend to infinity. Then, the limit:

$$\lim_{n\to\infty} f(\tau_n)$$

exists and is finite for each $f \in A(\Gamma_g)$

Proof. Let $f \in A(\Gamma_g)$ and let:

$$f(\tau_n) = \sum_{\substack{N\in Sym_g^s(\mathbb{Z}) \\ N\geq 0}} a(N) e^{2\pi i Tr(N\tau_n)} \qquad \forall n \in \mathbb{N}$$

be its Fourier expansion. By hypothesis, the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is contained in a set on which the Fourier series of f uniformly converges. Since:

$$Tr(N\tau) := \sum_{i=1}^{g} N_{ii}\tau_{ii} + 2 \sum_{1\leq i\leq j\leq g} N_{ij}\tau_{ij}$$

one has:

$$\lim_{n\to\infty} f(\tau_n) = \sum_{\substack{N\in Sym_k^s(\mathbb{Z}) \\ N\geq 0}} \lim_{n\to\infty} a(N) e^{2\pi i Tr(N\tau)} = \sum_{\substack{N'\in Sym_r^s(\mathbb{Z}) \\ N'\geq 0}} a\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i Tr(N'\tau')} \tag{2.12}$$

Then the thesis follows, since $\tau' \in \mathfrak{S}_k$ implies the convergence of the series on the right. □

One has to observe that for a given couple g, k with $1 \leq k \leq g$, and $\tau \in \mathfrak{S}_{\mathfrak{k}}$ fixed, there always exists a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathfrak{S}_g$ of the kind in (2.11) with $\tau' = \tau$. It will be convenient to denote by $\{\tau_n^\tau\}$ any sequence of points in $\mathfrak{S}_g$ satisfying such a property.

For each couple g, k with $1 \leq k \leq g$, Proposition 2.3 allows to define an operator acting on modular forms by setting:

$$(\Phi_{g,k} f)(\tau) := \lim_{n\to\infty} f(\tau_n^\tau) \qquad \forall f \in A(\Gamma_g) \tag{2.13}$$

Proposition 2.4. *The law (2.13) defines an operator $\Phi_{g,k} : A(\Gamma_g) \to A(\Gamma_k)$ which preserves the weight.*

Proof. Let $f \in A_h(\Gamma_g)$. It follows from (2.12) that for each $\tau \in \mathfrak{S}_k$, $(\Phi_{g,k}f)(\tau)$ does not depend on the choice of the sequence $\{\tau_n^\tau\} \subset \mathfrak{S}_g$; moreover, since the series in (2.12) is uniformly convergent on each compact, (2.13) defines a holomorphic function on $\mathfrak{S}_k$, which in case $g = 1$ is also bounded on the Siegel fundamental domain $\mathfrak{F}_1$. Moreover, by setting:

$$\gamma_\eta := \begin{pmatrix} a_\eta & 0 & b_\eta & 0 \\ 0 & 1_{g-k} & 0 & 0 \\ c_\eta & 0 & d_\eta & 0 \\ 0 & 0 & 0 & 1_{g-k} \end{pmatrix} \in \Gamma_g \qquad \forall \eta \in \Gamma_k$$

Since $f \in A_h(\Gamma_g)$, the following transformation law holds for each $\eta \in \Gamma_k$, $\tau' \in \mathfrak{S}_k$ and $\lambda > 0$:

$$f\begin{pmatrix} \eta\tau' & 0 \\ 0 & i\lambda 1_{g-k} \end{pmatrix} = \det(c_\eta \tau' + d_\eta)^h f\begin{pmatrix} \tau' & 0 \\ 0 & i\lambda 1_{g-k} \end{pmatrix}$$

Hence, when $\lambda \to \infty$, one has:

$$\Phi_{g,k} f(\eta\tau') = \det(c_\eta\tau' + d_\eta)^h \Phi_{g,k} f(\tau') \qquad \forall\, \eta \in \Gamma_k\,, \quad \forall \tau' \in \mathfrak{S}_k$$

and consequently $\Phi_{g,k} f \in A_h(\Gamma_k)$, which concludes the proof. □

Definition 2.4. *The operator defined in (2.13) is called the* **Siegel operator**.

As seen, the simplest way to describe the action of the Siegel operator is the following:

$$\Phi(f)(\tau) = \lim_{\lambda\to\infty} f\begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix} \qquad \forall f \in A(\Gamma_k),\ \ \forall \tau \in \mathfrak{S}_r \tag{2.14}$$

The Siegel operator can be used to define the so-called cusp forms:

Definition 2.5. *The elements of the complex vector space:*

$$S_k(\Gamma) := \{f \in A_k(\Gamma) \,|\, \Phi_{g,g-1}(\gamma^{-1}|_k f) = 0 \quad \forall \gamma \in \Gamma_g\}$$

are called **cusp forms** *of weight k.*

More generally, one can define the ideal of cusp forms with respect to a congruence subgroup Γ in $A(\Gamma)$:

Definition 2.6. *Let be Γ a congruence subgroup. The ideal $S(\Gamma) := \bigoplus_{k\geq 0} S_k(\Gamma) \subset A(\Gamma)$ is called the* **ideal of the cusp forms** *with respect to Γ.*

One has to remark that this definition characterizes the cusp forms with respect to Γ as the modular forms which vanish on the boundary of the Satake's compactification of A_Γ (see Appendix B, (B.10)).

As well as for the rings of modular forms, one clearly has:

$$S(\Gamma') \subset S(\Gamma) \qquad \text{whenever} \quad \Gamma' \subset \Gamma$$

Example 2.4. *The function* Δ *introduced in Example 2.3 is a modular form of weight* 12*. It is easily checked that:*

$$\lim_{t\to+\infty} \Delta(it) = 0$$

Therefore, Δ *is a cusp form.*

Chapter 3

Theta constants

3.1 Characteristics

This section will be devoted to introduce the notion of characteristic, which reveals itself to be involved in the parametrization of the so-called Theta constants.

Definition 3.1. *A g-characteristic (or simply a characteristic, when no misunderstanding is allowed) is a vector column* $\begin{bmatrix} m' \\ m'' \end{bmatrix}$ *with* $m', m'' \in \mathbb{Z}_2^g$.

Definition 3.2. *Let* $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$ *be a characteristic. The function:*

$$e(m) = (-1)^{{}^t m' m''} \tag{3.1}$$

is called the **parity** *of m. A characteristic m is called* **even** *if* $e(m) = 1$ *and* **odd** *if* $e(m) = -1$.

Henceforward, the symbol $C^{(g)}$ will stand for the set of g-characteristics. Needless to say, $C^{(g)}$ is isomorphic to $\mathbb{Z}_2^g \times \mathbb{Z}_2^g$ as a ring; in particular, each g-characteristic m satisfies $m + m = 0$.
The symbols $C_e^{(g)}$ and $C_o^{(g)}$ will respectively stand for the subset of even g-characteristics and the subset of odd g-characteristics. Their cardinalities are easily checked by introducing the notation:

$$\tilde{m}_\delta = \begin{bmatrix} m' & \delta' \\ m'' & \delta'' \end{bmatrix} \in C^{(g)} \qquad \forall\, m = \begin{bmatrix} m' \\ m'' \end{bmatrix} \in C^{(g-1)}, \quad \forall\, \delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in C^{(1)}$$

and by noting that, whenever m is even, $\tilde{m}_\delta$ is even or odd, depending on whether the 1-characteristic δ is respectively even or odd; on the other hand, whenever m is odd, $\tilde{m}_\delta$ is even or odd, depending on whether δ is respectively odd or even. Since there are three even 1-characteristics

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and only an odd one

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

one thus has:

$$|C_e^{(g)}| = 3|C_e^{(g-1)}| + |C_o^{(g-1)}|$$

$$|C_o^{(g)}| = |C_e^{(g-1)}| + 3|C_o^{(g-1)}|$$

Hence, $|C_e^{(g)}| - |C_o^{(g)}| = 2(|C_e^{(g-1)}| - |C_o^{(g-1)}|)$. Then, one obtains by induction $|C_e^{(g)}| - |C_o^{(g)}| = 2^g$, for $|C_e^{(1)}| - |C_o^{(1)}| = 2$. Since $|C_e^{(g)}| + |C_o^{(g)}| = 2^{2g}$, one has, therefore:

$$|C_e^{(g)}| = \frac{1}{2}(2^{2g} + 2^g) = 2^{g-1}(2^g + 1)$$

$$|C_o^{(g)}| = \frac{1}{2}(2^{2g} - 2^g) = 2^{g-1}(2^g - 1)$$

Thus, to sum up, there are $2^{g-1}(2^g + 1)$ even g-characteristics and $2^{g-1}(2^g - 1)$ odd g-characteristics.

An action of the modular group Γ_g on the set $C^{(g)}$ can be defined by setting:

$$\gamma \begin{bmatrix} m' \\ m'' \end{bmatrix} := \left[\begin{pmatrix} d_\gamma & -c_\gamma \\ -b_\gamma & a_\gamma \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} diag(c_\gamma {}^t d_\gamma) \\ diag(a_\gamma {}^t b_\gamma) \end{pmatrix} \right] mod 2 \tag{3.2}$$

It is easily verified by straightforward computations that:

$$\gamma(\gamma' m) = (\gamma \cdot \gamma')m, \qquad 1_g m = m$$

as well as it is plainly checked that $e(\gamma m) = e(m)$ for each γ in Γ_g. Hence, one can state:

Lemma 3.1. *The law in (3.2) defines an action on $C^{(g)}$, which preserves the parity of the characteristics. This action is, in particular, not transitive.*

More precisely, one has (cf. [I5] or [I3]):

Lemma 3.2. *$C^{(g)}$ is decomposed into two orbits by the action in (3.2). These two orbits consist of the set of even characteristics and the set of odd characteristics.*

The action defined in (3.2) is an affine transformation of $C^{(g)}$; the congruence subgroup $\Gamma_g(1, 2)$, in particular, acts linearly on $C^{(g)}$ by definition. Moreover:

Lemma 3.3. *The action of the principal congruence subgroup $\Gamma_g(2)$ on $C^{(g)}$ is trivial.*

Proof. Let $\gamma \in \Gamma_g(2)$. Then, with reference to the notation introduced in (1.3), one has indeed:

$$\gamma \begin{bmatrix} m' \\ m'' \end{bmatrix} = \left[\begin{pmatrix} 1_{2g} + 2d_M & -2c_M \\ -2b_M & 1_{2g} + 2a_M \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} \right] \mathrm{mod} 2 = \begin{bmatrix} m' \\ m'' \end{bmatrix}$$

□

Corollary 3.1. *An action of* $\Gamma_g/\Gamma_g(2) \cong Sp(g, \mathbb{Z}_2)$ *is defined on* $C^{(g)}$.

The parity function can be used to classify remarkable k-plets of characteristics according to the action introduced in (3.2). To pursue this purpose, a parity can be also introduced for triplets:

$$e(m_1, m_2, m_3) := e(m_1)e(m_2)e(m_3)e(m_1 + m_2 + m_3) \tag{3.3}$$

Definition 3.3. *A triplet* (m_1, m_2, m_3) *is called* **azygetic** *if* $e(m_i, m_j, m_h) = -1$ *and* **syzygetic** *if* $e(m_i, m_j, m_h) = 1$

Concerning the parity of the sum of an odd sequence of g-characteristics, the following formula holds (cf. [I7]):

$$e\left(\sum_{i=1}^{2k+1}\right) = \left(\prod_{i=1}^{2k+1} e(m_i)\right)\left(\prod_{1<i<j\leq 2k+1} e(m_1, m_i, m_j)\right) \tag{3.4}$$

The parity is involved in characterizing the orbits of K-plets of g-characteristics under the action described in (3.2), as the following Proposition states (cf. [I5]):

Proposition 3.1. *Let* $(m_1, \ldots m_K)$ *and* $(n_1, \ldots n_K)$ *be two ordered K-plets of g-characteristics. Then, there exists* $\gamma \in \Gamma_g$ *such that* $\gamma m_i = n_i$ *for each* $i = 1, \ldots n$ *if and only if:*

1. $e(m_i) = e(n_i)$ *for each* $i = 1, \ldots K$;
2. $e(m_i, m_j, m_k) = e(n_i, n_j, n_k)$ *for each* $1 \leq i < j < k \leq K$;
3. *Whenever* $\{m_1, \ldots, m_{2k}\} \subset \{m_1, \ldots, m_K\}$ *is such that* $m_1 + \cdots m_{2k} \neq 0$, *then* $n_1 + \cdots n_{2k} \neq 0$;

Corollary 3.2. *Let* $(m_1, \ldots m_g)$ *and* $(n_1, \ldots n_g)$ *be two different orderings of the set* $C_e^{(g)}$, *such that for each* l, r, s *one has:*

$$e(m_l, m_r, m_s) = e(n_l, n_r, n_s)$$

Then, there exists an element $[\gamma] \in \Gamma_g/\Gamma_g(2)$ *such that* $[\gamma]m_i = n_i$ *for each* $i = 1, \ldots g$.

Proof. Let $(m_1, \dots m_g)$ and $(n_1, \dots n_g)$ be as in the hypothesis, and let $\{m_1, \dots, m_{2h}\} \subset \{m_1, \dots, m_g\}$ be such that $m_1 + \cdots m_{2h} = 0$; then, by (3.4) one has:

$$1 = e(m_{2h}) = e\left(\sum_{i=1}^{2h-1} m_i\right) =$$
$$= \prod_{1<i<j\le 2h-1} e(m_1, m_i, m_j) = \prod_{1<i<j\le 2h-1} e(n_1, n_i, n_j) = e\left(\sum_{i=1}^{2h-1} n_i\right)$$

and, consequently, for each $1 \le l < r \le K$ one has:

$$e\left(n_l, n_r, \sum_{i=1}^{2h-1} n_i\right) = e\left(n_l + n_r + \sum_{i=1}^{2h-1} n_i\right) = e(n_1, n_l, n_r) \prod_{1<i<j\le 2h-1} e(n_1, n_i, n_j) =$$
$$= e(m_1, m_l, m_r) \prod_{1<i<j\le 2h-1} e(m_1, m_i, m_j) = e\left(m_l + m_r + \sum_{i=1}^{2h-1} m_i\right) =$$
$$= e\left(m_l, m_r, \sum_{i=1}^{2h-1} m_i\right) = e(m_l, m_r, m_{2k}) = e(n_l, n_r, n_{2k})$$

Then $\sum_{i=1}^{2h-1} n_i = n_{2h}$. Hence, whenever $\{m_1, \dots, m_{2h}\} \subset \{m_1, \dots, m_g\}$ is such that $m_1 + \cdots m_{2h} = 0$, $\{n_1, \dots, n_{2h}\} \subset \{n_1, \dots, n_g\}$ also verifies the condition $n_1 + \cdots n_{2h} = 0$. Then the thesis follows from Proposition 3.1. □

The notion of azygeticity can be also given for K-plets of characteristics:

Definition 3.4. *Let be$\{m_1, \dots, m_K\}$ a K-plet of characteristics. $\{m_1, \dots, m_K\}$ is called* **azygetic** *if each subtriple $\{m_i, m_j, m_h\}$ is azygetic.*

Since the case $g = 2$ will be relevant in ordrer to discuss the new results introduced in this thesis, a focus on the main properties of k-plets of even 2-characteristics under the action in (3.2) will be needed. The next section will thus be centered around this technical feature.

3.2 k-plets of even 2-characteristics

A specific notation will be introduced here for subsets of even 2-characteristics and conventionally used henceforward.
The symbols $C_1 := C_e^{(2)}$ and $\tilde{C}_1 := C_o^{(2)}$ will stand respectively for the set of even 2-characteristics and the set of odd 2-characteristics, so that $C^{(2)} = C_1 \cup \tilde{C}_1$ with $|C_1| = 10$ and $|\tilde{C}_1| = 6$.

The symbols C e $\tilde{C}$ will denote respectively the parts of C_1 and the parts of $\tilde{C}_1$. More in general, $C_k \in C$ will stand for the set of k-plets of even 2-characteristics and $\tilde{C}_k \in \tilde{C}$ for the set of k-plets of odd 2-characteristics.

For each subset $M \subset C_1$ its complementary set $C_1 - M$ in C_1 will be denoted as M^c. The so called symmetric difference will be represented by the classic notation:

$$M_i \vartriangle M_j := (M_i \cup M_j) - (M_i \cap M_j) \in C \qquad \forall\, M_i, M_j \in C$$

Therefore, for each couple M_i, $M_j \subset C_1$, C_1 can be written as a disjoint union of sets:

$$C_1 = \left({M_i}^c \cap {M_j}^c\right) \cup \left(M_i \cap M_j\right) \cap \left(M_i \vartriangle M_j\right) \tag{3.5}$$

The chief aim of this section is to point out some combinatorial features related to remarkable subsets of C_k, which will be needed in the last chapter; these subset arise as orbits under the action of $\Gamma_2/\Gamma_2(2)$.
In fact, when $g = 2$, by Corollary 3.1 and Lemma 3.2 the group $\Gamma_2/\Gamma_2(2)$ both acts on C_1 and $\tilde{C}_1$. By focusing on the action on $\tilde{C}_1$ one note that the group homomorphism naturally defined by:

$$\psi_P : \Gamma_2/\Gamma_2(2) \mapsto S_6 \tag{3.6}$$

is injective, for the identity in $\Gamma_2/\Gamma_2(2)$ is the only element which fix all the six odd 2-characteristics. Moreover, $\Gamma_2/\Gamma_2(2) \cong Sp(2, \mathbb{Z}_2)$ and S_6 have the same order, hence the homomorphism (3.6) is also surjective. Then:

$$\Gamma_2/\Gamma_2(2) \cong S_6 \tag{3.7}$$

An action of this group is naturally defined on each C_k, by the one defined on C_1. While the action on C_2 is transitive, each C_k for $k > 2$ turns out to be decomposed into orbits, which can be described using Corollary 3.2. Seeking a notation imported from [vGvS], one observes that the set C_3 is, in particular, decomposed into two orbits C_3^- and C_3^+, which consist respectively of azygetic and syzygetic triplets:

$$C_3^- = \{\,\{m_1, m_2, m_3\} \in C_3 \mid m_1 + m_2 + m_3 \in \tilde{C}_1\}$$

$$C_3^+ = \{\,\{m_1, m_2, m_3\} \in C_3 \mid m_1 + m_2 + m_3 \in C_1\}$$

with $|C_3^-| = |C_3^+| = 60$.

The set C_4 decomposes into three orbits C_4^+, C_4^* and C_4^-, where, in particular:

$$C_4^- = \{\,\{m_1, \ldots, m_4\} \in C_4 \mid \{m_i, m_j, m_k\} \in C_3^- \;\; \forall \{m_i, m_j, m_k\} \subset \{m_1, \ldots, m_4\}\,\}$$

$$C_4^+ = \{\,\{m_1, \ldots, m_4\} \in C_4 \mid \{m_i, m_j, m_k\} \in C_3^+ \;\; \forall \{m_i, m_j, m_k\} \subset \{m_1, \ldots, m_4\}\,\}$$

with $|C_4^-| = |C_4^+| = 15$.

The set C_5 also turns out to be decomposed into three orbits C_5^+, C_5^* and C_5^-, where, in particular:

$$C_5^- = \{\, \{m_1, \dots m_5\} \in C_5 \mid \{m_1, \dots m_5\} \text{ contains a unique element of } C_4^-\}$$

$$C_5^+ = \{\, \{m_1, \dots m_5\} \in C_5 \mid \{m_1, \dots m_5\} \text{ contains a unique element of } C_4^+\}\}$$

with $|C_5^-| = |C_5^+| = 90$.

The orbit decomposition for C_k when $k > 5$ can be described by taking the complementary sets. In particular, the following orbits ofC_6 will be needed:

$$C_6^- = \{\, \{m_1, \dots, m_6\} \in C_6 \mid \{m_1, \dots, m_6\}^c \in {C_4}^+\}$$

$$C_6^+ = \{\, \{m_1, \dots, m_6\} \in C_6 \mid \{m_1, \dots, m_6\}^c \in {C_4}^-\}$$

The behaviour of this action is completely displayed in a diagram in [vGvS]. Here, some specific properties will be briefly reviewed.

Pertaining to the orbits of C_3, one has the following:

Lemma 3.4. *Let $m_1, m_2 \in C_1$ be distinct. If $M = \{m_1, m_2\}^c \subset C_1$, then $M = \{n_1, n_2, n_3, n_4, h_1, h_2, h_3, h_4\}$, with $\{m_1, m_2, n_i\} \in C_3^-$ and $\{m_1, m_2, h_i\} \in C_3^+$ for each i.*

Concerning the orbits of C_4, one has, in particular:

Lemma 3.5. *Let $\{m_1, m_2, m_3\} \in C_3^-$. There is exactly one characteristic $n \in C_1$ such that $\{m_1, m_2, m_3, n\} \in C_4^-$.*

and:

Lemma 3.6. *Let $\{m_1, m_2, m_3\} \in C_3^+$. There is exactly one characteristic $n \in C_1$ such that $\{m_1, m_2, m_3, n\} \in C_4^+$, namely $n = m_1 + m_2 + m_3$.*

As a straight consequence of Lemma 3.6 one has the following:

Corollary 3.3. *$C_4^+ \subset C_4$ is the set of the 4-plets $\{m_1, m_2, m_3, m_4\}$ satisfying the condition $m_1 + m_2 + m_3 + m_4 = 0$.*

Concerning the existence of common couples of characteristics for elements in C_4^-, Lemma 3.4 implies the following:

Corollary 3.4. *Let $h_1, h_2 \in C_1$ be distinct characteristics. The couple $\{h_1, h_2\}$ appears in exactly two elements of C_4^-.*

More precisely, one has:

Lemma 3.7. *Let* $\{m_1, m_2, h, k\}, \{m_3, m_4, h, k\} \in C_4^-$ *be distinct* 4*-plets. Then:*

$$\{m_1, m_2, m_3, m_4\} \in C_4^-$$

Lemma 3.8. *Let* $\{m_1, m_2, m_3, n\}, \{m_4, m_5, m_6, n\} \in C_4^-$ *be such that* m_i, n *are all distinct. Then:*

$$\{m_7, m_8, m_9, n\} \in C_4^-$$

where $\{m_7, m_8, m_9\} = \{m_1, m_2, m_3, m_4, m_5, m_6, n\}^c$

The following property holds for C_5^-:

Proposition 3.2. *If* $M \notin C_5^-$, *M does not contain any element of* C_4^-.

For the orbits of C_6, one has:

Proposition 3.3. $M \in C_6^-$ *if and only if M contains exactly six elements of* C_5^-.

Moreover, Corollary 3.3 implies the following:

Corollary 3.5. $M \in C_6^-$ *if and only if* $\sum_{m \in M} m = 0$.

Then it follows that:

Lemma 3.9. $M \in C_6^-$ *if and only if M does not contain any element of* C_4^+.

Proof. A 4-plets $\{m_1, m_2, m_3, m_4\} \in C_4^-$ can not belong to an element of C_6^-, for$\sum_{i=1}^{4} m_i = 0$. □

By taking the complementary sets, one obtains:

Lemma 3.10. $M \in C_6^+$ *if and only if M does not contain any element of* C_4^-.

For elements M_1, M_2 belonging both to C_4^+ and C_6^- Corollaries 3.3 and 3.5 point out immediately a property satisfied by their symmetric difference:

$$0 = \sum_{m \in M_1} m + \sum_{m \in M_2} m = \sum_{m \in M_1 \triangle M_2} m \tag{3.8}$$

The behaviour of the symmetric difference of elements of C_4^- and C_6^+ can be described more precisely, using the results gathered throughout this section.

Proposition 3.4. *Let* $M_1, M_2 \in C_4^-$. *Then* $M_1 \cap M_2 \neq \emptyset$ *and:*

1) $M_1 \triangle M_2 \in C_6^+ \quad if \quad |M_1 \cap M_2| = 1$

2) $M_1 \triangle M_2 \in C_4^- \quad if \quad |M_1 \cap M_2| = 2$

3) $M_1 = M_2 \quad if \quad |M_1 \cap M_2| > 2$

Proof. Lemma 3.10 implies $M_2 \not\subset M_1^c$, hence $M_1 \cap M_2 \neq \emptyset$. The possible cases are thus to be checked:

1) If $M_1 \cap M_2 = \{n\}$, one can set $M_1 = \{m_1, m_2, m_3, n\}$ and $M_2 = \{m_4, m_5, m_6, n\}$. Then, by Lemma 3.8, $M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4, m_5, m_6\} = \{m_7, m_8, m_9, n\}^c \in C_6^+$.

2) If $M_1 \cap M_2 = \{h, k\}$, one can set $M_1 = \{m_1, m_2, h, k\}$ and $M_2 = \{m_3, m_4, h, k\}$. Then, by Lemma 3.7, $M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4\} \in C_4^-$.

3) As a straight consequence of Lemma 3.5, if M_1 and M_2 share at least 3 common characteristics, they must be the same element of C_4^- □

Proposition 3.5. *Let $M_1, M_2 \in C_6^+$. Then, the only possible cases are:*

1) $M_1 \vartriangle M_2 \in C_6^+ \quad if \quad |M_1 \cap M_2| = 3$

2) $M_1 \vartriangle M_2 \in C_4^- \quad if \quad |M_1 \cap M_2| = 4$

3) $M_1 = M_2 \qquad if \quad |M_1 \cap M_2| > 4$

Proof. Since $|C_1| = 10$, obviously $|M_1 \cap M_2| \geq 2$.
Moreover, the relation $12 = |M_1| + |M_2| = |M_1 \cup M_2| + |M_1 \cap M_2|$, combined with (3.5), implies:

$$|M_1 \cap M_2| - |M_1^c \cap M_2^c| = 2 \tag{3.9}$$

Therefore $|M_1 \cap M_2| > 2$, because $M_1^c \cap M_2^c \neq \emptyset$ by Proposition 3.4.

Now, the remaining cases are to be checked.

1) If $M_1 \cap M_2 = \{h, k, l\}$, one can set $M_1 = \{m_1, m_2, m_3, h, k, l\}$ and $M_2 = \{m_4, m_5, m_6, h, k, l\}$. Then, $M_1^c = \{m_4, m_5, m_6, n\} \in C_4^-$ and $M_2^c = \{m_1, m_2, m_3, n\} \in C_4^-$, where $n \neq m_i, h, k, l$. Therefore, by Lemma 3.8, $M_1 \vartriangle M_2 = \{h, k, l, n\}^c \in C_6^+$.

2) If $M_1 \cap M_2 = \{n, h, k, l\}$, one can set $M_1 = \{m_1, m_2, n, h, k, l\}$ and $M_2 = \{m_3, m_4, n, h, k, l\}$. Then, $M_1^c = \{m_3, m_4, i, j\} \in C_4^-$ and $M_2^c = \{m_1, m_2, i, j\} \in C_4^-$, where $i, j \neq m_i, , n, h, k, l$. Hence, by Lemma 3.7, $M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4\} \in C_4^-$.

3) If $|M_1 \cap M_2| > 4$, then (3.9) implies $|M_1^c \cap M_2^c| > 2$; therefore, $M_1^c = M_2^c$ by Proposition 3.4, and consequently $M_1 = M_2$. □

Proposition 3.6. *Let $M_1 \in C_6^+$ and $M_2 \in C_4^-$. If $M_1^c \neq M_2$ the only possible cases are:*

1) $M_1 \vartriangle M_2 \in C_4^- \quad if \quad |M_1 \cap M_2| = 3$

2) $M_1 \vartriangle M_2 \in C_6^+ \quad if \quad |M_1 \cap M_2| = 2$

Proof. Obviously $|M_1 \cap M_2| \leq 4$; moreover, Lemma 3.10 implies that $M_2 \not\subset M_1$, hence $|M_1 \cap M_2| \leq 3$. If $M_1^c \neq M_2$, Lemma 3.5 implies $|M_1^c \cap M_2| < 3$ and consequently $|M_1 \cap M_2| > 1$. Therefore, the only cases to be checked, when $M_1^c \neq M_2$, are the cases 1) and 2):

1) If $M_1 \cap M_2 = \{h,k,l\}$, one can set $M_1 = \{m_1, m_2, m_3, h, k, l\}$ and $M_2 = \{m_4, h, k, l\}$. Then $M_2^c = \{m_1, m_2, m_3, r, s, t\} \in C_6^+$ with $r, s, t \neq m_i, h, k, l$, and by Proposition 3.5 (case 1) one has:

$$M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4\} = \{h, k, l, r, s, t\}^c = (M_1 \vartriangle M_2^c)^c \in C_4^-$$

2) If $M_1 \cap M_2 = \{h,k\}$, one can set $M_1 = \{m_1, m_2, m_3, m_4, h, k\}$ and $M_2 = \{m_5, m_6, h, k\}$. Then $M_1^c = \{m_5, m_6, i, j\} \in C_4^-$ with $i, j \neq m_i, h, k$, and by Proposition 3.4 (case 2) one has:

$$M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4, m_5, m_6\} = \{i, j, h, k\}^c = (M_1^c \vartriangle M_2)^c \in C_6^+$$

□

3.3 Theta constants

Theta constants play an essential role in the construction of several modular forms. The aim of this section is to introduce these remarkable functions as well as to outline their main basic properties. The main references for this section will be Igusa's classical works [I5], [I3] and [I4], Freitag's book [F], Farkas and Kra's book [FK] and Mumford's lectures [Mf].

Definition 3.5. *For each $m = (m', m'') \in \mathbb{Z}^g \times \mathbb{Z}^g$, the function $\theta_m : \mathfrak{S}_g \times \mathbb{C}^g \to \mathbb{C}$, defined by:*

$$\theta_m(\tau, z) := \sum_{n \in \mathbb{Z}^g} exp\left\{ {}^t\left(n + \frac{m'}{2}\right)\tau\left(n + \frac{m'}{2}\right) + 2^t\left(n + \frac{m'}{2}\right)\left(z + \frac{m''}{2}\right)\right\} \qquad (3.10)$$

is called **Riemann Theta function with characteristic** *$m = (m', m'')$.*

The series in (3.10) is easily seen to be absolutely convergent and uniformly convergent on each compact of $\mathfrak{S}_g \times \mathbb{C}^g$; therefore, for each m it defines a holomorphic function. One can note that in the case $g = 1$ the series for $m = 0$ is the classical Jacobi Theta function.

By setting:

$$\tilde{\tau} := (\tau\, 1_g) \in Sym_{g, 2g(\mathbb{C})} \qquad \forall \tau \in \mathfrak{S}_g \qquad (3.11)$$

is plainly verified that for each $m = (m', m'')$, $n = (n', n'') \in \mathbb{Z}^g \times \mathbb{Z}^g$ the Theta function with characteristic m satisfies the equation [1]

$$\theta_m(\tau, z + \tilde{\tau} n) = (-1)^{{}^tm'n'' - {}^tm''n'} \exp\left\{2^t\left(-{}^tn'z - \frac{1}{2}{}^tn\tau n'\right)\right\}\theta_m(\tau, z) \qquad (3.12)$$

[1]One proves that any other holomorphic function $\theta : \mathbb{C}^n \to \mathbb{C}$ which is solution of (3.12) differs from the Theta function with characteristic by a multiplicative factor; for each $\tau \in \mathfrak{S}_g$ fixed, the function $z \mapsto \theta_m(\tau, z)$ is, therefore, characterized as an analytic function on $\mathbb{C}^n$, by being solution of the equation (3.12).

As proved through term by term differentiation, the Theta function with characteristic also satisfies the heat equation: [2].

$$\sum_{j,k=1}^{g} \sigma_{jk} \frac{\partial \theta_m}{\partial z_j z_k} = 2\pi i \sum_{j,k=1}^{g} \sigma_{jk} \frac{\partial \theta_m}{\partial \tau_{jk}} \tag{3.13}$$

With reference to the actions of Γ_g described in (1.10) and (3.2) , if $n = \gamma m$ with $\gamma \in \Gamma_g$, it easily turns out that:

$$\theta_n(\gamma\tau, {}^t(c_\gamma\tau + d_\gamma)^{-1}z) = K(m, \gamma, \tau) \exp\left\{2\left(-\frac{1}{2}{}^t z(c_\gamma\tau + d_\gamma)^{-1} c_\gamma z\right)\right\} \theta_m(\tau, z) \tag{3.14}$$

where $K : \mathbb{Z}^{2g} \times \Gamma_g \times \mathfrak{S}_g \to \mathbb{C}$ is a non-vanishing function, which does not depend on the variable z.

The g-characteristics parametrize distinct Theta functions. The definition (3.10) implies, indeed:

$$\theta_{m+2n}(\tau, z) = (-1)^{{}^t m' n''} \theta_m(\tau, z)$$

for each $m = (m', m'')$, $n = (n', n'') \in \mathbb{Z}^g \times \mathbb{Z}^g$. Hence, in order to study Theta functions one can focus on the only ones which are related to g-characteristics.

Definition 3.6. *For each characteristic m the holomorphic function $\theta_m : \mathfrak{S}_g \to \mathbb{C}$, defined by:*

$$\theta_m(\tau) := \theta_m(\tau, 0) \tag{3.15}$$

is called **Theta constant** *with characteristic m.*

From (3.10), in particular, it follows that:

$$\theta_m(\tau, -z) = e(m)\, \theta_m(\tau, z) \tag{3.16}$$

where $e(m)$ is the parity of the characteristic m introduced in (3.1) [3] By (3.16), Theta constants related to odd characteristics vanish. On the converse, one easily notes that $\lim_{\lambda\to\infty} \theta_0(i\lambda 1_g) = 1$; hence, the Theta constant θ_0, related to the null characteristic, does not vanish. Lemma 3.2 and the non-vanishing property of the function K imply that each Theta constant θ_m, related to an even characteristic m, does not vanish.

In order to determine an explicit expression for the function K, the following differential equation, derived form (3.13) for even characteristics, can be used:

$$\sum_{1\le j,k\le g} \sigma_{jk} \frac{\partial \log K}{\tau_{jk}} = \frac{1}{2} \sum_{1\le j,k\le g} \sigma_{jk}\mu_{jk}$$

[2]One also proves that as a holomorphic function $\theta_m : \mathfrak{S}_g \times \mathbb{C}^g \to \mathbb{C}$, the Theta function is characterized by being solution of (3.12) and (3.13)

[3]The formula (3.16) justifies the classical choice to define this function a parity.

As it can be shown, it implies:

$$K(m,\gamma,\tau) = C(m,\gamma)\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$$

where C is a function which does not depend on τ. In particular, one has to observe that, albeit the root $\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$ is not unique, it is well defined as an analytic function on $\mathfrak{S}_g$.
Now, for each $n = (n', n'') \in C^{(g)}$ e $\tilde{\tau} = (\tau\, 1_g)$ as in (3.11), one has:

$$\theta_m\left(\tau, z + \frac{1}{2}\tilde{\tau}n\right) = \exp\left\{2\left[-\frac{1}{8}{}^t n'\tau n' - \frac{1}{2}{}^t n'\left(z + \frac{1}{2}(m'' + n'')\right)\right]\right\}\theta_{m+n}(\tau, z)$$

Then, by selecting the branch of $\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$ whose sign turns to be positive when $Re\tau = 0$, and applying (3.14), one has for each $\gamma \in \Gamma_g$:

$$\theta_{\gamma 0}(\gamma\tau, {}^t(c_\gamma\tau + d_\gamma)^{-1}z) = C(m,\gamma)\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\exp\left\{2\left(\frac{1}{2}{}^t z(c_\gamma\tau + d_\gamma)^{-1}c_\gamma\right)\right\}\theta_0(\tau, z)$$

Operating for each even characteristic m the substitution $z \mapsto z + \frac{1}{2}\tilde{\tau}m$, the multiplicative factor of C, depending only on m, can be explicited; hence one is supplied with the following transformation law for the Theta function:

$$\begin{gathered}\theta_{\gamma m}(\gamma\tau, {}^t(c_\gamma\tau + d_\gamma)^{-1}z) = \kappa(\gamma)e^{\pi i\left\{\frac{1}{2}{}^t z\,[(c_\gamma\tau + d_\gamma)^{-1}c_\gamma]\,z + 2\phi_m(\gamma)\right\}}det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\theta_m(\tau, z) \\ \forall\gamma \in \Gamma_g, \quad \forall m \in C^{(g)}, \quad \forall\tau \in \mathfrak{S}_g, \quad \forall z \in \mathbb{C}^g\end{gathered} \tag{3.17}$$

where:

$$\begin{aligned}\phi_m(\gamma) = &-\frac{1}{8}({}^t m'{}^t b_\gamma d_\gamma m' + {}^t m''{}^t a_\gamma c_\gamma m'' - 2{}^t m'{}^t b_\gamma c_\gamma m'') + \\ &-\frac{1}{4}{}^t diag(a_\gamma{}^t b_\gamma)(d_\gamma m' - c_\gamma m'')\end{aligned} \tag{3.18}$$

and $\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$ is the root, which has been chosen in the described way.

In particular, the function:

$$\Phi(m,\gamma,\tau,z) := \kappa(\gamma)e^{\pi i\left\{\frac{1}{2}{}^t z\,[(c_\gamma\tau + d_\gamma)^{-1}c_\gamma]\,z + 2\phi_m(\gamma)\right\}}$$

is such that $\Phi|_{z=0}$ does not depend on τ. For this reason the functions θ_m defined in (3.15) are called Theta constants; the transformation law in (3.17) becomes for Theta constants:

$$\begin{gathered}\theta_{\gamma m}(\gamma\tau) = \kappa(\gamma)\chi_m(\gamma)det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\theta_m(\tau) \\ \forall\gamma \in \Gamma_g, \quad \forall m \in C^{(g)}, \quad \forall\tau \in \mathfrak{S}_g,\end{gathered} \tag{3.19}$$

where:

$$\chi_m(\gamma) := \Phi(m,\gamma,\tau,0) = e^{2\pi i \phi_m(\gamma)} \tag{3.20}$$

Concerning the function κ, an important property can be stated:

Proposition 3.7. *κ^2 is a character of $\Gamma_g(1,2)$.*

Proof. Since $\chi_0 = 1$ for the null characteristic $m = 0$ (by definition in (3.20) and (3.18)), by applying (3.19) to the Theta constant θ_0 related to the null characteristic, one obtains:

$$\theta^2_{\gamma 0}(\gamma\tau) = \kappa^2(\gamma) det(c_\gamma \tau + d_\gamma)\theta^2_0(\tau)$$

Since $\Gamma_g(1,2)$ acts linearly, $\gamma 0 = 0$ whenever $\gamma \in \Gamma_g(1,2)$; then:

$$det(c_\gamma\tau + d_\gamma)^{-1}\theta^2_0(\gamma\tau) = \kappa^2(\gamma)\theta^2_0(\tau) \qquad \forall\gamma \in \Gamma_g(1,2)$$

Hence, with reference to the action introduced in (2.1), one has:

$$\gamma^{-1}|_1\theta^2_0 = \kappa^2(\gamma)\theta^2_0 \qquad \forall\gamma \in \Gamma_g(1,2)$$

and consequently:

$$\kappa^2(\gamma\gamma')\theta^2_0(\tau) = \kappa^2(\gamma)\kappa^2(\gamma')\theta^2_0(\tau) \qquad \forall\gamma,\gamma' \in \Gamma_g(1,2)$$

Then $\kappa^2(\gamma\gamma') = \kappa^2(\gamma)\kappa^2(\gamma')$ and $\kappa^2(1_{1g}) = 1$, for θ_0 does not vanish; hence the thesis follows. □

The following Lemma provides an expression for the function κ for special elements of Γ_g:

Lemma 3.11. *Let:*

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{pmatrix} \in \Gamma_g$$

Then:

$$\kappa^2(\gamma) = \kappa^2({}^t\gamma^{-1}) = \det(d_\gamma)$$

Proof. The proof of the whole statement can be found in [I4]. Here it will be only shown that $\kappa^2(\gamma) = \det(d_\gamma)$.
Since $\chi_0 = 1$, the transformation law (3.19) implies in particular:

$$\theta^2_{\gamma 0}(\gamma\tau) = \kappa^2(\gamma)\det(d_\gamma)\theta^2_0(\tau)$$

$$\forall\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{pmatrix} \in \Gamma_g$$

As $\theta^2_{\gamma 0}(\gamma\tau) = \theta^2_0(\tau)$, one has $\kappa^2(\gamma) = \det(d_\gamma)$. □

As a consequence of this Lemma, one obtains a description for the possible values of the function κ:

Proposition 3.8. *For each $\gamma \in \Gamma$, $\kappa(\gamma)$ is an eight root of the unity.*

Proof. By definition in (3.20) and (3.18), one clearly has $\chi_m^8 = 1$. Then (3.19) implies:

$$det(c_\gamma \tau + d_\gamma)^{-4} \theta^8_{\gamma m}(\gamma \tau) = \kappa^8(\gamma) \theta^8_m(\tau)$$

$$\forall \gamma \in \Gamma_g, \quad \forall m \in C^{(g)}$$

Again with reference to the action introduced in (2.1), one has:

$$\gamma^{-1}|_4 \theta^8_{\gamma m} = \kappa^8(\gamma) \theta^8_m \qquad \forall \gamma \in \Gamma_g, \quad \forall m \in C^{(g)}$$

Hence:

$$\kappa^8(\gamma\gamma')\theta^8_m(\tau) = \kappa^8(\gamma)\kappa^8(\gamma')\theta^8_m(\tau) \qquad \forall \gamma, \gamma' \in \Gamma_g \quad \forall m \in C^{(g)}$$

Then, κ^8 is a character of Γ_g. Moreover, by Lemma 3.11, one has:

$$\kappa^4(\gamma) = 1 \qquad \kappa^4({}^t\gamma^{-1}) = 1$$

whenever $\gamma \in \Gamma_g$ is such that $c_\gamma = 0$; since by Corollary 1.1 these elements generates Γ_g, one has:

$$\kappa^8(\gamma) = 1 \qquad \forall \gamma \in \Gamma_g \tag{3.21}$$

□

An explicit expression for κ^4, which can be found in [I3], is recalled in the following statement:

Proposition 3.9. *For each $\gamma \in \Gamma_g$ the following expression holds:*

$$\kappa(\gamma)^4 = e^{\pi Tr({}^t b_\gamma c_\gamma) i} \tag{3.22}$$

A simple explicit expression for κ^2 on suitable congruence subgroups can be also found, using Proposition 3.7:

Proposition 3.10. *For each $\gamma \in \Gamma_g(2)$ the following expression holds:*

$$\kappa(\gamma)^2 = e^{\frac{\pi}{2} Tr(a_\gamma - 1_g) i} \tag{3.23}$$

Proof. Let $\gamma, \gamma' \in \Gamma_g(2)$. With reference to the notation in (1.3) one has:

$$a_{\gamma\gamma'} = 1_g + 2(a_M + a_{M'}) + 4a_M a_{M'} + 4b_M cM'$$

Therefore, by setting:

$$g(\gamma) := e^{\frac{\pi}{2}Tr(a_\gamma - 1_g)i}$$

one has:

$$g(\gamma\gamma') = e^{\frac{\pi}{2}2Tr(a_M + a_{M'})i} = e^{\frac{\pi}{2}2Tra_M i}e^{\frac{\pi}{2}2Tra_{M'})i} = \\ = e^{\frac{\pi}{2}Tr(a_\gamma - 1_g)i}e^{\frac{\pi}{2}Tr(a_{\gamma'} - 1_g)i} = g(\gamma)g(\gamma')$$

Consequently, the function g is a character of $\Gamma_g(2)$. However, κ^2 is also a character of $\Gamma_g(2)$, by Proposition 3.7. Then, by using Lemma 3.11, it can be directly checked that $g(\gamma) = \kappa^2(\gamma)$ and $g({}^t\gamma^{-1}) = \kappa^2({}^t\gamma^{-1})$, whenever γ is such that $c_\gamma = 0$; since by Proposition 1.4 the generators of $\Gamma_g(2)$ are such elements, the characters g and κ^2 coincide on $\Gamma_g(2)$. □

Thanks to (3.23) one has, in particular:

$$k^2(\gamma) = 1 \qquad \forall\, \gamma \in \Gamma(4,8) \tag{3.24}$$

and

$$\kappa^2(-1_{2g}) = \begin{cases} 1 & \text{if } g \text{ is even} \\ -1 & \text{if } g \text{ is odd} \end{cases} \tag{3.25}$$

The formula (3.25) implies, in particular, that the function κ^2 is well defined on the quotient group $\Gamma_g(2)/\{\pm 1_{2g}\}$ if g is even. Then, Proposition 3.7 implies:

Corollary 3.6. *When g is even, κ^2 is a character of $\Gamma_g(2)/\{\pm 1_{2g}\}$.*

Moreover, one has:

Corollary 3.7. *When g is even, κ^2 is a character of the group $\Gamma_g(2,4)/\{\pm\Gamma(4,8)\}$.*

Proof. When g is even, Corollary 3.6 implies, in particular, that κ^2 is a character of $\Gamma_g(2,4)/\{\pm 1_{2g}\}$; hence the thesis follows, since κ^2 is well defined on the quotient $\Gamma_g(2,4)/\{\pm\Gamma(4,8)\}$ due to (3.24). □

Concerning the function χ_m introduced in (3.20), a straightforward computation allows to verify a basic properties:

Lemma 3.12.

$$\chi_m(-1_{2g}) = \begin{cases} 1 & \textit{if } m \textit{ is even} \\ -1 & \textit{if } m \textit{ is odd} \end{cases}$$

Moreover, one has:

Lemma 3.13. *Let be $A_{ij}, B_{ij}, C_{ij} \in \Gamma_g(2,4)$ as in Proposition 1.4. Then, for each characteristic $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$ one has:*

$$\chi_m(A_{ij}) = (-1)^{m'_i m''_j}$$

$$\chi_m(B_{ij}) = (-1)^{m'_i m'_j}$$

$$\chi_m(C_{ij}) = (-1)^{m'''_i m''_j}$$

$$\chi_m(B^2_{ii}) = (-1)^{m'^2_i} \quad (i < j)$$

$$\chi_m(C^2_{ii}) = (-1)^{m''^2_i} \quad (i < j)$$

where m'_i and m''_i denote respectively the i-th coordinate of m' and the i-th coordinate of m''.

Proof. By applying (3.18) to the generic element in $\Gamma_g(2,4)$, one finds:

$$\begin{aligned} &\chi_m(\gamma) = e^{\frac{\pi}{2}{}^t m_1(-b_\gamma m' + a_\gamma m'' - m'')i} e^{-\frac{\pi}{4}({}^t m'{}^t b_\gamma d_\gamma m' + {}^t m''{}^t a_\gamma c_\gamma m'')i} \\ &\forall\, \gamma \in \Gamma_g(2,4) \end{aligned} \tag{3.26}$$

thanks to which the values in the statement can be computed. □

By Corollary 1.2, using Lemmas 3.12 and 3.13, the following statements can be straightly proved:

Lemma 3.14. $\chi^2_m(\gamma) = 1$ *for each* $\gamma \in \Gamma_g(2,4)$

Lemma 3.15. *For each characetristic $m \in C^{(g)}$, χ_m is a character of $\Gamma_g(2,4)$.*

Lemma 3.16. $\chi_m(\gamma) = 1$ *for each* $\gamma \in \Gamma_g(4,8)$.

By Lemmas 3.15 and 3.16 one has, in particular:

Corollary 3.8. *For each characteristic $m \in C^{(g)}$, the function χ_m is well defined on the quotient group:*

$$\chi_m : \Gamma_g(2,4)/\Gamma_g(4,8) \longrightarrow \mathbb{C}^*$$

Moreover χ_m is a character of $\Gamma_g(2,4)/\Gamma_g(4,8)$ [4].

Corollary 3.8 and Lemma 3.12 imply, in particular:

[4]Likewise, one can observe that Lemma 3.14, allows to define χ^2_m on the quotient group $\Gamma_g(2)/\Gamma_g(2,4)$, hence χ^2_m is a character of this group. (cf. [SM2])

Corollary 3.9. *Let $m, n \in C^{(g)}$. If m and n are both even or odd, the function $\chi_m\chi_n$ is well defined on the quotient group:*

$$\chi_m\chi_n : \Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\} \longrightarrow \mathbb{C}^*$$

and is a character of $\Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\}$ [5].

In general, one can prove the following:

Proposition 3.11. *The set $\{\chi_m\}$, parametrized by the even characteristics m, is a set of generators for thel group $\Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\}$.*

Proof. A proof can be found in [SM2]. □

As a consequence of (3.19), the action of the modular group Γ_g can be defined on couples of characters $\chi_m\chi_n$. In fact, if one defines for each $\gamma \in \Gamma_g$ the function:

$$\gamma \cdot \chi_m(\eta) := \chi_m(\gamma\eta\gamma^{-1}) \tag{3.27}$$

the following statement holds:

Proposition 3.12. *The law (3.27) defines an action of the modular group Γ_g on the products $\chi_m\chi_n$ satisfying the property:*

$$\gamma(\chi_m\chi_n) = (\gamma \cdot \chi_m)(\gamma \cdot \chi_n) = \chi_{\gamma^{-1}m}\chi_{\gamma^{-1}n} \tag{3.28}$$

Proof. With reference to the action introduced in (2.1), (3.18) and (3.19) imply for each $\gamma \in \Gamma$:

$$\gamma^{-1}|_1\theta_m\theta_n = \kappa^2(\gamma)e^{2\pi i(\phi_{\gamma^{-1}m}(\gamma)+\phi_{\gamma^{-1}n}(\gamma))}\theta_{\gamma^{-1}m}\theta_{\gamma^{-1}n} \tag{3.29}$$

Now, let be m, n both even. Then (3.29) implies:

$$\kappa^2(\gamma)\kappa^2(\gamma^{-1})e^{2\pi i(\phi_{\gamma^{-1}m}(\gamma)+\phi_{\gamma^{-1}n}(\gamma))}e^{2\pi i(\phi_{\gamma m}(\gamma^{-1})+\phi_{\gamma n}(\gamma^{-1}))} = 1 \tag{3.30}$$

Hence, for each $\gamma \in \Gamma_g$ and $\eta \in \Gamma_g(2,4)$:

$$((\gamma\eta\gamma^{-1})^{-1}|_1\theta_m\theta_n)(\tau) = \kappa^2(\gamma\eta\gamma^{-1})^2\chi_m(\gamma\eta\gamma^{-1})\chi_n(\gamma\eta\gamma^{-1})\theta_m(\tau)\theta_n(\tau) \tag{3.31}$$

But since

$$\gamma^{-1}|_1\theta_m\theta_n = \kappa^2(\gamma)\chi_m\chi_n\theta_{\gamma^{-1}m}\theta_{\gamma^{-1}n} \qquad \forall\gamma \in \Gamma_g(2,4) \tag{3.32}$$

one also has by (3.30):

$$((\gamma\eta\gamma^{-1})^{-1}|_1\theta_m\theta_n)(\tau) = \kappa^2(\eta)\chi_{\gamma^{-1}m}(\eta)\chi_{\gamma^{-1}n}(\eta)\theta_m(\tau)\theta_n(\tau) \tag{3.33}$$

[5] More in general $\chi_m\chi_n$ is proved to be a character of $\Gamma_g(2)/\{\pm 1_g\}$.

Then the thesis follows from (3.31) and (3.33), for (3.23) implies:

$$\kappa^2(\gamma\eta\gamma^{-1}) = \kappa^2(\eta) \qquad \forall\gamma \in \Gamma_g, \qquad \forall\eta \in \Gamma_g(2,4)$$

Concerning the other cases, since for each odd characteristic n the holomorphic maps:

$$\mathrm{grad}_z^0\theta_n := \mathrm{grad}_z\theta_n|_{z=0} = \left(\frac{\partial}{\partial z_1}\theta_n|_{z=0}, \ldots, \frac{\partial}{\partial z_g}\theta_n|_{z=0}\right)$$

satisfy the following transformation law:

$$\begin{aligned} &\mathrm{grad}_z^0\theta_n(\gamma\tau) = \det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}(c_\gamma\tau + d_\gamma)\cdot \mathrm{grad}_z^0\theta_n(\tau) \\ &\forall\gamma \in \Gamma_g(4,8), \quad \forall\tau \in \mathfrak{S}_g \end{aligned} \tag{3.34}$$

one can likewise use the same argument on $\theta_m\mathrm{grad}_z^0\theta_n$ and on $\mathrm{grad}_z^0\theta_m\mathrm{grad}_z^0\theta_n$ to prove the statement respectively when m is even and n is odd and when both m and n are odd. □

The transformation law in (3.34) means, in particular, that gradients of odd Theta functions can be regarded as modular forms with respect to $\Gamma_g(4,8)$ under the representation [6]:

$$T_0(A) := \det(A)^{1/2}\dot{A} \tag{3.35}$$

Such a behaviour allows to define the map, on which the thesis will focus in the next chapter.

As a consequence of the Proposition 3.12 Γ_g acts on even sequences of characters χ_m.

Thanks to some of the results gathered here, the modular property of the Theta constants can be discussed:

Proposition 3.13. *The product of two Theta constants $\theta_m\theta_n$ is a modular form of weight* 1 *with respect to* $\Gamma_g(4,8)$*.*

Proof. First of all, by Lemma 3.3, the law (3.19) turns out to be:

$$\begin{aligned} &\theta_m(\gamma\tau) = \kappa(\gamma)\chi_m(\gamma)\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\theta_m(\tau) \\ &\forall\gamma \in \Gamma_g(2), \quad \forall m \in C^{(g)}, \quad \forall\tau \in \mathfrak{S}_g, \end{aligned} \tag{3.36}$$

Then, by applying (3.24) and Lemma 3.16, one is supplied with the following transformation law:

$$\begin{aligned} &\theta_m\theta_n(\gamma\tau) = \det(c_\gamma\tau + d_\gamma)\theta_m\theta_n(\tau) \\ &\forall\gamma \in \Gamma_g(4,8), \quad \forall m \in C_g, \quad \forall\tau \in \mathfrak{S}_g, \end{aligned} \tag{3.37}$$

which proves the thesis. □

[6]The classical modular forms of weight k, which are the only ones discussed in this work, are indeed modular forms under the representation $T(A) := \det(A)^k$

A useful criterion of modularity with respect to $\Gamma_g(2,4)$ can be more generally stated for products of an even sequence of Theta constants:

Proposition 3.14. *Let $M = (m_1, \dots, m_{2k})$ be a sequence of even characteristics. The product $\theta_{m_1} \cdots \theta_{m_{2k}}$ is a modular form with respect to $\Gamma_g(2,4)$ if and only if the following condition is verified:*

$$M\,{}^tM \equiv k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; mod\, 2 \tag{3.38}$$

Proof. Let $M = (m_1, \dots, m_{2k})$ be a sequence of even characteristics; one can set:

$$m_h = \begin{bmatrix} m'_h \\ m''_h \end{bmatrix}$$

for each characteristic m_h appearing in the sequence M, and $m'_{h,(i)}$ and $m''_{h,(i)}$ respectively for the i-th coordinate of m'_h and m''_h.
Now, by (3.36) and (3.23), if k is even, $\theta_{m_1} \cdots \theta_{m_{2k}}$ is a modular form with respect to $\Gamma_g(2,4)$ if and only if the character $\chi_M := \chi_{m_1} \cdots \chi_{m_{2k}}$ is trivial on $\Gamma_g(2,4)$. Due to Corollary 1.2, this is true if and only if the elements A_{ij} (for $i, j \neq g$), B_{ij}, C_{ij} (for $i < j$), B^2_{ii}, C^2_{ii} belong to the kernel $Ker\chi_M$. Then, Lemma 3.13 translates this criterion into conditions for the sequence M:

a) $\sum_{h=1}^k (m'_{h,(i)})^2 = 0 \quad \forall\, i \qquad (\ \chi_M(B^2_{ii}) = 1\ )$

b) $\sum_{h=1}^k (m''_{h,(i)})^2 = 0 \quad \forall\, i \qquad (\ \chi_M(C^2_{ii}) = 1\ )$

c) $\sum_{h=1}^k m'_{h,(i)} m'_{h,(j)} = 0 \quad \forall\, i < j \qquad (\ \chi_M(B_{ij}) = 1\ )$

d) $\sum_{h=1}^k m''_{h,(i)} m''_{h,(j)} = 0 \quad \forall\, i < j \qquad (\ \chi_M(C_{ij}) = 1\ )$

e) $\sum_{h=1}^k m'_{h,(i)} m''_{h,(j)} = 0 \quad \forall\, i, j \neq g \qquad (\ \chi_M(A_{ij}) = 1\ )$

Therefore, for k even, one has the criterion:

$$M\,{}^tM \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \; \mathrm{mod}\, 2$$

Now, if k is odd, (3.36) and (3.23) imply that $\theta_{m_1} \cdots \theta_{m_{2k}}$ is a modular form with respect to $\Gamma_g(2,4)$ if and only if $\kappa^2\chi_M$ is trivial on $\Gamma_g(2,4)$. Using the definition of these elements (see Proposition 1.4), one can apply (3.23) and verify that the A_{ii} are the only elements on which the function κ^2 assumes the value -1 . Then conditions a) b) c) d) keep unchanged, while condition e) is substituted by a twofold condition:

e′) $\sum_{h=1}^k m'_{h,(i)} m''_{h,(j)} = 0 \quad \forall\, i \neq j \qquad (\ \kappa^2\chi_M(A_{ij}) = 1\ )$

e′′) $\sum_{h=1}^k m'_{h,(i)} m''_{h,(i)} = 1 \quad \forall\, i \qquad (\ \kappa^2\chi_M(A_{ij}) = 1\ )$

Hence, for k odd one has the criterion:

$$M\,{}^tM \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2$$

which concludes the proof. □

Concerning the product of all the non-vanishing Theta constants:

$$\chi^{(g)} := \prod_{m \text{ even}} \theta_m \tag{3.39}$$

one knows by Proposition 3.13 that for $g \geq 2$ $\chi^{(g)}$ is a modular form of weight $2^{g-2}(2^g+1)$ with respect to $\Gamma_g(4,8)$. However, for each $\gamma \in \Gamma_g$ one has by (3.17):

$$\begin{aligned}\prod_{m \text{ even}} \theta_m(\gamma\tau) &= \prod_{m \text{ even}} \theta_{\gamma m}(\gamma\tau) = \\ &= \left(\kappa^{2^{g-1}(2^g+1)}(\gamma) \prod_{m \text{ even}} \chi_m(\gamma)\right) \det(c_\gamma\tau + d_\gamma)^{2^{g-2}(2^g+1)} \prod_{m \text{ even}} \theta_m(\tau)\end{aligned}$$

and the character appearing is trivial whenever $g \geq 3$. Therefore, one has the following:

Proposition 3.15. *When $g \geq 3$, $\chi^{(g)}$ is a modular form of weight $2^{g-2}(2^g+1)$ with respect to Γ_g.*

The case $g = 2$ is a special one; the product

$$\chi_5 := \chi^{(2)} := \prod_{m \in C_1} \theta_m \tag{3.40}$$

satisfying the transformation law:

$$\begin{aligned}&\chi_5(\gamma\tau) = \kappa^2(\gamma) \textstyle\prod_{m \in C_1} \chi_m(\gamma) \det(c_\gamma\tau + d_\gamma)^5 \chi_5(\tau) \\ &\forall \gamma \in \Gamma_g\end{aligned} \tag{3.41}$$

is not a modular form with respect to Γ_2, since the character appearing is not trivial. However, the function:

$$\chi_{10} := \chi_5^2 = \left(\prod_{m \in C_1} \theta_m\right)^2 \tag{3.42}$$

is a modular form of weight 10 with respect to Γ_2.

Other remarkable modular forms with respect to suitable congruence subgroups can be built by the following notable functions:

Definition 3.7. *For each $m' \in \mathbb{Z}_2^g$ the holomorphic function $\Theta_{m'} : \mathfrak{S}_g \times \mathbb{C}^g \to \mathbb{C}$, defined by:*

$$\Theta_{m'}(\tau, z) := \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(2\tau, 2z) \tag{3.43}$$

is called a **second order Theta function**.

In particular,, one has the following:

Definition 3.8. *For each $m' \in \mathbb{Z}_2^g$ the holomorphic function:*

$$\Theta_{m'}(\tau) := \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(2\tau, 0) = \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(2\tau) \tag{3.44}$$

is named **second order Theta constant**.

A transformation law for second order Theta constants can be easily derived by focusing on the congruence subgroup $\Gamma_{g,0}(2)$ introduced in (1.5).
For each $\gamma \in \Gamma_g$ one can define:

$$\tilde{\gamma} := \begin{pmatrix} a_\gamma & 2b_\gamma \\ \frac{1}{2}c_\gamma & d_\gamma \end{pmatrix} \in Sp(g, \mathbb{Q})$$

Then, by definition in (1.10), one has $2\gamma\tau = \tilde{\gamma}2\tau$ for each $\tau \in \mathfrak{S}_g$. Since $\tilde{\gamma} \in \Gamma_g$ whenever $\gamma \in \Gamma_{g,0}(2)$, one can apply (3.19) to obtain [7]:

$$\begin{aligned} &\Theta_{m'}(\gamma\tau) = \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(\tilde{\gamma}2\tau) = \kappa(\tilde{\gamma})\chi_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(\tilde{\gamma}) \det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\Theta_{m'}(\tau) \\ &\forall\gamma \in \Gamma_{g,0}(2), \qquad \forall m' \in \mathbb{Z}_2^g, \qquad \forall\tau \in \mathfrak{S}_g \end{aligned} \tag{3.45}$$

With reference to the notation introduced in (1.3) one has:

$$\tilde{\gamma} = \begin{pmatrix} 1_g + 2a_M & 4b_M \\ c_M & 1_g + 2d_M \end{pmatrix} \qquad \forall\gamma \in \Gamma_g(2)$$

Then, by (3.18) and (3.20) one has:

$$\chi_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(\tilde{\gamma}) = e^{-\pi i[{}^t m'{}^t b_M(1+2d_M)m']} = e^{-\pi i[{}^t m'{}^t b_M m']} \qquad \forall\gamma \in \Gamma_g(2) \tag{3.46}$$

Hence, in particular, $\chi_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(\tilde{\gamma}) = 1$ whenever $\gamma \in \Gamma_g(2,4)$; therefore, (3.45) implies [8]:

$$\begin{aligned} &\Theta_{m'}\Theta_{n'}(\gamma\tau) = \kappa^2(\tilde{\gamma}) \det(c_\tau + d_\gamma)\Theta_{m'}\Theta_{n'}(\tau) \\ &\forall\gamma \in \Gamma_g(2,4), \qquad \forall m' \in \mathbb{Z}_2^g, \qquad \forall\tau \in \mathfrak{S}_g \end{aligned} \tag{3.47}$$

[7] A transformation law for second order Theta constants under the action of the whole Siegel modular group Γ_g can be also derived (see, for instance, [F]); however, such an action turns out not to be monomial, since it transforms a single second order Theta constant into a linear combination of second order Theta constants.

[8] The transformation law (3.47) means the product of two second order Theta constants is a modular form with respect to $\Gamma_g(2,4)$ with a multiplier.

Proposition 3.16. *Let be $g \geq 2$. Then, the product of all the second order Theta constants:*

$$F^{(g)} := \prod_{m' \in \mathbb{Z}^g} \Theta_{m'} \tag{3.48}$$

is a modular form of weight 2^{g-1} with respect to $\Gamma_g(2)$.

Proof. With reference to the notation introduced in (1.3) one has:

$$\tilde{\gamma} = \begin{pmatrix} 1_g + 2a_M & 4b_M \\ c_M & 1_g + 2d_M \end{pmatrix} \qquad \forall\, \gamma \in \Gamma_g(2)$$

Then, the thesis follows from (3.47) by applying (3.23), since there are 2^g second order Theta constants appearing in the product (3.48). □

The section ends recalling that second order Theta constants are related to Theta constants by a relation, which can be derived from the so-called Riemann's addition formula (see, for instance, [FK] or [I5]):

Proposition 3.17. *For each $h, k, m \in \mathbb{Z}_2^g$, $z, w \in \mathbb{C}^g$ and $\tau \in \mathfrak{S}_g$:*

$$\theta^2_{\left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right]}(\tau, 0) = \sum_{h \in \mathbb{Z}_2^g} (-1)^{{}^t(m'+h)m''} \theta_{\left[\begin{smallmatrix} m'+h \\ 0 \end{smallmatrix}\right]}(2\tau, 0)\theta_{\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right]}(2\tau, 0) \tag{3.49}$$

3.4 Jacobian determinants

By admitting half-integer weights in the graded ring $A(\Gamma_g(4,8))$ [9], the transformation law (3.34) for gradients of odd Theta functions can be regarded in terms of Jacobian determinants:

$$D(n_1, \ldots, n_g)(\tau) := \frac{1}{\pi^g} \begin{vmatrix} \frac{\partial}{\partial z_1}\theta_{n_1}|_{z=0}(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_1}|_{z=0}(\tau) \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_1}\theta_{n_g}|_{z=0}(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_g}|_{z=0}(\tau) \end{vmatrix}$$

Then, (3.34) implies that the functions $D(n_1, \ldots, n_g)$ are modular forms of weight $\frac{g}{2} + 1$ with respect to the congruence subgroup $\Gamma_g(4,8)$. More in general (3.17) implies the following:

Proposition 3.18. *Let $N = \{n_1, \ldots n_g\}$ be a sequence of odd g-characteristics and $D(N)$ the correspondent Jacobian determinant; then, one has:*

$$\begin{aligned} &D(\gamma \cdot N)(\gamma\tau) = \kappa(\gamma)^g e^{2\pi i \sum_{i=1}^g \phi_{n_i}(\gamma)} \det(c_\gamma \tau + d_\gamma)^{\frac{g}{2}+1} D(N)(\tau) \\ &\forall \gamma \in \Gamma_g, \quad \forall \tau \in \mathfrak{S}_g, \end{aligned} \tag{3.50}$$

[9] A ring can be generally graded by a monoid; for a detailed discussion on graded rings [E] can be consulted.

where the functions ϕ_{n_i}, are the ones defined in (3.18) corresponding to each characteristic n_i.

The transformation law (3.50) provides a criterion of modularity with respect to the congruence subgroup $\Gamma_g(2,4)$ for products of Jacobian determinants, which is similar to the one proved in (3.38) for Theta constants:

Proposition 3.19. *Let $N = (N_1, \dots, N_h)$ be a sequence of g-plets $N_i = \{n_1^i, \dots n_g^i\}$ of odd g-characteristics. The product $D(N_1)\cdots D(N_h)$ is, then, a modular form with respect to $\Gamma_g(2,4)$ if and only if the following condition is satisfied:*

$$N\,{}^tN \equiv h\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ mod\,2 \tag{3.51}$$

To point out more properties concerning the connection between the Theta constants and the Jacobian determinants, it is useful to introduce a function, which is defined on the parts $P(C_e^{(g)})$ of the set $C_e^{(g)}$ of even g-characteristics, by mapping sequences characteristics on the monomial given by the product of the correspondent Theta constants:

$$\begin{aligned} F : P(C_e^g) &\longrightarrow \mathbb{C}[\theta_m] \\ \{m_1, m_2, \dots, m_h\} &\longrightarrow \theta_{m_1}\theta_{m_2}\cdots\theta_{m_h} \end{aligned} \tag{3.52}$$

with $F(\emptyset) := 1$.

Then, the following important Theorem holds (cf. [F] or [I7]):

Theorem 3.1. (Generalized Jacobi's formula) *For $g \leq 5$, one has:*

$$D(n_1, \dots, n_g) = \sum_M \pm F(M)$$

where the summation is extended to all the sequences $M = \{m_1, \dots m_{g+2}\}$ of even characteristics such that $\{n_1, \dots, n_g, m_1, \dots m_{g+2}\}$ is azygetic.

Then, whenever $g \leq 5$, the Jacobian determinants are contained in the ring $\mathbb{C}[\theta_m]$. A similar statement has been conjectured for $g > 5$, but not proved yet.

Example 3.1. *(Case g=1)*
In this case, one obtains the classical Jacobi's derivative formula:

$$\theta'_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} = -\theta_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}\theta_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\theta_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}$$

Example 3.2. *(Case g=2) The 6 odd characteristics produce 15 distinct Jacobian determinants; by Jacobi's derivative formula, these determinants are monomials of degree 4 in the Theta constants. More precisely, for each $M = (m_1, m_2, m_3, m_4) \in C_4^-$ there exists a Jacobian determinant $D(n_i, n_j)$ such that $D(n_i, n_j) = \theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}$, and* *distinct Jacobian determinants correspond to distinct element of C_4^-. Therefore, the map:*

$$\begin{aligned} D : C_4^- &\to \mathbb{C}[\theta_m] \\ M = (m_1, m_2, m_3, m_4) &\longmapsto D(M) := \theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4} \end{aligned} \tag{3.53}$$

is a bijection which provides a parametrization for the Jacobian determinants. By suitably enumerating the 6 *odd characteristics:*

$$n^{(1)} := \begin{bmatrix} 01 \\ 01 \end{bmatrix} \quad n^{(2)} := \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad n^{(3)} := \begin{bmatrix} 01 \\ 11 \end{bmatrix}$$
$$n^{(4)} := \begin{bmatrix} 10 \\ 11 \end{bmatrix} \quad n^{(5)} := \begin{bmatrix} 11 \\ 01 \end{bmatrix} \quad n^{(6)} := \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

and the 10 *even characteristics:*

$$m^{(1)} := \begin{bmatrix} 00 \\ 00 \end{bmatrix} \quad m^{(2)} := \begin{bmatrix} 00 \\ 01 \end{bmatrix} \quad m^{(3)} := \begin{bmatrix} 00 \\ 10 \end{bmatrix} \quad m^{(4)} := \begin{bmatrix} 00 \\ 11 \end{bmatrix} \quad m^{(5)} := \begin{bmatrix} 01 \\ 00 \end{bmatrix}$$
$$m^{(6)} := \begin{bmatrix} 10 \\ 00 \end{bmatrix} \quad m^{(7)} := \begin{bmatrix} 11 \\ 00 \end{bmatrix} \quad m^{(8)} := \begin{bmatrix} 01 \\ 10 \end{bmatrix} \quad m^{(9)} := \begin{bmatrix} 10 \\ 01 \end{bmatrix} \quad m^{(10)} := \begin{bmatrix} 11 \\ 11 \end{bmatrix}$$

and by denoting the respective non-trivial Theta constants by $\theta_i := \theta_{m^{(i)}}$ for each $i = 1, \ldots 10$, one has:

$$D(n^{(1)}, n^{(2)}) = \theta_2\theta_3\theta_5\theta_6; \quad D(n^{(1)}, n^{(3)}) = -\theta_6\theta_7\theta_9\theta_{10}; \quad D(n^{(1)}, n^{(4)}) = \theta_1\theta_4\theta_5\theta_9$$

$$D(n^{(1)}, n^{(5)}) = -\theta_3\theta_4\theta_8\theta_{10}; \quad D(n^{(1)}, n^{(6)}) = \theta_1\theta_2\theta_7\theta_8; \quad D(n^{(2)}, n^{(3)}) = -\theta_1\theta_4\theta_6\theta_8$$

$$D(n^{(2)}, n^{(4)}) = -\theta_5\theta_7\theta_8\theta_{10}; \quad D(n^{(2)}, n^{(5)}) = -\theta_1\theta_3\theta_7\theta_9; \quad D(n^{(2)}, n^{(6)}) = \theta_2\theta_4\theta_9\theta_{10}$$

$$D(n^{(3)}, n^{(4)}) = \theta_2\theta_3\theta_8\theta_9; \quad D(n^{(3)}, n^{(5)}) = -\theta_1\theta_2\theta_5\theta_{10}; \quad D(n^{(3)}, n^{(6)}) = \theta_3\theta_4\theta_5\theta_7$$

$$D(n^{(4)}, n^{(5)}) = -\theta_2\theta_4\theta_6\theta_7; \quad D(n^{(4)}, n^{(6)}) = \theta_1\theta_3\theta_6\theta_{10}; \quad D(n^{(5)}, n^{(6)}) = \theta_5\theta_6\theta_8\theta_9$$

3.5 Classical structure theorems

This section aims to outline introduce some classical theorems, describe the structure of the ring of modular forms with respect to the Siegel modular group.

A classical result of the elliptic theory states that the modular forms with respect to Γ_1, namely the modular functions, are generated by the Eisenstein series E_4 and E_6 (see Example 2.1):

Theorem 3.2.

$$A(\Gamma_1) \equiv \mathbb{C}[E_4, E_6]$$

Proof. A proof can be found, for instance, in [FB] or [DS]. □

Pertaining to the ideal $S(\Gamma_1)$ of the cusp forms, one has, moreover:

Theorem 3.3.

$$S(\Gamma_1) \equiv \mathbb{C}(\Delta)$$

where Δ *is the cusp form introduced in Example 2.3.*

The structure of the ring $A(\Gamma_2)$ reveals itself to be more complicated. In fact, the respective generalized Eisenstein series E_4 and E_6 (see Example 2.2) are not enough, for instance, to generate the modular form χ_{10} introduced in (3.42); indeed, such form is generated by E_4, E_6 and E_{10} (cf. [I2] or [VdG]).
Moreover, by multiplying by a suitable normalization constant c'', is seen that the function:

In [I1] Juni-ichi Igusa found the generators of the even part $A(\Gamma_2)^{(e)}$ of the graded ring $A(\Gamma_2)$:

Theorem 3.4.

$$A(\Gamma_2)^{(e)} = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}]$$

where:

$$\chi_{12} = \frac{1}{2^{17}3} \sum_{\{m_{i_1},\ldots,m_{i_6}\}\in C_6^-} \pm(\theta_{m_{i_1}} \cdots \theta_{m_{i_6}})^4$$

is a modular form of weight 12.

In [I2], Igusa proved that only a modular form χ_{35} of odd weight has to be added, in order to generate the whole ring $A(\Gamma_2)$. With reference to the notation introduced in Section 3.2, a description of such a modular form can be given, as for χ_{12}, in terms of a symmetrization of products of Theta constants (cf. [I2] and [I6]):

$$\chi_{35} = -\frac{i}{2^{39}5^3}\left(\prod_{m\in C_1} \theta_m\right)\left(\sum_{\{m_i,m_j,m_k\}\in C_3^-} \pm(\theta_{m_i}\theta_{m_j}\theta_{m_k})^{20}\right) \tag{3.54}$$

Then, Igusa's structure theorem can be stated here:

Theorem 3.5. (Igusa's structure theorem)

$$A(\Gamma_2) = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}, \chi_{35}]$$

In [I2] Igusa also proved a structure theorem for the ideal $S(\Gamma_2)$ of the cusp forms:

Theorem 3.6.

$$S(\Gamma_2) = \mathbb{C}[\chi_{10}, \chi_{12}, \chi_{35}]$$

The modular form χ_{35}, in particular, admits a factor:

$$\chi_{30} = \frac{1}{8} \sum_{\{m_i,m_j,m_k\} \in C_3^-} \pm(\theta_{m_i}\theta_{m_j}\theta_{m_k})^5 \tag{3.55}$$

which is is a modular form of weight 30 with a non trivial character (cf. [I2]). Such a modular form is also involved as a generator in another structure theorem. By (3.7), the group $\Gamma_2/\Gamma_2(2)$ has only a non trivial irreducible representation of degree 1, corresponding to the sign of the permutation; then, one can denote the respective character by χ_P, so that:

$$\chi_P([\gamma]) := \begin{cases} 1 & \text{if} \quad \psi_P([\gamma]) \quad \text{is an even permutation} \\ -1 & \text{if} \quad \psi_P([\gamma]) \quad \text{is an odd permutation} \end{cases}$$

where ψ_P is the isomorphism described in (3.6). Then, by setting $\Gamma_2^+ \subset \Gamma_2$ the correspondent subgroup of index two, defined by the condition $\chi_P(\gamma) = 1$, one has the following structure theorem, proved by Igusa in [I2]:

Theorem 3.7.

$$A(\Gamma_2^+) = \mathbb{C}[E_4, E_6, \chi_5, \chi_{12}, \chi_{30}]$$

3.6 A new result: another description for χ_{30}

This section is devoted to the presentation of the first new result proposed by this thesis. A new description will be provided for the modular form χ_{30}, introduced in (3.55); to pursue such an aim, the section will need to focus on a particular construction, described by Bert Van Geemen and Duco Van Straten in their paper [vGvS].

When $g = 2$ there are four second order Theta constants, which can be conventionally enumerated for the sake of simplicity:

$$\Theta_1 := \Theta_{[0\,0]} \quad \Theta_2 := \Theta_{[0\,1]} \quad \Theta_3 := \Theta_{[1\,0]} \quad \Theta_4 := \Theta_{[1\,1]} \tag{3.56}$$

Then, (3.49) provides homogeneous relations between the ten first order Theta constants and these four ones; more precisely, with reference to the notation introduced in Example 3.2 for even Theta constants, one has:

$$\theta_1^2 = \Theta_1^2 + \Theta_2^2 + \Theta_3^2 + \Theta_4^2; \qquad \theta_2^2 = \Theta_1^2 - \Theta_2^2 + \Theta_3^2 - \Theta_4^2;$$

$$\theta_3^2 = \Theta_1^2 + \Theta_2^2 - \Theta_3^2 - \Theta_4^2; \qquad \theta_4^2 = \Theta_1^2 - \Theta_2^2 - \Theta_3^2 + \Theta_4^2;$$

$$\theta_5^2 = 2\Theta_1\Theta_2 + 2\Theta_3\Theta_4; \qquad \theta_6^2 = 2\Theta_1\Theta_3 + 2\Theta_2\Theta_4;$$

$$\theta_7^2 = 2\Theta_1\Theta_4 + 2\Theta_2\Theta_3; \qquad \theta_8^2 = 2\Theta_1\Theta_2 - 2\Theta_3\Theta_4;$$

$$\theta_9^2 = 2\Theta_1\Theta_3 - 2\Theta_2\Theta_4; \qquad \theta_{10}^2 = 2\Theta_1\Theta_4 - 2\Theta_2\Theta_3;$$

A quadratic form Q_m in the variables X_1, X_2, X_3, X_4 is, therefore, associated to each even 2-characteristic $m \in C_1$:

$$m \mapsto Q_m \qquad \theta_m^2 = Q_m(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \tag{3.57}$$

Hence, a related quadric V_m in P^3 turns out to be associated to each $m \in C_1$:

$$V_m := V(Q_m) = \{[X_1, X_2, X_3, X_4] \in P^3 \mid Q_m(X_1, X_2, X_3, X_4) = 0\} \tag{3.58}$$

Proposition 3.20. *With reference to the notation introduced in Section 3.2, for each 4-plet $M \in C_4^+$, the intersection:*

$$\bigcap_{m \in M^c} V_m \subset P^3 \tag{3.59}$$

is a set of four points in P^3.

Proof. Since C_4^+ is an orbit under the action of Γ_g, one has only to prove the statement for a specific element $M \in C_4^+$. By focus, in particular, on the 4-plet:

$$M_1 := \{m^{(1)}, m^{(2)}, m^{(3)}, m^{(4)}\} = \left\{ \begin{bmatrix} 00 \\ 00 \end{bmatrix}, \begin{bmatrix} 00 \\ 01 \end{bmatrix}, \begin{bmatrix} 00 \\ 10 \end{bmatrix}, \begin{bmatrix} 00 \\ 11 \end{bmatrix} \right\} \in C_4^+ \tag{3.60}$$

one has:

$$\bigcap_{m \in M^c} V_m = \{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$$

which concludes the proof. □

By Proposition 3.20, for a fixed $M \in C_4^+$ the four points described in (3.59) uniquely determine a configuration of four hyperplanes in P^3, which are characterized by the condition that each of them must pass through all except one of this points; hence, a collection of four linear forms, describing these four hyperplanes, is determined by each $M \in C_4^+$:

$$\begin{cases} \psi_1^M := \psi_1^M(X_1, X_2, X_3, X_4) \\ \psi_2^M := \psi_2^M(X_1, X_2, X_3, X_4) \\ \psi_3^M := \psi_3^M(X_1, X_2, X_3, X_4) \\ \psi_1^M := \psi_1^M(X_1, X_2, X_3, X_4) \end{cases} \tag{3.61}$$

A tetrahedron T_M in the projective space P^3 turns out thus to be uniquely associated to each 4-plet $M = (m_1, m_2, m_3, m_4) \in C_4^+$:

$$M = (m_1, m_2, m_3, m_4) \mapsto T_M \tag{3.62}$$

in such a way that the vertices are the four points in $\bigcap_{m \in M^c} V_m$ and all the points on the tetrahedron T_M are described by the vanishing set:

$$\begin{aligned} &\{ [X_1, X_2, X_3, X_4] \in P^3 \mid \tilde{F}_M(X_1, X_2, X_3, X_4) = 0 \} \\ &\text{where} \quad \tilde{F}_M := \textstyle\prod_{i=1}^4 \psi_i^M \end{aligned} \tag{3.63}$$

By a straightforward calculation for each $M \in C_4^+$, one is easily supplied with the linear forms appearing in the related configuration, as described in (3.61):

$$T_1 : \begin{cases} \psi_1^1 = X_1 \\ \psi_2^1 = X_2 \\ \psi_3^1 = X_3 \\ \psi_4^1 = X_4 \end{cases}$$

$$T_2 : \begin{cases} \psi_1^1 = X_1 - X_3 \\ \psi_2^1 = X_1 + X_3 \\ \psi_3^1 = X_2 - X_4 \\ \psi_4^1 = X_2 + X_4 \end{cases} \qquad T_3 : \begin{cases} \psi_1^1 = X_1 - X_2 \\ \psi_2^1 = X_1 + X_2 \\ \psi_3^1 = X_3 - X_4 \\ \psi_4^1 = X_3 + X_4 \end{cases} \qquad T_4 : \begin{cases} \psi_1^4 = X_1 - X_4 \\ \psi_2^4 = X_1 + X_4 \\ \psi_3^4 = X_2 - X_3 \\ \psi_4^4 = X_2 + X_3 \end{cases}$$

$$T_5 : \begin{cases} \psi_1^5 = X_1 + iX_4 \\ \psi_2^5 = X_1 - iX_4 \\ \psi_3^5 = X_2 + iX_3 \\ \psi_4^5 = X_2 - iX_3 \end{cases} \qquad T_6 : \begin{cases} \psi_1^6 = X_1 + iX_2 \\ \psi_2^6 = X_1 - iX_2 \\ \psi_3^6 = X_3 + iX_4 \\ \psi_4^6 = X_3 - iX_4 \end{cases} \qquad T_7 : \begin{cases} \psi_1^7 = X_1 - iX_3 \\ \psi_2^7 = X_1 + iX_3 \\ \psi_3^7 = X_2 + iX_4 \\ \psi_4^7 = X_2 - iX_4 \end{cases}$$

$$T_8 : \begin{cases} \psi_1^8 = X_1 - X_2 + X_3 - X_4 \\ \psi_2^8 = X_1 + X_2 - X_3 - X_4 \\ \psi_3^8 = X_1 - X_2 - X_3 + X_4 \\ \psi_4^8 = X_1 + X_2 - X_3 + X_4 \end{cases} \qquad T_9 : \begin{cases} \psi_1^9 = X_1 - X_2 - X_3 - X_4 \\ \psi_2^9 = X_1 - X_2 + X_3 + X_4 \\ \psi_3^9 = X_1 + X_2 + X_3 - X_4 \\ \psi_4^9 = X_1 + X_2 - X_3 + X_4 \end{cases}$$

$$T_{10} : \begin{cases} \psi_1^{10} = X_1 + iX_2 - X_3 - iX_4 \\ \psi_2^{10} = X_1 - iX_2 - X_3 + iX_4 \\ \psi_3^{10} = X_1 + iX_2 + X_3 + iX_4 \\ \psi_4^{10} = X_1 - iX_2 + X_3 - iX_4 \end{cases} \qquad T_{11} : \begin{cases} \psi_1^{11} = X_1 + iX_2 - X_3 + iX_4 \\ \psi_2^{11} = X_1 - iX_2 + X_3 + iX_4 \\ \psi_3^{11} = X_1 + iX_2 + X_3 - iX_4 \\ \psi_4^{11} = X_1 - iX_2 - X_3 - iX_4 \end{cases}$$

$$T_{12} : \begin{cases} \psi_1^{12} = X_1 + iX_2 + iX_3 + X_4 \\ \psi_2^{12} = X_1 + iX_2 - iX_3 - X_4 \\ \psi_3^{12} = X_1 - iX_2 + iX_3 - X_4 \\ \psi_4^{12} = X_1 - iX_2 - iX_3 + X_4 \end{cases} \qquad T_{13} : \begin{cases} \psi_1^{13} = X_1 + iX_2 - iX_3 + X_4 \\ \psi_2^{13} = X_1 + iX_2 + iX_3 - X_4 \\ \psi_3^{13} = X_1 - iX_2 + iX_3 + X_4 \\ \psi_4^{13} = X_1 - iX_2 - iX_3 - X_4 \end{cases}$$

$$T_{14} : \begin{cases} \psi_1^{14} = X_1 - X_2 + iX_3 + iX_4 \\ \psi_2^{14} = X_1 - X_2 - iX_3 - iX_4 \\ \psi_3^{14} = X_1 + X_2 + iX_3 - iX_4 \\ \psi_4^{14} = X_1 + X_2 - iX_3 + iX_4 \end{cases} \qquad T_{15} : \begin{cases} \psi_1^{15} = X_1 - X_2 + iX_3 - iX_4 \\ \psi_2^{15} = X_1 - X_2 - iX_3 + iX_4 \\ \psi_3^{15} = X_1 + X_2 - iX_3 - iX_4 \\ \psi_4^{15} = X_1 + X_2 + iX_3 + iX_4 \end{cases}$$

These 15 configurations, describing the tetrahedrons T_M, can be used to build a modular forms with respect to the Siegel modular group Γ_2. With reference to the notation introduced in (3.56), a holomorphic function can be associated to each $\tilde{F}_M$ described in (3.63), by:

$$F_M(\tau) := \tilde{F}_M(\,\Theta_1(\tau), \Theta_2(\tau), \Theta_3(\tau), \Theta_4(\tau)\,)$$

In particular, since T_1 is the tetrahedron associated to the 4-plet M_1 in (3.60), one has:

$$F_1(\tau) := F_{M_1}(\tau) = \Theta_1(\tau)\Theta_2(\tau)\Theta_3(\tau)\Theta_4(\tau) = F^{(2)}(\tau) \tag{3.64}$$

which is the modular form of weight 2 with respect to $\Gamma_2(2)$ introduced in (3.48). The product of all the 15 modular forms associated to the configurations is thus a good candidate to study.

To be more precise, one has to remark that $\Gamma_{2,0}(2)$, as defined in (1.5), is the stabilizer of M_1. In fact, by (3.2) one clearly has $\Gamma_{2,0}(2) \subset St_{M_1}$; on the other hand, $\gamma \in St_{M_1}$ implies:

$$diag({}^t c_\gamma d_\gamma) - c_\gamma m'' \equiv 0 \bmod 2 \qquad \forall m'' \in \mathbb{Z}_2^2$$

which implies $c_\gamma \equiv 0 \bmod 2$, hence $\gamma \in \Gamma_{2,0}(2)$.

Since C_4^+ is an orbit, one has, therefore:

$$[\Gamma_2 : \Gamma_{2,0}(2)] = |C_4^+| = 15$$

Consequently, there exists a bijection between $\Gamma_2/\Gamma_{2,0}(2)$ and the tetrahedrons, given by the map:

$$[\gamma] \mapsto T_{\gamma M_1}$$

and one can therefore study the product of all the conjugates of F_1 under the action described in (2.1):

$$\Psi(\tau) := \prod_{[\gamma]\in\Gamma_2/\Gamma_{2,0}(2)} (\gamma^{-1}|_2 F_1)(\tau)$$

The main theorem of the section can be stated now:

Theorem 3.8. *There exists $c \in \mathbb{C}$ such that:*

$$\Psi = c\chi_{30}$$

where χ_{30} is the modular form introduced in (3.55).

Proof. By definition of F_1 in (3.64) Ψ is a modular form of weight 30 with respect to $\Gamma_2(2)$. Moreover, for

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_{2,0}(2)$$

by (3.45), (3.20) and Lemma 3.11, one has:

$$F_1(\gamma_0\tau) = -F_1(\tau)$$

Then F_1 admits a non trivial character under the action of $\Gamma_{2,0}(2)$. Such a character extends to the unique non trivial character χ of Γ_2 (cf. [Ib]), hence $\chi(\gamma\gamma_0\gamma^{-1}) = \chi(\gamma_0) = -1$ whenever $\gamma \in \Gamma_2$. Moreover, with reference to the notation introduced in (2.2), $\gamma^{-1}|_2F_1 \in A(\gamma\Gamma_{2,0}(2)\gamma^{-1}, \chi)$ for each $\gamma \in \Gamma_2$, hence:

$$\gamma^{-1}\gamma_0^{-1}\gamma(\gamma^{-1}|_2F_1)(\tau) = \chi(\gamma\gamma_0\gamma^{-1})(\gamma^{-1}|_2F_1)(\tau) = \chi(\gamma_0)(\gamma^{-1}|_2F_1)(\tau) = -(\gamma^{-1}|_2F_1)(\tau)$$

Therefore:

$$(\gamma_0^{-1}|_{30}\Psi)(\tau) = \prod_{[\gamma]\in\Gamma_2/\Gamma_{2,0}(2)} (\gamma_0^{-1}\gamma^{-1}|_2F_1)(\tau) = \prod_{[\gamma]\in\Gamma_2/\Gamma_{2,0}(2)} (\gamma^{-1}\gamma\gamma_0^{-1}\gamma^{-1}|_2F_1)(\tau) =$$

$$= \prod_{[\gamma]\in\Gamma_2/\Gamma_{2,0}(2)} \chi(\gamma_0)(\gamma^{-1}|_2F_1)(\tau) = (-1)^{15} \prod_{[\gamma]\in\Gamma_2/\Gamma_{2,0}(2)} (\gamma^{-1}|_2F_1)(\tau) =$$

$$= (-1)^{15}\Psi(\tau)$$

and consequently $\Psi \in A(\Gamma_2, \chi^{15} = \chi)$. Hence Ψ must be a multiple of χ_{30}, which is the only modular form of weight 30 with a non-trivial character. □

Such an expression for χ_{30} has been also found by Aloys Krieg and Dominic Gehre in a completely different way, using quaternionic Theta constants (cf. [Kr]).

3.7 Rings of modular forms with levels

This section is devoted to recall important structure theorems for the rings of modular forms with respect to some of the congruence subgroups described in the Section 1.2.

Concerning modular forms with respect to the congruence subgroup $\Gamma_g(4,8)$, Theta constants with characteristics have already been described as a chief example; indeed their fundamental importance is soon revealed.

It is a remarkable fact that, for $g = 1, 2$, the only independent relations between the generators $\theta_m \theta_n$ are the so called Riemann's relations, which can be derived from (3.49) in a simple way.
For $g = 1$, (3.49) implies for the three non trivial Theta constants:

$$\theta^2_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(\tau,0) = \theta^2_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(2\tau,0) + \theta^2_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(2\tau,0)$$

$$\theta^2_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(\tau,0) = 2\theta_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(2\tau,0)\,\theta_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(2\tau,0)$$

$$\theta^2_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(\tau,0) = \theta^2_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(2\tau,0) - \theta^2_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(2\tau,0)$$

Therefore, one obtains the only non trivial Riemann's relation for $g = 1$:

$$\theta^4_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} - \theta^4_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} - \theta^4_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} = 0$$

Two kinds of Riemann's relations arise when $g = 2$, which can be likewise derived from (3.49); with reference to the notation introduced in Example 3.2 for even characteristics, one has 15 biquadratic Riemann's relations:

$$\theta_2^2\theta_3^2 = \theta_1^2\theta_4^2 - \theta_7^2\theta_{10}^2; \quad \theta_2^2\theta_5^2 = \theta_7^2\theta_9^2 + \theta_4^2\theta_8^2; \quad \theta_3^2\theta_5^2 = \theta_9^2\theta_{10}^2 + \theta_1^2\theta_8^2;$$

$$\theta_2^2\theta_6^2 = \theta_1^2\theta_9^2 + \theta_8^2\theta_{10}^2; \quad \theta_3^2\theta_6^2 = \theta_4^2\theta_9^2 + \theta_7^2\theta_8^2; \quad \theta_6^2\theta_5^2 = \theta_1^2\theta_7^2 - \theta_4^2\theta_{10}^2;$$

$$\theta_6^2\theta_7^2 = \theta_3^2\theta_8^2 - \theta_1^2\theta_5^2; \quad \theta_6^2\theta_{10}^2 = \theta_4^2\theta_5^2 - \theta_2^2\theta_8^2; \quad \theta_6^2\theta_9^2 = \theta_1^2\theta_2^2 - \theta_3^2\theta_4^2;$$

$$\theta_5^2\theta_9^2 = \theta_2^2\theta_7^2 - \theta_3^2\theta_{10}^2; \quad \theta_4^2\theta_6^2 = \theta_5^2\theta_{10}^2 + \theta_3^2\theta_9^2; \quad \theta_1^2\theta_6^2 = \theta_5^2\theta_7^2 - \theta_2^2\theta_9^2;$$

$$\theta_6^2\theta_8^2 = \theta_3^2\theta_7^2 - \theta_2^2\theta_{10}^2; \quad \theta_5^2\theta_8^2 = \theta_1^2\theta_3^2 - \theta_2^2\theta_4^2; \quad \theta_8^2\theta_9^2 = \theta_4^2\theta_7^2 - \theta_1^2\theta_{10}^2;$$

and 15 quartic Riemann's relations, amongst which there are only 5 independent relations:

$$\theta_1^4 - \theta_4^4 - \theta_5^4 - \theta_9^4 = 0; \quad \theta_2^4 - \theta_3^4 + \theta_5^4 - \theta_6^4 = 0; \quad \theta_2^4 - \theta_3^4 + \theta_8^4 - \theta_9^4 = 0;$$

$$\theta_1^4 - \theta_3^4 - \theta_6^4 - \theta_{10}^4 = 0; \quad \theta_1^4 - \theta_2^4 - \theta_7^4 - \theta_8^4 = 0;$$

The 15 biquadratic Riemann's relations correspond to the elements of C_6^+ by the bijective map:

$$M = (m_1, \dots m_6) \quad \longmapsto \quad R_2(M) \quad : \theta^2_{m_1}\theta^2_{m_2} \pm \theta^2_{m_3}\theta^2_{m_4} \pm \theta^2_{m_5}\theta^2_{m_6} \tag{3.65}$$

while the 15 quartic Riemann's relations correspond to the elements of C_4^- by the bijective map:

$$M = (m_1, \dots m_4) \quad \longmapsto \quad R_4(M) \quad : \theta^4_{m_1} \pm \theta^4_{m_2} \pm \theta^4_{m_3} \pm \theta^4_{m_4} \tag{3.66}$$

Denoting by I_R the ideal generated by Riemann's Relations induced on the variables $X_{mn} = \theta_m\theta_n$, one has, as already stated:

$$\mathbb{C}[\theta_m\theta_n] = \frac{\mathbb{C}[X_{mn}]}{I_R} \qquad (g = 1, 2) \tag{3.67}$$

For $g \geq 3$ Riemann's relations still provide independent relations between the generators $\theta_m\theta_n$; it is still not known, however, whether they are the only independent relations between generators for the ring $\mathbb{C}[\theta_m\theta_n]$ or not.
Concerning the ring in (3.67), one has:

Theorem 3.9. *The ring $\mathbb{C}[\theta_m\theta_n]$ is normal when $g = 1, 2$.*

Proof. The proof can be found in [I3] for the case $g = 1$ and in [I2] for the case $g = 2$. □

The ring $\mathbb{C}[\theta_m\theta_n]$ reveals itself to be strictly connected with the modular forms with respect to the congruence subgroup $\Gamma_g(4, 8)$ as a classical result proved by Igusa in [I3] states:

Theorem 3.10. (Igusa's structure theorem) *$A(\Gamma_g(4, 8))$ is the normalization of the ring $\mathbb{C}[\theta_m\theta_n]$.*

A straight consequence of Theorem 3.9 is the following:

Corollary 3.10. *$A(\Gamma_g(4, 8)) = \mathbb{C}[\theta_m\theta_n]$ when $g = 1, 2$.*

The section ends by recalling a structure theorem for cusp forms with respect to the subgroup $\Gamma_2(2, 4, 8)$ defined in (1.8), which will be used in the next chapter to prove new results; this structure theorem was proved by van Geemen and van Straten in [vGvS], also using the construction already described in Section 3.6.

Theorem 3.11. (van Geemen, van Straten) *A set of generators for the ideal $S(\Gamma_2(2, 4, 8))$ of the cusp forms with respect to $\Gamma_2(2, 4, 8)$ is given by:*

1. $D(M) \qquad \forall\, M \in C_4^-$

2. $\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}\theta_{m_5} \qquad \forall\, \{m_1 \ldots, m_5\} \notin C_5^+ \cup C_5^-$

3. $\theta_{m_1}\theta_{m_2}\theta_{m_3}\psi_i^M, \psi_j^M, \psi_k^M \qquad \forall\, \{m_1, m_2, m_3\} \in C_3^+ \qquad \forall\, i, j, k$

where, for each $\{m_1, m_2, m_3\} \in C_3^+$, $M \in C_4^+$ the only element such that $\{m_1, m_2, m_3\} \subset M$ and $\psi_i^M, \psi_j^M, \psi_k^M$ are three of the linear forms associated to T_M as in (3.61).

Chapter 4

The group Γ and the structure of $A(\Gamma)$ and $S(\Gamma)$

The aim of this chapter is to present the new results found, concerning a remarkable map made by the gradients of odd Theta functions in genus 2. The first section will be devoted to outline the general description of this map, thus pointing out the exceptional features pertaining to the case $g = 2$, on which the other sections will be focused.
Since every statement concerning the new results will pertain to the case $g = 2$, the obvious indication of the grade $g = 2$ will be conventionally omitted.
For the sake of simplicity, the group $\Gamma(2,4)/\{\pm\Gamma(4,8)\}$, which will turn out to be involved in the discussion, will be denoted by the symbol G, while the symbol $\hat{G}$ will stand for the group of characters of G.

4.1 The Theta Gradients Map

By the transformation formula (3.34) one is allowed to define a map on the quotient space $A_g^{4,8} := \mathfrak{S}_g/\Gamma_g(4,8)$:

$$\mathbb{P}grTh : A_g^{4,8} \longrightarrow \overbrace{\mathbb{C}^g \times \cdots \times \mathbb{C}^g}^{2^{g-1}(2^g-1)\text{times}} / T_0(Gl(g,\mathbb{C}))$$

$$\tau \longrightarrow \left\{ \text{grad}_z \theta_n |_{z=0} \right\}_{n \text{ odd}}$$

where T_0 is the representation defined in (3.35).

As a consequence of Lefschetz's theorem (cf. [GH]) the range of this map lies in the Grassmannian $Gr_{\mathbb{C}}(g, 2^{g-1}(2^g - 1))$, as proved by Riccardo Salvati Manni in [SM1]. The Jacobian determinants are the Plücker coordinates of this map.
In their work [GSM], Samuel Grushevsky and Riccardo Salvati Manni proved that this map is generically injective on $A_g^{4,8}$ when $g \geq 3$ and injective on the tangent spaces when $g \geq 2$.

The map was also conjectured to be injective whenever $g \geq 3$, albeit it has not been proved yet.

The case $g = 2$ reveals itself, indeed, to be a case of special interest. The six odd 2-characteristics originate fifteen Jacobian determinants, namely the ones enumerated in Example 3.2, which satisfy by (3.36) the following transformation law:

$$\begin{gathered} D(N)(\gamma\tau) = \kappa(\gamma)^2 \chi_N(\sigma) det(c_\gamma \tau + d_\gamma)^2 D(N)(\tau) \\ \forall\, \tau \in \mathfrak{S}_2 \qquad \forall\, \gamma \in \Gamma_2(2,4) \qquad \forall\, N = \{n_1, n_2\} \in \tilde{C}_2 \end{gathered} \tag{4.1}$$

where $\chi_N = \chi_{n_1}\chi_{n_2}$.
The respective Theta gradients map in these fifteen Plücker coordinates is:

$$\begin{aligned} \mathbb{P}grTh : A^{4,8} &\longrightarrow \mathbb{P}^{14} \\ \tau &\longrightarrow \{\, [\, D(N)(\tau)\,]\,\}_{N \in \tilde{C}_2} \end{aligned}$$

which can not be injective by (1.7), but is seen to be finite.

However, there exists a suitable intermediate congruence subgroup $\Gamma(4,8) \subset \Gamma \subset \Gamma(2,4)$ such that the Theta gradients map factors on the correspondent level moduli space $A_\Gamma := \mathfrak{S}_g/\Gamma$ as:

$$\mathbb{P}grTh^* : A_2^\Gamma \to \mathbb{P}^{14} \qquad A_2^\Gamma := \mathfrak{S}_2/\Gamma$$

and the new map $\mathbb{P}grTh^*$ turns out to be injective.

In the following section such a remarkable group will be described.

4.2 The congruence subgroup Γ

As already explained, the chief interest of this section is to locate between $\Gamma(4,8)$ and $\Gamma(2,4)$ the right congruence subgroup Γ, whose correspondent level moduli space $A_\Gamma := \mathfrak{S}_g/\Gamma$ is such that the Theta gradients map $\mathbb{P}grTh$ is still well defined on it and injective. The following description is provided for such a group:

Proposition 4.1.

$$\Gamma = \bigcap_{i=1}^{15} Ker\chi_{N_i} = \{\gamma \in \Gamma(2,4) \mid \kappa(\gamma)^2 \chi_{N_i}(\gamma) = 1 \quad \forall i = 1, \ldots, 15\} \tag{4.2}$$

where χ_{N_i} are the fifteen characters involved in the transformation law (4.1) for the fifteen Jacobian determinants.

Proof. As far as one knows by (4.1), an *a priori* description for the subgroup Γ is:

$$\Gamma := \Gamma^{(1)} \cup \Gamma^{(-1)} \tag{4.3}$$

where:

$$\Gamma^{(1)} := \{\gamma \in \Gamma(2,4) \mid \kappa(\gamma)^2 \chi_{N_i}(\gamma) = 1 \quad \forall i = 1, \ldots, 15\}$$

$$\Gamma^{(-1)} := \{\gamma \in \Gamma(2,4) \mid \kappa(\gamma)^2 \chi_{N_i}(\gamma) = -1 \quad \forall i = 1, \ldots, 15\}$$

Γ, as defined in (4.3) is plainly checked to be a subgroup of the Siegel modular group Γ_2, for Corollary 3.7 and Lemma 3.15 imply for each $i = 1, \ldots, 15$:

$$\left.\begin{aligned} &\kappa^2(\gamma\gamma')\chi_{N_i}(\gamma\gamma') = \kappa^2(\gamma)\chi_{N_i}(\gamma)\kappa^2(\gamma')\chi_{N_i}(\gamma') = 1 \\ &\kappa^2(\gamma^{-1})\chi_{N_i}(\gamma^{-1}) = [\, \kappa^2(\gamma)\chi_{N_i}(\gamma)\,]^{-1} = \kappa^2(\gamma)\chi_{N_i}(\gamma) \end{aligned}\right. \qquad \forall \gamma, \gamma' \in \Gamma$$

Moreover, by (3.22) and Lemma 3.16, one has $\Gamma(4,8) \subset \Gamma$; hence the subgroup Γ, as defined in (4.3), is indeed a congruence subgroup such that $\Gamma(4,8) \subset \Gamma \subset \Gamma(2,4)$.

The next step is to refine the definition of Γ, by detecting which elements of $\Gamma(2,4)$ belong to this subgroup.

Clearly $\bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i} \subset \Gamma$. Therefore, the reverse inclusion has only to be shown, to prove the first part of the statement.
Let thus $\gamma \in \Gamma$. Due to the definition in (4.3), either $\chi_{N_i}(\gamma) = 1$ for each $i = 1, \ldots, 15$ or $\chi_{N_i}(\gamma) = -1$ for each $i = 1, \ldots, 15$. However, if $\chi_{N_i}(\gamma) = -1$, for each i, an absurd statement turns up:

$$-1 = \chi_{(n,n_i)}(\gamma)\,\chi_{(n,n_j)}(\gamma)\,\chi_{(n,n_k)}(\gamma) = \chi_{(n_i,n_j)}(\gamma)\,\chi_{(n,n_k)}\,(\gamma) = 1$$

Hence, the only possible case is $\gamma \in \bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i}$.
Consequently, one has:

$$\Gamma = \bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i} \tag{4.4}$$

and this part of the statement is proved.
In order to obtain the second identity in the statement, one needs to prove that $k^2(\gamma) = 1$ whenever $\gamma \in \Gamma$. By the criterion described in Proposition 3.19, the products:

$$D := D(n_1, n_2)D(n_3, n_4)D(n_5, n_6)$$

with $n_1, \ldots n_6$ all distinct, are plainly checked to be modular forms with respect to $\Gamma(2,4)$. Moreover, by (4.1), the following transformation law is satisfied for each $\gamma \in \Gamma(2,4)$:

$$D(\gamma\tau) = k^2(\gamma)\chi_{n_1}\cdots\chi_{n_6} det(c_\gamma\tau + d_\gamma)^6 D(\tau)$$

Hence, by (3.14), one has for each $\gamma \in \Gamma(2,4)$:

$$k^2(\gamma) = \prod_{i=1}^{6} \chi_{n_i}(\gamma) = \chi_{(n_1,n_2)}(\gamma)\,\chi_{(n_3,n_4)}(\gamma)\,\chi_{(n_5,n_6)}(\gamma)$$

Therefore, whenever $\gamma \in \Gamma$, $k^2(\gamma) = 1$ by the characterization in (4.4). This shows $\Gamma^{(-1)}$ in (4.3) is indeed an empty set, and $\Gamma = \Gamma^{(1)}$. This concludes the proof.

□

Thanks to the description of the congruence subgroup Γ provided in Proposition 4.1, an important property can be immediately stated:

Proposition 4.2. *Γ is normal in* Γ_2*.*

Proof. One has to prove that:

$$\chi_{N_i}(\gamma^{-1}\eta\gamma) = 1 \qquad \forall\gamma \in \Gamma_2 \quad , \quad \forall\eta \in \Gamma \quad , \quad \forall i = 1,\ldots,15$$

By setting $N_i = (n_{1\,i}, n_{2\,i})$ for each $i = 1,\ldots,15$, one has:

$$\chi_{N_i}(\gamma^{-1}\eta\gamma) = \chi_{n_{1\,i}}(\gamma^{-1}\eta\gamma)\chi_{n_{2\,i}}(\gamma^{-1}\eta\gamma) = \gamma^{-1}(\chi_{n_{1\,i}}, \chi_{n_{2\,i}})(\eta) = \chi_{\gamma n_{1\,i}}(\eta)\chi_{\gamma n_{2\,i}}(\eta)$$

Since the action in (3.2) preserves the parity, for each $i = 1,\ldots,15$ there exists a j, depending on i and γ, such that $(\gamma n_{1\,i}, \gamma n_{2\,i}) = N_j$. Therefore:

$$\chi_{N_j}(\gamma^{-1}\eta\gamma) = \chi_{\gamma n_{1\,i}}(\eta)\chi_{\gamma n_{2\,i}}(\eta) = \chi_{N_j}(\eta) = 1$$

where the last equality on the right holds since $\eta \in \Gamma$. □

As a straight consequence, one has the following:

Corollary 4.1. $\mathfrak{S}_2$ *does not admit any fixed point for the action of* Γ*. In particular, the space* $\mathfrak{S}_2/\Gamma$ *is smooth.*

Proof. Since $\Gamma \subset \Gamma(2,4)$, the thesis follows from Corollary 1.5 and Proposition 4.4. □

A concrete description for the congruence subgroup Γ in terms of generators can be also a useful tool to work with. Corollary 3.9 suggests how to find such a description.
Since the functions χ_{N_i} are characters of the group $G = \Gamma(2,4)/\{\pm\Gamma(4,8)\}$, one can find the elements in $\Gamma(2,4)$, which belongs to $\bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i}$, by checking just the representative elements for the cosets in $\{\pm\Gamma(4,8)\}$ in $\Gamma(2,4)$.
For such a purpose, Proposition 1.3 and Corollary 1.2 can be applied, to obtain:

Proposition 4.3. *The group G is a 9-dimensional vector space on $\mathbb{Z}_2$. A basis is given by:*

$$A_{11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \quad A_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{11}^2 = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_{22}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_{12} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_{11}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{22}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix} \quad C_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Then, by setting $\chi_{i,j} := \chi_{(n^{(i)},n^{(j)})}$ for each i, j, with reference to the notation introduced in Example 3.2 for odd characteristics, a simple table can be redacted by a straightforward computation, using (3.23) and the values in Lemma 3.13:

	A_{11}	A_{12}	A_{21}	B_{12}	B_{11}^2	B_{22}^2	C_{12}	C_{11}^2	C_{22}^2
χ_{12}	-1	1	1	1	-1	-1	1	-1	-1
χ_{13}	1	1	-1	1	1	1	-1	-1	1
χ_{14}	-1	-1	1	1	-1	-1	-1	-1	1
χ_{15}	1	-1	1	-1	-1	1	1	1	1
χ_{16}	-1	1	-1	-1	-1	1	1	-1	-1
χ_{23}	-1	1	-1	1	-1	-1	-1	1	-1
χ_{24}	1	-1	1	1	1	1	-1	1	-1
χ_{25}	-1	-1	1	-1	1	-1	1	-1	-1
χ_{26}	1	1	-1	-1	1	-1	1	1	1
χ_{34}	-1	-1	-1	1	-1	-1	1	1	1
χ_{35}	1	-1	-1	-1	-1	1	-1	-1	1
χ_{36}	-1	1	1	-1	-1	1	-1	1	-1
χ_{45}	-1	1	1	-1	1	-1	-1	-1	1
χ_{46}	1	-1	-1	-1	1	-1	-1	1	-1
χ_{56}	-1	-1	-1	1	1	1	1	-1	-1

The independent elements satisfying the desired properties can be easily detected by this table, to state the following:

Proposition 4.4. *The group* Γ *is generated by* $\Gamma(4,8)$ *and the elements:*

$$A_{12}B_{11}^2C_{22}^2 = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}$$

$$^t(A_{12}B_{11}^2C_{22}^2) = A_{21}B_{22}^2C_{11}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 4 \\ 4 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{12}B_{11}^2B_{22}^2 = \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^t(B_{12}B_{11}^2B_{22}^2) = C_{12}C_{11}^2C_{22}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{pmatrix}$$

4.3 Structure of $A(\Gamma)$: generators

Since the Plücker coordinates $D(N)$ are known to be cusp forms, the Theta gradients map does not extend to the boundary of Satake's compactification [1] of the level moduli space $A^{4,8}$. The graded ring $A(\Gamma)$ of the modular forms with respect to the congruence subgroup Γ, as well as the ideal $S(\Gamma) \subset A(\Gamma)$ of the cusp forms, are needed to describe the Satake's compactification Proj$A(\Gamma)$ and the desingularization Proj$S(\Gamma)$ of the map on it. The respective even parts $A(\Gamma)^e$ and $S(\Gamma)^e$ are, indeed, the only relevant part in describing the Proj scheme [2]. This work is therefore interested in describing the structure of $A(\Gamma)^e$ and $S(\Gamma)^e$; as a first step, this section will aim, in particular, to find generators for $A(\Gamma)^e$ A structure theorem have to be proved first, in order to describe $A(\Gamma)^e$:

Proposition 4.5. *Let* $\chi_5 := \prod_{i=1}^{10} \theta_{m_i}$ *be the modular form introduced in (3.40). Then:*

$$A(\Gamma(4,8)) = \left(\bigoplus_{d\ even} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}} \right) \bigoplus \left(\bigoplus_{h=0,2,4} \mathbb{C}[\theta_m^2\theta_n^2] \frac{\chi_5}{\theta_{m_1}\cdots\theta_{m_{2h}}} \right)$$

where:

$$A(\Gamma(4,8))^e = \bigoplus_{d\ even} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}}$$

is the even part of the graded ring, and

[1]See Appendix B

[2]A detailed exposition of these topics can be found, for instance, in the classic book [Ht].

$$A(\Gamma(4,8))^o = \bigoplus_{h=0,2,4} \mathbb{C}[\theta_m^2\theta_n^2]\frac{\chi_5}{\theta_{m_1}\cdots\theta_{m_{2h}}}$$

is the odd part.

Proof. By Igusa's Theorem, $A(\Gamma(4,8) = \mathbb{C}[\theta_m\theta_n]$ (Corollary 3.10); the result of the decomposition of the ring under the action of $\Gamma(2,4)$, according to the general procedure described in (2.2), is then:

$$A(\Gamma(4,8)) = \bigoplus_{\chi\in\hat{G}} \mathbb{C}[\theta_m\theta_n, \chi]$$

Since monomials in Theta constants transforms into monomials under the action of $\Gamma(2,4)$ (by (3.36)), one needs to focus only on monomials in $\theta_m\theta_n$, in order to study the transformation law for elements of $\mathbb{C}[\theta_m\theta_n]$ under this action.

In particular, if $P_d = \theta_{m_1}\cdots\theta_{m_{2d}} \in \mathbb{C}[\theta_m\theta_n]_d$ is a monomial of degree d in the variables $\theta_m\theta_n$, (3.36) implies the following transformation law:

$$P_d(\gamma\tau) = \kappa^{2d}(\gamma)\chi_{m_1}\cdots\chi_{m_{2d}}\det(c_\gamma\tau + d_\gamma)^d P_d(\tau) \qquad \forall\gamma\in\Gamma(2,4)$$

If $d = 2l$, $P_d \in \mathbb{C}[\theta_m\theta_n, \chi_{m_1}\cdots\chi_{m_{2d}}]$, because $\kappa^4(\gamma) = 1$ for each $\gamma \in \Gamma(2,4)$ by (3.22). Moreover, Corollary 3.9 and Lemma 3.14 imply that for each couple of characteristics m, n, the product $\chi_m^2\chi_n^2$ is a trivial character of $\hat{G}$; the following decomposition arises, therefore, for the even part $A(\Gamma(4,8))^e$ of the ring:

$$A(\Gamma(4,8))^e = \bigoplus_{d\text{ even}} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}} \tag{4.5}$$

On the other hand, if $d = 2l+1$, $P_d \in \mathbb{C}[\theta_m\theta_n, \kappa^2\chi_{m_1}\cdots\chi_{m_{2d}}]$. Moreover, since $\chi_{m_1}\cdots\chi_{m_{10}} = 1$ on $\Gamma(2,4)$, (3.41) implies the following transformation law for χ_5:

$$\chi_5(\gamma\tau) = \kappa^2(\gamma)\det(c_\gamma\tau + d_\gamma)^5\chi_5(\tau) \qquad \forall\gamma\in\Gamma(2,4)$$

Therefore, for each sequence $M = \{m_1, \ldots m_{2h}\}$ of even characteristics, one has:

$$\frac{\chi_5}{\theta_{m_1}\cdots\theta_{m_{2h}}} \in \mathbb{C}[\theta_m\theta_n, \chi_{M^c}]$$

Since κ^2 is a character of $\hat{G}$, as stated in Corollary 3.7, it is indeed a product of the functions χ_m by Proposition 3.11; the following decomposition arises, then, for the odd part $A(\Gamma(4,8))^o$ of the ring:

$$A(\Gamma(4,8))^o = \bigoplus_{h=0,2,4} \mathbb{C}[\theta_m^2\theta_n^2]\frac{\chi_5}{\theta_{m_1}\cdots\theta_{m_{2h}}} \tag{4.6}$$

This concludes the proof.

□

Thanks to Proposition 4.5 a structure theorem can be finally stated for $A(\Gamma)^e$.

Theorem 4.1. $A(\Gamma)^e = \mathbb{C}[\theta_m^2\theta_n^2, D(N)]$.

Proof. By Proposition 4.1, $\Gamma/\{\pm\Gamma(4,8)\} \subset G$ is the dual subgroup corresponding to the subgroup $< \chi_{N_i} > \subset \hat{G}$ generated by the fifteen characters χ_{N_i} related to the Jacobian determinants $D(N_i)$. One has, therefore:

$$A(\Gamma) = \bigoplus_{\chi\in<\chi_{N_i}>} A(\Gamma(4,8),\chi)$$

Hence, the thesis follows from (4.5). □

4.4 Structure of $A(\Gamma)$: relations

The foregoing section has been devoted to the detection of the generators of $A(\Gamma)^e$, which have turned out to be $\theta_m^2\theta_n^2$ and $D(N)$. Some relations exist amongst these generators, most of which are induced by Riemann's relations. This section aims to provide them, by a combinatorial description. For this purpose a threefold investigation will be needed, in order to find the relations involving only the $\theta_m^2\theta_n^2$, the relations involving only the $D(N)$, and finally the ones between the $\theta_m^2\theta_n^2$ and the $D(N)$.

4.4.1 Relations among $\theta_m^2\theta_n^2$

The relations among $\theta_m^2\theta_n^2$ are completely described by Riemann's relations (see Section 3.7). Therefore, with reference to the notation introduced in (3.65) and (3.66), there are 15 independent biquadratic relations:

$$R_2(M) = 0 \qquad \forall M \in C_6^+ \tag{4.7a}$$

and 5 independent quartic relations:

$$R_4(M_i) = 0 \qquad i = 1,\dots 5 \tag{4.7b}$$

4.4.2 Relations among $D(N)$

The 15 relations of the kind:

$$D(M)^2 = \theta^2_{m_1}\theta^2_{m_2}\theta^2_{m_3}\theta^2_{m_4} \qquad \forall\, M = \{m_1, m_2, m_3, m_4\} \in C_4^-$$

induced by Jacobi's formula and described in Example 3.2, have to be used together with the ones in (4.7a) and (4.7b), to find all the desired relations amongst the fifteen Jacobian determinants $\{D(M)\}_{M\in C_4^-}$. These have been done by Professor Eberhard Freitag by means of a computer program written by him; here a combinatoric explanation follows for all the relations found.

1. For each even characteristic m, one can enumerate the six 4-plets $\{M_i^m\}_{i=1,\dots 6}$ in C_4^- containing m, in such a way that:

   $$M_1^m \cap M_2^m \cap M_3^m = \{m\} = M_4^m \cap M_5^m \cap M_6^m$$

   Then, one has:

   $$D(M_1^m)D(M_2^m)D(M_3^m) = \chi_5\theta^2_m = D(M_4^m)D(M_5^m)D(M_6^m) \tag{4.8a}$$

   which are obviously 10 relations, namely one for each choice of m.

2. For each $M = \{m_1, \dots m_6\} \in C_6^+$ there are eight 4-plets of C_4^-, containing exactly a triplet $\{m_i, m_j, m_k\} \subset M$; these 4-plets can be enumerated in such a way that:

   $$\begin{aligned} D(\tilde{M}_1)D(\tilde{M}_2)D(\tilde{M}_3)D(\tilde{M}_4) &= \chi_5 \textstyle\prod_{i=1}^6 \theta^2_{m_i} = \\ &= D(\tilde{M}_5)D(\tilde{M}_6)D(\tilde{M}_7)D(\tilde{M}_8) \end{aligned} \tag{4.8b}$$

   These are 15 relations, namely one for each choice of $M \in C_6^+$.

3. Let $M = \{m_1, \dots m_6\} \in C_6^+$ and let

   $$R_2(M) = \theta^2_{m_1}\theta^2_{m_2} \pm \theta^2_{m_3}\theta^2_{m_4} \pm \theta^2_{m_5}\theta^2_{m_6} = 0$$

   be the associated biquadratic Riemann's relation as in (3.65).
   For each couple of even characteristics $\{m_i, m_j\}$, one can denote by $M_1^{i,j}$ and $M_2^{i,j}$ the only two 4-plets of C_4^- containing $\{m_i, m_j\}$; then, one has:

   $$\begin{aligned} D(M_1^{1,2})D(M_2^{1,2}) \pm D(M_1^{3,4})D(M_2^{3,4}) \pm D(M_1^{5,6})D(M_2^{5,6}) &= \\ = \pm D(M')R_2(M) = 0 \end{aligned} \tag{4.8c}$$

These are 15 relations, since they correspond to the elements of C_6^+.

Otherwise, by choosing for each couple of even characteristics m_i, m_j only one of the two 4-plets $M_1^{i,j}$ e $M_2^{i,j}$, one has the following general identity:

$$D^2(M_\alpha^{1,2}) \pm D^2(M_\beta^{3,4}) \pm D^2(M_\epsilon^{5,6}) =$$

$$= \theta_{\alpha_1}^2 \theta_{\alpha_2}^2 \theta_{m_1}^2 \theta_{m_2}^2 \pm \theta_{\beta_1}^2 \theta_{\beta_2}^2 \theta_{m_3}^2 \theta_{m_4}^2 \pm \theta_{\epsilon_1}^2 \theta_{\epsilon_2}^2 \theta_{m_5}^2 \theta_{m_6}^2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \epsilon_1, \epsilon_2$ are characteristics in $M^c \in C_4^-$.

In particular, each triplet of determinants can be chosen in such a way that only three of the four characteristics appearing in the 4-plet M^c are involved in the respective identity; for combinatorial reasons, there is a unique way to build a relation among $D(M)$'s by multiplying each determinant of such a triplet by two other distinct determinants:

$$D_h\, D_k\, D^2(M_\alpha^{1,2}) \pm D_l\, D_r\, D^2(M_\beta^{3,4}) \pm D_s\, D_t\, D^2(M_\epsilon^{5,6}) = 0 \tag{4.8d}$$

One can observe that, for each choice of $M \in C_6^+$, there exist four distinct triplets of determinants $D(M_\alpha^{1,2}), D^2(M_\beta^{3,4}), D^2(M_\epsilon^{5,6})$ satisfying the desired condition; more precisely, each triplet corresponds to a choice for the characteristic in M^c which does not appear in the identity, and obviously there are four possible choices for such a characteristic. Therefore, the relations (4.8d) are in number of $15 \cdot 4 = 60$.

4. Let $M = \{m_1, \dots m_4\} \in C_4^-$ an let

$$R_4(M) = \theta_{m_1}^4 \pm \theta_{m_2}^4 \pm \theta_{m_3}^4 \pm \theta_{m_4}^4 = 0$$

be the associated quartic Riemann's relation as in (3.66). For each $m_i \in M$ there exist only 2 elements $M_1^i, M_2^i \in C_4^-$ containing m_i and such that:

$$M_1^i \vartriangle M_2^i = M^c = \{m_5, \dots m_{10}\}$$

One has, therefore:

$$\sum_{i=1}^{4} \pm D(M_1^i)^2\, D(M_2^i)^2 = \theta_{m_5} \cdots \theta_{m_{10}} R_4(M) = 0 \tag{4.8e}$$

It must be remarked that all the 15 quartic Riemann relations induce independent relations on the $D(M)$, even though they are not independent themselves; the relations in (4.8e) are, therefore, in number of 15.

5. Let $M = \{m_1, m_2, m_3\} \in C_3^-$ be fixed.
There exist only two distinct 6-plets in C_6^+, containing M. Then let $\tilde{M} = \{m_1, m_2, m_3, m'_1, m'_2, m'_3\}$ be such a 6-plet; the corresponding biquadratic Riemann's relation is easily seen to be such that:

$$R_2(\tilde{M}) = \theta^2_{m_1}\theta^2_{m'_1} \pm \theta^2_{m_2}\theta^2_{m'_2} \pm \theta^2_{m_3}\theta^2_{m'_3} = 0$$

with $\{m'_1, m'_2, m'_3\} \in C_3^-$.
Moreover, for each couple of characteristics $\{m_i, m_j\} \subset M$, there exists a unique $M_{i,j} \in C_6^+$, containing $\{m_i, m_j\}$ and satisfying:

$$R_2(M_{i,j}) = \pm\theta^2_{m_i}\theta^2_{m_j} + P_{ij} = 0$$

For combinatorial reasons, all the terms $\theta^2_{m'_i}P_{j,k}$ share the common addend $\theta^2_{m'_1}\theta^2_{m'_2}\theta^2_{m'_3}$; therefore, one has:

$$0 = \theta^2_{m_1}\theta^2_{m_2}\theta^2_{m_3}R_2(\tilde{M}) = \pm\theta^2_{m_1}\theta^2_{m'_1}P_{2,3} \pm \theta^2_{m_2}\theta^2_{m'_2}P_{1,3} \pm \theta^2_{m_3}\theta^2_{m'_3}P_{1,2} =$$

$$= \pm\theta^2_{m_4}\theta^2_{m'_1}\theta^2_{m'_2}\theta^2_{m'_3} \pm \theta^2_{m_1}\theta^2_{m'_1}\theta^2_{m_\alpha}\theta^2_{m_\beta} \pm \theta^2_{m_2}\theta^2_{m'_2}\theta^2_{m_\alpha}\theta^2_{m_\epsilon} \pm \theta^2_{m_3}\theta^2_{m'_3}\theta^2_{m_\beta}\theta^2_{m_\epsilon}$$

where m_4 is the unique even characteristic which completes $M = \{m_1, m_2, m_3\}$ to a 4-plet in C_4^- (to which a quartic Riemann's relations correspond, as in (3.66)), and $\{m_\alpha, m_\beta, m_\epsilon\} = \{m_1, m_2, m_3, m_4, m'_1, m'_2, m'_3\}^c$.
Then it is easily checked that:

$$0 = \left(\prod_{m\notin\{m_1,m_2,m_3,m_4\}} \theta_m\right)\theta^2_{m_1}\theta^2_{m_2}\theta^2_{m_3}R_2(\tilde{M}) = \sum_{i=1}^{4} \pm D(M_1^i)\,D(M_2^i)^3 \quad (4.8f)$$

where M_1^i and M_2^i are, for each $i = 1, \dots 4$, the 4-plets in C_4^- containing m_i and such that $M_1^i \triangle M_2^i = \{m_1, m_2, m_3, m_4\}^c$, already appeared in the relations (4.8e).
By selecting the other 6-plet $\tilde{M} = \{m_1, m_2, m_3, m_\alpha, m_\beta, m_\epsilon\}$ containing $M = \{m_1, m_2, m_3\}$, one obtains the same relations with interchanged exponents:

$$0 = \sum_{i=1}^{4} \pm D(M_1^i)^3\,D(M_2^i) \quad (4.8g)$$

One has to observe that triplets $M = \{m_1, m_2, m_3\}$ belonging to the same 4-plet in C_4^- produce exactly the same relation (essentially because the same related quartic Riemann relation comes substituted in the foregoing null expression). Therefore, these relations are parameterized by the elements in C_4^-; consequently, there are 15 relations of the type (4.8f) and 15 of the type (4.8g).

6. For each even characteristic $m \in C_1$, there are exactly six determinants $\{D_i^m\}_{i=1,\ldots 6}$ such that:

$$D_i^m = D(M) = \pm\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4} \qquad \text{with } m \in M = \{m_1, m_2, m_3, m_4\} \in C_4^-$$

Then, one has:

$$\sum_{i=1}^{6}(D_i^m)^4 = \theta_m^4[\theta_{n_1}^4(\theta_{\alpha_1}^4\theta_{\alpha_3}^4 \pm \theta_{\alpha_2}^4\theta_{\alpha_4}^4)\pm$$
$$\pm \theta_{n_2}^4(\theta_{\alpha_1}^4\theta_{\alpha_5}^4 \pm \theta_{\alpha_2}^4\theta_{\alpha_6}^4) \pm \theta_{n_3}^4(\theta_{\alpha_3}^4\theta_{\alpha_5}^4 \pm \theta_{\alpha_4}^4\theta_{\alpha_6}^4)]$$

where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\} \in C_4^-$.
Then, for each $i, j, k = 1, \ldots 6$ such that $\{\alpha_i, \alpha_j, \alpha_k\} \in C_3^-$, one can denote by the symbol M_{ij}^k the only 4-plet in C_4^- containing α_i, α_j, and not α_k, and by the symbol $P(M_{ij}^k)$ the polynomial:

$$P(M_{ij}^k) := R_4(M_{ij}^k) - \theta_{\alpha_j}^4$$

where $R_4(M_{ij}^k)$ is the quartic Riemann relation associated to M_{ij}^k as in (3.66). Then, in particular:

$$\sum_{i=1}^{6} \pm(D_i^m)^4 = \theta_m^4\{\theta_{n_1}^4[\theta_{\alpha_1}^4 P(M_{23}^1) \pm \theta_{\alpha_2}^4 P(M_{14}^2)]\pm$$
$$\pm \theta_{n_2}^4[\theta_{\alpha_1}^4 P(M_{25}^1) \pm \theta_{\alpha_2}^4 P(M_{16}^2)] \pm \theta_{n_3}^4[\theta_{\alpha_3}^4 P(M_{45}^3) \pm \theta_{\alpha_4}^4 P(M_{36}^4)]\}$$

where:

$$\begin{cases} M_{23}^1 \cap M_{14}^2 = \{n_2, n_3\} \\ M_{25}^1 \cap M_{16}^2 = \{n_1, n_3\} \\ M_{45}^3 \cap M_{36}^4 = \{n_1, n_2\} \end{cases}$$

Therefore, for a suitable choice of the relative signs, one has:

$$\sum_{i=1}^{6} \pm (D_i^m)^4 = \theta_m^4[\theta_{n_1}^4(\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)(\pm\theta_{n_2}^4 \pm \theta_{n_3}^4)\pm$$
$$\theta_{n_2}^4(\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)(\pm\theta_{n_1}^4 \pm \theta_{n_3}^4) \pm \theta_{n_3}^4(\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)(\pm\theta_{n_1}^4 \pm \theta_{n_2}^4)] =$$

$$= \theta_m^4(\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)[\theta_{n_1}^4(\pm\theta_{n_2}^4 \pm \theta_{n_3}^4) \pm \theta_{n_2}^4(\pm\theta_{n_1}^4 \pm \theta_{n_3}^4) \pm \theta_{n_3}^4(\pm\theta_{n_1}^4 \pm \theta_{n_2}^4)]$$

which is made null by a suitable choice for the remaining signs. To sum up, therefore, for each $m \in C_1$ one has the relation:

$$\sum_{i=1}^{6} \pm (D_i^m)^4 = 0 \tag{4.8h}$$

with a suitable choice of the signs, uniquely determined as explained. However, it is easily verified that only 6 of this ten relations are independent.

Now, the following definitive Proposition can be stated concerning the relations among the $D(N)$:

Proposition 4.6. *All the relations amongst the $D(N)$ are generated by:*

1. *The* 10 *relations in (4.8a);*
2. *The* 15 *relations in (4.8b);*
3. *The* 15 *relations in (4.8c);*
4. *The* 60 *relations in (4.8d);*
5. *The* 15 *relations in (4.8e);*
6. *The* 15 *relations in (4.8f);*
7. *The* 15 *relations in (4.8g);*
8. *The* 6 *relations in (4.8h);*

Proof. The statement has been proved computationally by elimination theory, thanks to a computer program created by Professor Eberhard Freitag. □

4.4.3 Relations among $D(N)$ and $\theta_m^2\theta_n^2$

There are of course the 15 relations induced by the Jacobi's formula:

$$D(M)^2 = \theta_{m_1}^2\theta_{m_2}^2\theta_{m_3}^2\theta_{m_4}^2 \qquad \forall\, M = \{m_1, m_2, m_3, m_4\} \in C_4^- \tag{4.9a}$$

as described in Example 3.2. Any other relation is clearly generated by the ones in (4.9a) and by all the relations of the kind:

$$\prod_{i=1}^{h} D(N_i) = P(\theta_m^2\theta_n^2)$$

where $P(\theta_m^2\theta_n^2)$ is a polynomial in $\theta_m^2\theta_n^2$ and the determinants $D(N_i)$ are all distinct.

Since for each couple of characteristics m, n $\theta_m^2\theta_n^2$ is a modular form with respect to $\Gamma(2,4)$ by (3.36) and (3.23), such a relation is satisfied if and only if $\prod_i D(N_i)$ itself is a modular form with respect to $\Gamma(2,4)$.
By Proposition 3.14 such a condition is equivalent to $M^tM \equiv 0\, mod 2$ for the $4\times 4h$ matrix $M = (M_1 \dots M_h)$ of even characteristics, associated to $\prod_i D(M_i)$, and also, by Proposition 3.19, to $N^tN \equiv 0\, mod 2$ on the $4 \times 2h$ matrix $N = (N_1 \dots N_h)$ of odd characteristics, associated to $\prod_i D(N_i)$. A necessary condition is therefore given by:

$$diag(M^tM) \equiv 0 \bmod 2$$

or, likewise, concerning odd characteristics, by:

$$diag(N^tN) \equiv 0 \bmod 2$$

As a first step, the discussion will be therefore focused on products $\prod_i D(N_i)$ of distinct Jacobian determinants, satisfying this condition. For such a purpose the following technical definition will be useful throughout this section:

Definition 4.1. *Let $1 \le h \le 15$, and let $D(N_1)\cdots D(N_h) = D(M_1)\cdots D(M_h)$ be a product of distinct Jacobian determinants. If the sum of all the even characteristics m appearing in the 4plets $M_i \in C_4^-$ (or, equivalently, the sum of all the odd characteristics n appearing in the couples $N \in \tilde{C}_2$), each counted with its multiplicity, is congruent to 0 mod 2, such a product will be called a* **remarkable factor** *of degree h. A remarkable factor which is product of remarkable factors will be named* **reducible**, *otherwise it will be called* **non-reducible**.

As already seen, if the product $D(N_1)\cdots D(N_h)$ of distinct determinants is a modular form with respect to $\Gamma(2,4)$, it is a remarkable factor, while the converse statement is not necessarily true.

Remarkable factors can be easily characterized.

Proposition 4.7. *P is a remarkable factor if and only if P is a monomial in the variables θ_m^2 and χ_5. More precisely:*

$$P = {\chi_5}^h \prod_m {\theta_m}^2 \qquad h = 0, 1$$

Proof. If P is a monomial in the variables θ_m^2 and χ_5, then P is clearly a remarkable factor; therefore, only the converse statement has to be proved .
For such a purpose, one can use the function, defined in (3.52):

$$F : C \longrightarrow \mathbb{C}[\theta_m]$$

$$\{m_1, m_2, \dots, m_h\} \longrightarrow \theta_{m_1}\theta_{m_2}\cdots\theta_{m_h}$$

$$F(\emptyset) := 1$$

Then, in particular:

$$F(\{m\}) = \theta_m; \qquad F(C_1) = \chi_5; \qquad F(M) = D(M) \quad \forall M \in C_4^-;$$

and moreover one has:

$$F(M_i)\,F(M_j) = F(M_1 \vartriangle M_j) \prod_{m \in M_i \cap M_j} {\theta_m}^2$$

If $P = F(M_1) \cdots F(M_h)$ is a remarkable factor, by using such identity and Propositions 3.4, 3.5 and 3.6, it necessarily follows that:

$$P = {\chi_5}^h \prod_m {\theta_m}^2$$

with $h = 0, 1$. □

In order to classify remarkable factors in terms of the Jacobian determinants appearing in the product, the law which associates to each couple of odd characteristics $N \in \tilde{C}_2$ their sum $S(N) \in \mathbb{Z}_2^4$ will be a useful tool:

$$N = \{n_1, n_2\} \longrightarrow S(N) := n_1 + n_2$$

In fact, a product $P = \prod_i D(N_i)$ of distinct Jacobian determinants is clearly a remarkable factor if and only if $\sum_i S(N_i) = 0$.

Lemma 4.1. *Remarkable factors of degree greater than* 5 *are reducible.*

Proof. Let $P = \prod_{i=1}^h D(N_i)$ be a remarkable factor with $h > 5$. The set $\{S(N_i)\}_{i=1,\dots h} \subset \mathbb{Z}_2^4$ necessarily contains at least two elements linearly dependent from the others. Since $S(N) \neq 0$ for each $N \in \tilde{C}_2$, the thesis follows. □

By Lemma 4.1, remarkable factors of degree at most 5 are the only ones to check, in order to find the non-reducible ones.

Proposition 4.8. *Non-reducible factors are:*

1. $D(n_i, n_j)D(n_j, n_k)D(n_k, n_i)$;
2. $D(n_i, n_j)D(n_k, n_l)D(n_s, n_t)$;
3. $D(n_i, n_j)D(n_j, n_k)D(n_k, n_l)D(n_l, n_i)$;
4. $D(n_i, n_j)D(n_i, n_k)D(n_i, n_l)D(n_s, n_t)$;
5. $D(n_i, n_j)D(n_j, n_k)D(n_k, n_l)D(n_l, n_r)D(n_r, n_i)$;
6. $D(n, n_j)D(n, n_k)D(n, n_l)D(m, n_r)D(m, n_s)$;

Proof. They can be plainly detected by the following table, which can be redacted with reference to the notation introduced in the Example 3.2 for odd characteristics:

□

$D(N)$	${}^tS(N)$
$D(n^{(1)}, n^{(2)})$	(1111)
$D(n^{(1)}, n^{(3)})$	(0010)
$D(n^{(2)}, n^{(3)})$	(1101)
$D(n^{(1)}, n^{(4)})$	(1110)
$D(n^{(2)}, n^{(4)})$	(0001)
$D(n^{(1)}, n^{(5)})$	(1000)
$D(n^{(2)}, n^{(5)})$	(0111)
$D(n^{(1)}, n^{(6)})$	(1011)
$D(n^{(2)}, n^{(6)})$	(0100)
$D(n^{(3)}, n^{(4)})$	(1100)
$D(n^{(3)}, n^{(5)})$	(1010)
$D(n^{(3)}, n^{(6)})$	(1001)
$D(n^{(4)}, n^{(5)})$	(0110)
$D(n^{(4)}, n^{(6)})$	(0101)
$D(n^{(5)}, n^{(6)})$	(0011)

By using (3.19), one can observe that only the factors of the type 2., 3. and 6. are modular forms with respect to $\Gamma(2,4)$. As a consequence of Proposition 4.7, χ_5 appears in such factors with even multiplicity, while it appears in the factors of type 1., 4. and 5. χ_5 with odd multiplicity.

The products $\prod_i D(N_i)$ of distinct determinants which are functions of $\theta_m^2\theta_n^2$, are thus necessarily products of the factors listed in Proposition 4.8.

Proposition 4.9. *The relations involving products of* 3 *determinants are:*

$$D(n_i, n_j)D(n_k, n_l)D(n_s, n_t) = \prod_{i=1}^{6} \theta_{m_i}{}^2 \tag{4.9b}$$

Proof. As already stated, the factors of type 2. are the only ones involved.
In order to prove that the six Theta constants appearing in the expression (4.9b) are all distinct, let P be a product of determinants of the type in (4.9b), and let $M_1, M_2, M_3 \in C_4^-$ be the 4-plets satisfying:

$$P = D(M_1)D(M_2)D(M_3)$$

Since P is a remarkable factor, if $M_i \vartriangle M_j \in C_6^+$ for any couple of this 4-plets, it would follow that $(M_1 \vartriangle M_2)^c = M_3$; then one would have $P = \theta_{m_1}{}^2\theta_{m_2}{}^2\theta_{m_3}{}^2\chi_5$, which is an absurd statement, because P is also a modular form with respect to $\Gamma_2(2,4)$. Then, by Proposition 3.4, $M_i \vartriangle M_j \in C_4^-$ for each distinct couple M_i, M_j, and the only possibility is $M_1 \vartriangle M_2 = M_3 \in C_4^-$, namely:

$$M_1 = \{m_1, m_2, m_3, m_4\} \qquad M_2 = \{m_1, m_2, m_5, m_6\} \qquad M_3 = \{m_3, m_4, m_5, m_6\}$$

Therefore, the thesis follows. ◻

Proposition 4.10. *The relations involving products of* 4 *determinants are:*

$$D(n_i, n_j)D(n_j, n_k)D(n_k, n_l)D(n_l, n_i) = \prod_{i=1}^{8} \theta_{m_i}{}^2 \tag{4.9c}$$

Proof. Remarkable factors of type 3. are the only ones involved.
To prove the eight Theta constants appearing in the expression (4.9c) are all distinct, let P be a product of determinants of the type in (4.9c), and let $M_1, M_2, M_3, M_4 \in C_4^-$ the 4-plets satisfying:

$$P = D(M_1)D(M_2)D(M_3)D(M_4)$$

If $M_1 \vartriangle M_2 \in C_4^-$, then $M_3 \vartriangle M_4 = M_1 \vartriangle M_2 \in C_4^-$, since P is a modular form with respect to $\Gamma_2(2,4)$. Due to $|M_1 \vartriangle M_2| = 4$, there are at least six distinct characteristics among the ones associated to the Theta constants appearing in the expression (each of them appearing with multiplicity 2); however, $M_3 \vartriangle M_4 = M_1 \vartriangle M_2$ and $|M_3 \cap M_4| = 2$, hence the eight characteristics, appearing with multiplicity 2, are all distinct, since the determinants involved $D(M_i)$ are all distinct.
If $M_1 \vartriangle M_2 \in C_6^+$, then $M_3 \vartriangle M_4 = M_1 \vartriangle M_2 \in C_6^+$. Since $|M_1 \cap M_2| = 1$, at least seven distinct characteristics appear, each with multiplicity 2. As before, $M_3 \vartriangle M_4 = M_1 \vartriangle M_2$ with $|M_3 \cap M_4| = 1$ and the common characteristic in M_3 and M_4 must be different from the other seven, since the $D(M_i)$ are all distinct. □

Proposition 4.11. *The relations involving products of* 5 *determinants are:*

$$D(n, n_j)D(n, n_k)D(n, n_l)D(m, n_r)D(m, n_s) = \prod_{m} \theta_m^2 \tag{4.9d}$$

Proof. As seen, the factors of type 6 are the only ones involved. By Proposition 4.7 χ_5 appears with even multiplicity. However, it can be plainly seen that the ten Theta constants in the expression (4.9d) are not necessarily all distinct in this case. □

Concerning the relations involving products of more than 5 determinants, one has to observe that the product of two non-reducible remarkable factors of type 2, 4 and 5 is indeed a modular form with respect to $\Gamma(2,4)$; therefore, if it does not factorize into a product of the factors already selected, it will induce new independent relations. One has, in particular, the following:

Proposition 4.12. *The relations involving products of* 6 *determinants are:*

$$D(n_i, n_j)D(n_j, n_k)D(n_k, n_i)D(n_l, n_r)D(n_r, n_s)D(n_s, n_l) = \chi_5{}^2\theta_m{}^2\theta_n{}^2 \tag{4.9e}$$

Proof. The only possible case rises from the product P of two distinct factors of type 1.:

$$Q_1 = D(n_i, n_j)D(n_j, n_k)D(n_k, n_i) = \chi_5\theta_m{}^2$$

$$Q_1' = D(n_i', n_j')D(n_j', n_k')D(n_k', n_i') = \chi_5\theta_n^2$$

Clearly $Q_1 \cdot Q_1'$ does not factorize into products of determinants which are in turn modular forms with respect to $\Gamma(2,4)$; therefore, the relations in (4.9e) are not generated by the previous ones. □

The following Proposition ends the investigation around these relations.

Proposition 4.13. *Let P be a product of more than 6 distinct determinants, which is a modular form with respect to* $\Gamma(2,4)$*. Then, each relation involving P is dependent from the ones in (4.9a), (4.9b), (4.9c), (4.9d) and (4.9e).*

Proof. The single cases have to be briefly discussed.

Let P be a product of 7 distinct determinants such that $P \in A(\Gamma(2,4))$.Then P is necessarily the product of a factor P_1 of type 1. and a factor P_4 of type 4. and the only possible cases are:

$$P_1 \cdot P_4 = [D(n_i,n_j)D(n_j,n_k)D(n_k,n_i)][D(n_l,n_i)D(n_l,n_j)D(n_l,n_k)D(n_r,n_s)]$$

$$P_1 \cdot P_4 = [D(n_i,n_j)D(n_j,n_k)D(n_k,n_i)][D(n_l,n_i)D(n_l,n_j)D(n_l,n_r)D(n_k,n_s)]$$

However, by using the relations (4.8a), it turns out that:

$$D(n_i,n_j)D(n_j,n_k)D(n_k,n_i) = D(n_l,n_r)D(n_r,n_s)D(n_s,n_l)$$

Therefore in both case at least a $D(N)^2$ appears, and the relations involving P are dependent from the ones which have been already found (the relations in (4.9a) hold, in particular).

Concerning products of more than 7 determinants:

$$P = \prod_{N \in C} D(N) \qquad C \subset \tilde{C}_2 \quad \text{s. t. } |C| > 7$$

it will be useful to study the product of the determinants, associated to the complementary couples N:

$$P^c := \prod_{N \notin C} D(N)$$

In fact, if $P \in A(\Gamma(2,4))$, then clearly $P^c \in A(\Gamma(2,4))$, so that the behavious of P^c pertains to the previous cases.

Let P be, therefore, a product of 8 distinct determinants such that $P \in A(\Gamma(2,4))$. P^c has thus degree 7; then, either $P^c = Q_1 \cdot Q_4$ with Q_1 of type 1. and Q_4 of type 4, or $P^c = Q_2 \cdot Q_3$ with Q_2 of type 2. and Q_3 of type 3.
If $P^c = Q_1 \cdot Q_4$the only two possible cases are the ones discussed before; then, it is easily verified that P always admits a factor of the type (4.9d):

$$D(n_i,n_k)D(n_i,n_r)D(n_i,n_s)D(n_s,n_j)D(n_s,n_l)$$

which does not appear in the product $Q_1 \cdot Q_4$. Then P factorizes into the product of two factors which are modular forms with respect to $A(\Gamma(2,4))$, and have been therefore already checked.
If $P^c = Q_2 \cdot Q_3$, then:

$$P^c = D(n_i, n_j)D(n_k, n_l)D(n_r, n_s)D(n'_i, n'_j)D(n'_j, n'_k)D(n'_k, n'_l)D(n'_l, n'_i)$$

Since four of the six odd characteristics appear with multiplicity 3 and the other two with multiplicity 1, P always contains a factor of the type (4.9c):

$$D(n, n_\alpha)D(n_\alpha, n_\beta)D(n_\beta, n_\gamma)D(n_\gamma, n)$$

Therefore, P factorizes again into the product of two factors which are modular forms with respect to $A(\Gamma(2,4))$.

Let P be the product of 9 distinct determinants such that $P \in A(\Gamma(2,4))$. $P^c \in A(\Gamma(2,4))$ has, therefore, degree 6; consequently, it is either the product $Q_2 \cdot Q'_2$ of two factors of type 2, or the product $Q_1 \cdot Q'_1$ of two factors of type 1. In the first case:

$$P^c = Q_2 \cdot Q'_2 = D(n_i, n_j)D(n_k, n_l)D(n_r, n_s)D(n'_i, n'_j)D(n'_k, n'_l)D(n'_r, n'_s)$$

all the characteristics appear with multiplicity 2; therefore, P always contains a factor of the type (4.9d).
In the second case P^c is of the type (4.9e); then, at least five characteristics appear in P^c with multiplicity 2. Hence, P always contains a factor of the type (4.9d).

If P is a product of 10 distinct determinants such that $P \in A(\Gamma(2,4))$, by Proposition 4.11 P^c is of the type (4.9d). Then, it can be easily checked that P necessarily contains a factor P_3 of the type (4.9c) both when $n \neq m$ and when $n = m$.

If P is a product of 11 distinct determinants such that $P \in A(\Gamma(2,4))$, by Proposition 4.10 P^c is of the type (4.9c). Then, $P = P_6 \cdot P'$ where P_6 is of the type (4.9d) and $P' = P_1 \cdot P_1$ is of the type (4.9e).

Finally, if P is a product of 12 distinct determinants such that $P \in A(\Gamma(2,4))$, by Proposition 4.9 P^c is of the type (4.9b). Then, $P = P_6 \cdot P_3 \cdot P_2$ where P_6 is of the type (4.9d), P_3 is of the type (4.9c) and P_2 is of the type (4.9b).

Obviously products of 13 or 14 Jacobian determinants can not be modular forms with respect to $\Gamma(2,4)$, while the product of all the 15 determinants trivially factorizes into factors already selected.

□

To sum up, one has:

Proposition 4.14. *A system of independent relations between the generators $D(M)$ and $\theta_m^2\theta_n^2$ is given by (4.9a), (4.9b), (4.9c), (4.9d) and (4.9e).*

The results concerning the description of $A(\Gamma)^{(e)}$ pursued throughout the foregoing sections can be, now, summarized as a structure theorem in terms of generators and relations:

Theorem 4.2. (Structure Theorem for $A(\Gamma)^{(e)}$) *The ring $A(\Gamma)^{(e)}$ is generated by $\theta_m^2\theta_n^2$ and $D(M)$. The ideal of the relations amongst them is generated by the ones in (4.7a), (4.7b), (4.8a), (4.8b), (4.8c), (4.8d), (4.8e), (4.8f), (4.8g), (4.8h), (4.9a), (4.9b), (4.9c), (4.9d) and (4.9e).*

4.5 The Ideal $S(\Gamma)^e$

The even part $S(\Gamma)^e$ of the ideal of the cusp forms with respect to the subgroup Γ can be described thanks to the results by Van Geemen and Van Straten concerning the generators of the cusp forms with respect to $\Gamma(2,4,8)$:

Theorem 4.3. *A system of generators for $S(\Gamma)^e$ is given by:*

1. $D(M) \qquad \forall\, M \in C_4^-$
2. $\theta_{m_1}^4\theta_{m_2}^2\theta_{m_3}^2\theta_{m_4}^2\theta_{m_5}^2 \qquad \forall\, \{m_1, m_2, m_3, m_4, m_5\} \notin C_5 + \cap C_5^-$

In particular, there are $15 + 5 \cdot 70 = 365$ generators for this ideal.

Proof. Since $\Gamma(2,4,8) \subset \Gamma(4,8) \subset \Gamma$, the inclusion $S(\Gamma) \subset S(\Gamma(2,4,8)$ is plainly verified; the generators of $S(\Gamma)^e$ can be therefore selected amongst the ones described in Theorem 3.11. Since, by Theorem 4.1, $S(\Gamma)^e \subset \mathbb{C}[\theta_m^2\theta_n^2, D(M)]$, only the types 1. and 2., enumerated in the statement of Theorem 3.11, generates $S(\Gamma)^e$; in fact, by using $\theta_m^2 = Q_m(\Theta_{m'})$, elements of type 3. are easily seen no to be in the ideal, being expressed as $P(\theta_m^2\theta_n^2)\Theta_{m'}$ where $\Theta_{m'}$ is a second order Theta constant. □

Appendix A

Elementary results of matrix calculus

Proposition A.1. *Let $k = {}^t(k_1, \dots k_g) \in \mathbb{Z}^g$ a column vector and $D := MCD(k_1, \dots k_g)$ the greatest common divisor of $k_1, \dots k_g$. There exists a matrix $M \in M_g(\mathbb{Z})$ with k as first column and* $\det M = D$.

Proof. The statement can be proved by induction on g. It is trivial indeed for $g = 1$. Then, let it be true for $g-1$ and let $k = (k_1, \dots k_g) \in \mathbb{Z}^g$. By the inductive hypothesis, a matrix $M' \in M_{g-1}(\mathbb{Z})$ exists with $k' = {}^t(k_1, \dots k_{g-1}) \in \mathbb{Z}^{g-1}$ as first column and $det(M') = D' = MCD(k_1, \dots k_{g-1})$. Now, let $p, q \in \mathbb{Z}$ satisfy the Bezout identity $pD' - qk_g = D$. Then the matrix M defined by:

$$M = \begin{pmatrix} & & & & \frac{k_1 q}{D'} \\ & & & & \frac{k_2 q}{D'} \\ & & M' & & \vdots \\ & & & & \frac{k_{g-1} q}{D'} \\ k_g & 0 & \cdots & 0 & p \end{pmatrix}$$

has the desired properties [1]. ◻

The following is an immediate application:

Corollary A.1. *Let $k = {}^t(k_1, \dots k_g) \in \mathbb{Z}^g$ a primitive column vector. Then, an unimodular matrix $M \in GL_g(\mathbb{Z})$ exists, such that k is its first column.*

Concerning matrices with entries in a field, decompositions into triangular factors are often useful to consider.

Definition A.1. *Let K be a field and $M \in GL_n(K)$. The matrix M is said to admit a LU decomposition if $M = LU$, where $L \in GL_n(K)$ is a lower triangular matrix with entries 1 on the diagonal and $U \in GL_n(K)$ is an upper triangular matrix.*

[1]Since the main ingredient of the proof is the Bezout identity, the statement generally holds for principal ideal domains with no crucial difference in the proof. The existence of such a matrix M is likewise proved for each distinct greater common divisor D.

Proposition A.2. *The LU decomposition is unique.*

Proof. Let $M \in GL_n(K)$ be such that $LU = M = L'U'$, where $L, L' \in GL_n(K)$ are lower triangular matrices with entries 1 on the diagonal, and $U, U' \in GL_n(K)$ are upper triangular matrices. Then, $(L')^{-1}L = U'U^{-1}$. However, since $(L')^{-1}L$ is lower triangular and $U'U^{-1}$ is upper triangular, both must be diagonal. Moreover, the entries of $(L')^{-1}L$'s diagonal are 1; hence $(L')^{-1}L = U'U^{-1} = 1_n$. Therefore, $L = L'$ and $U = U'$. □

Proposition A.3. *Let be K a field, and $M \in GL_n(K)$. Then M admits a LU decomposition if and only if all the leading principal minors are nonzero.*

Proof. Let $M \in GL_n(K)$ be such that $M = LU$ with $L \in GL_n(K)$ lower triangular matrix with entries 1 on the diagonal, and $U \in GL_n(K)$ upper triangular matrix. Then, for each $1 \leq h \leq n$ the $h \times h$ submatrix $M^{(h)}$ of M, corresponding to the $h \times h$ leading principal minor, clearly admits the decomposition $M^{(h)} = L^{(h)}U^{(h)}$, where $L^{(h)}$ and $U^{(h)}$ correspond to the $h \times h$ leading principal minor respectively of L and U; hence:

$$\det M^{(h)} = \det L^{(h)} \det U^{(h)} = \prod_{i=1}^{h} U_{ii} \neq 0$$

On the converse, let be $M \in GL_n(K)$ such that all the leading principal minors are nonzero; one can prove the LU decomposition for M, by induction on n. For $n = 1$ the statement is trivial. Therefore, let it be true for $n - 1$, and let $M \in GL_n(K)$ be such a matrix. M can be described in a suitable block notation as:

$$M = \begin{pmatrix} M_{(n-1)} & a \\ {}^t b & M_{nn} \end{pmatrix}$$

where $M_{(n-1)}$ is a $(n-1) \times (n-1)$ matrix, whose leading principal minors are nonzero. By the inductive hypothesis, $M_{(n-1)} = L_{(n-1)}U_{(n-1)}$, where $L_{(n-1)} \in GL_{n-1}(K)$ is a lower triangular matrix with entries 1 on the diagonal, and $U_{(n-1)} \in GL_{n-1}(K)$ is a upper triangular matrix. Then, the matrices L and U are:

$$L = \begin{pmatrix} L_{(n-1)} & 0 \\ {}^t x & 1 \end{pmatrix} \qquad U = \begin{pmatrix} U_{(n-1)} & y \\ 0 & U_{nn} \end{pmatrix}$$

where x and y are the unique solutions of the systems:

$$L_{(n-1)}y = a \qquad {}^t U_{(n-1)}x = b$$

and $U_{nn} = M_{nn} - {}^t xy$.

□

As a corollary, one has the following:

Proposition A.4. (Jacobi decomposition) *Let $M \in Sym_n(\mathbb{R})$ be a definite positive matrix. Then, one has a unique decomposition $M = {}^tUDU$, where $U \in GL_n(\mathbb{R})$ is an upper triangular matrix with entries* 1 *on the diagonal and D is a diagonal matrix with positive entries.*

Proof. Since M is a real symmetric definite positive matrix, all its leading principal minors are nonzero. By Proposition A.3, M admits a LU decomposition. Let be $M = L_0U_0$ such a unique decomposition. Since $\det U_0 \neq 0$, each diagonal entry U_{ii} of the matrix U_0 is nonzero; then, one has:

$$U_0 = \begin{pmatrix} U_{11} & 0 & \dots & 0 \\ 0 & U_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & U_{nn} \end{pmatrix} \begin{pmatrix} 1 & \frac{U_{12}}{U_{11}} & \dots & \frac{U_{1n}}{U_{11}} \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \frac{U_{n-1n}}{U_{n-1n-1}} \\ 0 & \dots & 0 & 1 \end{pmatrix} = DU$$

By transposing, one also obtains $M = {}^tUD\,{}^tL_0$. Since the LU decomposition is unique, ${}^tL_0 = U$, and consequently $M = {}^tUDU$. Since U is invertible, one also has $D = {}^tU^{-1}MU^{-1}$; hence, D is definite positive, which concludes the proof. □

A straightforward corollary is the following classical result:

Corollary A.2. (Cholesky Decomposition) *Let be $M \in Sym_n(\mathbb{R})$ a definite positive matrix. Then, one has a unique decomposition $M = L^tL$, where $L \in GL_n(\mathbb{R})$ is a lower triangular matrix with positive entries on the diagonal.*

Proof. Since the diagonal matrix D appearing in the Jacobi decomposition $M = {}^tUDU$ is definite positive, $D^{1/2}$ exists. Then, the thesis follows by setting $L = {}^tUD^{1/2}$. □

Appendix B

Satake's compactification

This appendix is designed to describe a singular compactification of the moduli space $A_g = \mathfrak{S}_g/\Gamma_g$, provided by Satake in [Sa], which is realized by adding cusps, namely orbits under the action of suitable subgroups of Γ_g; indeed, in this construction this cusps play the role of different directions to infinity to add to the space in order to make it compact.
This procedure can be done in different steps.

B.1 Realization of $\mathfrak{S}_g$ as a bounded domain

To compactify $\mathfrak{S}_g/\Gamma_g$ in the way described above, one needs to seek the right cusps to add; the first step consist of realizing$\mathfrak{S}_g$ as a bounded domain in $Sym_g(\mathbb{C})$, so that the points to add will be sought along the boundary. In case $g = 1$ by the *Cayley transformation*:

$$C_1(\tau) := \frac{\tau - i}{\tau + i} \quad \forall \tau \in \mathfrak{S}_1 \tag{B.1}$$

one is provided with the Poincaré model, which realizes the complex upper half-plane $\mathbb{H}$ as the open unit disk $D_1 := \{z \in \mathbb{C} \mid |z| < 1\}$. The Cayley transformation admits a generalization to the upper half-plane $\mathfrak{S}_g$:

$$\begin{aligned} &C_g : \mathfrak{S}_g \to D_g \\ &\tau \to (\tau - i1_g) \cdot (\tau + i1_g)^{-1} \end{aligned} \tag{B.2}$$

where $D_g = \{z \in Sym_g\mathbb{C} \mid z\bar{z} - 1 < 0\}$ is the natural generalization of the open unit disk D_1.

Proposition B.1. *The map C_g in (B.2) is an analytic isomorphism, whose inverse map is:*

$$C_g^{-1}(z) = i(1_g + z)(1_g - z)^{-1} \tag{B.3}$$

Proof. As proved in Proposition 1.5, $c_\gamma\tau + d_\gamma$ is invertible whenever $\tau \in \mathfrak{S}_g$ and $\gamma \in Sp(g, \mathbb{R})$. Hence, in particular $\det\tau \neq 0$ whenever $\tau \in \mathfrak{S}_g$; then, $\det\tau + i1_g \neq 0$ whenever $\tau \in \mathfrak{S}_g$ and the map C_g is consequently a well defined analytic map. Moreover, for each $\tau \in \mathfrak{S}_g$, one has:

$$\begin{aligned} C_g(\tau)\overline{C_g(\tau)} - 1_g &= (\tau - i1_g)(\tau + i1_g)^{-1}(\overline{\tau} + i1_g)(\overline{\tau} - i1_g)^{-1} = \\ &= {}^t\overline{(\overline{\tau} - i1_g)^{-1}}[(\tau - i1_g)(\overline{\tau} + i1_g) + (\tau + i1_g)(\overline{\tau} - i1_g)](\overline{\tau} - i1_g)^{-1} = \\ &= -4\,{}^t\overline{(\overline{\tau} - i1_g)^{-1}} Im\tau(\overline{\tau} - i1_g)^{-1} < 0 \end{aligned}$$

Therefore, C_g maps $\mathfrak{S}_g$ into D_g. Moreover, for each $z \in D_g$, $1_g - z$ is also invertible; in fact, if $w \in \mathbb{C}^g$ is such that $(1_g - z)w = 0$, then:

$${}^tw(1_g - z\overline{z})\overline{w} = 0$$

hence, $w = 0$ whenever $z \in D_g$; then, the map in (B.3) is also a well defined analytic map. Moreover, for each $z \in D_g$, one has:

$$\begin{aligned} ImC_g^{-1}(z) &= \frac{1}{2}[(1_g - z)^{-1}(1_g + z) + (1_g + \overline{z})(1_g - \overline{z})^{-1}] = \\ &= \frac{1}{2}{}^t\overline{(1_g - \overline{z})^{-1}}[(1_g + z)(1_g - \overline{z}) + (1_g - z)(1_g + \overline{z})](1_g - \overline{z})^{-1} = \\ &= {}^t\overline{(1_g - \overline{z})^{-1}}(1_g - z\overline{z})(1_g - \overline{z})^{-1} > 0 \end{aligned}$$

Then, C_g^{-1} maps D_g into $\mathfrak{S}_g$ and, as easily checked, is the inverse to C_g. □

In general, an embedding theorem proved by Borel and Harish-Chandra (cf. [AMRT]) states that every symmetric domain can be realized as a bounded domain in a complex affine space of the same dimension if and only if it does not admit a direct factor, which is isomorphic to $\mathbb{C}^n$ modulo a discrete group of translation. Since, $Sp(g, \mathbb{R})$ is a simple Lie group, (1.14) implies $\mathfrak{S}_g$ does not admit such a factor and consequently the Borel and Harish-Chandra theorem applies. It can be seen indeed that the Cayley transformation C_g provides the Harish-Chandra embedding.

Due to (1.10), an action of the symplectic group $Sp(g, \mathbb{R})$ is induced on the bounded domain D_g by the Cayley transform:

$$\begin{aligned} &Sp(g, \mathbb{R}) \times D_g \to D_g \\ &(\gamma, z) \to C_g\,\gamma\,C_g^{-1}z \end{aligned} \tag{B.4}$$

As easily seen, one has:

$$\gamma z := [\ (a_\gamma - ic_\gamma)(z+1_g) + i(b_\gamma - id_\gamma)(z-1_g)\ ] \cdot [\ (a_\gamma + ic_\gamma)(z+1_g) + i(b_\gamma + id_\gamma)(z-1_g)\ ]^{-1}$$

The bounded realization of the Siegel upper half-plane $\mathfrak{S}_g$, allows to seek the suitable cusps along the boundary. Indeed the symplectic group acts on the boundary of D_g as well, as the following Proposition states:

Proposition B.2. *Let* $\overline{D}_g := \{z \in Sym_g(\mathbb{C}|1_g - z\overline{z} \leq 0)\}$ *be the closure of* D_g *in* $Sym(g, \mathbb{C})$*. The action of* $Sp(g, \mathbb{R})$ *extends to* $\overline{D}_g$*.*

Proof. One has only to prove that the matrix:

$$M_\gamma^+(z) := (a_\gamma + ic_\gamma)(z + 1_g) + i(b_\gamma + id_\gamma)(z - 1_g)$$

is invertible whenever $\gamma \in Sp(g, \mathbb{R})$ and $z \in \overline{D}_g$, namely that $M_\gamma^+(z)$ is of maximum rank. By setting:

$$M_\gamma^-(z) := (a_\gamma - ic_\gamma)(z + 1_g) + i(b_\gamma - id_\gamma)(z - 1_g)$$

one has:

$$\begin{aligned} {}^t\overline{M_\gamma^\pm(z)}M_\gamma^\pm(z) = {} & (\overline{z} + 1_g)({}^ta_\gamma a_\gamma + {}^tc_\gamma c_\gamma)(z + 1_g) + (\overline{z} - 1_g)({}^tb_\gamma b_\gamma + {}^td_\gamma d_\gamma)(z - 1_g) + \\ & - i(\overline{z} - 1_g)({}^tb_\gamma a_\gamma + {}^td_\gamma c_\gamma)(z + 1_g) + i(\overline{z} + 1_g)({}^ta_\gamma b_\gamma + {}^tc_\gamma d_\gamma)(z - 1_g) + \\ & \pm 2(1_g - \overline{z}z) \end{aligned}$$

Hence:

$$ {}^t\overline{M_\gamma^+(z)}M_\gamma^+(z) = \frac{1}{2}({}^t\overline{M_\gamma^+(z)}M_\gamma^+(z) + {}^t\overline{M_\gamma^-(z)}M_\gamma^-(z)) + 2(1_g - \overline{z}z) \tag{B.5}$$

Now, since:

$$\begin{pmatrix} M_\gamma^-(z) \\ M_\gamma^+(z) \end{pmatrix} = \begin{pmatrix} 1_g & -i1_g \\ 1_g & i1_g \end{pmatrix} \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \begin{pmatrix} 1_g & 1_g \\ i1_g & -i1_g \end{pmatrix} \begin{pmatrix} z \\ 1_g \end{pmatrix}$$

one has:

$$\text{rank} \begin{pmatrix} M_\gamma^-(z) \\ M_\gamma^+(z) \end{pmatrix} = \text{rank} \begin{pmatrix} z \\ 1_g \end{pmatrix} = g$$

and consequently ${}^t\overline{M_\gamma^+(z)}M_\gamma^+(z) + {}^t\overline{M_\gamma^-(z)}M_\gamma^-(z) > 0$. Therefore, for each $z \in \overline{D}_g$, (B.5) implies:

$$ {}^t\overline{M_\gamma^+(z)}M_\gamma^+(z) > 0$$

Then, rank$M_\gamma^+(z) = g$. □

B.2 Boundary Components

The next step is decomposing $\overline{D}_g$ in such a way that the decomposition is preserved by the action of $Sp(g, \mathbb{R})$; the group will thus operate on the components of $\overline{D}_g$.

First of all, one can observe that an equivalence relation can be introduced in $\overline{D}_g$ by stating two points of $\overline{D}_g$ are equivalent if they are connected by finitely many holomorphic arcs.

Definition B.1. *For $z, w \in \overline{D}_g$, one has $z \; \rho \; w$, if and only if there exist finitely many holomorphic maps $f_1, \dots, f_k : D_1 \to \overline{D}_g$, such that $f_1(0) = z$, $f_k(0) = w$, and $f_i(D_1) \cap f_{i+1}(D_1) \neq \emptyset$ for each $i = 1, \dots, k$.*

The relation ρ is thus easily checked to be an equivalence relation on $\overline{D}_g$.

Definition B.2. *The equivalence classes of ρ are known as the* **boundary components** *of $\overline{D}_g$.*

In order to classify boundary components, a suitable map $\psi_z : \mathbb{R}^{2g} \to \mathbb{C}^g$ can be defined for any fixed $z \in \overline{D}_g$:

$$\psi_z(x) := x \begin{pmatrix} i(1_g + z) \\ 1_g - z \end{pmatrix} \tag{B.6}$$

The importance of such maps is related to the subspaces $Ker\psi_z$, which are invariants of the boundary components.

Proposition B.3. *Let $z \in \overline{D}_g$ and let ψ_z be the correspondent map as in (B.6). Then:*

1. *$Ker\psi_z$ is an isotropic subspace of $\mathbb{R}^{2g}$, namely $xJ_g{}^t y = 0$ whenever $x, y \in Ker\psi_z$;*
2. *$Ker\psi_z \neq \{0\}$ if and only if $z \notin D_g$;*
3. *$Ker\psi_{\gamma z} = Ker\psi_z \gamma^{-1}$ for each $\gamma \in Sp(g, \mathbb{R})$;*

Proof. One can identify $\mathbb{R}^{2g}$ with $\mathbb{C}^g$ by:

$$(x_1, \dots x_{2g}) \xrightarrow{\phi} (x_1 + ix_{g+1}, x_2 + ix_{g+2}, \dots x_g + ix_{2g})$$

Then, for each $z \in \overline{D}_g$:

$$\begin{aligned} \psi_z \phi^{-1}(w) &= \phi^{-1}(w) \begin{pmatrix} i1_g & i1_g \\ -1_g & -1_g \end{pmatrix} \begin{pmatrix} z \\ 1_g \end{pmatrix} = \\ &= (ix_1 - x_{g+1}, \dots ix_g - x_{2g}, ix_1 + x_{g+1}, \dots ix_g + x_{2g}) \begin{pmatrix} z \\ 1_g \end{pmatrix} = \\ &= i(wz + \overline{w}) \end{aligned}$$

Hence, whenever $x, y \in Ker\psi_z$, $\overline{\phi(x)} = -i\phi(x)z$ and $\overline{\phi(y)} = -i\phi(y)z$; then one has:

$$\begin{aligned} xJ_g{}^t y &= \sum_{i=1}^{g} x_i y_{g+i} - \sum_{i=1}^{g} x_{g+i} y_i = Im\left(\overline{\phi(x)}\,{}^t\phi(y)\right) = \\ &= \frac{1}{2}(\overline{\phi(x)}\,{}^t\phi(y) - \phi(x)^t\overline{\phi(y)}) = 0 \end{aligned}$$

and thus 1. is proved.
To prove 2, one observes that $Ker\psi_z \neq 0$ implies there exists $w \in \mathbb{C}^g$ such that $0 = wz + \overline{w}$. Then, $w(1_g - z\overline{z}) = w + \overline{w}\,\overline{z} = w - w = 0$, and consequently $1_g - z\overline{z}$ is not positive definite; therefore, $z \in \overline{D}_g - D_g$. On the other hand, $z \notin \overline{D}_g$ implies

there exists an eigenvector $w \in \mathbb{C}^{2g}$ such that $(z\bar{z})w = w$. If $w = -\overline{w}\,\overline{z}$, then w is a non null vector in $Ker\psi_z$; otherwise $(i\,w + i\,\overline{w}\,\overline{z})z = i(w\bar{z} + \overline{w}) = -(i\,w + i\,\overline{w}\,\overline{z})$ and $i\,w + i\,\overline{w}\,\overline{z}$ is thus a non null vector in $Ker\psi_z$.
Finally 3. follows by a straightforward computation.

□

Corollary B.1. *Let $z_1, z_2 \in \overline{D}_g$. If $z_1\,\rho\,z_2$, then $Ker\psi_{z_1} = Ker\psi_{z_2}$.*

Thanks to Corollary B.1, the following definition can be introduced:

Definition B.3. *Let F be a boundary component of D_g. The isotropic subspace associated to F is:*

$$U(F) := Ker\psi_z \qquad z \in F \tag{B.7}$$

By using the associated isotropic subspaces, the boundary components can be classified.

Proposition B.4. *The following subsets of $\overline{D}_g$:*

$$F_0 := \left\{1_g\right\} \tag{B.8a}$$

$$F_h := \left\{ \begin{pmatrix} \tau & 0 \\ 0 & 1_h \end{pmatrix} \;\middle|\; \tau \in D_h \right\} \cong D_h \quad \forall \quad 0 < h < g \tag{B.8b}$$

$$F_g := D_g \tag{B.8c}$$

are boundary components.

Proof. It is easily checked that the following isotropic subspaces of $\mathbb{R}^{2g}$:

$$U^{(h)} := \sum_{i=0}^{h} \mathbb{R}e_{g+i}$$

are such that $U^h = U(F_h)$ for each $h = 1, \dots g$. Therefore, each F_h must be union of boundary components; since F_h is connected by holomorphic arcs, then $F_h h$ is a boundary component itself. □

It is plainly checked that (B.7) defines indeed a one-to-one correspondence between boundary components of $\overline{D}_g$ and isotropic subspaces of $\mathbb{R}^{2g}$ (cf. [HKW]). As a consequence, $\overline{D}_g$ admits a decomposition into boundary components:

$$\overline{D}_g = \bigcup_{\substack{\gamma \in Sp(g,\mathbb{R}) \\ 0 \le k \le g}} \gamma(F_k)$$

However, the boundary components whose related isotropic spaces are $\mathbb{Q}$-generated are the only ones to consider for adding cusps in order to compactify $\mathfrak{S}_g/\Gamma$.

Definition B.4. *The following subset of* $\overline{D}_g$*:*

$$\overline{D}_g^{rc} = \bigcup_{\substack{\gamma \in Sp(g,\mathbb{Q}) \\ 0 \le k \le g}} \gamma(F_k)$$

is called the **rational closure** *of* D_g*.*

The rational closure is indeed the suitable set of points to add in order to obtain the desired compactification.

B.3 The cylindrical topology on D_g^{rc}

In this section, the so-called cylindrical topology on the rational closure D_g^{rc} will be briefly discussed. For such a purpose, by using a suitable block notation, the following maps can be defined:

$$\pi_{ji}: \quad \mathfrak{S}_j \to \mathfrak{S}_i \qquad\qquad \rho_{ji}: \quad \mathfrak{S}_j \to Sym^+_{j-i}(\mathbb{R})$$

$$\begin{pmatrix} \tau & {}^t w \\ w & \tau' \end{pmatrix} \to \tau' \qquad\qquad \begin{pmatrix} \tau & {}^t w \\ w & \tau' \end{pmatrix} \to Im\tau' - Imw\, Im\tau^{-1}\, Im^t w$$

Then, for each $U \subset \mathfrak{S}_i$ open and $S_{j-i} \in Sym^+_{j-i}(\mathbb{R})$ $(j \ge i)$ one can define a generalized open cylindrical neighbourhood:

$$N_{U,S_i} := \bigcup_{j \ge i} V_{U,S_{j-i}}$$

where:

$$V_{U,S_{j-i}} := \{\tau \in \mathfrak{S}_i \,|\, \pi_{ji}(\tau) \in U\,,\, \rho_{ji} - S_{J-i} > 0\}$$

A basis for a topology is then provided by the sets:

$$\tilde{N}_{U,S_i} := \bigcup_{\substack{\gamma \in Sp(g,\mathbb{Z}) \\ \gamma = \gamma_1\gamma_2\gamma_3}} \gamma N_{U,S_i}$$

and their translates under the action of $Sp(g,\mathbb{Q})$, where $\gamma_1, \gamma_2, \gamma_3$ are particular elements of the stabilizer P_i of the rational boundary component F_i:

$$P_i = P(F_i) = \left\{ \begin{pmatrix} a & 0 & b & \alpha_1 \\ \alpha_2 & u & \alpha_3 & \alpha_4 \\ c & 0 & d & \alpha_5 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in Sp(g,\mathbb{R}) \;\middle|\; \begin{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(i,\mathbb{R}) \\ \\ u \in GL(g-i,\mathbb{R}) \end{matrix} \right\}$$

and, more precisely, the ones with the following shape:

$$\gamma_1 = \begin{pmatrix} 1_i & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1_i & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \qquad u \in GL(g-i, \mathbb{R})$$

$$\gamma_2 = \begin{pmatrix} 1_i & 0 & 0 & {}^t n \\ m & 1_{g-i} & n & 0 \\ 0 & 0 & 1_i & -{}^t m \\ 0 & 0 & 0 & 1_{g-i} \end{pmatrix} \qquad m^t n \in Sym_{g-i}(\mathbb{R})$$

$$\gamma_3 = \begin{pmatrix} 1_i & 0 & 0 & 0 \\ 0 & 1_{g-i} & 0 & S \\ 0 & 0 & 1_i & 0 \\ 0 & 0 & 0 & 1_{g-i} \end{pmatrix} \qquad S \in Sym_{g-i}(\mathbb{R})$$

The topology generated by $\tilde{N}_{U,S_i}$ and its translates under the action of $Sp(g, \mathbb{Q})$ is called the **cylindrical topology**.
Then the main theorem of the Appendix can be stated:

Proposition B.5. *The cylindrical topology turns D_g^{rc}/Γ into a compact Hausdorff space containing D_g/Γ as a dense open subset.*

In particular, concerning the full modular group Γ_g, one can observe (cf. [Na]) that a sequence

$$\left\{ \tau^{(n)} := \begin{pmatrix} \tau_1^{(n)} & \tau_2^{(n)} \\ \tau_2^{(n)} & \tau_3^{(n)} \end{pmatrix} \right\}_{n \in \mathbb{N}} \subset \mathfrak{S}_2$$

such that $\{\tau_1^{(n)}\} \subset \mathfrak{S}_1$ converges in D_g^{rc} to an element:

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathfrak{S}_1 \cong F_1$$

in the cylindrical topology if and only if

$$\tau_1^{(n)} \xrightarrow[n\to\infty]{} \tau_1 \qquad \tau_3^{(n)} - Im\tau_2^{(n)} \left(Im\tau_1^{(n)} \right)^{-1} Im^t \tau_2^{(n)} \xrightarrow[n\to\infty]{} \infty \tag{B.9}$$

If the sequence $\{\tau_2^{(n)}\} \subset \mathbb{C}$ is bounded, by (B.9) the convergence is characterized by:

$$\tau_1^{(n)} \xrightarrow[n\to\infty]{} \tau_1 \quad Im\tau_3^{(n)} \xrightarrow[n\to\infty]{} \infty \tag{B.10}$$

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