Large Deviations for Stochastic Conservation Laws and Their Variational Counterparts

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Contents

CHAPTER 1

Large Deviations via Γ-convergence

1.1. Motivations

1.1.1. Interacting particles systems. The focus of this thesis is to provide an asymptotic analysis for some stochastic and variational models describing the evolution of "large" physical systems. The analysis of a system with a large number of degrees of freedom cannot in general be addressed by Newton-like differential models. Since the seminal works of Boltzmann in the XIX century, a statistical approach to the problem has been considered. While a mathematical framework to study systems at equilibrium has been provided in the last decades [**16**], a rigorous setting for the analysis of systems out of equilibrium is still missing. Several different approaches have been purposed, both stochastic and deterministic.

In the stochastic approach, a much used framework to understand Statistical Mechanics out of equilibrium is the one of *stochastic interacting particles systems* [**12, 13**]. Roughly speaking, a bunch of N particles jumping randomly on a countable lattice is considered. The dynamics is in general determined by assigning the jump rates of these particles. One is generally interested in analyzing the asymptotic behavior of the particles, when their number N diverges to infinity (hydrodynamical limit). This task is carried out by appropriately scaling the system, then identifying its characteristic quantities for each finite N, and finally evaluating the limit of these quantities as N diverges. A widely considered simplification consists in focusing on the density of the particles. In several models it has been shown that, in the hydrodynamical limit, the behavior of the density is deterministic, and it satisfies a differential evolution equation, usually called the *hydrodynamical equation*. By the way, the understanding of the properties of the system generally requires a much finer analysis of the involved quantities. In particular, it is well known that establishing large deviations principles for these quantities is a main step to provide a deeper insight into the system. However, while several techniques and results are available concerning large deviations of systems under the so called *diffusive scaling*, there is a little literature about large deviations for systems under *Euler scaling*, see [**11, 21, 3**] where some important partial results have been established for specific models.

1.1.2. Introducing the model. In this thesis we are concerned with a slightly different approach. Instead of dealing with discrete models, we focus on a continuous description of physical systems, and thus consider a "density" $u = u(t,x)$ as a real valued function, depending on the time variable t and the space variable x. Hopefully, a continuous model may inherit the key properties of the physical system, while allowing a more general investigation than a model-by-model based analysis.

In many situations, that is whenever the number of particles is conserved by the dynamics, it is natural to assume that the density u satisfies a continuity equation

$$\partial_t u + \operatorname{div}(J) = 0 \tag{1.1.1}$$

In several models, the *current* J takes into account the basic phenomena occurring in conservative physical systems: transport, diffusion, fluctuation, which in general appear as nonlinear terms in (1.1.1). We thus come with the crucial assumption

$$J = f(u) - \frac{1}{2} D(u)\operatorname{grad} u + \sigma(u)\alpha \tag{1.1.2}$$

where f represents the so called *flux* (related to transport phenomena), D is an elliptical matrix governing diffusion, and σ is a *fluctuation* matrix acting on the stochastic noise α.

Motivated by the stochastic particles systems setting, and in particular by the limiting behavior of systems under Euler scaling, we are interested in the asymptotic properties of the solution to (1.1.1) when the diffusion and fluctuation terms in (1.1.2) vanish simultaneously. Moreover, still motivated by particle systems and by quite general physical systems heuristics, we assume a natural hypotheses on the noise α. Namely, we suppose α to be *white* in time and to have a small (i.e. vanishing) correlation in space.

In a more precise mathematical framework, we come up the following Cauchy problem related to a stochastic partial differential equation, which has to be interpreted in the Itô sense [**8**]

$$\begin{aligned} du &= \Big[-\nabla\cdot f(u) + \tfrac{\varepsilon}{2}\nabla\cdot\big[D(u)\nabla u\big]\Big]\,dt + \varepsilon^\gamma \nabla\cdot\big[\sigma(u)\,dW^\varepsilon\big] \\ u(0,x) &= u_0(x) \end{aligned} \tag{1.1.3}$$

Here the parameter ε plays the role of the inverse number of particles, so that we are interested in the limit $\varepsilon \to 0$. For a given sequence of smooth mollifiers $\{\jmath^\varepsilon\}$ on $\mathbb{T}$ and a cylindrical Brownian motion W, the trace-class Brownian motion W^ε is defined as $W^\varepsilon := \jmath^\varepsilon * W$, where $*$ denotes convolution on $\mathbb{T}$. Moreover u_0 is a bounded measurable function on $\mathbb{T}$ and γ is a real parameter $\gamma > 1/2$, so that, as $\varepsilon \to 0$, the diffusion coefficient, the noise and the space-correlation of the noise itself vanish. The rate at which these quantities vanish depends on the (quite) arbitrary choices of γ and $\{\jmath^\varepsilon\}$. See Chapter 3 for a precise definition of (1.1.3) and the assumptions concerning f, D, σ and $\jmath^\varepsilon$.

1.1.3. The asymptotic $\varepsilon \to 0$. Existence and uniqueness results for (1.1.3) are established in the Appendix A of Chapter 3. There are quite a little results for fully nonlinear stochastic partial differential equations in the literature; in particular a problem similar to (1.1.3) is addressed in [**14, 15**] in a Hamilton-Jacobi context, although these papers deal with a finite dimensional noise. Let $\mathbb{P}^\varepsilon$ the law of the process u^ε satisfying (1.1.3). We will see in Chapter 3 that the sequence $\{\mathbb{P}^\varepsilon\}$ converges weakly (in a suitable topology) to the so called *entropic solution* (see Section 1.4 below) to the limiting equation obtained by informally setting $\varepsilon = 0$ in (1.1.3).

We are then left with the key issue of investigating large deviations principles for $\{\mathbb{P}^\varepsilon\}$. Note that u^ε is a diffusion Itô process in a infinite dimensional Banach space, with a drift term $-\nabla \cdot f(u) + \frac{\varepsilon}{2}\nabla \cdot \big[D(u)\nabla u\big]$ and a stochastic diffusion term $\varepsilon^\gamma \nabla \cdot \big[\sigma(u)\, dW^\varepsilon\big]$ which have a nontrivial behavior in the limit $\varepsilon \to 0$. We recall that, even in the finite dimensional case, large deviations techniques for Itô diffusions have been widely investigated in the "small noise" asymptotic. However, at our knowledge, there are no general results addressing the problem of large deviations for Itô diffusion processes with drift and diffusion coefficients depending arbitrarily on a parameter ε. In Section 1.3 we show that, even in the finite dimensional case, large deviations principles for diffusions processes are closely related to variational problems. Indeed, we first establish in Section 1.2.3 a general equivalence between large deviations principles for sequence of probability measures on a Polish space, and a so called Γ-*convergence problem* (see Section 1.2.2) for a corresponding sequence of "relative entropy" functionals. Then in Section 1.2.3, we show that a Γ-convergence result is also necessary to establish a large deviation principles for finite dimensional Itô diffusions. While the corresponding Γ-convergence problem is trivial in the classical "small noise" asymptotic, it can be a challenging issue in more general cases.

We are then left with the idea that, given a sequence of Itô diffusions, a Γ-convergence result should be investigated in order to understand the large deviations asymptotic. Roughly speaking, we may say that the main result of this thesis is to prove that this heuristic idea holds true not only in the finite dimensional case, but in the infinite dimensional case as well, at least as far as the laws of the solutions to (1.1.3) are concerned.

We thus address a Γ-convergence result related to (1.1.3) in Chapter 2. We study this problem in a slightly different setting than the one introduced in (1.1.3). This variational problem may have an independent interest by itself as, for instance, it allows a variational characterization of measure-valued and entropic, respectively viscosity, solutions to conservations laws, respectively Hamilton-Jacobi equations.

In Chapter 3 we then use the results of Chapter 2 to establish large deviations principles for $\{\mathbb{P}^\varepsilon\}$. The main difference w.r.t. the finite dimensional case is that the drift and diffusion coefficients involve derivatives w.r.t. the space variable x, and thus have no regularity properties in the natural topologies in which the convergence has to analyzed. Roughly speaking, this difficulty is solved by using the fact that the higher order part $\frac{\varepsilon}{2}\nabla \cdot \big[D(u)\nabla u\big]$ of the drift term has indeed a regularizing effect, which sharply compensates the "bad" noise effect. Once this is understood, everything cools down to investigate the stability of (1.1.3) w.r.t. small deterministic perturbations.

Before stating the main results of this thesis, we introduce some preliminary notions. In Section 1.2 we recall the main definitions concerning large deviations theory and Γ-convergence. We next state and prove two results connecting the two theories both from a theoretical and "operational" point of view. In particular we apply these results to investigate large deviations principles for finite dimensional Itô diffusions. In Section 1.4 we recall some basic statements concerning inviscous conservation laws; we also introduce a so called kinetic formulation for entropy-measures solutions to a conservation law, which is proved in Chapter 2. These results are used to link the main results stated in Section 1.5, and should help the understanding of the strategy of their proofs. There are a few minor results that are obtained as byproducts from the proofs of the Γ-convergence and large deviations principles. We briefly sketch some of them in Section 1.5.3

1.2. Large deviations theory and its variational counterpart

In this section we recall the basic definitions concerning large deviations and Γ-convergence theories, see [**10**] and [**4, 6**]. We next establish a connection between the two theories, showing that large deviations principles are equivalent to the Γ-convergence of relative entropies. Then we introduce some techniques to prove large deviations principles via Γ-convergence in a Markov processes framework. In particular, we prove a large deviations upper bound and lower bound for a wide class of finite-dimensional Itô diffusions.

Hereafter, for X a Polish space (that is a completely metrizable separable space), $\mathcal{P}(X)$ denotes the set of Borel probability measures on X, equipped with the vague topology. Recall that $\mathcal{P}(X)$ is a Polish space itself.

1.2.1. Large deviations. Let $\{a_\varepsilon\}$ be a sequence of positive reals such that $\lim_{\varepsilon\to 0} a_\varepsilon = 0$; let X be a Polish space and $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$ a sequence of Borel probability measures on X; let $I : X \to [0,+\infty]$ be a lower semicontinuous functional on X. The sequence $\{\mathbb{P}^\varepsilon\}$ is said to satisfy a *large deviations* *weak upper bound* with speed $\{a_\varepsilon^{-1}\}$ and rate I iff for each compact set $K \subset X$

$$\overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(K) \le - \inf_{v\in K} I(v) \tag{1.2.1}$$

$\{\mathbb{P}^\varepsilon\}$ satisfies a *large deviations (full) upper bound* with speed $\{a_\varepsilon^{-1}\}$ and rate I iff for each closed set $\mathcal{C} \subset X$

$$\overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(\mathcal{C}) \le - \inf_{v\in \mathcal{C}} I(v) \tag{1.2.2}$$

$\{\mathbb{P}^\varepsilon\}$ satisfies a *large deviations lower bound* iff for each open set $\mathcal{O} \subset X$

$$\underline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(\mathcal{O}) \ge - \inf_{v\in \mathcal{O}} I(v) \tag{1.2.3}$$

$\{\mathbb{P}^\varepsilon\}$ satisfies a *large deviation principle* iff an upper bound and a lower bound hold with the same speeds and rates. $\{\mathbb{P}^\varepsilon\}$ is called *exponentially tight* iff for each $\ell > 0$ there exist an $\varepsilon_0 > 0$ and compact $K \subset X$ such that $\mathbb{P}^\varepsilon(K_\ell^c) \le \exp\big(-\ell/a_\varepsilon\big)$ for each $\varepsilon \le \varepsilon_0$. Note that an exponentially tight family of probability measures satisfies a large deviations upper bound iff it satisfies a large deviations weak upper bound.

1.2.2. Γ-convergence. Let $\{I_\varepsilon\}$ be a sequence of functionals $I_\varepsilon : X \to [0,+\infty]$. We define two functionals $\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon$, $\Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon : X \to [0,+\infty]$ as

$$\begin{aligned}\big(\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon\big)(x) &:= \inf\Big\{\, \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(x^\varepsilon),\ x^\varepsilon \to x\Big\}\\ \big(\Gamma\text{–}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon\big)(x) &:= \inf\Big\{\, \overline{\lim}_{\varepsilon\to 0} F_\varepsilon(x^\varepsilon),\ x^\varepsilon \to x\Big\}\end{aligned}$$

Whenever $\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon(x) = \Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon(x) = I(x)$ we say that I_ε Γ-converges to I in x, and that Γ-convergence holds in X iff this equality holds true for all $x \in X$. The sequence $\{I_\varepsilon\}$ is called *equicoercive* iff for each N there exists an $\varepsilon_0 > 0$ and a compact $K \subset X$ such that $\cup_{\varepsilon\le\varepsilon_0}\{x \in X\,:\, I_\varepsilon(x) \le N\} \subset K$.

Note that $\Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon \ge \Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon$, and that these functionals are lower-semicontinuous [**6**]. We recall that, for $I : X \to [0,+\infty]$ a lower semicontinuous functional

- $\big(\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon\big)(x) \ge \underline{I}(x)$ iff for any sequence $x^\varepsilon \to x$ we have $\underline{\lim}_\varepsilon I_\varepsilon(x^\varepsilon) \ge \underline{I}(x)$ (Γ-*liminf inequality*);
- $\big(\Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon\big)(x) \ge \overline{I}(x)$ iff there exists a sequence $x^\varepsilon \to x$ such that $\overline{\lim}_\varepsilon I_\varepsilon(x^\varepsilon) \le \overline{I}(x)$ (Γ-*limsup inequality*).

Moreover for each compact set $K \subset X$ and each open set $\mathcal{O} \subset X$

$$\inf_{x\in K}(\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon)(x) \le \underline{\lim_{\varepsilon}} \inf_{x\in K} I_\varepsilon(x) \tag{1.2.4a}$$

$$\inf_{x\in \mathcal{O}}(\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon)(x) \ge \underline{\lim_{\varepsilon}} \inf_{x\in \mathcal{O}} I_\varepsilon(x) \tag{1.2.4b}$$

$$\inf_{x\in \mathcal{O}}(\Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon)(x) \ge \overline{\lim_{\varepsilon}} \inf_{x\in \mathcal{O}} I_\varepsilon(x) \tag{1.2.4c}$$

Note that if $\{I_\varepsilon\}$ is equicoercive, then the inequality (1.2.4a) also holds for closed sets $\mathcal{C} \subset X$.

1.2.3. Large deviations as Γ-convergence of relative entropies. We establish a preliminary lemma that will be used in the following. For X a Polish space, and φ a map $\varphi : X \to \bar{\mathbb{R}}$, we denote by φ^+, resp. φ^-, the positive, resp. negative, part of φ.

Lemma 1.2.1. *Let $\{\mathbb{Q}^\varepsilon\} \subset \mathcal{P}(X)$ and $\mathbb{Q} \in \mathcal{P}(x)$. The following are equivalent:*

(i) $\mathbb{Q}^\varepsilon \to \mathbb{Q}$ *in* $\mathcal{P}(X)$.

(ii) *For each sequence $\{\varphi_\varepsilon\}$ of Borel measurable functions $\varphi_\varepsilon : X \to \bar{\mathbb{R}}$ such that* $\lim_{M\to+\infty} \underline{\lim}_\varepsilon \mathbb{Q}^\varepsilon\big((\varphi_\varepsilon + M)^-\big) = 0$

$$\underline{\lim}_{\varepsilon} \mathbb{Q}^\varepsilon(\varphi_\varepsilon) \geq \mathbb{Q}(\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon)$$

where we understand $\mathbb{Q}(\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon) = +\infty$ *whenever* $\mathbb{Q}\big((\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon)^+\big) = +\infty$.

(iii) *For each sequence $\{\varphi_\varepsilon\}$ of Borel measurable functions $\varphi_\varepsilon : X \to \bar{\mathbb{R}}$ such that* $\lim_{M\to+\infty} \overline{\lim}_\varepsilon \mathbb{Q}^\varepsilon\big((\varphi_\varepsilon + M)^-\big) = 0$

$$\overline{\lim}_{\varepsilon} \mathbb{Q}^\varepsilon(\varphi_\varepsilon) \geq \mathbb{Q}(\Gamma\text{–}\overline{\lim}_\varepsilon \varphi_\varepsilon)$$

where we understand $\mathbb{Q}(\Gamma\text{–}\overline{\lim}_\varepsilon \varphi_\varepsilon) = +\infty$ *whenever* $\mathbb{Q}\big((\Gamma\text{–}\overline{\lim}_\varepsilon \varphi_\varepsilon)^+\big) = +\infty$.

Proof. The implication (ii) ⇒ (i) and (iii) ⇒ (i) are trivial. We next show (i) ⇒ (ii); the implication (i) ⇒ (iii) follows analogously. Let $\{K_\ell\}_{\ell=1}^\infty$ be an increasing sequence of compacts $K_\ell \subset X$ such that $\lim_\ell \mathbb{Q}(K_\ell) = 0$. It is not difficult to see that for each ℓ, $n \in \mathbb{N}$, there exists a finite family of pairwise disjoint Borel measurable sets $\{E^i_{n,\ell}\}_{i=1}^{N_{n,\ell}}$, such that $\cup_{i\geq 1} E^i_{n,\ell} \supset K_\ell$, and for $i = 1, \ldots, N_{n,\ell}$, $\mathbb{Q}(\partial E^i_{n,\ell}) = 0$, diameter$(E^i_{n,\ell}) \leq 1/n$. We also set $E^0_{n,\ell} := X \setminus \cup_{i\geq 1} E^i_{n,\ell}$. By a refinement procedure, we can assume the partition $\{E^i_{n,\ell}\}_{i=0}^{N_{n,\ell}}$ to be finer than $\{E^i_{n',\ell'}\}_{i=0}^{N_{n',\ell'}}$ for $n \geq n'$ and $\ell \geq \ell'$.

Let $\{\varphi_\varepsilon\}$ be as in the statement of the lemma. For $\varepsilon > 0$ and n, $\ell \in \mathbb{N}$, define $\varphi_{\varepsilon;n,\ell} : X \to \bar{\mathbb{R}}$ by $\varphi_{\varepsilon;n,\ell}(x) = \inf_{y\in E^i_{n,\ell}} \varphi_\varepsilon(y)$ for $x \in E^i_{n,\ell}$, and $\varphi_{n,\ell}(x) := \underline{\lim}_\varepsilon \varphi_{\varepsilon;n,\ell}(x)$. Note that $\varphi_{n,\ell}(x)$ increases pointwise to $\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon(x)$ as $n \to +\infty$

and $\ell \to +\infty$, so that for each $M > 0$

$$\begin{aligned}\mathbb{Q}\big((\Gamma\text{-}\underline{\lim}_\varepsilon\, \varphi_\varepsilon) \vee (-M)\big) &= \mathbb{Q}\big(\,\Gamma\text{-}\underline{\lim}_\varepsilon (\varphi_\varepsilon \vee (-M))\big)\\ &= \lim_\ell \lim_n \mathbb{Q}\big(\varphi_{n,\ell} \vee (-M)\big)\\ &= \lim_\ell \lim_n \sum_{i=0}^{N_{n,\ell}} \Big[\mathbb{Q}(E^i_{n,\ell})\ \underline{\lim}_\varepsilon \inf_{x\in E^i_{n,\ell}} \big(\varphi_{\varepsilon;n,\ell}(x) \vee (-M)\big)\Big]\\ = \lim_\ell \lim_n \underline{\lim}_\varepsilon \sum_{i=0}^{N_{n,\ell}} \mathbb{Q}^\varepsilon(E^i_{n,\ell}) &\inf_{x\in E^i_{n,\ell}} \varphi_{\varepsilon;n,\ell}(x) \vee (-M) \le \underline{\lim}_\varepsilon \mathbb{Q}^\varepsilon\big(\varphi_\varepsilon \vee (-M)\big)\end{aligned}$$

where in the first equality of the last line we used the required $\mathbb{Q}$-regularity of the sets $E^i_{n,\ell}$. The statement then follows by taking the limit $M \to +\infty$ on both sides. □

Recall that, given $\mathbb{P}$, $\mathbb{Q} \in \mathcal{P}(X)$, the *relative entropy* $H(\mathbb{Q}|\mathbb{P})$ of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is defined as

$$H(\mathbb{Q}|\mathbb{P}) := \sup\big\{\mathbb{Q}(\varphi) - \log\big(\mathbb{P}\big(\exp(\varphi)\big), \varphi \in C_{\mathrm{b}}(X)\big\} \tag{1.2.5}$$

As well known, the relative entropy admits an explicit representation in terms of the Radon-Nykodim derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$

$$H(\mathbb{Q}|\mathbb{P}) = \begin{cases} \int_X \mathbb{Q}(dx)\log\big(\frac{d\mathbb{Q}}{d\mathbb{P}}(x)\big) & \text{if } \mathbb{Q} \text{ is absolutely continuous w.r.t. } \mathbb{P} \\ & \text{and } \log(\frac{d\mathbb{Q}}{d\mathbb{P}}) \in L_1(d\mathbb{Q}) \\ +\infty & \text{otherwise} \end{cases} \tag{1.2.6}$$

We also recall a basic inequality involving the relative entropy. For each measurable set $A \subset X$

$$\mathbb{Q}(A) \le \frac{H(\mathbb{Q}|\mathbb{P}) + \log 2}{\log(1 + \mathbb{P}(A)^{-1})} \tag{1.2.7}$$

Given $\{a_\varepsilon\} \subset \mathbb{R}^+$ and $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$, let us introduce the sequence $\{H_\varepsilon\}$ of functionals $H_\varepsilon : \mathcal{P}(X) \to [0, +\infty]$ by defining

$$H_\varepsilon(\mathbb{Q}) := a_\varepsilon H(\mathbb{Q}|\mathbb{P}^\varepsilon) \tag{1.2.8}$$

Note that H_ε is convex, since $H(\mathbb{Q}|\mathbb{P})$ is a supremum of linear functionals in $\mathbb{Q}$.

It is well known that large deviations principles are deeply connected with convergence properties of relative entropies, see [**9, 11**]. The following propositions show that the Γ-convergence provides a suitable framework to exploit this connection.

In the remaining of this section, for $\mathbb{Q} \in \mathcal{P}(X)$ and E a Borel set such that $\mathbb{Q}(E) > 0$, we denote by $\mathbb{Q}_E \in \mathcal{P}(X)$ the probability measure defined by $\mathbb{Q}_E(E') := \mathbb{Q}(E \cap E')/\mathbb{Q}(E)$, for each Borel set E'.

PROPOSITION 1.2.2. *Let* $\{a_\varepsilon\} \subset \mathbb{R}^+$ *be such that* $\lim_\varepsilon a_\varepsilon = 0$, *and let* $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$. *The following are equivalent:*

$(A1)$ $\{\mathbb{P}^\varepsilon\}$ *is exponentially tight w.r.t.* $\{a_\varepsilon\}$.
$(A2)$ $\{H_\varepsilon\}$ *is equicoercive.*

PROPOSITION 1.2.3. *Let* $\{a_\varepsilon\} \subset \mathbb{R}^+$ *be such that* $\lim_\varepsilon a_\varepsilon = 0$, *and let* $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$. *Let furthermore* $I : X \to [0, +\infty]$ *be a lower semicontinuous functional. The following are equivalent:*

$(B1)$ $\{\mathbb{P}^\varepsilon\}$ *satisfies a large deviations weak upper bound with speed* $\{a_\varepsilon^{-1}\}$ *and rate* I.
$(B2)$ *For each* $x \in X$, $\big(\Gamma\text{–}\underline{\lim}_\varepsilon H_\varepsilon\big)(\delta_x) \geq I(x)$, *where* $\delta_x \in \mathcal{P}(X)$ *is the Dirac mass concentrated at* x.
$(B3)$ *For each* $\mathbb{Q} \in \mathcal{P}(X)$, $\big(\Gamma\text{–}\underline{\lim}_\varepsilon H_\varepsilon\big)(\mathbb{Q}) \geq \mathbb{Q}(I)$.
$(B4)$ *For each sequence* $\{\varphi_\varepsilon\}$ *of measurable maps* $\varphi_\varepsilon : X \to \overline{\mathbb{R}}$, *such that*
 (i) $\lim_{M\to+\infty} \overline{\lim}_\varepsilon \mathbb{P}^\varepsilon\big(\exp[(\varphi_\varepsilon + M)^-/a_\varepsilon]\big) = 0$.
 (ii) *There exists an increasing sequence* $\{K_\ell\}$ *of compact subsets of* X *such that* $\lim_\ell \overline{\lim}_\varepsilon \int_{K_\ell} \mathbb{P}^\varepsilon(dx)\, e^{\varphi_\varepsilon(x)/a_\varepsilon} = 0$.

the following inequality holds

$$\overline{\lim_\varepsilon}\, a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(-\varphi_\varepsilon/a_\varepsilon)\big) \leq \sup_{x\in X} \big\{ -\big(\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon\big)(x) - I(x)\big\}$$

provided we read $-\big(\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon\big)(x) - I(x) := -\infty$ *whenever* $I(x) = +\infty$.

Assume furthermore that $\{\mathbb{P}^\varepsilon\}$ *satisfies the equivalent conditions* $(A1) - (A2)$ *of Proposition 1.2.2. Then* $(B1) - (B4)$ *are also equivalent to*

$(B5)$ $\{\mathbb{P}^\varepsilon\}$ *satisfies a large deviations upper bound with speed* $\{a_\varepsilon^{-1}\}$ *and rate* I.
$(B6)$ *For each sequence* $\{\varphi_\varepsilon\}$ *of measurable maps* $\varphi_\varepsilon : X \to \overline{\mathbb{R}}$, *such that condition (i) in* $(B4)$ *holds:*

$$\overline{\lim_\varepsilon}\, a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(-\varphi_\varepsilon/a_\varepsilon)\big) \leq \sup_{x\in X} \big\{ -\big(\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon\big)(x) - I(x)\big\}$$

provided we read $-\big(\Gamma\text{–}\underline{\lim}_\varepsilon \varphi_\varepsilon\big)(x) - I(x) := -\infty$ *whenever* $I(x) = +\infty$.

PROPOSITION 1.2.4. *Let* $\{a_\varepsilon\} \subset \mathbb{R}^+$ *be such that* $\lim_\varepsilon a_\varepsilon = 0$, *and let* $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$. *Let furthermore* $I : X \to [0, +\infty]$ *be a lower semicontinuous functional. The following are equivalent:*

$(C1)$ $\{\mathbb{P}^\varepsilon\}$ *satisfies a large deviations lower bound with speed* $\{a_\varepsilon^{-1}\}$ *and rate* I.
$(C2)$ *For each* $x \in X$, $\big(\Gamma\text{–}\overline{\lim}_\varepsilon H_\varepsilon\big)(\delta_x) \leq I(x)$.
$(C3)$ *For each* $\mathbb{Q} \in \mathcal{P}(X)$, $\big(\Gamma\text{–}\overline{\lim}_\varepsilon H_\varepsilon\big)(\mathbb{Q}) \leq \mathbb{Q}(I)$.

(*C*4) *For each sequence* $\{\varphi_\varepsilon\}$ *of measurable maps* $\varphi_\varepsilon : X \to \overline{\mathbb{R}}$

$$\varliminf_{\varepsilon} a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(\varphi_\varepsilon/a_\varepsilon)\big) \geq \sup_{x\in X} \big\{ \big(\Gamma\text{–}\varliminf_\varepsilon \varphi_\varepsilon\big)(x) - I(x) \big\}$$

provided we read $\big(\Gamma\text{–}\varliminf_\varepsilon \varphi_\varepsilon\big)(x) - I(x) = -\infty$ *whenever* $I(x) = +\infty$.

Proof of Proposition 1.2.2. (*A*1) ⇒ (*A*2). For $\ell > 0$, let K_ℓ be a compact subset of X such that $\mathbb{P}^\varepsilon(K_\ell^c) \leq e^{-\ell/a_\varepsilon}$. By (1.2.7), for each $\mathbb{Q} \in \mathcal{P}(X)$

$$\mathbb{Q}(K_\ell^c) \leq \frac{a_\varepsilon H(\mathbb{Q}|\mathbb{P}^\varepsilon) + a_\varepsilon \log 2}{a_\varepsilon \log\big(1 + \mathbb{P}^\varepsilon(K_\ell^c)^{-1}\big)} \leq \frac{H_\varepsilon(\mathbb{Q}) + a_\varepsilon \log 2}{\ell}$$

Let ε_0 be such that $a_\varepsilon \leq 1$ for $\varepsilon \leq \varepsilon_0$. For each $N > 0$ and $\varepsilon \leq \varepsilon_0$ we get

$$\cup_{\varepsilon\leq\varepsilon_0}\big\{\mathbb{Q} \in \mathcal{P}(X) \,:\, H_\varepsilon(\mathbb{Q}) \leq N\big\} \subset \big\{\mathbb{Q} \in \mathcal{P}(X) \,:\, \forall \ell > 0,\, \mathbb{Q}(K_\ell^c) \leq \frac{N + \log 2}{\ell}\big\}$$

which is a tight set, and thus precompact in $\mathcal{P}(X)$.

(*A*2) ⇒ (*A*1). By (1.2.6), for $\mathbb{Q} \in \mathcal{P}(X)$ and a Borel set E such that $\mathbb{Q}(E) > 0$, $H(\mathbb{Q}_E|\mathbb{Q}) = -\log\big(\mathbb{Q}(E)\big)$. Therefore for each ε_0, $\ell > 0$

$$\begin{array}{rcl} \mathcal{Q}_{\varepsilon_0,\ell} & := & \cup_{\varepsilon\leq\varepsilon_0}\big\{\mathbb{P}^\varepsilon_{K^c},\, K \subset X \text{ is compact and } \mathbb{P}^\varepsilon(K^c) \geq e^{-\ell/a_\varepsilon}\big\} \\ & \subset & \cup_{\varepsilon\leq\varepsilon_0}\{\mathbb{Q} \in \mathcal{P}(X) \,:\, H_\varepsilon(\mathbb{Q}) \leq \ell\} \end{array}$$

By the equicoercivity assumption on H_ε, for each $\ell > 0$ there exists $\varepsilon_0(\ell)$ such that $\mathcal{Q}_{\varepsilon_0(\ell),\ell}$ is precompact in $\mathcal{P}(X)$, and thus tight. Therefore for each $\ell > 0$ there exists a compact set $K_\ell \subset X$ such that $\mathbb{P}^\varepsilon_{K^c}({K_\ell}^c) \leq 1/2$ for each $\varepsilon \leq \varepsilon_0(\ell)$ and each compact K such that $\mathbb{P}^\varepsilon(K^c) \geq e^{-\ell/a_\varepsilon}$. Since $\mathbb{P}^\varepsilon_{K^c}(K^c) = 1$ for each K with $\mathbb{P}^\varepsilon(K^c) > 0$, we necessarily have $K_\ell \neq K$ for each K such that $\mathbb{P}^\varepsilon(K^c) \geq e^{-\ell/a_\varepsilon}$ for some $\varepsilon \leq \varepsilon_0(\ell)$. Namely $\mathbb{P}^\varepsilon(K_\ell^c) \leq e^{-\ell/a_\varepsilon}$ for each $\ell > 0$ and $\varepsilon \leq \varepsilon_0(\ell)$. □

Proof of Proposition 1.2.3. (*B*1) ⇒ (*B*2). Let $x \in X$ and $\{\mathbb{Q}^\varepsilon\} \subset \mathcal{P}(X)$ be such that $\lim_{\varepsilon\to 0} \mathbb{Q}^\varepsilon = \delta_x$ in $\mathcal{P}(X)$. Multiplying (1.2.7) by a_ε, for each Borel set $E \subset X$

$$H_\varepsilon(\mathbb{Q}^\varepsilon) \geq a_\varepsilon \mathbb{Q}^\varepsilon(E) \log\Big(1 + \frac{1}{\mathbb{P}^\varepsilon(E)}\Big) - a_\varepsilon \log 2 \tag{1.2.9}$$

Let $\delta > 0$, and $\bar{B}_\delta(x)$ be the closed ball of radius δ centered at x. Since $\lim_\varepsilon \mathbb{Q}^\varepsilon(\bar{B}_\delta(x)) = 1$, taking $E = \bar{B}_\delta(x)$ in (1.2.9) and passing to the liminf

$$\begin{array}{rcl} \varliminf_\varepsilon H_\varepsilon(\mathbb{Q}^\varepsilon) & \geq & \varliminf_\varepsilon a_\varepsilon \log\big(1 + \frac{1}{\mathbb{P}^\varepsilon(\bar{B}_\delta(x))}\big) \\ & \geq & -\varlimsup_\varepsilon \log\big(\mathbb{P}^\varepsilon(\bar{B}_\delta(x)\big) \geq \inf_{y\in\bar{B}_\delta(x)} I(y) \end{array}$$

where we used the (*B*1) hypotheses in the last inequality. Taking the limit $\delta \to 0$, and recalling that I is lower-semicontinuous, we get (*B*2).

(*B*2) ⇒ (*B*3). Let $\mathbb{Q} \in \mathcal{P}(X)$ and let $\{K_\ell\}_{\ell=1}^\infty$ be an increasing sequence of compacts $K_\ell \subset X$ such that $\lim_\ell \mathbb{Q}(K_\ell) = 0$. It is not difficult to see that for each ℓ, $n \in \mathbb{N}$, there exists a finite family of pairwise disjoint Borel measurable sets $\{E^i_{n,\ell}\}_{i=1}^{N_{n,\ell}}$, such that $\cup_{i\geq 1} E^i_{n,\ell} \supset K_\ell$, and for $i = 1, \dots, N_{n,\ell}$, $\mathbb{Q}(\partial E^i_{n,\ell}) = 0$, diameter$(E^i_{n,\ell}) \leq 1/n$. We also set $E^0_{n,\ell} := X \setminus \cup_{i\geq 1} E^i_{n,\ell}$. By a refinement procedure, we can assume the partition $\{E^i_{n,\ell}\}_{i=0}^{N_{n,\ell}}$ to be finer than $\{E^i_{n',\ell'}\}_{i=0}^{N_{n',\ell'}}$ for $n \geq n'$ and $\ell \geq \ell'$.

Let $\{\mathbb{Q}^\varepsilon\}$ be a sequence converging to $\mathbb{Q}$ in $\mathcal{P}(X)$. We want to show $\underline{\lim}\, H_\varepsilon(\mathbb{Q}^\varepsilon) \geq \int \mathbb{Q}(dx) I(x)$. Let $\varepsilon > 0$, for $i = 0, \dots, N_{n,\ell}$ such that $\mathbb{Q}^\varepsilon(E^i_{n,\ell}) > 0$ define the probability measures $\mathbb{Q}^{\varepsilon;i}_{n,\ell} := \mathbb{Q}^\varepsilon_{E^i_{n,\ell}} \in \mathcal{P}(X)$. We have $\mathbb{Q}^\varepsilon = \sum_{i=0}^{N_{n,l}} \mathbb{Q}^\varepsilon(E^i_{n,\ell})\mathbb{Q}^{\varepsilon;i}_{n,\ell}$, where we understand that the terms in this sum vanish whenever $\mathbb{Q}^\varepsilon(E^i_{n,\ell}) = 0$. By (1.2.5), for each n, $\ell > 0$

$$\begin{aligned} H(\mathbb{Q}^\varepsilon|\mathbb{P}^\varepsilon) &= \sum_{i=0}^{N_{n,\ell}} \mathbb{Q}^\varepsilon(E^i_{n,\ell}) H(\mathbb{Q}^{\varepsilon;i}_{n,\ell}|\mathbb{P}^\varepsilon) + \mathbb{Q}^\varepsilon(E^i_{n,\ell}) \log \mathbb{Q}^\varepsilon(E^i_{n,\ell}) \\ &\geq -\log N_{n,\ell} + \sum_{i=0}^{N_{n,\ell}} \mathbb{Q}^\varepsilon(E^i_{n,\ell}) H(\mathbb{Q}^{\varepsilon;i}_{n,\ell}|\mathbb{P}^\varepsilon) \end{aligned} \tag{1.2.10}$$

where we meant $0 \log 0 \equiv 0$. Multiplying by a_ε and taking the liminf

$$\begin{aligned} \underline{\lim}_\varepsilon H_\varepsilon(\mathbb{Q}^\varepsilon) &\geq \sum_{i=0}^{N_{n,\ell}} \underline{\lim}_\varepsilon \left[\mathbb{Q}^\varepsilon(E^i_{n,\ell}) H_\varepsilon(\mathbb{Q}^{\varepsilon;i}_{n,\ell})\right] \\ &= \sum_{i=0}^{N_{n,\ell}} \mathbb{Q}(E^i_{n,\ell})\, \underline{\lim}_\varepsilon H_\varepsilon(\mathbb{Q}^{\varepsilon;i}_{n,\ell}) = \int \mathbb{Q}(dx)\, I_{n,\ell}(x) \end{aligned}$$

where we used the $\mathbb{Q}$-regularity of the sets $E^i_{n,\ell}$, and $I_{n,\ell}(x) := \underline{\lim}_\varepsilon a_\varepsilon H(\mathbb{Q}^{\varepsilon;i}_{n,\ell})$ for $x \in E^i_{n,\ell}$. Note that $I_{n,\ell}$ is increasing both in n and ℓ, since H_ε is convex and we assumed the partitions $\{E^i_{n,\ell}\}$ to be increasing. Therefore, by monotone convergence $\underline{\lim}_\varepsilon H_\varepsilon(\mathbb{Q}^\varepsilon) \geq \int \mathbb{Q}(dx) \sup_{n,\ell} I_{n,\ell}(x)$. On the other hand by $(B2)$ $\lim_\ell \lim_n I_{n,\ell}(x) \geq I(x)$ pointwise.

$(B3) \Rightarrow (B4)$. We prove that statement for a sequence $\{\varphi_\varepsilon\}$ of functions uniformly bounded from below. The general case is then easily obtained by the requirement (i). Consider the sequence $\{\mathbb{Q}^\varepsilon\} \subset \mathcal{P}(X)$ of probability measures defined as

$$\mathbb{Q}^\varepsilon(dx) := \frac{\exp(-\varphi_\varepsilon/a_\varepsilon)}{\mathbb{P}^\varepsilon\big(\exp(-\varphi_\varepsilon/a_\varepsilon)\big)} \mathbb{P}^\varepsilon(dx)$$

By (1.2.6)

$$a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(-\varphi_\varepsilon/a_\varepsilon)\big) = \mathbb{Q}^\varepsilon\big(-\varphi_\varepsilon\big) - H_\varepsilon(\mathbb{Q}^\varepsilon)$$

By requirement (ii), $\{\mathbb{Q}^\varepsilon\}$ is tight and thus precompact in $\mathcal{P}(X)$. Let $\mathbb{Q}$ be an arbitrary limit point of $\{\mathbb{Q}^\varepsilon\}$; taking the limsup, using Lemma 1.2.1 and $(B3)$

$$\begin{aligned} \overline{\lim}_\varepsilon a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(-\varphi_\varepsilon/a_\varepsilon)\big) &\leq -\underline{\lim}_\varepsilon \mathbb{Q}^\varepsilon\big(\varphi_\varepsilon\big) - \underline{\lim}_\varepsilon H_\varepsilon(\mathbb{Q}^\varepsilon) \\ &\leq -\mathbb{Q}(\Gamma\text{-}\underline{\lim}_\varepsilon \varphi_\varepsilon) - \mathbb{Q}(I) \leq \sup_{x\in X} \big\{ -(\Gamma\text{-}\underline{\lim}_\varepsilon \varphi_\varepsilon)(x) - I(x)\big\} \end{aligned}$$

$(B4) \Rightarrow (B1)$. Let K be a compact in X, and for ε, $M > 0$ consider the statement $(B4)$ for $\varphi_\varepsilon \equiv M 1\!\!1_{K^c}$. $\{\varphi_\varepsilon\}$ is lower semicontinuous and satisfies (i) and (ii) in $(B4)$. Therefore

$$\overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(K) \le \overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(-M\mathbb{1}_{K^c}/a_\varepsilon)\big) \le \sup_{x\in X}\{-M\mathbb{1}_{K^c}(x) - I(x)\}$$

and we get $(B1)$ letting $M \to +\infty$.

The implications $(B5) \Rightarrow (B1)$, $(B6) \Rightarrow (B5)$, and $\{(A1), (B1)\} \Rightarrow (B5)$ are trivial. On the other hand, once $(A1)$ is assumed, the implication $(B4) \Rightarrow (B6)$ follows from a standard cut-off argument. □

Proof of Proposition 1.2.4. $(C1) \Rightarrow (C2)$. Let $x \in X$, and for $\delta > 0$ let $B_\delta(x)$ the open ball of radius δ centered at x. For ε, $\delta > 0$, define $\mathbb{Q}^{\varepsilon,\delta} \in \mathcal{P}(X)$ as

$$\mathbb{Q}^{\varepsilon,\delta} := \begin{cases} \mathbb{P}^\varepsilon_{B_\delta(x)} & \text{if } \mathbb{P}^\varepsilon\big(B_\delta(x)\big) > 0 \\ \delta_x & \text{otherwise} \end{cases}$$

and note $H(\mathbb{Q}|\mathbb{P}) = -\log \mathbb{P}^\varepsilon\big(B_\delta(x)\big)$, where we understand $-\log(0) = +\infty$. By $(C1)$ we thus get for each $\delta > 0$

$$\overline{\lim_{\varepsilon}}\, H_\varepsilon(\mathbb{Q}^{\varepsilon,\delta_\varepsilon}) = -\underline{\lim_{\varepsilon}} \log \mathbb{P}^\varepsilon\big(B_\delta(x)\big) \le \inf_{y\in B_\delta(x)} I(y) \le I(x)$$

On the other hand $\lim_\delta \lim_\varepsilon \mathbb{Q}^{\varepsilon,\delta} = \delta_x$ in $\mathcal{P}(X)$, so that we there exists a sequence $\{\delta_\varepsilon\} \subset (0,1)$ such that $\lim_\varepsilon \mathbb{Q}^{\varepsilon,\delta_\varepsilon} = \delta_x$ and $\overline{\lim}_\varepsilon H_\varepsilon(\mathbb{Q}^{\varepsilon,\delta_\varepsilon}) \le I(x)$.

$(C2) \Rightarrow (C3)$. By the convexity of H_ε, $\Gamma\text{–}\overline{\lim}\, H_\varepsilon$ is also convex, and by $(C2)$ we have $(\Gamma\text{–}\overline{\lim}\, H_\varepsilon)(\delta_x) \le \int \delta_x(dy)\, I(y)$. $(C3)$ follows by convexification.

$(C3) \Rightarrow (C4)$. Let $Y := \big\{x \in X \,:\, (\Gamma\text{–}\underline{\lim}_\varepsilon\, \varphi_\varepsilon)(x) > -\infty\big\}$. By the definition of the Γ-liminf, for each $x \in Y$ there exists $\varepsilon_0(x)$ and $\delta(x) > 0$ such that $\inf_{y\in B_{\delta(x)}(x)} \varphi_\varepsilon(y) > -\infty$, for each $\varepsilon \le \varepsilon_0(x)$. For $x \in Y$, let $\{\mathbb{Q}^{\varepsilon;x}\}$ be a sequence converging to δ_x in $\mathcal{P}(X)$ and such that $\overline{\lim}\, H_\varepsilon(\mathbb{Q}^{\varepsilon,x}) \le I(x)$. Such a sequence exists by $(C3)$. Note that by (1.2.10) the sequence $\mathbb{Q}^{\varepsilon,x}_{B_{\delta(x)}(x)}$ enjoys these properties as well, so that we can assume $\mathbb{Q}^{\varepsilon,x}$ to be concentrated on $B_{\delta(x)}(x)$. By the definition (1.2.5), for each $\varphi \in C_{\mathrm{b}}(X)$

$$\log \mathbb{P}^\varepsilon(e^\varphi) \ge -H(\mathbb{Q}^{\varepsilon,x}|\mathbb{P}) + \mathbb{Q}^{\varepsilon,x}(\varphi) \tag{1.2.11}$$

By a limiting argument, this inequality holds true for each measurable $\varphi : X \to [-\infty, +\infty]$, provided we read the r.h.s. as $-\infty$ whenever $H(\mathbb{Q}^{\varepsilon,x}|\mathbb{P}) = +\infty$ or $\mathbb{Q}^{\varepsilon,x}(\varphi^-) = +\infty$. Evaluating (1.2.11) for $\varphi = \varphi^\varepsilon/a_\varepsilon$, taking the liminf and optimizing on $x \in Y$ we thus obtain

$$\underline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(\varphi^\varepsilon/a_\varepsilon)\big) \ge \sup_{x\in Y} \Big\{ -\overline{\lim_{\varepsilon}}\, H_\varepsilon(\mathbb{Q}^{\varepsilon,x}) + \underline{\lim_{\varepsilon}}\, \mathbb{Q}^{\varepsilon,x}(\varphi^\varepsilon) \Big\}$$

Since we assumed $\mathbb{Q}^{\varepsilon,x}$ concentrated on $B_{\delta(x)}(x)$, and since φ^ε is bounded from below on this ball, we can apply Lemma 1.2.1, so that

$$\begin{aligned}\underline{\lim}_\varepsilon\, a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(\varphi^\varepsilon/a_\varepsilon)\big) &\geq \sup_{x\in Y}\big\{-I(x) + \Gamma\text{–}\underline{\lim}_\varepsilon\, \varphi^\varepsilon)(x)\big\}\\ &= \sup_{x\in X}\big\{-I(x) + (\Gamma\text{–}\underline{\lim}_\varepsilon\, \varphi^\varepsilon)(x)\big\}\end{aligned}$$

$(C4) \Rightarrow (C1)$. For a Borel set $\mathcal{O} \subset X$ and $\varepsilon > 0$

$$\begin{aligned}a_\varepsilon \log \mathbb{P}^\varepsilon(\mathcal{O}) &= a_\varepsilon \log\Big[\tfrac{\mathbb{P}^\varepsilon(\mathcal{O})}{1+(e^{1/a_\varepsilon}-1)\mathbb{P}^\varepsilon(\mathcal{O})}\Big] + a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(\tfrac{1\!\!1_\mathcal{O}}{a_\varepsilon})\big)\\ &\geq -1 + a_\varepsilon \log \mathbb{P}^\varepsilon\big(\exp(\tfrac{1\!\!1_\mathcal{O}}{a_\varepsilon})\big)\end{aligned} \tag{1.2.12}$$

Take now $\mathcal{O}$ an open set, and consider the statement $(C4)$ with $\varphi^\varepsilon \equiv 1\!\!1_\mathcal{O}$. Since $1\!\!1_\mathcal{O}$ is lower semicontinuous, $\Gamma\text{–}\lim \varphi^\varepsilon = 1\!\!1_\mathcal{O}$, so that taking the liminf (1.2.12)

$$\underline{\lim_\varepsilon}\, a_\varepsilon \log \mathbb{P}^\varepsilon(\mathcal{O}) \geq -1 + \sup_{x\in X}\big(1\!\!1_\mathcal{O} - I(x)\big) \geq -\inf_{x\in\mathcal{O}} I(x)$$

□

As a byproduct, we get that, if $\{\mathbb{P}^\varepsilon\}$ is exponentially tight, there exist two “optimal” rate functionals for the large deviations upper and lower bounds, and they are given by $(\Gamma\text{–}\underline{\lim}_\varepsilon\, H_\varepsilon)(\delta.)$ and $(\Gamma\text{–}\overline{\lim}_\varepsilon\, H_\varepsilon)(\delta.)$ respectively. In particular, by well known compactness properties of Γ-convergence [**6**], we also have that, given a sequence $\{a_\varepsilon\}$ as above and an exponentially tight family $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$, there exists a subsequence $\{\mathbb{P}^{\varepsilon_k}\}$ satisfying a large deviations principle with speed $\{a_{\varepsilon_k}^{-1}\}$.

1.2.4. A large deviations bound for Feller processes. Beyond suggesting a general framework to fit large deviations theory in Γ-convergence theory, Propositions 1.2.2, 1.2.3, 1.2.4 do not provide any concrete tool to understand large deviations principles for a given family $\{\mathbb{P}^\varepsilon\}$ of probability measures. We next establish a more operative connection between large deviations and Γ-convergence in the setting of Feller processes, by proving a general large deviations upper bound for an arbitrary sequence of Feller processes. Although a more general treatment in the setting of cadlag Feller processes is possible, we restrict to the case of continuous processes. We refer to [**19**] for basic definitions concerning Markov generators and Feller processes.

Let X be a Polish space, let $T > 0$, and for L a Markov pregenerator on $C(X)$, let $D(L) \subset C_\mathrm{b}(X)$ be the domain of L, $\mathfrak{D}_L \subset C_\mathrm{b}([0,T]\times X)$ be the domain of $\partial_t + L$ and

$$\mathfrak{D}_{L,2} := \big\{\phi \in C_\mathrm{b}([0,T]\times X) \,:\, \phi \in \mathfrak{D}_L,\ \phi^2(t,\cdot) \in D(L) \text{ for each } t \in [0,T]\big\}$$

For $u_0 \in X$, we introduce the functional $I_{L,u_0} : C([0,T];X) \times \mathfrak{D}_{L,2} \to [-\infty,+\infty]$ as

$$I_{L,u_0}(u;\phi) := \begin{cases} \phi(T,u(T)) - \phi(0,u_0) - \int dt \Big[\big((\partial_t + L)\phi \big)(t,u(t)) \\ \quad -\frac{1}{2}\big(L(\phi^2)\big)(t,u(t)) + \phi(t,u(t))\,(L\phi)(t,u(t)) \Big] & \text{if } u(0)=u_0 \\ +\infty & \text{otherwise} \end{cases}$$

The integral in the r.h.s. of this formula is well defined, since for $\phi \in \mathfrak{D}_{L,2}$ and $t \in [0,T]$, $\big(L(\phi^2)\big)(t,u(t)) - 2\phi(t,u(t))\,(L\phi)(t,u(t)) \geq 0$.

Proposition 1.2.5. *Let X be a Polish space, let $T > 0$, and $\{L_\varepsilon\}$ be a sequence of Markov pregenerators on $C(X)$. For a given $u_0 \in X$, and for each $\varepsilon > 0$ suppose that there exists a solution $\mathbb{P}^\varepsilon \in \mathcal{P}\big(C([0,T];X)\big)$ to the martingale problem for L_ε with initial datum u_0. For $\Phi = \{\phi_\varepsilon\} \in \Pi_\varepsilon \mathfrak{D}_{L_\varepsilon,2}$, namely for a sequence $\Phi = \{\phi_\varepsilon\}$ such that $\phi_\varepsilon \in \mathfrak{D}_{L_\varepsilon,2}$, define the functional $I_{\varepsilon,u_0:\Phi} : C([0,T];X) \to (-\infty,+\infty]$ as*

$$I_{\varepsilon,u_0;\Phi}(u) := \varepsilon\, I_{L_\varepsilon,u_0}(u;\phi_\varepsilon). \tag{1.2.13}$$

and $I_{u_0} : C([0,T];X) \to [0,+\infty]$ as

$$I_{u_0}(u) := \sup_{\Phi \in \Pi_\varepsilon \mathfrak{D}_{L_\varepsilon,2}} (\Gamma\text{-}\underline{\lim}_\varepsilon\, I_{\varepsilon,u_0;\Phi})(u) \tag{1.2.14}$$

Then $\{\mathbb{P}^\varepsilon\}$ satisfies a weak large deviations upper bound with rate I_{u_0}, in the uniform topology of $C([0,T];X)$.

Proof. For each $\varepsilon > 0$ and $\phi \in \mathfrak{D}_{L_\varepsilon,2}$, the map

$$\begin{aligned} M_\phi \quad &: \quad [0,T] \times C\big([0,T];X\big) \to \mathbb{R} \\ M_\phi(t,u) \quad &:= \quad \phi(t,u(t)) - \phi(0,u(0)) - \int_{[0,t]} ds\, \big((\partial_s + L)\phi\big)(s,u(s)) \end{aligned}$$

is a continuous $\mathbb{P}^\varepsilon$-martingale with quadratic variation

$$\big[M_\phi(\cdot,u), M_\phi(\cdot,u)\big]_t = \int_{[0,t]} ds\, \big(L(\phi^2)\big)(s,u(s)) - 2\phi(s,u(s))\,(L\phi)(s,u(s))$$

Therefore its stochastic exponential

$$\begin{aligned} E_\phi \quad &: \quad [0,T] \times C\big([0,T];X\big) \to (0,+\infty) \\ E_\phi(t,u) \quad &:= \quad \exp\Big\{ \phi(t,u(t)) - \phi(0,u(0)) - \int_{[0,t]} ds \Big[\big((\partial_s + L)\phi\big)(s,u(s)) \\ &\qquad - \tfrac{1}{2}\big(L(\phi^2)\big)(s,u(s)) + \phi(s,u(s))\,(L\phi)(s,u(s)) \Big] \Big\} \end{aligned}$$

is a continuous $\mathbb{P}^\varepsilon$ supermartingale, with $E_\phi(0,u) = 1$.

Recall that $\mathbb{P}^\varepsilon$ is concentrated on the closed set $A_{u_0} := \{u \in C([0,T];X) : u(0) = u_0\}$. For each $\varepsilon > 0$, $\phi \in \mathfrak{D}_{L_\varepsilon,2}$ and each Borel set $K \subset C([0,T];X)$ we then have

$$\begin{aligned}\mathbb{P}^\varepsilon(K) &= \mathbb{P}^\varepsilon(K\cap A_{u_0}) = \mathbb{P}^\varepsilon\big(E_\phi^{-1}(T,\cdot)E_\phi(T,\cdot)1\!\!1_{K\cap A_{u_0}}(\cdot)\big)\\ &\le \sup\nolimits_{v\in K\cap A_{u_0}} E_\phi(T,v)^{-1}\mathbb{P}^\varepsilon\big(E_\phi(T,\cdot)1\!\!1_K(\cdot)\big)\\ &\le \sup\nolimits_{v\in K\cap A_{u_0}} E_\phi(T,v)^{-1} = -\inf\nolimits_{v\in K} I_{L_\varepsilon,u_0}(v,\phi)\end{aligned}$$

Taking the logarithm, optimizing over $\phi\in\mathfrak{D}_{L_\varepsilon,2}$

$$\varepsilon\log\mathbb{P}^\varepsilon(K)\le -\sup_{\phi\in\mathfrak{D}_{L_\varepsilon,2}}\inf_{v\in K}\varepsilon I_{L_\varepsilon,u_0}(v,\phi)$$

Therefore for each sequence $\Phi\in\Pi_\varepsilon\mathfrak{D}_{L_\varepsilon,2}$

$$\overline{\lim}\,\varepsilon\log\mathbb{P}^\varepsilon(K)\le -\underline{\lim}_{\varepsilon}\inf_{v\in K} I_{\varepsilon,u_0;\Phi}(v)$$

For K a compact set, by (1.2.4a)

$$\overline{\lim}\,\varepsilon\log\mathbb{P}^\varepsilon(K)\le -\underline{\lim}_{\varepsilon}\inf_{v\in K} I_{\varepsilon,u_0;\Phi}(v)\le -\inf_{u\in K}\big(\Gamma\text{-}\underline{\lim}_\varepsilon\, I_{\varepsilon,u_0;\Phi}\big)(v)$$

We then conclude by optimizing on Φ and using the minimax lemma [**15**]. □

1.3. Large deviations for finite dimensional diffusion processes

In this section we prove some results concerning finite dimensional Itô processes. For sake if simplicity, we develop the one dimensional case, although it is immediate to extend the results to the the finite dimensional setting.

Let $\{b^\varepsilon\}$, $\{d^\varepsilon\}\subset C(\mathbb{R})$. Consider the stochastic differential equation in the Itô sense

$$\begin{aligned} dx &= b^\varepsilon(x)dt + d^\varepsilon(x)dW\\ x(0) &= x_0\end{aligned}\tag{1.3.1}$$

where W is a one dimensional Brownian motion and $x_0\in\mathbb{R}$. Suppose that there exists a sequence $\{\mathbb{P}^\varepsilon\}\subset\mathcal{P}\big(C([0,T])\big)$ such that, for each $\varepsilon>0$, $\mathbb{P}^\varepsilon$ is a martingale solution to (1.3.1). We are here interested in establishing large deviations principles for $\{\mathbb{P}^\varepsilon\}$. The classical Freidlin-Wentcell computation [**10**] deals with the case in which $b^\varepsilon\equiv b$ does not depend on ε, and $d^\varepsilon=\sqrt{\varepsilon}d$. In such a case, under suitable hypotheses on b and d, $\{\mathbb{P}^\varepsilon\}$ satisfies a large deviations principle with speed ε and rate

$$I^{FW}(x) := \begin{cases}\frac12\int_{[0,T]}dt\frac{|\dot x(t)-b((x(t))|^2}{d(x(t))^2} & \text{if } x\in H^1([0,T]) \text{ and } x(0)=x_0\\ +\infty & \text{otherwise}\end{cases}$$

Back to the general case (1.3.1), define $I^\varepsilon:\mathbb{C}([0,T])\times C^\infty([0,T]\times\mathbb{R})\to\mathbb{R}$ as

$$I^\varepsilon(x;\phi) := \begin{cases}\varepsilon\Big\{\phi(T,x(T))-\phi(0,x_0)-\int_0^T dt\,\Big[\partial_t\phi(t,x(t))\\ \quad -b(x(t))\phi'(t,x(t))-\frac12 d(x(t))\phi'(t,x(t))^2\Big]\Big\} & \text{if } x(0)=x_0\\ +\infty & \text{otherwise}\end{cases}$$

Then by Proposition 1.2.5, $\{\mathbb{P}^\varepsilon\}$, satisfies a weak large deviations upper bound with speed ε^{-1} and rate

$$\underline{I} := \sup_{\{\phi^\varepsilon\}\subset C^\infty([0,T]\times\mathbb{R})} \Gamma\text{–}\underline{\lim}_\varepsilon\, I^\varepsilon(\cdot;\phi^\varepsilon) \tag{1.3.2}$$

Note that in the Friedlin-Wentcell case, $\underline{I}(\cdot) = \sup_\phi I^\varepsilon(\cdot;\phi) = I^{FW}(\cdot)$.

While it is not difficult to give conditions on $\{b^\varepsilon\}$ and $\{d^\varepsilon\}$ that guarantee exponential tightness of $\{\mathbb{P}^\varepsilon\}$ and thus a full large deviations upper bound, here we focus on a technique to establish a lower bound.

Proposition 1.3.1. *Assume*

(a) *There exists* $C > 0$ *such that* $|b^\varepsilon(x) - b^\varepsilon(y)| \le C|x-y|$ *for each* $x, y \in \mathbb{R}$.
(b) *Let* α_ε *be defined by* $\alpha_\varepsilon^2 := \sup_{x\in\mathbb{R}} |d^\varepsilon(x)|^2$. *Then* $\lim_\varepsilon \alpha_\varepsilon = 0$.
(c) *Let* α_ε *as in (b). Then for each* $C' > 0$

$$\overline{\lim_\varepsilon} \sup_{|x-y|\le C'\alpha_\varepsilon} \frac{d^\varepsilon(y)^2 - d^\varepsilon(x)^2}{d^\varepsilon(x)^2} = 0$$

and define $I^\varepsilon : C([0,T]) \to [0,+\infty]$ *as*

$$I^\varepsilon(x) := \begin{cases} \frac{\varepsilon}{2}\int_{[0,T]} dt \frac{|\dot{x}(t) - b^\varepsilon((x(t))|^2}{d^\varepsilon(x(t))^2} & \text{if } x \in H^1([0,T]) \text{ and } x(0) = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

Then $\{\mathbb{P}^\varepsilon\}$ *satisfies a large deviations lower bound with speed* ε^{-1} *and rate* $\overline{I} := \Gamma\text{–}\overline{\lim}\, I^\varepsilon$.

Note that $\overline{I} \ge \underline{I}$, where $\underline{I}$ is defined in (1.3.2), and that in the Friedlin-Wentcell case $\overline{I} = I^{FW} = \underline{I}$.

Proof. Let us fix $y \in C([0,T])$ such that $\overline{I}(y) < +\infty$. We will exhibit a sequence $\{\mathbb{Q}^\varepsilon\} \subset \mathcal{P}\big(C([0,T])\big)$ such that $\mathbb{Q}^\varepsilon \to \delta_y$ and such that $\overline{\lim}_\varepsilon H_\varepsilon(\mathbb{Q}^\varepsilon) \le \overline{I}(y)$, where H_ε is defined as in (1.2.8) with $a_\varepsilon = \varepsilon$. We then conclude by Proposition 1.2.4.

Recall that there exists a standard filtered probability space $\big(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\in[0,T]}, \mathbb{P}\big)$ on which a Brownian motion W generating the filtration $\mathfrak{F}_t$ is defined. Moreover there exists an adapted process $x^\varepsilon : \Omega \to C([0,T])$ such the $\mathbb{P}^\varepsilon = \mathbb{P}\circ(x^\varepsilon)^{-1}$, see [**20**]. As usual, in this proof, we denote by ω the generic element of Ω, and for a process $\zeta : [0,T]\times\Omega \to \mathbb{R}$ we use the equivalent notations $z(t,\omega) \equiv z_t(\omega)$, depending on which aspect of the process we want to emphasize.

By the definition of the Γ-limsup, for each $y \in C([0,T])$ there exists a sequence $\{y^\varepsilon\} \subset C([0,T])$ such that $y^\varepsilon \to y$ in $C([0,T])$ and $\overline{\lim}\, I^\varepsilon(y^\varepsilon) \le (\Gamma\text{–}\overline{\lim}_\varepsilon I^\varepsilon)(y)$. With no loss of generality we can assume $I^\varepsilon(y^\varepsilon) < +\infty$ for each ε small enough, so that $y^\varepsilon \in H^1([0,T])$. We define the $\{\mathfrak{F}_t\}$-adapted measurable map $\varphi^\varepsilon : [0,T]\times\Omega \to \mathbb{R}$ as

$$\varphi^\varepsilon(t,\omega) := -\frac{\dot{y}^\varepsilon(t) - b^\varepsilon(y(t))}{d^\varepsilon(x^\varepsilon(t,\omega))}$$

For $\zeta > 0$ we also define the $\{\mathfrak{F}_t\}$ stopping time

$$\tau^{\varepsilon,\zeta}(\omega) := \inf\big\{t \le T \, : \, \frac{\varepsilon}{2}\int_{[0,t]} ds\,[\varphi^\varepsilon(s,\omega)]^2 \le I_\varepsilon(y^\varepsilon) + \zeta\big\}$$

We also introduce the $\mathbb{P}$-martingale $\{M_t^\varepsilon\}$ on Ω as

$$M_t^{\varepsilon,\zeta} := \int_0^{t\wedge\tau^{\varepsilon,\zeta}} \varphi^\varepsilon(s)\,dW_s$$

We have that $\big[M^{\varepsilon,\zeta},M^{\varepsilon,\zeta}\big]_T \le 2\varepsilon^{-1}(I_\varepsilon(y^\varepsilon)+\zeta)$ is bounded uniformly on Ω. Therefore the process $E^{\varepsilon,\zeta}$ on Ω defined as

$$E_t^{\varepsilon,\zeta} := \exp\big(M_t^\varepsilon - \frac{1}{2}\big[M^{\varepsilon,\zeta},M^{\varepsilon,\zeta}\big]_t\big)$$

is also a $\mathbb{P}$-martingale. We define $\tilde{\mathbb{Q}}^{\varepsilon,\zeta}$ on $\mathcal{P}(\Omega)$ by $\tilde{\mathbb{Q}}^{\varepsilon,\zeta}(d\omega) = E_T^{\varepsilon,\zeta}(\omega)\mathbb{P}(d\omega)$, and $\mathbb{Q}^{\varepsilon,\zeta}$ on $\mathcal{P}\big(C([0,T])\big)$ as $\mathbb{Q}^{\varepsilon,\zeta} := \tilde{\mathbb{Q}}^{\varepsilon,\zeta}\circ(x^\varepsilon)^{-1}$. By definition (1.2.8), applying the change of variable $\Omega \ni \omega \mapsto x_\varepsilon(\omega) \in C([0,T])$

$$\begin{aligned} H_\varepsilon(\mathbb{Q}^{\varepsilon;\zeta}) &= \varepsilon H(\tilde{\mathbb{Q}}^{\varepsilon,\zeta}|\mathbb{P}) = \varepsilon\int_\Omega \tilde{\mathbb{Q}}^{\varepsilon,\zeta}(d\omega)\big(M_T^\varepsilon(\omega) - \tfrac{1}{2}\big[M^\varepsilon,M^\varepsilon\big]_T(\omega)\big) \\ &\le \varepsilon\int_\Omega \tilde{\mathbb{Q}}^{\varepsilon,\zeta}(d\omega)\big(M_T^\varepsilon(\omega) - \big[M^\varepsilon,M^\varepsilon\big]_T(\omega)\big) + I^\varepsilon(y^\varepsilon) + \zeta \\ &\le I^\varepsilon(y^\varepsilon) + \zeta \end{aligned} \tag{1.3.3}$$

where in the last line we used $\int_\Omega \tilde{\mathbb{Q}}^\varepsilon(d\omega)\big(M_T^\varepsilon(\omega) - \big[M^\varepsilon,M^\varepsilon\big]_T(\omega)\big) = 0$ since, by Girsanov theorem, $M_t^\varepsilon - \big[M^\varepsilon,M^\varepsilon\big]_t$ is a $\tilde{\mathbb{Q}}^{\varepsilon,\zeta}$-martingale.

Still by Girsanov Theorem, there exists a $\tilde{\mathbb{Q}}^{\varepsilon,\zeta}$ Brownian motion $\tilde{W}^{\varepsilon,\zeta}$ such that

$$\begin{aligned} x^\varepsilon(t\wedge\tau^{\varepsilon,\zeta}) - y^\varepsilon(t\wedge\tau^{\varepsilon,\zeta}) &= \int_{[0,t\wedge\tau^{\varepsilon,\zeta}]} ds\,\big[b^\varepsilon(x^\varepsilon(s)) - b^\varepsilon(y^\varepsilon(s))\big] \\ &\quad + \int_{[0,t\wedge\tau^{\varepsilon,\zeta}]} d^\varepsilon(x^\varepsilon(s))d\tilde{W}_s^{\varepsilon,\zeta} \end{aligned}$$

Squaring, and using (a), (b) and Doob maximal inequality

$$\begin{aligned} \tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le\tau^{\varepsilon,\eta}}|x^\varepsilon(t)-y^\varepsilon(t)|^2\big) &\le 2C^2\,T^2\,\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|^2\big) \\ &\quad + C_1\alpha_\varepsilon^2 \end{aligned}$$

for some constant $C_1 > 0$. With no loss of generality, we can assume T small enough, namely such that $C\,T \le 1/2$, by standard iterative disintegration arguments. We gather

$$\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|^2\big) \le 2C_1\alpha_\varepsilon^2 \tag{1.3.4}$$

Note that

$$\frac{\varepsilon}{2}\int_{[0,\tau^{\varepsilon,\zeta}]}[\varphi^\varepsilon(s,\omega)]^2 \le I_\varepsilon(y^\varepsilon)+I_\varepsilon(y^\varepsilon)\sup_{t\le\tau^{\varepsilon,\zeta}}\frac{d^\varepsilon(y^\varepsilon)^2-d^\varepsilon(x^\varepsilon(t,\omega))^2}{d^\varepsilon(x^\varepsilon(t,\omega))^2}$$

$I_\varepsilon(y^\varepsilon)$ is uniformly bounded, since $\overline{I}(y)<+\infty$. Therefore, by assumption (c), for each $C'>0$ there exists $\varepsilon_0=\varepsilon_0(C')$ such that for each $\varepsilon\le\varepsilon_0$

$$\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\tau^{\varepsilon,\zeta}<T\big)\le\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|\le C'\alpha_\varepsilon\big)$$

Therefore for each ξ, $C'>0$ and ε small enough

$$\begin{aligned}&\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le T}|x^\varepsilon(t)-y^\varepsilon(t)|>\xi\big)\\&\quad\le\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|>\xi\big)+\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\tau^{\varepsilon,\zeta}<T\big)\\&\quad\le\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|>\xi\big)\\&\qquad+\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|\le C'\alpha_\varepsilon\big)\\&\quad\le 2\tilde{\mathbb{Q}}^{\varepsilon,\zeta}\big(\sup\nolimits_{t\le\tau^{\varepsilon,\zeta}}|x^\varepsilon(t)-y^\varepsilon(t)|\le C'\alpha_\varepsilon\big)\le 2\tfrac{4C_1}{C'^2}\end{aligned}$$

where in the last line we used (1.3.4) and Tchebyshev inequality. Taking the limit $\varepsilon\to0$ and then sending $C'\to+\infty$, we obtain that, for each $\zeta>0$, $x^\varepsilon\to y$ in $\tilde{Q}^{\varepsilon,\zeta}$ probability, since $y^\varepsilon\to y$ in $C([0,T])$. Thus $\mathbb{Q}^{\varepsilon,\zeta}\to\delta_y$ in $\mathcal{P}\big(C([0,T])\big)$ for each $\zeta>0$. Since (1.3.3) holds, we can extract a subsequence $\zeta_\varepsilon\to0$, such that $\mathbb{Q}^{\varepsilon,\zeta_\varepsilon}\to\delta_y$ and $\overline{\lim}_\varepsilon H_\varepsilon(Q^{\varepsilon,\zeta_\varepsilon})\le\overline{\lim}\,I_\varepsilon(y^\varepsilon)+\zeta_\varepsilon\le(\Gamma\text{–}\overline{\lim}\,I_\varepsilon)(y)=\overline{I}(y)$. □

1.4. Conservation Laws

In this section we introduce some basics notions concerning the limiting equation of (1.1.3), obtained by informally setting $\varepsilon=0$. As we restrict our analysis to the $1+1$ dimensional case, we denote space derivatives with a subscript x. We think of x as a variable on the one-dimensional torus, although the results in this section can be straightforwardly stated also in the case $x\in\mathbb{R}$, that is also considered in Chapter 2. The time variable t is restricted to a finite time horizon $t\in[0,T]$ for some $T>0$.

We refer to [**5, 18**] for the precise statements and proofs concerning conservation laws. Consider the Cauchy problem

$$\begin{aligned}\partial_t u+f(u)_x&=0\\u(0,x)&=u_0(x)\end{aligned}\tag{1.4.1}$$

where we assume f to be smooth and u_0 bounded. As well known, even if the initial datum u_0 is smooth, the flow (1.4.1) may develop singularities for some positive time. In general, these singularities appear as discontinuities of u and are called *shocks*. It is then natural to interpret (1.4.1) in weak sense. In this weak formulation an additional condition is needed to guarantee uniqueness of the solutions to (1.4.1).

More precisely, a bounded measurable map $u : [0,T] \times \mathbb{T} \to \mathbb{R}$ is a *weak solution* to (1.4.1) iff for each smooth $\varphi : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that $\varphi(T,x) = 0$

$$-\langle u_0, \varphi(0,\cdot)\rangle - \langle\langle u, \partial_t \varphi\rangle\rangle - \langle\langle f(u), \varphi_x\rangle\rangle = 0$$

Here $\langle \cdot,\cdot\rangle$ denotes duality in $L_2(\mathbb{T})$ and $\langle\langle \cdot,\cdot\rangle\rangle$ duality in $L_2([0,T]\times\mathbb{T})$. Given a differentiable function η, called *entropy*, the conjugated *entropy flux* q is defined up to an additive constant by $q' = \eta' f'$. A weak solution to (1.4.1) is called *entropic* iff for each entropy – entropy flux pair (η, q) with η convex, the inequality $\eta(u)_t + q(u)_x \le 0$ holds in distribution sense. Note that the entropy condition is always satisfied for smooth solutions to (1.4.1). The classical theory, see e.g. [**18**], shows existence and uniqueness in $C([0,T]; L_1(\mathbb{T}))$ of the entropic solution $\bar u$ to the Cauchy problem associated to (1.4.1). $\bar u$ is also called the Kruzkov solution with initial datum u_0. While the flow (1.4.1) is invariant w.r.t. $(t,x) \mapsto (-t,-x)$, the entropy condition breaks such invariance and selects the “physical” direction of time.

In this section we are concerned with various classes of solutions to (1.4.1). For sake of simplicity, let us assume that u_0 takes values in $[0,1]$, and that we restrict our attention to $[0,1]$-valued solution to (1.4.1). We introduce the space $\mathcal{M}$ of Young measures as follows. Let $\mathcal{P}([0,1])$ be the set of probability measures on $[0,1]$, $\imath : [0,1] \to [0,1]$ the identity map, and U the set of measurable functions $v : \mathbb{T} \to [0,1]$. U is a (metrizable) space if regarded as a subset of the set of measures on $\mathbb{T}$ equipped with the $*$-weak topology. We define $\mathcal{M}$ the set of maps $\mu : [0,T]\times\mathbb{T} \to \mathcal{P}([0,1])$, $\mu : (t,x) \mapsto \mu_{t,x}(d\lambda)$, such that $\mu_{\cdot,\cdot}(\imath) \in C([0,T]; U)$. Hereafter for a Borel measure μ and a continuous function F on some Polish space X, $\mu(F)$ denotes the integral of F w.r.t. μ. A $\mu \in \mathcal{M}$ is a *measure-valued solution* to (1.4.1) iff for each smooth function φ on $[0,T]\times\mathbb{T}$

$$\langle \mu(\imath)_{T,\cdot}, \varphi(T,\cdot)\rangle - \langle u_0, \varphi(0,\cdot)\rangle - \langle\langle \mu(\imath), \partial_t\varphi\rangle\rangle - \langle\langle \mu(f), \varphi_x\rangle\rangle = 0$$

If u is a weak solution to (1.4.1), then $\mu_{t,x}(d\lambda) := \delta_{u(t,x)}(d\lambda)$ is a measure-valued solution; on the other hand there exist measure-valued solutions that do not have this form.

Let u a $[0,1]$-valued weak solution to (1.4.1), $\eta : [0,1] \to \mathbb{R}$ a twice differentiable map, and q its conjugated flux. We introduce the distribution $\wp_{\eta,u}$ acting on $C^\infty_{\mathrm{c}}\big((0,T)\times\mathbb{T}\big)$ as

$$\wp_{\eta,u}(\varphi) = -\langle\langle \eta(u), \partial_t\varphi\rangle\rangle - \langle\langle q(u), \varphi_x\rangle\rangle$$

Note that a weak solution u to (1.4.1) is entropic iff $\wp_{\eta,u} \le 0$ for each convex η. We say that a $[0,1]$-valued weak solution u to (1.4.1) is an *entropy-measure* solution iff $\wp_{\eta,u}$ can be extended to a Radon measure on $(0,T)\times\mathbb{T}$. In Chapter 2 we show that entropy-measure solutions have some regularity properties, and in particular we establish a so called kinetic formulation for these solutions.

Namely, suppose that u is an entropy-measure solution to (1.4.1), then there exists a Radon measure P_u on $[0,1]\times[0,T]\times\mathbb{T}$ such that for each $\eta \in C^2([0,1])$, $\varphi \in C_c^\infty\big((0,T)\times\mathbb{T}\big)$

$$\wp_{\eta,u}(\varphi) = P_u(\eta''\,\varphi) \tag{1.4.2}$$

1.5. A sketch of the main results

In this Section we roughly sketch the main results concerning large deviations principles for the probability laws of the solution u^ε to the Cauchy problem (1.1.3), see also the discussion in Section 1.1.3.

1.5.1. Statements of the results. We recall that U denotes the set of measurable maps $v : \mathbb{T} \to [0,1]$ equipped with the (metrizable) relative topology it inherits from the weak* topology of measures on $\mathbb{T}$. There are two metrics that we will consider on $C([0,T];U)$. The first one is its natural uniform topology; if $C([0,T];U)$ is equipped with this metric we denote it by $\mathcal{U}$ and by $d_{\mathcal{U}}$ the metric itself. Since $C([0,T];U)$ can be regarded as a suitable set of measurable maps $u : [0,T]\times\mathbb{T} \to [0,1]$, it can also be equipped with the strong $L_1([0,T]\times\mathbb{T})$ distance, and we denote by $d_{\mathcal{X}}$ the distance given by the sum of $d_{\mathcal{U}}$ and the L_1 distance; when endowed with $d_{\mathcal{X}}$, we denote $C([0,T];U)$ by $\mathcal{X}$.

We also recall that the set of Young measures $\mathcal{M}$ has been defined in Section 1.4. We endow $\mathcal{M}$ with the metric

$$d_{\mathcal{M}}(\mu,\nu) := d_{*\mathrm{w}}(\mu,\nu) + d_{\mathcal{U}}\big(\mu(\imath),\nu(\imath)\big)$$

where $d_{*\mathrm{w}}$ is a distance generating the relative topology on $\mathcal{M}$ regarded as a subset of the finite measures on $[0,T]\times\mathbb{T}\times[0,1]$ equipped with the vague topology. $(\mathcal{U}, d_{\mathcal{U}})$, $(\mathcal{X}, d_{\mathcal{X}})$, $(\mathcal{M}, d_{\mathcal{M}})$ are Polish spaces. We remark in particular that $\mathcal{X}$ can be regarded as a subset of $\mathcal{M}$ endowed with the relative topology.

In Section 3.5 it is shown, under suitable general hypotheses on f, D, σ and $\jmath^\varepsilon$, that for each $\varepsilon > 0$ small enough there exists a unique solution $u^\varepsilon \in C([0,T];U)\cap L_2\big([0,T];H^1(\mathbb{T})\big)$ to (1.1.3). A first result states that, as $\varepsilon \to 0$, the process u^ε converges in probability on $\mathcal{X}$ to the unique entropic solution to (1.4.1). We thus turn our focus to large deviations principles for the probability law $\mathbb{P}^\varepsilon$ of u^ε.

We want to regard u^ε as a process on $\mathcal{M}$. We thus introduce an $\mathcal{M}$-valued random process $\mu^\varepsilon := \delta_{u^\varepsilon}$, and with a little abuse of notation we denote by $\{\mathbf{P}^\varepsilon\}$ the law of μ^ε on $\mathcal{M}$. The following statement is proved in Chapter 3, under suitable hypotheses on $\jmath^\varepsilon$

Theorem 1.5.1. *$\{\mathbf{P}^\varepsilon\}$ is exponentially tight on $\mathcal{M}$ on the scale $\varepsilon^{-2\gamma}$, and satisfies a large deviations principle with speed $\varepsilon^{-2\gamma}$ and rate $\mathcal{I} : \mathcal{M} \to [0,+\infty]$*

defined by

$$\mathcal{I}(\mu) := \sup_{\varphi \in C^\infty([0,T]\times\mathbb{T})} \Big\{ \langle \mu(\imath)_{T,\cdot}, \varphi(T,\cdot)\rangle - \langle u_0, \varphi(0,\cdot)\rangle - \langle\langle \mu(\imath), \partial_t \varphi\rangle\rangle \\ - \langle\langle \mu(f), \varphi_x \rangle\rangle - \tfrac{1}{2} \langle\langle \mu(\sigma)\varphi_x, \varphi_x\rangle\rangle \Big\}$$

As it will be clearer in the following, $\mathcal{I}(\mu)$ represents a suitable Hilbert norm of $\partial_t \mu(\imath) + \mu(f)_x$, and $\mathcal{I}(\mu) = 0$ iff μ is a measure-valued solution to (1.4.1).

We know that $\mathbb{P}^\varepsilon$ converges in probability to the deterministic entropic solution to (1.4.1); on the other hand in general there exists infinitely many measure-valued solutions to (1.4.1), namely infinitely many zeros of $\mathcal{I}$. It is then natural to study the large deviations of $\mathbb{P}^\varepsilon$ on a finer scale, which roughly speaking correspond to the analysis of the Γ-development, see [**4**], of the functional H_ε as defined in (1.2.8). We are thus concerned with a large deviations principle on the scale $\varepsilon^{-2\gamma+1}$, for which we can only prove partial results. We show that the sequence $\{\mathbb{P}^\varepsilon\}$ is exponentially tight on $\mathcal{X}$ w.r.t. the scale $\varepsilon^{-2\gamma+1}$. In particular, since the topology of $\mathcal{X}$ coincides with the relative topology induced by the immersion of $\mathcal{X}$ in $\mathcal{M}$ via the map $\mathcal{X} \ni u \mapsto \delta_i \in \mathcal{M}$, once a large deviations principle is established for $\{\mathbb{P}^\varepsilon\}$ on $\mathcal{X}$, it is immediate to get a large deviations principle for $\{\mathbf{P}^\varepsilon\}$ on $\mathcal{M}$.

Recall that if u is an entropy-measure solution to (1.4.1), then there exists a Radon measure P_u on $[0,1]\times[0,T]\times\mathbb{T}$ such that (1.4.2) holds. We denote by P_u^+ its positive part. We define $H : \mathcal{X} \to [0,+\infty]$ by

$$H(u) := \begin{cases} \displaystyle\int P_u^+(dv; dt, dx)\, \frac{D(v)}{\sigma(v)} & \text{if is an entropy measure solution to (1.4.1)} \\ +\infty & \text{otherwise} \end{cases}$$

In Chapter 3 we also define a suitable set $\mathcal{S}$ of "entropy-splittable" solutions, which are entropy-measure solutions to (1.4.1), such that the supports of the positive and negative part of P_u have some nice properties. Then we set

$$\overline{H}(u) := \sup_{\substack{\mathcal{O}\ni u \\ \mathcal{O}\text{ open}}} \ \inf_{v \in \mathcal{O}\cap\mathcal{S}} H(v)$$

THEOREM 1.5.2. $\{\mathbb{P}^\varepsilon\}$ *is exponentially tight on* $\mathcal{X}$ *on the scale* $\varepsilon^{-2\gamma+1}$, *and satisfies a large deviations upper bound with rate* H *and speed* $\varepsilon^{-2\gamma+1}$, *and a large deviations lower bound with rate* $\overline{H}$ *and speed* $\varepsilon^{-2\gamma+1}$.

In order to prove the full large deviations principle, one would need to show that $\overline{H} = H$. While it is easy to see $\overline{H} \geq H$, the converse inequality is equivalent to the so called H-density of $\mathcal{S}$ in $\mathcal{X}$. This issue is briefly discussed in Section 3.2 below and appears to be linked to much hard open problems.

The physical interpretation of H and its connections with Einstein diffusion-fluctuation relation is described in Section 2.2.

1.5.2. Outline of the proofs. Exponential tightness of $\{\mathbb{P}^\varepsilon\}$ on $\mathcal{M}$ is easy, due to strong compactness properties of the space itself. Exponential tightness of $\mathbb{P}^\varepsilon$ on $\mathcal{X}$ is obtained via a compensated compactness argument, once a sharp estimate for the behavior of ∇u^ε is obtained.

Both the large deviations upper bounds are gathered by building on the result in Proposition 1.2.5, and the Γ-convergence results proved in Chapter 3.

Both the large deviations lower bounds are obtained following a strategy similar to the one used to prove Proposition 1.3.1, and the computation of the Γ-limits in Chapter 2. Additional stability estimate for (1.1.3) are then needed to conclude.

1.5.3. Other results. The results obtained in this paper have been motivated by the investigation of large deviations principles for the solution to (1.1.3). However we believe that some these results may have an independent interest. Here we list some results that are quite independent from the large deviations problem; they are discussed in more generality in Chapters 2 and 3.

The Γ-convergence problem investigated in Chapter 2 is largely independent of the large deviations issue. It provides a variational characterization of measure-valued and entropic solutions to (1.4.1). In particular it gives a sharp stability bound for the viscous approximation to conservation laws under H^{-1}-like perturbations. In Appendix B of Chapter 2 a similar Γ-convergence results is also established for Hamilton-Jacobi equations, providing the correspondent variational characterization of measure-valued and viscosity solutions.

Corollary 2.2.2 can be regarded as a negative-Sobolev version of classical results, see [**6**, Chap. 3], for the relaxation of integral functionals in weak L_p spaces.

In Lemma 3.2.2 a generalization of the classical Bernestein inequality [**17**] is provided. We remark that it is possible to generalize this inequality to the cadlag case.

The correspondence between large deviations and Γ-convergence established in Proposition 1.2.2, 1.2.3, 1.2.4 suggests various connections between the two theories. For instance, in Proposition 2.4.6 we prove the Γ-convergence analogous statement to the so called *contraction principle* for large deviations (indeed, the contraction principle is a straightforward consequence of Propositions 2.4.6, 1.2.2, 1.2.3, 1.2.4).

1.5.4. Open problems and developments. As mentioned above, the H-density of $\mathcal{S}$ in $\mathcal{X}$ is open. As discussed in Chapter 2, this issue is related to fine structure analysis of entropy-measure solutions to (1.4.1). In particular it seems that an important step in proving the H-density is to provide a chain rule formula for divergence-measure field. Chain rule formulas out of the BV setting are subject of recent research investigation; in particular, as far as divergence-measure fields are concerned, the attempt to understand their differential properties can be tracked back to De Giorgi and Anzellotti [**2**], and more recently to [**7**].

We believe that the variational techniques introduced in this paper are a useful tool to provide large deviations principle in other settings. In particular we mention the possibility to apply this methods to other classes of stochastic partial differential equations (like degenerate parabolic diffusion equations and the $2D$ Navier-Stokes equation).

As better explained in Section 2.2, the functional H comes as a natural generalization of the large deviations functional introduced in [**11, 21**]. We thus hope it is possible to extend the variational techniques here introduced to establish large deviations principles for various classes of asymmetric particles systems.

Finally, an enhanced investigation of large deviations principles for the general finite dimensional diffusion (1.3.1) may be interesting.

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CHAPTER 2

Γ-entropy cost functional for scalar conservation laws

The results in this chapter have been obtained jointly with G. Bellettini, L. Bertini, and M. Novaga, see also [**3**].

2.1. Introduction

We are concerned with the scalar one-dimensional conservation law

$$u_t + f(u)_x = 0 \tag{2.1.1}$$

where, given $T > 0$, $u = u(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$, subscripts denote partial derivatives, and the *flux* f is a Lipschitz function. As well known, even if the initial datum $u(0) = u(0,\cdot)$ is smooth, the flow (2.1.1) may develop singularities for some positive time. In general, these singularities appear as discontinuities of u and are called *shocks*. It is therefore natural to interpret (2.1.1) weakly; in the weak formulation uniqueness is however lost, if no further conditions are imposed. Given a function η, called *entropy*, the conjugated *entropy flux* q is defined up to an additive constant as $q(u) = \int^u dv\, \eta'(v)\, f'(v)$. A weak solution to (2.1.1) is called *entropic* iff for each entropy – entropy flux pair (η, q) with η convex, the inequality $\eta(u)_t + q(u)_x \le 0$ holds in the sense of distributions. Note that the entropy condition is always satisfied for smooth solutions to (2.1.1). The classical theory, see e.g. [**6, 16**], shows existence and uniqueness in $C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$ of the entropic solution to the Cauchy problem associated to (2.1.1). While the flow (2.1.1) is invariant w.r.t. $(t,x) \mapsto (-t,-x)$, the entropy condition breaks such invariance and selects the "physical" direction of time.

In the conservation law (2.1.1) the viscosity effects are neglected. This approximation is no longer valid if the gradients become large as it happens when shocks appear. A more accurate description is then given by the parabolic equation

$$u_t + f(u)_x = \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \tag{2.1.2}$$

in which $(t,x) \in [0,T] \times \mathbb{R}$, D, assumed uniformly positive, is the diffusion coefficient and $\varepsilon > 0$ is the viscosity. In this context of scalar conservation laws, it is also well known that, as $\varepsilon \to 0$, equibounded solutions to (2.1.2) converge in $L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$ to entropic solutions to (2.1.1), see e.g. [**6, 16**]. This approximation result shows that the entropy condition is relevant.

Perhaps less well known, at least in the hyperbolic literature, is the fact that entropic solutions to (2.1.1) can be obtained as scaling limit of discrete stochastic models of lattice gases, see e.g. [**12**, Ch. 8]. In a little more detail, consider particles living on a one-dimensional lattice and randomly jumping to their neighboring sites. It is then proved that, under hyperbolic scaling, the empirical density of particles converges in probability to entropic solutions to (2.1.1). A much studied example is the *totally asymmetric simple exclusion process*, where there is at most one particle in each site and only jumps heading to the right are allowed. In this case, the empirical density takes values in $[0, 1]$ and its scaling limit is given by (2.1.1) with flux $f(u) = u(1 - u)$. In this stochastic framework, it is also worth looking at the large deviations asymptotic associated to the aforementioned law of large numbers. Basically, this amounts to estimate the probability that the empirical density lies in a neighborhood of a given trajectory. In general this probability is exponentially small, and the corresponding decay rate is called the large deviations rate functional. For the totally asymmetric simple exclusion process, this issue has been analyzed in [**10, 18**]. It is there shown that the large deviations rate functional is infinite off the set of weak solutions to (2.1.1); on such solutions the rate functional is given by the total positive mass of the entropy production $h(u)_t + g(u)_x$ where h is the Bernoulli entropy, i.e. $h(u) = -u \log u - (1 - u) \log(1 - u)$ and g is its conjugated entropy flux.

A stochastic framework can also be naturally introduced in a partial differential equations' setting by adding to (2.1.2) a random perturbation, namely

$$u_t + f(u)_x = \frac{\varepsilon}{2}\big(D(u)u_x\big)_x + \sqrt{\gamma}\big(\sqrt{\sigma(u)}\alpha_\gamma\big)_x \qquad (t, x) \in (0, T) \times \mathbb{R} \quad (2.1.3)$$

where $\sigma(u) \geq 0$ is a conductivity coefficient and α_γ is a Gaussian random forcing term white in time and with spatial correlations on a scale much smaller than γ. Let $u^{\varepsilon,\gamma}$ be the corresponding solution; if $\gamma \ll \varepsilon$ then $u^{\varepsilon,\gamma}$ still converges in probability to the entropic solution to (2.1.1) and the large deviations asymptotic becomes a relevant issue. Referring to [**14**] for this analysis, here we formulate the problem from a purely variational point of view quantifying, in terms of the parabolic problem (2.1.2), the asymptotic *cost* of non-entropic solutions to (2.1.1). Introducing in (2.1.2) a *control* $E \equiv E(t, x)$ we get

$$u_t + f(u)_x = \frac{\varepsilon}{2}\big(D(u)u_x\big)_x - \big(\sigma(u)E\big)_x \qquad (t, x) \in (0, T) \times \mathbb{R} \qquad (2.1.4)$$

If we think of u as a density of charge, then E can be naturally interpreted as the 'controlling' external electric field and $\sigma(u) \geq 0$ as the conductivity. The flow (2.1.4) conserves the *total charge* $\int dx\, u(t, x)$, whenever it is well defined.

The *cost functional* I_ε associated with (2.1.2) can be now informally defined as the work done by the optimal controlling field E in (2.1.4), namely

$$I_\varepsilon(u) = \inf_E \frac{1}{2}\int_{[0,T]} dt\, dx\, \sigma(u)E^2 = \inf_E \frac{1}{2}\int_{[0,T]} dt\, \big\|E\big\|^2_{L_2(\mathbb{R},\sigma(u)dx)} \tag{2.1.5}$$

where the infimum is taken over the controls E such that (2.1.4) holds. For a suitable choice of the random perturbation α_γ, I_ε is the large deviations rate functional of the process $u^{\varepsilon,\gamma}$ solution to (2.1.3), when ε is fixed and $\gamma \to 0$. To avoid the technical problems connected to the possible unboundedness of the density u, we assume that the conductivity σ has compact support. In this case, if u is such that $I_\varepsilon(u) < +\infty$ then u takes values in the support of σ, see Proposition 2.3.4 for the precise statement. For the sake of simplicity, we assume that σ is supported by $[0,1]$. The case of strictly positive σ also fits in the description below, provided however that the analysis is a priori restricted to equibounded densities u.

In this chapter we analyze the variational convergence of I_ε as $\varepsilon \to 0$. Our first result holds for a Lipschitz flux f, and identifies the so called Γ-limit of I_ε, which is naturally studied in a Young measures setting. The limiting cost of a Young measure $\mu \equiv \mu_{t,x}(d\lambda)$ is

$$\mathcal{I}(\mu) = \frac{1}{2}\int_{[0,T]} dt\, \Big\| \big[\mu(\lambda)\big]_t + \big[\mu(f(\lambda))\big]_x \Big\|^2_{H^{-1}(\mathbb{R},\mu(\sigma(\lambda))dx)}$$

where, for $F \in C([0,1])$, $[\mu(F(\lambda))](t,x) = \int \mu_{t,x}(d\lambda)\, F(\lambda)$ and, with a little abuse of notation, $\|\varphi\|_{H^{-1}(\mathbb{R},\mu_{t,\cdot}(\sigma(\lambda))dx)}$ is the dual norm to $\big[\int dx\, \mu_{t,x}(\sigma(\lambda))\, \varphi_x^2\big]^{1/2}$.

Note that $\mathcal{I}(\mu)$ vanishes iff μ is a *measure-valued* solution to (2.1.1). Hence we can obtain such solutions as limits of solutions to (2.1.4) with a suitable sequence E_ε with vanishing cost. On the other hand, if we set in (2.1.4) $E = 0$ we obtain, in the limit $\varepsilon \to 0$, an entropic solution to (2.1.1). If the flux f is nonlinear, the set of measure-valued solutions to (2.1.1) is larger than the set of entropic solutions; it is thus natural to study the Γ-convergence of the rescaled cost functional $H_\varepsilon := \varepsilon^{-1} I_\varepsilon$, which formally corresponds to the scaling in [**10, 18**]. Our second result concerns the Γ-convergence of H_ε which is studied under the additional hypotheses that the flux f is smooth and such that there are no intervals in which f is affine. A compensated compactness argument shows that H_ε has enough coercivity properties to force its convergence in a functions setting and not in a Young measures' one.

To informally define the candidate Γ-limit of H_ε, we first introduce some preliminary notions. We say that a weak solution u to (2.1.1) is *entropy-measure* iff for each *smooth* entropy η the distribution $\eta(u)_t + q(u)_x$ is a Radon measure on $(0,T)\times\mathbb{R}$. If u is an entropy-measure solution to (2.1.1), then there exists a measurable map ϱ_u from $[0,1]$ to the set of Radon measures on $(0,T)\times\mathbb{R}$, such that for each $\eta \in C^2([0,1])$ and $\varphi \in C^\infty_{\mathrm{c}}\big((0,T)\times$

$\mathbb{R}\big)$, $-\int dt\,dx\,\big[\eta(u)\varphi_t + q(u)\varphi_x\big] = \int dv\,\varrho_u(v;dt,dx)\eta''(v)\varphi(t,x)$, see Proposition 2.2.3. The candidate Γ-limit of H_ε is the functional H defined as follows. If u is not an entropy-measure solution to (2.1.1) then $H(u) = +\infty$. Otherwise $H(u) = \int\, dv\,\varrho_u^+(v;dt,dx)D(v)/\sigma(v)$, where ϱ_u^+ denotes the positive part of ϱ_u. Note that while I_ε and $\mathcal{I}$ are nonlocal functionals, H is local. On the other hand, while I_ε, resp. $\mathcal{I}$, quantifies in a suitable squared Hilbert norm the violation of equation (2.1.2), resp. (2.1.1), this quadratic structure is lost in H. In Proposition 2.2.6 we show that H is a coercive lower semicontinuous functional, this matching the necessary properties for being the Γ-limit of a sequence of equicoercive functionals. Note also that H depends on the diffusion coefficient D and the conductibility coefficient σ only through their ratio, which is an expected property of well-behaving driven diffusive systems, in hydrodynamical-like limits. We discuss this issue in Remark 2.2.11, where a link between the functional H and the large deviations rate functional introduced in [**10, 18**] is also investigated. In particular, H comes as a natural generalization of the functional introduced in [**10, 18**], whenever the flux f is neither convex nor concave.

In this chapter we prove that for each sequence $u^\varepsilon \to u$ in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$ we have $\underline{\lim}_\varepsilon H_\varepsilon(u^\varepsilon) \ge H(u)$, namely Γ–$\underline{\lim}\, H_\varepsilon \ge H$. Since the functional H vanishes only on entropic solutions to (2.1.1), its zero-level set coincides with the limit points of the minima of I_ε. Concerning the Γ-limsup inequality, for each weak solution u to (2.1.1) in a suitable set $\mathcal{S}_\sigma$, see Definition 2.2.4, we construct a sequence $u^\varepsilon \to u$ such that $H_\varepsilon(u^\varepsilon) \to H(u)$. The above statements imply (Γ–$\lim H_\varepsilon)(u) = H(u)$ for $u \in \mathcal{S}_\sigma$. To complete the proof of the Γ-convergence of H_ε to H on the whole set of entropy-measure solutions, an additional density argument is needed. This seems to be a difficult problem, as Varadhan [**18**] puts it: "...one does not see at the moment how to produce a 'general' non-entropic solution, partly because one does not know what it is."

The above results show in particular that if u^ε solves (2.1.4) for some control E^ε such that $\varepsilon^{-1}\int_{[0,T]} dt\, \|E^\varepsilon\|^2_{L_2(\mathbb{R},\sigma(u^\varepsilon)dx)}$ vanishes as $\varepsilon \to 0$, then any limit point of u^ε is an entropic solution to (2.1.1). This statement is sharp in the sense that there are sequences $\{E^\varepsilon\}$ with $\underline{\lim}_\varepsilon\, \varepsilon^{-1}\int_{[0,T]} dt\, \|E^\varepsilon\|^2_{L_2(\mathbb{R},\sigma(u^\varepsilon)dx)} > 0$ such that any limit point of the corresponding u^ε is not an entropic solutions to (2.1.1). More generally, the variational description of conservation laws here introduced allows the following point of view. Measure-valued solutions to (2.1.1) are the points in the zero-level set of the Γ-limit of I_ε, while entropic weak solutions are the points in the zero-level set of the Γ-limit of $\varepsilon^{-1}I_\varepsilon$. In Appendix 2.7 we introduce a sequence $\{J_\varepsilon\}$ of functionals related to the viscous approximation of Hamilton-Jacobi equations. In [**15**] a Γ-limsup inequality for a related family of functionals has been independently investigated in a

BV setting. Following closely the proofs of the Γ-convergence of $\{I_\varepsilon\}$, we establish the corresponding Γ-convergence results, thus obtaining a variational characterization of measure-valued and viscosity solutions to Hamilton-Jacobi equations. Although this "variational" point of view is consistent with the standard concepts of solution in the current setting of scalar conservation laws and Hamilton-Jacobi equations, it might be helpful for less understood model equations.

2.2. Notation and results

Hereafter in this chapter, we assume that f is a Lipschitz function on $[0,1]$, D and σ are continuous functions on $[0,1]$, with D uniformly positive and σ strictly positive on $(0,1)$. We understand that these assumptions are supposed to hold in every statement below.

We also let $\langle\cdot,\cdot\rangle$ denote the inner product in $L_2(\mathbb{R})$, for $T>0$ $\langle\langle\cdot,\cdot\rangle\rangle$ stands for the inner product in $L_2([0,T]\times\mathbb{R})$, and for O an open subset of $\mathbb{R}^n$, $C_c^\infty(O)$ denotes the collection of compactly supported infinitely differentiable functions on O.

Scalar conservation law

Our analysis will be restricted to equibounded densities u that take values in $[0,1]$. Let U denote the compact separable metric space of measurable functions $u:\mathbb{R}\to[0,1]$, equipped with the following H^{-1}_{loc}-like metric d_U. For $L>0$, set

$$\|u\|_{-1,L} := \sup\big\{\langle u,\varphi\rangle,\ \varphi\in C_c^\infty\big((-L,L)\big),\ \langle\varphi_x,\varphi_x\rangle=1\big\}$$

and define the metric d_U in U by

$$d_U(u,v) := \sum_{N=1}^{\infty} 2^{-N}\frac{\|u-v\|_{-1,N}}{1+\|u-v\|_{-1,N}} \tag{2.2.1}$$

Given $T>0$, let $\mathcal{U}$ be the set $C\big([0,T];U\big)$ endowed with the uniform metric

$$d_{\mathcal{U}}(u,v) := \sup_{t\in[0,T]} d_U\big(u(t),v(t)\big) \tag{2.2.2}$$

An element $u\in\mathcal{U}$ is a *weak solution* to (2.1.1) iff for each $\varphi\in C_c^\infty\big((0,T)\times\mathbb{R}\big)$ (in particular $\varphi(0)=\varphi(T)=0$) it satisfies

$$\langle\langle u,\varphi_t\rangle\rangle + \langle\langle f(u),\varphi_x\rangle\rangle = 0$$

We also introduce a suitable space $\mathcal{M}$ of Young measures and recall the notion of measure-valued solution to (2.1.1). Consider the set $\mathcal{N}$ of measurable maps μ from $[0,T]\times\mathbb{R}$ to the set $\mathcal{P}([0,1])$ of Borel probability measures on $[0,1]$. The set $\mathcal{N}$ can be identified with the set of positive Radon measures μ on $[0,1]\times[0,T]\times\mathbb{R}$ such that $\mu([0,1],\,dt,\,dx)=dt\,dx$. Indeed, by existence of a regular version of conditional probabilities, for such measures μ there exists a measurable kernel $\mu_{t,x}(d\lambda) \in \mathcal{P}([0,1])$ such that $\mu(d\lambda,\, dt,\, dx) = dt\, dx\, \mu_{t,x}(d\lambda)$. For $\imath : [0,1] \to [0,1]$ the identity map, we set

$$\mathcal{M} := \big\{\mu \in \mathcal{N} \,:\, \text{the map } [0,T] \ni t \mapsto \mu_{t,\cdot}(\imath) \text{ is in } \mathcal{U}\big\} \tag{2.2.3}$$

in which, for a bounded measurable function $F : [0,1] \to \mathbb{R}$, the notation $\mu_{t,x}(F)$ stands for $\int_{[0,1]} \mu_{t,x}(d\lambda) F(\lambda)$. We endow $\mathcal{M}$ with the metric

$$d_{\mathcal{M}}(\mu,\nu) := d_{\mathrm{vag}}(\mu,\nu) + d_{\mathcal{U}}\big(\mu(\imath),\nu(\imath)\big) \tag{2.2.4}$$

where d_{vag} is a distance generating the relative topology on $\mathcal{N}$ regarded as a subset of the finite Borel measures on $[0,1] \times [0,T] \times \mathbb{R}$ equipped with the vague topology. $(\mathcal{M}, d_{\mathcal{M}})$ is a complete separable metric space.

An element $\mu \in \mathcal{M}$ is a *measure-valued solution* to (2.1.1) iff for each $\varphi \in C_{\mathrm{c}}^\infty((0,T) \times \mathbb{R})$ it satisfies

$$\langle\langle \mu(\imath), \varphi_t \rangle\rangle + \langle\langle \mu(f), \varphi_x \rangle\rangle = 0$$

If $u \in \mathcal{U}$ is a weak solution to (2.1.1), then $\delta_{u(t,x)}(d\lambda) \in \mathcal{M}$ is a measure-valued solution. On the other hand, there exist measure-valued solutions which do not have this form.

Parabolic cost functional

We next give the definition of the parabolic cost functional informally introduced in (2.1.5). Given $u \in \mathcal{U}$ we write $u_x \in L_{2,\mathrm{loc}}([0,T] \times \mathbb{R})$ iff u admits a locally square integrable weak x-derivative. For $\varepsilon > 0$, $u \in \mathcal{U}$ such that $u_x \in L_{2,\mathrm{loc}}([0,T] \times \mathbb{R})$, and $\varphi \in C_{\mathrm{c}}^\infty((0,T) \times \mathbb{R})$ we set

$$\ell_\varepsilon^u(\varphi) := -\langle\langle u, \varphi_t \rangle\rangle - \langle\langle f(u), \varphi_x \rangle\rangle + \frac{\varepsilon}{2} \langle\langle D(u) u_x, \varphi_x \rangle\rangle \tag{2.2.5}$$

and define $I_\varepsilon : \mathcal{U} \to [0,+\infty]$ by

$$I_\varepsilon(u) := \begin{cases} \sup\limits_{\varphi \in C_{\mathrm{c}}^\infty((0,T)\times\mathbb{R})} \Big[\ell_\varepsilon^u(\varphi) - \dfrac{1}{2}\langle\langle \sigma(u)\varphi_x, \varphi_x \rangle\rangle\Big] & \text{if } u_x \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases} \tag{2.2.6}$$

$I_\varepsilon(u)$ vanishes iff $u \in \mathcal{U}$ is a weak solution to (2.1.2); more generally, by Riesz representation theorem, it is not difficult to prove the connection of I_ε with the perturbed parabolic problem (2.1.4), see Lemma 2.3.1 below for the precise statement.

In order to discuss the behavior of I_ε as $\varepsilon \to 0$ we lift it to the space of Young measures $(\mathcal{M}, d_{\mathcal{M}})$, see (2.2.3), (2.2.4). We thus define $\mathcal{I}_\varepsilon : \mathcal{M} \to [0,+\infty]$ by

$$\mathcal{I}_\varepsilon(\mu) := \begin{cases} I_\varepsilon(u) & \text{if} \quad \mu_{t,x} = \delta_{u(t,x)} \text{ for some } u \in \mathcal{U} \\ +\infty & \text{otherwise} \end{cases} \tag{2.2.7}$$

Asymptotic parabolic cost

As well known, a most useful notion of variational convergence is the so called Γ-convergence which, together with some compactness estimates, implies convergence of the minima. Let X be a complete separable metrizable space; recall that a sequence of functionals $F_\varepsilon : X \to [-\infty, +\infty]$ is *equicoercive* on X iff for each $M > 0$ there exists a compact set K_M such that for any $\varepsilon \in (0, 1]$ we have $\{x \in X \,:\, F_\varepsilon(x) \le M\} \subset K_M$. We briefly recall the basic definitions of the Γ-convergence theory, see e.g. [**4, 7**]. Given $x \in X$ we define

$$\begin{aligned}\left(\Gamma\text{–}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon\right)(x) &:= \inf\left\{ \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(x^\varepsilon),\ \{x^\varepsilon\} \subset X \,:\, x^\varepsilon \to x\right\}\\ \left(\Gamma\text{–}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon\right)(x) &:= \inf\left\{ \overline{\lim}_{\varepsilon\to 0} F_\varepsilon(x^\varepsilon),\ \{x^\varepsilon\} \subset X \,:\, x^\varepsilon \to x\right\}\end{aligned}$$

Whenever $\Gamma\text{–}\underline{\lim}\, F_\varepsilon = \Gamma\text{–}\overline{\lim}\, F_\varepsilon = F$ we say that F_ε Γ-converges to F in X. Equivalently, F_ε Γ-converges to F iff for each $x \in X$ we have:

– for any sequence $x^\varepsilon \to x$ we have $\underline{\lim}_\varepsilon F_\varepsilon(x^\varepsilon) \ge F(x)$ (Γ-*liminf inequality*);
– there exists a sequence $x^\varepsilon \to x$ such that $\overline{\lim}_\varepsilon F_\varepsilon(x^\varepsilon) \le F(x)$ (Γ-*limsup inequality*).

Equicoercivity and Γ-convergence of a sequence $\{F_\varepsilon\}$ imply an upper bound of infima over open sets, and a lower bound of infima over closed sets, see e.g. [**4**, Prop. 1.18], and therefore it is the relevant notion of variational convergence in the control setting introduced above.

Theorem 2.2.1. *The sequence* $\{\mathcal{I}_\varepsilon\}$ *defined in* (2.2.6), (2.2.7) *is equicoercive on* $\mathcal{M}$ *and, as* $\varepsilon \to 0$, Γ-*converges in* $\mathcal{M}$ *to*

$$\mathcal{I}(\mu) := \sup_{\varphi \in C_c^\infty((0,T)\times\mathbb{R})} \Big\{ -\langle\langle \mu(\imath), \varphi_t\rangle\rangle - \langle\langle \mu(f), \varphi_x\rangle\rangle - \frac{1}{2}\,\langle\langle \mu(\sigma)\varphi_x, \varphi_x\rangle\rangle \Big\} \tag{2.2.8}$$

$\mathcal{I}(\mu) = 0$ iff μ is a measure-valued solution to (2.1.1).

From Theorem 2.2.1 we deduce the Γ-limit of I_ε, see (2.2.6), on $\mathcal{U}$ by projection.

Corollary 2.2.2. *The sequence of functionals* $\{I_\varepsilon\}$ *is equicoercive on* $\mathcal{U}$ *and, as* $\varepsilon \to 0$, Γ-*converges in* $\mathcal{U}$ *to the functional* $I : \mathcal{U} \to [0, +\infty]$ *defined by*

$$\begin{aligned} I(u) := \inf \Big\{ &\textstyle\int dt\, dx\, R_{f,\sigma}\big(u(t,x), \Phi(t,x)\big),\\ &\Phi \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R}) \,:\, \Phi_x = -u_t \ \textit{weakly} \Big\}\end{aligned}$$

where $R_{f,\sigma} : [0,1]\times\mathbb{R} \to [0,+\infty]$ *is defined by*

$$R_{f,\sigma}(w,c) := \inf\{\big(\nu(f) - c\big)^2/\nu(\sigma),\ \nu \in \mathcal{P}([0,1]) \,:\, \nu(\imath) = w\}$$

in which we understand $(c-c)^2/0 = 0$.

From the proof of Corollary 2.2.2 it follows $I(\cdot) \le \mathcal{I}(\delta_\cdot)$, and the equality holds iff f is linear. If we restrict to *stationary* u's, namely to the case $u_t = 0$, Corollary 2.2.2 can be regarded as a negative-Sobolev version of classical relaxation results for integral functionals in weak topology. More precisely, from the proofs of Theorem 2.4.1 and Corollary 2.2.2 it follows that if we define the functional $\tilde{F} : U \to [0, +\infty]$ by

$$\tilde{F}(u) := \inf_{c \in \mathbb{R}} \int dx \, \frac{\big[f(u(x)) - c\big]^2}{\sigma(u(x))}$$

then its lower semicontinuous envelope w.r.t. the d_U-distance (2.2.1) is given by

$$F(u) := \inf_{c \in \mathbb{R}} \int dx \, R_{f,\sigma}(u(x), c)$$

Note also that $R_{f,\sigma}$ can be explicitly calculated in some cases. Let $\underline{f}, \overline{f} : [0,1] \to \mathbb{R}$ be respectively the convex and concave envelope of f. Then, in the case $\sigma = 1$, we have $R_{f,1}(w,c) = \big[\text{distance}(c, [\underline{f}(w), \overline{f}(w)])\big]^2$. In the case $f = \sigma$ (which include the example mentioned in the introduction $f(u) = \sigma(u) = u(1-u)$) then

$$R_{f,f}(w,c) = \begin{cases} 2(|c| - c) & \text{if } |c| \in [\underline{f}(w), \overline{f}(w)] \\ \frac{(\overline{f}(w) - c)^2}{\overline{f}(w)} & \text{if } |c| > \overline{f}(w) \\ \frac{(\underline{f}(w) - c)^2}{\underline{f}(w)} & \text{if } |c| < \underline{f}(w) \end{cases}$$

Entropy-measure solutions

Recalling (2.2.2), we let $\mathcal{X}$ be the same set $C([0,T]; U)$ endowed with the metric

$$d_{\mathcal{X}}(u,v) := \sum_{N=1}^{\infty} \frac{1}{2^N} \|u - v\|_{L_1([0,T]\times[-N,N])} + d_{\mathcal{U}}(u,v) \tag{2.2.9}$$

Convergence in $\mathcal{X}$ is equivalent to convergence in $\mathcal{U}$ *and* in $L_{p,\mathrm{loc}}([0,T] \times \mathbb{R})$ for $p \in [1, +\infty)$.

Let $C^2([0,1])$ be the set of twice differentiable functions on $(0,1)$ whose derivatives are continuous up to the boundary. A function, resp. a convex function, $\eta \in C^2([0,1])$ is called an *entropy*, resp. a *convex entropy*, and its *conjugated entropy flux* $q \in C([0,1])$ is defined up to a constant by $q(u) := \int^u dv \, \eta'(v) f'(v)$. For u a weak solution to (2.1.1), for (η, q) an entropy – entropy flux pair, the *η-entropy production* is the distribution $\wp_{\eta,u}$ acting on $C_c^\infty\big((0,T) \times \mathbb{R}\big)$ as

$$\wp_{\eta,u}(\varphi) := -\langle\langle \eta(u), \varphi_t \rangle\rangle - \langle\langle q(u), \varphi_x \rangle\rangle \tag{2.2.10}$$

Let $C_c^{2,\infty}\big([0,1] \times (0,T) \times \mathbb{R}\big)$ be the set of compactly supported maps $\vartheta : [0,1] \times (0,T) \times \mathbb{R} \ni (v,t,x) \mapsto \vartheta(v,t,x) \in \mathbb{R}$, that are twice differentiable in the v variable, with derivatives continuous up to the boundary of $[0,1] \times (0,T) \times \mathbb{R}$, and that are infinitely differentiable in the (t,x) variables. For $\vartheta \in C^{2,\infty}_{\mathrm{c}}\big([0,1] \times (0,T) \times \mathbb{R}\big)$ we denote by ϑ' and ϑ'' its partial derivatives w.r.t. the v variable. We say that a function $\vartheta \in C^{2,\infty}_{\mathrm{c}}\big([0,1] \times (0,T) \times \mathbb{R}\big)$ is an *entropy sampler*, and its *conjugated entropy flux sampler* $Q : [0,1] \times (0,T) \times \mathbb{R}$ is defined up to an additive function of (t,x) by $Q(u,t,x) := \int^u dv\, \vartheta'(v,t,x) f'(v)$. Finally, given a weak solution u to (2.1.1), the *ϑ-sampled entropy production* $P_{\vartheta,u}$ is the real number

$$P_{\vartheta,u} := -\int dt\, dx \Big[\big(\partial_t \vartheta\big)\big(u(t,x),t,x\big) + \big(\partial_x Q\big)\big(u(t,x),t,x\big) \Big] \tag{2.2.11}$$

If $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some entropy η and some $\varphi \in C^\infty_{\mathrm{c}}\big((0,T) \times \mathbb{R}\big)$, then $P_{\vartheta,u} = \wp_{\eta,u}(\varphi)$.

The next proposition introduces a suitable class of solutions to (2.1.1) which will be needed in the following. We denote by $M\big((0,T) \times \mathbb{R}\big)$ the set of Radon measures on $(0,T) \times \mathbb{R}$ that we consider equipped with the vague topology. In the following, for $\varrho \in M\big((0,T) \times \mathbb{R}\big)$ we denote by $\varrho^\pm$ the positive and negative part of ϱ. For u a weak solution to (2.1.1) and η an entropy, recalling (2.2.10) we set

$$\|\wp_{\eta,u}\|_{\mathrm{TV},L} := \sup \big\{ \wp_{\eta,u}(\varphi),\, \varphi \in C^\infty_{\mathrm{c}}\big((0,T) \times (-L,L)\big),\, |\varphi| \le 1 \big\} \tag{2.2.12}$$

$$\|\wp^+_{\eta,u}\|_{\mathrm{TV},L} := \sup \big\{ \wp_{\eta,u}(\varphi),\, \varphi \in C^\infty_{\mathrm{c}}\big((0,T) \times (-L,L)\big),\, 0 \le \varphi \le 1 \big\}$$

PROPOSITION 2.2.3. *Let $u \in \mathcal{X}$ be a weak solution to* (2.1.1). *The following statements are equivalent:*

(i) *There exists $c > 0$ such that $\|\wp^+_{\eta,u}\|_{\mathrm{TV},L} < +\infty$ for each $L > 0$ and $\eta \in C^2([0,1])$ with $0 \le \eta'' \le c$.*

(ii) *For each entropy η, the η-entropy production $\wp_{\eta,u}$ can be extended to a Radon measure on $(0,T) \times \mathbb{R}$, namely $\|\wp_{\eta,u}\|_{\mathrm{TV},L} < +\infty$ for each $L > 0$.*

(iii) *There exists a bounded measurable map $\varrho_u : [0,1] \ni v \to \varrho_u(v; dt, dx) \in M\big((0,T) \times \mathbb{R}\big)$ such that for any entropy sampler ϑ*

$$P_{\vartheta,u} = \int dv\, \varrho_u(v; dt, dx)\, \vartheta''(v,t,x) \tag{2.2.13}$$

A weak solution $u \in \mathcal{X}$ that satisfies any of the equivalent conditions in Proposition 2.2.3 is called an *entropy-measure solution* to (2.1.1). We denote by $\mathcal{E} \subset \mathcal{X}$ the set of entropy-measure solutions to (2.1.1). Proposition 2.2.3 establishes a so called *kinetic formulation* for entropy-measure solutions, see also [**8**, Prop. 3.1] for a similar result. If $f \in C^2([0,1])$ is such that there are no intervals in which f is affine, using the results in [**5**] we show that entropy-measure solutions have some regularity properties, see Lemma 2.5.1.

A weak solution $u \in \mathcal{X}$ to (2.1.1) is called an *entropic solution* iff for each convex entropy η the inequality $\wp_{\eta,u} \le 0$ holds in distribution sense, namely $\|\wp^+_{\eta,u}\|_{\mathrm{TV},L} = 0$ for each $L > 0$. In particular entropic solutions are entropy-measure solutions such that $\varrho_u(v; dt, dx)$ is a negative Radon measure for each $v \in [0,1]$. It is well known, see e.g. [**6, 16**], that for each $u_0 \in U$ there exists a unique entropic solution $\bar{u} \in C([0,T]; L_{1,\mathrm{loc}}(\mathbb{R}))$ to (2.1.1) such that $\bar{u}(0) = u_0$. Such a solution $\bar{u}$ is called the *Kruzkov solution* with initial datum u_0.

Γ-*entropy cost of non-entropic solutions*

We next introduce a rescaled cost functional and prove in particular that entropic solutions are the only ones with vanishing rescaled asymptotic cost. Recalling that I_ε has been introduced in (2.2.6), the *rescaled cost functional* $H_\varepsilon : \mathcal{X} \to [0, +\infty]$ is defined by

$$H_\varepsilon(u) := \varepsilon^{-1} I_\varepsilon(u) \tag{2.2.14}$$

In the Γ-convergence theory, the asymptotic behavior of the rescaled functional H_ε is usually referred to as the development by Γ-convergence of I_ε, see e.g. [**4**, §1.10]. In our case, while we lifted I_ε to the space of Young measures $\mathcal{M}$, we can consider the rescaled cost functional H_ε on $\mathcal{X}$. In fact, as shown below, H_ε has much better compactness properties than I_ε and it is equicoercive on $\mathcal{X}$. Therefore the Γ-convergence of the lift of H_ε to $\mathcal{M}$ can be immediately retrieved from the Γ-convergence of H_ε on $\mathcal{X}$. Indeed, since $\delta_{u_\varepsilon} \to \delta_u$ in $\mathcal{M}$ implies $u_\varepsilon \to u$ in $\mathcal{X}$, the metric (2.2.9) generates the relative topology of $\mathcal{X}$ regarded as a subset of $\mathcal{M}$.

Recall that $\mathcal{E} \subset \mathcal{X}$ denotes the set of entropy-measure solutions to (2.1.1), and that for $u \in \mathcal{E}$ there exists a bounded measurable map $\varrho_u : [0,1] \to M\big((0,T) \times \mathbb{R}\big)$ such that (2.2.13) holds. Let ϱ_u^+ be the positive part of ϱ_u, and define $H : \mathcal{X} \to [0, +\infty]$ by

$$H(u) := \begin{cases} \displaystyle\int dv\, \varrho_u^+(v; dt, dx)\, \frac{D(v)}{\sigma(v)} & \text{if } u \in \mathcal{E} \\ +\infty & \text{otherwise} \end{cases} \tag{2.2.15}$$

As shown in the proof of Theorem 2.2.5, if u is a weak solution to (2.1.1) and $H(u) < +\infty$, then $H(u) = \sup_\vartheta P_{\vartheta,u}$, where the supremum is taken over the entropy samplers ϑ such that $0 \le \sigma(v)\vartheta''(v,t,x) \le D(v)$, for each $(v,t,x) \in [0,1] \times [0,T] \times \mathbb{R}$.

Definition 2.2.4. *An entropy-measure solution $u \in \mathcal{E}$ is* entropy-splittable *iff there exist two closed sets $E^+, E^- \subset [0,T] \times \mathbb{R}$ such that*

(i) *For a.e. $v \in [0,1]$, the support of $\varrho_u^+(v; dt, dx)$ is contained in E^+, and the support of $\varrho_u^-(v; dt, dx)$ is contained in E^-.*

(ii) *For each* $L > 0$, *the set* $\Big\{t \in [0,T] \,:\, \big(\{t\} \times [-L,L]\big) \cap E^+ \cap E^- \neq \emptyset\Big\}$ *is nowhere dense in* $[0,T]$.

The set of entropy-splittable solutions to (2.1.1) *is denoted by* $\mathcal{S}$. *An entropy-splittable solution* $u \in \mathcal{S}$ *such that* $H(u) < +\infty$ *and*

(iii) *For each* $L > 0$ *there exists* $\delta_L > 0$ *such that* $\sigma(u(t,x)) \geq \delta_L$ *for a.e.* $(t,x) \in [0,T] \times [-L,L]$.

is called nice w.r.t. σ. *The set of nice (w.r.t.* σ*) solutions to* (2.1.1) *is denoted by* $\mathcal{S}_\sigma$.

Note that $\mathcal{S}_\sigma \subset \mathcal{S} \subset \mathcal{E} \subset \mathcal{X}$, and that, if σ is uniformly positive on $[0,1]$, then $\mathcal{S}_\sigma = \mathcal{S}$. In Remark 2.2.9 we exhibit a few classes of entropy-splittable solutions to (2.1.1).

In the next theorem we state our results concerning the Γ-convergence of the rescaled functional H_ε, see (2.2.6) and (2.2.14), to the functional H defined in (2.2.15).

THEOREM 2.2.5. (i) *The sequence of functionals* $\{H_\varepsilon\}$ *satisfies the* Γ*-liminf inequality* $\Gamma\text{-}\varliminf_\varepsilon H_\varepsilon \geq H$ *on* $\mathcal{X}$.

(ii) *Assume that there is no interval where* f *is affine. Then the sequence of functionals* $\{H_\varepsilon\}$ *is equicoercive on* $\mathcal{X}$.

(iii) *Assume furthermore that* $f \in C^2([0,1])$, *and* $D, \sigma \in C^\alpha([0,1])$ *for some* $\alpha > 1/2$. *Define*

$$\overline{H}(u) := \inf\big\{\varliminf H(u_n),\, \{u_n\} \subset \mathcal{S}_\sigma \,:\, u_n \to u \text{ in } \mathcal{X}\big\}$$

Then the sequence of functionals $\{H_\varepsilon\}$ *satisfies the* Γ*-limsup inequality* $\Gamma\text{-}\varlimsup_\varepsilon H_\varepsilon \leq \overline{H}$ *on* $\mathcal{X}$.

From the lower semicontinuity of H on $\mathcal{X}$, see Proposition 2.2.6, it follows that $\overline{H} \geq H$ on $\mathcal{X}$ and $\overline{H} = H$ on $\mathcal{S}_\sigma$, namely the Γ-convergence of H_ε to H holds on $\mathcal{S}_\sigma$. To get the full Γ-convergence on $\mathcal{X}$, the inequality $H(u) \geq \overline{H}(u)$ is required also for $u \notin \mathcal{S}_\sigma$. This amounts to show that $\mathcal{S}_\sigma$ is H-dense in $\mathcal{X}$, namely that for $u \in \mathcal{X}$ such that $H(u) < +\infty$ there exists a sequence $\{u^n\} \subset \mathcal{S}_\sigma$ converging to u in $\mathcal{X}$ such that $H(u^n) \to H(u)$. As mentioned at the end of the introduction, this appears to be a difficult problem. A preliminary step in this direction is to obtain a chain rule formula for bounded vector fields on $[0,T] \times \mathbb{R}$ the divergence of which is a Radon measure (divergence-measure fields). This is a classical result for locally BV fields [**2**]. However, while entropic solutions to (2.1.1) are in $BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$ [**1**, Corollary 1.3], as shown in Example 2.2.8 below, the set $\{u \in \mathcal{X} \,:\, H(u) < +\infty\}$ is not contained in $BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$; see [**9**] for similar examples including estimates in Besov norms. Chain rule formulas out of the BV setting have been investigated in several recent papers; in particular in [**8**], a chain rule formula for divergence-measure fields is addressed, providing some partial results. In the remaining of this section we discuss some properties of H, and some issues related to the H-density of $\mathcal{S}_\sigma$.

In the following proposition we show that H is lower semicontinuous, and that it is coercive under the same hypotheses used for the equicoercivity of $\{H_\varepsilon\}$. Moreover, we prove that the minimizers of H are limit points of the minimizers of I_ε as $\varepsilon \to 0$, so that no further rescaling of $\{I_\varepsilon\}$ has to be investigated.

PROPOSITION 2.2.6. *H is lower semicontinuous on $\mathcal{X}$, and $H(u) = 0$ iff u is an entropic solution to* (2.1.1).

Assume that there are no intervals where f is affine. Then H is coercive on $\mathcal{X}$.

Assume furthermore that $f \in C^2([0,1])$ and let $u \in \mathcal{X}$. Then $H(u) = 0$ iff u is a limit point of a sequence $\{u^\varepsilon\} \subset \mathcal{X}$ such that $I_\varepsilon(u^\varepsilon) = 0$. In particular the map that associates to a given $u_0 \in U$ the Kruzkov solution to (2.1.1) *with initial datum u_0 is bijective on the zero-level set of H.*

If $u \in \mathcal{X}$ is a weak solution with locally bounded variation, Vol'pert chain rule, see [**2**], gives a formula for $H(u)$ in terms of the normal traces of u on its jump set.

REMARK 2.2.7. *Let $u \in \mathcal{X} \cap BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$ be a weak solution to* (2.1.1). *Denote by $J_u \subset [0,T] \times \mathbb{R}$ its jump set, by $\mathcal{H}^1 \llcorner J_u$ the one-dimensional Hausdorff measure restricted to J_u, by $n = \big(n^t, n^x\big)$ a unit normal to J_u (which is well defined $\mathcal{H}^1 \llcorner J_u$ a.e.), and by $u^\pm$ the normal traces of u on J_u w.r.t. n. Then the Rankine-Hugoniot condition $(u^+ - u^-)n^t + \big(f(u^+) - f(u^-)\big)n^x = 0$ holds. In particular we can choose n so that n^x is uniformly positive, and thus u^+ is the right trace of u and u^- is the left trace of u. Then $u \in \mathcal{E}$ and*

$$\varrho_u(v; dt, dx) = \frac{d\mathcal{H}^1 \llcorner J_u}{\big\{(u^+ - u^-)^2 + [f(u^+) - f(u^-)]^2\big\}^{1/2}} \rho(v, u^+, u^-)$$

where, denoting by $u^- \wedge u^+$ and $u^- \vee u^+$ respectively the minimum and maximum of $\{u^-, u^+\}$, $\rho : [0,1]^3 \to \mathbb{R}$ is defined by

$$\rho(v, u^+, u^-) := \big[f(u^-)(u^+ - v) + f(u^+)(v - u^-) - f(v)(u^+ - u^-)\big]\, \mathbb{1}_{[u^- \wedge u^+, u^- \vee u^+]}(v)$$

Hence, denoting by ρ^+ the positive part of ρ

$$\begin{aligned} H(u) &= \int_{J_u} \frac{d\mathcal{H}^1}{\big\{(u^+ - u^-)^2 + [f(u^+) - f(u^-)]^2\big\}^{1/2}} \int dv\, \rho^+(v, u^+, u^-)\, \frac{D(v)}{\sigma(v)} \\ &= \int_{J_u} d\mathcal{H}^1 |n^x| \int dv\, \frac{\rho^+(v, u^+, u^-)}{|u^+ - u^-|} \frac{D(v)}{\sigma(v)} \end{aligned} \tag{2.2.16}$$

Note $\rho(v, u^+, u^-) \le 0$ iff $\frac{f(v) - f(u^-)}{v - u^-} \ge \frac{f(u^+) - f(v)}{u^+ - v}$. This corresponds to the well known geometrical secant condition for entropic solutions, see e.g. [**6, 16**].

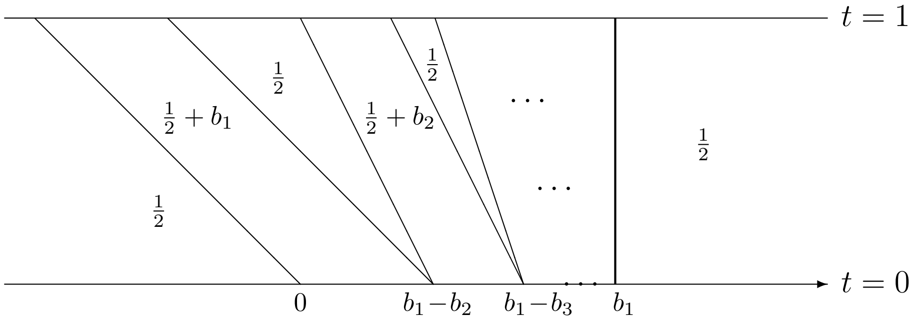

FIGURE 1. The values of u in Example 2.2.8 for $T = 1$.

Therefore $H(u)$ quantifies the violation of the entropy condition along the non-entropic shocks of u.

In the following Example 2.2.8 we show that neither the domain of H, neither the H-closure of $\mathcal{S}_\sigma$ are contained in $BV_{\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$.

EXAMPLE 2.2.8. *Let $f(u) = u(1-u)$ and pick a decreasing sequence $\{b_i\}$ of positive reals such that $b_1 < 1/2$, $\sum_i b_i = +\infty$ and $\sum_i b_i^3 < +\infty$. Let u be defined by*

$$u(t,x) := \begin{cases} 1/2 + b_i & \text{if } T(b_1 - b_i) < x + b_i\, t < T\big(b_1 - b_{i+1}\big) \text{ for some } i \\ 1/2 & \text{otherwise} \end{cases}$$

Then $H(u) = \frac{T}{2}\sum_i \int_{[0,b_i]} dv\, \frac{D(1/2+v)}{\sigma(1/2+v)} v(b_i - v) < +\infty$. Note that, even if the initial datum is in $BV(\mathbb{R})$ and f is concave, $u \notin BV_{\mathrm{loc}}([0,T]\times\mathbb{R})$. However $H(u) = \overline{H}(u)$. Indeed the sequence $\{u^n\} \subset \mathcal{S}_\sigma$ defined by

$$u^n(t,x) := \begin{cases} u(t,x) & \text{if } x + b_n\, t < T(b_1 - b_{n+1}) \\ 1/2 & \text{otherwise} \end{cases}$$

is such that $u^n \to u$ in $\mathcal{X}$ and $\lim_n H(u^n) = H(u)$.

In the following remarks we identify some classes of entropy-splittable solutions to (2.1.1), see Definition 2.2.4.

REMARK 2.2.9. *Weak solutions to* (2.1.1) *such that, for each convex entropy η, $\wp_{\eta,u} \le 0$ (entropic solutions) or $\wp_{\eta,u} \ge 0$ (anti-entropic solutions) are entropy-splittable. Indeed they are entropy-measure solutions (see Proposition 2.2.6) and they fit in Definition 2.2.4 with the choice $E^- = [0,T]\times\mathbb{R}$ and $E^+ = \emptyset$ (for entropic solutions), and respectively $E^+ = [0,T]\times\mathbb{R}$ and $E^- = \emptyset$ (for anti-entropic solutions).*

Let $u \in BV_{\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$ be a weak solution to (2.1.1)*. In the same setting of Remark 2.2.7, let us define $J_u^\pm := \mathrm{Closure}\big(\{(t,x) \in J_u \,:\, \exists v \in [0,1] \,:\, \pm\varrho(v; u^+, u^-) > 0\}\big)$. Suppose that for each $L > 0$ the set $\{t \in [0,T] \,:\, \big(\{t\} \times [-L,L]\big) \cap J_u^+ \cap J_u^-\}$ is nowhere dense in $[0,T]$. Then u is an entropy-splittable* *solution. If f is convex or concave the sign of $\rho(v, u^+, u^-)$ does not depend on $v \in [u^- \wedge u^+, u^- \vee u^+]$. Therefore, under this convexity hypothesis, weak solutions to* (2.1.1) *with locally bounded variations and with a jump set J_u consisting of a locally finite number of Lipschitz curves, intersecting each other at a locally finite number of points are entropy splittable.*

For a general (possibly neither convex nor concave) flux f, even piecewise constant solutions to (2.1.1) may fail to be entropy-splittable. However, in the following Example 2.2.10 we introduce a family of weak solutions u to (2.1.1) that are not entropy-splittable, and show that they are in the H-closure of $\mathcal{S}_\sigma$, and thus $\overline{H}(u) = H(u)$. However, while Example 2.2.10 can be widely generalized to prove $\overline{H}(u) = H(u)$ for u in suitable classes of piecewise smooth solutions, it does not seem that the ideas suggested by this example may work in the general setting of entropy-measure solutions $u \in \mathcal{E}$.

Example 2.2.10. *Let $\gamma : [0, T] \to \mathbb{R}$ be a Lipschitz map, let u be a weak solution of bounded variation to* (2.1.1)*, and suppose that the jump set of u coincides with γ. Let $u^- \equiv u^-(t)$ and $u^+ \equiv u^+(t)$ be the traces of u on γ, and suppose that there exists $u^0 \in (0, 1)$ such that $u^-(t) < u^0 < u^+(t)$ for each t and $\frac{f(v)-f(u^-)}{v-u^-} \geq \frac{f(u^+)-f(v)}{u^+-v}$ for $v \in [u^-, u^0]$ and $\frac{f(v)-f(u^-)}{v-u^-} \leq \frac{f(u^+)-f(v)}{u^+-v}$ for $v \in [u^-, u^0]$. Then, if these inequalities are strict at some v and t, u is not entropy-splittable. However defining $u^n \in \mathcal{X}$ by*

$$u^n(t, x) := \begin{cases} u(t, x + n^{-1}) & \text{if } x \leq \gamma(t) - n^{-1} \\ u^0 & \text{if } \gamma(t) - n^{-1} < x < \gamma(t) + n^{-1} \\ u(t, x - n^{-1}) & \text{if } x \leq \gamma(t) + n^{-1} \end{cases}$$

we have that $u^n \in \mathcal{S}$, $u^n \to u$ in $\mathcal{X}$ and $H(u^n) = H(u)$. In particular, if $\sigma(u)$ is uniformly positive on compact subsets of $[0, T] \times \mathbb{R}$, then $\overline{H}(u) = H(u)$. It is easy to extend this example to the case in which the jump set of u consists of a locally finite number of Lipschitz curves non-intersecting each other, provided that on each curve the quantity $\frac{f(v)-f(u^-)}{v-u^-} - \frac{f(u^+)-f(v)}{u^+-v}$ changes its sign a finite number of times for $v \in [u^+ \wedge u^-, u^+ \vee u^-]$.

We next discuss the link between this paper and [**10, 18**]. In the introduction we informally described the connection between the problem (2.1.4) and stochastic particles systems under Euler scaling. It is interesting to note that such a quantitative connection can also be established for the limiting functionals. The key point is that we expect the functional H defined in (2.2.15) to coincide with the large deviations rate functional introduced in [**10, 18**], provided the functions f, D and σ are chosen correspondingly. Unfortunately, we cannot establish such an identification off the set of weak solutions to (2.1.1) with locally bounded variation.

REMARK 2.2.11. *Let* $H' : \mathcal{X} \to [0, +\infty]$ *be defined by*

$$H'(u) := \begin{cases} \sup\big\{\|\wp^+_{\eta,u}\|_{\mathrm{TV},L},\ L>0,\ \eta \in C^2([0,1])\ :\ 0 \le \sigma\,\eta'' \le D\big\} & \textit{if } u \in \mathcal{E} \\ +\infty & \textit{otherwise} \end{cases}$$

Then $H \ge H'$ *and* $H(u) = H'(u)$ *whenever there exists a Borel set* $E^+ \subset [0,T] \times \mathbb{R}$ *such that the measure* $dv\, \varrho^+_u(v; dt, dx)$ *is concentrated on* $[0,1] \times E^+$ *and* $dv\, \varrho^-_u(v; dt, dx) = 0$ *on* $[0,1] \times E^+$*. In particular if* f *is convex or concave and* $u \in BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$*, then* $H(u) = H'(u)$*. If* f *is neither convex nor concave, then there exists* $u \in \mathcal{X}$ *such that* $H(u) > H'(u)$*.*

A general connection between dynamical transport coefficients and thermodynamic potentials in driven diffusive systems is the so called *Einstein relation*, see e.g. [**17**, II.2.5]. For a physical model described by (2.1.4), this relation states that the *Einstein entropy* $h \in C^2((0,1)) \cap C([0,1]; [0,+\infty])$ defined by

$$\sigma(v)h''(v) = D(v) \qquad v \in (0,1)$$

is a physically relevant entropy in the limit $\varepsilon \to 0$. We let $g(u) := \int_{1/2}^u dv\, h'(v)f'(v)$ be the conjugated flux of h. Note that h, g may be unbounded if σ vanishes at the boundary of $[0,1]$ and that $g \le C_1 + C_2\, h$ for some constants $C_1, C_2 \ge 0$. If u is a weak solution to (2.1.1) such that $h(u) \in L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$ and such that the distribution $h(u)_t + g(u)_x$ acts as a Radon measure on $(0,T) \times \mathbb{R}$, we let $\|\wp^+_{h,u}\|_{\mathrm{TV}}$ be the total variation of the positive part of such a measure. By monotone convergence $H'(u) \ge \|\wp^+_{h,u}\|_{\mathrm{TV}}$ for such a u, and if f is convex or concave and u has locally bounded variation, then indeed $H'(u) = \|\wp^+_{h,u}\|_{\mathrm{TV}}$. If f is convex or concave, we do not know whether $H(u) = H'(u) = \|\wp^+_{h,u}\|_{\mathrm{TV}}$ for all $u \in \mathcal{X}$, since a chain rule formula for divergence-measure fields is missing.

The problem investigated in [**10, 18**] formally corresponds to the case $f(u) = \sigma(u) = u(1-u)$ and $D(u) = 1$, so that the Einstein entropy h coincides with the Bernoulli entropy $h(u) = -u \log u - (1-u)\log(1-u)$. The (candidate) large deviations rate functional H^{JV} introduced in [**10, 18**] is defined as $+\infty$ off the set of weak solutions to (2.1.1), while $H^{JV}(u) = \|\wp^+_{h,u}\|_{\mathrm{TV}}$ for u a weak solution (this is well defined, since h is bounded). We thus have $H \ge H^{JV}$, and in view of the Γ-liminf inequality, H comes as a natural generalization of H^{JV} for diffusive systems with no convexity assumptions on the the flux f.

Outline of the proofs

Standard parabolic a priori estimates on u in terms of $I_\varepsilon(u)$ imply equicoercivity of $\mathcal{I}_\varepsilon$ on $\mathcal{M}$. Equicoercivity of H_ε on $\mathcal{X}$ is obtained by the same bounds and a classical compensated compactness argument.

The Γ-liminf inequality in Theorem 2.2.1 follows from the variational definition (2.2.6) of I_ε. The Γ-liminf inequality in Theorem 2.2.5 still follows from (2.2.6) by choosing test functions of the form $\varepsilon\vartheta(u^\varepsilon(t,x), t, x)$, with $\sigma\vartheta'' \le D$.

The Γ-limsup inequality in Theorem 2.2.1 is not difficult if $\mu_{t,x} = \delta_{u(t,x)}$ for some smooth u; the general result is obtained by taking lower semicontinuous envelope. The Γ-limsup statement in Theorem 2.2.5 is proved by building, for each $u \in \mathcal{S}_\sigma$, a recovery sequence $\{u^\varepsilon\}$ such that *a priori* $H_\varepsilon(u^\varepsilon) \to H(u)$. The convergence $u^\varepsilon \to u$ is then obtained by a stability analysis of the parabolic equation (2.1.4) w.r.t small variations of the control E.

Eventually, in Appendix 2.7 we apply our results to Hamilton-Jacobi equations.

2.3. Representation of I_ε and a priori bounds

Given a bounded measurable function $a \geq 0$ on $[0,T] \times \mathbb{R}$ let $\mathcal{D}^1_a$ be the Hilbert space obtained by identifying and completing the functions $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$ w.r.t. the seminorm $\langle\langle \varphi_x, a\,\varphi_x\rangle\rangle^{1/2}$. Let $\mathcal{D}^{-1}_a$ be its dual space. The corresponding norms are denoted respectively by $\|\cdot\|_{\mathcal{D}^1_a}$ and $\|\cdot\|_{\mathcal{D}^{-1}_a}$.

We first establish the connection between the cost functional I_ε and the perturbed parabolic problem (2.1.4). The following lemma is a standard tool in large deviations theory, see e.g. [**12**, Lemma 10.5.3]. We however detail its proof for sake of completeness.

Lemma 2.3.1. *Fix $\varepsilon > 0$ and let $u \in \mathcal{U}$. Then $I_\varepsilon(u) < +\infty$ iff there exists $\Psi^{\varepsilon,u} \in \mathcal{D}^1_{\sigma(u)}$ such that u is a weak solution to* (2.1.4) *with $E = \Psi^{\varepsilon,u}_x$, namely for each $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$*

$$\langle u(T),\varphi(T)\rangle - \langle u(0),\varphi(0)\rangle - \Big[\langle\langle u,\varphi_t\rangle\rangle + \Big\langle\Big\langle f(u) - \frac{\varepsilon}{2}D(u)u_x + \sigma(u)\Psi^{\varepsilon,u}_x, \varphi_x\Big\rangle\Big\rangle\Big] = 0 \tag{2.3.1}$$

In such a case $\Psi^{\varepsilon,u}$ is unique and

$$I_\varepsilon(u) = \frac{1}{2}\Big\| u_t + f(u)_x - \frac{\varepsilon}{2}\big(D(u)u_x\big)_x\Big\|^2_{\mathcal{D}^{-1}_{\sigma(u)}} = \frac{1}{2}\,\|\Psi^{\varepsilon,u}\|^2_{\mathcal{D}^1_{\sigma(u)}} \tag{2.3.2}$$

Proof. Fix $\varepsilon > 0$ and $u \in \mathcal{U}$ such that $I_\varepsilon(u) < +\infty$. The functional ℓ^ε_u defined in (2.2.5) can be extended to a linear functional on $C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$ by setting

$$\begin{aligned}\ell^u_\varepsilon(\varphi) \;=\; & \langle u(T),\varphi(T)\rangle - \langle u(0),\varphi(0)\rangle - \langle\langle u,\varphi_t\rangle\rangle - \langle\langle f(u),\varphi_x\rangle\rangle \\ & + \tfrac{\varepsilon}{2}\langle\langle D(u)u_x,\varphi_x\rangle\rangle\end{aligned} \tag{2.3.3}$$

Since for any $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$ the map $[0,T] \ni t \mapsto \langle u(t),\varphi(t)\rangle \in \mathbb{R}$ is continuous, it is easily seen that

$$I_\varepsilon(u) = \sup_{\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})} \Big\{ \ell^u_\varepsilon(\varphi) - \frac{1}{2}\langle\langle \sigma(u)\varphi_x,\varphi_x\rangle\rangle \Big\}$$

We claim that ℓ_ε^u defines a bounded linear functional on $\mathcal{D}^1_{\sigma(u)}$. Indeed, since $I_\varepsilon(u) < +\infty$

$$\ell_\varepsilon^u(\varphi) \le I_\varepsilon(u) + \frac{1}{2}\|\varphi\|^2_{\mathcal{D}^1_{\sigma(u)}}$$

which shows that $\ell_\varepsilon^u(\varphi) = 0$ whenever $\|\varphi\|_{\mathcal{D}^1_{\sigma(u)}} = 0$, as $\ell_\varepsilon^u(\cdot)$ is 1-homogeneous. We also get that ℓ_ε^u is bounded in $\mathcal{D}^1_{\sigma(u)}$, and it can therefore be extended by compatibility and density to a continuous linear functional on $\mathcal{D}^1_{\sigma(u)}$. Still denoting by ℓ_ε^u such a functional we get

$$I_\varepsilon(u) = \sup_{\varphi\in\mathcal{D}^1_{\sigma(u)}} \Big\{ \ell_\varepsilon^u(\varphi) - \frac{1}{2}\langle\langle \sigma(u)\varphi_x, \varphi_x\rangle\rangle \Big\} \tag{2.3.4}$$

which is equivalent to the first equality in (2.3.2). By Riesz representation theorem we now get existence and uniqueness of $\Psi^{\varepsilon,u} \in \mathcal{D}^1_{\sigma(u)}$ such that $\ell_\varepsilon^u(\varphi) = \big(\Psi^{\varepsilon,u}, \varphi\big)_{\mathcal{D}^1_{\sigma(u)}}$ for any $\varphi \in \mathcal{D}^1_{\sigma(u)}$, which implies (2.3.1). Riesz representation also yields $I_\varepsilon(u) = \frac{1}{2}\|\Psi^{\varepsilon,u}\|^2_{\mathcal{D}^1_{\sigma(u)}}$. The converse statements are obvious. □

In the following lemma we give some regularity results for $u \in \mathcal{U}$ with finite cost, and we prove some a priori bounds.

LEMMA 2.3.2. *Let $\varepsilon > 0$ and $u \in \mathcal{U}$ be such that $I_\varepsilon(u) < +\infty$. Then $u \in C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. Moreover for each entropy – entropy flux pair (η, q), each $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$, and each $t \in [0,T]$*

$$\begin{aligned}
&\langle \eta(u(t)), \varphi(t)\rangle - \langle \eta(u(0)), \varphi(0)\rangle - \textstyle\int_{[0,t]} ds \,\big[\langle \eta(u), \varphi_s\rangle + \langle q(u), \varphi_x\rangle\big] \\
&\quad = -\tfrac{\varepsilon}{2} \textstyle\int_{[0,t]} ds \Big[\langle \eta''(u) D(u) u_x, \varphi\, u_x\rangle + \langle \eta'(u) D(u) u_x, \varphi_x\rangle\Big] \\
&\qquad + \textstyle\int_{[0,t]} ds \Big[\langle \eta''(u)\sigma(u)\, u_x, \Psi_x^{\varepsilon,u}\,\varphi\rangle + \langle \eta'(u)\sigma(u)\,\Psi_x^{\varepsilon,u}, \varphi_x\rangle\Big]
\end{aligned} \tag{2.3.5}$$

where $\Psi^{\varepsilon,u}$ is as in Lemma 2.3.1. Finally, there exists a constant $C > 0$ depending only on f, D and σ such that for any ε, $L > 0$

$$\varepsilon \int dt \int_{[-L,L]} dx\, u_x^2 \le C\big[\varepsilon^{-1} I_\varepsilon(u) + L + 1\big] \tag{2.3.6}$$

PROOF. Recall that the linear functional ℓ_ε^u on $\mathcal{D}^1_{\sigma(u)}$ is defined as the extension of (2.3.3). Let $\theta := -f(u) + \frac{\varepsilon}{2} D(u)\, u_x - \sigma(u)\,\Psi_x^{\varepsilon,u} \in L_{2,\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$; by (2.3.1) $u_t = \theta_x$ holds weakly. Since $I_\varepsilon(u) < +\infty$ we also have $u_x \in L_{2,\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$, so that $u \in C\big([0,T]; L_{2,\mathrm{loc}}(\mathbb{R})\big)$ by standard interpolations arguments, see e.g. [**13**]. Since u is bounded, this is equivalent to the statement $u \in C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$.

This fact implies that integrations by parts are allowed in the first line on the r.h.s. of (2.3.3), namely for each measurable compactly supported ϕ :

$[0,T]\times\mathbb{R}\to\mathbb{R}$ with $\phi_x \in L_2\big([0,T]\times\mathbb{R}\big)$

$$\ell^u_\varepsilon(\phi) = \langle\langle u_t, \phi\rangle\rangle + \langle\langle f(u)_x, \phi\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u) u_x, \phi_x\rangle\rangle \tag{2.3.7}$$

Since u_x is locally square integrable, if $\eta \in C^2([0,1])$ and $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$, then $\eta'(u)\varphi$ has compact support and its weak x-derivative is square integrable. We can thus evaluate (2.3.7) with ϕ replaced by $\eta'(u)\varphi$, and since $\ell^u_\varepsilon(\eta'(u)\varphi) = \big(\Psi^{\varepsilon,u}, \eta'(u)\varphi\big)_{\mathcal{D}^1_{\sigma(u)}}$ and $u \in C\big([0,T]; L_{2,\mathrm{loc}}(\mathbb{R})\big)$ we get (2.3.5).

To prove the last statement, consider an entropy – entropy flux pair (η, q). By (2.3.4) and (2.3.7)

$$\begin{aligned} I_\varepsilon(u) \geq\ & \ell^u_\varepsilon(\varepsilon\, \eta'(u)\, \varphi) - \tfrac{\varepsilon^2}{2}\big\langle\big\langle \big(\eta'(u)\varphi\big)_x, \sigma(u)\big(\eta'(u)\varphi\big)_x\big\rangle\big\rangle \\ =\ & \varepsilon\langle \eta(u(T)), \varphi(T)\rangle - \varepsilon\langle\eta(u(0)), \varphi(0)\rangle \\ & - \varepsilon\big[\langle\langle \eta(u), \varphi_t\rangle\rangle + \langle\langle q(u), \varphi_x\rangle\rangle\big] \\ & + \tfrac{\varepsilon^2}{2}\Big[\langle\langle D(u)\eta''(u) u_x^2, \varphi\rangle\rangle + \langle\langle \eta'(u) D(u) u_x, \varphi_x\rangle\rangle \\ & \qquad - \langle\langle \sigma(u)\eta''(u)^2 u_x^2, \varphi^2\rangle\rangle - \langle\langle \sigma(u)\eta'(u)^2\varphi_x, \varphi_x\rangle\rangle \\ & \qquad - 2\langle\langle \sigma(u)\eta''(u)\eta'(u) u_x, \varphi\,\varphi_x\rangle\rangle\Big] \end{aligned}$$

We now choose $\eta \geq 0$, uniformly convex and such that $\sigma\eta'' \leq D$, and for such a η we let $\alpha := \max_v \big[D(v)\eta'(v)^2/\eta''(v)\big]$, so that $\sigma\,(\eta')^2 \leq \alpha$. By Cauchy-Schwarz inequality

$$\begin{aligned} 2\big|\langle\langle \sigma(u)\eta''(u)\eta'(u)u_x, \varphi\,\varphi_x\rangle\rangle\big| \ &\leq\ \langle\langle \sigma(u)\eta''(u)^2 u_x^2, \varphi^2\rangle\rangle + \langle\langle \sigma(u)\eta'(u)^2, \varphi_x\varphi_x\rangle\rangle \\ &\leq\ \langle\langle D(u)\eta''(u)u_x^2, \varphi^2\rangle\rangle + \alpha\langle\langle \varphi_x, \varphi_x\rangle\rangle \end{aligned}$$

Letting $\zeta : [0,1]\to\mathbb{R}$ be such that $\zeta' = \eta'\, D$, and integrating by parts we get $\langle \eta'(u)D(u)u_x, \varphi_x\rangle = -\langle \zeta(u), \varphi_{xx}\rangle$. Collecting all the bounds

$$\begin{aligned} & \langle \eta(u(T)), \varphi(T)\rangle + \tfrac{\varepsilon}{2}\langle\langle D(u)\eta''(u)u_x^2, \varphi - 2\varphi^2\rangle\rangle \\ & \quad \leq \varepsilon^{-1} I_\varepsilon(u) + \langle \eta(u(0)), \varphi(0)\rangle + \langle\langle \eta(u), \varphi_t\rangle\rangle + \langle\langle q(u), \varphi_x\rangle\rangle \\ & \quad\ + \tfrac{\varepsilon}{2}\langle\langle \zeta(u), \varphi_{xx}\rangle\rangle + \varepsilon\,\alpha\,\langle\langle \varphi_x, \varphi_x\rangle\rangle \end{aligned}$$

We now choose φ independent of t and such that $\varphi(x) = 1/4$ for $|x| \leq L$, $0 \leq \varphi(x) \leq 1/4$ for $L \leq |x| \leq L+1$, $\varphi(x) = 0$ for $|x| \geq L+1$, and $\langle \varphi_x, \varphi_x\rangle + \langle \varphi_{xx}, \varphi_{xx}\rangle \leq 2$. Since q, ζ are bounded and $\eta \geq 0$, estimate (2.3.6) easily follows. □

Lemma 2.3.3. *The sequence of functionals $\{I_\varepsilon\}$ is equicoercive on $\mathcal{U}$.*

Proof. Let $u \in \mathcal{U}$ be such that $I_\varepsilon(u) < +\infty$ and $\Psi^{\varepsilon,u}$ be as in Lemma 2.3.1. By (2.3.1), (2.3.2) and the bound (2.3.6), for each $s,t \in [0,T]$, each $L > 0$, each $\varphi \in C_c^\infty(\mathbb{R})$ supported by $[-L, L]$

$$\begin{aligned}|\langle u(t) - u(s), \varphi\rangle| &= \Big| \int_{[s,t]} dr \, \big\langle f(u) - \tfrac{\varepsilon}{2} D(u) u_x + \sigma(u) \Psi_x^{\varepsilon,u}, \varphi_x \big\rangle \Big| \\ &\le \Big\{ 2 \int_{[s,t]\times[-L,L]} dr\, dx \Big[f(u)^2 + \tfrac{\varepsilon^2}{4} D(u)^2 u_x^2 \Big] \Big\}^{1/2} \Big[|t-s| \langle \varphi_x, \varphi_x \rangle \Big]^{1/2} \\ &\quad + \Big[\int_{[s,t]} dr \langle \sigma(u) \Psi_x^{\varepsilon,u}, \Psi_x^{\varepsilon,u} \rangle \Big]^{1/2} \Big[|t-s| \langle \sigma(u) \varphi_x, \varphi_x \rangle \Big]^{1/2} \\ &\le C \Big[1 + L + I_\varepsilon(u) \Big]^{1/2} |t-s|^{1/2} \langle \varphi_x, \varphi_x \rangle^{1/2}\end{aligned}$$

for a suitable constant C depending only on f, D, and σ. Since (U, d_U) is compact, see (2.2.1), recalling (2.2.2) and the Ascoli-Arzelá theorem, the equicoercivity of $\{I_\varepsilon\}$ on $\mathcal{U}$ follows. □

As mentioned in the introduction, the assumption that σ is supported by $[0, 1]$ allows us to consider only functions u that take values in $[0, 1]$. More precisely, consider a cost functional $\hat{I}_\varepsilon$ analogous to I_ε but defined on $L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$. We next prove that, if $u \in L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$ is such that $\hat{I}_\varepsilon(u) < +\infty$ and satisfies some growth conditions, then u takes values in $[0, 1]$.

Proposition 2.3.4. *Let f, D, $\sigma : \mathbb{R} \to \mathbb{R}$; assume f Lipschitz, σ and D continuous and bounded, with $\sigma \ge 0$ and D uniformly positive. Let $\hat{I}_\varepsilon : L_{1,\mathrm{loc}}([0, T]\times\mathbb{R}) \to [0, +\infty]$ be defined as follows. If $f(u) \in L_{2,\mathrm{loc}}([0, T]\times\mathbb{R})$, we define $\hat{I}_\varepsilon(u)$ as in (2.2.6), and we set $\hat{I}_\varepsilon(u) = +\infty$ otherwise. Suppose that $u \in L_{1,\mathrm{loc}}([0, T]\times\mathbb{R})$ is such that $\hat{I}_\varepsilon(u) < +\infty$. Then $u \in C\big([0, T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. Moreover, if σ is supported by $[0, 1]$, and u is such that $u(0) \in U$ and $\int dt\, dx\, |u(t, x)| e^{-r|x|} < +\infty$ for some $r > 0$, then u takes values in $[0, 1]$, hence $u \in \mathcal{U}$.*

Proof. Let $u \in L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$ be such that $\hat{I}_\varepsilon(u) < +\infty$. By the same arguments of Lemma 2.3.2, since $f(u) \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$, $u_t = \theta_x$ holds weakly for some $\theta \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$. Hence, as in Lemma 2.3.2, $u \in C\big([0, T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. Suppose now that σ is supported by $[0, 1]$. Pick a sequence of positive entropies $\eta_n \in C^2(\mathbb{R})$ such that: $\big|\eta_n'(u)\big|$, $\eta_n''(u) \le C_n$ for some $C_n > 0$; for $u \in (0, 1)$, $\eta_n(u)$ does not depend on n and satisfies $0 < c \le \eta_n''(u) \le D(u)/\sigma(u)$; for $u \notin [0, 1]$ the sequence $\{\eta_n(u)\}$ increases pointwise to $+\infty$ as $n \to \infty$. Still following the proof of Lemma 2.3.2, for $t \in [0, T]$ and $\varphi \in C_c^\infty(\mathbb{R})$

$$\begin{aligned}\langle \eta_n(u(t)), \varphi\rangle + \tfrac{\varepsilon}{2} \int_{[0,t]} ds \, \langle D(u) \eta_n''(u) u_x^2, \varphi - 2\varphi^2 \rangle &\le \varepsilon^{-1} \hat{I}_\varepsilon(u) \\ + \langle \eta_n(u(0)), \varphi\rangle + \int_{[0,t]} ds \Big[\langle q_n(u), \varphi_x \rangle &+ \tfrac{\varepsilon}{2} \langle \zeta_n(u), \varphi_{xx} \rangle + \varepsilon\, \alpha \, \langle \varphi_x, \varphi_x \rangle \Big]\end{aligned}$$

where q_n and ζ_n are defined (up to a constant) by $q_n(v) = \int^v dw \, \eta_n'(w)\, f'(w)$ and $\zeta_n' = \eta_n' D$, and $\alpha := \max_{u\in[0,1]} D(u) \eta_n'(u)^2 / \eta_n''(u)$ is a constant independent of n, since σ is supported by $[0, 1]$. Since f is Lipschitz and D is bounded, it is possible to choose the arbitrary constants in the definition of q_n and ζ_n such that $|q_n|$, $|\zeta_n| \le C\eta_n$ for some constant $C > 0$ independent of n. In particular ζ_n, $q_n \in L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$; for each $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \le \varphi(x) \le 1/2$

$$\begin{aligned}\langle\langle \eta_n(u), \varphi \rangle\rangle \;\le\;& T\,\varepsilon^{-1} \hat{I}_\varepsilon(u) + T\,\langle \eta_n(u(0)), \varphi\rangle \\ &+ \int_{[0,T]} dt \int_{[0,t]} ds \Big[\langle q_n(u), \varphi_x\rangle + \tfrac{\varepsilon}{2}\langle \zeta_n(u), \varphi_{xx}\rangle + \varepsilon\,\alpha\,\langle \varphi_x, \varphi_x\rangle\Big]\end{aligned}$$

Let now r be such that $\int dt\, dx\, e^{-r|x|} |u(t,x)| < +\infty$. By a limiting procedure, the above bound holds for any $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1/2$ and $\sup_{x\in\mathbb{R}} e^{r|x|} \big[|\varphi(x)| + |\varphi_x(x)| + |\varphi_{xx}(x)|\big] < +\infty$. For such φ, by the choice of q_n, ζ_n

$$\begin{aligned}\tfrac{1}{T}\langle\langle \eta_n(u), \varphi\rangle\rangle \;\le\;& \varepsilon^{-1}\hat{I}_\varepsilon(u) + \langle \eta_n(u(0)), \varphi\rangle \\ &+ \varepsilon\tfrac{\alpha T}{2}\langle \varphi_x, \varphi_x\rangle + C\langle\langle \eta_n(u), |\varphi_x| + \tfrac{\varepsilon}{2}|\varphi_{xx}|\rangle\rangle\end{aligned}$$

It is easy to verify that, given $L > 0$ large enough, we can choose φ such that $\varphi(x) = 1/2$ for $|x| \le L$, $\varphi(x) = \frac{1}{2}e^{-r|x-L|}$ for $|x| > 2\,L$ and $|\varphi_{xx}(x)| \le r|\varphi_x(x)| \le r^2\,\varphi(x) \le r^2/2$ for $|x| > L$. Moreover, with no loss of generality, we can assume that $\frac{1}{T} - C\,(r + \frac{\varepsilon}{2}r^2) > 0$, otherwise we can suppose T small enough and iterate this proof. Therefore

$$\begin{aligned}\Big[\tfrac{1}{T} - C\big(r + \tfrac{\varepsilon}{2}r^2\big)\Big] & \int_{[0,T]\times[-L,L]} dt\, dx\, \eta_n(u) \\ &\le \varepsilon^{-1}\hat{I}_\varepsilon(u) + \langle \eta_n(u(0)), \varphi\rangle + \varepsilon\tfrac{\alpha T}{2}\langle \varphi_x, \varphi_x\rangle\end{aligned}$$

If $u(0) \in U$ the r.h.s. of this formula is finite and independent of n, and therefore the l.h.s. is bounded uniformly in n. Taking the limit $n \to \infty$, by the choice of η_n necessarily $u(t,x) \in [0,1]$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}$. □

The following result is not used in the sequel, but together with Lemma 2.3.1 and Proposition 2.3.4, motivates the choice of I_ε as the cost functional related to (2.1.2).

Proposition 2.3.5. *For each $\varepsilon > 0$ the functional $I_\varepsilon : \mathcal{U} \to [0,+\infty]$ is lower semicontinuous.*

Proof. Let $\{u^n\} \subset \mathcal{U}$ be a sequence converging to u in $\mathcal{U}$, and such that $I_\varepsilon(u^n)$ is bounded uniformly in n. By (2.3.6), for each $L > 0$ we have that $\int_{[0,T]\times[-L,L]} dt\, dx\, (u^n_x)^2$ is also bounded uniformly in n. Therefore, recalling definition (2.2.6), the lower semicontinuity of I_ε is established once we show that u^n converges to u strongly in $L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$. Fix $L > 0$ and pick $\chi_L \in C_c^\infty(\mathbb{R})$ such that $0 \le \chi_L \le 1$ with $\chi_L(x) = 1$ for $x \in [-L,L]$. We show that $u^{n,L} := u^n\,\chi_L$ converges to $u^L := u\,\chi_L$ in $L_2([0,T] \times \mathbb{R})$. Choose a sequence of mollifiers $\jmath_k : \mathbb{R} \to \mathbb{R}^+$ with $\int dx\, \jmath_k(x) = 1$, then

$$\begin{aligned}\big\|u^{n,L} - u^L\big\|_{L_2([0,T]\times\mathbb{R})} &\le \big\|u^{n,L} - \jmath_k * u^{n,L}\big\|_{L_2([0,T]\times\mathbb{R})} \\ &+ \big\|\jmath_k * u^{n,L} - \jmath_k * u^L\big\|_{L_2([0,T]\times\mathbb{R})} + \big\|\jmath_k * u^L - u^L\big\|_{L_2([0,T]\times\mathbb{R})}\end{aligned}$$

where the convolution is only in the space variable. For each k the second term on the r.h.s. above vanishes as $n \to \infty$ by the convergence $u^n \to u$ in $\mathcal{U}$. Since the third term vanishes as $k \to \infty$ it remains to show that the first one vanishes as $k \to \infty$ uniformly in n. Integration by parts and Young inequality for convolutions yield

$$\left\| u^{n,L} - \jmath_k * u^{n,L} \right\|_{L_2([0,T]\times\mathbb{R})} \le \left\| 1\!\!1_{[0,+\infty)} - \int_{-\infty}^{\cdot} dy\, \jmath_k(y) \right\|_{L_1(\mathbb{R})} \left\| u_x^{n,L} \right\|_{L_2([0,T]\times\mathbb{R})}$$

The uniform boundedness of $\int_{[0,T]\times[-L,L]} dt\, dx\, (u_x^n)^2$, (2.3.6) and the choice of χ_L imply that the second term on the r.h.s. is bounded uniformly in n, while the first term vanishes as $k \to \infty$. □

2.4. Γ-convergence of $\mathcal{I}_\varepsilon$

In this section we prove the Γ-convergence of the parabolic cost functional $\mathcal{I}_\varepsilon$ as $\varepsilon \to 0$, see Theorem 2.2.1. Some technical steps are postponed in Appendix 2.6.

Proof of Theorem 2.2.1: equicoercivity of $\mathcal{I}_\varepsilon$. Recall that $(\mathcal{M}, d_{\mathcal{M}})$ has been defined in (2.2.3), (2.2.4) and note that $(\mathcal{N}, d_{\mathrm{vag}})$ is compact. By Lemma 2.3.3, for each $C > 0$ there exists a compact $K_C \subset \mathcal{U}$, such that for any ε small enough $\{\mu \in \mathcal{M} : \mathcal{I}_\varepsilon(\mu) \le C\} \subset \{\mu \in \mathcal{M} : \mu_{t,x} = \delta_{u(t,x)} \text{for some } u \in K_C\} =: \mathcal{K}_C$. In order to prove that $\mathcal{K}_C$ is compact in $(\mathcal{M}, d_{\mathcal{M}})$, consider a sequence $\{\mu^n = \delta_{u^n}\} \subset \mathcal{K}_C$. Then there exists a subsequence $\{\mu^{n_j}\}$ such that, for some $\mu \in \mathcal{N}$ and $u \in \mathcal{U}$, $\mu^{n_j} \to \mu$ in $(\mathcal{N}, d_{\mathrm{vag}})$, and $\mu^{n_j}(\imath) = u^{n_j} \to u$ in $\mathcal{U}$, hence $\mu(\imath) = u$. Therefore $\mu \in \mathcal{M}$ and $\mu^{n_j} \to \mu$ in $(\mathcal{M}, d_{\mathcal{M}})$. □

Proof of Theorem 2.2.1: Γ-liminf inequality. Let $\{\mu^\varepsilon\} \subset \mathcal{M}$ be a sequence converging to μ in $\mathcal{M}$. In order to prove $\underline{\lim}_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\mu^\varepsilon) \ge \mathcal{I}(\mu)$, it is not restrictive to assume $\mathcal{I}_\varepsilon(\mu^\varepsilon) < +\infty$, and therefore $\mu^\varepsilon_{t,x} = \delta_{u^\varepsilon(t,x)}$ for some $u^\varepsilon \in \mathcal{U}$. For each $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$, recalling definition (2.2.6)

$$\begin{aligned} \mathcal{I}_\varepsilon(\mu^\varepsilon) &\ge \ell_\varepsilon^{u^\varepsilon}(\varphi) - \tfrac{1}{2}\|\varphi\|^2_{\mathcal{D}^1_{\sigma(u^\varepsilon)}} \\ &= -\langle\langle \mu^\varepsilon(\imath), \varphi_t \rangle\rangle - \langle\langle \mu^\varepsilon(f), \varphi_x \rangle\rangle - \tfrac{1}{2}\langle\langle \mu^\varepsilon(\sigma)\varphi_x, \varphi_x \rangle\rangle + \tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon) u_x^\varepsilon, \varphi_x \rangle\rangle \end{aligned}$$

Let $d \in C^1([0,1])$ be such that $d'(u) = D(u)$. Then $D(u^\varepsilon)u_x^\varepsilon = d(u^\varepsilon)_x$, and an integration by parts shows that the last term on the r.h.s. of the previous formula vanishes as $\varepsilon \to 0$. Hence

$$\underline{\lim}_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\mu^\varepsilon) \ge -\langle\langle \mu(\imath), \varphi_t \rangle\rangle - \langle\langle \mu(f), \varphi_x \rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\varphi_x, \varphi_x \rangle\rangle$$

By optimizing over $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$ the Γ-liminf inequality follows. □

PROOF OF THEOREM 2.2.1: Γ-LIMSUP INEQUALITY. Let

$$\begin{aligned}\mathcal{M}_g \;:=\; \Big\{\mu \in \mathcal{M} \,:\, \mathcal{I}(\mu) < +\infty,\, \exists r,\, L > 0,\, \exists \mu_\infty \in \mathcal{P}([0,1]) \quad \text{such that} \\ \mu(\imath),\, \mu(\sigma) \geq r,\; \mu_{t,x} = \mu_\infty \text{ for } |x| > L\Big\}\end{aligned} \tag{2.4.1}$$

$$\mathcal{M}_0 := \Big\{\mu \in \mathcal{M}_g \,:\, \mu = \delta_u \text{ for some } u \in C^1\big([0,T]\times\mathbb{R};[0,1]\big)\Big\} \tag{2.4.2}$$

and define $\tilde{\mathcal{I}} : \mathcal{M} \to [0,+\infty]$ by

$$\widetilde{\mathcal{I}}(\mu) := \begin{cases}\mathcal{I}(\mu) & \text{if } \mu \in \mathcal{M}_0 \\ +\infty & \text{otherwise}\end{cases} \tag{2.4.3}$$

We claim that for $\mu \in \mathcal{M}_0$, a recovery sequence is simply given by $\mu^\varepsilon = \mu$. Indeed, if $\mu = \delta_u$ for some $u \in C^1\big([0,T]\times\mathbb{R};[0,1]\big)$, we have

$$\begin{aligned}\mathcal{I}_\varepsilon(\mu^\varepsilon) \;&=\; I_\varepsilon(u) = \tfrac{1}{2}\Big\| u_t + f(u)_x - \tfrac{\varepsilon}{2}\big(D(u)u_x\big)_x\Big\|^2_{\mathcal{D}^{-1}_{\sigma(u)}} \\ &\leq\; \tfrac{1+\varepsilon}{2}\Big\| u_t + f(u)_x\Big\|^2_{\mathcal{D}^{-1}_{\sigma(u)}} + \tfrac{1+\varepsilon^{-1}}{2}\Big\|\tfrac{\varepsilon}{2}\big(D(u)u_x\big)_x\Big\|^2_{\mathcal{D}^{-1}_{\sigma(u)}}\end{aligned}$$

As $\mu \in \mathcal{M}_g$, u is constant for $|x|$ large enough, in particular $u_x \in L_2([0,T]\times\mathbb{R})$. Since we have also $\sigma(u) \geq r > 0$, the last term in the above formula vanishes as $\varepsilon \to 0$. Hence $\Gamma\text{-}\overline{\lim}_\varepsilon \mathcal{I}_\varepsilon \leq \tilde{\mathcal{I}}$. As well known, see e.g. [**4**, Prop. 1.28], any Γ-limsup is lower semicontinuous; the proof is then completed by Theorem 2.4.1 below. □

The relaxation of the functional $\widetilde{\mathcal{I}}$ on $\mathcal{M}$ defined in (2.4.3) might have an independent interest; in the following result we show it coincides with $\mathcal{I}$, as defined in (2.2.8).

THEOREM 2.4.1. *$\mathcal{I}$ is the lower semicontinuous envelope of $\widetilde{\mathcal{I}}$.*

The following representation of $\mathcal{I}$ is proved similarly to Lemma 2.3.1.

LEMMA 2.4.2. *Let $\mu \in \mathcal{M}$. Then $\mathcal{I}(\mu) < +\infty$ iff there exists $\Psi^\mu \in \mathcal{D}^1_{\mu(\sigma)}$ such that μ is a measure-valued solution to $u_t + f(u)_x = -\big(\sigma(u)\Psi^\mu_x\big)_x$, namely*

$$\mu(\imath)_t + \mu(f)_x = -\big(\mu(\sigma)\Psi^\mu_x\big)_x \tag{2.4.4}$$

holds weakly. In such a case Ψ^μ is unique and

$$\mathcal{I}(\mu) = \frac{1}{2}\Big\|\mu(\imath)_t + \mu(f)_x\Big\|^2_{\mathcal{D}^{-1}_{\mu(\sigma)}} = \frac{1}{2}\|\Psi^\mu\|^2_{\mathcal{D}^1_{\mu(\sigma)}}$$

Furthermore, suppose that $\mu(\sigma) \geq r$ for some constant $r > 0$. Then $\mathcal{I}(\mu) < +\infty$ iff there exists $G^\mu \in L_2([0,T]\times\mathbb{R})$ such that

$$\mu(\imath)_t + \mu(f)_x = -G^\mu_x \tag{2.4.5}$$

holds weakly. In such a case Ψ^μ_x *can be identified with a function in* $L_2([0,T]\times\mathbb{R})$*, and*

$$G^\mu = \mu(\sigma)\Psi^\mu_x, \qquad \mathcal{I}(\mu) = \frac{1}{2}\int dt\,dx \frac{\big(G^\mu(t,x)\big)^2}{\mu_{t,x}(\sigma)} \tag{2.4.6}$$

The following remark is a consequence of Lemma 2.4.2.

REMARK 2.4.3. *Let* $\{\mu^k\}\subset\mathcal{M}$ *be such that* $\mu^k\to\mu$ *in* $\mathcal{M}$, $\mathcal{I}(\mu^k)<+\infty$ *and* $\mu^k(\sigma)\ge r$ *for some* $r>0$. *Let also* G^{μ^k} *be defined as in Lemma 2.4.2. If* $\mu^k(\sigma)\to\mu(\sigma)$ *strongly in* $L_{1,\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$ *and* $\{G^{\mu^k}\}$ *is strongly precompact in* $L_2\big([0,T]\times\mathbb{R}\big)$*, then* $\mathcal{I}(\mu^k)\to\mathcal{I}(\mu)$.

Throughout the proof of Theorem 2.4.1, approximation of Young measures by piecewise smooth measures is a much used procedure. In particular we will refer repeatedly to the following result, which is a simple restatement of the Rankine-Hugoniot condition for the divergence-free vector field $(\mu(\imath),\,\mu(f)+G^\mu)$ on $(0,T)\times\mathbb{R}$.

LEMMA 2.4.4. *Let* $\gamma:(0,T)\to\mathbb{R}$ *be a Lipschitz map with a.e. derivative* $\dot\gamma$*, and let* $O^\mp\subset(0,T)\times\mathbb{R}$ *be a left, resp. a right, open neighborhood of the graph of* γ*; namely* $\mathrm{Graph}(\gamma)\subset\mathrm{Closure}(O^-)\cap\mathrm{Closure}(O^+)$*, and for all* $(t,x)\in O^-$*, resp.* $(t,x)\in O^+$*, the inequality* $x<\gamma(t)$*, resp.* $x>\gamma(t)$*, holds. Let also* $O:=O^+\cup O^-\cup\mathrm{Graph}(\gamma)$*. Suppose that a Young measure* $\mu\in\mathcal{M}$ *is such that, for each continuous function* $F\in C([0,1])$ *the map* $(t,x)\mapsto\mu_{t,x}(F)$ *is continuously differentiable in* $O^-\cup O^+$*, and denote by* $\mu^\mp(F)$ *the respective traces of* $\mu(F)$ *on the graph of* γ*. Then there exists a map* $G:O\to\mathbb{R}$*, defined up to an additive measurable function of the* t *variable, which is continuous in* $O^-\cup O^+$*, and such that* (2.4.5) *holds weakly in* O*. Moreover the Rankine-Hugoniot condition holds for a.e.* $t\in[0,T]$*, namely*

$$G^+-G^- = \big[\mu(\imath)^+-\mu(\imath)^-\big]\dot\gamma - \big[\mu(f)^+-\mu(f)^-\big] \tag{2.4.7}$$

where $G^\mp$ *are the traces of* G *on* γ *evaluated on the neighborhoods* $O^\mp$ *of* γ.

PROOF OF THEOREM 2.4.1. Since $\mathcal{I}$ is lower semicontinuous, it is enough to prove that $\mathcal{M}_0$, as defined in (2.4.2), is $\mathcal{I}$-dense in $\mathcal{M}$, namely that for each $\mu\in\mathcal{M}$ with $\mathcal{I}(\mu)<+\infty$, there exists a sequence $\{\mu^k\}\subset\mathcal{M}_0$ such that $\mu^k\to\mu$ in $\mathcal{M}$ and $\overline{\lim}_k\,\mathcal{I}(\mu^k)\le\mathcal{I}(\mu)$ (we will also say that μ^k $\mathcal{I}$-converges to μ). We split the proof in several steps.

Step 1. Here we show that $\mathcal{M}_0$ is $\mathcal{I}$-dense in the set of Young measures which are a finite convex combination of Dirac masses for a.e. (t,x). More precisely, recalling definition (2.4.1), we set

$$\begin{aligned}\mathcal{M}_1^n := \Big\{\mu\in\mathcal{M}_g\,:\,\mu=\textstyle\sum_{i=1}^n\alpha^i\delta_{u^i}\ \text{for some}\ \alpha^i\in L_\infty\big([0,T]\times\mathbb{R};[0,1]\big)\\ \text{with}\ \textstyle\sum_{i=1}^n\alpha^i=1\ \text{and}\ u^i\in L_\infty\big([0,T]\times\mathbb{R};[0,1]\big)\Big\}\end{aligned}$$

and

$$\mathcal{M}_1 := \bigcup_{n=1}^{\infty} \mathcal{M}_1^n$$

In this step, we prove that $\mathcal{M}_0$ is $\mathcal{I}$-dense in $\mathcal{M}_1$. We proceed by induction on n; to this aim, for $n \geq 1$, we introduce the auxiliary sets

$$\overline{\mathcal{M}}_1^n := \Big\{ \mu \in \mathcal{M}_g \,:\, \exists r > 0 \,:\, \mu = \textstyle\sum_{i=1}^n \alpha^i \delta_{u^i}, \text{ for some } \alpha^i \in L_\infty\big([0,T] \times \mathbb{R}; [r,1]\big) \text{ with } \sum_{i=1}^n \alpha^i = 1 \text{ and } u^i \in C^0\big([0,T] \times \mathbb{R}; [0,1]\big) \Big\}$$

$$\widetilde{\mathcal{M}}_1^n := \Big\{ \mu \in \mathcal{M}_g \,:\, \exists r > 0 \,:\, \mu = \textstyle\sum_{i=1}^n \alpha^i \delta_{u^i} \text{ for some } \alpha^i \in C^1\big([0,T] \times \mathbb{R}; [r,1]\big) \text{ with } \sum_{i=1}^n \alpha^i = 1 \text{ and } u^i \in C^1\big([0,T] \times \mathbb{R}; [r,1-r]\big), \text{ such that } u^{i+1} \geq u^i + r \Big\}$$

Note that $\widetilde{\mathcal{M}}_1^n \subset \overline{\mathcal{M}}_1^n \subset \mathcal{M}_1^n$, and $\widetilde{\mathcal{M}}_1^1 \subset \mathcal{M}_0$. We claim that for each $n \geq 1$, $\widetilde{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\overline{\mathcal{M}}_1^n$, that $\overline{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\mathcal{M}_1^n$, and that $\mathcal{M}_1^n$ is $\mathcal{I}$-dense in $\widetilde{\mathcal{M}}_1^{n+1}$. The $\mathcal{I}$-density of $\mathcal{M}_0$ in $\mathcal{M}_1$ then follows by induction. The previous claims are proved in Appendix 2.6.

Step 2. In this step we prove that $\mathcal{M}_1$ is $\mathcal{I}$-dense in $\mathcal{M}_g$, see (2.4.1). We use the following elementary extension of the mean value theorem.

LEMMA 2.4.5. *Let X be a connected compact separable metric space, $F_1, \dots, F_d \in C(X)$ be continuous functions on X, and $\mathbb{P} \in \mathcal{P}(X)$ be a Borel probability measure on X. Then there exist $\alpha^1, \dots, \alpha^d \geq 0$ with $\sum_i \alpha^i = 1$, $x^1, \dots, x^d \in X$ such that $\mathbb{P}(F^i) = \sum_{j=1}^d \alpha^j F^i(x^j)$, $i = 1, \dots, d$. Furthermore there exists a sequence $\{\mathbb{P}^n\} \subset \mathcal{P}(X)$ converging weakly* to $\mathbb{P}$, such that each $\mathbb{P}^n$ is a finite convex combination of Dirac masses, $\mathbb{P}^n(F^i) = \mathbb{P}(F^i)$ for $i = 1, \dots, d$, and for each n the map $\mathcal{P}(X) \ni \mathbb{P} \mapsto \mathbb{P}^n \in \mathcal{P}(X)$ is Borel measurable w.r.t. the weak* topology.*

PROOF. It is easy to see that the point $\mathbb{P}(F) := \big(\mathbb{P}(F_1), \dots, \mathbb{P}(F_d)\big) \in \mathbb{R}^d$ belongs to the closed convex hull of the set $B := \{\big(F_1(x), \dots, F_d(x)\big), x \in X\} \subset \mathbb{R}^d$. Since B is compact and connected, Caratheodory theorem implies that $\mathbb{P}(F)$ is a convex combination of at most d points in B, namely the first statement of the lemma holds. Since X is compact, for each integer $n \geq 1$, there exist an integer $k = k(n)$ and pairwise disjoint measurable sets $A_1^n, \dots, A_k^n \subset X$, such that $\mathbb{P}(X \setminus \cup_{l=1}^k A_l^n) = 0$, $\mathbb{P}(A_l^n) > 0$, and diameter$(A_l^n) \leq n^{-1}$, $l = 1, \dots, k$. For $l = 1, \dots, k$, let $\mathbb{P}(\cdot | A_l^n) \in \mathcal{P}(X)$ be defined by $\mathbb{P}(B|A_l^n) := \mathbb{P}(A_l^n \cap B)/\mathbb{P}(A_l^n)$ for any Borel set $B \subset X$. By the first part of the lemma, there exists a probability measure $\mathbb{P}_l^n \in \mathcal{P}(X)$, which is a convex combination of d Dirac masses, such that $\mathbb{P}_l^n(F_i) = \mathbb{P}(F_i|A_l^n)$. The sequence $\{\mathbb{P}^n\}$ defined as $\mathbb{P}^n(\cdot) := \sum_{l=1}^k \mathbb{P}(A_l^n)\mathbb{P}_l^n(\cdot)$ satisfies the requirements of the lemma. □

Let $\mu \in \mathcal{M}_g$. By Lemma 2.4.5, there exists a sequence $\{\mu^n\} \subset \mathcal{M}$ converging to μ in $\mathcal{M}$ such that $\mu_{t,x}$ is a convex combination of Dirac masses (t,x) for a.e. (t,x), and $\mu^n(\imath) = \mu(\imath)$, $\mu^n(f) = \mu(f)$, $\mu^n(\sigma) = \mu(\sigma)$. Hence $\mathcal{I}(\mu^n) = \mathcal{I}(\mu)$ and $\mu^n \in \mathcal{M}_1$.

Step 3. Recall Lemma 2.4.2 and set

$$\begin{aligned}\mathcal{M}_3 := \Big\{\mu \in \mathcal{M} \,:\, &\mathcal{I}(\mu) < +\infty,\ \exists r > 0 \text{ such that } \mu(\imath),\ \mu(\sigma) \geq r,\\ &G^\mu \in C^1\big([0,T]\times\mathbb{R}\big) \cap L_\infty\big([0,T]\times\mathbb{R}\big),\\ &\text{for each } F \in C([0,1])\ \ \mu(F) \in C^1\big([0,T]\times\mathbb{R}\big)\Big\}\end{aligned}$$

In this step we prove that $\mathcal{M}_g$ is $\mathcal{I}$-dense in $\mathcal{M}_3$.

Let $\mu \in \mathcal{M}_3$, and choose a constant $u_\infty > 0$ such that $\mu(\imath) - u_\infty > \delta$ for some $\delta > 0$. Define the maps $\gamma^k_\pm \in C([0,T]) \cap C^1((0,T))$ as the solutions to the Cauchy problems

$$\begin{cases}\dot\gamma(t) = \frac{G^\mu(t,\gamma(t)) + \mu_{t,\gamma(t)}(f) - f(u_\infty)}{\mu_{t,\gamma(t)}(\imath) - u_\infty}\\ \gamma(0) = \pm k\end{cases}$$

$\gamma^k_\pm$ are well-defined by the smoothness hypotheses on μ and G^μ. On the other hand, since we assumed G^μ to be uniformly bounded, $|\gamma^k_\pm(t) \mp k| \leq C$, for some constant $C > 0$ not depending on k. We define, for $k > C$, μ^k by $\mu^k_{t,x} = \mu_{t,x}$ if $\gamma^k_-(t) < x < \gamma^k_+(t)$ and $\mu^k_{t,x} = \delta_{u_\infty}$ otherwise. Clearly $\mu^k \to \mu$ in $\mathcal{M}$ as $k \to \infty$. We also let $G^{\mu^k}(t,x) = G^\mu(t,x)$ if $\gamma^k_-(t) < x < \gamma^k_+(t)$, and $G^{\mu^k}(t,x) = 0$ otherwise. By (2.4.7) and the definition of $\gamma^k_\pm$, the equation $\mu^k(\imath)_t + \mu^k(f)_x = -G^{\mu^k}_x$ holds weakly in $(0,T)\times\mathbb{R}$. In particular, by Lemma 2.4.2, $\mathcal{I}(\mu^k) \leq \mathcal{I}(\mu)$.

Step 4. Here we prove that $\mathcal{M}_3$ is $\mathcal{I}$-dense in

$$\mathcal{M}_4 := \{\mu \in \mathcal{M} \,:\, I(\mu) < +\infty,\ \exists r > 0 \text{ such that } \mu(\imath),\ \mu(\sigma) \geq r\}$$

Let $\mu \in \mathcal{M}_4$ and $\{\jmath^k\}_{k\geq 1} \subset C^\infty_{\mathrm{c}}(\mathbb{R}\times\mathbb{R})$ be a sequence of smooth mollifiers supported by $[-T/k, T/k] \times [-1,1]$. For $k \geq 1$, let us define the rescaled time-space variables $b^k : [0,T]\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ by

$$b^k(t,x) := \Big(\frac{t + T/k}{1 + 2T/k}, \frac{x}{1 + 2T/k}\Big) \tag{2.4.8}$$

For $k \geq 1$ we also define the Young measure μ^k by setting for $F \in C([0,1])$ and $(t,x) \in [0,T]\times\mathbb{R}$

$$\mu^k_{t,x}(F) := \int dy\, ds\, \jmath^k(t-s, x-y)\, \mu_{b^k(s,y)}(F)$$

It is immediate to see that $\mu^k \in \mathcal{M}_3$. Moreover, as $k \to \infty$, $\mu^k \to \mu$ in $\mathcal{M}$ and $\mu^k(F) \to \mu(F)$ strongly in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$ for each $F \in C([0,1])$.

Let us also define $G^{\mu^k} \in L_2([0,T]\times\mathbb{R})$ by

$$G^{\mu^k}_{t,x} := \int dy\, ds\, \jmath^k(t-s,x-y) G^\mu\big(b^k(s,y)\big)$$

Then $\mu^k(\imath)_t + \mu^k(f)_x = -G^{\mu^k}_x$ holds weakly, and $G^{\mu^k} \to G^\mu$ in $L_2([0,T]\times\mathbb{R})$ as $k\to\infty$. The proof is then achieved by Remark 2.4.3.

Step 5. $\mathcal{M}_4$ *is* $\mathcal{I}$*-dense in* $\mathcal{M}$. For $\mu \in \mathcal{M}$ with $\mathcal{I}(\mu) < +\infty$, we define $\mu^k := (1-k^{-1})\mu + k^{-1}\delta_{1/2}$. Clearly $\mu^k \to \mu$ in $\mathcal{M}$, and $\mu^k(\imath) \ge k^{-1}/2$, $\mu^k(\sigma) \ge k^{-1}\sigma(1/2)$. Therefore $\mu^k \in \mathcal{M}_4$. From (2.2.8) it follows that $\mathcal{I}$ is convex, and since $\mathcal{I}(\delta_{1/2}) = 0$, we have $\mathcal{I}(\mu^k) \le (1-k^{-1})\mathcal{I}(\mu)$. □

The following proposition is easily proved, and will be used in the proof of Corollary 2.2.2

Proposition 2.4.6. *Let X, Y be complete separable metrizable spaces, and let $\omega : X \to Y$ be continuous. Let also $\{\mathcal{F}_\varepsilon\}$ be a family of functionals $\mathcal{F}_\varepsilon : Y \to [-\infty,+\infty]$. Let us define $F_\varepsilon : X \to [-\infty,+\infty]$ by*

$$F_\varepsilon(x) = \inf_{y\in\omega^{-1}(x)} \mathcal{F}_\varepsilon(y)$$

Then

$$\big(\Gamma\text{-}\overline{\lim}_\varepsilon F_\varepsilon\big)(x) \le \inf_{y\in\omega^{-1}(x)} \big(\Gamma\text{-}\overline{\lim}_\varepsilon \mathcal{F}_\varepsilon\big)(y)$$

Furthermore if $\{\mathcal{F}_\varepsilon\}$ is equicoercive on Y then $\{F_\varepsilon\}$ is equicoercive on X. In such a case

$$\big(\Gamma\text{-}\underline{\lim}_\varepsilon F_\varepsilon\big)(x) \le \inf_{y\in\omega^{-1}(x)} \big(\Gamma\text{-}\underline{\lim}_\varepsilon \mathcal{F}_\varepsilon\big)(y)$$

Proof of Corollary 2.2.2. Since the map $\mathcal{M} \ni \mu \mapsto \mu(\imath) \in \mathcal{U}$ is continuous, by Proposition 2.4.6 we have that I_ε is equicoercive on $\mathcal{U}$ (which we already knew from Lemma 2.3.3) and Γ-converges to $I : \mathcal{U} \to [0,+\infty]$ defined by

$$I(u) = \inf_{\mu\in\mathcal{M}\,:\,\mu(\imath)=u} \mathcal{I}(\mu)$$

Recall that, if $\mathcal{I}(\mu) < +\infty$, Ψ^μ_x has been defined in Lemma 2.4.2. Equality (2.4.4) yields

$$\begin{aligned} I(u) = \inf\Big\{ &\langle\langle \mu(\sigma)\Psi^\mu_x, \Psi^\mu_x\rangle\rangle,\ \Phi \in L_{2,\mathrm{loc}}\big([0,T]\times\mathbb{R}\big),\ \mu\in\mathcal{M}\ :\\ &\mathcal{I}(\mu) < +\infty,\ \mu(\imath) = u,\ \Phi_x = -\mu(\imath)_t \text{ weakly},\ \mu(\sigma)\Psi^\mu_x = \Phi - \mu(f)\Big\}\end{aligned}$$

The corollary then follows by direct computation. □

2.5. Γ-convergence of H_ε

Proof of Proposition 2.2.3. (i) ⇒ (ii). We first show that $\|\wp_{\eta,u}\|_{\mathrm{TV},L}$ is finite for each η such that $0 \le \eta'' \le c$. It is easily seen that for each $\varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times(-L,L);[0,1]\big)$ there exists $\bar\varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times(-L,L);[0,1]\big)$ such that $\bar\varphi \ge \varphi$ and $\||\bar\varphi_t|+|\bar\varphi_x|\|_{L_1} \le 2(2L+T)$. Therefore

$$\begin{aligned}\wp_{\eta,u}(-\varphi) &= \wp_{\eta,u}(\bar\varphi-\varphi)-\wp_{\eta,u}(\bar\varphi) \le \|\wp^+_{\eta,u}\|_{\mathrm{TV},L}+\langle\langle \eta(u),\bar\varphi_t\rangle\rangle+\langle\langle q(u),\bar\varphi_x\rangle\rangle\\ &\le \|\wp^+_{\eta,u}\|_{\mathrm{TV},L}+2(\|\eta\|_\infty+\|q\|_\infty)(2L+T)\end{aligned}$$

and thus $\|\wp_{\eta,u}\|_{\mathrm{TV},L} \le 2\|\wp^+_{\eta,u}\|_{\mathrm{TV},L}+2(\|\eta\|_\infty+\|q\|_\infty)(2L+T)$.

Let now $\tilde\eta(v) := c\,v^2/2$, and for $\eta \in C^2([0,1])$ arbitrary, let $\alpha := c^{-1}\max_v|\eta''(v)|$. Then $\wp_{\eta,u} = -\alpha\wp_{\tilde\eta-\eta/\alpha,u}+\alpha\wp_{\tilde\eta,u}$. Since both $\tilde\eta-\eta/\alpha$ and $\tilde\eta$ are convex with second derivative bounded by c, $\wp_{\eta,u}$ is a linear combination of Radon measures, and thus a Radon measure itself.

(ii) ⇒ (iii). Throughout this proof, we say that η_1, $\eta_2 \in C^2([0,1])$ are equivalent, and we write $\eta_1 \sim \eta_2$, iff $\eta_1''=\eta_2''$. We identify $C^2([0,1])/\sim$ with $C([0,1])$, which we equip with the topology of uniform convergence. For $u \in \mathcal{X}$ a weak solution to (2.1.1), for $\varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times\mathbb{R}\big)$, the linear mapping $C^2([0,1]) \ni \eta \mapsto \wp_{\eta,u}(\varphi) \in \mathbb{R}$ is compatible with $\sim$, and it thus defines a linear mapping $P_{\varphi,u} : C([0,1]) \to \mathbb{R}$. It is immediate to see that $P_{\varphi,u}$ is continuous, and by (ii) for each $\eta \in C^2([0,1])$ and $L>0$

$$\sup\big\{P_{\varphi,u}(\eta''),\ \varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times(-L,L)\big),\ |\varphi|\le 1\big\} = \|\wp_{\eta,u}\|_{TV,L} < +\infty$$

By Banach-Steinhaus theorem

$$\sup\big\{P_{\varphi,u}(e),\ \varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times(-L,L)\big),\ |\varphi|\le 1,\ e\in C([0,1]),\ |e|\le 1\big\} < +\infty$$

Therefore the linear mapping $P_u^L : C([0,1]) \times C_{\mathrm{c}}^\infty\big((0,T)\times(-L,L)\big) \to \mathbb{R}$, $P_u^L(e,\varphi) := P_{\varphi,u}(e)$ can be extended to a finite Borel measure on $[0,1]\times(0,T)\times(-L,L)$. The collection $\{P_u^L\}_L$ defines a unique Radon measure P_u on $[0,1]\times(0,T)\times\mathbb{R}$, since two elements of this collection coincide on the intersection of their domains. Recalling (2.2.10), we thus gather for each $\eta \in C^2([0,1])$, for each $\varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times\mathbb{R}\big)$ and for some constant $C>0$ depending only on f

$$\Big|\int P_u(dv,dt,dx)\,\eta''(v)\varphi(t,x)\Big| = \Big|\wp_{\eta,u}(\varphi)\Big| \le C\|\varphi\|_{C^1\left((0,T)\times\mathbb{R}\right)}\int dv\,|\eta''(v)|$$

P_u thus defines a linear continuous functional on $L_1([0,1]) \times C^1_{\mathrm{c}}\big((0,T)\times\mathbb{R}\big)$. This implies that the Radon measure P_u can be disintegrated as $P_u = dv\,\varrho_u(v;dt,dx)$, for some bounded measurable map $\varrho_u : [0,1] \to M\big((0,T)\times\mathbb{R}\big)$. From the definition of P_u, we obtain for $\eta \in C^2([0,1])$, $\varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times\mathbb{R}\big)$

and $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$

$$P_{\vartheta,u} = \wp_{\eta,u}(\varphi) = \int P_u(dv,dt,dx)\eta''(v)\varphi(t,x) = \int dv\, \varrho_u(v;dt,dx)\, \vartheta''(v,t,x)$$

By linearity and density (2.2.13) holds for each entropy sampler ϑ.

(iii) ⇒ (i). It follows by choosing $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ in equation (2.2.13) for $\varphi \in C_c^\infty\big((0,T)\times\mathbb{R};[0,1]\big)$ and $\eta \in C^2([0,1])$ with $0 \le \eta'' \le c$ for an arbitrary $c > 0$. □

Proof of Theorem 2.2.5 item (ii): equicoercivity of H_ε. The equicoercivity of H_ε w.r.t. the topology generated by the $d_{\mathcal{U}}$-distance (2.2.2) follows from Lemma 2.3.3. It remains to show that, if u^ε is such that $H_\varepsilon(u^\varepsilon)$ is bounded uniformly in ε, then $\{u^\varepsilon\}$ is precompact in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$. By equicoercivity of $\{\mathcal{I}_\varepsilon\}$, the sequence $\{\mu^\varepsilon\}$ defined by $\mu^\varepsilon_{t,x} = \delta_{u^\varepsilon(t,x)}$ is precompact in $\mathcal{M}$. Therefore we have only to show that any limit point $\mu \in \mathcal{M}$ of $\{\mu^\varepsilon\}$ has the form $\mu_{t,x} = \delta_{u(t,x)}$ for some $u \in \mathcal{X}$, to obtain the existence of limit points for $\{u^\varepsilon\}$ in $\mathcal{X}$. This is implied by a compensated compactness argument due to Tartar, see [**16**, Ch. 9], provided that there is no interval where f is affine, and that, for any entropy - entropy flux pair (η, q), the sequence $\{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x\}$ is precompact in $H^{-1}_{\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$. Let us show the latter. By (2.3.5), there exists $C > 0$ such that for each $\varphi \in C_c^\infty\big((0,T)\times(-L,L)\big)$

$$\begin{aligned}
\big|\langle\langle \eta(u^\varepsilon)_t + q(u^\varepsilon)_x, \varphi\rangle\rangle\big| \;&\le\; \tfrac{\varepsilon}{2}\big|\langle\langle \eta''(u^\varepsilon)D(u^\varepsilon)u_x, \varphi\, u^\varepsilon_x\rangle\rangle\big| + \tfrac{\varepsilon}{2}\big|\langle\langle \eta'(u^\varepsilon)D(u^\varepsilon)u^\varepsilon_x, \varphi_x\rangle\rangle\big| \\
&\quad + \big|\langle\langle \eta''(u^\varepsilon)\sigma(u^\varepsilon)\, u^\varepsilon_x, \Psi^{\varepsilon,u^\varepsilon}_x \varphi\rangle\rangle\big| + \big|\langle\langle \eta'(u^\varepsilon)\sigma(u^\varepsilon)\, \Psi^{\varepsilon,u^\varepsilon}_x, \varphi_x\rangle\rangle\big| \\
&\le\; C\big[1 + H_\varepsilon(u^\varepsilon)\big]\Big[\varepsilon \textstyle\int_{[0,T]\times[-L,L]} dt\, dx\, (u^\varepsilon_x)^2\Big] \|\varphi\|_{L_\infty([0,T]\times\mathbb{R})} \\
&\quad + C\Big[\varepsilon H_\varepsilon(u^\varepsilon) + \varepsilon^2 \textstyle\int_{[0,T]\times[-L,L]} dt\, dx\, (u^\varepsilon_x)^2\Big]^{1/2} \|\varphi_x\|_{L_2([0,T]\times\mathbb{R})}
\end{aligned}$$

By the bound (2.3.6), $\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$ is the sum of a term bounded in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$ and a term vanishing in $H^{-1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$ as $\varepsilon \to 0$. By Sobolev compact embedding and boundedness of η, q, the sequence $\{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x\}$ is compact in $H^{-1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$. □

Proof of Theorem 2.2.5 item (i): Γ-liminf inequality. Let $\{u^\varepsilon\}$ be a sequence converging to u in $\mathcal{X}$. If u is not a weak solution to (2.1.1), by Theorem 2.2.1 we have $\underline{\lim}_{\varepsilon\to 0} I_\varepsilon(u^\varepsilon) \ge \mathcal{I}(\delta_u) > 0$, and therefore $\underline{\lim}_{\varepsilon\to 0} H_\varepsilon(u^\varepsilon) = +\infty$. Let now u be a weak solution to (2.1.1). With no loss of generality we can suppose $H_\varepsilon(u^\varepsilon) \le C_H$. We now consider an entropy sampler – entropy sampler flux pair (ϑ, Q) such that

$$0 \le \sigma(v)\vartheta''(v,t,x) \le D(v), \qquad (v,t,x) \in [0,1]\times(0,T)\times\mathbb{R} \tag{2.5.1}$$

We also let $\varphi^\varepsilon(t,x) = \varepsilon\vartheta'(u^\varepsilon(t,x),t,x)$, and introduce the short hand notation $\big(\vartheta'(u^\varepsilon)\big)(t,x) \equiv \vartheta'(u^\varepsilon(t,x),t,x)$, $\big(\vartheta''(u^\varepsilon)\big)(t,x) \equiv \vartheta''(u^\varepsilon(t,x),t,x)$, $\big((\partial_x\vartheta')(u^\varepsilon)\big)(t,x) \equiv \big(\partial_x\vartheta'\big)(u^\varepsilon(t,x),t,x)$. As we assumed $H_\varepsilon(u^\varepsilon) < +\infty$, u^ε_x is locally square integrable, see (2.2.6), and since ϑ is compactly supported we have $\varphi^\varepsilon_x = \varepsilon\vartheta''(u^\varepsilon)\, u^\varepsilon_x + \varepsilon(\partial_x\vartheta')(u^\varepsilon) \in L_2([0,T]\times\mathbb{R})$. The representation (2.3.7) of $\ell_\varepsilon^{u^\varepsilon}(\varphi^\varepsilon)$ thus holds, and recalling (2.2.11) we get

$$\begin{aligned}
&H_\varepsilon(u^\varepsilon) \ge \varepsilon^{-1}\ell_\varepsilon^{u^\varepsilon}(\varphi^\varepsilon) - \tfrac{\varepsilon^{-1}}{2}\|\varphi^\varepsilon\|^2_{\mathcal{D}^1_{\sigma(u^\varepsilon)}} = \langle\langle u^\varepsilon_t, \vartheta'(u^\varepsilon)\rangle\rangle + \langle\langle f(u^\varepsilon)_x, \vartheta'(u^\varepsilon)\rangle\rangle\\
&+\tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x, \vartheta''(u^\varepsilon)u^\varepsilon_x\rangle\rangle + \tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x, \big(\partial_x\vartheta'\big)(u^\varepsilon)\rangle\rangle - \tfrac{\varepsilon}{2}\langle\langle\sigma(u^\varepsilon)\vartheta''(u^\varepsilon)u^\varepsilon_x, \vartheta''(u^\varepsilon)u^\varepsilon_x\rangle\rangle\\
&-\varepsilon\langle\langle\sigma(u^\varepsilon)\vartheta''(u^\varepsilon)u^\varepsilon_x, \big(\partial_x\vartheta'\big)(u^\varepsilon)\rangle\rangle - \tfrac{\varepsilon}{2}\langle\langle\sigma(u^\varepsilon)\big(\partial_x\vartheta'\big)(u^\varepsilon), \big(\partial_x\vartheta'\big)(u^\varepsilon)\rangle\rangle\\
&= -\int dt\,dx\,\Big[\big(\partial_t\vartheta\big)\big(u^\varepsilon(t,x),t,x\big) + \big(\partial_x Q\big)\big(u^\varepsilon(t,x),t,x\big)\Big]\\
&+\tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon) - \sigma(u^\varepsilon)\vartheta''(u^\varepsilon), \vartheta''(u^\varepsilon)(u^\varepsilon_x)^2\rangle\rangle + \tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x, \big(\partial_x\vartheta'\big)(u^\varepsilon)\rangle\rangle\\
&-\varepsilon\langle\langle\sigma(u^\varepsilon)\vartheta''(u^\varepsilon)u^\varepsilon_x, \big(\partial_x\vartheta'\big)(u^\varepsilon)\rangle\rangle - \tfrac{\varepsilon}{2}\langle\langle\sigma(u^\varepsilon)\big(\partial_x\vartheta'\big)(u^\varepsilon), \big(\partial_x\vartheta'\big)(u^\varepsilon)\rangle\rangle
\end{aligned}$$

By the bound (2.3.6), the last three terms in the above formula vanish as $\varepsilon \to 0$, while $\langle\langle[D(u^\varepsilon) - \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)]u^\varepsilon_x, \vartheta''(u^\varepsilon)u^\varepsilon_x\rangle\rangle \ge 0$ for each entropy sampler ϑ satisfying (2.5.1). Therefore, taking the limit $\varepsilon \to 0$ and optimizing over ϑ

$$\varliminf_{\varepsilon\to0} H_\varepsilon(u^\varepsilon) \ge \sup_\vartheta \varliminf_{\varepsilon\to0} -\int dt\,dx\,\Big[\big(\partial_t\vartheta\big)\big(u^\varepsilon(t,x),t,x\big)+\big(\partial_x Q\big)\big(u^\varepsilon(t,x),t,x\big)\Big] = \sup_\vartheta P_{\vartheta,u}$$

where the supremum is taken on the $\vartheta \in C^{2,\infty}_{\mathrm{c}}\big([0,1]\times(0,T)\times\mathbb{R}\big)$ satisfying (2.5.1). Recalling that we assumed the l.h.s. of this formula to be finite, we next show that this inequality implies that $u \in \mathcal{E}$, and that the r.h.s. is equal to $H(u)$. By taking $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some $\varphi \in C^\infty_{\mathrm{c}}\big([0,T]\times\mathbb{R};[0,1]\big)$ and entropy η such that $0 \le \sigma(v)\eta''(v) \le D(v)$, we get $\wp_{\eta,u}(\varphi) \le \varliminf_\varepsilon H_\varepsilon(u^\varepsilon)$. Optimizing over φ it follows that u fulfills condition (i) in Proposition 2.2.3 with $c = \min_v D(v)/\sigma(v) > 0$, and thus $u \in \mathcal{E}$. By (iii) in Proposition 2.2.3 and monotone convergence we then get

$$\begin{aligned}
\varliminf_{\varepsilon\to0} H_\varepsilon(u^\varepsilon) &\ge \sup_\vartheta P_{\vartheta,u} = \sup_\vartheta \int dv\, \varrho_u(v;dt,dx)\,\vartheta''(v,t,x)\\
&= \int dv\, \varrho^+_u(v;dt,dx)\,\frac{D(v)}{\sigma(v)} = H(u)
\end{aligned} \tag{2.5.2}$$

□

LEMMA 2.5.1. *Let $f \in C^2([0,1])$, and assume that there is no interval where f is affine. Then entropy-measure solutions to* (2.1.1) *are in* $C\big([0,T];L_{1,\mathrm{loc}}(\mathbb{R})\big)$.

Let furthermore

$$V_f^+ := \max_{v\in[0,1]} f'(v) \qquad V_f^- := \min_{v\in[0,1]} f'(v) \tag{2.5.3}$$

Then for each $u \in \mathcal{E}$, $x \in \mathbb{R}$, $V > V_f^+$ or $V < V_f^-$

$$\lim_{\zeta\to0}\int dt\big|u\big(t,x+\zeta+V\,t\big) - u\big(t,x+V\,t\big)\big| = 0 \tag{2.5.4}$$

PROOF. With the same hypotheses of this lemma, in [**5**, Sect. 4] it is shown that if a weak solution u to (2.1.1) is such that $\wp_{f,u}$ is a Radon measure, then, for each $L > 0$ and $t \in [0,T)$, $\lim_{s\downarrow t}\int_{[-L,L]}|u(s,x) - u(t,x)|dx = 0$.

Therefore, by item (ii) in Proposition 2.2.3, entropy-measure solutions enjoy this property. Since the set $\mathcal{E}$ of entropy-measure solutions is invariant under the symmetry $(t,x) \mapsto (-t,-x)$, the same holds true also for $s \uparrow t$, and thus $\mathcal{E} \subset C\big([0,T];L_{1,\mathrm{loc}}(\mathbb{R})\big)$.

If u is an entropy-measure solution to the conservation law (2.1.1), then $u^{V,\pm}(t,x) := u(\pm t, x \pm Vt)$ is an entropy-measure solution to the conservation law with flux $f^{\pm}$, where $f^{\pm}(w) = f(w) \mp Vw$. With no loss of generality, we can thus prove (2.5.4) only in the case $V = 0$ with the assumption $V_f^- > 0$. In this case f is invertible on its range $[a,b]$, and we let $g \in C^2([a,b])$ be its inverse. We define $v : \mathbb{R} \times [0,T] \mapsto [a,b]$ by $v(x,t) = f(u(t,x))$. Then v satisfies

$$v_x + g(v)_t = 0 \tag{2.5.5}$$

Furthermore, if $l, m \in C^2([a,b])$ satisfy $m' = l'g'$, then by chain rule $l(v)_x + m(v)_t = \wp_{\eta,u}$, where $\eta(w) := \int^w dz\, l'(f(z))$. Therefore v is an entropy-measure solution to (2.5.5), and by the first part of this lemma

$$\lim_{\zeta \to 0} \int ds\, \big|v(x+\zeta,s) - v(x,s)\big| = 0$$

The result then follows by recalling $u(t,x) = g(v(x,t))$. □

Proof of Theorem 2.2.5 item (iii): Γ-limsup inequality. Given an nice (w.r.t. σ) solution $\tilde{u} \in \mathcal{S}_\sigma$, let $E^{\pm}$ be as in Definition 2.2.4. We want to construct a recovery sequence $\{u^\varepsilon\} \subset \mathcal{X}$ that converges to $\tilde{u}$ in $\mathcal{X}$ as $\varepsilon \to 0$, and such that $\overline{\lim}_\varepsilon H_\varepsilon(u^\varepsilon) \le H(\tilde{u})$. We split the proof in four steps. In *Step 1* we build a suitable family of rectangles contained in $[0,T] \times \mathbb{R}$. In *Step 2*, for ε, δ, $L \ge 1$, we introduce two collections $\{v^{\varepsilon,\delta,L,\pm}\}$ of auxiliary functions on $[0,T] \times \mathbb{R}$. In *Step 3*, for $N \in \mathbb{N}$ we define a collection $\{u^{\varepsilon,\delta,N,L}\} \subset \mathcal{X}$, and we prove

$$\overline{\lim_{\delta \to 0}}\, \overline{\lim_{\varepsilon \to 0}}\, H_\varepsilon(u^{\varepsilon,\delta,N,L}) \le H(\tilde{u}) \tag{2.5.6}$$

In particular $\{u^{\varepsilon,\delta,N,L}\}$ is precompact in $\mathcal{X}$. In *Step 4* we show that any limit point of $\{u^{\varepsilon,\delta,N,L}\}$ coincides with $\tilde{u}$ in $\mathcal{X}$, provided we consider the limit in ε, δ, N, L in a suitable order. More precisely we show

$$\lim_{L \to \infty} \lim_{N \to \infty} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} u^{\varepsilon,\delta,N,L} = \tilde{u} \tag{2.5.7}$$

By (2.5.6) and (2.5.7) it follows that there exist subsequences $\{\delta^\varepsilon\}$, $\{L^\varepsilon\} \subset (0,+\infty)$ and $\{N^\varepsilon\} \subset \mathbb{N}$ such that $u^\varepsilon := u^{\varepsilon,\delta^\varepsilon,N^\varepsilon,L^\varepsilon}$ provides the required recovery sequence for $\tilde{u}$.

Throughout this proof, we assume f' to be uniformly positive in $[0,1]$, namely that V_f^-, as defined in (2.5.3), is positive. As noted in the proof of Lemma 2.5.1, this assumption is not restrictive. Note also that the calculations carried out below make sense also if $E^+ = \emptyset$ or $E^- = \emptyset$.

Step 1. For each t such that $\big(\{t\}\times[-L,L]\big)\cap E^+\cap E^- = \emptyset$, the compact sets $\big(\{t\}\times[-L,L]\big)\cap E^\pm$ are disjoint, hence strictly separated. By (ii) in Definition 2.2.4, there exists a countable collection of pairwise disjoint time intervals $\{(s_i^L,t_i^L)\}_{i\in\mathbb{N}}$, with $(s_i^L,t_i^L)\subset(0,T)$ such that $\tau^L := \cup_i(s_i^L,t_i^L)$ is dense in $[0,T]$, and for each $i\in\mathbb{N}$ the two sets $E_i^{L,\pm} := \big((s_i^L,t_i^L)\times[-L,L]\big)\cap E^\pm$ are strictly separated. By splitting each of these intervals in a finite number of intervals, with no loss of generality we can assume

$$t_i^L - s_i^L < \frac{1}{4+4\,V_f^+}\,\mathrm{distance}\big(E_i^{L,+},E_i^{L,-}\big) \tag{2.5.8}$$

where V_f^+ is defined in (2.5.3), and it coincides with the Lipschitz constant of f since we supposed $V_f^->0$.

For $i\in\mathbb{N}$ let $n_i^L\in\mathbb{N}$ be such that

$$\frac{L}{n_i^L}\le\frac{1}{4}\min\big\{1,\mathrm{distance}\big(E_i^{L,+},E_i^{L,-}\big)\big\} \tag{2.5.9}$$

and consider the rectangles $R_{i,j}^L := (s_i^L,t_i^L)\times(\frac{j}{n_i^L}L,\frac{j+1}{n_i^L}L)$, for $j=-n_i^L,\,-n_i^L+1,\dots,\,n_i^L-1$. By the definition (2.5.9) of n_i^L and condition (2.5.8), for each $|j|\le n_i^L-1$

$$\mathrm{diameter}(R_{i,j-1}^L\cup R_{i,j}^L\cup R_{i,j+1}^L) < \frac{1}{2}\mathrm{distance}\big(E_i^{L,+},E_i^{L,-}\big) \tag{2.5.10}$$

In particular each $R_{i,j}^L$ has nonempty intersection with at most one of the sets E^+, E^-. We define

$$R_i^{L,\pm} := \bigcup_{\substack{j\,:\,|j|\le n_i^L-1,\\ (R_{i,j-1}^L\cup R_{i,j}^L\cup R_{i,j+1}^L)\cap E^\mp=\emptyset}} R_{i,j}^L \tag{2.5.11}$$

and for $N\in\mathbb{N}$

$$R^{N,L,\pm} := \cup_{i=1}^N R_i^{L,\pm} \qquad\qquad R^{L,\pm} := \cup_N R^{N,L,\pm} \tag{2.5.12}$$

Note that by (2.5.8) and (2.5.9)

$$R_{i,j}^L\subset\Big\{(r,x)\,:\,s_i^L<r<t_i^L,\quad \frac{j-1}{n_i^L}L+V_f^+r\le x\le\frac{j+1}{n_i^L}L-V_f^+r\Big\} \tag{2.5.13}$$

and by (2.5.10)

$$R^{L,+}\cup R^{L,-} = \bigcup_i\bigcup_{j=-n_i^L+1}^{n_i^L-1} R_{i,j}^L \tag{2.5.14}$$

Step 2. For $L \geq 1$ and $\delta \in (0, 1/2)$, let $\tilde{u}^{\delta,L} \in \mathcal{X}$ be defined by

$$\tilde{u}^{\delta,L}(t,x) := \begin{cases} \tilde{u}(t,x) & \text{if } |x| \leq L \text{ and } \tilde{u}(t,x) \in [\delta, 1-\delta] \\ \delta & \text{if } |x| \leq L \text{ and } \tilde{u}(t,x) \leq \delta \\ 1-\delta & \text{if } |x| \leq L \text{ and } \tilde{u}(t,x) \geq 1-\delta \\ 1/2 & \text{if } |x| > L \end{cases} \tag{2.5.15}$$

For $\varepsilon > 0$, $i \in \mathbb{N}$, we define $v_i^{\varepsilon,\delta,L,-} : (s_i^L, t_i^L) \times \mathbb{R} \to \mathbb{R}$ as the solution to the forward-parabolic Cauchy problem

$$\begin{cases} v_t + f(v)_x = \frac{\varepsilon}{2}\big(D(v)v_x\big)_x \\ v(s_i^L) = \tilde{u}^{\delta,L}(s_i^L) \end{cases} \tag{2.5.16}$$

and $v_i^{\varepsilon,\delta,L,+} : (s_i^L, t_i^L) \times \mathbb{R} \to \mathbb{R}$ as the solution to the backward-parabolic Cauchy problem

$$\begin{cases} v_t + f(v)_x = -\frac{\varepsilon}{2}\big(D(v)v_x\big)_x \\ v(t_i^L) = \tilde{u}^{\delta,L}(t_i^L) \end{cases} \tag{2.5.17}$$

We also define $v^{\varepsilon,\delta,L,\pm} : \tau^L \times \mathbb{R} \to \mathbb{R}$ by requiring $v^{\varepsilon,\delta,L,\pm}(r,x) = v_i^{\varepsilon,\delta,L,\pm}(r,x)$ for $r \in (s_i^L, t_i^L)$. Note that $v^{\varepsilon,\delta,L,\pm} \in C\big(\tau^L; U\big)$ and $v^{\varepsilon,\delta,L,\pm}(t,x) \in [\delta, 1-\delta]$ by maximum principle. Furthermore $v_x^{\varepsilon,\delta,L,\pm} \in L_{2,\mathrm{loc}}(\tau^L \times \mathbb{R})$, and indeed by standard parabolic estimates

$$\varepsilon \int_{R^{N,L,\pm}} dr\, dx\, \big(v_x^{\varepsilon,\delta,L,\pm}(r,x)\big)^2 \leq \sum_{i=1}^{N} \varepsilon \int_{[s_i^L, t_i^L] \times [-L,L]} dr\, dx\, \big(v_x^{\varepsilon,\delta,L,\pm}(r,x)\big)^2 \leq C^{N,L} \tag{2.5.18}$$

for some constant $C^{N,L} > 0$ independent of ε and δ.

We claim

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{R^{N,L,\pm}} dr\, dx\, \big|v^{\varepsilon,\delta,L,\pm}(r,x) - \tilde{u}(r,x)\big| = 0 \tag{2.5.19}$$

We show (2.5.19) for $v^{\varepsilon,\delta,L,-}$. The analogous statement for $v^{\varepsilon,\delta,L,+}$ follows by the fact that the set $\mathcal{S}_\sigma$ is invariant w.r.t. the symmetry $(t,x) \mapsto (-t,-x)$, while the supports of $\varrho_u^{\pm}$ are exchanged under this symmetry. By the well known results of convergence of the vanishing viscosity approximations to conservation laws (and as it also follows from the Γ-liminf inequality in Theorem 2.2.5 item (i))

$$\lim_{\varepsilon \to 0} \int_{[s_i^L, t_i^L] \times [-L,L]} dr\, dx\, \big|v^{\varepsilon,\delta,L,-}(r,x) - \bar{u}_i^{\delta,L}(r - s_i^L, x)\big| = 0 \tag{2.5.20}$$

where $\bar{u}_i^{\delta,L}$ is the Kruzkov solution to (2.1.1) with initial condition $\bar{u}_i^{\delta,L}(0,\cdot) = \tilde{u}^{\delta,L}(s_i^L,\cdot)$. On the other hand, by the definition (2.5.11) of $R_i^{L,-}$, if j is such that $R_{i,j}^L \subset R_i^{L,-}$, then $\tilde{u}$ is entropic in the rectangle $(s_i^L, t_i^L) \times (\frac{j-1}{n_i^L}L, \frac{j+1}{n_i^L}L)$, namely $\wp_{\eta,\tilde u}(\varphi) \le 0$ for each convex entropy η and each positive test function φ compactly supported in $(s_i^L, t_i^L)\times(\frac{j-1}{n_i^L}L, \frac{j+1}{n_i^L}L)$. Therefore, by Kruzkov theorem [**16**]

$$\begin{aligned}&\overline{\lim}_{\delta\to 0}\sup_{s_i^L\le r\le t_i^L}\int_{\frac{j-1}{n_i^L}L+V_f^+ r}^{\frac{j+1}{n_i^L}L-V_f^+ r} dx\big|\bar u_i^{\delta,L}(r-s_i^L,x)-\tilde u(r,x)\big|\\ &\le \overline{\lim}_{\delta\to 0}\int_{\frac{j-1}{n_i^L}L}^{\frac{j+1}{n_i^L}L} dx\big|\bar u_i^{\delta,L}(0,x)-\tilde u(s_i^L,x)\big| = \overline{\lim}_{\delta\to 0}\int_{\frac{j-1}{n_i^L}L}^{\frac{j+1}{n_i^L}L} dx\big|\tilde u^{\delta,L}(s_i^L,x)-\tilde u(s_i^L,x)\big| = 0\end{aligned}$$

and thus, fixed $N\in\mathbb N$, by (2.5.13) the convergence claimed in (2.5.19) holds on each $R_{i,j}^L$ for each $i\le N$ and each j such that $R_{i,j}^L\subset R_i^{L,-}$, and therefore on $R^{N,L,-}$ itself.

Next we claim that for each $L\ge 1$, $N\in\mathbb N$ and $\varphi\in C_c^\infty\big(R^{N,L,+};[0,1]\big)$

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{\varepsilon}{2}\langle\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+},\varphi\frac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})}v_x^{\varepsilon,\delta,L,+}\rangle\rangle \le H(\tilde u) \qquad (2.5.21)$$

Note that the l.h.s. of this formula is well defined, since $\delta\le v^{\varepsilon,\delta,L,+}\le 1-\delta$ and thus $\sigma(v^{\varepsilon,\delta,L,+})$ is uniformly positive. For each $\varphi\in C_c^\infty\big(([0,T]\times\mathbb R)\setminus E^-;[0,1]\big)$ and $\eta\in C^2\big([0,1]\big)$ such that $\sigma\eta''\le D$ we have

$$H(\tilde u)\ge\int dw\,\varrho_{\tilde u}(w;dt,dx)\,\eta''(w)\varphi(t,x)=\wp_{\eta,\tilde u}(\varphi) \qquad (2.5.22)$$

By (2.5.17) and (2.5.18) for each $\eta\in C^2([0,1])$, $N\in\mathbb N$ and $\varphi\in C_c^\infty\big(R^{N,L,+}\big)$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\langle\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+},\varphi\eta''(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}\rangle\rangle=\wp_{\eta,\tilde u}(\varphi) \qquad (2.5.23)$$

This implies (2.5.21) if σ is uniformly positive on $[0,1]$, since we can evaluate (2.5.23) on an entropy η such that $\eta''=D/\sigma$ and use the trivial bound (2.5.22). On the other hand, if $\sigma(0)=0$, resp. if $\sigma(1)=0$, then by condition (iii) in Definition 2.2.4, we have that $\tilde u(t,x)\ge\zeta_L$, resp. $\tilde u(t,x)\le 1-\zeta_L$, for a.e. $(t,x)\in(0,T)\times(-L,L)$ and for some $\zeta_L>0$. By the definition of $\tilde u^{\delta,L}$ and maximum principle, we have also $v^{\varepsilon,\delta,L,+}\ge\zeta_L$, resp. $v^{\varepsilon,\delta,L,+}\le 1-\zeta_L$, and thus (2.5.19) follows by evaluating (2.5.23) on an entropy η such that $\eta''(w)=D(w)/\sigma(w)$ for all $w\ge\zeta_L$, resp. $w\le 1-\zeta_L$.

Step 3. In this step, with a little abuse of notation, we denote by f and D two bounded continuous functions on $\mathbb R$, such they their restrictions to $[0,1]$ coincide with f and D, and f is uniformly Lipschitz and D uniformly positive. We also let $\sigma^\delta\in C^\alpha([0,1])$ be such that $\sigma^\delta(w)=\sigma(w)$ for $w\in[\delta,1-\delta]$, $\sigma^\delta(w)\le\sigma(w)$ for $w\in[0,1]$, and $\sigma^\delta(w)=0$ for $w\le\delta/2$ or $w\ge 1-\delta/2$.

For $L \ge 1$ and $N \in \mathbb{N}$, let $\Xi^{N,L} \in C_c^\infty\big(R^{N,L,+};[0,1]\big)$, and define

$$\begin{aligned} P^{N,L,+} &:= \mathrm{Interior}\Big(\big\{(t,x)\in R^{N,L,+} \,:\, \Xi^{N,L}(t,x)=1\big\}\Big)\\ P^{N,L,-} &:= \mathrm{Interior}\Big(\big\{(t,x)\in R^{N,L,-} \,:\, \Xi^{N,L}(t,x)=0\big\}\Big)\end{aligned} \tag{2.5.24}$$

For each fixed $L \ge 1$, we require the sequence $\{\Xi^{N,L}\}$ to be increasing in N and such that

$$\cup_N P^{N,L,+} = R^{L,+} \tag{2.5.25}$$

For δ, $L \ge 1$ and $N \in \mathbb{N}$ define $u^{\varepsilon,\delta,N,L} : [0,T]\times\mathbb{R}\to\mathbb{R}$ as the solution to the Cauchy problem

$$\begin{cases} u_t + f(u)_x = \frac{\varepsilon}{2}\big(D(u)u_x\big)_x - \varepsilon\Big[\Xi^{N,L}\frac{\sqrt{\sigma^\delta(u)}}{\sqrt{\sigma(v^{\varepsilon,\delta,L,+})}}D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}\Big]_x\\ u(0,x) = \tilde u^{\delta,L}(0,x) \qquad x\in\mathbb{R}\end{cases} \tag{2.5.26}$$

Note that the term in square brackets in (2.5.26) is well-defined since $v^{\varepsilon,\delta,L,+}$ is well-defined on the support of $\Xi^{N,L}$, and since $\delta \le v^{\varepsilon,\delta,L,+} \le 1-\delta$, $\sigma(v^{\varepsilon,\delta,L,+})$ is uniformly positive.

It is easily seen that the problem (2.5.26) admits at least a solution $u^{\varepsilon,\delta,N,L} \in L_\infty\big([0,T]\times\mathbb{R}\big)$ with $u_x^{\varepsilon,\delta,N,L} \in L_{2,\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$. By (2.5.26) we also gather

$$\begin{aligned} &\big\|u_t^{\varepsilon,\delta,N,L} + f(u^{\varepsilon,\delta,N,L})_x - \tfrac{\varepsilon}{2}\big(D(u^{\varepsilon,\delta,N,L})u_x^{\varepsilon,\delta,N,L}\big)_x\big\|^2_{\mathcal{D}^1_{\sigma^\delta(u)}}\\ &= \varepsilon^2\langle\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, (\Xi^{N,L})^2\tfrac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})}v_x^{\varepsilon,\delta,L,+}\rangle\rangle < +\infty\end{aligned}$$

Therefore, replacing σ with σ^δ in the statement of Proposition 2.3.4, we have $\delta \le u^{\varepsilon,\delta,N,L} \le 1-\delta$ and $u^{\varepsilon,\delta,N,L} \in \mathcal{X}$. Since $(\Xi^{N,L})^2 \in C_c^\infty(R^{N,L,+};[0,1])$, by the same estimate and (2.5.21)

$$\begin{aligned} &\overline{\lim}_\delta\,\overline{\lim}_\varepsilon\, H_\varepsilon(u^{\varepsilon,\delta,N,L})\\ &\quad= \overline{\lim}_\delta\,\overline{\lim}_\varepsilon\,\tfrac{\varepsilon}{2}\langle\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, (\Xi^{N,L})^2\tfrac{\sigma^\delta(u^{\varepsilon,\delta,N,L})}{\sigma(u^{\varepsilon,\delta,N,L})}\tfrac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})}v_x^{\varepsilon,\delta,L,+}\rangle\rangle\\ &\quad\le \overline{\lim}_\delta\,\overline{\lim}_\varepsilon\,\tfrac{\varepsilon}{2}\langle\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, (\Xi^{N,L})^2\tfrac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})}v_x^{\varepsilon,\delta,L,+}\rangle\rangle \le H(\tilde u)\end{aligned}$$

so that (2.5.6) holds.

Step 4. Since $\{H_\varepsilon\}$ is equicoercive on $\mathcal{X}$ and (2.5.6) holds, there exist δ_0, $\varepsilon_0 \equiv \varepsilon_0(\delta_0)$ small enough and a compact set $\mathcal{K}_0 \subset \mathcal{X}$ such that $u^{\varepsilon,\delta,N,L} \in \mathcal{K}_0$ for each $\varepsilon < \varepsilon_0$, $\delta < \delta_0$, $N \in \mathbb{N}$ and $L \ge 1$. In this step we show that any limit point u of $\{u^{\varepsilon,\delta,N,L}\}$ coincide with $\tilde u$, provided the limits in ε, δ, N and L are taken in a suitable order, see (2.5.7). This will conclude the proof.

Let $z^{\varepsilon,\delta,N,L,\pm} : \tau^L \times \mathbb{R} \to [-1,1]$, $z^{\varepsilon,\delta,N,L,\pm} := u^{\varepsilon,\delta,N,L} - v^{\varepsilon,\delta,L,\pm}$. By (2.3.6), (2.5.6) and (2.5.18), for each $N \in \mathbb{N}$

$$\varepsilon \int_{R^{N,L,\pm}} dt\,dx\,(z_x^{\varepsilon,\delta,N,L,\pm})^2 \le \tilde C^{N,L} \tag{2.5.27}$$

for some constant $\tilde{C}^{N,L} > 0$ independent of ε and δ.

Since we will first perform the limit $\varepsilon \to 0$, we now fix δ, N, L as above, and we drop for a few lines these indexes, thus writing $u^\varepsilon \equiv u^{\varepsilon,\delta,N,L}$, $v^{\varepsilon,\pm} \equiv v^{\varepsilon,\delta,L,\pm}$, $z^{\varepsilon,\pm} \equiv z^{\varepsilon,\delta,N,L,\pm}$, $\Xi \equiv \Xi^{N,L}$. Recalling the definition (2.5.24), by (2.5.26) and (2.5.16), we have weakly on $P^{N,L,-}$

$$z_t^{\varepsilon,-} + \big(f(u^\varepsilon) - f(v^{\varepsilon,-})\big)_x = \frac{\varepsilon}{2}\big(D(u^\varepsilon) z_x^{\varepsilon,-}\big)_x + \frac{\varepsilon}{2}\big(\big[D(u^\varepsilon) - D(v^{\varepsilon,-})\big] v_x^{\varepsilon,-}\big)_x$$

Let now $l \in C^2([-1,1])$ and $\varphi \in C_{\mathrm{c}}^\infty\big(P^{N,L,-}\big)$. It follows

$$\begin{aligned}
&-\langle\langle l(z^{\varepsilon,-}), \varphi_t\rangle\rangle - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,-}), l'(z^{\varepsilon,-})\varphi_x\rangle\rangle \\
&-\langle\langle f(u^\varepsilon) - f(v^{\varepsilon,-}), l''(z^{\varepsilon,-}) z_x^{\varepsilon,-}\varphi\rangle\rangle \\
&\qquad = -\tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,-}, l''(z^{\varepsilon,-}) z_x^{\varepsilon,-}\varphi\rangle\rangle - \tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,-}, l'(z^{\varepsilon,-})\varphi_x\rangle\rangle \\
&\qquad\quad - \tfrac{\varepsilon}{2}\langle\langle \big[D(u^\varepsilon) - D(v^{\varepsilon,-})\big] v_x^{\varepsilon,-}, l''(z^{\varepsilon,-}) z_x^{\varepsilon,-}\varphi\rangle\rangle \\
&\qquad\quad - \tfrac{\varepsilon}{2}\langle\langle \big[D(u^\varepsilon) - D(v^{\varepsilon,-})\big] v_x^{\varepsilon,-}, l'(z^{\varepsilon,-})\varphi_x\rangle\rangle
\end{aligned} \tag{2.5.28}$$

In the same fashion, by (2.5.17), weakly on $P^{N,L,+}$

$$\begin{aligned}
z_t^{\varepsilon,+} + \big(f(u^\varepsilon) - f(v^{\varepsilon,+})\big)_x &= \tfrac{\varepsilon}{2}\big(D(u^\varepsilon) z_x^{\varepsilon,+}\big)_x + \tfrac{\varepsilon}{2}\big(\big[D(u^\varepsilon) - D(v^{\varepsilon,+})\big] v_x^{\varepsilon,+}\big)_x \\
&\quad + \varepsilon\Big(\big[\sqrt{\sigma(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\big]\tfrac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}\Big)_x
\end{aligned}$$

Since $v^{\varepsilon,+}$ takes values in $[\delta, 1-\delta]$, we have $\sigma^\delta(v^{\varepsilon,+}) = \sigma(v^{\varepsilon,+})$ and thus, in the same fashion as above, for each $l \in C^2([-1,1])$ and $\varphi \in C_{\mathrm{c}}^\infty\big(P^{N,L,+}\big)$

$$\begin{aligned}
&-\langle\langle l(z^{\varepsilon,+}), \varphi_t\rangle\rangle - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,+}), l'(z^{\varepsilon,+})\varphi_x\rangle\rangle - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,+}), l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\varphi\rangle\rangle \\
&\quad = -\tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,+}, l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\varphi\rangle\rangle - \tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x\rangle\rangle \\
&\qquad - \tfrac{\varepsilon}{2}\langle\langle \big[D(u^\varepsilon) - D(v^{\varepsilon,+})\big] v_x^{\varepsilon,+}, l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\varphi\rangle\rangle \\
&\qquad - \tfrac{\varepsilon}{2}\langle\langle \big[D(u^\varepsilon) - D(v^{\varepsilon,+})\big] v_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x\rangle\rangle \\
&\qquad - \varepsilon\langle\langle \big[\sqrt{\sigma^\delta(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\big]\tfrac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}, \varphi l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\rangle\rangle \\
&\qquad - \varepsilon\langle\langle \big[\sqrt{\sigma^\delta(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\big]\tfrac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x\rangle\rangle
\end{aligned} \tag{2.5.29}$$

For l convex and φ nonnegative, the first term in the second lines of (2.5.28) and (2.5.29) is nonpositive. With these assumptions on l and φ we thus define $B_l \equiv B_{l,\varphi}^{\varepsilon,\delta,N,L,\pm} := \big[\langle\langle D(u^\varepsilon) z_x^{\varepsilon,\pm}, l''(z^{\varepsilon,\pm}) z_x^{\varepsilon,\pm}\varphi\rangle\rangle\big]^{1/2}$, and let for $F \in C([0,1])$

$$C_{F,l}^\delta := \max\{l''(z)|F(v+z) - F(v)|^2 \,:\, v \in [\delta, 1-\delta],\, z \in [-1,1],\, v+z \in [0,1]\} \tag{2.5.30}$$

Since $v_x^{\varepsilon,\pm}$, $z_x^{\varepsilon,\pm} \in L_{2,\mathrm{loc}}\big(P^{N,L,\pm}\big)$, by Cauchy-Schwarz inequality and the fact that D is uniformly positive, we have for each nonnegative $\varphi^\pm \in C_{\mathrm{c}}^\infty\big(P^{N,L,\pm}\big)$, and for some constant $C \equiv C_{\varphi^\pm}^{\varepsilon,\delta,N,L}$ independent of l

$$\begin{aligned}\big|\langle\langle f(u^\varepsilon)-f(v^{\varepsilon,\pm}), l''(z^{\varepsilon,\pm})z_x^{\varepsilon,\pm}\varphi^\pm\rangle\rangle\big| + \big|\tfrac{\varepsilon}{2}\langle\langle\big[D(u^\varepsilon)-D(v^{\varepsilon,\pm})\big]v_x^{\varepsilon,\pm}, l''(z^{\varepsilon,\pm})z_x^{\varepsilon,\pm}\varphi^\pm\rangle\rangle\big| \\ +\big|\varepsilon\langle\langle\big[\sqrt{\sigma^\delta(v^{\varepsilon,+})}-\sqrt{\sigma^\delta(u^\varepsilon)}\big]\tfrac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}}v_x^{\varepsilon,+}, \varphi^+ l''(z^{\varepsilon,+})z_x^{\varepsilon,+}\rangle\rangle\big| \\ \le C\,\Big[\sqrt{C_{f,l}^\delta}+\sqrt{C_{D,l}^\delta}+\sqrt{C_{\sqrt{\sigma^\delta},l}^\delta}\Big]B_l\end{aligned}$$

We also let $C_l := \max_{z\in[-1,1]}|l'(z)|$ and note that, in view of (2.5.18) and (2.5.27), for any nonnegative $\varphi^\pm \in C_c^\infty\big(P^{N,L,\pm}\big)$ and for some constant $\tilde C = \tilde C_{\varphi^\pm}^{\delta,N,L}$ independent of ε and l

$$\begin{aligned}\big|\tfrac{\varepsilon}{2}\langle\langle D(u^\varepsilon)z_x^{\varepsilon,\pm}, l'(z^{\varepsilon,\pm})\varphi_x^\pm\rangle\rangle\big| + \big|\tfrac{\varepsilon}{2}\langle\langle\big[D(u^\varepsilon)-D(v^{\varepsilon,\pm})\big]v_x^{\varepsilon,\pm}, l'(z^{\varepsilon,\pm})\varphi_x^\pm\rangle\rangle\big| \\ +\big|\varepsilon\langle\langle[\sqrt{\sigma(v^{\varepsilon,+})}-\sqrt{\sigma(u^\varepsilon)}]\tfrac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}}v_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x^+\rangle\rangle\big| \le \tilde C C_l\sqrt{\varepsilon}\end{aligned}$$

Patching all together, for each nonnegative $\varphi^\pm \in C_c^\infty\big(P^{N,L,\pm}\big)$ we gather

$$\begin{aligned}-\langle\langle l(z^{\varepsilon,\pm}),\varphi_t^\pm\rangle\rangle - \langle\langle f(u^\varepsilon)-f(v^{\varepsilon,\pm}), l'(z^{\varepsilon,\pm})\varphi_x^\pm\rangle\rangle \\ \le -\tfrac{\varepsilon}{2}B_l^2 + C\,\Big[\sqrt{C_{f,l}^\delta}+\sqrt{C_{D,l}^\delta}+\sqrt{C_{\sqrt{\sigma^\delta},l}^\delta}\Big]B_l + \tilde C C_l\sqrt{\varepsilon} \\ \le \tfrac{3}{2\varepsilon}C^2\big[C_{f,l}^\delta + C_{D,l}^\delta + C_{\sqrt{\sigma^\delta},l}^\delta\big] + \tilde C C_l\sqrt{\varepsilon}\end{aligned} \tag{2.5.31}$$

It is then easily seen that we can take a sequence of convex smooth functions $\{l_n\}\subset C^2([-1,1])$ such that $|l_n'(z)|\le 1$, $l_n(z)\to|z|$, $zl_n'(z)\to|z|$ uniformly on $[-1,1]$, and such that, by the Hölder continuity hypotheses on D and σ

$$\lim_{n\to\infty}\big(C_{f,l_n}^\delta + C_{D,l_n}^\delta + C_{\sqrt{\sigma^\delta},l_n}^\delta\big)=0$$

Evaluating (2.5.31) for $l\equiv l_n$, taking the limit $n\to\infty$, and recalling that we assumed f' to be positive on $[0,1]$, we gather for each nonnegative $\varphi^\pm \in C_c^\infty\big(P^{N,L,\pm}\big)$

$$-\langle\langle|u^\varepsilon - v^{\varepsilon,\pm}|,\varphi_t^\pm\rangle\rangle - \langle\langle\big|f(u^\varepsilon)-f(v^{\varepsilon,\pm})\big|,\varphi_x^\pm\rangle\rangle \le C'\sqrt{\varepsilon} \tag{2.5.32}$$

We now reintroduce the dropped indexes δ, N, L, and recall that for $\delta\le\delta_0$, $\varepsilon\le\varepsilon_0(\delta_0)$, $N\in\mathbb{N}$ and $L\ge 1$ we have $u^{\varepsilon,\delta,N,L}\in\mathcal{K}_0$ for some compact $\mathcal{K}_0\subset\mathcal{X}$. Let $u^{N,L}\in\mathcal{K}_0$ be a generic limit point of $\{u^{\varepsilon,\delta,N,L}\}$ in $\mathcal{X}$ as $\varepsilon\to 0$ and successively $\delta\to 0$. By (2.5.19) and (2.5.32), for each nonnegative $\varphi\in C_c^\infty\big(P^{N,L,-}\cup P^{N,L,+}\big)$

$$-\langle\langle|u^{N,L}-\tilde u|,\varphi_t\rangle\rangle - \langle\langle\big|f(u^{N,L})-f(\tilde u)\big|,\varphi_x\rangle\rangle \le 0 \tag{2.5.33}$$

Since $u^{N,L}\in\mathcal{K}_0$, there exist $u^L\in\mathcal{X}$ and a subsequence $\{N_k\}\subset\mathbb{N}$ such that $u^{N_k,L}\to u^L$ in $\mathcal{X}$ as $k\to+\infty$. By (2.5.25) and (2.5.33), it follows that for each nonnegative $\varphi\in C_c^\infty\big(R^{L,-}\cup R^{L,+}\big)$

$$-\langle\langle|u^L-\tilde u|,\varphi_t\rangle\rangle - \langle\langle\big|f(u^L)-f(\tilde u)\big|,\varphi_x\rangle\rangle \le 0 \tag{2.5.34}$$

Since τ^L is dense in $[0,T]$, by (2.5.14) and (2.5.9) we have that, for $L \ge 1$, $R^{L,+} \cup R^{L,-}$ is dense in $[0,T] \times \big[-L+\frac{1}{4L}, L-\frac{1}{4L}\big]$. Note also that $\tilde u \in \mathcal{S}_\sigma \subset \mathcal{E}$ by hypotheses. Furthermore, since u^L is a limit point of a sequence with uniformly bounded H_ε-cost, we also have $u^L \in \mathcal{E}$ by item (ii) in Theorem 2.2.5, namely $\tilde u$ and u^L are entropy-measure solutions to (2.1.1). By Lemma 2.5.1, $\tilde u$, $u^L \in C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. By the same Lemma 2.5.1 and the assumption $V_f^- > 0$ we have that the maps $x \mapsto \tilde u(t,x)$ and $x \mapsto u^L(t,x)$ are continuous from $\mathbb{R}$ to $L_1\big([0,T]\big)$. Therefore, since the boundaries of $R^{L,+}$ and $R^{L,-} \setminus R^{L,+}$ are countable unions of segments parallel to the x and t axes, we have that (2.5.34) holds for each nonnegative $\varphi \in C_c^\infty\big((0,T) \times (-L+\frac{1}{4L}, L-\frac{1}{4L})\big)$.

Recalling $\{u^L\} \subset \mathcal{K}_0$, let u be a limit point of $\{u^L\}$ along a subsequence $L_k \to \infty$. From (2.5.34) we get for each nonnegative $\varphi \in C_c^\infty\big((0,T) \times \mathbb{R}\big)$

$$-\langle\langle |u-\tilde u|, \varphi_t\rangle\rangle - \langle\langle \big|f(u)-f(\tilde u)\big|, \varphi_x\rangle\rangle \le 0 \tag{2.5.35}$$

Reasoning as above, we also have $u \in \mathcal{E}$, and thus setting $z := u - \tilde u$, by Lemma 2.5.1, u, $\tilde u$, $z \in C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. By (2.5.35), it is then easily seen that for each bounded nonnegative Lipschitz function φ on $[0,T]\times\mathbb{R}$ such that $\int dt\, dx\, [|\varphi| + |\varphi_t| + |\varphi_x|] < +\infty$, and for each $t \in [0,T]$

$$\begin{aligned}\langle |z(t)|, \varphi(t)\rangle - \langle |z(0)|, \varphi(0)\rangle - \textstyle\int_{[0,t]} dr \big[\langle |z|(r), \varphi_r(r)\rangle \\ + \langle \big|f(\tilde u(r) + z(r)) - f(\tilde u(r))\big|, \varphi_x(r)\rangle\big] \le 0\end{aligned} \tag{2.5.36}$$

Fixed $L \ge 1$, we evaluate the inequality (2.5.36) for $\varphi(t,x) \equiv \varphi^L(x)$ defined as

$$\varphi^L(x) := \begin{cases} e^{-(L-x)} & \text{if } x < -L \\ 1 & \text{if } -L \le x \le L \\ e^{-(x-L)} & \text{if } x > L \end{cases}$$

so that setting $Z^L(t) := \langle |z(t)|, \varphi^L\rangle$ we have

$$Z^L(t) - Z^L(0) \le V_f^+ \int_{[0,t]} dr\, \langle |z|(r), |\varphi_x^L|\rangle \le V_f^+ \int_{[0,t]} dr\, Z^L(r)$$

By Gronwall inequality, for each $L\ ge 1$ and $t \in [0,T]$, $Z^L(r) \le \exp[V_f^+\, t] Z^L(0)$. Note that $u(0,x) = \tilde u(0,x)$ by (2.5.15) and the definition of convergence in $\mathcal{X}$. Therefore $Z^L(0) = 0$, and thus $Z^L(t) = 0$ for each $t \in [0,T]$ and $L \ge 1$. Hence $u = \tilde u$. □

Proof of Proposition 2.2.6. In order to show that H is lower semicontinuous, first note that the set of weak solutions is closed in $\mathcal{X}$. Moreover for each entropy sampler ϑ the map $X \ni u \mapsto P_{\vartheta,u} \in \mathbb{R}$ is continuous. On the other hand, if u is a weak solution to (2.1.1) then the equalities in (2.5.2) holds, and thus H is a supremum of continuous maps.

Since $D(\cdot)/\sigma(\cdot)$ is uniformly positive on $[0,1]$, $H(u) = 0$ iff $u \in \mathcal{E}$ and $\varrho_u^+ = 0$, thus u is entropic. Conversely, entropic solutions u are in $\mathcal{E}$ by item (i) in Proposition 2.2.3, and the entropic condition is thus equivalent to $\varrho_u^+ = 0$.

The coercivity of H follows from the Tartar's method of compensated compactness, that we already applied in the proof of Theorem 2.2.5 item (ii). Suppose indeed that we are given a sequence $\{u^n\} \subset \mathcal{X}$ such that $H(u^n) \le C_H < +\infty$ for each n. Then each u^n is an entropy-measure solution to (2.1.1) by the definition of H. For each entropy η, each n, $L > 0$, by the same bound in the proof of Proposition 2.2.3, $\|\wp_{\eta,u^n}\|_{\mathrm{TV},L} \le 2\|\wp^+_{\eta,u^n}\|_{\mathrm{TV},L} + 2\,(\|\eta\|_\infty + \|q\|_\infty)(2L+T)$. On the other hand, for each $\eta \in C^2([0,1])$ such that $\sigma\eta'' \le D$, $\|\wp^+_{\eta,u^n}\|_{\mathrm{TV},L} \le H(u^n)$ and therefore $\|\wp_{\eta,u^n}\|_{\mathrm{TV},L}$ is bounded uniformly in n. Since η and q are bounded, we have that $\{\eta(u^n)_t + q(u^n)_x\}$ is precompact in $H^{-1}_{\mathrm{loc}}([0,T] \times \mathbb{R})$. As we already noted in the proof of Theorem 2.2.5 item (ii), see [**16**, Ch. 9], this yields the compactness of $\{u^n\}$ in $\mathcal{X}$.

The last statement follows by the first part of proposition, Lemma 2.5.1 and Kruzkov uniqueness in $C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$ of entropic solutions to (2.1.1), see e.g. [**6, 16**]. □

Proof of Remark 2.2.7. By well known properties of functions of locally bounded variation, for each entropy η and $u \in \mathcal{X} \cap BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$ we have that $\wp_{\eta,u}$ is a Radon measure on $(0,T) \times \mathbb{R}$. If u is a weak solution to (2.1.1), by Vol'pert chain rule [**2**], the absolutely continuous and Cantor parts of $\wp_{\eta,u}$ w.r.t. the Lebesgue measure on $(0,T) \times \mathbb{R}$ vanish, and we get

$$d\wp_{\eta,u} = \Big\{ \big[\eta(u^+) - \eta(u^-)\big] n^t + \big[q(u^+) - q(u^-)\big] n^x \Big\}\, d\mathcal{H}^1 \llcorner J_u$$

On the other hand the Rankine-Hugoniot condition $\big[u^+ - u^-\big] n^t + \big[f(u^+) - f(u^-)\big] n^x = 0$ holds. The statement of the remark follows by direct calculation. □

Proof of Remark 2.2.11. For $u \in \mathcal{E}$ we have

$$H'(u) = \sup \big\{ \wp_{\eta,u}(\varphi), \varphi \in C^\infty_{\mathrm{c}}\big((0,T)\times\mathbb{R}; [0,1]\big),\, \eta \in C^2([0,1]) \,:\, 0 \le \sigma\eta'' \le D \big\}$$

so that the inequality $H \ge H'$ follows from the equalities in (2.5.2). The same inequality yields $H(u) = H'(u)$ if there exists a set E^+ as in the statement of the remark. If f is convex or concave and u has locally bounded variation, we can take $E^+ = \{(t,x) \in J_u \,:\, \exists v \in [0,1] \,:\, \rho(v, u^+, u^-) > 0\}$, where J_u, $u^\pm$ and ρ are defined as in Remark 2.2.7.

If f is neither convex nor concave, then there exist u^-, u^+, v', $v'' \in (0,1)$ such that $\rho(v', u^+, u^-) > 0$ and $\rho(v'', u^+, u^-) < 0$, where ρ is defined as in

Remark 2.2.7. Let $V := \frac{f(u^+)-f(u^-)}{u^+-u^-}$, and define $u : [0,T] \times \mathbb{R} \to [0,1]$ by

$$u(t,x) := \begin{cases} u^+ & \text{for } x < V\,t \\ u^- & \text{for } x > V\,t \end{cases}$$

Then $u \in \mathcal{E}$ and by direct computation $H(u) > H'(u)$. □

2.6. Appendix A: $\mathcal{I}$-approximation of atomic Young measures

Here we prove the claims stated in the proof of Theorem 2.4.1, *Step 1*, where the sets $\mathcal{M}_n^1$, $\widetilde{\mathcal{M}}_n^1$, $\overline{\mathcal{M}}_n^1$ are defined.

Claim 1: $\widetilde{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\overline{\mathcal{M}}_1^n$. For $n \geq 1$, let $\mu \in \overline{\mathcal{M}}_1^n$, let G^μ be defined as in Lemma 2.4.2. Let also r, α^i, u^i be as in the definition of $\overline{\mathcal{M}}_1^n$ and L, μ_∞ be as in the definition (2.4.1) of $\mathcal{M}_g$. With no loss of generality, we can assume that $u^{i+1} \geq u^i$, $i = 1, \dots, n-1$, since we can reorder the $u^i(t,x)$ for all (t,x) preserving continuity of the u^i and measurability of the α^i. Analogously it is not restrictive to assume, for $|x| > L$, $u^i(t,x) = u^i_\infty$, $\alpha^i(t,x) = \alpha^i_\infty$ for some constants u^i_∞, $\alpha^i_\infty \in (0,1]$; in particular $\mu_\infty = \sum_i \alpha^i_\infty \delta_{u^i_\infty}$.

Let now $\{\jmath^k\} \subset C_c^\infty(\mathbb{R} \times \mathbb{R})$ be a sequence of smooth mollifiers supported by $[-T/k, T/k] \times [-1, 1]$, and recall the definition (2.4.8) of b^k. For $i = 1, \dots, n$ and $h, k \geq 1$ define $\alpha^{i;k} \in C^1\big([0,T] \times \mathbb{R}; [r,1]\big)$, and $u^{i;h,k} \in C^1\big([0,T] \times \mathbb{R}; [h^{-1}, 1-h^{-1}]\big)$ by

$$\begin{aligned} &\alpha^{i;k}(t,x) := \int dy\, ds\, \jmath^k(t-s, x-y) \alpha^i\big(b^k(s,y)\big) \\ &u^{i;h,k}(t,x) := h^{-1}\Big[1 + \tfrac{i}{n \sum_{i'} i' \alpha^{i';k}(t,x)}\Big] \\ &\qquad + \tfrac{1-3\,h^{-1}}{\alpha^{i;k}(t,x)} \int dy\, ds\, \jmath^k(t-s, x-y)\, \alpha^i\big(b^k(s,y)\big)\, u^i\big(b^k(s,y)\big) \end{aligned} \tag{2.6.1}$$

Clearly $\alpha^{i;k}$ and $u^{i;h,k}$ are smooth, with $\alpha^{i;k} \geq r$, $\sum_i \alpha^{i;k} = 1$, and $\alpha^{i;k}$, $u^{i;h,k}$ are constant for $|x| > L+1$. Furthermore for $i = 1, \dots, n-1$ and $(t,x) \in [0,T] \times \mathbb{R}$

$$\begin{aligned} &\lim_{k\to\infty} \big[u^{i+1;h,k}(t,x) - u^{i;h,k}(t,x) - \tfrac{h^{-1}}{n^2}\big] \\ &\geq \lim_{k\to\infty} \Big[u^{i+1;h,k}(t,x) - u^{i;h,k}(t,x) - \tfrac{1}{n \sum_{i'} i' \alpha^{i';k}(t,x)}\Big] \\ &= \big[1 - 3h^{-1}\big]\big[u^{i+1;h,k}(t,x) - u^{i;h,k}(t,x)\big] \end{aligned}$$

Since the u^i are continuous, it is not difficult to see that convergence in the last line above is uniform on compact subsets of $[0,T] \times \mathbb{R}$. On the other hand, since the u^i and $u^{i;h,k}$ are constant for $|x| > L+1$, we have that convergence is indeed uniform on $[0,T] \times \mathbb{R}$. It follows that for each $h > 1$ there exists $K^h \geq 1$ such that $u^{i+1;h,k} \geq u^{i;h,k} + h^{-1} n^{-2}$ for each $k \geq K^h$. Therefore, defining $\mu^{h,k} \in \mathcal{M}$ by

$$\mu^{h,k}_{t,x} := \sum_{i=1}^n \alpha^{i;k}(t,x) \delta_{u^{i;h,k}(t,x)}$$

we get, for $k \ge K^h$, $\mu^{h,k} \in \widetilde{\mathcal{M}}_1^n$ provided $\mathcal{I}(\mu^{h,k}) < +\infty$. Recalling Lemma 2.4.2, this follows by the existence of $G^{\mu^{h,k}} \in L_2([0,T] \times \mathbb{R})$ satisfying weakly on $(0,T) \times \mathbb{R}$:

$$\mu^{h,k}(\imath)_t + \mu^{h,k}(f)_x = -G_x^{\mu^{h,k}}$$

Indeed $G^{\mu^{h,k}}$ can be computed explicitly as

$$\begin{aligned} G^{\mu^{h,k}}(t,x) \ := \ & \big(1-3h^{-1}\big) \int ds\, dy\, \jmath^k(t-s, x-y) G^{\mu}\big(b^k(s,y)\big) \\ & + \big(1-3h^{-1}\big) \int ds\, dy\, \jmath^k(x-y, t-s)\, \mu_{b^k(s,y)}(f) \\ & - \mu^{h,k}(f) - (1-3h^{-1})\mu_\infty(f) + \mu_\infty^{h,k}(f) \end{aligned}$$

where

$$\mu_\infty^{h,k}(f) := \sum_i \alpha_\infty^i f\Big(h^{-1} + \frac{i\, h^{-1}}{n + \sum_{i'} i' \alpha_\infty^{i'}} + \big(1-3h^{-1}\big) u_\infty^i\Big)$$

It immediately follows that $\lim_{h\to\infty} \lim_{k\to\infty} \|G^{\mu^{h,k}} - G^{\mu}\|_{L_2([0,T]\times\mathbb{R})} = 0$, and it is also straightforward to see that, for each $F \in C([0,1])$

$$\lim_{h\to\infty} \lim_{k\to\infty} \mu^{h,k}(F) = \mu(F) \qquad \text{strongly in } L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$$

By Remark 2.4.3, we can extract a subsequence $\{\mu^k\}$ from $\{\mu^{h,k}\}$ that $\mathcal{I}$-converges to μ.

Claim 2: $\overline{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\mathcal{M}_1^n$. For $n \ge 1$, let $\mu \in \mathcal{M}_1^n$. Let also α^i, u^i and L be as in the definition of $\mathcal{M}_1^n$ and $\mathcal{M}_g$. With no loss of generality, we can assume that $\alpha^i > 0$, since we do not require the u^i to be distinct. As in *Claim 1* above, we can also assume that, for $|x| > L$, $u^i(t,x) = u_\infty^i$, $\alpha^i(t,x) = \alpha_\infty^i$ for some constants u_∞^i, $\alpha_\infty^i \in [0,1]$.

With these assumptions, for h, $k \ge 1$ and $i = 1, \dots, n$, let us define $\alpha^{i;k}$ as in (2.6.1), and $u^{i;k}$ by

$$u^{i;k}(t,x) := \tfrac{1}{\alpha^{i;k}} \int dy\, ds\, \jmath^k(t-s, x-y) \alpha^i\big(b^k(s,y)\big) u^i\big(b^k(s,y)\big)$$

Letting

$$\mu_{t,x}^k := \sum_{i=1}^n \alpha^{i;k}(t,x) \delta_{u^{i;k}(t,x)}$$

we gather $\mu^k \in \overline{\mathcal{M}}_1^n$. A computation similar to the one carried out in *Claim 1* shows that μ^k $\mathcal{I}$-converges to μ as $k \to \infty$.

Claim 3: $\mathcal{M}_1^n$ is $\mathcal{I}$-dense in $\widetilde{\mathcal{M}}_1^{n+1}$. This is the key step in the proof of Theorem 2.4.1. For $n \ge 1$, let $\mu \in \widetilde{\mathcal{M}}_1^{n+1}$, and let G^{μ} be defined as in Lemma 2.4.2. Let also r, α^i, u^i be as in the definition of $\widetilde{\mathcal{M}}_1^{n+1}$, and L, μ_∞ as in the definition (2.4.1) of $\mathcal{M}_g$. Note that for $|x| > L$, $\alpha^i(t,x) = \alpha_\infty^i$ and $u^i(t,x) = u_\infty^i$ for some constants $\alpha_\infty^i \in [r, 1-r]$, $u_\infty^i \in [r, 1-r]$, with $u_\infty^{i+1} \ge u_\infty^i + r$, $i = 1, \dots, n$.

Let us define the Young measures ν^1, $\nu^0 \in \widetilde{\mathcal{M}}_1^n$ by

$$\nu^1_{t,x} := \delta_{u^{n+1}(t,x)} \qquad \nu^0_{t,x} := \sum_{i=1}^n \frac{\alpha^i(t,x)}{1-\alpha^{n+1}(t,x)} \delta_{u^i(t,x)}$$

so that, letting $\beta(t,x) := \alpha^{n+1}(t,x) \le 1-r$

$$\mu_{t,x} = \beta(t,x)\nu^1_{t,x} + \big(1-\beta(t,x)\big)\nu^0_{t,x}$$

The basic idea is to build up a sequence $\{\mu^k\}$ $\mathcal{I}$-converging to μ, as follows: we first slice up $[0,T]\times\mathbb{R}$ in small strips, alternating a strip of width $\beta\, k^{-1}$ with a strip of width $(1-\beta)k^{-1}$; we then set $\mu^k_{t,x} = \nu^1_{t,x}$ for (t,x) in the first family of strips, and $\mu^k_{t,x} = \nu^0_{t,x}$ for (t,x) in the second family of strips. As we let $k\to\infty$, we easily get $\mu^k \to \mu$; however, to get also $\mathcal{I}(\mu^k)\to\mathcal{I}(\mu)$, we will have to carefully define these strips.

For $j\in\mathbb{Z}$ and $k\in\mathbb{N}$, let us consider the maps $\gamma^k_j : [0,T]\to\mathbb{R}$ solutions to

$$\begin{cases} \dot\gamma = \frac{\nu^1_{t,\gamma}(f)-\nu^0_{t,\gamma}(f)}{\nu^1_{t,\gamma}(\imath)-\nu^0_{t,\gamma}(\imath)} \\ \gamma(0) = \frac{j}{k} \end{cases} \tag{2.6.2}$$

These equations are well-posed since $\nu^1(f)$, $\nu^0(f)$, $\nu^1(\imath)$, $\nu^0(\imath)$ are Lipschitz functions in the (t,x) variables, and $\nu^1(\imath)-\nu^0(\imath)\ge r$, by the definition of $\widetilde{\mathcal{M}}_1^{n+1}$. Furthermore, by standard theory for (2.6.2), $\gamma^k_j \in C^0([0,T])\cap C^1((0,T))$; $|\dot\gamma^k_j| \le 2r^{-1}\max_{v\in[0,1]}|f(v)|$; $\gamma^k_{j+1} > \gamma^k_j$; and $\gamma^k_{j+1}(t)-\gamma^k_j(t)\le Ck^{-1}$ for some constant C independent of k, j and t.

We next define the maps $\beta^k_j : [0,T]\to\mathbb{R}$ by

$$\int_{\gamma^k_j(t)}^{\gamma^k_j(t)+\beta^k_j(t)} dx\,\big[\nu^1_{t,x}(\imath)-\nu^0_{t,x}(\imath)\big] = \int_{\gamma^k_j(t)}^{\gamma^k_{j+1}(t)} dx\,\beta(t,x)\big[\nu^1_{t,x}(\imath)-\nu^0_{t,x}(\imath)\big] \tag{2.6.3}$$

Since $\nu^1_{t,x}(\imath)-\nu^0_{t,x}(\imath)\ge r>0$, for any fixed $t\in[0,T]$ the l.h.s. of this equation is increasing in $\beta^k_j(t)$. Since it vanishes for $\beta^k_j(t)=0$ and it is larger than the r.h.s. for $\beta^k_j(t)=\gamma^k_{j+1}(t)-\gamma^k_j(t)$ (recall $\beta(t,x)\in[r,1-r]$), there exists a unique $0<\beta^k_j(t)<\gamma^k_{j+1}(t)-\gamma^k_j(t)$ satisfying (2.6.3). Furthermore, since β and the γ^k_j are smooth, we have $\beta^k_j\in C^0([0,T])\cap C^1((0,T))$. The mean value theorem then implies

$$\Big|\beta^k_j(t) - \int_{\gamma^k_j(t)}^{\gamma^k_{j+1}(t)} dx\,\beta(t,x)\Big| \le C\big[\gamma^k_{j+1}(t)-\gamma^k_j(t)\big]^2 \le C' k^{-2} \tag{2.6.4}$$

for suitable constants C, C'. For h and k two positive integers, we next define the Young measure $\mu^{h,k}\in\mathcal{M}$ by

$$\mu^{h,k}_{t,x} := \begin{cases} \nu^0_{t,x} & \text{if } \exists j\in\mathbb{Z},\ |j|\le h\,k \text{ such that } \gamma^k_j(t)+\beta^k_j(t) < x < \gamma^k_{j+1}(t) \\ \nu^1_{t,x} & \text{otherwise} \end{cases}$$

Since $\nu^1_{t,x}$ is constant for $|x|$ sufficiently large, we have $\mu^{h,k} \in \mathcal{M}_g$ for h large enough. Furthermore, since convergence in $\mathcal{M}$ is *local*, (2.6.4) yields $\lim_{h\to\infty} \lim_{k\to\infty} \mu^{h,k} = \mu$ in $\mathcal{M}$, and for each $F \in C([0,1])$

$$\lim_{h\to\infty} \lim_{k\to\infty} \mu^{h,k}(F) = \mu(F) \qquad \text{strongly in } L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$$

We next prove that $\mathcal{I}(\mu^{h,k}) < +\infty$ and $\lim_h \lim_k G^{\mu^{h,k}} = G$ in $L_2([0,T]\times\mathbb{R})$; so that, reasoning as in the proof of *Claim 1*, by Remark 2.4.3 we get the existence of a subsequence $\{\mu^k\}$ $\mathcal{I}$-converging to μ. For each $F \in C([0,1])$, $(t,x) \mapsto \mu^{h,k}_{t,x}(F)$ is smooth outside the graph of the curves γ^k_j. Therefore by Lemma 2.4.4 there exists $G^{h,k} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$, such that $\mu^{h,k}(\imath)_t + \mu^{h,k}(f)_x = -G^{h,k}_x$ holds weakly. First we show that we can choose $G^{h,k}$ to be compactly supported, so that $G^{h,k} \in L_2([0,T]\times\mathbb{R})$, and thus $\mathcal{I}(\mu^{h,k}) < +\infty$ with $G^{h,k} = G^{\mu^{h,k}}$ according to the definition given in Lemma 2.4.2.

Since $G^{h,k}$ is defined up to a measurable function of t, and $G^{h,k}_x(t,x) = 0$ for $x < \gamma^k_{-hk}(t)$ (we are considering h large enough as above), we can assume $G^{h,k}(t,x) = G^\mu(t,x) = 0$ for $x < \gamma^k_{-hk}(t)$. Furthermore, by (2.4.7) and (2.6.2), for each $j \in \mathbb{Z}$, $G^{h,k}$ is continuous in the regions $\{(t,x) \,:\, \gamma^k_j(t) + \beta^k_j(t) < x < \gamma^k_{j+1}(t) + \beta^k_{j+1}(t)\}$. Let now $j \in \mathbb{Z}$ with $|j| \le hk$, and $t \in [0,T]$; by (2.4.7) and (2.6.2)

$$\begin{aligned} -\big[G^{h,k}\big(t, [\gamma^k_j(t) + \beta^k_j(t)]^-\big) - G^{h,k}\big(t, \gamma^k_j(t)\big)\big] \;&=\; \nu^1_{t,\gamma^k_j(t)+\beta^k_j(t)}(f) - \nu^1_{t,\gamma^k_j(t)}(f) \\ &\quad + \int_{\gamma^k_j(t)}^{\gamma^k_j(t)+\beta^k_j(t)} dx\, \big[\nu^1_{t,x}(\imath)\big]_t \end{aligned}$$

and

$$\begin{aligned} -\big[G^{h,k}\big(t, \gamma^k_{j+1}(t)\big) - G^{h,k}\big(t, [\gamma^k_j(t) + \beta^k_j(t)]^-\big)\big] &= \nu^0_{t,\gamma^k_{j+1}(t)}(f) - \nu^0_{t,\gamma^k_j(t)+\beta^k_j(t)}(f) \\ + \int_{\gamma^k_j(t)+\beta^k_j(t)}^{\gamma^k_{j+1}(t)} dx\, \big[\nu^0_{t,x}(\imath)\big]_t &+ \big[\nu^0_{t,\gamma^k_j(t)+\beta^k_j(t)}(f) - \nu^1_{t,\gamma^k_j(t)+\beta^k_j(t)}(f)\big] \\ - \big[\nu^0_{t,\gamma^k_j(t)+\beta^k_j(t)}(\imath) - \nu^1_{t,\gamma^k_j(t)+\beta^k_j(t)}(\imath)\big] &\big[\dot\gamma^k_j(t) + \dot\beta^k_j(t)\big] \end{aligned}$$

By (2.6.3) and simple algebraic manipulations

$$\begin{aligned} G^{h,k}\big(t, \gamma^k_{j+1}(t)\big) - G^{h,k}\big(t, \gamma^k_j(t)\big) \;&=\; -\big[\mu_{t,\gamma^k_{j+1}(t)}(f) - \mu_{t,\gamma^k_j(t)}(f)\big] \\ - \int_{\gamma^k_j(t)}^{\gamma^k_{j+1}(t)} dx\, \big[\mu_{t,x}(\imath)\big]_t \;&=\; G^\mu\big(t, \gamma^k_{j+1}(t)\big) - G^\mu\big(t, \gamma^k_j(t)\big) \end{aligned}$$

Since $G^{h,k}(t, \gamma^k_{-hk}(t)) = G^\mu(t, \gamma^k_{-hk}(t)) = 0$, we have $G^{h,k}(t, \gamma^k_j(t)) = G^\mu(t, \gamma^k_j(t))$ for any $j \in \mathbb{Z}$. In particular, since $G^\mu(t, \gamma^k_{hk}(t)) = 0$ and $G^{h,k}_x(t,x) = G^\mu_x(t,x) = 0$ for $x > \gamma^k_{hk}(t)$, we have $G^{h,k}(t,x) = G^\mu(t,x) = 0$ for $x > \gamma^k_{hk}(t)$ and $x < \gamma^k_{-hk}(t)$. That is, $G^{h,k}$ and G^μ are compactly supported. Thus $\mathcal{I}(\mu^{h,k}) < +\infty$ and $G^{h,k} = G^{\mu^{h,k}}$.

Finally, by the definition of G^μ and $G^{\mu^{h,k}}$, recalling $G^{h,k}(t,\gamma_j^k(t)) = G^\mu(t,\gamma_j^k(t))$ we have

$$\|G^{\mu^{h,k}} - G^\mu\|^2_{L_2([0,T]\times\mathbb{R})} = \sum_{j=-hk}^{hk}\int_{[0,T]} dt \int_{\gamma_j^k(t)}^{\gamma_{j+1}^k(t)} dx \left(G^{\mu^{h,k}}(t,x) - G^\mu(t,x)\right)^2$$

$$= \sum_{j=-hk}^{hk}\int_{[0,T]} dt \Bigg\{ \int_{\gamma_j^k(t)}^{\gamma_j^k(t)+\beta_j^k(t)} dx \Big[\int_{\gamma_j^k(t)}^x dy [\nu^1_{t,y}(\imath) - \mu_{t,y}(\imath)]_t + [\nu^1_{t,y}(f) - \mu_{t,y}(f)]_y \Big]^2$$

$$+ \int_{\gamma_j^k(t)+\beta_j^k(t)}^{\gamma_{j+1}^k(t)} dx \left[\int_{\gamma_{j+1}^k(t)}^x dy \, [\nu^0_{t,y}(\imath) - \mu_{t,y}(\imath)]_t + [\nu^0_{t,y}(f) - \mu_{t,y}(f)]_y \right]^2 \Bigg\}$$

Since all the integrands in the are last two lines of this formula are bounded uniformly in h and k, each term of the sum is bounded by $C\,k^{-3}$ for some constant $C > 0$. Therefore the sum itself is bounded by $2\,C\,h\,k^{-2}$, and we get $\lim_{h\to\infty}\lim_{k\to\infty}\mathcal{I}(\mu^{h,k}) = \mathcal{I}(\mu)$.

2.7. Appendix B: Γ-viscosity cost for scalar Hamilton-Jacobi equations

In this appendix we establish a Γ-convergence result for a sequence of functionals associated with the Hamilton-Jacobi equation (2.1.1)

$$b_t + f(b_x) = 0 \tag{2.7.1}$$

which is related to (2.1.1) via the transformation $u = b_x$. In (2.7.1) we understand $(t,x) \in [0,T]\times\mathbb{R}$ and $b(t,x) \in \mathbb{R}$. As usual, we assume f to be a Lipschitz function on $[0,1]$, D and σ continuous functions on $[0,1]$, with D uniformly positive and σ strictly positive on $(0,1)$. We will just sketch most of the proofs, since they are similar to the proofs of the corresponding statements for (2.1.1).

We introduce the equivalence $\sim$ on $C\big([0,T];L_{2,\mathrm{loc}}(\mathbb{R})\big)$ by setting $b^1 \sim b^2$ iff $b^1 - b^2$ is constant in $[0,T]\times\mathbb{R}$. We let $\mathcal{B}$ be the set of functions $b \in C\big([0,T];L_{2,\mathrm{loc}}(\mathbb{R})\big)/\sim$ such that $b_x \in \mathcal{U}$. The requirement $b_x \in \mathcal{U}$ is clearly compatible with $\sim$, so that $\mathcal{B}$ is well defined. We equip $\mathcal{B}$ with the metric

$$d_{\mathcal{B}}(b^1,b^2) := d_{\mathcal{U}}(b^1_x,b^2_x) + \inf_{c\in\mathbb{R}} \sup_{t\in[0,T]} \sum_{N=1}^\infty \frac{1}{2^N}\|b^1(t,\cdot) - b^2(t,\cdot) + c\|_{L_2\big([-N,N]\big)} \tag{2.7.2}$$

Note that the second term in the r.h.s. of (2.7.2) is the projection of the $C\big([0,T];L_{2,\mathrm{loc}}(\mathbb{R})\big)$-distance w.r.t. the $\sim$ equivalence. $(\mathcal{B},d_{\mathcal{B}})$ is a complete separable metric space.

For $b \in \mathcal{B}$ such that $b_{xx} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$ and $\varepsilon > 0$ we next define the linear functional a^b_ε on $C^\infty_c\big((0,T)\times\mathbb{R}\big)$ by

$$a^b_\varepsilon(\varphi) := -\langle\langle b,\varphi_t\rangle\rangle + \langle\langle f(b_x),\varphi\rangle\rangle - \frac{\varepsilon}{2}\langle\langle D(b_x)b_{xx},\varphi\rangle\rangle \tag{2.7.3}$$

and the functional $J_\varepsilon : \mathcal{B} \mapsto [0, +\infty]$ by

$$J_\varepsilon(b) := \begin{cases} \sup_{\varphi \in C_c^\infty((0,T)\times\mathbb{R})} \Big[a_\varepsilon^b(\varphi) - \frac{1}{2} \langle\langle \sigma(b_x)\varphi,)\varphi \rangle\rangle \Big] & \text{if } b_{xx} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases} \tag{2.7.4}$$

We want to study the Γ-convergence of $\{J_\varepsilon\}$. As shown below, this problem is strictly related to the Γ-convergence of $\{I_\varepsilon\}$ defined in (2.2.6).

We introduce the set $\mathcal{A} := \big\{(b,\mu) \in \mathcal{B} \times \mathcal{M} \,:\, b_x = \mu(\imath)\big\}$ which we equip with the metric

$$d_{\mathcal{A}}\big((b^1,\mu^1),(b^2,\mu^2)\big) := d_{\mathcal{B}}(b^1,b^2) + d_{\mathcal{M}}(\mu^1,\mu^2) \tag{2.7.5}$$

We say that $(b,\mu) \in \mathcal{A}$ is a measure-valued solution to (2.7.1) iff $b_t + \mu(f) = 0$ weakly in $(0,T)\times\mathbb{R}$. We lift J_ε to a functional $\mathcal{J}_\varepsilon : \mathcal{A} \to [0,+\infty]$ by setting

$$\mathcal{J}_\varepsilon(b,\mu) := \begin{cases} J_\varepsilon(b) & \text{if} \quad \mu_{t,x} = \delta_{b_x(t,x)} \\ +\infty & \text{otherwise} \end{cases} \tag{2.7.6}$$

THEOREM 2.7.1. *The sequence $\{\mathcal{J}_\varepsilon\}$ is equicoercive on $\mathcal{A}$ and Γ-converges to*

$$\mathcal{J}\big((b,\mu)\big) := \sup_{\varphi \in C_c^\infty((0,T)\times\mathbb{R})} \Big\{ -\langle\langle b, \varphi_t\rangle\rangle + \langle\langle \mu(f), \varphi\rangle\rangle - \frac{1}{2}\,\langle\langle \mu(\sigma)\varphi,\varphi\rangle\rangle \Big\} \tag{2.7.7}$$

Note that $\mathcal{J}\big((b,\mu)\big) = 0$ iff (b,μ) is a measure-valued solution to (2.7.1).

On the set $\mathcal{B}$ we next introduce the metric $d_{\mathcal{Y}}$

$$d_{\mathcal{Y}}(b^1,b^2) := d_{\mathcal{X}}(b^1_x,b^2_x) + \inf_{c\in\mathbb{R}} \sup_{t\in[0,T]} \sum_{N=1}^{\infty} \frac{1}{2^N} \|b^1(t) - b^2(t) + c\|_{L_2\big([-N,N]\big)} \tag{2.7.8}$$

and denote by $(\mathcal{Y}, d_{\mathcal{Y}})$ the complete separable metric space consisting of the same set $\mathcal{B}$ equipped with the distance $d_{\mathcal{Y}}$. We say that $b \in \mathcal{Y}$ is a weak solution to (2.7.1) iff $-\langle\langle b,\varphi_t\rangle\rangle + \langle\langle f(b_x),\varphi\rangle\rangle = 0$ for each $\varphi \in C_c^\infty\big((0,T)\times\mathbb{R}\big)$. We denote by $\mathcal{W} \subset \mathcal{Y}$ the set of weak solutions to (2.7.1). We rescale the functional J_ε defining $K_\varepsilon : \mathcal{Y} \to [0,+\infty]$ as

$$K_\varepsilon := \varepsilon^{-1} J_\varepsilon$$

THEOREM 2.7.2. (i) *The sequence of functionals $\{K_\varepsilon\}$ satisfies the Γ-liminf inequality*

$$\big(\Gamma\text{-}\underline{\lim}_\varepsilon\, K_\varepsilon\big)(b) \geq \begin{cases} H(b_x) & \textit{if } b \in \mathcal{W} \\ +\infty & \textit{otherwise} \end{cases}$$

(ii) *Assume there is no interval where f is affine. Then the sequence $\{K_\varepsilon\}$ is equicoercive on $\mathcal{Y}$.*

(iii) *Suppose furthermore* $f \in C^2([0,1])$ *and* $D, \sigma \in C^\alpha([0,1])$ *for some* $\alpha > 1/2$. *Then*

$$\big(\Gamma\text{–}\overline{\lim}_\varepsilon K_\varepsilon\big)(b) \le \begin{cases} \overline{H}(b_x) & \text{if } b \in \mathcal{W} \\ +\infty & \text{otherwise} \end{cases}$$

Since $b(0,\cdot)$ is bounded and Lipschitz, by a well known connection between entropic solutions to (2.1.1) and viscosity solutions to (2.7.1), see e.g. [**11**, Theorem 1.1], we gather $(\Gamma\text{–}\underline{\lim}_\varepsilon K_\varepsilon)(b) = 0$ iff b is a viscosity solutions to (2.7.1). It follows that if b^ε satisfies the equation

$$b_t + f(b_x) = \frac{\varepsilon}{2} D(b_x) b_{xx} - \sigma(b_x) E^\varepsilon \tag{2.7.9}$$

for some $E^\varepsilon \in L_2\big([0,T]\times\mathbb{R}, \sigma(b_x) dt\, dx\big)$ such that $\lim_\varepsilon \varepsilon \|E^\varepsilon\|^2_{L_2([0,T]\times\mathbb{R},\sigma(b_x)dt\,dx)} = 0$, then limit points of $\{b^\varepsilon\}$ are viscosity solutions to (2.7.1). On the other hand if b^ε solves (2.7.9) for some E^ε with $\varepsilon\|E^\varepsilon\|^2_{L_2([0,T]\times\mathbb{R},\sigma(b_x)dt\,dx)}$ uniformly bounded, then limit points b of $\{b^\varepsilon\}$ are such that $b_x \in \mathcal{E}$.

In order to prove Theorem 2.7.1 and Theorem 2.7.2 we first establish some preliminary results. Given a measurable map $a : [0,T] \times \mathbb{R} \to [0,+\infty]$, we let $\mathcal{L}_a$ be the Hilbert space obtained by identifying and completing the set $\big\{\varphi \in C^\infty_c\big((0,T)\times\mathbb{R}\big) \,:\, \langle\langle a\varphi, \varphi\rangle\rangle < +\infty\big\}$ w.r.t. the seminorm $\langle\langle a\varphi,\varphi\rangle\rangle$.

LEMMA 2.7.3. *Let* $\varepsilon > 0$ *and* $b \in \mathcal{B}$ *be such that* $J_\varepsilon(b) < +\infty$. *Then there exists* $E^{\varepsilon,b} \in \mathcal{L}_{\sigma(b_x)}$ *such that*

$$b_t + f(b_x) = \frac{\varepsilon}{2} D(b_x)\, b_{xx} - \sigma(b_x)\, E^{\varepsilon,b} \tag{2.7.10}$$

holds weakly on $(0,T)\times\mathbb{R}$ *and* $J_\varepsilon(b) = \frac{1}{2}\|E^{\varepsilon,b}\|^2_{\mathcal{L}_{\sigma(b_x)}}$. *Furthermore* $I_\varepsilon(b_x) < +\infty$ *and there exists* $\gamma^{\varepsilon,b} \in \mathcal{L}_{\sigma(b_x)^{-1}}$ *such that* $\gamma^{\varepsilon,b}_x = 0$ *and* $\sigma(b_x)E^{\varepsilon,b} = \sigma(b_x)\Psi^{\varepsilon,b_x} + \gamma^{\varepsilon,b}$, *where* Ψ^{ε,b_x} *is defined as in Lemma 2.3.1. In particular*

$$J_\varepsilon(b) = \frac{1}{2}\|\Psi^{\varepsilon,b_x}\|^2_{\mathcal{D}^1_{\sigma(b_x)}} + \frac{1}{2}\|\gamma^{\varepsilon,b}\|^2_{\mathcal{L}_{\sigma(b_x)}} = I_\varepsilon(b_x) + \frac{1}{2}\langle\langle \sigma(b_x)^{-1}\gamma^{\varepsilon,b}, \gamma^{\varepsilon,b}\rangle\rangle \tag{2.7.11}$$

PROOF. The existence of $E^{\varepsilon,b}$, (2.7.10) and the equality $J_\varepsilon(b) = \frac{1}{2}\|E^{\varepsilon,b}\|^2_{\mathcal{L}_{\sigma(b_x)}}$ are achieved as in Lemma 2.3.1. We also have

$$\begin{aligned} J_\varepsilon(b) &= \sup_{\varphi\in C^\infty_c((0,T)\times\mathbb{R})} \big\{a^b_\varepsilon(\varphi) - \tfrac{1}{2}\langle\langle\sigma(b_x)\varphi,\varphi\rangle\rangle\big\} \\ &\ge \sup_{\phi\in C^\infty_c((0,T)\times\mathbb{R})} \big\{a^\varepsilon_b(\phi_x) - \tfrac{1}{2}\langle\langle\sigma(b_x)\phi_x,\phi_x\rangle\rangle\big\} \\ &= \sup_{\phi\in C^\infty_c((0,T)\times\mathbb{R})} \big\{\ell^\varepsilon_{b_x}(\phi) - \tfrac{1}{2}\langle\langle\sigma(b_x)\phi_x,\phi_x\rangle\rangle\big\} = I_\varepsilon(b_x) \end{aligned}$$

By (2.3.1) and (2.7.10) there exists $\Psi^{\varepsilon,b_x} \in \mathcal{D}^1_{\sigma(b_x)}$ such that $\big(\sigma(b_x)E^{\varepsilon,b}\big)_x = \big(\sigma(b_x)\Psi^{\varepsilon,b_x}_x\big)_x$, namely $\sigma(b_x)E^{\varepsilon,b} = \sigma(b_x)\Psi^{\varepsilon,b_x}_x + \gamma^{\varepsilon,b}(t)$ for some measurable map $\gamma^{\varepsilon,b} : [0,T] \to [-\infty,+\infty]$. It is then easy to check (2.7.11). □

The following lemma is proved analogously.

LEMMA 2.7.4. *Let $(b,\mu) \in \mathcal{A}$ be such that $\mathcal{J}\big((b,\mu)\big) < +\infty$. Then there exists $E^{(b,\mu)} \in \mathcal{L}_{\mu(\sigma)}$ such that*

$$b_t + \mu(f) = -\mu(\sigma)E^{(b,\mu)}$$

and $\mathcal{J}\big((b,\mu)\big) = \frac{1}{2}\|E^{(b,\mu)}\|^2_{\mathcal{L}_{\mu(\sigma)}}$. Furthermore $\mathcal{I}(\mu) < +\infty$ and there exists $\gamma^{(b,\mu)} \in \mathcal{L}_{\mu(\sigma)^{-1}}$ such that $\gamma^{(b,\mu)}_x = 0$ and

$$\mathcal{J}\big((b,\mu)\big) = \frac{1}{2}\|\Psi^\mu\|^2_{\mathcal{D}^1_{\mu(\sigma)}} + \frac{1}{2}\|\gamma^{(b,\mu)}\|^2_{\mathcal{L}_{\mu(\sigma)}} = \mathcal{I}(\mu) + \frac{1}{2}\langle\langle \mu(\sigma)^{-1}\gamma^{(b,\mu)}, \gamma^{(b,\mu)}\rangle\rangle \tag{2.7.12}$$

where Ψ^μ is defined as in Lemma 2.4.2.

LEMMA 2.7.5. *The sequence of functional $\{J_\varepsilon\}$ is equicoercive on $(\mathcal{B}, d_{\mathcal{B}})$.*

PROOF. Let $\{b^\varepsilon\} \subset \mathcal{B}$ be such that $J_\varepsilon(b^\varepsilon) \le C_J$ for some $C_J < +\infty$. By (2.7.11) $I_\varepsilon(b^\varepsilon_x) \le C_J$, and thus $\{b^\varepsilon_x\}$ is precompact in $\mathcal{U}$ by Lemma 2.3.3. We are left with the proof of the compactness of $\{b^\varepsilon\}$ w.r.t. the second term on the r.h.s. of (2.7.2). By (2.7.11) and (2.3.6) we have that for any $N > 0$, $\varepsilon^2 \int_{[0,T]\times[-N,N]} dt\, dx\, (b^\varepsilon_{xx})^2 \le C(C_J + \varepsilon N + 1)$ for some constant $C > 0$ depending only on f and D. It then follows by (2.7.10) that for each $N > 0$, $\|b^\varepsilon_t\|_{L_2([0,T]\times[-N,N]}$ is bounded uniformly in ε. Since $b_x \in \mathcal{U}$ for each $b \in \mathcal{B}$, we also have $0 \le b^\varepsilon_x \le 1$. Recalling that elements in b are defined up to a constant, the conclusion follows by these bounds on b^ε_t, b^ε_x and compact Sobolev embedding. □

The following remark follows by Proposition 2.3.3 and Lemma 2.7.3 and the definition (2.7.2) of $d_{\mathcal{B}}$.

REMARK 2.7.6. *For each $\varepsilon > 0$, J^ε is lower semicontinuous on $(\mathcal{B}, d_{\mathcal{B}})$.*

LEMMA 2.7.7. *For each $u \in \mathcal{U}$ such that $I_\varepsilon(u) < +\infty$ there exists $b^{\varepsilon,u} \in \mathcal{B}$ such that $b^{\varepsilon,u}_x = u$ and $J_\varepsilon(b^{\varepsilon,u}) = I_\varepsilon(u)$. Furthermore if $b \in \mathcal{B}$ is such that $b_x = u$ and $J_\varepsilon(b) < +\infty$, then $b_t = b^{\varepsilon,u}_t + \gamma^{\varepsilon,b}$, where $\gamma^{\varepsilon,b} \in \mathcal{L}_{\sigma(u)^{-1}}$ is defined as in Lemma 2.7.3. Conversely, given $\gamma \in \mathcal{L}_{\sigma(u)^{-1}}$ with $\gamma_x = 0$, there exists a unique $b \in \mathcal{B}$ such that $b_x = u$ and $b_t = b^{\varepsilon,u}_t + \gamma$.*

PROOF. From the definitions (2.2.6) and (2.7.4), it is not difficult to gather

$$I_\varepsilon(u) = \inf_{b\in\mathcal{B}\,:\,b_x=u} J_\varepsilon(b)$$

Since J_ε is coercive and lower semicontinuous on $\mathcal{B}$, and $\{b \in \mathcal{B} \,:\, b_x = u\}$ is a closed subset of $\mathcal{B}$, there exists a b^u on which the infimum is attained.

If b is such that $b_x = u$ and $J_\varepsilon(b) < +\infty$, then by the decomposition of $E^{\varepsilon,\cdot}$ in Lemma 2.7.3 we have $(b - b^u)_t = \sigma(u)(E^{\varepsilon,b} - E^{\varepsilon,b^u}) = \gamma^{\varepsilon,b}$. The converse statement follows by choosing $b(t,x) = b^u(t,x) + \int^t ds\gamma(s)$, which identifies a unique $b \in \mathcal{B}$. □

PROOF OF THEOREM 2.7.1. Equicoercivity follows by (2.7.11), the equicoercivity statement in Theorem 2.2.1 and Lemma 2.3.3.

In order to prove the Γ-liminf inequality, let $\{(b^\varepsilon, \mu^\varepsilon)\} \subset \mathcal{A}$ converge to some $(b,\mu) \in \mathcal{A}$. It is not restrictive to assume $J_\varepsilon(b^\varepsilon) < +\infty$, and thus $b^\varepsilon_{xx} \in L_{2,\mathrm{loc}}\big([0,T]\times\mathbb{R}\big)$ and $\mu^\varepsilon = \delta_{b^\varepsilon_x}$. Then for each $\varphi \in C^\infty_{\mathrm{c}}\big((0,T)\times\mathbb{R}\big)$

$$\mathcal{J}_\varepsilon\big((b^\varepsilon,\mu^\varepsilon)\big) = J_\varepsilon(b^\varepsilon) \ge -\langle\langle b^\varepsilon, \varphi_t\rangle\rangle + \langle\langle f(b^\varepsilon_x), \varphi\rangle\rangle - \frac{\varepsilon}{2}\langle\langle D(b^\varepsilon_x) b^\varepsilon_{xx}, \varphi\rangle\rangle - \frac{1}{2}\langle\langle \varphi, \sigma(b^\varepsilon_x)\varphi\rangle\rangle$$

As in the proof of the Γ-liminf inequality in Theorem 2.2.1, an integration by parts shows that the third term in the l.h.s. vanishes as $\varepsilon \to 0$. Hence

$$\varliminf_{\varepsilon} \mathcal{J}_\varepsilon\big((b^\varepsilon,\mu^\varepsilon)\big) \ge -\langle\langle b, \varphi_t\rangle\rangle + \langle\langle \mu(f), \varphi\rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\varphi, \varphi\rangle\rangle$$

and the Γ-liminf inequality is achieved by optimizing over φ.

Let $(b,\mu) \in \mathcal{A}$ be such that $\mathcal{J}\big((b,\mu)\big) < +\infty$. By Lemma 2.7.4 $\mathcal{I}(\mu) < +\infty$ and by the Γ-limsup inequality in Theorem 2.2.1 there exists a sequence $\{u^\varepsilon\} \subset \mathcal{U}$ such that $\delta_{u^\varepsilon} \to \mu$ in $\mathcal{M}$ and $\varlimsup I_\varepsilon(u^\varepsilon) = \varlimsup \mathcal{I}_\varepsilon(\delta_{u^\varepsilon}) \le \mathcal{I}(\mu)$. By Corollary 2.7.7 there exists $b^{\varepsilon,u^\varepsilon} \in \mathcal{B}$ such that $b^{\varepsilon,u^\varepsilon}_x = u^\varepsilon$ and $J_\varepsilon(b^{\varepsilon,u^\varepsilon}) = I_\varepsilon(u^\varepsilon)$. Letting $\gamma^{(b,\mu)}$ be defined as in Lemma 2.7.4, it is also easily seen that there exists a sequence $\gamma^\varepsilon \in \mathcal{L}_{\sigma(u^\varepsilon)^{-1}}$ such that $\gamma^\varepsilon_x = 0$, $\gamma^\varepsilon \to \gamma^{(b,\mu)}$ weakly in $L_2([0,T])$, and $\|\gamma^\varepsilon\|_{\mathcal{L}_{\sigma(u^\varepsilon)^{-1}}} \to \|\gamma^{(b,\mu)}\|_{\mathcal{L}_{\mu(\sigma)^{-1}}}$. Recalling Corollary 2.7.7, we define the sequence b^ε by the requirements $b^\varepsilon_x = u^\varepsilon$ and $b^\varepsilon_t = b^{\varepsilon,u^\varepsilon}_t + \gamma^\varepsilon$. We have

$$\begin{aligned} \varlimsup_\varepsilon \mathcal{J}_\varepsilon\big(b^\varepsilon, \delta_{b^\varepsilon_x}\big) &= \varlimsup_\varepsilon J_\varepsilon(b^{\varepsilon,u^\varepsilon}) + \tfrac{1}{2}\langle\langle \sigma(u^\varepsilon)\gamma^\varepsilon, \gamma^\varepsilon\rangle\rangle \\ &\le \mathcal{I}(\mu) + \tfrac{1}{2}\langle\langle \mu(\sigma)\gamma^{(b,\mu)}, \gamma^{(b,\mu)}\rangle\rangle = \mathcal{J}\big((b,\mu)\big) \end{aligned}$$

On the other hand $\delta_{b^\varepsilon_x} \to \mu$ in $\mathcal{M}$, and it is not difficult to check $b^\varepsilon_t \to b_t$ weakly. Therefore any limit point in $\mathcal{A}$ of $\{(b^\varepsilon, \delta_{b^\varepsilon_x})\}$ coincides with (b,μ). □

PROOF OF THEOREM 2.7.2. If $b \in \mathcal{Y}$ is such that (b, δ_{b_x}) is a measure-valued solution to (2.7.1), then $b \in \mathcal{W}$. By the Γ-liminf inequality in Theorem 2.7.1 we thus obtain $(\Gamma\text{-}\varliminf_\varepsilon K_\varepsilon)(b) = +\infty$ if $b \notin \mathcal{W}$. The Γ-liminf inequality on $\mathcal{W}$ follows immediately by (i) in Theorem 2.2.5 and (2.7.11).

Equicoercivity is a consequence of (ii) in Theorem 2.2.5 and Lemma 2.7.5.

In order to prove the Γ-limsup inequality, let $b \in \mathcal{W}$ be such that $\overline{H}(b_x) < +\infty$. By (iii) in Theorem 2.2.5 there exists a sequence $\{u^\varepsilon\} \subset \mathcal{X}$ converging to $u := b_x$ in $\mathcal{X}$ and such that $\varlimsup H_\varepsilon(u^\varepsilon) \le \overline{H}(u)$. Let $b^\varepsilon := b^{\varepsilon,u^\varepsilon}$; by Corollary 2.7.7 $\varlimsup K_\varepsilon(b^{\varepsilon,u^\varepsilon}) \le K(b)$. Furthermore, by (i) and (ii) proved above, $\{b^\varepsilon\}$ is precompact in $\mathcal{Y}$ and its limit points are in $\mathcal{W}$. Let $\tilde b \in \mathcal{W}$ be a limit point of $\{b^\varepsilon\}$. Then $\tilde b_x = b_x$, since $b^\varepsilon_x = u^\varepsilon \to u = b_x$ in $\mathcal{X}$; on the other hand $b_t + f(b_x) = 0 = \tilde b_t + f(\tilde b_x)$, so that we also gather $b_t = \tilde b_t$. It follows $\tilde b = b$. □

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CHAPTER 3

Large Deviations for stochastic conservation laws

In this chapter we are concerned with the asymptotic behaviour of the solution u^ε to(1.1.3) as $\varepsilon \to 0$. The analysis is restricted to the 1+1 dimensional case. Here we work in the same setting of Chapter 1, namely the space variable x lives on a torus and the initial data is fixed. Moreover, for technical reasons, we equip the set $\mathcal{U}$ (see (2.2.6)) with a sligtly weaker topology than the one used in 2. However, the results given in Chapter 2 can be transported to this case with minor modifications. These modifications require some "extra" technical hypotheses only for Theorem 3.1.3, as explained in its proof.

3.1. Main results

3.1.1. Basic hypotheses. We let $\mathbb{T}$ denote one-dimensional torus, $\langle \cdot, \cdot \rangle$ denote the inner product in $L_2(\mathbb{T})$, for $T > 0$ $\langle\langle \cdot, \cdot \rangle\rangle$ stands for the inner product in $L_2([0,T] \times \mathbb{T})$, and for E a closed set, $C^\infty(E)$ denotes the collection of infinitely differentiable functions on E, that are continuous up to the boundary. Throughout this paper ∂_t denotes derivative w.r.t. the time variable t, ∇ and $\nabla \cdot$ derivatives w.r.t. the space variable x, ∂_u derivative w.r.t. the state variable u. For a function ϑ explicitly depending on the x variable, ∂_x denotes the partial derivative w.r.t. such a variable. Namely, given the smooth functions $u : \mathbb{T} \to [0,1]$ and $\vartheta : [0,1] \times \mathbb{T} \to \mathbb{R}$, we understand $(\nabla \vartheta(u(x), x)) = (\partial_u \vartheta)(u(x), x) \nabla u(x) + (\partial_x \vartheta)(u(x), x)$.

In the following, when a martingale $M : [0,T] \times \mathcal{X} \to C([0,T])$ is given, we write equivalently $M_t \equiv M(t,v)$, depending on which aspect of the process we want to emphasize. For $T > 0$, B a real Banach space and $\{M_t\}_{t \in [0,T]}$ a B-valued martingale, for each $\phi \in B^*$ we denote by $\{\langle M_t, \phi \rangle\}_{t \in [0,T]}$ the real–valued martingale obtained by the dual action of M_t on B. In the following *martingale* will always stand for *continuous martingale*.

In this chapter, the following hypotheses will be always assumed

H1) $f : [0,1] \to \mathbb{R}$ is a Lipschitz function.
H2) $D : [0,1] \to \mathbb{R}$ is a uniformly positive Lipschitz function.
H3) $a \in C^2([0,1])$ is such that $a(0) = a(1) = 0$, and $a(v) \neq 0$ for $v \in (0,1)$.
H4) $\{\jmath^\varepsilon\}_{\varepsilon > 0} \subset H^1(\mathbb{T})$ and $\int dx\, \jmath^\varepsilon(x) = 1$ is a sequence of positive mollifiers weakly converging to the Dirac mass centered at 0.
H5) $u_0 : \mathbb{T} \to [0,1]$ is a Borel measurable function.

3.1.2. Stochastic scalar conservation laws. We refer to [**12**] for a general theory of stochastic equations in infinite dimensions. Let us fix a *standard* filtered probability space $\big(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\le t\le T}, \mathbb{P}\big)$, on which an $L_2(\mathbb{T})$–valued cylindrical Brownian motion W is defined. Namely, W is a continuous, Gaussian, $L_2(\mathbb{T})$–valued martingales $\{W_t, 0 \le t \le T\}$ with quadratic variation:

$$\big[\langle W, \phi\rangle, \langle W, \psi\rangle\big]_t = \langle \phi, \psi\rangle t \tag{3.1.1}$$

for each $\phi, \psi \in L_2(\mathbb{T})$. For $\varepsilon > 0$, we consider the following stochastic Cauchy problem:

$$\begin{aligned} &du^\varepsilon = \big[-\nabla \cdot f(u^\varepsilon) + \tfrac{\varepsilon}{2}\nabla \cdot (D(u^\varepsilon)\nabla u^\varepsilon)\big]\, dt + \varepsilon^\gamma\, \nabla \cdot \big[a(u^\varepsilon)(\jmath^\varepsilon * dW)\big] \\ &u^\varepsilon(0,x) = u_0(x) \end{aligned} \tag{3.1.2}$$

Here the writing $\nabla\cdot\big[a(u)(\jmath^\varepsilon * dW)\big]$ stands for the martingale differential acting on $H^1(\mathbb{T})$ as

$$\big\langle \nabla \cdot \big[a(u)(\jmath^\varepsilon * dW)\big]_t, \psi\big\rangle = -\langle dW_t, \jmath^\varepsilon * [a(u)\nabla\psi]\rangle \tag{3.1.3}$$

The following theorem is an immediate consequence of Proposition 3.5.7 in the appendix, where we also recall how solutions to (3.1.2) are defined.

PROPOSITION 3.1.1. *Assume* $\lim_\varepsilon \varepsilon^\gamma \|\jmath^\varepsilon\|_{L_2(\mathbb{T})} = 0$. *Then there is an* $\varepsilon_0 > 0$ *small enough such that, for each* $\varepsilon < \varepsilon_0$, *there exists a unique* $\{\mathfrak{F}_t\}$*-adapted process* $u^\varepsilon : \Omega \to \mathcal{U} \cap L_2\big([0,T]; H^1(\mathbb{T})\big)$ *solving* (3.1.2).

Note that the total mass of u^ε is conserved a.s. by the stochastic flow (3.1.2), namely for each $t \in [0,T]$ we have $\int dx\, u^\varepsilon(t) = \int dx\, u_0$. Consider the formal limiting equation for (3.1.2)

$$\begin{aligned} &\partial_t u + \nabla \cdot f(u) = 0 \\ &u(0,x) = u_0(x) \end{aligned} \tag{3.1.4}$$

Recall the notion of Kruzkov solution to (3.1.4) given in Section 1.4. The following statement follows from item (i) in Theorem 3.1.7 below and by a simple adaptation of Proposition 2.2.6 to the setting of this chapter, namely to the case $x \in \mathbb{T}$.

PROPOSITION 3.1.2. *Assume that* $f \in C^2([0,1])$ *is such that there is no interval in which* f *is affine. Assume also* $\gamma > 1/2$ *and* $\lim_\varepsilon \varepsilon^{2(\gamma-1)}\big[\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \varepsilon\|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\big] = 0$. *Let* $\bar u$ *be the unique Kruzkov solution to* (3.1.4). *Then* $\mathbb{P}^\varepsilon \to \delta_{\bar u}$ *weakly in probability, w.r.t. both the topology of* $\mathcal{U}$ *and the strong* $L_p([0,T]\times\mathbb{T})$ *topology.*

Proposition 3.1.2 establishes a convergence result for laws of the processes u^ε solutions to (3.1.2), as $\varepsilon \to 0$. We are then interested in large deviations principles for these laws. In the next sections, we first introduce some preliminary notions and state a first a large deviations principle. We then introduce some additional preliminaries and state the second large deviations principle. Note that most of the definitions are similar to the ones introduced in Section 2.2; however, for convenience of the reader, we next detail the adaptated versions of these definitions in the current setting.

3.1.3. Scalar conservation laws. Let U denote the compact separable metric space of measurable functions $u : \mathbb{T} \to [0,1]$, equipped with the metric inherited by the $*$-weak topology of the finite measures on $\mathbb{T}$. Namely, for $\{\phi_N\}_{N\in\mathbb{N}} \subset C(\mathbb{T})$ a dense subset in $C(\mathbb{T})$ containing the constant function $1\!\!1$, define the metric d_U on U as

$$d_U(u,v) := \sum_{N=1}^{\infty} 2^{-N} \frac{\big|\langle u-v, \varphi_N\rangle\big|}{1+\big|\langle u-v,\varphi_N\rangle\big|} \tag{3.1.5}$$

Given $T>0$, let $\mathcal{U}$ be the set $C\big([0,T];U\big)$ endowed with the uniform metric

$$d_{\mathcal{U}}(u,v) := \sup_{t\in[0,T]} d_U\big(u(t),v(t)\big) \tag{3.1.6}$$

An element $u \in \mathcal{U}$ is a *weak solution* to (3.1.4) iff for each $\varphi \in C^\infty([0,T]\times\mathbb{T})$ it satisfies

$$\langle u(T),\varphi(T)\rangle - \langle u_0,\varphi(0)\rangle - \langle\langle u,\partial_t\varphi\rangle\rangle - \langle\langle f(u),\nabla\varphi\rangle\rangle = 0$$

We also introduce a suitable space $\mathcal{M}$ of Young measures and recall the notion of measure-valued solution to (3.1.4). Consider the set $\mathcal{N}$ of measurable maps μ from $[0,T]\times\mathbb{T}$ to the set $\mathcal{P}([0,1])$ of Borel probability measures on $[0,1]$. The set $\mathcal{N}$ can be identified with the set of positive finite Borel measures μ on $[0,T]\times\mathbb{T}\times[0,1]$ such that $\mu(dt,\,dx,\,[0,1]) = dt\,dx$. Indeed, by existence of a regular version of conditional probabilities, for such measures μ there exists a measurable kernel $\mu_{t,x}(d\lambda) \in \mathcal{P}([0,1])$ such that $\mu(dt,\,dx,\,d\lambda) = dt\,dx\,\mu_{t,x}(d\lambda)$. For $\imath : [0,1] \to [0,1]$ the identity map, we set

$$\mathcal{M} := \big\{\mu \in \mathcal{N} \;:\; \text{the map } [0,T] \ni t \mapsto \mu_{t,\cdot}(\imath) \text{ is in } \mathcal{U}\big\} \tag{3.1.7}$$

in which, for a bounded measurable function $F : [0,1] \to \mathbb{R}$, the notation $\mu_{t,x}(F)$ stands for $\int_{[0,1]}\mu_{t,x}(d\lambda)F(\lambda)$. We endow $\mathcal{M}$ with the metric

$$d_{\mathcal{M}}(\mu,\nu) := d_{*\mathrm{w}}(\mu,\nu) + d_{\mathcal{U}}\big(\mu(\imath),\nu(\imath)\big) \tag{3.1.8}$$

where $d_{*\mathrm{w}}$ is a distance generating the relative topology on $\mathcal{N}$ regarded as a subset of the finite Borel measures on $[0,T]\times\mathbb{T}\times[0,1]$ equipped with the $*$-weak topology. $(\mathcal{M}, d_{\mathcal{M}})$ is a complete separable metric space.

An element $\mu \in \mathcal{M}$ is a *measure-valued solution* to (3.1.4) iff for each $\varphi \in C^\infty\big([0,T]\times\mathbb{T})$ it satisfies

$$\langle \mu_{T,\cdot}(\imath),\varphi(T)\rangle - \langle u_0,\varphi(0)\rangle - \langle\langle \mu(\imath),\partial_t\varphi\rangle\rangle - \langle\langle \mu(f),\nabla\varphi\rangle\rangle = 0$$

If $u \in \mathcal{U}$ is a weak solution to (3.1.4), then $\delta_{u(t,x)}(d\lambda) \in \mathcal{M}$ is a measure-valued solution. On the other hand, there exist measure-valued solutions which do not have this form.

3.1.4. First order large deviations. Recall that we defined the Polish space $(\mathcal{M}, d_{\mathcal{M}})$ in Section 3.1.3, and consider the process $\mu^\varepsilon : \Omega \to \mathcal{M}$ defined by $\mu^\varepsilon_{t,x} := \delta_{u^\varepsilon(t,x)}$. We let $\mathbf{P}^\varepsilon := \mathbb{P} \circ (\mu^\varepsilon)^{-1} \in \mathcal{P}(\mathcal{M})$ be the law of μ^ε on $\mathcal{M}$. In Section 3.3 we prove

THEOREM 3.1.3. (i) *Assume $\gamma > 1/2$. Then the sequence $\{\mathbf{P}^\varepsilon\} \subset \mathcal{P}(\mathcal{M})$ satisfies a large deviations upper bound on $\mathcal{M}$ with speed $\varepsilon^{-2\gamma}$ and rate functional $\mathcal{I} : \mathcal{M} \to [0, +\infty]$ defined as*

$$\begin{aligned}\mathcal{I}(\mu) := \sup_{\varphi \in C^\infty\left([0,T]\times\mathbb{T}\right)} \Big\{ & \langle \mu_{T,\cdot}(\imath), \varphi(T)\rangle - \langle u_0, \varphi(0)\rangle - \langle\langle \mu(\imath), \partial_t \varphi\rangle\rangle \\ & - \langle\langle \mu(f), \nabla\varphi\rangle\rangle - \tfrac{1}{2}\langle\langle \mu(a^2)\nabla\varphi, \nabla\varphi\rangle\rangle \Big\}\end{aligned} \tag{3.1.9}$$

(ii) *Assume $\gamma > 3/2$, $\lim_\varepsilon \left[\varepsilon^{2\gamma-1}\|\nabla \jmath^\varepsilon\|^2_{L_2} + \varepsilon^{2\gamma-3}\|\jmath^\varepsilon\|^2_{L_2}\right] = 0$, and that $\zeta \le u_0 \le 1-\zeta$ for some $\zeta > 0$. Then $\{\mathbf{P}^\varepsilon\} \subset \mathcal{P}(\mathcal{M})$ satisfies a large deviations upper bound on $\mathcal{M}$ with speed $\varepsilon^{-2\gamma}$ and rate functional $\mathcal{I}$.*

We denote by $\mathbb{P}^\varepsilon := \mathbb{P} \circ (u^\varepsilon)^{-1} \in \mathcal{P}(\mathcal{U})$ the law of u^ε on the Polish space $(\mathcal{U}, d_{\mathcal{U}})$. By contraction principle [**13**] we get

COROLLARY 3.1.4. *Under the same hypotheses of Theorem 3.1.3, the sequence $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(\mathcal{U})$ satisfies a large deviations principle on $\mathcal{U}$ with speed $\varepsilon^{-2\gamma}$ and rate functional $I : \mathcal{U} \to [0, +\infty]$ defined as*

$$\begin{aligned} I(u) := \inf\Big\{ & \textstyle\int dt\, dx\, R_{f,a^2}\big(u(t,x), \Phi(t,x)\big), \\ & \Phi \in L_2([0,T]\times\mathbb{T}) \;:\; \nabla\Phi = -\partial_t u \textit{ weakly}\Big\}\end{aligned}$$

where $R_{f,\sigma} : [0,1]\times\mathbb{R} \to [0,+\infty]$ is defined by

$$R_{f,\sigma}(w,c) := \inf\{\big(\nu(f) - c\big)^2/\nu(\sigma),\, \nu \in \mathcal{P}([0,1]) \;:\; \nu(\imath) = w\}$$

in which we understand $(c-c)^2/0 = 0$.

Note that, if $\mathcal{I}(\mu) < +\infty$, then $\mu_{0,x}(\imath) = u_0(x)$ and analogously $I(u) < +\infty$ implies $u(0,x) = u_0(x)$. On the other hand, if $\mathcal{I}(\mu) = 0$ then μ is a measure-valued solution to (3.1.4). However, as it follows from Corollary 3.1.4, if f is nonlinear in general we have $I(u) < \mathcal{I}(\delta_u)$, so that I vanishes on a set wider than the set of weak solutions to (3.1.4).

In general there exist infinitely many measure-valued solutions to (3.1.4), but Proposition 3.1.2 implies that $\{\mathbb{P}^\varepsilon\}$ converges in probability on $\mathcal{M}$ to the unique entropic solution $\bar{u}$ to (3.1.4) (more precisely, to the Young measure $\bar{\mu}$

defined by $\bar{\mu}_{t,x} = \delta_{\bar{u}(t,x)}$). We thus expect that other nontrivial large deviations principles may hold on scales finer than $\varepsilon^{-2\gamma}$.

3.1.5. Entropy-measure solutions to conservation laws. Recalling (3.1.6), we let $\mathcal{X}$ be the same set $C([0,T];U)$ endowed with the metric

$$d_{\mathcal{X}}(u,v) := \|u-v\|_{L_1([0,T]\times\mathbb{T})} + d_{\mathcal{U}}(u,v) \tag{3.1.10}$$

Convergence in $\mathcal{X}$ is equivalent to convergence in $\mathcal{U}$ *and* in $L_p([0,T]\times\mathbb{T})$ for $p\in[1,+\infty)$. Note that $\mathcal{X}$ can be identified with the subset $\{\mu\in\mathcal{M} : \exists u\in\mathcal{X},\, \mu=\delta_u\}$ of $\mathcal{M}$, and $d_{\mathcal{X}}$ is a distance generating the relative topology induced by $d_{\mathcal{M}}$ on $\mathcal{X}$.

Let $C^2([0,1])$ be the set of twice differentiable functions on $(0,1)$ whose derivatives are continuous up to the boundary. A function, resp. a convex function, $\eta\in C^2([0,1])$ is called an *entropy*, resp. a *convex entropy*, and its *conjugated entropy flux* $q\in C([0,1])$ is defined up to a constant by $q(u) := \int^u dv\, \eta'(v) f'(v)$. For u a weak solution to (3.1.4), for (η,q) an entropy – entropy flux pair, the η*-entropy production* is the distribution $\wp_{\eta,u}$ acting on $C^\infty_{\mathrm{c}}\big([0,T)\times\mathbb{T}\big)$ as

$$\wp_{\eta,u}(\varphi) := -\langle\eta(u_0),\varphi(0)\rangle - \langle\langle\eta(u),\partial_t\varphi\rangle\rangle - \langle\langle q(u),\nabla\varphi\rangle\rangle \tag{3.1.11}$$

Let $C^{2,\infty}_{\mathrm{c}}\big([0,1)\times[0,T]\times\mathbb{T}\big)$ be the set of compactly supported maps $\vartheta : [0,1]\times(0,T)\times\mathbb{T}\ni(v,t,x)\mapsto\vartheta(v,t,x)\in\mathbb{R}$, that are twice differentiable in the v variable, with derivatives continuous up to the boundary of $[0,1]\times[0,T)\times\mathbb{T}$, and that are infinitely differentiable in the (t,x) variables. For $\vartheta\in C^{2,\infty}_{\mathrm{c}}\big([0,1]\times[0,T)\times\mathbb{T}\big)$ we denote by ϑ' and ϑ'' its partial derivatives w.r.t. the v variable. We say that a function $\vartheta\in C^{2,\infty}_{\mathrm{c}}\big([0,1]\times[0,T)\times\mathbb{T}\big)$ is an *entropy sampler*, and its *conjugated entropy flux sampler* $Q : [0,1]\times[0,T]\times\mathbb{T}$ is defined up to an additive function of (t,x) by $Q(u,t,x) := \int^u dv\, \vartheta'(v,t,x) f'(v)$. Finally, given a weak solution u to (3.1.4), the ϑ*-sampled entropy production* $P_{\vartheta,u}$ is the real number

$$\begin{aligned} P_{\vartheta,u} \;:=\; & -\textstyle\int dx\,\vartheta(u_0(x),0,x)] \\ & -\textstyle\int dt\,dx\,\Big[\big(\partial_t\vartheta\big)\big(u(t,x),t,x\big) + \big(\partial_x Q\big)\big(u(t,x),t,x\big)\Big] \end{aligned} \tag{3.1.12}$$

If $\vartheta(v,t,x)=\eta(v)\varphi(t,x)$ for some entropy η and some $\varphi\in C^\infty_{\mathrm{c}}\big([0,T)\times\mathbb{T}\big)$, then $P_{\vartheta,u}=\wp_{\eta,u}(\varphi)$.

The next proposition introduces a suitable class of solutions to (3.1.4) which will be needed in the following. We denote by $M\big([0,T)\times\mathbb{T}\big)$ the set of Radon measures on $[0,T)\times\mathbb{T}$ that we consider equipped with the vague topology. In the following, for $\varrho\in M\big([0,T)\times\mathbb{T}\big)$ we denote by $\varrho^\pm$ the positive and negative part of ϱ. For u a weak solution to (3.1.4) and η an entropy, recalling (3.1.11) we set

$$\|\wp_{\eta,u}\|_{\mathrm{TV}} := \sup\big\{\wp_{\eta,u}(\varphi),\ \varphi\in C^\infty_{\mathrm{c}}\big([0,T)\times\mathbb{T}\big),\ |\varphi|\le 1\big\} \tag{3.1.13}$$

$$\|\wp_{\eta,u}^{+}\|_{\mathrm{TV}} := \sup\big\{\wp_{\eta,u}(\varphi),\ \varphi\in C_{\mathrm{c}}^{\infty}\big([0,T)\times\mathbb{T}\big),\ 0\le\varphi\le 1\big\}$$

The following result is a restatement of Proposition 2.2.3 in the slightly different setting of this chapter.

PROPOSITION 3.1.5. *Let $u\in\mathcal{X}$ be a weak solution to* (3.1.4). *The following statements are equivalent:*

(i) *There exists $c>0$ such that $\|\wp_{\eta,u}^{+}\|_{\mathrm{TV}}<+\infty$ for each $\eta\in C^2([0,1])$ with $0\le\eta''\le c$.*
(ii) *For each entropy η, the η-entropy production $\wp_{\eta,u}$ can be extended to a Radon measure on $[0,T)\times\mathbb{T}$, namely $\|\wp_{\eta,u}\|_{\mathrm{TV}}<+\infty$ for each entropy η.*
(iii) *There exists a bounded measurable map $\varrho_u:[0,1]\ni v\to\varrho_u(v;dt,dx)\in M\big([0,T)\times\mathbb{T}\big)$ such that for any entropy sampler ϑ*

$$P_{\vartheta,u}=\int dv\,\varrho_u(v;dt,dx)\,\vartheta''(v,t,x) \tag{3.1.14}$$

A weak solution $u\in\mathcal{X}$ that satisfies any of the equivalent conditions in Proposition 3.1.5 is called an *entropy-measure solution* to (3.1.4). We denote by $\mathcal{E}\subset\mathcal{X}$ the set of entropy-measure solutions to (3.1.4). Recall that in Chapter 2 we have discussed regularity properties of the entropy measure solutions. In particular, if $f\in C^2([0,1])$ is such that there are no intervals in which f is affine, then $\mathcal{E}\subset C([0,T];L_1(\mathbb{T}))$.

A weak solution $u\in\mathcal{X}$ to (3.1.4) is called an *entropic solution* iff for each convex entropy η the inequality $\wp_{\eta,u}\le 0$ holds in distribution sense, namely $\|\wp_{\eta,u}^{+}\|_{\mathrm{TV}}=0$. In particular entropic solutions are entropy-measure solutions such that $\varrho_u(v;dt,dx)$ is a negative Radon measure for each $v\in[0,1]$.

Up to minor adaptations, the following class of solutions have been also introduced in Section 2.2, where some examples of such solutions are are also given.

DEFINITION 3.1.6. *An entropy-measure solution $u\in\mathcal{E}$ is* entropy-splittable *iff there exist two closed sets $E^+,E^-\subset[0,T]\times\mathbb{T}$ such that*

(i) *For a.e. $v\in[0,1]$, the support of $\varrho_u^{+}(v;dt,dx)$ is contained in E^+, and the support of $\varrho_u^{-}(v;dt,dx)$ is contained in E^-.*
(ii) *For each $L>0$, the set $\big\{t\in[0,T]\,:\,\big(\{t\}\times[-L,L]\big)\cap E^+\cap E^-\neq\emptyset\big\}$ is nowhere dense in $[0,T]$.*
(iii) *There exists $\delta>0$ such that $\delta\le u\le 1-\delta$.*

The set of entropy-splittable solutions to (3.1.4) *is denoted by $\mathcal{S}$.*

Note that $\mathcal{S}\subset\mathcal{E}\subset\mathcal{X}$, and that we require u_0 to be uniformly far from 0, 1, so that $\mathcal{S}$ is nonempty.

3.1.6. Second order large deviations. We still denote with $\mathbb{P}^\varepsilon := \mathbb{P} \circ (u^\varepsilon)^{-1} \in \mathcal{P}(\mathcal{X})$ the law of u^ε on the Polish space $(\mathcal{X}, d_\mathcal{X})$. Since $\int dx\, \jmath^\varepsilon(x) = 1$ (see hypothesis **H4)**), we have that $\jmath^\varepsilon - 1$ is the derivative of some smooth function J on $\mathbb{T}$, defined up to an additive constant. We define $\|\jmath - 1\!\!1\|_{W^{-1,1}(\mathbb{T})}$ as the minimum of $\|J\|_{L_1(\mathbb{T})}$ on the set of functions J such that $\nabla \cdot J = \jmath^\varepsilon - 1$. We have the following

THEOREM 3.1.7. (i) *Assume that f is such that there is no interval in which f is affine. Assume also $\gamma > 1/2$ and $\lim_\varepsilon \varepsilon^{2(\gamma-1)} \big[\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \varepsilon \|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\big] = 0$. Then the sequence $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$ satisfies a large deviations upper bound on $(\mathcal{X}, d_\mathcal{X})$ with speed $\varepsilon^{-2\gamma+1}$ and rate functional $H : \mathcal{X} \to [0, +\infty]$ defined as*

$$H(u) := \begin{cases} \int dv\, \varrho_u^+(v; dt, dx)\, \frac{D(v)}{a^2(v)} & \text{if } u \in \mathcal{E} \\ +\infty & \text{otherwise} \end{cases} \tag{3.1.15}$$

(ii) *Assume that $f \in C^2([0,1])$ is such that there is no interval in which f is affine. Assume also $\lim_\varepsilon \varepsilon^{-3/2} \|\jmath^\varepsilon - 1\!\!1\|_{W^{-1,1}(\mathbb{T})} = 0$ and $\lim_\varepsilon \big[\varepsilon^{2\gamma-1} \|\nabla \jmath^\varepsilon\|^2_{L_2} + \varepsilon^{2\gamma-3} \|\jmath^\varepsilon\|^2_{L_2}\big] = 0$. Then the sequence $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$ satisfies a large deviations lower bound on $(\mathcal{X}, d_\mathcal{X})$ with speed $\varepsilon^{-2\gamma+1}$ and rate functional $\overline{H} : \mathcal{X} \to [0, +\infty]$ defined as*

$$\overline{H}(u) := \sup_{\substack{\mathcal{O} \ni u \\ \mathcal{O} \text{ open}}} \inf_{v \in \mathcal{O} \cap \mathcal{S}} H(v)$$

Since H is lower semicontinuous on $\mathcal{X}$, we have $\overline{H} \geq H$ on $\mathcal{X}$ and $\overline{H} = H$ on $\mathcal{S}$, namely a large deviations principle holds on $\mathcal{S}$. In order to obtain a full large deviations principle, one needs to show a $H(u) \geq \overline{H}(u)$ also for $u \notin \mathcal{S}$. This amounts to show that $\mathcal{S}$ is H-dense in $\mathcal{X}$, namely that for $u \in \mathcal{X}$ such that $H(u) < +\infty$ there exists a sequence $\{u^n\} \subset \mathcal{S}$ converging to u in $\mathcal{X}$ such that $H(u^n) \to H(u)$. This issue was briefly discussed in Section 2.2. The main difficulties arise from the lacking of a chain rule formula connecting the measures $\wp_{\eta,u}$ to the structure of u itself. If u has bounded variation, Vol'pert chain rule [**3**] allows an explicit representation for $\wp_{\eta,u}$ and thus $H(u)$, see Remark 2.2.7. On the other hand, there exists $u \in \mathcal{X}$ with infinite variation such that $H(u) < +\infty$, see Example 2.2.8. While chain rule formulas out of the BV setting are subject to current research investigation, see e.g. [**11, 2**], only partial results are available.

3.2. Convergence and bounds

In the following we use the notation

$$\sigma(v) := a(v)^2$$

LEMMA 3.2.1 (Itô formula). *Let $(\vartheta; Q)$ be an entropy sampler –entropy sampler flux pair for the equation* (3.1.4). *Then (hereafter, for sake of readability, we possibly omit the explicit dependence of ϑ w.r.t. the (t,x) variables):*

$$\begin{aligned}
&-\int dx\, \vartheta(u_0(x),0,x) - \int dt\, dx\, \big[\big(\partial_t \vartheta\big)\big(u^\varepsilon(t,x),t,x\big) + \big(\partial_x Q\big)\big(u^\varepsilon(t,x),t,x\big)\big] \\
&\quad = -\tfrac{\varepsilon}{2}\langle\langle \vartheta''(u^\varepsilon)\nabla u^\varepsilon, D(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle - \tfrac{\varepsilon}{2}\langle\langle \partial_x\theta'(u^\varepsilon), D(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \\
&\qquad + \tfrac{\varepsilon^{2\gamma}}{2}\|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle \vartheta''(u^\varepsilon)a(u^\varepsilon), a(u^\varepsilon)\rangle\rangle \\
&\qquad + \tfrac{\varepsilon^{2\gamma}}{2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle \vartheta''(u^\varepsilon)\nabla u^\varepsilon, [a'(u^\varepsilon)]^2\nabla u^\varepsilon\rangle\rangle + N_T^{\varepsilon;\vartheta}
\end{aligned} \tag{3.2.1}$$

where $\{N_t^{\varepsilon;\vartheta},\ t\in[0,T]\}$ is the martingale

$$N_t^{\varepsilon;\vartheta} := -\varepsilon^\gamma \int_{[0,t]} \langle \jmath^\varepsilon * \big[a(u^\varepsilon)\vartheta''(u^\varepsilon)\nabla u^\varepsilon + a(u^\varepsilon)\partial_x\vartheta'(u^\varepsilon)\big], dW_s\rangle \tag{3.2.2}$$

Moreover the quadratic variation of $N^{\varepsilon,\vartheta}$ is bounded by

$$\big[N^{\varepsilon;\vartheta}, N^{\varepsilon;\vartheta}\big]_t \le \varepsilon^{2\gamma}\int_{[0,t]} ds\, \big\langle \sigma(u^\varepsilon)\big[\vartheta''(u^\varepsilon)\nabla u^\varepsilon + \partial_x\vartheta'(u^\varepsilon)\big], \vartheta''(u^\varepsilon)\nabla u^\varepsilon + \partial_x\vartheta'(u^\varepsilon)\big\rangle \tag{3.2.3}$$

PROOF. Equation (3.2.1) follows, up to minor manipulations, from Itô formula [**12**] for the map

$$\begin{aligned} F^\vartheta : \mathcal{U} &\ \to\ \mathbb{R} \\ F^\vartheta : u &\ \to\ \int dt\, dx\, \vartheta(u(t,x),t,x) \end{aligned}$$

By (3.2.2) and (3.1.1), the quadratic variation of $N^{\varepsilon;\vartheta}$ is given by

$$\begin{aligned}
&\big[N^{\varepsilon;\vartheta}, N^{\varepsilon;\vartheta}\big]_t = \varepsilon^{2\gamma}\int_{[0,t]} ds \\
&\quad \big\langle \jmath^\varepsilon * \big\{a(u^\varepsilon)\big[\vartheta''(u^\varepsilon)\nabla u^\varepsilon + \partial_x\vartheta'(u^\varepsilon)\big]\big\}, \jmath^\varepsilon * \{a(u^\varepsilon)\vartheta''(u^\varepsilon)\nabla u^\varepsilon + \partial_x\vartheta'(u^\varepsilon)\}\big\rangle
\end{aligned}$$

so that the inequality stated in the lemma follows by Young inequality for convolutions and hypothesis **H4)**. □

LEMMA 3.2.2. *Let ζ, $T>0$ and $\{X_t;\ t\in[0,T]\}$ be a local continuous real-valued L^2–supermartingale starting from* 0*, and $\tau \le T$ a stopping time. Let $F:\mathbb{R}\to\mathbb{R}^+$ be such that:*

$$\frac{F(x)}{F(\zeta)} \le 2\frac{x}{\zeta} - 1, \quad \textit{for all } x > \zeta. \tag{3.2.4}$$

Then:

$$\mathbb{P}\left(\sup_{0\le t\le\tau} X_t \ge \zeta,\ \big[X,X\big]_\tau \le F(\sup_{t\le\tau} X_t)\right) \le \exp\left\{-\frac{\zeta^2}{2F(\zeta)}\right\}; \tag{3.2.5}$$

PROOF. Hypotheses on F imply that the map $G_\zeta : x \to \frac{\zeta}{F(\zeta)} x - \frac{1}{2}\frac{\zeta^2}{F(\zeta)^2} F(x)$ has the property $G_\zeta(x) \geq G_\zeta(\zeta) = \frac{\zeta^2}{2F(\zeta)}$ for all $x \geq \zeta$. Therefore:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\leq\tau} X_t \geq \zeta,\ \big[X,X\big]_\tau \leq F(\sup_{t\leq\tau} X_t)\right)\\
&\leq \mathbb{P}\left(e^{\frac{\zeta}{F(\zeta)}\sup_{t\leq\tau} X_t - \frac{1}{2}\frac{\zeta^2}{F(\zeta)^2} F(\sup_{t\leq\tau} X_t)} \geq e^{\frac{1}{2}\frac{\zeta^2}{F(\zeta)}},\ \big[X,X\big]_\tau \leq F(\sup_{t\leq\tau} X_t)\right)\\
&\leq \mathbb{P}\left[\sup_{t\leq T} e^{\frac{\zeta}{F(\zeta)} X_t - \frac{1}{2}\frac{\zeta^2}{F(\zeta)^2}\big[X,X\big]_t} \geq e^{\frac{1}{2}\frac{\zeta^2}{F(\zeta)}}\right] \leq e^{-\frac{\zeta^2}{2F(\zeta)}}.
\end{aligned}$$

where in the last line we used maximal inequality for positive supermartingales, see [**17**]. □

Note that the hypotheses (3.2.4) on F are satisfied by any nonincreasing function, and by functions with affine or subaffine behaviour. The lemma is a generalization of the well known Bernstein inequality [**17**].

COROLLARY 3.2.3. *Assume* $\gamma > 1/2$, $\lim_\varepsilon \varepsilon^{2\gamma-1}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} = 0$ *and* $\lim_\varepsilon \varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} = 0$. *Then there exists* C, $\varepsilon_0 > 0$ *depending only on* $\{\jmath^\varepsilon\}$, f, D *and* σ *such that for each* $\varepsilon < \varepsilon_0$*:*

$$\varepsilon\langle\langle\nabla u^\varepsilon, \nabla u^\varepsilon\rangle\rangle \leq C + N^\varepsilon_T \tag{3.2.6}$$

where $\{N^\varepsilon_t\}_{t\in[0,T]}$ *is a martingale starting from* 0 *and satisfying*

$$\mathbb{P}\big(\sup_{t\leq T} N^\varepsilon_t > \zeta\big) \leq \exp\Big\{-\frac{\zeta^2}{\varepsilon^{2\gamma-1}C(1+\zeta)}\Big\} \tag{3.2.7}$$

PROOF. Let $\chi \in C^\infty_{\mathrm{c}}([0,T))$ be a smooth decreasing function such that $\chi(0) = 1$. Evaluating Itô formula (3.2.1) for $\vartheta(u,t,x) = \bar\vartheta(u,t,x) := u^2\chi(t)$

$$\begin{aligned}
&-\langle u_0, u_0\rangle + \varepsilon\langle\langle\nabla u^\varepsilon, D(u^\varepsilon)\nabla u^\varepsilon\chi\rangle\rangle - \langle\langle u^\varepsilon, u^\varepsilon\partial_t\chi\rangle\rangle =\\
&N^{\varepsilon;\bar\vartheta}_T + \varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle a(u^\varepsilon), a(u^\varepsilon)\chi\rangle\rangle + \varepsilon^{2\gamma}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle\nabla u^\varepsilon, [a'(u^\varepsilon)]^2\nabla u^\varepsilon\chi\rangle\rangle
\end{aligned}$$

By **H2)**, **H3)** and the hypotheses of this lemma, there exist $\varepsilon_0 > 0$ such that, for each $\varepsilon \leq \varepsilon_0$ and $v \in [0,1]$, $\varepsilon^{2\gamma}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}[a'(v)]^2 \leq \frac{1}{2}\varepsilon\, D(v)$ and $\varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\sigma(v) \leq 1$. Therefore, since also $\langle u_0, u_0\rangle \leq 1$, $\langle\langle u^\varepsilon, u^\varepsilon\partial_t\chi\rangle\rangle \leq 0$

$$\frac{\varepsilon}{2}\langle\langle\nabla u^\varepsilon, D(u^\varepsilon)\nabla u^\varepsilon\chi\rangle\rangle \leq 2 + N^{\varepsilon;\bar\vartheta}_T$$

As we send χ to the indicator function of $[0,T)$ uniformly on compact subsets, it is not difficult to obtain the inequality

$$\frac{\varepsilon}{2}\langle\langle\nabla u^\varepsilon, D(u^\varepsilon)\nabla u^\varepsilon\chi\rangle\rangle \leq 2 + N^{\varepsilon;2}_T \tag{3.2.8}$$

where $N^{\varepsilon;2}_t := \int_{[0,t]}\langle\jmath^\varepsilon * [\sigma(u^\varepsilon)\nabla u^\varepsilon], dW\rangle$. Since D is uniformly positive, there exists $C_1 > 0$ such that $\sigma(v) \leq C_1 D(v)/8$ for each $v \in [0,1]$. Therefore, applying again Young inequality for convolutions to bound the quadratic variation of the martingale $\{N^{\varepsilon;2}_t\}$, we gather for $\varepsilon \le \varepsilon_0$

$$\begin{aligned}\big[N^{\varepsilon;2}_{\cdot}, N^{\varepsilon;2}_{\cdot}\big]_T &\le 4\,\varepsilon^{2\gamma}\langle\langle \nabla u^\varepsilon, \sigma(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \\ &\le \tfrac{\varepsilon^{2\gamma}C_1}{2}\langle\langle \nabla u^\varepsilon, D(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \le C_1\,\varepsilon^{2\gamma-1}\big[2+N^{\varepsilon;2}_T\big]\end{aligned}$$

Applying Lemma 3.2.2 to the martingale $\{N^{\varepsilon;\bar\vartheta}_t\}$

$$\mathbb{P}\big(\sup_{t\le T} N^{\varepsilon,\bar\vartheta}_T \ge \zeta\big) \le \exp\Big[-\frac{\zeta^2}{2C_1\varepsilon^{2\gamma-1}(2+\zeta)}\Big]$$

Since D is uniformly positive, for a suitable choice of C we conclude by (3.2.8). □

LEMMA 3.2.4. *There exists an increasing sequence $\{K_\ell\}$ of compact subsets of $\mathcal{U}$ such that*

$$\lim_{\ell}\overline{\lim_{\varepsilon}}\,\varepsilon^{2\gamma}\mathbb{P}^\varepsilon(K_\ell^c) = -\infty$$

PROOF. Let $d \in C^1([0,1])$ be a map such that $d'(v) = D(v)$ for $v \in [0,1]$. Then, integrating twice by parts the diffusive term in the weak formulation of (3.1.2), for each $\varphi \in C^\infty(\mathbb{T})$ and $s,t \in [0,T]$

$$\begin{aligned}|\langle u^\varepsilon(t) - u^\varepsilon(s), \varphi\rangle| &\le \big|\textstyle\int_{[s,t]} dr\,\langle f(u^\varepsilon), \nabla\varphi\rangle\big| \\ &\quad +\tfrac{\varepsilon}{2}\big|\textstyle\int_{[s,t]} dr\,\langle d(u^\varepsilon), \nabla(\nabla\varphi)\rangle\big| + \varepsilon^\gamma\big|\textstyle\int_{[s,t]}\langle a(u^\varepsilon)\,\jmath^\varepsilon * \nabla\varphi, dW_r\rangle\big| \\ &\le C'_\varphi|t-s| + \varepsilon^\gamma\big|\textstyle\int_{[s,t]}\langle a(u^\varepsilon)\,\jmath^\varepsilon * \nabla\varphi, dW_r\rangle\big|\end{aligned}$$

for some constant C'_φ depending only on f, d and φ. On the other hand, by Young inequality for convolutions, the martingale term in the last line of the above formula enjoys the bound (3.2.3) evaluated for $\vartheta(v,t,x) = v\,\varphi(t,x)$, so that by Bernstein inequality, there exists a constant $C''_\varphi > 0$ depending only on a and φ such that for each ξ, $\zeta > 0$ and $s \in [0,T]$

$$\mathbb{P}\Big(\varepsilon^\gamma \sup_{t\,:\,|t-s|\le\xi}\Big|\int_{[s,t]}\langle a(u^\varepsilon)\,\jmath^\varepsilon * \nabla\varphi, dW_r\rangle\Big| \ge \zeta\Big) \le \exp\Big(-\frac{\zeta^2}{C''_\varphi\,\varepsilon^{2\gamma}\xi}\Big)$$

We thus obtain, for each $\varphi \in C^\infty(\mathbb{T})$, $\zeta > 0$, $s \in [0,T]$ and ξ, ε small enough, and for some constant C_φ depending only on φ, f, D, a

$$\mathbb{P}^\varepsilon\big(\sup_{t\,:\,|t-s|\le\xi}\big|\langle u(t) - u(s), \varphi\rangle\big| \le \zeta\big) \le \exp\Big(-\frac{\zeta^2}{C_\varphi\,\varepsilon^{2\gamma}\xi}\Big)$$

Since (U, d_U) is compact, this inequality implies the exponential tightness of $\{\mathbb{P}^\varepsilon\}$ on $\mathcal{U} = C\big([0,T];U\big)$ by standard tightness arguments for probability measures on spaces of continuous functions [**6**]. □

3.3. Large deviations on the scale $\varepsilon^{-2\gamma}$

We recall a well known method to prove large deviations lower bound, see e.g. [**9, 14**], which can be easily restated in terms of Γ-convergence. A more general statement connecting large deviations principle and Γ-convergence of the relative entropy can also be found in the introductive chapter of this thesis, see Section 1.2.3. In order to avoid confusion between the functional H defined in (3.1.15), in this chapter, for $\mathbb{P}$, $\mathbb{Q}$ two Borel probability measures on a Polish space, we denote by $\mathrm{Ent}(\mathbb{Q}|\mathbb{P})$ the relative entropy of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ (so we will not use the standard notation $H(\mathbb{Q}|\mathbb{P})$ for the same quantity).

LEMMA 3.3.1. *Let X be a Polish space, let $\{\alpha_\varepsilon\} \subset \mathbb{R}^+$ be a sequence such that $\lim_\varepsilon \alpha_\varepsilon = 0$, and let $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$. For $\varepsilon > 0$ define the functionals $\mathrm{Ent}_\varepsilon : \mathcal{P}(X) \to [0, +\infty]$ as*

$$\mathrm{Ent}_\varepsilon(\mathbb{Q}) := \alpha_\varepsilon \mathrm{Ent}(\mathbb{Q}|\mathbb{P}^\varepsilon)$$

and the functional $I : X \to [0, +\infty]$ as

$$I(x) := (\Gamma\text{–}\overline{\lim}_\varepsilon \, \mathrm{Ent}_\varepsilon)(\delta_x)$$

where δ_x denotes the Dirac measure concentrated at x. Then $\{\mathbb{P}^\varepsilon\}$ satisfies a large deviations lower bound with speed $\{\alpha_\varepsilon^{-1}\}$ and rate I. Conversely, suppose that $\{\mathbb{P}^\varepsilon\}$ satisfies a large deviations principle with lower semicontinuous rate functional I. Then $(\Gamma\text{–}\overline{\lim}_\varepsilon \, \mathrm{Ent}_\varepsilon)(\mathbb{Q}) \le \int_X \mathbb{Q}(dx)\, I(x)$.

LEMMA 3.3.2. *There exists an increasing sequence $\{\mathcal{K}_\ell\}$ of compact subsets of $\mathcal{M}$ such that*

$$\lim_\ell \overline{\lim_\varepsilon} \, \varepsilon^{2\gamma} \mathbf{P}^\varepsilon(\mathcal{K}_\ell^c) = -\infty \tag{3.3.1}$$

PROOF. Let the sequence $\{K_\ell\}$ of compact subsets of $\mathcal{U}$ be as in Lemma 3.2.4. For $\ell > 0$ consider the set

$$\mathcal{K}_\ell := \{\mu \in \mathcal{M} \, : \, \mu_{t,x} = \delta_{u(t,x)} \text{ for some } u \in K_\ell\}$$

Then $\mathbf{P}^\varepsilon(\mathcal{K}_\ell) = \mathbb{P}^\varepsilon(K_\ell)$ and by Lemma 3.2.4, (3.3.1) holds. On the other hand $\mathcal{K}_\ell$ is precompact in $(\mathcal{M}, d_{\mathcal{M}})$. □

PROOF OF THEOREM 3.1.3: UPPER BOUND. Let $d \in C^2([0,1])$ be such that $d' = D$. For $\varepsilon > 0$ and $\varphi \in C^\infty([0,T] \times \mathbb{T})$, since $\mathbb{P}^\varepsilon$ solves (3.1.2), the map

$$\begin{aligned} N^{\varepsilon;\varphi} \;&:\; [0,T] \times C([0,T]; X) \to \mathbb{R} \\ N^{\varepsilon;\varphi} \;&:\; (t,v) \to \langle v(t), \varphi(t)\rangle - \langle u_0, \varphi(0)\rangle - \int_{[0,t]} ds \, \big[\langle v, \partial_s \varphi\rangle \\ &\qquad + \langle f(v), \nabla\varphi\rangle + \tfrac{\varepsilon}{2}\langle d(v), \nabla(\nabla\varphi)\rangle\big] \end{aligned} \tag{3.3.2}$$

is a $\mathbb{P}^\varepsilon$-martingale. Let $\mathcal{M}_e := \{\mu \in \mathcal{M} \,:\, \mu = \delta_u \text{ for some } u \in \mathcal{U}\}$. Note that, by its definition, $\mathbf{P}^\varepsilon$ is concentrated on $\mathcal{M}_e$ by definition. Therefore the map

$$\begin{array}{rcl} \mathcal{N}^{\varepsilon;\varphi} & : & [0,T] \times \mathcal{M} \to \mathbb{R} \\ \mathcal{N}^{\varepsilon;\varphi} & : & (t,\mu) \to \langle \mu_{T,\cdot}(\imath), \varphi(T)\rangle - \langle u_0, \varphi(0)\rangle \\ & & - \int_{[0,t]} ds \left[\langle \mu(\imath), \partial_t \varphi\rangle - \langle \mu(f), \nabla\varphi\rangle + \frac{\varepsilon}{2}\langle \mu(d), \nabla(\nabla\varphi)\rangle \right] \end{array}$$

is a $\mathbf{P}^\varepsilon$ martingale, since $\mathcal{N}^{\varepsilon;\varphi}(t,\delta_v) = N^{\varepsilon;\varphi}(t,v)$. By the same reason, and by inequality (3.2.3) the map

$$\begin{array}{rcl} \mathcal{Q}^{\varepsilon;\varphi} & : & [0,T] \times \mathcal{M} \to \mathbb{R} \\ \mathcal{Q}^{\varepsilon;\varphi} & : & (t,\mu) \to \exp\left\{ \mathcal{N}^{\varepsilon;\varphi}(t,\mu) - \frac{\varepsilon^{2\gamma}}{2} \int_{[0,t]} ds\, \langle \mu(\sigma)\nabla\varphi, \nabla\varphi\rangle \right\} \end{array}$$

is a continuous, strictly positive, $\mathbf{P}^\varepsilon$ supermartingale, with $\mathcal{Q}^{\varepsilon;\varphi}(0,\mu) = 1$. For a compact subset $\mathcal{K} \subset \mathcal{M}$ we then have

$$\begin{array}{rcl} \mathbf{P}^\varepsilon(\mathcal{K}) & = & \mathbf{P}^\varepsilon\big(\mathbb{1}_{\mathcal{K}}(\cdot) \mathcal{Q}^{\varepsilon;\varphi}(T,\cdot) [\mathcal{Q}^{\varepsilon;\varphi}(T,\cdot)]^{-1} \big) \\ & \le & \sup_{\mu\in\mathcal{K}} [\mathcal{Q}^{\varepsilon;\varphi}(T,\mu)]^{-1} \mathbb{P}^\varepsilon\big(\mathbb{1}_{\mathcal{K}}(\cdot) \mathcal{Q}^{\varepsilon;\varphi}(T,\cdot) \big) \le \inf_{\mu\in\mathcal{K}} [\mathcal{Q}^{\varepsilon;\varphi}(T,\mu)]^{-1} \end{array}$$

Since this inequality holds for each φ, we can evaluate it replacing φ with $\varepsilon^{-2\gamma}\varphi$ obtaining

$$\begin{array}{rcl} \log \mathbf{P}^\varepsilon(\mathcal{K}) & \le & -\varepsilon^{-2\gamma} \inf_{\mu\in\mathcal{K}} \Big\{ \langle \mu_{T,\cdot}(\imath), \varphi(T)\rangle - \langle u_0, \varphi(0)\rangle - \langle\langle \mu(\imath), \partial_t\varphi\rangle\rangle \\ & & \qquad - \langle\langle \mu(f), \nabla\varphi\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \mu(d), \nabla(\nabla\varphi)\rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\nabla\varphi, \nabla\varphi\rangle\rangle \Big\} \\ & \le & -\varepsilon^{-2\gamma} \inf_{\mu\in\mathcal{K}} \Big\{ \langle \mu_{T,\cdot}(\imath), \varphi(T)\rangle - \langle u_0, \varphi(0)\rangle - \langle\langle \mu(\imath), \partial_t\varphi\rangle\rangle \\ & & \qquad - \langle\langle \mu(f), \nabla\varphi\rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\nabla\varphi, \nabla\varphi\rangle\rangle \Big\} + \varepsilon^{-2\gamma+1} C_{d,\varphi} \end{array}$$

for some constant $C_{d,\varphi}$ depending only on the maximum value of d on $[0,1]$ and on φ. Multiplying by $\varepsilon^{2\gamma}$ and taking the limsup for $\varepsilon \to 0$, the last term vanishes. Optimizing on φ:

$$\begin{array}{rcl} \overline{\lim}_\varepsilon\, \varepsilon^{2\gamma} \log \mathbf{P}^\varepsilon(\mathcal{K}) & \le & -\sup_{\varphi\in C^\infty([0,T]\times\mathbb{T})} \inf_{\mu\in\mathcal{K}} \Big\{ \langle \mu_{T,\cdot}(\imath), \varphi(T)\rangle - \langle u_0, \varphi(0)\rangle \\ & & \qquad - \langle\langle \mu(\imath), \partial_t\varphi\rangle\rangle - \langle\langle \mu(f), \nabla\varphi\rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\nabla\varphi, \nabla\varphi\rangle\rangle \Big\} \end{array}$$

Bu minimax lemma we gather that upper bound with rate $\mathcal{I}$, see (3.1.9), holds on each compact subset $\mathcal{K} \subset \mathcal{M}$. By Lemma 3.3.2, it holds on each closed subset of $\mathcal{M}$. □

Proof of Theorem 3.1.3: lower bound. We will prove the lower bound by the means of Lemma 3.3.1. More precisely, consider the set

$$\mathcal{M}_0 := \Big\{ \mu \in \mathcal{M} \,:\, \exists r > 0 \,:\, \mu = \delta_u \text{ for some } u \in C^2\big([0,T]\times\mathbb{T}; [r, 1-r]\big) \Big\}$$

Here we prove that for each $\mu \in \mathcal{M}_0$ there exists a sequence of probability measure $\{\mathbf{Q}^\varepsilon\} \subset \mathcal{P}(\mathcal{M})$ such that $\mathbf{Q}^\varepsilon \to \delta_\mu$ and $\overline{\lim}\, \varepsilon^{2\gamma}\mathrm{Ent}(\mathbf{Q}^\varepsilon|\mathbf{P}^\varepsilon) \le \mathcal{I}(\mu)$.

By Lemma 3.3.1 this will yield a large deviations lower bound with rate $\tilde{\mathcal{I}} : \mathcal{M} \to [0, +\infty]$ defined as

$$\tilde{\mathcal{I}} := \begin{cases} I(\mu) & \text{if } \mu \in \mathcal{M}_0 \\ +\infty & \text{otherwise} \end{cases}$$

By a standard diagonal argument, or as it follows from Proposition 1.2.4, the lower bound then also holds with the lower semicontinuous envelope of $\tilde{I}$ on $\mathcal{M}$ as rate functional. In Theorem 2.4.1 it is shown, in a slightly different setting, that the lower semicontinuous envelope of $\tilde{\mathcal{I}}$ is indeed $\mathcal{I}$. By the assumption $r \le u_0 \le 1 - r$ (which is equivalent to the requirement that $\sigma(u_0)$ is uniformly positive), it is not difficult to adapt the arguments in the proof of Theorem 2.4.1, to obtain the analogous result in this case. We are thus left with the proof of the lower bound on $\mathcal{M}_0$.

Let $\mu = \delta_u \in \mathcal{M}_0$ be such that $\mathcal{I}(\mu) < \infty$. Then necessarily $u(0, x) = u_0(x)$, and by the definition of $\mathcal{I}$ and the smoothness of u

$$\begin{aligned} \mathcal{I}(\mu) = \mathcal{I}(\delta_u) &= \sup_{\varphi \in C^\infty\left([0,T]\times\mathbb{T}\right)} \Big\{ - \langle\langle \partial_t u + \nabla \cdot f(u), \varphi \rangle\rangle - \tfrac{1}{2} \langle\langle \sigma(u) \nabla\varphi, \nabla\varphi \rangle\rangle \Big\} \\ &\ge \sup_{\varphi \in C^\infty\left([0,T]\times\mathbb{T}\right)} \Big\{ - \langle\langle \partial_t u + \nabla \cdot f(u), \varphi \rangle\rangle - \tfrac{\zeta}{2} \langle\langle \nabla\varphi, \nabla\varphi \rangle\rangle \Big\} \end{aligned}$$

where $\zeta > 0$ is a real constant such that $\sigma(u) \ge \zeta$ on $[0, T] \times \mathbb{T}$. Such a constant exists, since we assumed u to be uniformly far from the zeros of σ. Note that the supremum in the last line of the above formula is finite iff there exists $\Psi^u \in L_2\big([0, T]; H^1(\mathbb{T})\big)$ such that

$$\partial_t u + \nabla \cdot f(u) = -\nabla \cdot [\sigma(u) \nabla \Psi^u] \tag{3.3.3}$$

holds weakly. In such a case

$$\mathcal{I}(\mu) = \langle\langle \sigma(u) \nabla\Psi^u, \nabla\Psi^u \rangle\rangle \tag{3.3.4}$$

Note that, as we assumed u smooth and $\sigma(u)$ uniformly positive, Ψ^u is also smooth by standard regularity results for (3.3.3), say $\Psi^u \in C^2([0, T] \times \mathbb{T})$. Recall the definition (3.3.2) of the martingale $N^{\varepsilon;\varphi}$. It is immediate to extend the definition of $N^{\varepsilon;\varphi}$ to the case $\varphi \in C^2([0, T] \times \mathbb{T})$. For $\varepsilon > 0$ we define the real-valued $\mathbb{P}^\varepsilon$-martingale $M_t^{\varepsilon;u}(v) := \varepsilon^{-\gamma} N^{\varepsilon;\Psi^u}$. Note that

$$\big[M^{\varepsilon;u}, M^{\varepsilon;u}\big]_T = \varepsilon^{-2\gamma} \langle\langle \jmath^\varepsilon * [a(u)\nabla\Psi^u], \jmath^\varepsilon * [a(u)\nabla\Psi^u] \rangle\rangle \le 2\varepsilon^{-2\gamma} I(\mu) \tag{3.3.5}$$

by Young inequality for convolutions and (3.3.4). Since the quadratic variation of $M^{\varepsilon;u}$ uniformly bounded, its stochastic exponential $E_t^{\varepsilon;u} := \exp\big(M_t^{\varepsilon;u} - \frac{1}{2}[M^{\varepsilon;u}, M^{\varepsilon;u}]_t\big)$ is also a $\mathbb{P}^\varepsilon$-martingale. For $\varepsilon > 0$ we define $\mathbb{Q}^{\varepsilon;u} \in \mathcal{P}(\mathcal{U})$ by its derivative as

$$\mathbb{Q}^{\varepsilon;u}(dv) := E_T^{\varepsilon;u}(v) \mathbb{P}^\varepsilon(dv)$$

and we let $\mathbf{Q}^{\varepsilon;u}$ be the pushforward of $\mathbb{Q}^{\varepsilon;u}$ w.r.t. the map $\mathcal{U} \ni u \mapsto \delta_u \in \mathcal{U}$. Then

$$\begin{aligned} \varepsilon^{2\gamma}\mathrm{Ent}(\mathbf{Q}^{\varepsilon;u}|\mathbf{P}^{\varepsilon;u}) &= \varepsilon^{2\gamma}\mathrm{Ent}(\mathbb{Q}^{\varepsilon;u}|\mathbb{P}^{\varepsilon;u}) = \varepsilon^{2\gamma}\int \mathbb{Q}^{\varepsilon;u}(dv)\log E_T^{\varepsilon;u}(v) \\ &= \varepsilon^{2\gamma}\int \mathbb{Q}^{\varepsilon;u}(dv)\big(M_T^{\varepsilon;u}(v) - [M^{\varepsilon;u}(v), M^{\varepsilon;u}(v)]_T\big) \\ &\quad + \tfrac{\varepsilon^{2\gamma}}{2}\int \mathbb{Q}^{\varepsilon;u}(dv)[M^{\varepsilon;u}(v), M^{\varepsilon;u}(v)]_T \\ &= \tfrac{1}{2}\langle\langle \jmath^\varepsilon * [a(u)\nabla\Psi^u], \jmath^\varepsilon * [a(u)\nabla\Psi^u]\rangle\rangle \\ &\le \tfrac{1}{2}\langle\langle \sigma(u)\nabla\Psi^u, \nabla\Psi^u\rangle\rangle = I(\mu) \end{aligned} \tag{3.3.6}$$

where we used the Girsanov theorem, stating that $M_t^{\varepsilon;u} - [M^{\varepsilon;u}(v), M^{\varepsilon;u}(v)]_t$ is a $\mathbb{Q}^\varepsilon$-martingale, and thus has vanishing $\mathbb{Q}^{\varepsilon;u}$ expectaction.

By (3.3.6), (3.3.1) and Proposition 1.2.2, the sequence $\{\mathbf{Q}^{\varepsilon;u}\}_\varepsilon$ is tight in $\mathcal{P}(\mathcal{M})$, and in view of (3.3.6) it remains to show that any limit point of $\{\mathbf{Q}^{\varepsilon;u}\}$ coincides with δ_μ. Still by Girsanov theorem, $\mathbb{Q}^{\varepsilon;u}$ is the law of a process v^ε that takes values in $\mathcal{U} \cap L_2\big([0,T]; H^1(\mathbb{T})\big)$ and is a solution to the martingale problem associated with the stochastic partial differential equation

$$\begin{aligned} dv &= \Big[-\nabla\cdot f(v) + \tfrac{\varepsilon}{2}\nabla\cdot\big[D(v)\nabla v - a(v)\big((\jmath^\varepsilon * \jmath^\varepsilon) * (a(u)\nabla\Psi^u)\big)\big]\Big]\,dt \\ &\quad + \varepsilon^\gamma\,\nabla\cdot\big[a(v)(\jmath^\varepsilon * dW)\big] \\ v(0,x) &= u_0(x) \end{aligned} \tag{3.3.7}$$

where we used the same notation of (3.1.2). We will use (3.3.7) to show that $\int_{\mathcal{U}} \mathbb{Q}^{\varepsilon;u}(dv)\sup_t \int dx|v-u|$ converges to 0 as we send $\varepsilon \to 0$. In view of (3.3.6) this will conclude the proof.

Let $l \in C^2([-1,1])$. Applying Itô formula to the map $[-1,1]\times[0,T]\times\mathbb{T} \ni (v,t,x) \mapsto l(v-u(t,x))$, and denoting $z^\varepsilon = v^\varepsilon - u$, some direct computations show that for each $t \in [0,T]$

$$\begin{aligned} &\int dx\,[l(z^\varepsilon(t,x)) - l(0)] \\ &= \int_{[0,t]} ds\Big\{ -\tfrac{\varepsilon}{2}\langle l''(z^\varepsilon)\nabla z^\varepsilon, D(v^\varepsilon)\nabla z^\varepsilon\rangle + \langle l''(z^\varepsilon)\nabla z^\varepsilon, f(u+z^\varepsilon) - f(u)\rangle \\ &\qquad + \langle l''(z^\varepsilon)\nabla z^\varepsilon, a(v^\varepsilon)\big(\jmath^\varepsilon * j^\varepsilon * [a(u)\nabla\Psi^u] - a(u)\big(\jmath^\varepsilon * j^\varepsilon * [a(u)\nabla\Psi^u]\big)\rangle \\ &\qquad - \langle l'(z^\varepsilon), \nabla\big\{a(u)\big(\jmath^\varepsilon * j^\varepsilon * [a(u)\nabla\Psi^u] - a(u)a(u)\nabla\Psi^u]\big)\big\}\rangle \\ &\qquad + \tfrac{\varepsilon^{2\gamma}}{2}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle l''(z^\varepsilon)a(v^\varepsilon), a(v^\varepsilon)\rangle \\ &\qquad + \tfrac{\varepsilon^{2\gamma}}{2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle l''(z^\varepsilon)\nabla v^\varepsilon, [a'(v^\varepsilon)]^2\nabla v^\varepsilon\rangle\Big\} + N_t^{\varepsilon;l} \end{aligned} \tag{3.3.8}$$

for some square-integrable martingale $\{N_t^{\varepsilon;l}\}$. Let us define $B^\ell := \{v \in \mathcal{U} : \langle\langle \nabla v, \nabla v\rangle\rangle \le \ell\}$. By Corollary 3.2.3, (3.3.6) and the inequality (1.2.7), we have

$$\lim_{\ell\to+\infty}\varliminf_{\varepsilon}\mathbb{Q}^{\varepsilon;u}(B^{\varepsilon^{-2}\ell}) = 1 \tag{3.3.9}$$

Let us now assume l convex. Then we can define, for ε, $\ell > 0$, $t \in [0,T]$

$$\begin{array}{lcl} A^{1,l} & := & \max_{z\in[0,1]} |l'(z)| \\ A^{2,l} & := & \max_{z\in[0,1]} |l''(z)| \\ A^{3,l} & := & \sqrt{\max_{z\in[0,1]} |l''(z)\, z^2|} \\ R^{\varepsilon,l}(t) & := & \sqrt{\int_{[0,t]} ds\, \langle l''(z^\varepsilon)\nabla z^\varepsilon, D(v^\varepsilon)\nabla z^\varepsilon\rangle} \\ A^{\varepsilon,u} & := & \int dt ds \big|\nabla\big\{a(u)\big(\jmath^\varepsilon * \jmath^\varepsilon * [a(u)\nabla\Psi^u] - a(u)a(u)\nabla\Psi^u]\big)\big\}\big| \end{array} \tag{3.3.10}$$

For $v^\varepsilon \in B^{\varepsilon^{-2}\ell}$, by (3.3.8), Cauchy-Schwarz inequality and using the smoothness of u and Ψ^u, there exists a constant C independent of ε, such that for each $t \in [0,T]$

$$\begin{aligned} &\int dx\, [l(z^\varepsilon(t,x)) - l(0)] \le -\tfrac{\varepsilon}{2}(R^{\varepsilon;\ell}(t))^2 + CA^{3,l}R^{\varepsilon,l}(t) + CA^{1,l}A^{\varepsilon,u} \\ &\quad + C\varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}A^{2,l} + C\varepsilon^{2\gamma-2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}A^{2,l} + N_t^{\varepsilon;l} \\ &\le \frac{3(A^{3,l}+A^{1,l}A^{\varepsilon,u})^2C^2}{2\varepsilon} + C\varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}A^{2,l} + C\varepsilon^{2\gamma-2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}A^{2,l}\ell + N_t^{\varepsilon;l} \end{aligned}$$

Assume now also $l(0) = 0$. Integrating in dt, taking the $\mathbb{Q}^{\varepsilon;u}$ expected value on v, and redefining the constant C

$$\begin{aligned} &\mathbb{Q}^{\varepsilon;u}\Big(1\!\!1_{B^{\varepsilon^{-2}\ell}}(v)\int dt\, dx\, l(v - u(t,x))\Big) \\ &\qquad \le C\big[(A^{3,l} + A^{1,l}A^{\varepsilon,u})^2\varepsilon^{-1} \\ &\qquad + \big(\varepsilon^{2\gamma-1}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \varepsilon^{2\gamma-3}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\ell\big)\varepsilon A^{2,l}\big] \end{aligned} \tag{3.3.11}$$

Note now that, by the smoothness of u and Ψ^u, since $\jmath^\varepsilon * \jmath^\varepsilon$ is a sequence of mollifiers converging to the identity, $\lim_\varepsilon A^{\varepsilon;u} \to 0$. By the assumptions of this theorem, the term in round brackets in the last line of (3.3.11) also vanishes. It is the easy to see that there exists a sequence $\{l^\varepsilon\}$ such that the $l^\varepsilon(\cdot) \to |\cdot|$ uniformly on $[-1,1]$, and the r.h.s. of (3.3.11) vanishes as $\varepsilon \to 0$. We then conclude by (3.3.9). □

Proof of Corollary 3.1.4. The proof is achieved by following closely Corollary 2.2.2. □

3.4. Large deviations on the scale $\varepsilon^{-2\gamma+1}$

The next proposition is a convenient restatement in this setting of Tartar compensated compactness method. We refer to [**18**, Chapter 9] for the proof.

Proposition 3.4.1 (Tartar). *Assume that f is such that there are no intervals where f is affine. Let $K \subset \mathcal{U}$ be a compact w.r.t. $d_\mathcal{U}$. Suppose that for each $\eta \in C^2([0,1])$ there exists a compact (w.r.t. the strong $H^{-1}([0,T]\times\mathbb{T})$ topology) set $K_\eta \subset H^{-1}([0,T]\times\mathbb{T})$ such that $\partial_t\eta(u) + \nabla\cdot q(u) \in K_\eta$, for each $u \in K$. Then K is strongly compact in $\mathcal{X}$.*

LEMMA 3.4.2. *Under the same hypotheses of Theorem 3.1.7 item (i), there exists an increasing sequence $\{K_\ell\}$ of compact subsets of $\mathcal{X}$ such that*

$$\lim_{\ell} \overline{\lim_{\varepsilon}}\, \varepsilon^{-2\gamma+1} \log \mathbb{P}^\varepsilon(K_\ell) = -\infty$$

PROOF. Let η be an entropy, and let q be its conjugated flux. For $\varphi \in C_c^\infty((0,T)\times\mathbb{T})$, by Itô formula (3.2.1) applied to the function $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$

$$\begin{aligned}
-\langle\langle \eta(u^\varepsilon), \partial_t\varphi\rangle\rangle - \langle\langle q(u^\varepsilon), \nabla\varphi\rangle\rangle = & -\tfrac{\varepsilon}{2}\langle\langle \eta''(u^\varepsilon)\nabla u^\varepsilon\varphi, D(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \\
& -\tfrac{\varepsilon}{2}\langle\langle \eta'(u^\varepsilon)\nabla\varphi, D(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \\
& +\tfrac{\varepsilon^{2\gamma}}{2}\|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle \eta''(u^\varepsilon)a(u^\varepsilon), a(u^\varepsilon)\varphi\rangle\rangle \\
& +\tfrac{\varepsilon^{2\gamma}}{2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle \eta''(u^\varepsilon)\varphi\nabla u^\varepsilon, [a'(u^\varepsilon)]^2\nabla u^\varepsilon\rangle\rangle + N_T^{\varepsilon;\eta\varphi}
\end{aligned} \tag{3.4.1}$$

where $\{N_t^{\varepsilon,\eta\varphi}, t\in[0,T]\}$ is the martingale $N_t^{\varepsilon,\eta\varphi} := \varepsilon^\gamma \int_{[0,t]} \langle \jmath^\varepsilon * [\sigma(u^\varepsilon)\nabla(\eta'(u^\varepsilon)\varphi)], dW\rangle$. For ε, $\ell>0$ let us define the stopping time $\tau^{\varepsilon,\ell} : \mathcal{X}\to[0,T]$ as

$$\tau^{\varepsilon,\ell}(v) := \inf\big\{t\in[0,T] \,:\, \varepsilon\int_{[0,t]} ds\, \langle\nabla v(s),\nabla v(s)\rangle \ge \ell\big\} \tag{3.4.2}$$

where we understand $\langle\nabla v(s),\nabla v(s)\rangle = +\infty$ if $v(s)\notin H^1(\mathbb{T})$. Note that by (3.2.3)

$$\begin{aligned}
\big[N^{\varepsilon;\eta\varphi}, N^{\varepsilon;\eta\varphi}\big]_t \le 2\varepsilon^{2\gamma}\int_{[0,t]} ds \Big[& \big\|\sigma(u^\varepsilon(s))\eta''(u^\varepsilon(s))\varphi(s)\nabla u^\varepsilon\big\|^2_{L_2(\mathbb{T})} \\
& + \big\|\sigma(u^\varepsilon(s))\eta'(u^\varepsilon(s))\nabla\varphi(s)\big\|^2_{L_2(\mathbb{T})}\Big]
\end{aligned}$$

so that, for some constant $C_1>0$ depending only on σ and η

$$\big[N^{\varepsilon;\eta\varphi}, N^{\varepsilon;\eta\varphi}\big]_{t\wedge\tau^{\varepsilon,\ell}} \le C_1\varepsilon^{2\gamma-1}\int_{[0,t]} ds\Big[\ell\big\|\varphi(s)\big\|^2_{L_\infty(\mathbb{T})} + \varepsilon\big\|\nabla\varphi(s)\big\|^2_{L_2(\mathbb{T})}\Big]$$

By the Sobolev embedding of $L_1(\mathbb{T})$ in $H^{-1}(\mathbb{T})$, we gather that, for each $\ell>0$, the law of the $H^{-1}(\mathbb{T})$-valued martingale $\{\bar N_t^{\varepsilon,\ell;\eta},\, t\in[0,T]\}$ defined as $\bar N_t^{\varepsilon,\ell;\eta}(\psi) := N^{\varepsilon;\eta\psi}_{t\wedge\tau^{\varepsilon,\ell}}$ for $\psi\in H^1(\mathbb{T})$, is exponentially tight in $H^{-1}([0,T]\times\mathbb{T})$ as $\varepsilon\to0$, see e.g. [**12**, Chapter 12].

By (3.4.1) we get for some constant C depending only on η, f, D and σ

$$\begin{aligned}
\big|\langle\langle \eta(u^\varepsilon),\partial_t\varphi\rangle\rangle + \langle\langle q(u^\varepsilon),\nabla\varphi\rangle\rangle\big| \le & \;\varepsilon C\|\varphi\|_{L_\infty([0,T]\times\mathbb{T})}\langle\langle\nabla u^\varepsilon,\nabla u^\varepsilon\rangle\rangle \\
& +\varepsilon C\langle\langle\nabla\varphi,\nabla\varphi\rangle\rangle^{1/2}\langle\langle\nabla u^\varepsilon,\nabla u^\varepsilon\rangle\rangle^{1/2} + |N_T^{\varepsilon;\eta\varphi}| \\
& +\varepsilon^{2\gamma}C\|\varphi\|_{L_\infty([0,T]\times\mathbb{T})}\big[\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle\langle\nabla u^\varepsilon,\nabla u^\varepsilon\rangle\rangle\big]
\end{aligned} \tag{3.4.3}$$

Therefore, for u^ε in the set $\{\tau^{\varepsilon,\ell}\ge T\}$ we have

$$\begin{aligned}
&\big|\langle\langle \eta(u^\varepsilon),\partial_t\varphi\rangle\rangle + \langle\langle q(u^\varepsilon),\nabla\varphi\rangle\rangle\big| \\
&\qquad \le C\big(\ell + \varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \varepsilon^{2\gamma}\ell\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\big)\|\varphi\|_{L_\infty([0,T]\times\mathbb{T})} \\
&\qquad\quad + C\sqrt{\varepsilon\ell}\|\nabla\varphi\|^2_{L_2([0,T]\times\mathbb{T})} + \big|\int_{[0,t]} d\bar N_s^{\varepsilon,\ell;\eta}(\varphi(s))\big|
\end{aligned}$$

Since the term in square brackets in the first line of the r.h.s. is bounded uniformly in ε, by the compactness result on $\{\bar{N}_t^{\varepsilon,\ell;\eta}\}$ and the the compact embedding of $L_1([0,T]\times\mathbb{T})$ in $H^{-1}([0,T]\times\mathbb{T})$, we get for each $\ell>0$ the existence of a sequence $\{K_{\ell,n}\}_n$ of compact subsets of $H^{-1}([0,T]\times\mathbb{T})$ such that

$$\lim_n \overline{\lim_\varepsilon}\, \varepsilon^{2\gamma-1}\mathbb{P}^\varepsilon\big(\eta(u)_t+q(u)_x\in K_{\ell,n}^c,\, \tau^{\varepsilon,\ell}(u)\geq T\big)=-\infty$$

Since, by Corollary 3.2.3

$$\lim_\ell \overline{\lim_\varepsilon}\, \varepsilon^{2\gamma-1}\log\mathbb{P}^\varepsilon(\tau^{\varepsilon,\ell}\geq T)=-\infty$$

we get that the law of $\eta(u^\varepsilon)_t+q(u^\varepsilon)_x$ is exponentially tight in $H^{-1}([0,T]\times\mathbb{T})$. The statement of the lemma then follows by Lemma 3.2.4 and Proposition 3.4.1. □

PROOF OF THEOREM 3.1.7: UPPER BOUND. Recall Proposition 1.2.3 proved in the introduction. Let $\underline{H}:\mathcal{M}\to[0,+\infty]$ be the (lower semicontinuous) optimal rate functional for the large deviations upper bound principle of $\{\mathbf{P}^\varepsilon\}$ with speed $\varepsilon^{-2\gamma+1}$. Such an optimal rate exists, as we characterized it as the Γ-liminf of the functional (1.2.8). Since $\{\mathbf{P}^\varepsilon\}$ satisfies a large deviations upper bound with speed $\varepsilon^{-2\gamma}$ and with a rate functional $\mathcal{I}:\mathcal{M}\to[0,+\infty]$ which is strictly positive off the set of measure-valued solutions to (3.1.4), $\underline{H}$ is infinite off the set of measure-valued solutions. On the other hand, since the topology generated by $d_{\mathcal{X}}$ on $\mathcal{X}$ coincides with topology induced on $\mathcal{X}$ by the immersion map $\mathcal{X}\ni u\mapsto\delta_u\in\mathcal{X}$, Lemma 3.4.2 implies that $\underline{H}$ is infinite off the set $\mathcal{M}_e=\{\mu\in\mathcal{M}\,:\,\mu=\delta_u \text{ for some } u\in\mathcal{X}\}$. Thus, since $\mathbf{P}^\varepsilon$ is the pushforward of $\mathbb{P}^\varepsilon$ w.r.t. this immersion map, and since a measure-valued solutions μ to (3.1.4) that are in $\mathcal{M}_e$ have necessarily the form $\mu=\delta_u$ for some weak solution $u\in\mathcal{X}$, the optimal rate functional for the upper bound of $\{\mathbb{P}^\varepsilon\}$ is infinite off the closed set $\mathcal{W}$ of weak solutions to (3.1.4). Therefore, in view of Lemma 3.4.2, we have to prove the large deviations upper bound inequality for $\{\mathbb{P}^\varepsilon\}$ with speed $\varepsilon^{-2\gamma+1}$ only on compact subsets K of $\mathcal{X}$, that are contained in $\mathcal{W}$.

Let (ϑ,Q) be an entropy sampler-entropy sampler pair. By Lemma 3.2.1 it is not difficult to see that the map $[0,T]\ni t\mapsto\int dx\vartheta(u(t,x),t,x)\in\mathbb{R}$ is continuous. With the same notation of Lemma 3.2.1, consider the exponential supermartingale obtained as the stochastic exponential of $\{N_t^{\varepsilon;\vartheta}\}$. By the bound (3.2.3) and the just stated continuity property of u, we have that for each $\varepsilon > 0$ the map

$$
\begin{aligned}
E^{\varepsilon;\vartheta} \ &: \ [0,T] \times \mathcal{X} \to \mathbb{R}^+ \\
E^{\varepsilon;\vartheta} \ &: \ (t,u) \to \exp\Big\{ \textstyle\int dx\, \vartheta(u(t),t,x) - \int dx\, \vartheta(u_0,0,x) \\
&\qquad - \textstyle\int_{[0,t]\times\mathbb{T}} ds\, dx\, \big[\big(\partial_s \vartheta\big)\big(u(s,x),s,x\big) + \big(\partial_x Q\big)\big(u(s,x),s,x\big) \big] \\
&\qquad + \textstyle\int_{[0,t]} ds\, \big[\frac{\varepsilon}{2} \langle \vartheta''(u)\nabla u, D(u)\nabla u \rangle + \frac{\varepsilon}{2} \langle \partial_x \theta'(u), D(u)\nabla u \rangle \\
&\qquad - \tfrac{\varepsilon^{2\gamma}}{2} \|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})} \langle \vartheta''(u) a(u), a(u) \rangle \\
&\qquad - \tfrac{\varepsilon^{2\gamma}}{2} \|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} \langle \vartheta''(u)\nabla u, [a'(u)]^2 \nabla u \rangle \big] \\
&\qquad - \tfrac{\varepsilon^{2\gamma}}{2} \textstyle\int_{[0,t]} ds\, \big\langle \sigma(u)\big[\vartheta''(u)\nabla u + \partial_x \vartheta'(u)\big], \vartheta''(u)\nabla u + \partial_x\vartheta'(u) \big\rangle \Big\}
\end{aligned}
$$

is a continuous strictly positive $\mathbb{P}^\varepsilon$-supermartingale, with $E^\vartheta_0 = 1$, $\mathbb{P}^\varepsilon$ almost surely. For $\ell > 0$ let $B^\ell := \{u \in \mathcal{X} \cap L_2\big([0,T]; H^1(\mathbb{T})\big) \ : \ \langle\langle \nabla u, \nabla u \rangle\rangle \le \ell\}$. Given a compact subset $K \subset \mathcal{X}$ we have, for C, ε_0 as in Corollary 3.2.3 and $\ell > C$, $\varepsilon \le \varepsilon_0$

$$
\begin{aligned}
\mathbb{P}^\varepsilon(K) \ &\le \ \mathbb{P}^\varepsilon\big(E^{\varepsilon;\frac{\vartheta}{\varepsilon^{2\gamma-1}}}(T,u) [E^{\varepsilon;\frac{\vartheta}{\varepsilon^{2\gamma-1}}}(T,u)]^{-1} 1\!\!1_{K\cap B^{\ell/\varepsilon}}(u) \big) + \mathbb{P}^\varepsilon(B^{\ell/\varepsilon}) \\
&\le \ \sup_{u \in K \cap B^{\ell/\varepsilon}} [E^{\varepsilon;\frac{\vartheta}{\varepsilon^{2\gamma-1}}}(T,v)]^{-1} + \exp\big(- \tfrac{(\ell - C)^2}{C\varepsilon^{2\gamma-1}(\ell+1)} \big)
\end{aligned}
\tag{3.4.4}
$$

where in the last line we used the supermartingale property of $E^{\varepsilon;\vartheta}$ and Corollary 3.2.3. On the other hand, by Cauchy-Schwartz inequality, for each $u \in B^{\ell/\varepsilon}$

$$
\begin{aligned}
\varepsilon^{2\gamma-1} \log E^{\varepsilon;\frac{\vartheta}{\varepsilon^{2\gamma-1}}}(T,u) &= - \textstyle\int dx\, \vartheta(u_0(x),0,x) \\
&\quad - \textstyle\int ds\, dx\, \big[\big(\partial_s\vartheta\big)\big(u(s,x),s,x\big) + \big(\partial_x Q\big)\big(u(s,x),s,x\big) \big] \\
&\quad + \tfrac{\varepsilon}{2} \langle\langle \vartheta''(u)\nabla u, D(u)\nabla u \rangle\rangle + \tfrac{\varepsilon}{2} \langle\langle \partial_x \theta'(u), D(u)\nabla u \rangle\rangle \\
&\quad - \tfrac{\varepsilon^{2\gamma}}{2} \|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})} \langle\langle \vartheta''(u)a(u), a(u) \rangle\rangle \\
&\quad - \tfrac{\varepsilon^{2\gamma}}{2} \|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} \langle\langle \vartheta''(u)\nabla u, [a'(u)]^2 \nabla u \rangle\rangle \\
&\quad - \tfrac{\varepsilon}{2} \langle\langle \sigma(u)\vartheta''(u)\nabla u, \vartheta''(u)\nabla u \rangle\rangle - \tfrac{\varepsilon}{2} \langle\langle \sigma(u)\partial_x\vartheta'(u), \partial_x\vartheta'(u) \rangle\rangle \\
&\quad - \varepsilon \langle\langle \sigma(u)\vartheta''(u)\nabla u, \partial_x\vartheta'(u) \rangle\rangle \\
&\ge - \textstyle\int dx\, \vartheta(u_0(x),0,x) \\
&\quad - \textstyle\int ds\, dx\, \big[\big(\partial_s\vartheta\big)\big(u(s,x),s,x\big) + \big(\partial_x Q\big)\big(u(s,x),s,x\big) \big] \\
&\quad + \tfrac{\varepsilon}{2} \langle\langle \vartheta''(u)\nabla u, \big(D(u) - \sigma(u)\vartheta''(u)\big)\nabla u \rangle\rangle - C_\vartheta \sqrt{\varepsilon\ell} \\
&\quad - C_\vartheta \varepsilon^{2\gamma} \|\nabla \jmath^\varepsilon\|^2_{L_2(\mathbb{T})} - C_\vartheta \varepsilon^{2\gamma-1} \ell \|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} - C_\vartheta\varepsilon - \sqrt{\varepsilon\ell}\, C_\vartheta
\end{aligned}
$$

for a suitable constant $C_\vartheta > 0$ depending only on ϑ, D and σ. The key point now is that, if the entropy sampler ϑ satisfies

$$
\sigma(u)\vartheta''(u,t,x) \le D(u) \qquad \forall\, u \in [0,1],\ t \in [0,T],\ x \in \mathbb{T} \tag{3.4.5}
$$

then the term $\langle\langle \vartheta''(u)\nabla u, \big(D(u) - \sigma(u)\vartheta''(u)\big)\nabla u \rangle\rangle$ is positive, namely the largest term related to the quadratic variation of $N^{\varepsilon;\vartheta}_t$ is controlled by the positive parabolic term related to the deterministic diffusion. Therefore, by the hypotheses assumed on $\jmath^\varepsilon$, for each entropy sampler ϑ satisfying (3.4.5),

$u \in B^{\ell/\varepsilon}$ and up to redefining the constant C_ϑ, there exists a sequence $\alpha_\varepsilon \to 0$ such that

$$\begin{aligned}\varepsilon^{2\gamma-1} \log E^{\varepsilon; \frac{\vartheta}{\varepsilon^{2\gamma-1}}}(T,u) \geq & - \int dx\, \vartheta(u_0(x),0,x) - C_\vartheta(1+\ell)\alpha_\varepsilon \\ & - \int ds\, dx\, \big[\big(\partial_s \vartheta\big)\big(u(s,x),s,x\big) + \big(\partial_x Q\big)\big(u(s,x),s,x\big)\big]\end{aligned}$$

By (3.4.4), taking the logarithm, multiplying by $\varepsilon^{2\gamma-1}$ optimizing on $\ell > 0$ and ϑ satisfying (3.4.5), and passing to the limit $\varepsilon \to 0$

$$\begin{aligned}\overline{\lim}_\varepsilon\, \varepsilon^{2\gamma-1} \mathbb{P}^\varepsilon(K) \leq - \sup_\vartheta \inf_{u \in K} \big\{ & - \int dx\, \vartheta(u_0(x),0,x) \\ & - \int ds\, dx\, \big[\big(\partial_s \vartheta\big)\big(u(s,x),s,x\big) + \big(\partial_x Q\big)\big(u(s,x),s,x\big)\big]\big\}\end{aligned}$$

where the supremum is taken on the entropy samplers ϑ satisfying (3.4.5). It is immediate to see that the map $\mathcal{X} \ni u \mapsto - \int dt\, dx\, \big[\big(\partial_t \vartheta\big)\big(u(t,x),t,x\big) + \big(\partial_x Q\big)\big(u(t,x),t,t\big)\big] \in \mathbb{R}$ is lower semicontinuous in $\mathcal{X}$ for each ϑ. Therefore the minimax lemma yields

$$\begin{aligned}\overline{\lim}_\varepsilon\, \varepsilon^{2\gamma-1} \mathbb{P}^\varepsilon(K) \leq - \inf_{u \in K} \sup_\vartheta \big\{ & - \int dx\, \vartheta(u_0(x),0,x) \\ & - \int ds\, dx\, \big[\big(\partial_s \vartheta\big)\big(u(s,x),s,x\big) + \big(\partial_x Q\big)\big(u(s,x),s,x\big)\big]\big\}\end{aligned} \tag{3.4.6}$$

As noted at the beginning of this proof, we need to show the upper bound only for compact sets K contained in the set of weak solutions to (3.1.4). On the other hand, for such a K, (3.4.6) reads

$$\overline{\lim_\varepsilon}\, \varepsilon^{2\gamma-1} \mathbb{P}^\varepsilon(K) \leq - \inf_{u \in K} \sup_\vartheta P_{\vartheta,u}$$

where, as usual, the supremum is taken over the entropy samplers ϑ satisfying (3.4.5). In the proof of Theorem 2.2.5, it was shown that a weak solution u to (3.1.4) such that $\sup_\vartheta P_{\vartheta,u} < +\infty$- is indeed an entropy-measure solution $u \in \mathcal{E}$, and $\sup_\vartheta P_{\vartheta,u} = H(u)$. □

Proof of Theorem 3.1.7: lower bound. We will use the entropy method suggested by Lemma 3.3.1, as we did in the proof of Theorem 3.1.3 item (ii). Given $u \in \mathcal{S}$, we need to show that there exists a sequence $\{\mathbb{Q}^\varepsilon\} \subset \mathcal{P}(\mathcal{X})$ such that $\overline{\lim}\, \varepsilon^{2\gamma-1} H(\mathbb{Q}^\varepsilon|\mathbb{P}^\varepsilon) \leq H(u)$ and $\mathbb{Q}^\varepsilon \to \delta_u$ weakly* in $\mathcal{P}(\mathcal{X})$. Still following the proof of Theorem 3.1.3 item (ii), we can construct such a sequence $\{\mathbb{Q}^{\varepsilon;u}\}$ using exponential martingales. By the calculation in (3.3.6), everything boils down to find a sequence of martingales $\{M^{\varepsilon;u}\}_\varepsilon$ such that

$$\overline{\lim_\varepsilon}\, \frac{\varepsilon^{2\gamma-1}}{2} \big[M^{\varepsilon;u}, M^{\varepsilon;u}\big]_T \leq \mathcal{H}(u) \tag{3.4.7}$$

and such that any sequence of martingale solutions $\{v^\varepsilon\}$ to the problem

$$\begin{aligned}& dv = \big[- \nabla \cdot f\,(v) + \tfrac{\varepsilon}{2} \nabla \cdot (D(v)) \nabla v) - \mathrm{Gir} \big]\, dt + \varepsilon^\gamma\, \nabla \cdot \big[a(v)(\jmath^\varepsilon * dW)\big] \\ & u^\varepsilon(0,x) = u_0(x)\end{aligned} \tag{3.4.8}$$

converges to u in $\mathcal{X}$. Here Gir stands for the Girsanov term, namely the cross quadratic variation of the martingale $M^{\varepsilon;u}$ with the martingale term $\varepsilon^\gamma \nabla\cdot\big[a(u^\varepsilon)(\jmath^\varepsilon * dW)\big]$.

With minor adaptations from Theorem 2.2.5, we have that the following statement holds. For each sequence $\alpha_\varepsilon \to 0$ and each each $u\in\mathcal{S}$, there exists a sequence a $\{w^\varepsilon\}\subset\mathcal{X}$ and a sequence $\{\Psi^\varepsilon\}\subset L_2\big([0,T];H^2(\mathbb{T})\big)$ such that:

(a) $w^\varepsilon\to u$ in $\mathcal{X}$, and $w^\varepsilon(0,x)=u_0(x)$.
(b) $\varepsilon\langle\langle\nabla w^\varepsilon,\nabla w^\varepsilon\rangle\rangle\le C$ for some $C>0$ independent of ε.
(c) $\overline{\lim}_\varepsilon\,\varepsilon^{-1}\langle\langle\sigma(w^\varepsilon)\nabla\Psi^\varepsilon,\nabla\Psi^\varepsilon\rangle\rangle = H(u)$.
(d) $\alpha_\varepsilon\langle\langle\nabla[a(w^\varepsilon)\nabla\Psi^\varepsilon],\nabla[a(w^\varepsilon)\nabla\Psi^\varepsilon]\rangle\rangle\le C\,\varepsilon^{-1}$, for some $C>0$ independent of ε.
(e) The equation
$$\partial_t w^\varepsilon+\nabla\cdot f(v)-\frac{\varepsilon}{2}\nabla\cdot\big(D(w^\varepsilon)\nabla w^\varepsilon\big)=-\nabla\cdot\big(\sigma(w^\varepsilon)\,\nabla\Psi^\varepsilon\big)$$
holds weakly.

We let $\alpha_\varepsilon:=\varepsilon^{-3/2}\|\jmath^\varepsilon-1\!\!1\|_{W^{-1,1}(\mathbb{T})}$, and let $\{w^\varepsilon\}$, $\{\Psi^\varepsilon\}$ be chosen correspondingly. Note that with this choice and by the assumption on $\|\jmath^\varepsilon-1\!\!1\|_{W^{-1,1}(\mathbb{T})}$, the quantity
$$\beta_\varepsilon:=\varepsilon^{-2}\int_0^t ds\,\|\jmath^\varepsilon*\jmath^\varepsilon*[a(w^\varepsilon)\nabla\Psi^\varepsilon]-a(w^\varepsilon)\nabla\Psi^\varepsilon\|^2_{L_2(\mathbb{T})} \tag{3.4.9}$$
converges to 0 as $\varepsilon\to 0$.

We define the martingale $M^{\varepsilon;u}$ as
$$M^{\varepsilon;u}_t:=\varepsilon^{-\gamma}\int_{[0,t]}\langle\jmath^\varepsilon*[a(w^\varepsilon)\nabla\Psi^\varepsilon],dW_s\rangle$$
Then by Young inequality for convolutions:
$$\frac{1}{2}\big[M^{\varepsilon;u},M^{\varepsilon;u}\big]_t\le\frac{\varepsilon^{-2\gamma}}{2}\langle\langle\sigma(w^\varepsilon)\Psi^\varepsilon,\Psi^\varepsilon\rangle\rangle$$
so that by property (c) we have that (3.4.7) is satisfied for this choice of $M^{\varepsilon;u}$.

Moreover for this choice of $M^{\varepsilon;u}$ the equation (3.4.8) reads
$$\begin{aligned} dv=&\big[-\nabla\cdot f(v)+\tfrac{\varepsilon}{2}\nabla\cdot(D(v)\nabla v)\big]\,dt\\ &+\big[\nabla\cdot a(v)(\jmath*\jmath*(a(w^\varepsilon)\nabla\Psi^\varepsilon)\big]\,dt+\varepsilon^\gamma\,\nabla\cdot\big[a(v)(\jmath^\varepsilon*dW)\big]\\ v^\varepsilon(0,x)=&\,u_0(x)\end{aligned} \tag{3.4.10}$$
For $\varepsilon>0$ let $v^\varepsilon\in\mathcal{X}\cap L_2([0,T];H^1(\mathbb{T})$ be the canonical process for a generic martingal solution to (3.4.10).

For $l\in C^2([-1,1])$, let us apply Itô formula for the map $\mathcal{X}\cap L_2([0,T];H^1(\mathbb{T})\in v\mapsto\int dt\,dx\,l(v-w^\varepsilon(t,x))$. Direct computation and the equation in (e) above yield, denoting $z^\varepsilon = v^\varepsilon - w^\varepsilon$

$$\begin{aligned}
&\int dx\, [l(z^\varepsilon(t,x)) - l(0)] \\
&= \int_{[0,t]} ds \Big\{ -\tfrac{\varepsilon}{2}\langle l''(z^\varepsilon)\nabla z^\varepsilon, D(v^\varepsilon)\nabla z^\varepsilon\rangle + \langle l''(z^\varepsilon)\nabla z^\varepsilon, f(v^\varepsilon) - f(w^\varepsilon)\rangle \\
&\qquad - \tfrac{\varepsilon}{2}\int_{[0,t]} ds \langle l''(z^\varepsilon)\nabla z^\varepsilon, [D(v^\varepsilon) - D(w^\varepsilon)]\nabla w^\varepsilon\rangle \\
&\qquad + \langle l''(z^\varepsilon)\nabla z^\varepsilon, [a(v^\varepsilon) - a(w^\varepsilon)]\big(\jmath^\varepsilon * \jmath^\varepsilon * [a(w^\varepsilon)\nabla\Psi^\varepsilon]\big)\rangle \\
&\qquad - \langle l''(z^\varepsilon)\nabla z^\varepsilon, a(w^\varepsilon)\big(\jmath^\varepsilon * \jmath^\varepsilon * [a(w^\varepsilon)\nabla\Psi^\varepsilon] - a(w^\varepsilon)a(w^\varepsilon)\nabla\Psi^\varepsilon]\big)\rangle \\
&\qquad + \tfrac{\varepsilon^{2\gamma}}{2}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle l''(z^\varepsilon)a(v^\varepsilon), a(v^\varepsilon)\rangle \\
&\qquad + \tfrac{\varepsilon^{2\gamma}}{2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\langle l''(z^\varepsilon)\nabla v^\varepsilon, [a'(v^\varepsilon)]^2\nabla v^\varepsilon\rangle\Big\} + N_t^{\varepsilon;l}
\end{aligned} \tag{3.4.11}$$

for each $t \in [0,T]$. Here $N^{\varepsilon;l}$ is a square-integrable martingale with vanishing $\mathbb{Q}^{\varepsilon;u}$ mean. Recall the definition $B^\ell := \{u \in \mathcal{X} \cap L_2\big([0,T];H^1(\mathbb{T})\big) : \langle\langle \nabla u, \nabla u\rangle\rangle \le \ell\}$. With the same notation of (3.3.10), we gather by Cauchy-Schwartz inequality, for each $\ell > 0$ and $v^\varepsilon \in B^{\ell/\varepsilon}$

$$\begin{aligned}
&\int dx\, [l(z^\varepsilon(t,x)) - l(0)] \le -\tfrac{\varepsilon}{2}(R^{\varepsilon;\ell}(t))^2 + C_1 A^{3,l} R^{\varepsilon,l}(t) \\
&\quad + \sqrt{\varepsilon} C_1 \big[\varepsilon \textstyle\int_0^t ds\, \langle \nabla w^\varepsilon, \nabla w^\varepsilon\rangle\big]^{1/2} A^{3,l} R^{\varepsilon,l}(t) \\
&\quad + C_1 \big[\textstyle\int_0^t ds\, \|\jmath^\varepsilon * \jmath^\varepsilon * [a(w^\varepsilon)\nabla\Psi^\varepsilon]\|^2_{L_2(\mathbb{T})}\big]^{1/2} A^{3,l} R^{\varepsilon,l}(t) \\
&\quad + C_1 \big[\textstyle\int_0^t ds\, \|\jmath^\varepsilon * \jmath^\varepsilon * [a(w^\varepsilon)\nabla\Psi^\varepsilon] - a(w^\varepsilon)\nabla\Psi^\varepsilon\|^2_{L_2(\mathbb{T})}\big]^{1/2} \sqrt{A^{2,l}} R^{\varepsilon,l}(t) \\
&\quad + C_1 \varepsilon^{2\gamma}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} A^{2,l} + C_1 \varepsilon^{2\gamma-2}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} A^{2,l}\ell + N_t^{\varepsilon;l}
\end{aligned} \tag{3.4.12}$$

for some constant $C_1 > 0$. The terms in square brackets in the second and third lines of (3.4.12) are bounded uniformly in ε by properties (b) and (c) respectively. Therefore, recalling (3.4.9), maximizing the r.h.s. of (3.4.12) as $R^{\varepsilon;\ell}(t)$ runs on $\mathbb{R}$, and assuming l such that $l(0) = 0$, we get for $v^\varepsilon \in B^{\ell/\varepsilon}$ and for some $C_2 > 0$

$$\begin{aligned}
&\int dx\, l(z^\varepsilon(t,x)) \le \varepsilon^{-1} C_2 (A^{3,l})^2 + \varepsilon C_2 A^{2,l}\beta_\varepsilon \\
&\quad + C_2\big[\varepsilon^{2\gamma-1}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \varepsilon^{2\gamma-3}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\ell\big]\varepsilon A^{2,l} + N_t^{\varepsilon;l}
\end{aligned}$$

Integrating in dt, taking the expectation w.r.t. $\mathbb{Q}^{\varepsilon;u}$

$$\begin{aligned}
&\mathbb{Q}^{\varepsilon;u}\Big(1\!\!1_{B^{\varepsilon^{-1}\ell}}(v) \int dx\, l(v - u^\varepsilon(t,x))\Big) \le \varepsilon^{-1} C_3 (A^{3,l})^2 + \varepsilon C_3 A^{2,l}\beta_\varepsilon \\
&\quad + C_3\big[\varepsilon^{2\gamma-1}\|\nabla\jmath^\varepsilon\|^2_{L_2(\mathbb{T})} + \varepsilon^{2\gamma-3}\|\jmath^\varepsilon\|^2_{L_2(\mathbb{T})}\ell\big]\varepsilon A^{2,l}
\end{aligned} \tag{3.4.13}$$

for some $C_3 > 0$. The term in square brackets in the last line of (3.4.13) vanishes by assumption. On the other hand, it is easy to see that there exists a sequence $\{l^\varepsilon\}$ such that $l^\varepsilon(\cdot) \to |\cdot|$ uniformly on $[-1,1]$, and the r.h.s. of (3.4.13) vanishes as $\varepsilon \to 0$. By the entropy bound (3.4.7) on $\varepsilon^{2\gamma-1}\mathrm{Ent}(\mathbb{Q}^{\varepsilon;u}|\mathbb{P}^{\varepsilon;u})$, by

(1.2.7) and Corollary 3.2.3 we have

$$\lim_{\ell\to+\infty}\underline{\lim}_{\varepsilon}\mathbb{Q}^{\varepsilon;u}(B^{\varepsilon^{-1}\ell})=1$$

so that we can conclude by (3.4.13) and property (a).

□

3.5. Appendix A: Existence and uniqueness results for fully nonlinear parabolic SPDEs with conservative noise

In this appendix, we are concerned with existence and uniqueness results for the Cauchy problem in the unknown $u\equiv u(t,x)$, $t\in[0,T]$, $x\in\mathbb{T}$

$$\begin{array}{lcl} du &=& \big[-\nabla\cdot f(u)+\frac{1}{2}\nabla\cdot\big(D(u)\nabla u\big)\big]\,dt+\nabla\cdot\big[a(u)(\jmath * dW)\big]\\ u(0,x) &=& u_0(x)\end{array}\tag{3.5.1}$$

Although we assume the space-variable x to run on a one-dimensional torus $\mathbb{T}$, it is not difficult to extend the results given below to the case $x\in\mathbb{T}^d$ or $x\in\mathbb{R}^d$ for $d\geq 1$. The assumptions on the quantities involved in (3.5.1) are given below.

We assume that a standard filtered probability space $\big(\Omega,\mathfrak{F},\mathfrak{F}_t,\mathbb{P}\big)$ is given, and that W is a cylindrical Brownian motion on this space. Hereafter we set

$$Q(v):=\big[(\partial_u a)(v)\big]^2\int dx\,|\nabla\jmath(x)|^2$$

We will assume the following hypotheses:

A1) f and D are uniformly Lipschitz on $\mathbb{R}$.
A2) $a\in C^2(\mathbb{R})$ is uniformly bounded.
A3) $\jmath\in H^1(\mathbb{T})$ and, with no loss of generality, $\int dx\,|\jmath(x)|=1$.
A4) D is uniformly positive, and there exists $c>0$ such that $D\geq Q+c$.
A5) u_0 is $\mathfrak{F}_0$ measurable and satisfies $\mathbb{E}\langle u_0,u_0\rangle<+\infty$.

We introduce the Polish space $Y:=C\big([0,T];H^{-2}(\mathbb{T})\big)\cap L_2\big([0,T];H^1(\mathbb{T})\big)\cap L_\infty\big([0,T];L_2(\mathbb{T})\big)$. A probability measure $\bar{\mathbb{P}}$ on Y is a *martingale solution* to (3.5.1) iff the law of $u(0)$ under $\bar{\mathbb{P}}$ in $L_2(\mathbb{T})$ is the same of the law of u_0, and for each $\varphi\in C^\infty\big([0,T]\times\mathbb{T}\big)$

$$L^u_t(\varphi):=\langle u(t),\varphi(t)\rangle-\langle u(0),\varphi(0)\rangle-\int_{[0,t]}ds\,\langle u,\partial_t\varphi\rangle-\langle f(u)-\frac{1}{2}D(u)\nabla u,\nabla\varphi\rangle\tag{3.5.2}$$

is a continuous square-integrable martingale (w.r.t. $\bar{\mathbb{P}}(du)$) with quadratic variation

$$\big[L^u_\cdot(\varphi),L^u_\cdot(\varphi)\big]_t=\int_{[0,t]}ds\,\langle\jmath*(a(u)\nabla\varphi),\jmath*(a(u)\nabla\varphi)\rangle\tag{3.5.3}$$

We say that an $\{\mathfrak{F}_t\}$-adapted process $u : \Omega \to Y$ is a *strong solution* to (3.5.1) iff $u(0) = u_0$ $\mathbb{P}$-a.s. and for each $\varphi \in C^\infty\big([0,T] \times \mathbb{T}\big)$

$$L_t^u(\varphi) = \int_{[0,t]} \langle \jmath * \big(a(u)\nabla\varphi\big), dW\rangle \tag{3.5.4}$$

In this appendix we prove

THEOREM 3.5.1. *Assume **A1)**–**A5)**. Then there exists a unique strong solution u to* (3.5.1)*. Furthermore, if u_0 takes values in* $[0,1]$ *and a is supported by* $[0,1]$*, then u takes values in* $[0,1]$ *a.s..*

By compactness estimates we will prove that there exists a solution to the martingale problem related to (3.5.1). Then we will prove that there exists a most one strong solution u to (3.5.1) using a stability result similar to the one used in the proof of Theorem 3.1.7. By Yamada-Watanabe theorem we get the existence and uniqueness stated in Theorem 3.5.1.

LEMMA 3.5.2. *Let $0 \le t' < t'' \le T$, let u', v be two $\mathfrak{F}_{t'}$ measurable random functions on $L_2(\mathbb{T})$ such that $\mathbb{E}\||u'| + |v| + |\nabla v|\|^2_{L_2(\mathbb{T})} < +\infty$. Then the stochastic Cauchy problem*

$$\begin{aligned} dw &= \big[-\nabla\cdot f(v) + \tfrac{1}{2}\nabla\cdot\big(D(v)\nabla w\big)\big]\,dt + \nabla\cdot\big[a(v)(\jmath * dW)\big] \\ w(t',x) &= u'(x) \end{aligned} \tag{3.5.5}$$

admits a unique strong solution u with values in $L_2\big([t',t''];H^1(\mathbb{T})\big)\cap C\big([t',t''],H^{-1}(\mathbb{T})\big)$. Such a solution u satisfies

$$\begin{aligned} \langle u(t),u(t)\rangle + \textstyle\int_{[t',t]} ds\langle D(v)\nabla u,\nabla u\rangle &= N(t,t') + \langle u',u'\rangle \\ &\quad + \textstyle\int_{[t',t]} ds\,\big[\langle Q(v)\nabla v,\nabla v\rangle + \int dx\,S(v)\big] \end{aligned} \tag{3.5.6}$$

where $N(t,t') := 2\int_{[t',t]}\langle \jmath*\big(a(v)\nabla u\big),dW\rangle$. Furthermore $\mathbb{E}\sup_{t\in[t',t'']}\|u(t)\|^2_{L_2(\mathbb{T})} < +\infty$.

PROOF. Existence and uniqueness follows by explicit representation, see e.g. [**12**]. Applying Itô formula to the map $w \mapsto \langle w,w\rangle$ acting o on $L_2(\mathbb{T})$ we get (3.5.6). Note that by Doob inequality, for a suitable constant $C > 0$

$$\begin{aligned} \mathbb{E}\sup_{t\in[t',t'']}|N(t,t')| &\le 2\mathbb{E}\big[N(\cdot,t'),N(\cdot,t')\big]^{1/2}_{t''} \\ &= 4\Big[\textstyle\int_{[t',t'']} ds\,\langle \jmath*\big(a(v)\nabla u\big),\jmath*\big(a(v)\nabla u\big)\rangle\Big]^{1/2} \\ &\le 4\Big[\textstyle\int_{[t',t'']} ds\,\langle a(v)\nabla u,a(v)\nabla u\rangle\Big]^{1/2} \\ &\le C\Big[\textstyle\int_{[t',t'']} ds\,\langle D(v)\nabla u,\nabla u\rangle\Big]^{1/2} \end{aligned}$$

so that the bound on $\mathbb{E}\sup_{t\in[t',t'']}\|u(t)\|^2_{L_2(\mathbb{T})}$ is easily obtained by (3.5.6). ☐

We next introduce a sequence $\{u^n\}$ of adapted Y-valued processes. We will gather existence of a weak solution to (3.5.1) by tightness of the laws $\{\mathbb{P}^n\}$ of this sequence.

For $n \in \mathbb{N}$ and $i = 0, \ldots, 2^n$ let $t_i^n := i 2^{-n} T$, and let $\{\imath^n\}$ be a sequence of smooth mollifiers on $\mathbb{T}$ such that $\lim_n 2^{-n} \|\imath\|^2_{L_1(\mathbb{T})} = 0$. We define a process u^n on Y and the auxiliary random functions $\{v_i^n\}_{i=0}^{2^n}$ on $\mathbb{T}$ as follows. For $i = 0$ we set

$$\begin{aligned} u(0) &= u_0 \\ v_0^n &:= \imath^n * u_0 \end{aligned}$$

and for $i = 1, \ldots, 2^n - 1$ and $t \in [t_i^n, t_{i+1}^n]$, we let $u^n(t)$ be the solution to the problem (3.5.5) with $u' = u(t_i^n)$ and $v = v_i^n$, and we set

$$v_i^n := \frac{2^n}{T} \int_{[t_{i-1}^n, t_i^n]} ds\, u^n(s)$$

By Lemma 3.5.2, these definitions are recursively well-posed, and indeed u^n takes values in Y. We also define a sequence of $D\big([0,T]; L_2(\mathbb{T})\big)$ cadlag processes $\{v^n\}$ by requiring $v^n(t) = v_i^n$ for $t \in [t_i^n, t_{i+1}^n)$.

LEMMA 3.5.3. *There exists a constant $C > 0$ independent of n such that*

$$\mathbb{E} \sup_{t \in [0,T]} \langle u^n(t), u^n(t) \rangle + \mathbb{E} \langle\langle \nabla u^n, \nabla u^n \rangle\rangle \le C \tag{3.5.7}$$

and for each $\varphi \in H^1(\mathbb{T})$ such that $\langle \nabla\varphi, \nabla\varphi \rangle \le 1$, for each $\delta > 0$ and $r \in (0,1)$

$$\mathbb{P}\Big(\sup_{s,t \in [0,T]\,:|s-t| \le \delta} \big| \langle u^n(t) - u^n(s), \varphi \rangle \big| > r \Big) \le C\, \delta\, r^{-2} \tag{3.5.8}$$

Furthermore for each $\delta > 0$

$$\lim_{n \to \infty} \mathbb{P}\big(\langle\langle u^n - v^n, u^n - v^n \rangle\rangle > \delta \big) = 0 \tag{3.5.9}$$

PROOF. Writing Itô formula (3.5.6) for u^n in the intervals $[t_i^n, t_{i+1}^n]$ and summing over i, we get for each $t \in [0,T]$

$$\begin{aligned} \langle u^n(t), u^n(t) \rangle + \textstyle\int_{[0,t]} ds\, \langle D(v^n) \nabla u^n, \nabla u^n \rangle &= N^n(t) + \langle u_0, u_0 \rangle \\ &+ \textstyle\int_{[0,t]} ds \big[\langle Q(v^n) \nabla v^n, \nabla v^n \rangle + \int dx\, S(v^n) \big] \end{aligned}$$

where, by the same means of Lemma 3.5.2, the martingale $N^n(t) := 2 \int_{[0,t]} \langle \jmath * \big(a(v^n) \nabla u^n\big), dW \rangle$ enjoys the bound $\mathbb{E} \sup_{s \in [0,T]} |N^n(t)| \le C_1 \langle\langle D(v^n) \nabla u^n, \nabla u^n \rangle\rangle^{1/2}$ for some $C_1 > 0$ depending on D and a. Note that, by the definition of the v_i^n, hypotheses **A5)** and Young inequality for convolutions

$$\begin{aligned} \textstyle\int_{[0,t]} ds\, \langle Q(v^n) \nabla v^n \nabla v^n \rangle &\le \textstyle\int_{[0,t]} ds\, \langle Q(v^n) \nabla u^n, \nabla u^n \rangle + C_2 \int_{[0,t_1^n]} ds\, \langle \imath^n * u_0, \imath^n * u_0 \rangle \\ &\le \textstyle\int_{[0,t]} ds\, \langle (D(v^n) - c) \nabla u^n, \nabla u^n \rangle + 2^{-n} T\, C_2\, \|\imath^n\|^2_{L_1(\mathbb{T})} \langle u_0, \langle u_0 \rangle \end{aligned}$$

for some constant C_2 depending only on a. Patching all together

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\langle u^n(t),u^n(t)\rangle + c\langle\langle D(v^n)\nabla u^n,\nabla u^n\rangle\rangle \\ \le +\big(1+2^{-n}T\,C_2\|\imath^n\|^2_{L_1(\mathbb{T})}\big)\mathbb{E}\langle u_0,u_0\rangle \\ + C_1\langle\langle D(v^n)\nabla u^n,\nabla u^n\rangle\rangle^{1/2} + \int_{[0,t]\times\mathbb{T}} ds\,dx\,S(v^n)\end{aligned}$$

Since $S(\cdot)$ is bounded by a constant depending only on $\jmath$ and a, it is not difficult to gather (3.5.7).

Since u satisfies (3.5.5) in each interval $[t_i^n,t_{i+1}^n]$

$$\begin{aligned}\big|\langle u^n(t)-u^n(s),\varphi\rangle\big| \;&\le\; C_3\big(1+\langle\langle\nabla u^n,\nabla u^n\rangle\rangle^{1/2}\big)|t-s|^{1/2}\langle\nabla\varphi,\nabla\varphi\rangle^{1/2} \\ &\quad+\big|\int_{[s,t]}\langle \jmath*(a(v)\nabla\varphi),dW\rangle\big|\end{aligned}$$

for a suitable constant C_3 depending only on f and D. (3.5.8) then follows from the first part of the lemma.

In order to prove (3.5.9), by (3.5.7) it is enough to show that for each $\ell>0$

$$\lim_{n\to\infty}\mathbb{P}\big(\langle\langle u^n-v^n,u^n-v^n\rangle\rangle>\delta,\langle\langle\nabla u^n,\nabla u^n\rangle\rangle\le\ell\big)=0$$

Let $\kappa\in C^\infty(\mathbb{T})$ be such that $\int dx\,\kappa(x)=1$, and note

$$\begin{aligned}\|\kappa-\mathrm{id}\|_{-1,1}:=\sup\Big\{\int dx\big|\int dy\,\kappa^{\jmath}(x-y)\varphi(y)-\varphi(x)\big|, \\ \varphi\in C^\infty(\mathbb{T}),\ \sup_x|\nabla\varphi(x)|\le1\Big\}<+\infty\end{aligned}$$

It is easily seen that exists κ such that $\|\kappa-\mathrm{id}\|_{-1,1}\le\frac{\zeta}{8\ell}$, as this quantity vanishes as we let κ converge weakly to the Dirac mass centered at 0. Then

$$\begin{aligned}\|u^n-v^n\|_{L_2([t_1^n,T]\times\mathbb{T})} \;&\le\; +\|u^n-\kappa*u^n\|_{L_2([t_1^n,T]\times\mathbb{T})} \\ &\quad+\|v^n-\kappa*v^n\|_{L_2([t_1^n,T]\times\mathbb{T})}+\|\kappa*u^n-\kappa*v^n\|_{L_2([t_1^n,T]\times\mathbb{T})} \\ &\le\; \|\kappa-\mathrm{id}\|_{-1,1}\int_{[t_1^n,T]}dt\,\big[\langle\nabla u^n,\nabla u^n\rangle+\langle\nabla v^n,\nabla v^n\rangle\big] \\ &\quad+\int_{[t_1^n,T]}dt\,\langle\kappa*(u^n-v^n),\kappa*(u^n-v^n)\rangle\end{aligned}$$

By the definition of v^n, $\int_{[t_1^n,T]}ds\,\nabla v^n,\nabla v^n\rangle\le\int_{[t_1^n,T]}ds\,\langle\nabla u^n,\nabla u^n\rangle$. Moreover

$$\begin{aligned}&\int_{[t_1^n,T]}dt\,\langle\kappa*(u^n-v^n),\kappa*(u^n-v^n)\rangle \\ &\qquad=\frac{2^n}{T}\sum_{i=1}^{2^n-1}\int_{[t_i^n,t_{i+1}^n]}dt\,\int_{[t_{i-1}^n,t_i^n]}ds\,\langle\kappa*(u^n(t)-u^n(s)),\kappa*(u^n(t)-u^n(s))\rangle \\ &\qquad\le\sup_{|t-s|\le2^{-n+1}T}\langle\kappa*(u^n(t)-u^n(s)),\kappa*(u^n(t)-u^n(s))\rangle\end{aligned}$$

Therefore

$$\begin{aligned}\|u^n-v^n\|_{L_2([t_1^n,T]\times\mathbb{T})}\le\frac{\zeta}{4\ell}\int_{[t_1^n,T]}dt\,\langle\nabla u^n,\nabla u^n\rangle+ \\ +\sup_{|t-s|\le2^{-n+1}T}\|\kappa*(u^n(t)-u^n(s))\|^2_{L_2(\mathbb{T})}\end{aligned}\tag{3.5.10}$$

so that

$$\begin{aligned}&\overline{\lim}_{n\to\infty}\mathbb{P}\big(\langle\langle u^n-v^n,u^n-v^n\rangle\rangle>\zeta,\langle\langle\nabla u^n,\nabla u^n\rangle\rangle\le\ell\big) \\ &\qquad\le\overline{\lim}_{n\to\infty}\mathbb{P}\big(2\,\|u^n-v^n\|_{L_2([0,t_1^n]\times\mathbb{T})}>\tfrac{\zeta}{4}\big) \\ &\qquad\quad+\mathbb{P}\big(2\|u^n-v^n\|_{L_2([t_1^n,T]\times\mathbb{T})}>\tfrac{3\zeta}{4},\langle\langle\nabla u^n,\nabla u^n\rangle\rangle\le\ell\big)\end{aligned}$$

The first term in the r.h.s. of this formula vanishes by the bound on the $L_\infty\big([0,T];L_2(\mathbb{T})\big)$ in (3.5.7). By (3.5.10), the second term in the r.h.s. is bounded by $\mathbb{P}\big(\sup_{|t-s|\le 2^{-n+1}T}\|\kappa*(u^n(t)-u^n(s))\|^2_{L_2(\mathbb{T})}\rangle \ge \zeta/4\big)$, which also vanishes by (3.5.8). □

We define $\mathbb{P}^n$ to be the law of u^n, considered as a stochastic process on $C\big([0,T],H^{-2}(\mathbb{T})\big)\supset Y$.

COROLLARY 3.5.4. *$\{\mathbb{P}^n\}$ is tight, and thus compact, on $C\big([0,T],H^{-2}(\mathbb{T})\big)$ equipped with the uniform topology. Furthermore each limit point $\bar{\mathbb{P}}$ of $\{\mathbb{P}^n\}$ is concentrated on Y and satisfies*

$$\bar{\mathbb{E}}\sup_t\langle u(t),u(t)\rangle+\bar{\mathbb{E}}\langle\langle\nabla u,\nabla u\rangle\rangle<+\infty \tag{3.5.11}$$

PROOF. The estimate (3.5.8) implies that for each $\varphi\in H^1(\mathbb{T})$ the laws of the processes $t\mapsto\langle u^n(t),\varphi\rangle$ are tight in $C\big([0,T];\mathbb{R}\big)$ as n runs on $\mathbb{N}$, see [**6**, pag. 83]. A standard application of Mitoma's theorem (see [**21**, Cap. 6, Corollary 6.16]) implies that $\{\mathbb{P}^n\}$ is tight on $C\big([0,T],H^{-2}(\mathbb{T})\big)$. (3.5.11) follows by (3.5.7). □

The following statement is derived following closely the proof of Proposition 2.3.5.

PROPOSITION 3.5.5. *Let $K\subset\mathcal{U}$ be a compact w.r.t. $d_{\mathcal{U}}$. Suppose that each $u\in K$ has a weak x-derivative $\nabla u\in L_2([0,T]\times\mathbb{T})$, and suppose that exists $\zeta>0$ such that $\langle\langle\nabla u,\nabla u\rangle\rangle\le\zeta$. Then K is strongly compact in $\mathcal{X}$.*

PROPOSITION 3.5.6. *Each limit point $\bar{\mathbb{P}}$ of $\{\mathbb{P}^n\}$ is concentrated on Y and is a weak solution to* (3.5.1).

PROOF. Let $\bar{\mathbb{P}}$ be a limit point of $\{\mathbb{P}^n\}$ along a subsequence n_k. It is easily seen that the law of $u(0)$ under $\bar{\mathbb{P}}$ coincides with the law of u_0.

For $u\in Y$, $v\in D\big([0,T);L_2(\mathbb{T})\big)$ and $\varphi\in C^\infty\big([0,T]\times\mathbb{T}\big)$ let

$$L^{u,v}_t(\varphi):=\langle u(t),\varphi(t)\rangle-\langle u(0),\varphi(0)\rangle-\int_{[0,t]}ds\,\langle u,\partial_t\varphi\rangle-\langle f(v)-\frac{1}{2}D(v)\nabla u,\nabla\varphi\rangle$$

By (3.5.9), (3.5.7), and Proposition 3.5.5, the law of $L^{u^n,v^n}_T(\varphi)$ under $\mathbb{P}^n$ converges, along the subsequence n_k, to the law under $\bar{\mathbb{P}}$ of $L^{u,u}_T(\varphi)=L^u_\cdot(\varphi)$.

On the other hand, for each n and φ, $L^{u^n,v^n}_\cdot(\varphi)$ is a martingale w.r.t. $\mathbb{P}^n$, with quadratic variation $\big[L^{u,v}_\cdot(\varphi),L^{u,v}_\cdot(\varphi)\big]_t=\int_{[0,t]}ds\,\langle\jmath*(a(v^n)\nabla\varphi),\jmath*(a(v^n)\nabla\psi)\rangle$. Still by (3.5.9), (3.5.7), and compactness in $L_2\big([0,T]\times\mathbb{T}\big)$, we have that $L^u_\cdot(\varphi)$ is a martingale under $\bar{\mathbb{P}}$, with quadratic variation given by (3.5.3). □

PROPOSITION 3.5.7. *There exists at most one strong solution to* (3.5.1).

Proof. Let u, v be to strong solutions to equation (3.5.1). By Ito formula, for $l \in C^2(\mathbb{R})$ with bounded derivatives

$$\begin{aligned} &\int dx\, l(u-v)(t) - l(0) + \tfrac{1}{2}\int_{[0,t]} ds\, \langle D(u) l''(u-v)\nabla(u-v), \nabla(u-v)\rangle \\ &\quad = X(t) + \int_{[0,t]} ds \langle l''(u-v)\nabla(u-v), f(u)-f(v)\rangle \\ &\quad - \tfrac{1}{2}\int_{[0,t]} ds\, \langle l''(u-v)\nabla(u-v), [D(u)-D(v)]\nabla v\rangle \\ &\quad + \tfrac{1}{2}\int_{[0,t]} ds\, \langle l''(u-v), \|\nabla \jmath\|^2_{L^2(\mathbb{T})}\big(a(u)-a(v)\big)^2 \\ &\quad + \|\jmath\|^2_{L^2(\mathbb{T})}\big((\partial_u a)(u)\nabla u - (\partial_u a)(v)\nabla v\big)^2\rangle \end{aligned}$$

where, as usual, the quadratic variation of the martingale $X(t)$ is bounded by $\int_{[0,t]} \|l''(u-v)\nabla(u-v)\,(a(u)-a(v))\|^2_{L^2(\mathbb{T})}$. Introducing

$$R := \Big[\mathbb{E}\int_{[0,t]} ds\, \langle l''(u-v)\nabla(u-v), \nabla(u-v)\rangle\Big]^{1/2}$$

and using Holder inequality, assumptions **A2)** and **A5)** and the bound (3.5.11), we get for a suitable constant $C > 0$

$$\begin{aligned} &\mathbb{E}\sup_{t\le T}\int dx\, l(u-v)(t) + cR^2 \\ &\quad \le l(0) + C\big[\mathbb{E}\|l''(u-v)|u-v|^2\|_{L^\infty([0,T]\times\mathbb{T})}\big]^{1/2} R \\ &\quad + C\mathbb{E}\int_{[0,t]} ds\, \langle l''(u-v)|u-v|, |u-v|\rangle \end{aligned}$$

For any $\delta > 0$, we can choose l so that $|z| \le l(z) \le |z| + \delta$, $l(z) = |z|$ for $|z| \ge \delta$, and $|l''(z)| \le 3\delta^{-1}$. Therefore

$$\begin{aligned} \mathbb{E}\sup_t \|u-v\|_{L^1(\mathbb{T})} &\le \mathbb{E}\sup_t \int dx\, l(u-v)(t) \le \delta - cR^2 + C\sqrt{\delta}R + C\delta \\ &\le \big(\tfrac{C^2}{4c} + C + 1\big)\delta \end{aligned}$$

Since this holds for any $\delta > 0$, $u = v$. □

Proof of Theorem 3.5.1. Existence and uniqueness of a strong solution to (3.5.1) is a g consequence of Proposition 3.5.6, Proposition 3.5.7 and Yamada-Watanabe theorem [**16**, Cap. 5, Corollary 3.23]. The fact that u takes values supported by $[0,1]$ is provided in the same fashion of Corollary 3.2.3. Let $\{l^n\}$ be a sequence of infinitely differentiable convex functions on $\mathbb{R}$ with bounded derivatives. We can choose $\{l_n\}$ such that for $v \in [0,1]$ $(\big(\partial_{u,u} l_n\big)(v) \le D(v)\, a^{-2}(v)$ and $l_n(v) \le C_n(1+v^2)$ (for some $C_n > 0$), while $l_n(v) \uparrow +\infty$ for $n \to +\infty$ pointwise for $v \notin [0,1]$. By Itô formula (3.2.1)

$$\begin{aligned} &\int dx\, \big[l_n(u(t)) - l_n(u_0)\big] + \tfrac{1}{2}\int_{[0,t]} ds\, \big\langle \big(\partial_{u,u} l_n\big)(u) D(u)\,\nabla u, \big(\partial_{u,u} l_n\big)(u)\,\nabla u\big\rangle \\ &= \tfrac{1}{2}\int_{[0,t]} ds\, \big\langle \big(\partial_{u,u} l_n\big)(u)\nabla u, Q(u)\,\nabla u\big\rangle + \int_{[0,t]} ds \int dx\, \big(\partial_{u,u} l_n\big)(u)\, S(u) + N_n(t) \end{aligned}$$

where $N_n(t)$ is a martingale, and by Young inequality its quadratic variation bounded by $\big[N_n(\cdot), N_n(\cdot)\big]_t \le \int_{[0,t]} \langle a(u)\big(\partial_{u,u} l_n\big)(u)\, a(u)\,\nabla u, \big(\partial_{u,u} l_n\big)(u)\,\nabla u\rangle$.

Following closely the proof of Corollary 3.2.3, we gather for some constant C independent of n

$$\mathbb{E} \sup_{t \leq T} \int dx \, l_n(u(t)) \leq \mathbb{E} \int dx \, l_n(u_0) + C$$

As we let $n \to \infty$, the l.h.s. stays bounded, and since $l_n \to +\infty$ pointwise off $[0,1]$, necessarily $u(t,x) \in [0,1]$ $dt\,dx\,d\mathbb{P}$-a.s.. □

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