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From microscopic dynamics to macroscopic equations: scaling limits for the Lorentz gas

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Contents

Chapter 1

Introduction

The problem of deriving macroscopic evolution equations from the microscopic laws of motion governed by Newton's laws of classical mechanics is one of the most important keystones in mathematical physics. Here we consider a simple microscopic model, namely a gas of non-interacting particles in a fixed random configuration of scatterers. This dynamical system is usually referred to as the Lorentz gas since it was proposed by H. A. Lorentz in 1905, see [L], to explain the motion of electrons in metals applying the methods of the kinetic theory of gases. More precisely, at the turn of the twentieth century, Paul Drude developed a qualitative theory for electrical conduction in metals. With the purpose of establishing a more solid basis for the Drude theory, Hendrik Lorentz suggested an idealized model for this electron transport converting a many-body problem into an effective single-particle system that consists of a "test" particle and a static background.

Even though this model is quite simple, it is still paradigmatic. It represents a rare source of exact results in kinetic theory, providing a concrete example where microscopic reversibility can be reconciled with macroscopic irreversibility. Indeed complexities and interesting features come up in the analysis showing new and unexpected macroscopic phenomena.

The Lorentz gas consists of a particle moving through infinitely heavy, randomly distributed scatterers. The interaction between the Lorentz particle and the scatterers is specified by a central potential of finite range. Hence the motion of the Lorentz particle is defined through the solution of Newton's equation of motion. Lorentz's idea was to view electrons as a gas of light particles colliding with the metallic atoms; neglecting collisions between electrons, Lorentz described the interaction of electrons with the metallic atoms by a collision integral analogous to Boltzmann's. The original system is Hamiltonian, the only stochasticity being that of the positions of the scatterers. This randomness is absolutely necessary to obtain the correct kinetic description. Indeed, for this system, one can prove, under suitable scaling limits, a rigorous validation of linear kinetic equations and, from this, of diffusion equations.

We can argue in terms of stochastic processes. The motion of the Lorentz particle is a stochastic process which is non Markovian. The scaling limit procedure can be understood as a Markovian approximation which leads to a Markov process whose forward equation is a suitable kinetic equation.

The microscopic dynamics is governed by the following system of ordinary differential equations

$$\begin{cases} \dot{x}(t) = v \\ \dot{v}(t) = \sum_{i=1}^{N} F(x - c_i) \end{cases} , \tag{1.0.1}$$

with F a conservative force field, i.e. $F = -\nabla_x \phi$ being ϕ a suitable interaction potential. Here $(c_1, \dots, c_N)$ is a configuration of scatterers centers. Throughout this thesis we assume that the distribution of the scatterers is random, more precisely we choose a Poisson distribution. We remind that the Poisson distribution of a random variable $\mathbf{c} = (c_1, \dots, c_N)$, on a subset of finite measure $A \subset \mathbb{R}^d$, with parameter $\mu > 0$ and with values in $\cup_{N \in \mathbb{N}} A^N$ satisfies the following properties: the distribution of N is a Possion law of intensity $\mu|A|$, namely $\mathbb{P}(N = n) = \exp^{(-\mu|A|)} \frac{(\mu|A|)^n}{n!}$ and for N fixed, the law of $c_1 \dots c_N$ is proportional to the Lebesque measure $dc_1 \dots dc_N$ on A^N. We introduce a small scale parameter $\varepsilon \to 0$ which expresses the ratio between the macroscopic and the microscopic scales. The scaling limits we are considering consist of a kinetic scaling of space and time, namely $t \to \varepsilon t$, $x \to \varepsilon x$ and a suitable rescaling of the density of the obstacles μ and the intensity of the interaction potential ϕ. Accordingly to the resulting frequency of collisions, the mean free path of the particle can have or not macroscopic length and different kinetic equations arise. Typical examples are the linear Boltzmann equation and the linear Landau equation.

The first scaling one could consider is the Boltzmann-Grad limit or low density limit. To deal with a low density regime of scatterers we fix our scale parameter in such a way that the density is equal to $\varepsilon\mu$. As a consequence the typical variations, in space and time, of the distribution of the light particle are on the order of a mean free time, namely $1/\varepsilon\mu$. This suggests to scale space and time as $x \to \varepsilon x$, $t \to \varepsilon t$. For this model, it could be more phisically intuitive to transfer the scaling on the background medium. Indeed we consider the equivalent scaling that keep time and space fixed and rescale instead the range of the interaction and the density of the scatterers, i.e.

$$\begin{aligned} &\phi_\varepsilon(x) = \phi(\tfrac{x}{\varepsilon}) \\ &\mu_\varepsilon = \varepsilon^{-d+1}\mu \end{aligned} . \tag{1.0.2}$$

Consequently, the equations of motions (1.0.1) become

$$\begin{cases} \dot{x}(t) = v \\ \dot{v}(t) = -\frac{1}{\varepsilon} \sum_{i=1}^{N} \nabla_x \phi(\frac{x - c_i}{\varepsilon}) \end{cases} .$$

According to the rescaling (1.0.2) we have that the fraction of volume occupied by the obstacles is given by $\varepsilon^d \varepsilon^{-d+1} = \varepsilon$ and tends to zero in the limit $\varepsilon \to 0$. Moreover the mean free path of the Lorentz particle remains constant under the above scaling. In this regime the stochastic process of the Lorentz particle, converges to a limiting process such that the velocity process is a Markov jump process and the position is an additive functional of the velocity process. The corresponding forward equation is the following linear Boltzmann equation

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) = \mathrm{L} f(x, v, t) \tag{1.0.3}$$

with

$$\mathrm{L} f(x, v, t) = \pi\mu|v| \int dv'\, k(v'|v)(f(x, v', t) - f(x, v, t)).$$

Here $k(v'|v)dv'$ is the probability that the velocity of the Lorentz particle jumps instantaneously from v to v' in a collision and it is proportional to the scattering cross section of the interaction potential. We remind that, due to the energy conservation in a collision, the transition kernel $k(v'|v)$ contains the δ function $\delta(|v'| - |v|)$. (See Appendix 6.1 for a detailed discussion on the linear Boltzmann equation.) For instance, in the case of hard core potential $k(v'|v)dv'$ is the normalized uniform distribution on the sphere with radius $|v|$. The first result concerning the rigorous derivation of the linear Boltzmann equation was obtained by Gallavotti in 1969, see [G], who showed the validity of the linear Boltzmann equation starting from a random distribution of fixed hard scatterers in the Boltzmann-Grad limit (low density), namely when the number of collisions is small, thus the mean free path of the particle is macroscopic. We want to underline that Gallavotti's paper, before Lanford's fundamental result, must be regarded as an essential step in the understanding of kinetic theory. Moreover the assumption of independent heavy particles exclude the possibility that potential correlations among heavy particles may influence the light particle dynamics, hence the validity of the Boltzmann's Stosszahlansatz. Gallavotti's result was improved and extended to more general distribution by Spohn [S]. In [BBS] Boldrighini, Bunimovich and Sinai proved that the limiting Boltzmann equation holds for almost every scatterer configuration drawn from a Poisson distribution. Indeed they prove the almost sure convergence and the techniques used are different from those ones used to prove the convergence in mean of the test particle density, averaging over obstacle configurations.

As we already pointed out we remind that the randomness of the distribution of the scatterers is essential in the derivation of the linear Boltzmann equation. In fact for a periodic configuration, when the heavy particles are located at the vertices of some lattice in the Euclidean space, we face the maximum amount of correlation between the heavy particles. This entails a dramatic change in the structure of the equation that one obtains under the same scaling limit that would otherwise lead to a linear Boltzmann equation. Indeed the linear Boltzmann equation fails for this model (see [CG1]) and the random flight process that emerges in the Boltzmann-Grad limit is substantially more complicated. The first complete proof of the Boltzmann-Grad limit of the periodic Lorentz gas, valid for all lattices and in all space dimensions, can be found in [MS]. The mathematical properties of the generalized linear Boltzmann equation derived are analyzed in [CG2].

Another scaling of interest is the weak coupling limit. The idea of the weak coupling limit is that, by some kind of central limit effect, very many but weak collisions should lead to a diffusion type evolution. Therefore the strength of the interaction between the light particle and the scatterers becomes small as

$$\sqrt{\varepsilon}\phi(q)$$

which fixes our scale parameter. The average change of velocity is zero and the typical change in a collision is order $\sqrt{\varepsilon}$. To have ε^{-1} collisions per unit time interval, the time t is scaled as $t \to \varepsilon t$. In a t time interval the Lorentz particle travels over a large distance. Not to lose sight of the particle, we rescale the space according to $x \to \varepsilon x$. We observe that the initial position is scaled but this scaling is such that the free motion remains invariant. This define the weak coupling limit for a constant density of scatterers. However, also in this case, we could perform an equivalent scaling, transferring the scaling on the background medium. Then x, v, t remain unscaled and, in d dimension, we rescale the interaction potential and the density of the scatterers according to

$$\begin{aligned} &\phi_\varepsilon(x) = \sqrt{\varepsilon}\phi(\tfrac{x}{\varepsilon}) \\ &\mu_\varepsilon = \varepsilon^{-d}\mu. \end{aligned} \tag{1.0.4}$$

Consequently, the equations of motion (1.0.1) become

$$\begin{cases} \dot{x}(t) = v \\ \dot{v}(t) = -\frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{N} \nabla_x \phi(\frac{x-c_i}{\varepsilon}) \end{cases} .$$

According to (1.0.4) we have that a finite fraction of the volume remains filled with the obstacles, hence the light particle travels freely a time span ε and then interacts with a scatterer for another time span ε in which its momentum is deflected on the order $\sqrt{\varepsilon}$. We want to compute the total momentum variation for a unit time. The force acting on the Lorentz particle in a single collision is order $\varepsilon^{-\frac{1}{2}}$ on the time interval $O(\varepsilon)$. Therefore the change in velocity due to a single scatterer, i.e. the momentum variation, is $O(\sqrt{\varepsilon})$ since

$$\Delta v \simeq \nabla_x \phi_\varepsilon \cdot \Delta t \simeq \varepsilon^{-\frac{1}{2}}\varepsilon \simeq \varepsilon^{\frac{1}{2}}.$$

The number of scatterers met by the test particle is $O(\varepsilon^{-1})$, hence the total momentum variation for unit time is $O(\sqrt{\varepsilon}^{-1})$. Thanks to the symmetry of the force and the homogeneous gas this variation is zero in average. Assuming that the effect of the scatterers is approximately additive if we compute the total variance we get $\frac{1}{\varepsilon}O(\sqrt{\varepsilon})^2 = O(1)$. This central limit type argument suggests a Wiener like diffusion in the velocity variable. Moreover, by conservation of energy in a single collision we have that $|v|$ is preserved. Since no particular direction is singled out due to the Poisson distribution, we expect a diffusion on the sphere with radius $|v|$. At this point we are interested in determining the diffusion coefficient B. The same argument tell us that the diffusion coefficient B should be proportional to the average of the square of the change in velocity for the unscaled process with $\sqrt{\varepsilon}\, F(\cdot) = -\sqrt{\varepsilon}\, \nabla_x \phi(\cdot)$. Indeed

$$\begin{aligned}(v(t) - v(0))^2 =& \varepsilon \int_0^t ds \int_0^t ds' \, \langle F(x(s)) \cdot F(x(s')) \rangle \\ \simeq& \, \varepsilon \int_0^t ds \int_0^t ds' \, \langle F(vs) \cdot F(vs') \rangle \\ \simeq& \, \varepsilon \, t \, 2\, \pi \mu \int d^d k |k|^2 |\hat{\phi}(k)|^2 \, \delta(k \cdot v) \equiv \varepsilon \, t \, 2 \, (d-1) \, B(|v|),\end{aligned}$$

where $\hat{\phi}$ is the Fourier transform of ϕ. For further details see [S1] and [S2]. Therefore the correct kinetic equation which is derived in this scaling limit is the Linear Landau equation

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = B \Delta_{|v|} f(x, v, t),$$

where $\Delta_{|v|}$ is the Laplace-Beltrami operator on the d-dimensional sphere of radius $|v|$. It is a Fokker-Planck equation for the limiting stochastic process where the velocity process is a Brownian motion on the (kinetic) energy sphere, and the position is an additive functional of the velocity process. The first result concerning the rigorous derivation of the linear Landau equation was obtained by Kesten and Papanicolau in 1980. They proved the convergence in law towards a Brownian motion for the stochastic process of the test particle in $\mathbb{R}^d$, $d \geq 3$, in a weak mean zero random force field. This model is referred to as the stochastic acceleration problem, see [KP]. Later, in 1987, Dürr, Goldstein and Lebowitz proved that in $\mathbb{R}^2$ the velocity process of the light particle converges in distribution to a Brownian motion on a surface of constant speed. Their result holds for sufficiently smooth interaction potentials and for a Poisson distribution of obstacles, see [DGL]. More recently, Komorowski and Ryzhik handled the stochastic acceleration problem in two dimensions [KR] complementing in this way the limit theorem of Kesten and Papanicolaou proved in dimensions $d \geq 3$.

The linear Landau equation appears also in an intermediate scale between the low density and the weak-coupling regime, namely when the (smooth) interaction potential ϕ rescales according to $\phi \to \varepsilon^\alpha \phi$, $\alpha \in (0, 1/2)$ and we consider the rescaled density $\mu_\varepsilon = \varepsilon^{-2\alpha-(d-1)} \mu$. See for instance

[DR], [K]. We observe that in this regime the scatterer configuration is still dilute since $\mu_\varepsilon \varepsilon = \varepsilon^{-2\alpha-(d-1)}\varepsilon^{(d-1)} \to \infty$ and $\mu_\varepsilon \varepsilon^2 = \varepsilon^{-2\alpha-(d-1)}\varepsilon^{d} \to 0$. The limiting cases $\alpha = 0$ and $\alpha = 1/2$ correspond respectively to the low density limit and the weak-coupling limit.

The situation changes radically if we take into account the analogous quantum model (not discussed in this thesis). Here in the weak-coupling limit, as well in the low density limit, the kinetic equation approached is the linear Boltzmann equation (the only difference concerns the scattering cross section which changes according to the two different scalings). The kinetic equation is classical and the only quantum macroscopic effect appears in the cross-section which is computed in terms of the quantum scattering problem. In the weak coupling limit, in contrast with the classical case where we get a diffusion, due to a macroscopic tunnel effect we get a Markov jump process. This results from the asymptotics of a single scattering. In a quantum setting there is a finite probability of having any scattering angle while for a classical particle we have surely a small deviation from the free motion. The first result, concerning the derivation of the linear Boltzmann equation in a weak-coupling limit, has been obtained in [S4] for a potential given by a field of Gaussian random variables. This result holds for short times, we refer to [EY] for the extension to arbitrary times. Moreover the technique of [EY] can be applied to deal with a Poisson distribution of obstacles. The cross section appearing in the Boltzmann equation is that computed in the Born approximation. For what concerns the low-density regime we refer to [EE] where a linear Boltzmann equation with the full cross-section is approached in the limit.

The rigorous derivation of hydrodynamical equations grounds on the heuristic idea that after a few mean free times the Lorentz gas is already very close to the local equilibrium. Namely the system reaches relatively fast local equilibrium characterized by the hydrodynamic fields, corresponding to the locally conserved quantities. These fields subsequently evolve relatively slowly. Therefore the hydrodynamic limits involve the long time behavior of the system.

Roughly speaking the macroscopic dynamics is essentially generated by the conservation law fulfilled by the microscopic dynamics. Moreover there is a correspondence between conservation laws and transport coefficients, appearing in the non reversible equations of the macroscopic dynamics as coefficients. For a system of interacting particles governed by Newton's equations there are three classical conservation laws, namely the conservation of mass, momentum and energy. The corresponding transport coefficients are called coefficients of diffusion, viscosity and thermal conductivity. For this system instead the momentum is not conserved since the scatterers do not move and the conservation of energy is equivalent to the conservation of mass because there is no redistribution of energy between particles. Hence the only conserved quantity is the mass and the only hydrodynamic field is the spatial density $\rho(x,t)$. Consequently the only hydrodynamic equation for the Lorentz gas is the diffusion equation which is the analogue of the Navier Stokes equations in the nonlinear setting. For the hydrodynamic limits one considers only the spatial component of the stochastic motion, i.e.

$$x(t) = x + \int_0^t ds\, v(s).$$

To obtain the analogue of the Euler equations (for this model it is just $\partial_t \rho(x,t) = 0$) we have to look at the evolution of the Lorentz particle over distances on the order ε^{-1} and over times on the order ε^{-1}. To obtain instead the diffusion equation one has to consider the evolution over distances on the order ε^{-1} and over times on the order ε^{-2}. Hence we would expect that, under the previous diffusive scaling, the distribution density of the test particle converges to that of a diffusion process. Unfortunately, the rigorous derivation of the heat equation from the mechanical system given by the Lorentz gas is actually a very difficult and still unsolved problem. The only rigorous result in this direction was achieved by Bunimovich and Sinai in 1980 (see [BS]) by showing that such diffusive limit holds when the scatterers are fixed and periodically distributed. More precisely they consider a point particle elastically colliding with a periodic distribution of scatterers in the case of finite horizon, namely when the length of a free path of the moving particle is bounded. The centers of the hard disks are located at the points of a two dimensional triangular lattice and the initial conditions are distributed according to a probability measure, absolutely continuous with respect to the Lebesgue measure. Hence the only stochasticity is in the initial condition being the dynamics completely deterministic. The rescaled trajectory reads as

$$x_\varepsilon(t) = \sqrt{\varepsilon}\, x(t/\varepsilon),\ t \in [0,1] \tag{1.0.5}$$

and belongs to $C([0,1];\mathbb{R}^2)$. The initial measure induces a probability distribution on the space of all continuous trajectories which converges weakly to a Wiener measure as $\varepsilon \to 0$. In other words they proved the validity of a central limit theorem for the displacement of the test particle at large times t, by showing the convergence of the stochastic process given by (1.0.5) towards a Wiener process with diffusion coefficient D. According to Einstein formula, which relates D to the time integral of the time autocorrelation function of the mass current, it results

$$D = \frac{2}{d} \int_0^\infty \langle v(0), v(t)\rangle\, dt,$$

where $\langle \cdot, \cdot \rangle$ is the expectation with respect to the invariant measure. The above Einstein formula is the first one in the hierarchy of the Green-Kubo formulas for the transport coefficients. Hence we have the existence of a non degenerate transport coefficient D and the validity of the diffusion equation. As a consequence this result is one of the most important in the transition from the microscopic to the macroscopic description.

Nonetheless one can handle this problem by deriving the diffusion equation from the correct kinetic equation which arises, according to the suitable kinetic scaling performed, from the random Lorentz gas. We remark, however, that the hydrodynamics for the Lorentz model is not equivalent to the hydrodynamics for the kinetic equation.

In this direction, in the second chapter where we present [BNP], we provide a rigorous derivation of the heat equation from the particle system (the Lorentz model) using the linear Landau equation as a bridge between our original mechanical system and the diffusion equation. It works once having an explicit control of the error in the kinetic limit (see also [DP], where the set of bad configurations are explicitly estimated). The diffusive limit can be achieved since the control of memory effects still holds for a longer time scale.

Moreover, since it is well known how important and challenging is the characterization of stationary nonequilibrium states exhibiting transport phenomena in the rigorous approach to nonequilibrium Statistical Mechanics, we are also interested in considering the Lorentz model out of equilibrium. Energy or mass transport in non equilibrium macroscopic systems are described phenomenologically by Fourier's and Fick's law respectively. We notice that there are very few rigorous results in this direction in the current literature (see for instance [LS], [LS1], [LS2]). In the third chapter, where in the first section we present [BNPP], we give a contribution in this direction, by validating the Fick's law for the Lorentz model in a low density regime. Moreover further recent developments, strictly connected to this problem, are faced in the same chapter. Indeed in the last section we observe that the strategy used to prove the above result, suitably modified, allow to validate the Fick's Law in a weak coupling regime.

In the previous chapters we analyzed diffusive limits for the Lorentz Gas while in the last two chapters we focus instead on kinetic limits. More precisely in the fourth chapter we observe that, also in a linear case, the notion of Propagation of Chaos can be established. Indeed we validate the Propagation of Chaos for the full wind-tree model, proposed originally by Gallavotti in [G], in the low density limit. Namely we look at the evolution of the j (light) particle correlation functions and we prove that the factorization property is recovered in the limit.

In the last chapter instead we present [N1]. We deal with the rigorous derivation of linear kinetic equations from the microscopic model given by the Lorentz gas in presence of an external field, in particular we consider a uniform magnetic field. The interest for this problems emerges from recent results, see for instance [BMHH1], [BMHH2], where it has been proved that the presence of an external magnetic field is not innocent in the derivation of the linear Boltzmann equation in a low density regime. In contrast to this results we show that, in a weak coupling limit, the Lorentz particle distribution behaves according to the linear Landau equation with the magnetic field. Moreover we show that, in the low density limit, when each obstacles generates an inverse power law potential, the particle distribution behaves according to the linear Boltzmann equation with the magnetic field.

This thesis contains the following papers: the published paper [BNP] written in collaboration with G. Basile and M. Pulvirenti, which is presented in Chapter 2; the submitted paper [BNPP], available on arXiv:1404.4186, written in collaboration with G. Basile, F. Pezzotti and M. Pulvirenti, presented in Chapter 3, Section 3.1; my recent preprint [N3] available on arXiv:1411.6474, presented in Chapter 3, Section 3.2; my recent work in progress [N1] presented in Chapter 4.

Chapter 2

A diffusion limit for a test particle in a Random Distribution of scatterers [*J. Stat. Phys.* 155, 1087-1111, 2014]

In the present Chapter we present [BNP].

A diffusion limit for a test particle in a random distribution of scatterers

Abstract. We consider a point particle moving in a random distribution of obstacles described by a potential barrier. We show that, in a weak-coupling regime, under a diffusion limit suggested by the potential itself, the probability distribution of the particle converges to the solution of the heat equation. The diffusion coefficient is given by the Green-Kubo formula associated to the generator of the diffusion process dictated by the linear Landau equation.

2.1 Introduction

The evolution of the density of a test particle moving in a configuration of obstacles, usually called Lorentz gas [L], is described at mesoscopic level by linear kinetic equations. They are obtained from the microscopic Hamiltonian dynamics under a kinetic scaling of space and time, namely $t \to \varepsilon t$, $x \to \varepsilon x$ and a suitable rescaling of the density of the obstacles and the intensity of the interaction. Accordingly to the resulting frequency of collisions, the mean free path of the particle can have or not macroscopic length and different kinetic equations arise. Typical examples are the linear Boltzmann equation and the linear Landau equation.

The first rigorous result appeared in 1969 in the paper of Gallavotti [G], who derived a linear Boltzmann equation starting from a random distribution of fixed hard scatterers in the Boltzmann-Grad limit (low density), namely when the number of collisions is small, thus the mean free path of the particle is macroscopic. The result was improved by Spohn [S]. In [BBS] Boldrighini, Bunimovich and Sinai proved that the limiting Boltzmann equation holds for typical configurations.

In the weak-coupling regime, when there are very many but weak collisions, a linear Landau equation appears

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = B \Delta_{|v|} f(x, v, t), \tag{2.1.1}$$

where $\Delta_{|v|}$ is the Laplace-Beltrami operator on the d-dimensional sphere of radius $|v|$. It is a Fokker-Planck equation for the stochastic process $(V(t), X(t))$, where the velocity process V is a Brownian motion on the (kinetic) energy sphere, and the position X is an additive functional of V. The velocity diffusion follows from the facts that there are many elastic collisions. The diffusion coefficient B is proportional to the variance of the transferred momentum in a single collision and depends on the shape of the interaction potential. The first result in this direction was obtained by Kesten and Papanicolau for a particle in $\mathbb{R}^3$ in a weak mean zero random force field [KP]. Dürr, Goldstein and Lebowitz proved that in $\mathbb{R}^2$ the velocity process converges in distribution to Brownian motion on a surface of constant speed for sufficiently smooth interaction potentials [DGL].

The linear Landau equation appears also in an intermediate scale between low density and weak-coupling regime, namely when the (smooth) interaction potential ϕ rescales according to $\phi \to \varepsilon^{\alpha}\phi$, $\alpha \in (0, 1/2)$ and the density of the obstacles is of order $\varepsilon^{-2\alpha-(d-1)}$ ([DR], [K]). The limiting cases $\alpha = 0$ and $\alpha = 1/2$ correspond respectively to the low density limit and the weak-coupling limit.

In the present paper we consider an interaction potential given by a circular potential barrier, in dimension two (dimension three is easier). We want to investigate the limit $\varepsilon \to 0$ in the intermediate case, namely when $\alpha > 0$ but sufficiently small. In this case, due to lack of smoothness of the potential, new features emerge at mesoscopic level.

The physical interest of this problem is connected to the geometric optics since the trajectory of the test particle is that of a light ray traveling in a medium (say water) in presence of circular drops of a different substance with smaller refractive index (say air). The opposite situation, namely drops of water in a medium of air, can be described as well by the circular well potential. Our analysis applies also to this case with minor modifications, yielding the same result, but we consider only the case of potential barrier for sake of concreteness.

The novelty of this choice is that in this case the diffusion coefficient B diverges logarithmically. Roughly speaking, the asymptotic equation for the density of the Lorentz particle reads

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) \sim |\log \varepsilon| \, B \Delta_{|v|} f(x, v, t), \tag{2.1.2}$$

which suggests to look at a longer time scale $t \to |\log \varepsilon| t$. As expected, a diffusion in space arises.

The proof follows the original constructive idea, due to Gallavotti [G], for the low-density limit of a hard-sphere system. This approach is based on a suitable change of variables which leads to a Markovian approximation described by a linear Boltzmann equation. This presents some technical difficulties since some of the random configurations lead to trajectories that "remember" too much preventing the Markov property of the limit. In the two-dimensional case the probability of those bad behaviors producing memory effects (correlation between the past and the present) is nontrivial. Thus we need to control the unphysical trajectories: we estimate explicitly the set of bad configurations of the scatterers (such as the set of configurations yielding recollisions or interferences) showing that it is negligible in the limit (see [DP], where the set of bad configurations are explicitly estimated). The control of memory effects still holds for a longer time scale $|\log \varepsilon|$ which allows to get the heat equation from the rescaled linear Boltzmann equation.

We remark that the diffusive limit analyzed in the present paper is suggested by the divergence of the diffusion coefficient for the particular choice of the potential we are considering. However the same techniques could work in presence of a smooth, radial, short-range potential ϕ. Also in this case we obtain a diffusive equation as longer time scale limit of a linear Boltzmann equation (Section 5). Same spirit as in [KR] and [LE].

2.2 Main results

Consider a point particle of mass one in $\mathbb{R}^2$, moving in a random distribution of fixed scatterers whose center are denoted by $c_1, \ldots, c_N \in \mathbb{R}^2$. The equation of motion are

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\sum_{i=1}^{N} \nabla \phi(|x - c_i|) \end{cases} , \tag{2.2.1}$$

where (x, v) denote position and velocity of the test particle, t the time and, as usual, $\dot{A} = \frac{dA}{dt}$ indicates the time derivative for any time dependent variable A.
Finally $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a given spherically symmetric potential.

To outline a kinetic behavior of the particle, we usually introduce a scale parameter $\varepsilon > 0$, indicating the ratio between the macroscopic and the microscopic variables, and rescale according to

$$x \to \varepsilon x, \ t \to \varepsilon t, \ \phi \to \varepsilon^{\alpha} \phi$$

with $\alpha \in [0, 1/2]$. Then Eq.ns (2.2.1) become

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\varepsilon^{\alpha-1} \sum_i \nabla \phi(\frac{|x - c_i|}{\varepsilon}) \end{cases} . \tag{2.2.2}$$

We assume the scatterers $\mathbf{c}_N = (c_1, \ldots, c_N)$ distributed according to a Poisson distribution of intensity $\mu_\varepsilon = \mu\varepsilon^{-\delta}$, where $\delta = 1 + 2\alpha$. This means that the probability density of finding N obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^2$ is given by

$$\mathbb{P}_\varepsilon(\,d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|}\frac{\mu_\varepsilon^N}{N!}\,dc_1 \ldots dc_N, \tag{2.2.3}$$

where $|\Lambda| = \text{meas}\Lambda$.

Now let $T^t_{\mathbf{c}_N}(x, v)$ be the Hamiltonian flow solution of Eq.n (2.2.2) with initial datum (x, v) in a given sample $\mathbf{c}_N = (c_1, \ldots, c_N)$ of obstacles (skipping the ε dependence for notational simplicity) and, for a given initial probability distribution $f_0 = f_0(x, v)$, consider the quantity

$$f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_N}(x, v))], \tag{2.2.4}$$

where $\mathbb{E}_\varepsilon$ is the expectation with respect to the measure $\mathbb{P}_\varepsilon$ given by (2.2.3).

In the limit $\varepsilon \to 0$ we expect that the probability distribution (2.2.4) solves a linear kinetic equation depending on the value of α. More precisely if $\alpha = 0$ (low-density or Boltzmann-Grad limit) then f_ε converges to f, the solution of the following linear Boltzmann equation

$$(\partial_t + v \cdot \nabla_x)f(x, v, t) = \mathrm{L}f(x, v, t), \tag{2.2.5}$$

where

$$\mathrm{L}f(x, v, t) = \mu|v|\int_{-1}^{1} d\rho\{f(v') - f(v)\}, \tag{2.2.6}$$

and where

$$v' = v - 2(\omega \cdot v)\omega. \tag{2.2.7}$$

Here we are assuming ϕ of range one i.e. $\phi(r) = 0$ if $r > 1$, and $\omega = \omega(\rho, |v|)$ is the unit vector obtained by solving the scattering problem associated to ϕ (see Figure 2.1). This result was proven and discussed in [BBS],[DP],[G],[S]. On the other hand, if $\alpha = 1/2$, the corresponding limit, called weak-coupling limit, yields the linear Landau equation (see [DGL] and [K])

$$(\partial_t + v \cdot \nabla_x)f(x, v, t) = \mathcal{L}f(x, v, t), \tag{2.2.8}$$

where for every $g \in C_0(S_{|v|})$

$$\mathcal{L}g(v) = B\Delta_{|v|}g(v), \tag{2.2.9}$$

and

$$B = \frac{\pi\mu}{|v|}\int_0^\infty \zeta^2\hat{\phi}(\zeta)^2\,d\zeta, \tag{2.2.10}$$

where $\hat{\phi}$, the Fourier transform of the potential ϕ, is real and spherically symmetric.

In the present paper we want to investigate the limit $\varepsilon \to 0$, in case $\alpha > 0$ sufficiently small, when the diffusion coefficient B given by (2.2.10) is diverging. Actually we consider the specific example

$$\phi(r) = \begin{cases} 1 & \text{if } r < 1 \\ 0 & \text{otherwise} \end{cases}, \tag{2.2.11}$$

namely a circular potential barrier.

For a potential of the form (2.2.11) a simple computation shows that B defined in (2.2.10) diverges since $\zeta^2 \hat{\phi}(\zeta)^2$ is not integrable. Therefore we are interested in characterizing the asymptotic behavior of $f_\varepsilon(x, v, t)$, given by (2.2.4), under the scaling illustrated above. The main results of the present paper can be summarized in the following theorems.

Theorem 2.2.1. *Suppose $f_0 \in C_0(\mathbb{R}^2 \times \mathbb{R}^2)$ a continuous, compactly supported initial probability density. Suppose also that $|D_x^k f_0| \le C$, where D_x is any partial derivative with respect to x and $k = 1, 2$. Assume $\mu_\varepsilon = \varepsilon^{-2\alpha-1}$, with $\alpha \in (0, 1/8)$. Then the following statement holds*

$$\lim_{\varepsilon \to 0} f_\varepsilon(x, v, t) = \langle f_0 \rangle := \frac{1}{2\pi} \frac{1}{|v|} \int_{S_{|v|}} f_0(x, v)\, dv, \tag{2.2.12}$$

$\forall t \in (0, T]$, $T > 0$. The convergence is in $L^2(\mathbb{R}^2 \times S_{|v|})$.

Moreover, define $F_\varepsilon(x, v, t) := f_\varepsilon(x, v, t|\log \varepsilon|)$. Then, for all $t \in [0, T)$, $T > 0$,

$$\lim_{\varepsilon \to 0} F_\varepsilon(x, v, t) = \rho(x, t),$$

where ρ solves the following heat equation

$$\begin{cases} \partial_t \varrho = D \Delta \varrho \\ \varrho(x, 0) = \langle f_0 \rangle, \end{cases} \tag{2.2.13}$$

with D given by the Green-Kubo formula

$$D = \frac{2}{\mu} |v| \int_{S_{|v|}} v \cdot \big(-\Delta_{|v|}^{-1} \big) v\, dv = \frac{2\pi}{\mu} |v|^2 \int_0^\infty \mathbb{E}\big[v \cdot V(t, v) \big]\, dt, \tag{2.2.14}$$

where $V(t, v)$ is the stochastic process generated by $\Delta_{|v|}$ starting from v and $\mathbb{E}[\cdot]$ denotes the expectation with respect to the invariant measure, namely the uniform measure on $S_{|v|}$. The convergence is in $L^2(\mathbb{R}^2 \times S_{|v|})$.

Some comments are in order. As we shall prove in Sections 3 and 4, the asymptotic behavior of the mechanical system we are considering is the same as the Markov process ruled by the linear Landau equation with a diverging factor in front of $\mathcal{L}$. This is equivalent to consider the limit in the Euler scaling of the linear Landau equation, which is trivial. The system quickly thermalizes to the local equilibrium just given by $\langle f_0 \rangle$, see (2.2.12).

To detect something non-trivial we have to exploit longer times in which the local equilibrium starts to evolve (according to the diffusion equation), see (2.2.13).

Note however that, rescaling differently the density of the Poisson process, we can recover the kinetic picture given by Landau equation (with a renormalized diffusion coefficient B) as in [DR]. This is stated in the next Theorem.

Theorem 2.2.2. *Suppose $f_0 \in C_0(\mathbb{R}^2 \times \mathbb{R}^2)$ a continuous, compactly supported initial probability density. Suppose also that $|D_x^k f_0| \le C$, where D_x is any partial derivative with respect to x and $k = 1, 2$. Assume $\mu_\varepsilon = \frac{\varepsilon^{-2\alpha-1}}{|\log \varepsilon|}$, with $\alpha \in (0, 1/8)$. The following statement holds*

$$\lim_{\varepsilon \to 0} f_\varepsilon(x, v, t) = f(x, v, t),$$

$\forall t \in (0, T]$, $T > 0$. Here f solves the Landau equation (2.2.8) *with a renormalized diffusion coefficient*

$$B := \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon}{2} \varepsilon \, |v| \int_{-1}^{1} \theta_\varepsilon^2(\rho) \, d\rho, \tag{2.2.15}$$

where θ_ε is the scattering angle defined in (A.6), (A.5). *The convergence is in $L^2(\mathbb{R}^2 \times S_{|v|})$.*

We finally remark that the results of the above theorems are made possible because the recollisions set (see below for the precise definition) is negligible, as established in Section 5. We believe that the present result could be recovered also in high-density regimes $\alpha \in \left(\frac{1}{8}, \frac{1}{2}\right]$, namely also when the recollisions are not negligible anymore. However in this case different ideas and techniques are indeed necessary.

The plan of the paper is the following. In the next Section we illustrate our strategy and establish some preliminary results. In Section 4 we prove Theorem 2.2.1 and 2.2.2. Finally in Section 5 we prove a basic Lemma showing that our non-Markovian system can indeed be approximated by a Markovian one, easier to handle with.

2.3 Strategy

We follow the explicit approach in [G], [DP] and [DR].
By (2.2.4) we have, for $(x, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $t > 0$,

$$f_\varepsilon(x, v, t) = e^{-\mu_\varepsilon |B_t(x,v)|} \sum_{N \ge 0} \frac{\mu_\varepsilon^N}{N!} \int_{B_t(x,v)^N} d\mathbf{c}_N \, f_0(T_{\mathbf{c}_N}^{-t}(x, v)), \tag{2.3.1}$$

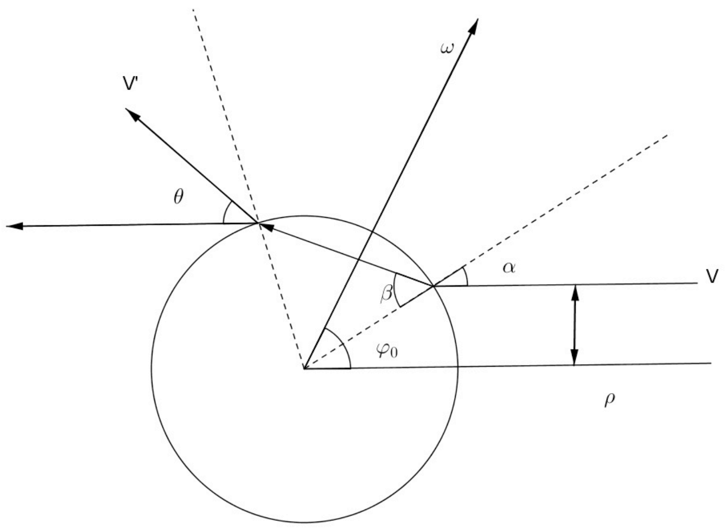

Figure 2.1: Scattering

where $T^t_{\mathbf{c}_N}(x, v)$ is the Hamiltonian flow generated by the Hamiltonian

$$\frac{1}{2}v^2 + \varepsilon^\alpha \sum_j \phi\left(\frac{|x - c_j|}{\varepsilon}\right), \tag{2.3.2}$$

where ϕ is given by (2.2.11), and initial datum (x, v). For this choice of the potential $\nabla\phi$ is not well defined. However the explicit solution of the equation of motion is obtained by solving the single scattering problem by using the energy and angular momentum conservation (see Figure 2.1). Finally we define $B_t(x, v) := B(x, |v|t)$, where here and in the following, $B(x, R)$ denotes the disk of center x and radius R.

Here we represent the scattering of a particle entering in the ball

$$B(0, 1) = \{x \text{ s.t. } |x| < 1\}$$

toward a potential barrier of intensity $\phi(x) = \varepsilon^\alpha$.

We have an explicit expression for the refractive index

$$n_\varepsilon = \frac{\sin\alpha}{\sin\beta} = \frac{|\bar{v}|}{|v|} = \sqrt{1 - \frac{2\varepsilon^\alpha}{v^2}}, \tag{2.3.3}$$

where v is the initial velocity, $\bar{v}$ the velocity inside the barrier, α the angle of incidence and β the angle of refraction. The scattering angle is $\Theta = \pi - 2\varphi_0 = 2(\beta - \alpha)$ and the impact parameter is $\rho = \sin\alpha$. (See Appendix 2.7 for a detailed analysis of the scattering problem.)

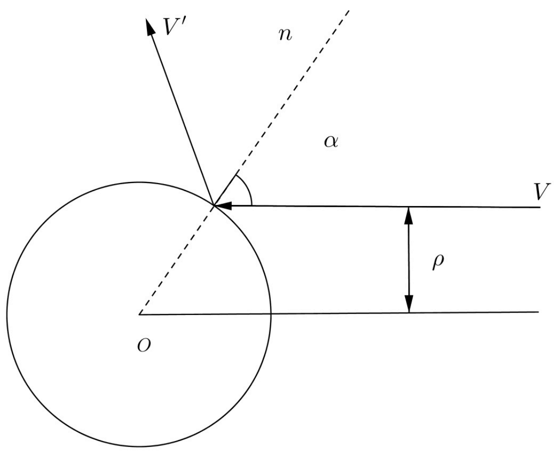

Figure 2.2: Elastic reflection

Remark 2.3.1. *Formula* (2.3.3) *makes sense if* $\frac{2\varepsilon^\alpha}{v^2} < 1$ *or* $\rho = \sin\alpha < \sqrt{1 - \frac{2\varepsilon^\alpha}{v^2}}$.
When one of such two inequalities is violated, the outgoing velocity is the one given by the elastic reflection.

After the scaling
$$x \to \varepsilon x, \ t \to \varepsilon t$$
the scattering process takes place in a disk of radius ε, but the velocities (and hence the angles) are invariant. A picture of a typical trajectory is given as in Figure 3. Here we are not considering possible overlappings of obstacles. The scattering process can be solved in this case as well. However, as we shall see, this event is negligible because of the moderate densities we are considering.

Coming back to Eq.n (2.3.1), we distinguish the obstacles of the configuration $\mathbf{c}_N = c_1 \dots c_N$ which, up to the time t, influence the motion, called internal obstacles, and the external ones. More precisely c_i is internal if
$$\inf_{-t \le s \le 0} |x_\varepsilon(s) - c_i| < \varepsilon, \tag{2.3.4}$$
while c_i is external if
$$\inf_{-t \le s \le 0} |x_\varepsilon(s) - c_i| \ge \varepsilon. \tag{2.3.5}$$
Here $(x_\varepsilon(s), v_\varepsilon(s)) = T^s_{\mathbf{c}}(x, v)$.

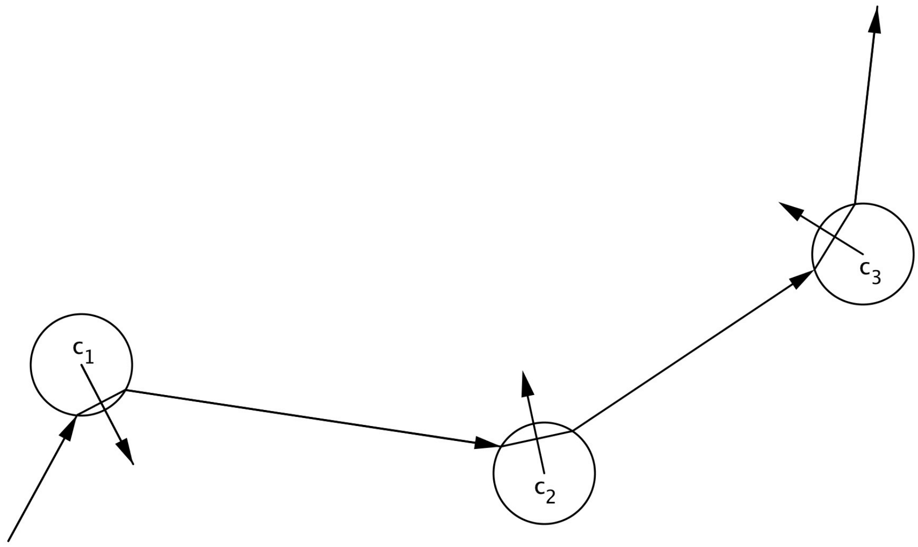

Figure 2.3: A typical trajectory

Note that the integration over the external obstacles can be performed so that

$$f_\varepsilon(x,v,t) = \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B_t(x,v)^Q} d\mathbf{b}_Q \, e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} f_0(T^{-t}_{\mathbf{b}_Q}(x,v)) \\ \chi(\{\text{the } \mathbf{b}_Q \text{ are internal}\}). \tag{2.3.6}$$

Here and in the sequel $\chi(\{\dots\})$ is the characteristic function of the event $\{\dots\}$.
Moreover $\mathcal{T}(\mathbf{b}_Q)$ is the tube:

$$\mathcal{T}(\mathbf{b}_Q) = \{y \in B_t(x,v) \text{ s.t. } \exists s \in (-t,0) \text{ s.t. } |y - x_\varepsilon(s)| < \varepsilon\}. \tag{2.3.7}$$

Note that

$$|\mathcal{T}(\mathbf{b}_Q)| \leq 2\varepsilon |v| t. \tag{2.3.8}$$

Instead of considering f_ε we introduce

$$\tilde{f}_\varepsilon(x,v,t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B_t(x,v)^Q} d\mathbf{b}_Q \\ \chi(\{\text{the } \mathbf{b}_Q \text{ are internal}\}) \chi_1(\mathbf{b}_Q) f_0(T^{-t}_{\mathbf{b}_Q}(x,v)), \tag{2.3.9}$$

where

$$\chi_1(\mathbf{b}_Q) = \chi\{\mathbf{b}_Q \text{ s.t. } b_i \notin B(x,\varepsilon) \text{ and } b_i \notin B(x_\varepsilon(-t),\varepsilon) \text{ for all } i = 1,\dots,Q\}. \tag{2.3.10}$$

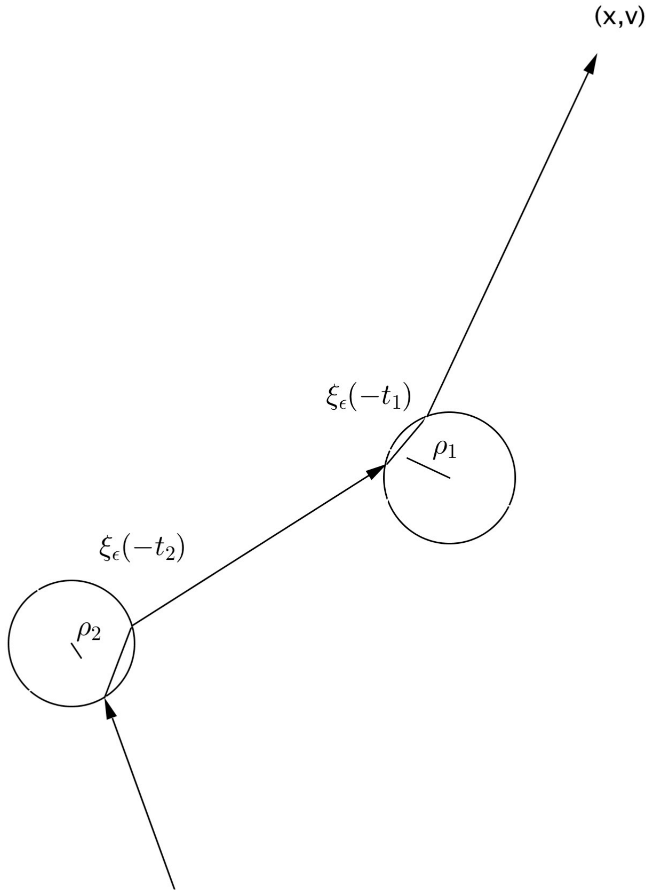

Figure 2.4: The change of variables

Obviously

$$f_\varepsilon \geq \tilde{f}_\varepsilon. \tag{2.3.11}$$

Following [G],[DP],[DR] we would like to perform the following change of variables

$$0 \leq t_1 < t_2 < \cdots < t_Q \leq t$$

$$b_1, \dots, b_Q \to \rho_1, t_1, \dots, \rho_Q, t_Q, \tag{2.3.12}$$

where, after ordering the obstacles $b_1, \dots, b_Q$ according to the scattering sequence, ρ_i and t_i are the impact parameter and the entrance time of the light particle in the protection disk around b_i, i.e. $B(b_i, \varepsilon)$.

More precisely, fixed an impact parameter ρ and an entrance time t, by using elementary geometrical arguments, we can construct uniquely $b = b(\rho, t)$, the center of the obstacle. Then we perform the backward scattering and iterate the procedure to construct a trajectory $(\xi_\varepsilon(s), \eta_\varepsilon(s))$, $s \in [-t, 0]$.

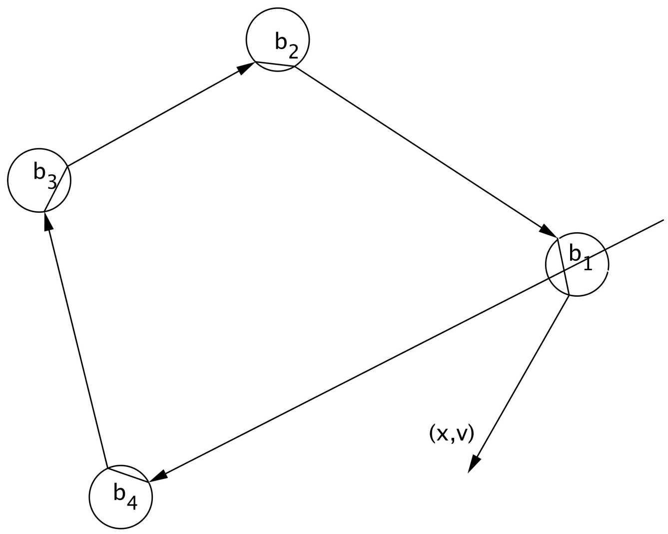

Figure 2.5: Recollisions (backward)

However $(\xi_\varepsilon(s), \eta_\varepsilon(s)) = (x_\varepsilon(s), v_\varepsilon(s))$ (therefore the mapping (2.3.12) is one-to-one) only outside the following pathological situations.
i) **Overlapping**.
If b_i and b_j are both internal then $B(b_i, \varepsilon) \cap B(b_j, \varepsilon) \neq \emptyset$.
ii) **Recollisions**.
There exists b_i such that for $\tilde{s} \in (t_j, t_{j+1})$, $j > i$, $\xi_\varepsilon(-\tilde{s}) \in B(b_i, \varepsilon)$.
iii) **Interferences**.
There exists b_i such that $\xi_\varepsilon(-\tilde{s}) \in B(b_i, \varepsilon)$ for $\tilde{s} \in (t_j, t_{j+1})$, $j < i$.

We simply skip such events by setting

$$\chi_{ov} = \chi(\{\mathbf{b}_Q \text{ s.t. i) is realized}\}),$$
$$\chi_{rec} = \chi(\{\mathbf{b}_Q \text{ s.t. ii) is realized}\}),$$
$$\chi_{int} = \chi(\{\mathbf{b}_Q \text{ s.t. iii) is realized}\}),$$

and defining

$$\bar{f}_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1$$
$$\int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \, \chi_1 (1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int}) f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t)). \tag{2.3.13}$$

Note that $\bar{f}_\varepsilon \leq \tilde{f}_\varepsilon \leq f_\varepsilon$. Note also that in (2.3.13) we have used the change of variables (2.3.12) for which, outside the pathological sets i), ii),

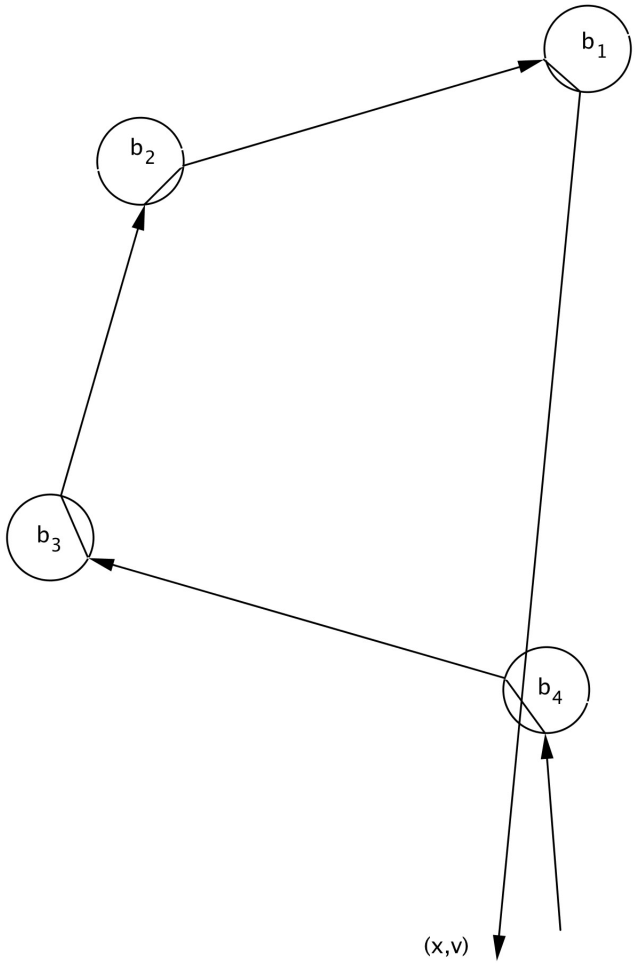

Figure 2.6: Interferences (backward)

iii), $T_{\mathbf{b}_Q}^{-t}(x,v) = (\xi_\varepsilon(-t), \eta_\varepsilon(-t))$.

Next we remove $\chi_1(1-\chi_{ov})(1-\chi_{rec})(1-\chi_{int})$ by setting

$$\begin{aligned}\bar{h}_\varepsilon(x,v,t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q\geq 0} \mu_\varepsilon^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1 \\ \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \, f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t)).\end{aligned} \tag{2.3.14}$$

Proposition 2.3.2. *We have*

$$\bar{f}_\varepsilon(t) = \bar{h}_\varepsilon(t) + \varphi_1(\varepsilon, t),$$

where $\|\varphi_1(\varepsilon,t)\|_{L^1} \to 0$ *as* $\varepsilon \to 0$ *for all* $t \in [0,T]$.

Remark 2.3.3. *Proposition 2.3.2 still holds for longer times, namely:*

$$\|\varphi_1(\varepsilon,t)\|_{L^1} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \forall t \in [0, |\log\varepsilon| T], \quad T>0.$$

We postpone the proof of the above Proposition to the last Section. Next we consider the limiting trajectory $\bar{\xi}_\varepsilon(s), \bar{\eta}_\varepsilon(s)$ obtained by considering the collision as instantaneous.
More precisely, for the sequence $t_1, \dots, t_Q$ $\rho_1, \dots \rho_Q$ consider the sequence $v_1, \dots, v_Q$ of incoming velocities before the Q collisions. Then

$$\begin{cases} \bar{\xi}_\varepsilon(-t) = x - vt_1 - v_1(t_2 - t_1) \cdots - v_Q(t - t_Q) \\ \bar{\eta}_\varepsilon(-t) = v_Q. \end{cases} \tag{2.3.15}$$

We define

$$\begin{aligned} h_\varepsilon(x,v,t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q\geq 0} \mu_\varepsilon^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1 \\ \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \, f_0(\bar{\xi}_\varepsilon(-t), \bar{\eta}_\varepsilon(-t)). \end{aligned} \tag{2.3.16}$$

Due to the Lipschitz continuity of f_0 we can assert that

$$\bar{h}_\varepsilon(x,v,t) = h_\varepsilon(x,v,t) + \varphi_2(x,v,t), \tag{2.3.17}$$

where

$$\sup_{x,v,t\in[0,T]} |\varphi_2(x,v,t)| \leq C\varepsilon^{1-2\alpha}T. \tag{2.3.18}$$

For more details see [DP], Section 3. As matter of facts, since we realize that h_ε is the solution of the following Boltzmann equation

$$(\partial_t + v\cdot\nabla_x) h_\varepsilon(x,v,t) = \mathrm{L}_\varepsilon h_\varepsilon(x,v,t), \tag{2.3.19}$$

where

$$\mathrm{L}_\varepsilon h(v) = \mu\varepsilon^{-2\alpha}|v| \int_{-1}^{1} d\rho \{h(v') - h(v)\}, \tag{2.3.20}$$

we have reduced the problem, thanks to Proposition 2.3.2, to the analysis of a Markov process which is an easier task.

2.4 Proof of the main theorems

Let be $\eta_\varepsilon = |\log\varepsilon|$. We rewrite the linear Boltzmann equation (2.3.19) in the following way

$$\big(\partial_t + v\cdot\nabla_x\big) h_\varepsilon(x,v,t) = \eta_\varepsilon \, \tilde{\mathrm{L}}_\varepsilon h_\varepsilon(x,v,t), \tag{2.4.1}$$

where $\tilde{L}_\varepsilon = L/\eta_\varepsilon$, namely

$$\tilde{\mathrm{L}}_\varepsilon f(v) = \mu|v| \frac{\varepsilon^{-2\alpha}}{|\log \varepsilon|} \int_{-1}^{1} d\rho \big[f(v') - f(v) \big]. \tag{2.4.2}$$

We will show that for $\eta_\varepsilon \to \infty$ we get a trivial result (Theorem 2.2.1), then we should look at the solution for times $\eta_\varepsilon t$, so that we obtain the diffusive scaling for space and time. Denoting by $\tilde{h}_\varepsilon := h_\varepsilon(x, v, \eta_\varepsilon t)$, where h_ε solves (2.4.1), $\tilde{h}_\varepsilon$ solves

$$\big(\partial_t + \eta_\varepsilon\, v \cdot \nabla_x\big) \tilde{h}_\varepsilon = \eta_\varepsilon^2\, \tilde{\mathrm{L}}_\varepsilon \tilde{h}_\varepsilon. \tag{2.4.3}$$

It is convenient to introduce the Cauchy problem associated to the following rescaled Landau equation:

$$\begin{cases} \big(\partial_t + \eta\, v \cdot \nabla_x\big) g_\eta(x, v, t) = \eta^2 \mathcal{L} g_\eta(x, v, t), \\ g_\eta(t = 0) = f_0. \end{cases} \tag{2.4.4}$$

where $\mathcal{L} = \frac{\mu}{2} \frac{1}{|v|} \Delta_{|v|}$ and $\eta = \eta_\varepsilon$.

We observe preliminarily that eq. (2.4.4) propagates the regularity of the derivatives with respect to the x variable and, due to the presence of $\mathcal{L}$, gains regularity with respect to the transverse component of the velocity. Indeed, for any fixed $|v|$, denoting by $S_{|v|}$ the circle of radius $|v|$, under the hypothesis of Theorem 2.2.1 on f_0, the solution $g_\eta : \mathbb{R}^2 \times S_{|v|} \to \mathbb{R}^+$ satisfies the bounds

$$|D_x^k g_\eta(x, v)| \le C, \quad |D_v^h g_\eta(x, v)| \le C \quad \forall k \le 2,\, h \ge 0, \tag{2.4.5}$$

$\forall t \in (0, T]$, where $C = C(f_0, T)$ and D_v is the derivative with respect to the transverse component of the velocity. In particular the solutions of (2.4.4) we are considering are classical.

Before analyzing the asymptotic behavior of the solution of (2.4.4) we first need a preliminary Lemma.

Lemma 2.4.1. *Let $\langle g_\eta \rangle$ be the average of g_η with respect to the invariant measure ν, namely $\langle g_\eta \rangle := \frac{1}{2\pi} \frac{1}{|v|} \int_{S_{|v|}} dv\, g_\eta(x, v)$. Under the hypothesis of Theorem 2.2.1*

(1) $\quad g_\eta - \langle g_\eta \rangle \underset{\eta \to \infty}{\longrightarrow} 0 \quad$ *in* $\quad L^\infty((0, T]; L^2(\mathbb{R}^2 \times S_{|v|}))$.

Moreover, setting $t_\eta = \frac{1}{\eta^\omega}$ for $\omega > 2$ then

(2) $\quad g_\eta(t_\eta) - \langle f_0 \rangle \underset{\eta \to \infty}{\longrightarrow} 0$ *in* $\quad L^2(\mathbb{R}^2 \times S_{|v|})$,

where $\langle f_0 \rangle = \frac{1}{2\pi} \frac{1}{|v|} \int_{S_{|v|}} dv\, f_0$.

Proof. Let $R_\eta = g_\eta - \langle g_\eta \rangle$. We have

$$\big(\partial_t + \eta\, v \cdot \nabla_x\big) R_\eta(x,v,t) = \eta^2 \mathcal{L} R_\eta(x,v,t) + \varphi, \tag{2.4.6}$$

where

$$\begin{aligned}\varphi &= -\big(\eta\, v \cdot \nabla_x \langle g_\eta \rangle + \partial_t \langle g_\eta \rangle\big) \\ &= \eta\Big(\frac{1}{2\pi}\frac{1}{|v|}\int_{S_{|v|}} v' \cdot \nabla_x g_\eta \, dv' - v \cdot \nabla_x \langle g_\eta \rangle\Big).\end{aligned} \tag{2.4.7}$$

We can estimate the last quantity by (2.4.5):

$$\sup_{t \le T} \|\varphi\|_{L^2} \le C\eta \|\nabla_x g_\eta\|_{L_2} \le C\eta.$$

Therefore by (2.4.6) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|R_\eta(t)\|_{L^2}^2 &= \eta^2 (R_\eta, \mathcal{L} R_\eta) + (R_\eta, \varphi) \\ &\le -\eta^2 \lambda \|R_\eta\|_{L^2}^2 + \|R_\eta\|_{L^2} \|\varphi\|_{L^2},\end{aligned}$$

where λ is the first positive eigenvalue of $\mathcal{L}$. Here we used that $R_\eta \perp 1$ in L^2. Hence

$$\begin{aligned}\|R_\eta(t)\|_{L^2} &\le e^{-\eta^2\lambda t}\|R_\eta(0)\|_{L^2} + \int_0^t ds\, e^{-\eta^2\lambda(t-s)}\|\varphi(s)\|_{L^2} \\ &\le e^{-\eta^2\lambda t}\|R_\eta(0)\|_{L^2} + \frac{C}{\eta}(1 - e^{-\eta^2\lambda t}),\end{aligned}$$

so that (1) is proven.

To prove (2) observe that, thanks to the fact $\mathcal{L}$ is negative, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g_\eta(t) - f_0\|_{L^2}^2 &\le -\eta(g_\eta - f_0, v \cdot \nabla_x f_0) + \eta^2 (g_\eta - f_0, \mathcal{L} f_0) \\ &\le \|g_\eta - f_0\|_{L^2}\big(\eta|v|\,\|\nabla_x f_0\| + \eta^2 \|\mathcal{L} f_0\|\big).\end{aligned}$$

Therefore

$$\|g_\eta(t_\eta) - f_0\|_{L^2} \le \frac{1}{\eta^\omega}\big(\eta|v|\,\|\nabla_x f_0\| + \eta^2\|\mathcal{L} f_0\|\big), \tag{2.4.8}$$

which vanishes as $\eta \to \infty$. Finally, recalling that $\langle f_0 \rangle = \frac{1}{2\pi}\frac{1}{|v|}\rho_0$, we have

$$\begin{aligned}\|g_\eta(t_\eta) - \langle f_0 \rangle\|_{L^2} &\le \sup_{t \in (0,T]} \|g_\eta - \langle g_\eta \rangle\|_{L^2} + \|\langle g_\eta(t_\eta)\rangle - \langle f_0 \rangle\|_{L^2} \\ &\le \sup_{t \in (0,T]} \|g_\eta - \langle g_\eta \rangle\|_{L^2} + c\|g_\eta(t_\eta) - f_0\|_{L^2}.\end{aligned}$$

By (2.4.8) and (1) we conclude the proof. □

Lemma 2.4.2. *Let g_η be the solution of* (2.4.4)*. Under the hypothesis of Theorem 2.2.1 for the initial datum f_0, for $\eta \to \infty$ g_η converges to the solution of the diffusion equation*

$$\begin{cases} \partial_t \varrho = D\Delta\varrho \\ \varrho(x,0) = \langle f_0 \rangle, \end{cases} \tag{2.4.9}$$

where $\langle f_0 \rangle = \frac{1}{2\pi}\frac{1}{|v|}\int_{S_{|v|}} dv\, f_0$ and

$$D = \frac{2}{\mu}|v| \int_{S_{|v|}} v \cdot \big(-\Delta_{|v|}^{-1} \big) v\, dv. \tag{2.4.10}$$

Convergence is in $L^\infty([0,T]; L^2(\mathbb{R}^2 \times S_{|v|}))$.

Proof. The proof of the above Lemma is rather straightforward (see e.g. [EP]).

Suppose for the moment that the initial datum depends only on the position variables, namely the initial datum has the form of a local equilibrium. We assume that g_η has the following form

$$g_\eta(x,v,t) = g^{(0)}(x,t) + \frac{1}{\eta} g^{(1)}(x,v,t) + \frac{1}{\eta^2} g^{(2)}(x,v,t) + \frac{1}{\eta} R_\eta,$$

where $g^{(i)}$, $i = 0,1,2$ are the first three coefficient of a Hilbert expansion in η, and R_η is the reminder. Comparing terms of the same order in η we obtain the following equations:

$$\begin{aligned}
&(i)\; v \cdot \nabla_x g^{(0)} = \frac{\mu}{2}\frac{1}{|v|}\,\Delta_{|v|} g^{(1)} \\
&(ii)\; \partial_t\, g^{(0)} + v \cdot \nabla_x g^{(1)} = \frac{\mu}{2}\frac{1}{|v|}\,\Delta_{|v|} g^{(2)} \\
&(iii)\; \big(\partial_t + \eta\, v \cdot \nabla_x\big) R_\eta = \eta^2 \frac{\mu}{2}\frac{1}{|v|}\,\Delta_{|v|} R_\eta - A_\eta(t),
\end{aligned}$$

with $A_\eta(t) = A_\eta(x,v,t) = \partial_t g^{(1)} + \frac{1}{\eta}\partial_t g^{(2)} + v \cdot \nabla_x g^{(2)}$. Since $v \cdot \nabla_x g^{(0)}$ is an odd function of v, the integral with respect to v of the left hand side of (i) vanishes. Then we can invert the operator $\Delta_{|v|}$ and set $g^{(1)} = \frac{2}{\mu}|v|\Delta_{|v|}^{-1} v \cdot \nabla_x g^{(0)}$, where $g^{(1)}$ is an odd function of the velocity. Now we integrate the second equation with respect to the velocity. By observing that $\int_{S_{|v|}} dv\, \Delta_{|v|} g^{(2)} = 0$, since $dv|_{S_{|v|}}$ is proportional the invariant measure, we obtain

$$\partial_t\, g^{(0)} + \frac{2}{\mu}|v| \int_{S_{|v|}} dv\, v \cdot \nabla_x \big(\Delta_{|v|}^{-1} v \cdot \nabla_x g^{(0)} \big) = 0.$$

We define the 2×2 matrix D as $D_{ij} = -\frac{2}{\mu}|v| \int_{S_{|v|}} v_i \Delta_{|v|}^{-1} v_j$ and we observe that $D_{ij} = 0$ for $i \neq j$ and $D_{11} = D_{22} = D$, where

$$D = \frac{2}{\mu}|v| \int_{S_{|v|}} dv\, v \cdot \big(- \Delta_{|v|}^{-1} \big) v.$$

Therefore

$$\partial_t\, g^{(0)} - D\Delta_x g^{(0)} = 0,$$

where $g^{(0)}$ satisfies the initial condition $g^{(0)}(x,0) = g(t=0)$. Moreover, the L^2-norm of $g^{(1)}$ is bounded. If we show that also the L^2-norm of $g^{(2)}$ and R_η are bounded, we deduce that g_η converges to $g^{(0)}$ for $\eta \to \infty$.

From equation (ii) and the diffusion equation for $g^{(0)}$ we derive that the integral with respect to v of the left hand side of (ii) vanishes. Therefore we can invert the operator $\Delta_{|v|}$ and obtain

$$\begin{aligned} g^{(2)} =& \frac{2}{\mu}|v|\, \Delta_{|v|}^{-1} \Big(\partial_t g^{(0)} + \frac{2}{\mu}|v|\, v \cdot \nabla_x (\Delta_{|v|}^{-1} v \cdot \nabla_x g^{(0)}) \Big) \\ =& \frac{2}{\mu}|v| \sum_{i,j} \partial_{x_i}\partial_{x_j} g^{(0)}\, \Delta_{|v|}^{-1} \Big[\frac{2}{\mu}\Delta_{|v|}^{-1} v_i v_j + D_{ij} \Big]. \end{aligned}$$

Therefore the L_2-norm of $g^{(2)}$ is bounded.

We derive from equation (iii)

$$\frac{1}{2}\partial_t \|R_\eta\|^2 = -\eta^2 \left(R_\eta,\, -\Delta_{|v|} R_\eta \right) - \big(R_\eta,\, A_\eta(t) \big),$$

where $(\cdot,\,\cdot)$ denotes the scalar product in L^2. Using positivity of $-\Delta_{|v|}$ and Cauchy-Schwartz we deduce $\partial_t \|R_\eta\| \leq \|A_\eta\|$. Recall the explicit expression for A_η, namely $A_\eta = \partial_t g^{(1)} + \frac{1}{\eta}\partial_t g^{(2)} + v \cdot \nabla_x g^{(2)}$. By direct computation

$$\partial_t g^{(1)} = \frac{2}{\mu}|v|\Delta_{|v|}^{-1} \Big[v \cdot \nabla_x \big(D\Delta_x g^{(0)} \big) \Big],$$

from which we deduce that the L^2-norm of $\partial_t g^{(1)}$ is bounded. We deduce similar estimates for $\partial_t g^{(2)}$ and $v \cdot \nabla_x g^{(2)}$, then $\|A_\eta\|$ is uniformly bounded in $[0,T]$ and $\|R_\eta\| \leq CT$.

To complete the proof we consider more general initial data f_0 depending also on the velocity variable. Let $A := L - \eta v \cdot \nabla_x$. We compare g_η with $\bar{g}_\eta$, the solution (2.4.4) with initial datum $\langle f_0 \rangle$. By the same argument as in Lemma 2.4.1, item (2), we have that $\forall t \geq t_\eta$

$$\|\bar{g}_\eta(t - t_\eta) - \bar{g}_\eta(t)\|_{L^2} \leq \frac{C}{\eta^{\omega - 2}},$$

where C depends on the L^2-norm of $\langle f_0 \rangle$ and $\nabla \langle f_0 \rangle$. Since $g_\eta(t) = e^{At} f_0 = e^{A(t - t_\eta)} g_\eta(t_\eta)$ and $\bar{g}_\eta(t - t_\eta) = e^{A(t - t_\eta)} \langle f_0 \rangle$ we derive

$$\|g_\eta(t) - \bar{g}_\eta(t - t_\omega)\|_{L^2} \leq C\|g(t_\eta) - \langle f_0 \rangle\|_{L^2}.$$

Thus, by Lemma 2.4.1, item (2), we obtain that $g_\eta(t)$ and $\bar{g}_\eta(t)$ have the same asymptotics and this concludes the proof of Lemma (2.4.2). □

Proposition 2.4.3. *Let f_0 be an initial datum for $\tilde{h}_\varepsilon$ solution of (2.4.3). Under the hypothesis of Theorem 2.2.1 $\tilde{h}_\varepsilon$ converges to ϱ as $\varepsilon \to 0$, where $\varrho : \mathbb{R}^2 \times [0,T] \to \mathbb{R}_+$ is the solution of the diffusion equation*

$$\begin{cases} \partial_t \varrho = D\Delta\varrho \\ \varrho(x,0) = \langle f_0 \rangle, \end{cases} , \tag{2.4.11}$$

with $\langle f_0 \rangle = \frac{1}{2\pi}\frac{1}{|v|}\int_{S_{|v|}} dv\, f_0$. The diffusion coefficient is D given by the Green-Kubo formula. Convergence is in $L^2(\mathbb{R}^2 \times S_{|v|})$ uniformly in $t \in (0,T]$.

Proof. Let g_{η_ε} be solution of (2.4.4) with $\eta = \eta_\varepsilon := |\log\varepsilon|$ and initial condition f_0. We look at the evolution of $\tilde{h}_\varepsilon - g_{\eta_\varepsilon}$, namely

$$\big(\partial_t + \eta_\varepsilon\, v \cdot \nabla_x\big)\big(\tilde{h}_\varepsilon - g_{\eta_\varepsilon}\big) = \eta_\varepsilon^2\Big(\tilde{\mathrm{L}}_\varepsilon \tilde{h}_\varepsilon - \mathcal{L} g_{\eta_\varepsilon}\Big),$$

where $\mathcal{L} := \frac{\mu}{2}\frac{1}{|v|}\Delta_{|v|}$. Then we obtain

$$\begin{aligned} \frac{1}{2}\,\partial_t \|\tilde{h}_\varepsilon - g_{\eta_\varepsilon}\|^2 =& -\eta_\varepsilon^2\big(\tilde{h}_\varepsilon - g_{\eta_\varepsilon},\, -\tilde{\mathrm{L}}_\varepsilon\big[\tilde{h}_\varepsilon - g_{\eta_\varepsilon}\big]\big) \\ &+ \eta_\varepsilon^2\big(\tilde{h}_\varepsilon - g_{\eta_\varepsilon},\, \big[\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big] g_{\eta_\varepsilon}\big), \end{aligned}$$

from which, using positivity of $-\tilde{\mathrm{L}}_\varepsilon$ and Cauchy-Schwartz,

$$\frac{1}{2}\,\partial_t \|\tilde{h}_\varepsilon - g_{\eta_\varepsilon}\| \le \eta_\varepsilon^2 \big\|\big(\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big) g_{\eta_\varepsilon}\big\|.$$

Recalling that

$$\tilde{\mathrm{L}}_\varepsilon g_{\eta_\varepsilon} = \mu |v| \frac{\varepsilon^{-2\alpha}}{|\log\varepsilon|}\int_{-1}^{1} d\rho\, \big[g_{\eta_\varepsilon}(x,v',t) - g_{\eta_\varepsilon}(x,v,t)\big],$$

we set

$$\begin{aligned} & g_{\eta_\varepsilon}(v') - g_{\eta_\varepsilon}(v) \\ &= (v'-v)\cdot \nabla_{|S_{|v|}} g_{\eta_\varepsilon}(v) \\ &\quad + \frac{1}{2}(v'-v)\otimes(v'-v)\nabla_{|S_{|v|}}\nabla_{|S_{|v|}} g_{\eta_\varepsilon}(v) \\ &\quad + \frac{1}{6}(v'-v)\otimes(v'-v)\otimes(v'-v)\nabla_{|S_{|v|}}\nabla_{|S_{|v|}}\nabla_{|S_{|v|}} g_{\eta_\varepsilon}(v) + R_{\eta_\varepsilon}, \end{aligned}$$

with $R_{\eta_\varepsilon} = \mathcal{O}(|v-v'|^4)$. Integrating with respect to v and using symmetry arguments we obtain

$$\tilde{\mathrm{L}}_\varepsilon g_{\eta_\varepsilon} = \mu|v|\frac{\varepsilon^{-2\alpha}}{|\log\varepsilon|}\Big\{\frac{1}{2}\Delta_{|v|} g_{\eta_\varepsilon}\int_{-1}^{1} d\rho\, |v'-v|^2 + \int_{-1}^{1} d\rho\, R_{\eta_\varepsilon}\Big\}.$$

Observe that $|v' - v|^2 = 4\sin^2\frac{\theta(\rho)}{2}$, then by direct computation (see Appendix)

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{-2\alpha}}{|\log\varepsilon|}\int_{-1}^{1} d\rho\,|v'-v|^2 = 2\frac{\alpha}{|v|^4}$$

and

$$\frac{\varepsilon^{-2\alpha}}{|\log\varepsilon|}\int_{-1}^{1} d\rho\,|v-v'|^4 = \varepsilon^{\alpha}|\log\varepsilon|^{\beta}, \qquad -1<\beta<\frac{5}{2}\alpha-1.$$

Therefore $\left\|\big(\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big)g_{\eta_\varepsilon}\right\| \le \varepsilon^{\alpha}|\log\varepsilon|^{\beta}\,\|\Delta^2_{|S_{|v|}}g_{\eta_\varepsilon}\| \le \varepsilon^{\alpha}|\log\varepsilon|^{\beta}\,C$, which vanishes for $\varepsilon\to 0$.

□

In order to complete the proof of Theorem 2.2.1, namely equation (2.2.1), we need to show that $f_\varepsilon(\eta_\varepsilon t)$ converges to $\tilde{h}_\varepsilon(t)$ in $L^2(\mathbb{R}^2\times\mathbb{R}^2)$, for every $t\in[0,T]$. By Proposition 2.3.2 and Remark 2.3.3 we have that $\bar{f}_\varepsilon(\eta_\varepsilon t)$ defined in (2.3.13) converges to $\bar{h}_\varepsilon(\eta_\varepsilon t)$, (2.3.14), in $L^1(\mathbb{R}^2\times\mathbb{R}^2)$, for every $t\in[0,T]$. Moreover, using (2.3.18) and the fact that the initial datum has compact support, we have that $\bar{h}_\varepsilon(\eta_\varepsilon t)$ converges to $\tilde{h}_\varepsilon(t)$ in $L^1(\mathbb{R}^2\times\mathbb{R}^2)$, for every $t\in[0,T]$. Under hypotheses of Theorem 2.2.1, $\bar{f}_\varepsilon(\eta_\varepsilon t)$ and $\tilde{h}_\varepsilon(t)$ are uniformly bounded for every $t\in[0,T]$, therefore convergence in L^1 implies convergence in L^2. Since $\bar{f}_\varepsilon\le f_\varepsilon$ and using the fact that at $t=0$ the equality holds and the linear Boltzmann equation 2.4.3 preserves the total mass, then also $f_\varepsilon(\eta_\varepsilon t)$ converges to $\tilde{h}_\varepsilon(t)$ in $L^2(\mathbb{R}^2\times S_{|v|})$, for every $t\in[0,T]$.

Now we go back to equation (2.4.1). Using the same strategy of the proof of Proposition 2.4.3 we can replace $\tilde{L}_\varepsilon$ with $\mathcal{L}$, and we denote $\tilde{g}_\eta$ the solution of

$$\big(\partial_t + v\cdot\nabla_x\big)\tilde{g}_\eta = \eta\,\mathcal{L}\tilde{g}_\eta,$$

with initial datum f_0. By the same arguments as in Lemma 2.4.1, item (i), one can prove that for $\eta\to\infty$ $\tilde{g}_\eta\to\langle\tilde{g}_\eta\rangle$ and $\nabla_x\tilde{g}_\eta\to\nabla_x\langle\tilde{g}_\eta\rangle$. We observe that

$$\partial_t\langle\tilde{g}_\eta\rangle + \nabla_x\int dv\,\big(\tilde{g}_\eta - \langle\tilde{g}_\eta\rangle\big)v = 0,$$

therefore $\langle\tilde{g}_\eta\rangle$ converges to $\langle\tilde{f}_0\rangle$ as $\eta\to\infty$, which concludes the proof of item 1).

The proof of Theorem 2.2.2 is included in the proof of Proposition 2.4.3.

2.5 The control of the pathological sets

In this section we prove Proposition 2.3.2.
Clearly

$$1-\chi_1(1-\chi_{ov})(1-\chi_{rec})(1-\chi_{int}) \le (1-\chi_1)+\chi_{ov}+\chi_{rec}+\chi_{int} \quad (2.5.1)$$

and we estimate separately all the events in the right hand side of (2.5.1).

We denote by $\xi_\varepsilon(s), \eta_\varepsilon(s)$ the backward Markov process defined, for $s \in (-t, 0)$, in Section 2 and we set

$$\begin{aligned} \mathbb{E}_{x,v}(u) = e^{-2|v|\varepsilon^{-2\alpha}t} \sum_{Q\geq 0} (2|v|\mu_\varepsilon)^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1 \\ \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \, u(\xi_\varepsilon, \eta_\varepsilon), \end{aligned} \tag{2.5.2}$$

for any measurable function u of the process $(\xi_\varepsilon, \eta_\varepsilon)$. We have

$$\begin{aligned} &\mathbb{E}_{x,v}((1-\chi_1) f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t))) \\ &\leq \frac{2\varepsilon}{|v|} e^{-2|v|\varepsilon^{-2\alpha}t} \|f_0\|_{L^\infty} \sum_{Q>0} \frac{(2|v|\varepsilon^{-2\alpha})^Q}{(Q-1)!} t^{Q-1} \\ &\leq 4\|f_0\|_{L^\infty} \varepsilon^{1-2\alpha} \leq C\varepsilon^\gamma, \end{aligned} \tag{2.5.3}$$

for $\gamma > 0$, $\alpha < 1/2$ and ε sufficiently small.
Here and in the sequel t is allowed to behave as $c|\log(\varepsilon)|$.

Estimate (2.5.3) is obvious. Indeed if $\chi_1 = 0$ the last or the first collision must satisfy either $|t - t_Q| \leq 2\varepsilon/|v|$ or $t_1 \leq 2\varepsilon/|v|$. Hence (2.5.3) follows easily.
A similar argument can be used to estimate χ_{ov}. Indeed if $\chi_{ov} = 1$ it must be $t_{i+1} - t_i \leq 2\varepsilon/|v|$ for some $i = 1, \dots, (Q-1)$. Therefore proceeding as before

$$\begin{aligned} &\mathbb{E}_{x,v}(\chi_{ov} f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t))) \\ &\leq \frac{2\varepsilon}{|v|} e^{-2|v|\varepsilon^{-2\alpha}t} \|f_0\|_{L^\infty} \sum_{Q>1} (Q-1) \frac{(2|v|\varepsilon^{-2\alpha})^Q}{(Q-1)!} t^{Q-1} \\ &\leq \frac{2\varepsilon}{|v|} \|f_0\|_{L^\infty} t (2|v|\varepsilon^{-2\alpha})^2 \leq C|v|\varepsilon^\gamma t, \end{aligned} \tag{2.5.4}$$

for some $\gamma > 0$, $\alpha < 1/4$ and ε sufficiently small.

Next we pass to the control of the recollision event. We proceed similarly as in [DP] and in [DR]. Let t_i the first time the light particle hits the i-th scattering (backward trajectory), v_i^- the incoming velocity, v_i^+ the outgoing velocity and t_i^+ the exit time. Moreover we fix the axis in such a way that v_i^- is parallel to the x axis (see Figure 2.7). We have

$$\chi_{rec} \leq \sum_{i=1}^{Q} \sum_{j>i} \chi_{rec}^{i,j}, \tag{2.5.5}$$

where $\chi_{rec}^{i,j} = 1$ if and only if b_i (constructed via the sequence $t_1, \rho_1, \dots, t_i, \rho_i$) is recollided in the time interval $(-t_{j+1}, -t_j^-)$.

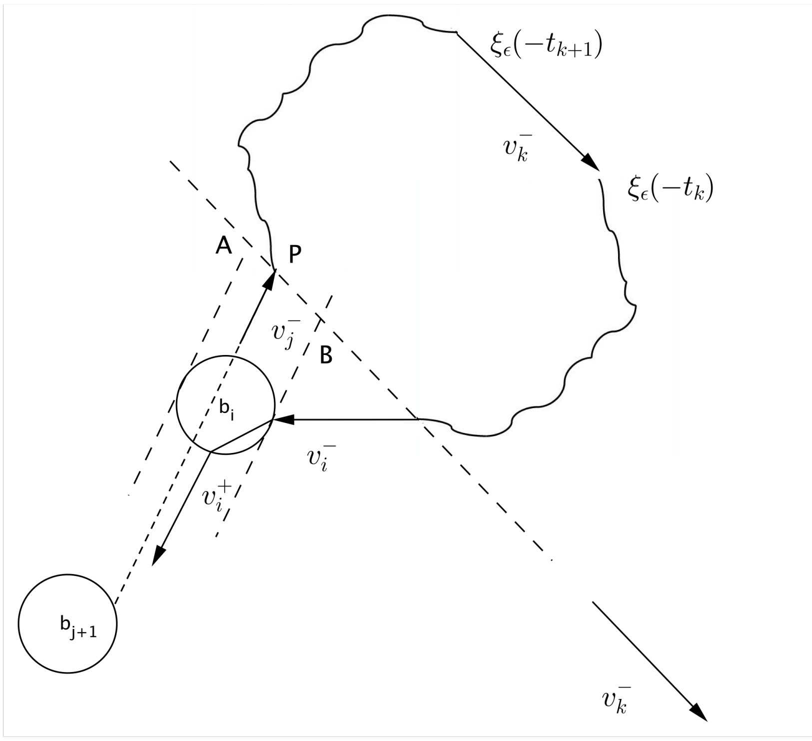

Figure 2.7: Backward recollision

Note that, since $|\theta_i| \leq C\varepsilon^\alpha$, where θ_i is the i-th scattering angle, in order to have a recollision it must be an intermediate velocity v_k, $k = i+1, \ldots, j-1$ such that

$$|v_k^- \cdot v_j^-| \leq C\varepsilon^\alpha |v|^2, \tag{2.5.6}$$

namely v_k^- is almost orthogonal to v_j^- (see Figure 2.7). Then

$$\chi_{rec} \leq \sum_{i=1}^{Q}\sum_{j=1}^{Q}\sum_{k=i+1}^{j-1} \chi_{rec}^{i,j,k}, \tag{2.5.7}$$

where $\chi_{rec}^{i,j,k} = 1$ if and only if $\chi_{rec}^{i,j} = 1$ and (2.5.6) is fulfilled.

Fix now all the parameters $\rho_1, \ldots, \rho_Q$, $t_1, \ldots, t_Q$ but t_{k+1} and perform such a time integration. The two branches of the trajectory l_1, l_2 are rigid so that, if the recollision happen the time integration with respect to t_{k+1} is restricted to a time interval proportional to AB. More precisely it is bounded by

$$\frac{2\varepsilon}{|v|\cos C\varepsilon^\alpha} \leq \frac{4\varepsilon}{|v|}.$$

Performing all the other integrations and summing over i, j, k we obtain

$$\begin{aligned} &\mathbb{E}_{x,v}(\chi_{rec} f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t))) \\ &\leq \frac{4\varepsilon}{|v|} e^{-2|v|\varepsilon^{-2\alpha}t} \|f_0\|_{L^\infty} \sum_{Q\geq 3} (Q-1)(Q-2)(Q-3) \frac{(2|v|\varepsilon^{-2\alpha})^Q}{(Q-1)!} t^{Q-1} \\ &\leq C|v|^3 t^3 \varepsilon^{1-8\alpha} \leq C|v|^3 \varepsilon^\gamma t^3, \end{aligned} \tag{2.5.8}$$

for some $\gamma > 0$, $\alpha < 1/8$ and ε sufficiently small.

We finally estimate the event χ_{int}. To do this we fix a sequence of parameters $\rho_1, \ldots, \rho_Q$, $t_1, \ldots, t_Q$. For instance consider the case in Figure 2.8 in which we exhibit an unphysical trajectory.

Consider the integral

$$I = \int_{B(0,M)} f_0(\xi(-t), \eta(-t)) \chi_{int} \, dx dv. \tag{2.5.9}$$

Here $\chi_{int} = 1$ for those values of x, v for which an interference takes place and

$$B(0, M) := \{(x, v) \in \mathbb{R}^2 \times \mathbb{R}^2; \text{ s.t. } |x|^2 + |v|^2 < M\}.$$

By the Liouville Theorem we can integrate over the variables $\big(\xi_\varepsilon(-t), \eta_\varepsilon(-t)\big) = (x_0, v_0)$ as independent variables

$$I = \int_{B_t(0,M)} f_0(x_0, v_0) \chi_{rec} \, dx_0 dv_0, \tag{2.5.10}$$

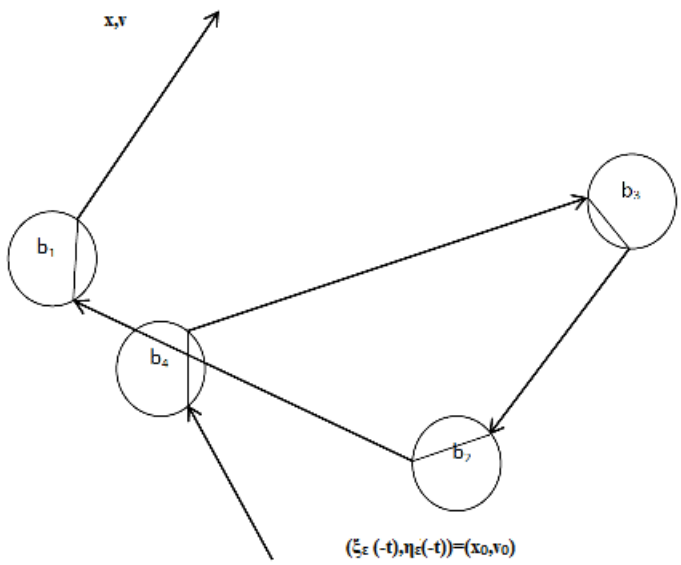

Figure 2.8:

where

$$B_t(0,M) = \left\{ \left(\xi_\varepsilon(-t), \eta_\varepsilon(-t)\right) \text{ s.t. } (x,v) \in B(0,M) \right\}.$$

Note that

$$\chi_{int}(x,v) = \chi_{rec}(x_0,v_0),$$

since a backward interference is a forward recollision. Clearly

$$B_t(0,M) \subset B(0,M(1+t)), \tag{2.5.11}$$

where $B(0,M(1+t))$ is the ball of radius $M(1+t)$ in $\mathbb{R}^4$.
Thus

$$I \leq \int_{B(0,M(1+t))} f_0(x_0,v_0)\chi_{rec}\, dx_0 dv_0. \tag{2.5.12}$$

Therefore, by using estimate (2.5.8) and (2.5.12)

$$\begin{aligned} &\int_{B(0,M)} \mathbb{E}_{x,v}(\chi_{int} f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t)))\, dx dv \\ &\leq C\varepsilon^\gamma M^7 (1+t)^7. \end{aligned} \tag{2.5.13}$$

This concludes the proof of Proposition 2.3.2.

2.6 Concluding Remarks

The diffusive limit analyzed in the present paper is suggested by the divergence of B for the particular choice of the potential we are considering. However the same techniques could work in presence of a smooth, radial, short-range potential ϕ. In this case we recover the previous logarithmic divergence (see (2.1.2)) by increasing suitably the density of the scatterers. We notice that we can implement this program thanks to the explicit estimates of the set of bad configurations, which allow stronger divergence in time and density.

Theorem 2.6.1. *Under the same hypothesis of Theorem 2.2.1, assume $\phi \in C^2([0,1])$. Scale the variables, the density and the potential according to*

$$\begin{cases} x \to \varepsilon x \\ t \to \varepsilon^{\lambda}\varepsilon t \\ \mu_\varepsilon = \varepsilon^{-(2\alpha+\lambda+1)}\mu \\ \phi \to \varepsilon^{\alpha}\phi. \end{cases} \tag{2.6.1}$$

Then, for $t > 0$ and $\varepsilon \to 0$, there exists $\lambda_0 = \lambda(\alpha)$ s.t. for $\lambda < \lambda(\alpha)$

$$f_\varepsilon(x, v, t) \to \rho(x, t)$$

solution of the heat equation

$$\begin{cases} \partial_t \varrho = D\Delta\varrho \\ \varrho(x, 0) = \langle f_0 \rangle, \end{cases} \tag{2.6.2}$$

with D given by the Green-Kubo formula

$$D = \frac{2}{\mu}|v| \int_{S_{|v|}} v \cdot \big(-\Delta^{-1}_{|v|}\big) v \, dv. \tag{2.6.3}$$

The convergence is in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$.

The significance and the proof of the above theorem is clear. The kinetic regime describes the system for kinetic times $O(1)$ (i.e. $\lambda = 0$). One can go further to diffusive times provided that $\lambda > 0$ is not too large. Indeed the distribution function f_ε "almost" solves

$$\begin{aligned} \big(\varepsilon^{\lambda}\partial_t + v \cdot \nabla_x\big) f_\varepsilon &\approx \varepsilon^{-2\alpha-\lambda} \mathrm{L}_\varepsilon f_\varepsilon \\ &\approx \varepsilon^{-\lambda} c \Delta_{|v|} f_\varepsilon, \end{aligned} \tag{2.6.4}$$

for which the arguments of Section 3 do apply. In other words there is a scale of time for which the system diffuses. However such times are not so large to prevent the Markov property. Obviously the diffusion coefficient is computed in terms of the limiting Markov process. We can give an estimate, certainly not optimal, of the coefficient λ appearing in (2.6.1). Estimating recollisions and interferences as in Section 4, setting $\gamma = 1 - 8(\alpha + \frac{\lambda}{2})$, the condition on λ is (see (2.5.13))

$$\gamma - 7\lambda > 0 \quad \text{i.e.} \quad \lambda < \frac{1 - 8\alpha}{11}.$$

Although the scaling we are considering in Theorem 2.6.1 is quite particular, the aim is the same as in [LE] where the same problem has been approached for the weak-coupling limit ($\alpha = \frac{1}{2}$) of a quantum system.

Recently we were aware of a result concerning the diffusion limit of a test particle of a hard-core system at thermal equilibrium [BGS-R]. Also in this case the quantitative control of the pathological trajectories allows to reach larger times in which a diffusive regime is outlined.

Acknowledgments.
We are indebted to S. Simonella and H. Spohn for illuminating discussions.

2.7 Appendix (on the scattering problem associated to a circular potential barrier)

The potential energy for a finite potential barrier is given by

$$\phi(r) = \begin{cases} u_0 & \text{if } r \leq 1 \\ 0 & \text{if } r > 1. \end{cases} \tag{A.1}$$

The light particle, of unitary mass, moves in a straight line with energy $E = \frac{1}{2}v^2 > u_0$, where $u_0 > 0$. Let ρ be the impact parameter. For small impact parameters the particle will pass through the barrier, for large ones the particle will be reflected. Inside the barrier the velocity is a constant $v = \bar{v}$ ($\bar{v} < v$). The complete trajectory of the light particle which passes through the barrier consists of three straight lines and is symmetrical about a radial line perpendicular to the interior path. For a general reference to the scattering problem, see [LL], chapter 4.

Let α be the angle of incidence (the inside angle between the trajectory and a radial line to the point of contact with the barrier at $r = 1$) and β the angle of refraction (the corresponding external angle). We assume that the radius of the circle is $r = 1$. According to the geometry of the problem α and β are such that

$$\sin\beta = \frac{v}{\bar{v}} \sin\alpha,$$

where $\sin\alpha = \rho$.

The angle of deflection is $\theta = 2(\beta - \alpha)$. Thanks to the energy and angular momentum conservation the expression for the refractive index becomes

$$n = \frac{\sin\alpha}{\sin\beta} = \frac{\bar{v}}{v} = \sqrt{1 - \frac{2u_0}{v^2}} \tag{A.2}$$

and so we have a scattering angle defined in the following way:

$$\theta(\rho) = \begin{cases} 2\left(\arcsin\left(\frac{\rho}{n}\right) - \arcsin(\rho)\right) & \text{if } \rho \leq n \\ 2\arccos(\rho) & \text{if } \rho > n. \end{cases} \tag{A.3}$$

In the first case the particle passes through the barrier (for $\rho \leq n$), and in the second one the particle is reflected (for $\rho > n$). The maximum scattering angle $\theta_{\max} = 2\arccos(n)$ is the angle at which the particle scatters tangentially to the barrier. The differential scattering cross section

$$\Psi(\theta) = \left|\frac{\partial\rho}{\partial\theta}\right|$$

is then:

$$\Psi(\theta) = \begin{cases} \frac{n[\cos(\theta/2)-n][1-n\cos(\theta/2)]}{(1+n^2-2n\cos(\theta/2))^{3/2}} & \text{if } \theta \leq 2\arccos(n) \\ [1-\cos^2(\theta/2)]^{1/2} & \text{otherwise .} \end{cases} \tag{A.4}$$

Scaling now the potential as $\phi(r) \to \varepsilon^\alpha \phi(r)$, the previous formulas still hold. Thus, according to this scaling, the refractive index becomes

$$n_\varepsilon = \sqrt{1 - \frac{2\varepsilon^\alpha u_0}{v^2}} \tag{A.5}$$

to replace into (A.4). The scattering angle (A.3) reads now

$$\theta_\varepsilon(\rho) = \begin{cases} 2\left(\arcsin\left(\frac{\rho}{n_\varepsilon}\right) - \arcsin(\rho)\right) & \text{if } \rho \leq n_\varepsilon \\ 2\arccos(\rho) & \text{if } \rho > n_\varepsilon. \end{cases} \tag{A.6}$$

2.8 Appendix (on the diffusion coefficient)

In this section we show that the diffusion coefficient is divergent for the circular potential barrier (A.1). At this level we assume that $u_0 = 1$ to simplify the following expressions.

We need to compute

$$\tilde{B} := \lim_{\varepsilon\to 0} \frac{\mu\varepsilon^{-2\alpha}}{2}|v| \int_{-1}^{1} \theta_\varepsilon^2(\rho)\, d\rho. \tag{B.1}$$

Thanks to the symmetry for the scattering problem

$$\varepsilon^{-2\alpha}\int_{-1}^{1}\theta_\varepsilon^2(\rho)\,d\rho = 2\varepsilon^{-2\alpha}\int_0^1\theta_\varepsilon^2(\rho)\,d\rho. \tag{B.2}$$

According to (A.3):

$$2\varepsilon^{-2\alpha}\int_0^1\theta_\varepsilon^2(\rho)\,d\rho = 2\varepsilon^{-2\alpha}\left(\int_0^{n_\varepsilon}\theta_\varepsilon^2(\rho)\,d\rho + \int_{n_\varepsilon}^1\theta_\varepsilon^2(\rho)\,d\rho\right). \tag{B.3}$$

Our aim is to perform a Taylor expansion of the first branch of $\theta_\varepsilon(\rho)$ for $\rho \geq 0$,
$\rho/n_\varepsilon < (1-\delta)$, with $\delta > 0$. We have

$$\arcsin\left(\rho/n_\varepsilon\right) = \arcsin(\rho) + \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\rho}{n_\varepsilon}-\rho\right) + R_1\left(\rho/n_\varepsilon\right),$$

where

$$R_1\left(\rho/n_\varepsilon\right) = \frac{\bar\rho}{2(1-\bar\rho^2)^{\frac{3}{2}}}\left(\frac{\rho}{n_\varepsilon}-\rho\right)^2 \quad \rho < \bar\rho < \frac{\rho}{n_\varepsilon}. \tag{B.4}$$

Then, looking at the first integral in the r.h.s of (B.3), we have to split it as

$$\varepsilon^{-2\alpha}\int_0^{n_\varepsilon}\theta_\varepsilon^2(\rho)\,d\rho = \underbrace{\varepsilon^{-2\alpha}\int_0^{n_\varepsilon(1-\delta)}\theta_\varepsilon^2(\rho)\,d\rho}_{A} + \underbrace{\varepsilon^{-2\alpha}\int_{n_\varepsilon(1-\delta)}^{n_\varepsilon}\theta_\varepsilon^2(\rho)\,d\rho}_{B}.$$

Thus

$$\begin{aligned}
A &= \varepsilon^{-2\alpha}\int_0^{n_\varepsilon(1-\delta)}\left[2(\arcsin\left(\rho/n_\varepsilon\right) - \arcsin(\rho))\right]^2 d\rho \\
&\leq 4\varepsilon^{-2\alpha}\left[\int_0^{n_\varepsilon(1-\delta)}\frac{(1-n_\varepsilon)^2}{n_\varepsilon^2}\frac{\rho^2}{(1-\rho^2)}\,d\rho + \int_0^{n_\varepsilon(1-\delta)}R_1\left(\rho/n_\varepsilon\right)^2\,d\rho\right] + \\
&4\varepsilon^{-2\alpha}\left[\left(\int_0^{n_\varepsilon(1-\delta)}R_1\left(\rho/n_\varepsilon\right)^2\right)^{\frac{1}{2}}\left(\int_0^{n_\varepsilon(1-\delta)}\frac{(1-n_\varepsilon)^2}{n_\varepsilon^2}\frac{\rho^2}{(1-\rho^2)}\,d\rho\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{B.5}$$

It is sufficient to compute the first two integrals . Let A_1 and A_2 be the first and the second integrals respectively. We have

$$\begin{aligned}
A_1 &= \varepsilon^{-2\alpha}\frac{(1-n_\varepsilon)^2}{n_\varepsilon^2}\int_0^{n_\varepsilon(1-\delta)}\frac{\rho^2}{1-\rho^2}\,d\rho \\
&= -\frac{\varepsilon^{-2\alpha}}{2}\frac{(1-n_\varepsilon)^2}{n_\varepsilon^2}[2n_\varepsilon(1-\delta) + \log(1-n_\varepsilon(1-\delta)) - \log(1+n_\varepsilon(1-\delta))].
\end{aligned} \tag{B.6}$$

Using that $n_\varepsilon = 1 - \frac{\varepsilon^\alpha}{|v|^2} + o(\varepsilon^{2\alpha})$, from (B.6) it is clear that

$$A_1 \simeq -\frac{\varepsilon^{-2\alpha}}{2}(1-n_\varepsilon)^2(\log(1-n_\varepsilon(1-\delta))) = -\frac{\varepsilon^{-2\alpha}}{2}\left(\frac{\varepsilon^{2\alpha}}{|v|^4}\right)\log(\varepsilon^\alpha(1-\delta)+\delta).$$

A straightforward computation shows that the right hand side of the previous expression is

$$\begin{aligned}
&-\frac{\varepsilon^{-2\alpha}}{2}\frac{\varepsilon^{2\alpha}}{|v|^4}\Big(\log(\varepsilon^\alpha) + \log(1-\delta+\frac{\delta}{\varepsilon^\alpha})\Big)\\
&= -\frac{1}{2|v|^4}\Big(\log(\varepsilon^\alpha) + \delta(1-\frac{1}{\varepsilon^\alpha})\Big)\\
&= \frac{\alpha}{2|v|^4}|\log(\varepsilon)|\Big(1+\frac{\delta(1-\frac{1}{\varepsilon^\alpha})}{|\log(\varepsilon^\alpha)|}\Big).
\end{aligned} \tag{B.7}$$

Choosing $\delta = \frac{\varepsilon^\alpha}{|\log\varepsilon|^\gamma}$ with $\gamma \in (0,\alpha/2)$, it follows $\delta/\varepsilon^\alpha \underset{\varepsilon\to 0}{\longrightarrow} 0$.

In order to compute A_2, we need the following estimate for the remainder term

$$|R_1(\rho/n_\varepsilon)| \le \frac{1}{2}\frac{\rho}{n_\varepsilon}\frac{1}{(1-\frac{\rho^2}{n_\varepsilon^2})^{\frac{3}{2}}}\left(\frac{\rho}{n_\varepsilon}-\rho\right)^2. \tag{B.8}$$

Then

$$\begin{aligned}
A_2 &\le \frac{\varepsilon^{-2\alpha}}{4}\int_0^{n_\varepsilon(1-\delta)}\frac{\rho^2}{n_\varepsilon^2}\frac{1}{(1-\frac{\rho^2}{n_\varepsilon^2})^3}\left(\frac{\rho}{n_\varepsilon}-\rho\right)^4 d\rho\\
&\underset{u=\frac{\rho}{n_\varepsilon}}{=} \varepsilon^{-2\alpha}n_\varepsilon\int_0^{1-\delta}\frac{u^2}{2(1-u^2)^3}u^4(1-n_\varepsilon)^4\,du\\
&\underset{v=1-u}{=} \frac{\varepsilon^{-2\alpha}n_\varepsilon}{2}\int_\delta^1\frac{(1-v)^6}{v^3}(1-n_\varepsilon)^4\,dv \simeq \frac{\varepsilon^{-2\alpha}n_\varepsilon}{2}\frac{(1-n_\varepsilon)^4}{\delta^2}.
\end{aligned} \tag{B.9}$$

Also in this case, the only significant contribution is given by

$$\frac{\varepsilon^{-2\alpha}(1-n_\varepsilon)^4}{\delta^2} \simeq \frac{\varepsilon^{-2\alpha}\varepsilon^{4\alpha}}{\delta^2} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

again for $\delta = \frac{\varepsilon^\alpha}{|\log\varepsilon|^\gamma}$ with $\gamma \in (0,\alpha/2)$. This shows that

$$A = A_1(1+\mathcal{O}(\varepsilon)).$$

Now we compute B in (B.3), namely

$$\begin{aligned}
B &= \varepsilon^{-2\alpha}\int_{n_\varepsilon(1-\delta)}^{n_\varepsilon}[2(\arcsin(\rho/n_\varepsilon) - \arcsin(\rho))]^2\,d\rho\\
&= \varepsilon^{-2\alpha}\int_{n_\varepsilon(1-\delta)}^{n_\varepsilon}\left(\int_\rho^{\frac{\rho}{n_\varepsilon}}dx\frac{1}{\sqrt{1-x^2}}\right)^2 d\rho.
\end{aligned} \tag{B.10}$$

Since

$$\begin{aligned}
\int_{\rho}^{\frac{\rho}{n_\varepsilon}} dx \frac{1}{\sqrt{1-x^2}} &= \int_{\rho}^{\frac{\rho}{n_\varepsilon}} dx \frac{1}{\sqrt{(1-x)}\sqrt{(1+x)}} \\
&\le \frac{1}{1+\rho} \int_{\rho}^{\frac{\rho}{n_\varepsilon}} dx \frac{1}{\sqrt{(1-x)}} \underset{u=1-x}{=} \int_{1-\frac{\rho}{n_\varepsilon}}^{1-\rho} \frac{1}{\sqrt{u}} \\
&= \frac{1}{\sqrt{(1-\rho)}} \left(\sqrt{(1-\rho)} - \sqrt{(1-\rho/n_\varepsilon)} \right),
\end{aligned}$$

in (B.10) we have

$$\begin{aligned}
B &= \varepsilon^{-2\alpha} \int_{n_\varepsilon(1-\delta)}^{n_\varepsilon} \frac{1}{(1-\rho)} \left(\sqrt{(1-\rho)} - \sqrt{(1-\rho/n_\varepsilon)} \right)^2 d\rho \\
&\underset{(1-\frac{\rho}{n_\varepsilon}<1-\rho)}{\le} \varepsilon^{-2\alpha} \int_{n_\varepsilon(1-\delta)}^{n_\varepsilon} (\rho/n_\varepsilon - \rho)\, d\rho \\
&= \varepsilon^{-2\alpha} n_\varepsilon (1-n_\varepsilon)[1-(1-\delta)^2] \simeq \varepsilon^{-2\alpha}\varepsilon^{2\alpha}\delta.
\end{aligned}$$

Again, with the previous choice for δ, this term vanishes in the limit for $\varepsilon \to 0$.

The second integral in the right hand side of (B.3) reads

$$\begin{aligned}
&\varepsilon^{-2\alpha} \int_{n_\varepsilon}^{1} \theta_\varepsilon^2(\rho)\, d\rho = \varepsilon^{-2\alpha} \int_{n_\varepsilon}^{1} (\pi - 2\arcsin(\rho))^2\, d\rho \\
&\simeq \varepsilon^{-2\alpha}(1-n_\varepsilon)^2 \simeq \varepsilon^{-2\alpha} \frac{\varepsilon^{2\alpha}}{|v|^2} = \frac{1}{|v|^2}.
\end{aligned} \tag{B.11}$$

Therefore the only contribution in the limit is the one given by (B.7) and we obtain

$$\tilde{B} := \lim_{\varepsilon\to 0} \frac{\mu\varepsilon^{-2\alpha}}{2} |v| \int_{-1}^{1} \theta_\varepsilon^2(\rho)\, d\rho = \lim_{\varepsilon\to 0} \mu \left[\frac{2\alpha}{|v|^3} |\log(\varepsilon)| \right] = +\infty, \tag{B.12}$$

and finally

$$B := \lim_{\varepsilon\to 0} \frac{\mu\varepsilon^{-2\alpha}}{2|\log\varepsilon|} |v| \int_{-1}^{1} \theta_\varepsilon^2(\rho)\, d\rho = \frac{2\alpha}{|v|^3}\mu.$$

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Chapter 3

3.1 Derivation of the Fick's law for the Lorentz model in a low density regime [arXiv: 1404.4186] [To appear in *Comm. Math. Phys.*]

In the present Chapter we present [BNPP].

Derivation of the Fick's law for the Lorentz model in a low density regime

Abstract. We consider the Lorentz model in a slab with two mass reservoirs at the boundaries. We show that, in a low density regime, there exists a unique stationary solution for the microscopic dynamics which converges to the stationary solution of the heat equation, namely to the linear profile of the density. In the same regime the macroscopic current in the stationary state is given by the Fick's law, with the diffusion coefficient determined by the Green-Kubo formula.

3.1.1 Introduction

One of the most important and challenging problem in the rigorous approach to non-equilibrium Statistical Mechanics is the characterization of stationary nonequilibrium states exhibiting transport phenomena such as energy or mass transport, which are macroscopically described by Fourier's and Fick's law respectively. A simple microscopic model to validate the Fick's Law is the Lorentz gas, namely a system of non interacting light particles in a distribution of scatterers, in contact with two mass reservoirs. One expects that under a suitable space-time scaling (hydrodynamical limit) the stationary mass current is proportional to the gradient of the density. However the rigorous proof of that is a difficult and still open problem.

In this paper we propose a contribution in this direction in a situation of low density. The system we study is the following. Consider the two-dimensional strip $\Lambda = (0, L) \times \mathbb{R}$. In the left and in the right of the boundaries, $\{0\} \times \mathbb{R}$ and $\{L\} \times \mathbb{R}$ respectively, there are two mass reservoirs constituted by free point particles at equilibrium at different densities ρ_1, ρ_2. Inside the strip there is a random distribution of hard disks of radius ε, distributed according to a Poisson law with density μ_ε. Here ε is a small scale parameter and we let it go to zero. In the mean time μ_ε is diverging in such a way that $\mu_\varepsilon \varepsilon \to \infty$ and $\mu_\varepsilon \varepsilon^2 \to 0$. Therefore the scatterer configuration is dilute.

The light particles are flowing through the boundaries, from right with density ρ_2 and from left with density ρ_1 They are not interacting among themselves, but are elastically reflected by the obstacles. Their mean free paths vanish as $\varepsilon \to 0$, but not too quickly. More precisely they can vanish at most as $\varepsilon^{1-\delta}$, $0 < \delta < 1$, in order to have a dilute configuration of scatterers.

We expect that there exists a stationary state for which

$$J \approx -D\nabla\rho \tag{3.1.1}$$

where J is the mass current, ρ is the mass density and $D > 0$ is the diffusion coefficient. Formula (3.1.1) is the well known Fick's law which we want to prove in the present context.

We underline preliminary that our result holds in a low-density regime. This means that we can use the linear Boltzmann equation as a bridge between our original mechanical system and the diffusion equation. This basic idea has been used in [ESY] [BGS-R] [BNP] to obtain the heat equation from a particle system in different contexts. It works once having an explicit control of the error in the kinetic limit, which suggests the scale of times for which the diffusive limit can be achieved. As a consequence the diffusion coefficient D is given by the Green-Kubo formula for the kinetic equation at hand (namely linear Quantum Boltzmann for [ESY], linear Boltzmann for [BGS-R], linear Landau for [BNP]). In the present paper we work in a stationary situation for which we face new problems which will be discussed later on.

The idea of using the linear Boltzmann equation for the Lorentz gas in not new. In [LS] the authors consider exactly our system but with two thermal reservoirs at different temperatures at the boundaries. The aim was to study the energy flux in a stationary regime. However, as pointed out in [LS], due to the energy conservation of a single elastic collision, the energy is not diffused, there is no local equilibrium and hence the local temperature is not defined. As a consequence the Fourier's law fails to hold, at least in the conventional sense.

This is the reason why we consider here the mass transport, being the heat equation for the mass density the unique hydrodynamical equation.

It may be worth to mention that, for a suitable stochastic dynamics, the Fourier's law can indeed be derived, see [KMP], [GKMP].

Concerning the Fick's law we mention the papers [LS1], [LS2], for the self-diffusion of a tagged particle in a gas at equilibrium.

Our paper is organized as follows. The starting point is the transition from the mechanical system to the Boltzmann equation in a low density regime. We follow the classical analysis due to Gallavotti [G], complemented by an explicit analysis of the bad events preventing the Markovianity, in the same spirit of [DP], [DR] . This is necessary to reach a diffusive behavior on a longer time scale as in [BGS-R], [BNP].

Moreover we point out that our initial boundary value problem presents a new feature due to the presence of the first exit (stopping) time. This difficulty is handled by an extension procedure which essentially reduces our problem to the corresponding one in the whole space.

The transition from the mechanical system to the linear Boltzmann regime is presented in Section 3.1.7.

However we are interested in a stationary problem. This is handled, more conveniently, in terms of a Neumann series to overcome problems connected with the exchange of the limits $t \to \infty$, $\varepsilon \to 0$. To the best of our knowledge this is a new tool. This analysis is presented in Section 3.1.3. The basic idea is that the explicit solution of the heat equation and the control of the time dependent problem allow us to characterize the stationary solution of the linear Boltzmann equation and this turns out to be the basic tool to obtain the stationary solution of the mechanical system which is the basic object of our investigation.

Finally the transition from Boltzmann to the diffusion equation is classical and ruled out by the Hilbert expansion method which is presented in Section 3.1.4. This step is discussed in detail, not only for completeness, but also because we need an apparently new analysis in L^∞, for the time dependent problem (needed for the control of the Neumann series) and a L^2 analysis for the stationary problem.

3.1.2 The model and main results

Let $\Lambda \subset \mathbb{R}^2$ be the strip $(0, L) \times \mathbb{R}$. We consider a Poisson distribution of fixed hard disks (scatterers) of radius ε in Λ and denote by $c_1, \dots, c_N \in \Lambda$ their centers. This means that, given $\mu > 0$, the probability density of finding N obstacles in a bounded measurable set $A \subset \Lambda$ is

$$\mathbb{P}(\, d\mathbf{c}_N) = e^{-\mu|A|} \frac{\mu^N}{N!} \, dc_1 \dots \, dc_N \tag{3.1.2}$$

where $|A| = \text{meas}A$ and $\mathbf{c}_N = (c_1, \dots, c_N)$.

A particle in Λ moves freely up to the first instant of contact with an obstacle. Then it is elastically reflected and so on. Since the modulus of the velocity of the test particle is constant, we assume it to be equal to one, so that the phase space of our system is $\Lambda \times S_1$.

We rescale the intensity μ of the obstacles as

$$\mu_\varepsilon = \varepsilon^{-1}\eta_\varepsilon\mu,$$

where, from now on, $\mu > 0$ is fixed and η_ε is slowly diverging as $\varepsilon \to 0$. More precisely we make the following assumption.

Assumption 1. As $\varepsilon \to 0$, η_ε diverges in such a way that

$$\varepsilon^{\frac{1}{2}}\eta_\varepsilon^6 \to 0. \tag{3.1.3}$$

The behaviour (3.1.3) is dictated mostly by the recollision estimates in Section 3.1.10.

We denote by $\mathbb{P}_\varepsilon$ the probability density (3.1.2) with μ replaced by μ_ε. $\mathbb{E}_\varepsilon$ will be the expectation with respect to the measure $\mathbb{P}_\varepsilon$ restricted on those configurations of the obstacles whose centers do not belong to the disk of center x and radius ε.

For a given configuration of obstacles $\mathbf{c}_N$, we denote by $T^{-t}_{\mathbf{c}_N}(x, v)$ the (backward) flow with initial datum $(x, v) \in \Lambda \times S_1$ and define $t - \tau$, $\tau = \tau(x, v, t, \mathbf{c}_N)$, as the first (backward) hitting time with the boundary. We use the notation $\tau = 0$ to indicate the event such that the trajectory $T^{-s}_{\mathbf{c}_N}(x, v)$, $s \in [0, t]$, never hits the boundary. For any $t \geq 0$ the one-particle correlation function reads

$$f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_B(T^{-(t-\tau)}_{\mathbf{c}_N}(x, v))\chi(\tau > 0)] + \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_N}(x, v))\chi(\tau = 0)], \tag{3.1.4}$$

where $f_0 \in L^\infty(\Lambda \times S_1)$ and the boundary value f_B is defined by

$$f_B(x, v) := \begin{cases} \rho_1 M(v) & \text{if} \quad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \\ \rho_2 M(v) & \text{if} \quad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \end{cases}$$

with $M(v)$ the density of the uniform distribution on S_1 and $\rho_1, \rho_2 > 0$. Here v_1 denotes the horizontal component of the velocity v. Without loss of generality we assume $\rho_2 > \rho_1$. Since $M(v) = \frac{1}{2\pi}$, from now on we will absorb it in the definition of the boundary values ρ_1, ρ_2. Therefore we set

$$f_B(x, v) := \begin{cases} \rho_1 & \text{if} \quad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \\ \rho_2 & \text{if} \quad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0. \end{cases} \tag{3.1.5}$$

Remark. Here we allow overlapping of scatterers, namely the Poisson measure is that of a free gas. It would also be possible to consider the Poisson measure restricted to non-overlapping configurations, namely the Gibbs measure for a systems of hard disks in the plane. However the two measures are asymptotically equivalent and the result does hold also in the last case.

Note also that the dynamics $T^t_{\mathbf{c}_N}$ is well defined only almost everywhere with respect to $\mathbb{P}_\varepsilon$.

We are interested in the stationary solutions f^S_ε of the above problem. More precisely for any $t \geq 0$ $f^S_\varepsilon(x,v)$ solves

$$f^S_\varepsilon(x,v) = \mathbb{E}_\varepsilon[f_B(T^{-(t-\tau)}_{\mathbf{c}_N}(x,v))\chi(\tau > 0)] + \mathbb{E}_\varepsilon[f^S_\varepsilon(T^{-t}_{\mathbf{c}_N}(x,v))\chi(\tau = 0)]. \tag{3.1.6}$$

The main result of the present paper can be summarized in the following theorem.

Theorem 3.1.1. *For ε sufficiently small there exists a unique L^∞ stationary solution f^S_ε for the microscopic dynamics (i.e. satisfying* (3.1.6)*). Moreover, as $\varepsilon \to 0$*

$$f^S_\varepsilon \to \varrho^S, \tag{3.1.7}$$

where ϱ^S is the stationary solution of the heat equation with the following boundary conditions

$$\begin{cases} \varrho^S(x) = \rho_1, & x \in \{0\} \times \mathbb{R}, \\ \varrho^S(x) = \rho_2, & x \in \{L\} \times \mathbb{R}. \end{cases} \tag{3.1.8}$$

The convergence is in $L^2((0,L) \times S_1)$.

Some remarks on the above Theorem are in order. The boundary conditions of the problem depend on the space variable only through the horizontal component. As a consequence, the stationary solution f^S_ε of the microscopic problem, as well as the stationary solution ϱ^S of the heat equation, inherits the same feature. This justifies the convergence in $L^2((0,L) \times S_1)$ instead of in $L^2(\Lambda \times S_1)$. The explicit expression for the stationary solution ϱ^S reads

$$\varrho^S(x) = \frac{\rho_1(L - x_1) + \rho_2 x_1}{L}, \tag{3.1.9}$$

where x_1 is the horizontal component of the space variable x. We note that in order to prove Theorem 3.1.1 it is enough to assume that $\varepsilon^{\frac{1}{2}}\eta^5_\varepsilon \to 0$. The stronger Assumption 1 is needed to prove Theorem 3.1.2 below.

Next we discuss the Fick's law by introducing the stationary mass flux

$$J^S_\varepsilon(x) = \eta_\varepsilon \int_{S_1} v\, f^S_\varepsilon(x,v)\, dv, \tag{3.1.10}$$

and the stationary mass density

$$\varrho^S_\varepsilon(x) = \int_{S_1} f^S_\varepsilon(x,v)\, dv. \tag{3.1.11}$$

Note that J^S_ε is the total amount of mass flowing through a unit area in a unit time interval. Although in a stationary problem there is no typical time scale, the factor η_ε appearing in the definition of J^S_ε, is reminiscent of the time scaling necessary to obtain a diffusive limit.

Theorem 3.1.2 (**Fick's law**)**.** *We have*

$$J^S_\varepsilon + D\nabla_x \varrho^S_\varepsilon \to 0 \tag{3.1.12}$$

as $\varepsilon \to 0$*. The convergence is in* $\mathcal{D}'(0,L)$ *and* $D > 0$ *is given by the Green-Kubo formula (see* (3.1.26) *below). Moreover*

$$J^S = \lim_{\varepsilon \to 0} J^S_\varepsilon(x), \tag{3.1.13}$$

where the convergence is in $L^2(0,L)$ *and*

$$J^S = -D\,\nabla \varrho^S = -D\,\frac{\rho_2 - \rho_1}{L}, \tag{3.1.14}$$

where ϱ^S *is the linear profile* (3.1.9).

Observe that, as expected by physical arguments, the stationary flux J^S does not depend on the space variable. Furthermore the diffusion coefficient D is determined by the behavior of the system at equilibrium and in particular it is equal to the diffusion coefficient for the time dependent problem.

Remark (The scaling)**.** We have formulated our result in macroscopic variables x, t. Another point of view is to argue in terms of microscopic variables.

Let us set our problem in these variables denoted by (q, t'). This means that the radius of the disks is unitary while the strip, seen in micro-variables, is $(0, \varepsilon^{-1}L) \times \mathbb{R}$.

To deal with a low density situation, we rescale the density as $\eta_\varepsilon \varepsilon \mu$, $\mu > 0$ where η_ε is gently diverging. Note that in the usual Boltzmann-Grad limit $\eta_\varepsilon = 1$. At times of order ε^{-1}, one particle has an average number of collisions of order η_ε. At larger times, namely of order $\eta_\varepsilon \varepsilon^{-1}$, we expect a diffusive behavior. Actually this emerges from the linear Boltzmann equation (see equation (3.1.24) and Proposition 3.1.4 below) which is derived from the microscopic dynamics through the scaling $x = \varepsilon q$ and $t = \varepsilon \eta_\varepsilon^{-1} t'$.

In this paper we consider a two dimensional case but our techniques apply in higher dimensions as well since in this case the pathological events are less likely. Moreover we consider the easier geometrical setting. However we believe that there are no serious obstructions to extend our results to more general geometries.

3.1.3 Proofs

In this section we prove Theorems 3.1.1 and 3.1.2, postponing the technical details to the next sections. In order to prove Theorem 3.1.1 our strategy is the following. We introduce the stationary linear Boltzmann equation

$$\begin{cases} (v \cdot \nabla_x) h^S_\varepsilon(x,v) = \eta_\varepsilon \, \mathcal{L} h^S_\varepsilon(x,v), \\ h^S_\varepsilon(x,v) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \\ h^S_\varepsilon(x,v) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \end{cases} \tag{3.1.15}$$

where $\mathcal{L}$ is the linear Boltzmann operator defined as

$$\mathcal{L} f(v) = \mu \int_{-1}^{1} d\rho \big[f(v') - f(v) \big], \qquad f \in L^1(S_1) \tag{3.1.16}$$

with

$$v' = v - 2(n \cdot v) n \tag{3.1.17}$$

and $n = n(\rho)$ the outward normal to the hard disk (see Figure 3.1). Here ρ is the impact parameter, namely $\rho = \sin \alpha$ with α the angle of incidence.

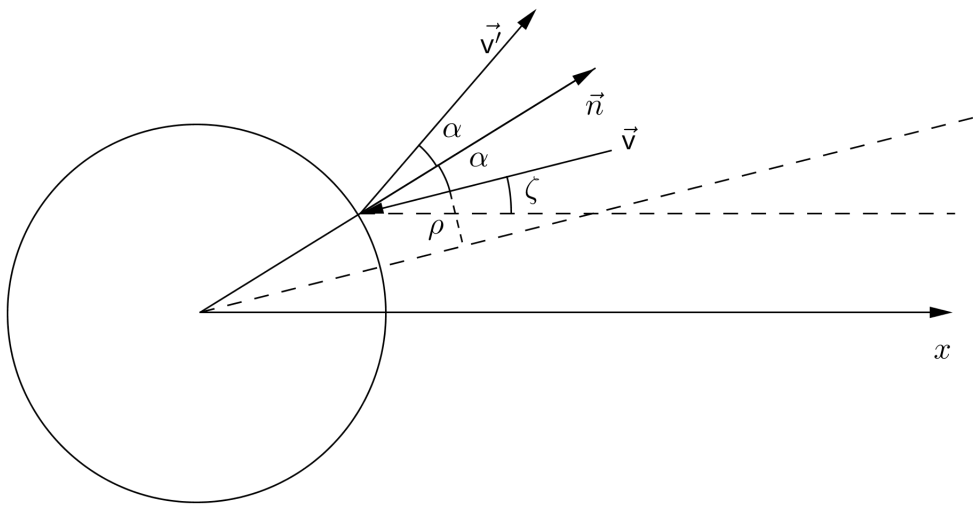

Figure 3.1:

Since the boundary conditions depend on the space variable only through the horizontal component, the stationary solution h^S_ε inherits the same feature, as well as f^S_ε and ϱ^S.

The strategy of the proof consists of two steps. First we prove that there exists a unique h^S_ε which converges, as $\varepsilon \to 0$, to ϱ^S given by (3.1.9). See Proposition 3.1.5 below. Secondly we show that there exists a unique f^S_ε

asymptotically equivalent to h^S_ε. See Proposition 3.1.8 below. This result is achieved by showing that the memory effects of the mechanical system, preventing the Markovianity, are indeed negligible.

Let h_ε be the solution of the problem

$$\begin{cases} \big(\partial_t + v\cdot\nabla_x\big)h_\varepsilon(x,v,t) = \eta_\varepsilon\,\mathcal{L}h_\varepsilon(x,v,t), \\ h_\varepsilon(x,v,0) = f_0(x,v), \qquad f_0 \in L^\infty(\Lambda\times S_1), \\ h_\varepsilon(x,v,t) = \rho_1, \qquad x\in\{0\}\times\mathbb{R}, \quad v_1>0, \quad t\geq 0, \\ h_\varepsilon(x,v,t) = \rho_2, \qquad x\in\{L\}\times\mathbb{R}, \quad v_1<0, \quad t\geq 0. \end{cases} \tag{3.1.18}$$

Then h_ε has the following explicit representation

$$\begin{aligned} h_\varepsilon(x,v,t) = &\sum_{N\geq 0}(\mu_\varepsilon\varepsilon)^N\int_0^t dt_1\dots\int_0^{t_{N-1}}dt_N \\ &\int_{-1}^1 d\rho_1\dots\int_{-1}^1 d\rho_N\,\chi(\tau<t_N)\chi(\tau>0)\,e^{-2\mu_\varepsilon\varepsilon\,(t-\tau)}\,f_B(\gamma^{-(t-\tau)}(x,v))+ \\ &+\sum_{N\geq 0}e^{-2\mu_\varepsilon\varepsilon\,t}\,(\mu_\varepsilon\varepsilon)^N\int_0^t dt_1\dots\int_0^{t_{N-1}}dt_N \\ &\int_{-1}^1 d\rho_1\dots\int_{-1}^1 d\rho_N\,\chi(\tau=0)\,f_0(\gamma^{-t}(x,v)), \end{aligned} \tag{3.1.19}$$

with f_B defined in (3.1.5). Given $x, v,\ t_1\dots t_N,\ \rho_1\dots\rho_N,\ \gamma^{-t}(x,v)$ denotes the trajectory whose position and velocity are

$$(x - v(t-t_1) - v_1(t_1-t_2)\cdots - v_N t_N, v_N).$$

The transitions $v\to v_1\to v_2\cdots\to v_N$ are obtained by means of a scattering with an hard disk with impact parameter ρ_i via (3.1.17). As before $t-\tau$, $\tau = \tau(x,v,t_1\dots,t_N,\rho_1\dots\rho_N)$, is the first (backward) hitting time with the boundary. We remind that $\mu_\varepsilon\varepsilon = \mu\eta_\varepsilon$.

In formula (3.1.19) $h_\varepsilon(t)$ results as the sum of two contributions, one due to the backward trajectories hitting the boundary and the other one due to the trajectories which never leave Λ. Therefore we set

$$h_\varepsilon(x,v,t) = h^{out}_\varepsilon(x,v,t) + h^{in}_\varepsilon(x,v,t),$$

where h^{out}_ε and h^{in}_ε are respectively the first and the second sum on the right hand side of (3.1.19). Observe that h^{out}_ε solves

$$\begin{cases} \big(\partial_t + v\cdot\nabla_x\big)h^{out}_\varepsilon(x,v,t) = \eta_\varepsilon\,\mathcal{L}h^{out}_\varepsilon(x,v,t), \\ h^{out}_\varepsilon(x,v,0) = 0, \qquad x\in\Lambda, \\ h^{out}_\varepsilon(x,v,t) = \rho_1, \qquad x\in\{0\}\times\mathbb{R}, \quad v_1>0, \quad t\geq 0, \\ h^{out}_\varepsilon(x,v,t) = \rho_2, \qquad x\in\{L\}\times\mathbb{R}, \quad v_1<0, \quad t\geq 0. \end{cases} \tag{3.1.20}$$

We denote by $S^0_\varepsilon(t)$ the Markov semigroup associated to the second sum, namely

$$(S^0_\varepsilon(t)\ell)(x,v) = \sum_{N\geq 0} e^{-2\mu_\varepsilon \varepsilon\, t}\,(\mu_\varepsilon\varepsilon)^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \\ \int_{-1}^{1} d\rho_1 \dots \int_{-1}^{1} d\rho_N\, \chi(\tau=0)\,\ell(\gamma^{-t}(x,v)),$$

with $\ell \in L^\infty(\Lambda\times S_1)$. In particular

$$h^{in}_\varepsilon(t) = S^0_\varepsilon(t) f_0.$$

We observe that h^S_ε, solution of (3.1.15), satisfies, for $t_0>0$

$$h^S_\varepsilon = h^{out}_\varepsilon(t_0) + S^0_\varepsilon(t_0)h^S_\varepsilon,$$

so that we can formally express h^S_ε as the Neumann series

$$h^S_\varepsilon = \sum_{n\geq 0}(S^0_\varepsilon(t_0))^n h^{out}_\varepsilon(t_0). \tag{3.1.21}$$

Remark. Note that h^S_ε is a fixed point of the map $f_0 \to h_\varepsilon(t_0)$ solution to (3.1.18). Hence h^S_ε belongs to a periodic orbit, of period t_0, of the flow $f_0 \to h_\varepsilon(t)$. But this orbit consists of a single point because the Neumann series, being convergent, identifies a single element. This implies that h^S_ε is constant with respect to the flow (3.1.18) and hence stationary.

We now establish existence and uniqueness of h^S_ε by showing that the Neumann series (3.1.21) converges. In order to do it we need to extend the action of the semigroup $S^0_\varepsilon(t)$ to the space $L^\infty(\mathbb{R}^2\times S_1)$, namely

$$S^0_\varepsilon(t)\ell_0(x,v) = \chi_\Lambda(x)\sum_{N\geq 0} e^{-2\mu_\varepsilon \varepsilon\, t}\,(\mu_\varepsilon\varepsilon)^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \\ \int_{-1}^{1} d\rho_1 \dots \int_{-1}^{1} d\rho_N\, \chi(\tau=0)\,\ell_0(\gamma^{-t}(x,v)), \tag{3.1.22}$$

for any $\ell_0(x,v)\in L^\infty(\mathbb{R}^2\times S_1)$. Here χ_Λ is the characteristic function of Λ.

Proposition 3.1.3. *There exists $\varepsilon_0>0$ such that for any $\varepsilon<\varepsilon_0$ and for any $\ell_0\in L^\infty(\mathbb{R}^2\times S_1)$ we have*

$$||S^0_\varepsilon(\eta_\varepsilon)\ell_0||_\infty \leq \alpha\,||\ell_0||_\infty, \quad \alpha<1. \tag{3.1.23}$$

As a consequence there exists a unique stationary solution $h^S_\varepsilon\in L^\infty(\Lambda\times S_1)$ satisfying (3.1.15).

To prove Proposition 3.1.3 we have first to exploit the diffusive limit of the linear Boltzmann equation in a L^∞ setting and in the whole space. We introduce $\tilde{h}_\varepsilon : \mathbb{R}^2 \times S_1 \times [0,T] \to \mathbb{R}^+$ the solution of the following rescaled linear Boltzmann equation

$$\begin{cases} \big(\partial_t + \eta_\varepsilon\, v \cdot \nabla_x\big)\tilde{h}_\varepsilon = \eta_\varepsilon^2\, \mathcal{L}\tilde{h}_\varepsilon \\ \\ \tilde{h}_\varepsilon(x,v,0) = \varrho_0(x), \end{cases} \tag{3.1.24}$$

with ϱ_0 is a smooth function of the variable x only (local equilibrium).

We can prove

Proposition 3.1.4. *Let $\tilde{h}_\varepsilon$ be the solution of* (3.1.24)*, with an initial datum $\varrho_0 \in C_0^\infty(\mathbb{R}^2)$. Then, as $\varepsilon \to 0$, $\tilde{h}_\varepsilon$ converges to the solution of the heat equation*

$$\begin{cases} \partial_t\varrho - D\Delta\varrho = 0 \\ \\ \varrho(x,0) = \varrho_0(x), \end{cases} \tag{3.1.25}$$

where D is given by the Green-Kubo formula

$$D = \frac{1}{4\pi}\int_{S_1} dv\, v \cdot \big(-\mathcal{L}\big)^{-1} v. \tag{3.1.26}$$

The convergence is in $L^\infty([0,T]; L^\infty(\mathbb{R}^2 \times S_1))$.

We postpone the proof of Proposition (3.1.4) to Section 3.1.5. The proof relies on the Hilbert expansion and, to make it work, we need smoothness of the initial datum ϱ_0.

Proof of Proposition 3.1.3. We can rewrite (3.1.22) as

$$S_\varepsilon^0(t)\ell_0(x,v) = \chi_\Lambda(x)\sum_{N\geq 0} e^{-2\mu_\varepsilon\varepsilon\, t}\,(\mu_\varepsilon\varepsilon)^N \int_0^t dt_1 \ldots \int_0^{t_{N-1}} dt_N$$
$$\int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_N\, \chi(\tau = 0)\, \ell_0(\gamma^{-t}(x,v))\chi_\Lambda(\gamma_v^{-t}(x)),$$

where $\gamma_v^{-t}(x) = x - v(t-t_1) - v_1(t_1 - t_2)\cdots - v_N t_N$ is the first component of $\gamma^{-t}(x,v)$. Note that the insertion of $\chi_\Lambda(\gamma_v^{-t}(x))$ is due to the constraint $\chi(\tau = 0)$. Therefore

$$S_\varepsilon^0(t)\ell_0 \leq ||\ell_0||_\infty \sum_{N\geq 0} e^{-2\mu_\varepsilon\varepsilon\, t}\,(\mu_\varepsilon\varepsilon)^N \int_0^t dt_1 \ldots \int_0^{t_{N-1}} dt_N$$
$$\int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_N\, \chi_\Lambda(\gamma_v^{-t}(x)).$$

We denote by χ_Λ^δ a mollified version of χ_Λ, namely $\chi_\Lambda^\delta \in C_0^\infty(\mathbb{R}^2)$, $\chi_\Lambda^\delta(x) \leq 1$, $\chi_\Lambda^\delta \geq \chi_\Lambda$ and $supp(\chi_\Lambda^\delta) \subset (-\delta, L+\delta) \times \mathbb{R}$. Therefore

$$\begin{aligned} S_\varepsilon^0(t)\ell_0 \leq ||\ell_0||_\infty \sum_{N\geq 0} e^{-2\mu_\varepsilon \varepsilon\, t}\, (\mu_\varepsilon \varepsilon)^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \\ \int_{-1}^1 d\rho_1 \dots \int_{-1}^1 d\rho_N\, \chi_\Lambda^\delta(\gamma_v^{-t}(x)). \end{aligned} \tag{3.1.27}$$

The series on the right hand side of (3.1.27) defines a function F which solves

$$\begin{cases} (\partial_t + v \cdot \nabla_x) F(x,v,t) = \eta_\varepsilon \mathcal{L} F(x,v,t), \\ F(x,v,0) = \chi_\Lambda^\delta(x). \end{cases}$$

Moreover, defining $G_\varepsilon(x,v,t) := F(x,v,\eta_\varepsilon t)$ then G_ε solves (3.1.24) with initial datum $\varrho_0 = \chi_\Lambda^\delta$. By virtue of Proposition 3.1.4

$$\|G_\varepsilon(1) - \varrho^\delta(1)\|_\infty \leq \omega(\varepsilon)$$

where $\varrho^\delta(t)$ is the solution of (3.1.25) with initial datum χ_Λ^δ. Here and in the sequel $\omega(\varepsilon)$ denotes a positive function vanishing with ε. On the other hand

$$\varrho^\delta(x,1) = \int_{\mathbb{R}^2} dy\, \frac{1}{4\pi D}\, e^{-\frac{|x-y|^2}{4D}}\, \chi_\Lambda^\delta(y) = \int_{-\delta}^{L+\delta} dy_1\, \frac{1}{\sqrt{4\pi D}}\, e^{-\frac{|x_1-y_1|^2}{4D}} < 1.$$

Therefore for ε small enough

$$\begin{aligned} ||S_\varepsilon^0(\eta_\varepsilon)\ell_0||_\infty &\leq ||\ell_0||_\infty ||S_\varepsilon^0(\eta_\varepsilon)\chi_\Lambda^\delta||_\infty \\ &\leq ||\ell_0||_\infty (\|G_\varepsilon(1) - \varrho^\delta(1)\|_\infty + ||\varrho^\delta(1)||_\infty) \\ &\leq ||\ell_0||_\infty (\omega(\varepsilon) + ||\varrho^\delta(1)||_\infty) < \alpha ||\ell_0||_\infty, \quad \alpha < 1. \end{aligned}$$

We are using (3.1.27) for $t = \eta_\varepsilon$.

Finally, since $\alpha < 1$, by (3.1.21) we get

$$||h_\varepsilon^S||_\infty \leq \frac{1}{(1-\alpha)}\, ||h_\varepsilon^{out}(\eta_\varepsilon)||_\infty \leq \frac{1}{(1-\alpha)}\, \rho_2.$$

□

As we will discuss later on, we find convenient to obtain the stationary solution h_ε^S via the Neumann series (3.1.21) rather than as the limit of $h_\varepsilon(t)$ as $t \to \infty$. For further details see Remark 3.1.9.

Remark (L^∞ vs. L^2)**.** The control of the Neumann series (3.1.21) in a L^∞ setting seems quite natural. This is provided by the bound (3.1.23). It basically means that for a time η_ε the probability of a backward trajectory to fall out of Λ is strictly positive. To prove rigorously this rather intuitive fact, we use Proposition 3.1.4 and explicit properties of the solution of the heat equation. The price we pay is to develop an L^∞ Hilbert expansion analysis (see Section 3.1.5) which is, however, interesting in itself. On the other hand the use of the well known L^2 version of Proposition 3.1.4 requires a L^2 control of the Neumann series which seems harder, weaker and less natural.

The last step is the proof of the convergence of h_ε^S to the stationary solution of the diffusion problem

$$\begin{cases} \partial_t \varrho - D\Delta\varrho = 0 \\ \varrho(x,t) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad t \geq 0 \\ \varrho(x,t) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad t \geq 0, \end{cases} \tag{3.1.28}$$

with the diffusion coefficient D given by the Green-Kubo formula (3.1.26). We remind that the stationary solution ϱ^S to the problem (3.1.28) has the following explicit expression

$$\varrho^S(x) = \frac{\rho_1(L - x_1) + \rho_2 x_1}{L}, \tag{3.1.29}$$

where $x = (x_1, x_2)$.

By using again the Hilbert expansion technique (this time in L^2) we can prove

Proposition 3.1.5. *Let $h_\varepsilon^S \in L^\infty((0,L) \times S_1)$ be the solution to the problem* (3.1.15)*. Then*

$$h_\varepsilon^S \to \varrho^S \tag{3.1.30}$$

as $\varepsilon \to 0$, where $\varrho^S(x)$ is given by (3.1.29)*. The convergence is in $L^2((0,L) \times S_1)$.*

The proof is postponed to Section 3.1.6.

This concludes our analysis of the Markov part of the proof.

Recalling the expression (3.1.4) for the one-particle correlation function f_ε, we introduce a decomposition analogous to the one used for $h_\varepsilon(t)$, namely

$$f_\varepsilon^{out}(x,v,t) := \mathbb{E}_\varepsilon[f_B(T_{\mathbf{c}_N}^{-(t-\tau)}(x,v))\chi(\tau > 0)] \tag{3.1.31}$$

and

$$f_\varepsilon^{in}(x,v,t) := \mathbb{E}_\varepsilon[f_0(T_{\mathbf{c}_N}^{-t}(x,v))\chi(\tau = 0)], \tag{3.1.32}$$

so that

$$f_\varepsilon(x,v,t) = f_\varepsilon^{out}(x,v,t) + f_\varepsilon^{in}(x,v,t).$$

Here f_ε^{out} is the contribution due to the trajectories that do leave Λ at times smaller than t, while f_ε^{in} is the contribution due to the trajectories that stay internal to Λ. We introduce the flow $F_\varepsilon^0(t)$ such that

$$(F_\varepsilon^0(t)\ell)(x,v) = \mathbb{E}_\varepsilon[\ell(T_{\mathbf{c}_N}^{-t}(x,v))\chi(\tau=0)], \quad \ell \in L^\infty(\Lambda \times S_1)$$

and remark that F_ε^0 is just the dynamics "inside" Λ. In particular $f_\varepsilon^{in}(t) = F_\varepsilon^0(t) f_0$.

To detect the stationary solution f_ε^S for the microscopic dynamics we proceed as for the Boltzmann evolution (see (3.1.6)) by setting, for $t_0 > 0$,

$$f_\varepsilon^S = f_\varepsilon^{out}(t_0) + F_\varepsilon^0(t_0) f_\varepsilon^S$$

and we can formally express the stationary solution as the Neumann series

$$f_\varepsilon^S = \sum_{n\geq 0} (F_\varepsilon^0(t_0))^n f_\varepsilon^{out}(t_0). \tag{3.1.33}$$

To show the convergence of the series (3.1.33) and hence existence of f_ε^S we first need the following two Propositions.

Proposition 3.1.6. *Let $T > 0$. For any $t \in (0,T]$*

$$\|f_\varepsilon^{out}(t) - h_\varepsilon^{out}(t)\|_{L^\infty(\Lambda\times S_1)} \leq C\varepsilon^{\frac{1}{2}}\, \eta_\varepsilon^3\, t^2, \tag{3.1.34}$$

where h_ε^{out} solves (3.1.20).

Proposition 3.1.7. *For every $\ell_0 \in L^\infty(\Lambda \times S_1)$*

$$|| \left(F_\varepsilon^0(t) - S_\varepsilon^0(t)\right) \ell_0||_\infty \leq C||\ell_0||_\infty\, \varepsilon^{\frac{1}{2}} \eta_\varepsilon^3 t^2, \quad \forall t \in [0,T]. \tag{3.1.35}$$

The proof of the above two Propositions is postponed to Section 3.1.7. As a corollary we can prove

Proposition 3.1.8. *For ε sufficiently small there exists a unique stationary solution $f_\varepsilon^S \in L^\infty(\Lambda \times S_1)$ satisfying* (3.1.6)*. Moreover*

$$\|h_\varepsilon^S - f_\varepsilon^S\|_\infty \leq C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5. \tag{3.1.36}$$

Proof. We prove the existence and uniqueness of the stationary solution by showing that the Neumann series (3.1.33) converges, namely

$$||F_\varepsilon^0(\eta_\varepsilon) f_0||_\infty \leq \alpha'\, ||f_0||_\infty, \quad \alpha' < 1. \tag{3.1.37}$$

This implies

$$||f_\varepsilon^S||_\infty \leq \frac{1}{(1-\alpha')}\, ||f_\varepsilon^{out}(\eta_\varepsilon)||_\infty \leq \frac{1}{(1-\alpha')}\, \rho_2, \quad \alpha' < 1.$$

In fact, since

$$||F_\varepsilon^0(\eta_\varepsilon)f_0||_\infty \le ||\left(F_\varepsilon^0(\eta_\varepsilon) - S_\varepsilon^0(\eta_\varepsilon)\right) f_0||_\infty + ||S_\varepsilon^0(\eta_\varepsilon)f_0||_\infty,$$

thanks to Propositions 3.1.3 and 3.1.7 we get

$$\begin{aligned} ||F_\varepsilon^0(\eta_\varepsilon)f_0||_\infty &\le ||f_0||_\infty C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5 + ||S_\varepsilon^0(\eta_\varepsilon)f_0||_\infty \\ &\le (C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5 + \alpha)||f_0||_\infty \le \alpha'||f_0||_\infty, \end{aligned} \tag{3.1.38}$$

with $\alpha' < 1$, for ε sufficiently small (remind that $\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5 \to 0$ as $\varepsilon \to 0$). This guarantees the existence and uniqueness of the microscopic stationary solution f_ε^S.

In order to prove (3.1.36) we compare the two Neumann series representing f_ε^S and h_ε^S,

$$\begin{aligned} \|f_\varepsilon^S - h_\varepsilon^S\|_\infty &= \|\sum_{n\ge 0}\left((F_\varepsilon^0(\eta_\varepsilon))^n f_\varepsilon^{out}(\eta_\varepsilon) - (S_\varepsilon^0(\eta_\varepsilon))^n h_\varepsilon^{out}(\eta_\varepsilon)\right)\|_\infty \\ &\le \sum_{n\ge 0}\|(F_\varepsilon^0(\eta_\varepsilon))^n(f_\varepsilon^{out}(\eta_\varepsilon) - h_\varepsilon^{out}(\eta_\varepsilon))\|_\infty \\ &+ \sum_{n\ge 0}\|\left((F_\varepsilon^0(\eta_\varepsilon))^n - (S_\varepsilon^0(\eta_\varepsilon))^n\right)h_\varepsilon^{out}(\eta_\varepsilon)\|_\infty. \end{aligned} \tag{3.1.39}$$

By (3.1.38), using Proposition 3.1.6, the first sum on the right hand side of (3.1.39) is bounded by

$$\frac{1}{1-\alpha'}\|f_\varepsilon^{out}(\eta_\varepsilon) - h_\varepsilon^{out}(\eta_\varepsilon)\|_\infty \le C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5.$$

As regard to the second sum on the right hand side of (3.1.39) we have

$$\begin{aligned} &\sum_{n\ge 0}\|\left((F_\varepsilon^0(\eta_\varepsilon))^n - (S_\varepsilon^0(\eta_\varepsilon))^n\right)h_\varepsilon^{out}(\eta_\varepsilon)\|_\infty \\ &\le \sum_{n\ge 0}\sum_{k=0}^{n-1}\|(F_\varepsilon^0(\eta_\varepsilon))^{n-k-1}\left(F_\varepsilon^0(\eta_\varepsilon) - S_\varepsilon^0(\eta_\varepsilon)\right)(S_\varepsilon^0(\eta_\varepsilon))^k h_\varepsilon^{out}(\eta_\varepsilon)\|_\infty \\ &\le \sum_{k,\ell\ge 0}\|(F_\varepsilon^0(\eta_\varepsilon))^\ell\left(F_\varepsilon^0(\eta_\varepsilon) - S_\varepsilon^0(\eta_\varepsilon)\right)(S_\varepsilon^0(\eta_\varepsilon))^k h_\varepsilon^{out}(\eta_\varepsilon)\|_\infty \\ &\le C\,\|h_\varepsilon^{out}(\eta_\varepsilon)\|_\infty\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5, \end{aligned}$$

by virtue of (3.1.23), (3.1.38) and (3.1.35). This concludes the proof of Proposition 3.1.8.

□

At this point the proof of Theorem 3.1.1 follows from Propositions 3.1.5 and 3.1.8.

Remark 3.1.9. *One could try to characterize h^S_ε and f^S_ε in terms of the long (macroscopic) time asymptotics of $h_\varepsilon(t)$ and $f_\varepsilon(t)$. The trick of expressing both stationary states by means of Neumann series avoids the problem of controlling the convergence rates, as $t \to \infty$, with respect to the scale parameter ε.*

We conclude by proving Theorem 3.1.2 which actually is a Corollary of the previous analysis.

Proof of Theorem 3.1.2. By standard computations (see e.g. Section 3.1.6) we have

$$h^S_\varepsilon = \varrho^S + \frac{1}{\eta_\varepsilon} h^{(1)} + \frac{1}{\eta_\varepsilon} R_{\eta_\varepsilon},$$

where

$$h^{(1)}(v) = \mathcal{L}^{-1}(v \cdot \nabla_x \varrho^S) = \frac{\rho_2 - \rho_1}{L} \mathcal{L}^{-1}(v_1)$$

and, as we shall see in Section 3.1.6, $R_{\eta_\varepsilon} = O(\frac{1}{\sqrt{\eta_\varepsilon}})$ in $L^2((0,L) \times S_1)$. Therefore, since $\int_{S_1} v \varrho^S dv = 0$,

$$\eta_\varepsilon \int_{S_1} v h^S_\varepsilon(x,v) dv = -D \nabla_x \varrho^S + O(\frac{1}{\sqrt{\eta_\varepsilon}}), \tag{3.1.40}$$

where D is given by (3.1.26). By Theorem 3.1.1 the right hand side of (3.1.40) is close to $D\nabla_x \varrho^S_\varepsilon$ in $\mathcal{D}'((0,L) \times S_1)$, where ϱ^S_ε is given by (3.1.11). On the other hand, by Proposition 3.1.8 and Assumption 1, the left hand side of (3.1.40) is close in $L^\infty((0,L) \times S_1)$ to $J^S_\varepsilon(x)$ defined in (3.1.10). This concludes the proof of (3.1.12). Moreover (3.1.13) and (3.1.14) follow by (3.1.40). □

3.1.4 The Hilbert expansions

3.1.5 Proof of Proposition 3.1.4

Let $\tilde{h}_\varepsilon : \mathbb{R}^2 \times S_1 \times [0,T] \to \mathbb{R}^+$ be the solution of the problem (3.1.24) that we recall here for the reader's convenience

$$\begin{cases} \big(\partial_t + \eta_\varepsilon\, v \cdot \nabla_x\big) \tilde{h}_\varepsilon = \eta^2_\varepsilon\, \mathcal{L} \tilde{h}_\varepsilon \\ \tilde{h}_\varepsilon(x,v,0) = \varrho_0(x), \end{cases} \tag{3.1.41}$$

where ϱ_0 is a smooth function of the variable x only. We will prove that $\tilde{h}_\varepsilon$ converges to the solution of the heat equation by using the Hilbert expansion technique (see e.g. [EP] and [CIP]), namely we assume that $\tilde{h}_\varepsilon$ has the following form

$$\tilde{h}_\varepsilon(x,v,t) = h^{(0)}(x,t) + \sum_{k=1}^{+\infty} \left(\frac{1}{\eta_\varepsilon}\right)^k h^{(k)}(x,v,t),$$

where the coefficients $h^{(k)}$ are independent of η_ε. The well known idea is to determine them recursively, by imposing that $\tilde{h}_\varepsilon$ is a solution of (3.1.41). Comparing terms of the same order we get

$$\begin{aligned} &v \cdot \nabla_x h^{(0)} = \mathcal{L}\, h^{(1)} \\ &\partial_t\, h^{(k)} + v \cdot \nabla_x h^{(k+1)} = \mathcal{L}\, h^{(k+2)}, \qquad k \geq 0. \end{aligned}$$

We require $h^{(0)}$ to satisfy the same initial condition as the whole solution $\tilde{h}_\varepsilon$, namely

$$h^{(0)}(x,0) = \varrho_0(x).$$

First we will show that each coefficient $h^{(k)}(t) \in L^\infty(\mathbb{R}^2 \times S_1)$. We discuss in detail the cases $k = 0, 1, 2$. The same procedure can be iterated for any k. The determination of the other coefficients $h^{(k)}$ is standard and we do not discuss it further. Then we will show that, in the truncated expansion at order η_ε^{-2}, namely

$$\tilde{h}_\varepsilon(x,v,t) = h^{(0)}(x,t) + \frac{1}{\eta_\varepsilon} h^{(1)}(x,v,t) + \frac{1}{\eta_\varepsilon^2} h^{(2)}(x,v,t) + \frac{1}{\eta_\varepsilon} R_{\eta_\varepsilon}(x,v,t), \tag{3.1.42}$$

the remainder R_{η_ε} is uniformly bounded in L^∞. Therefore $\tilde{h}_\varepsilon$ converges to $h^{(0)}$ in L^∞ for $\eta_\varepsilon \to \infty$.

In order to prove that $h^{(k)}(t) \in L^\infty(\mathbb{R}^2 \times S_1)$ we need the following Lemma.

Lemma 3.1.10. *Let $\mathcal{L}$ be the linear Boltzmann operator defined in* (3.1.16)*. Then for any $g \in L^\infty(S_1)$ such that $\int_{S_1} dv\, g(v) = 0$*

$$||\mathcal{L}^{-1} g||_\infty \leq C ||g||_\infty, \tag{3.1.43}$$

with $C > 0$.

Proof. We want to solve the equation $\mathcal{L}h = g$, with $\int_{S_1} dv\, g(v) = 0$. The operator $\mathcal{L}$ can be written as $\mathcal{L} = 2\mu(K - I)$, where

$$(Kf)(v) := \frac{1}{2} \int_{-1}^{1} d\rho\, f(v')$$

is self-adjoint in $L^2(S_1)$. Therefore

$$h = -\frac{g}{2\mu} + Kh$$

and, by iterating,

$$h = -\frac{g}{2\mu} - \frac{Kg}{2\mu} - \cdots - \frac{K^n g}{2\mu} + K^{n+1} h, \quad \forall\, n \geq 0.$$

Then $\mathcal{L}^{-1}$ can be formally defined through the Neumann series

$$h = \mathcal{L}^{-1} g := -\frac{1}{2\mu} \sum_{n=0}^{\infty} K^n g.$$

In order to prove that the series converges we need to show that

$$||Kg||_\infty \leq \beta \, ||g||_\infty, \quad \beta < 1, \tag{3.1.44}$$

$$\int_{S_1} dv \, (Kg)(v) = 0, \tag{3.1.45}$$

for any $g \in L^\infty(S_1)$ such that $\displaystyle\int_{S_1} dv \, g(v) = 0$. Indeed (3.1.44) and (3.1.45) imply

$$||\mathcal{L}^{-1} g||_\infty \leq \frac{1}{2\mu(1-\beta)} \, ||g||_\infty.$$

The self-adjointness of K and the fact that $K1 = 1$ imply (3.1.45).

We focus on the proof of (3.1.44). For any given v, fix a reference system in such a way that $v = (-\cos\zeta, -\sin\zeta)$, with $\zeta \in [-\pi, \pi)$ (see Figure 3.1). Then for every bounded function g with zero average we have

$$\begin{aligned}
(Kg)(v) &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \, \frac{d\rho}{d\alpha} \; g(\cos(\zeta + 2\alpha), \sin(\zeta + 2\alpha)) \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \; \cos\alpha \; g(\cos(\zeta + 2\alpha), \sin(\zeta + 2\alpha)),
\end{aligned}$$

where we used that $\rho = \sin\alpha$. Observe that for any $\gamma \in [-\pi, \pi)$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \; g(\cos(\gamma + 2\alpha), \sin(\gamma + 2\alpha)) = \frac{1}{2} \int_{-\pi}^{\pi} d\alpha \; g(\cos\alpha, \sin\alpha) = 0.$$

Then we can write

$$(Kg)(v) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \; (\cos\alpha - 1) \; g(\cos(\zeta + 2\alpha), \sin(\zeta + 2\alpha)),$$

which implies

$$|(Kg)(v)| \leq ||g||_\infty \, \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \; (1 - \cos\alpha) = \beta \; ||g||_\infty, \qquad \beta < 1.$$

$\square$

Next we consider the first two equations arising from the Hilbert expansion, namely

(i) $v \cdot \nabla_x h^{(0)} = \mathcal{L}\, h^{(1)}$,

(ii) $\partial_t h^{(0)} + v \cdot \nabla_x h^{(1)} = \mathcal{L}\, h^{(2)}$.

We remind that the linear Boltzmann operator $\mathcal{L}$ on $L^2(S_1)$ is selfadjoint and has the form $\mathcal{L} = 2\mu(K - I)$ where K is a compact operator. Therefore, by the Fredholm alternative, equation (i) has a solution if and only if the left hand side belongs to $(Ker\mathcal{L})^{\perp}$. Since the null space of $\mathcal{L}$ is constituted by the constant functions, it follows that

$$(Ker\mathcal{L})^{\perp} = \{g \in L^2(S_1) : \int_{S_1} g(v)\, dv = 0\}$$

and, in order to solve equation (i), we have to show that $v \cdot \nabla_x h^{(0)} \in (Ker\mathcal{L})^{\perp}$. This follows by the fact that $v \cdot \nabla_x h^{(0)}$ is an odd function of v. Then we can invert the operator $\mathcal{L}$ and set

$$h^{(1)}(x, v, t) = \mathcal{L}^{-1}(v \cdot \nabla_x h^{(0)}(x, t)) + \xi^{(1)}(x, t), \tag{3.1.46}$$

where $\xi^{(1)}(x, t)$ belongs to the kernel of the operator $\mathcal{L}$. On the other hand, since $\mathcal{L}^{-1}$ preserves the parity (see e.g. [EP]), $\mathcal{L}^{-1}(v \cdot \nabla_x h^{(0)})$ is an odd function of the velocity.

We integrate equation (ii) with respect to the uniform measure on S_1. Since $\int_{S_1} dv\, \mathcal{L}\, h^{(2)} = 0$, using equation (3.1.46), we obtain

$$\partial_t\, h^{(0)} + \frac{1}{2\pi} \int_{S_1} dv\, v \cdot \nabla_x \big(\mathcal{L}^{-1} v \cdot \nabla_x h^{(0)}\big) = 0.$$

Notice that the term $\xi^{(1)}(x, t)$ gives no contribution since $\int_{S_1} dv\, v{\cdot}\nabla_x \xi^{(1)}(x, t) = 0$. We define the 2×2 matrix $D_{ij} = \frac{1}{2\pi} \int_{S_1} dv\, v_i (-\mathcal{L})^{-1} v_j$ and we observe that $D_{ij} = 0$ for $i \neq j$ and $D_{11} = D_{22} = D > 0$, where

$$D = \frac{1}{4\pi} \int_{S_1} dv\, v \cdot \big(-\mathcal{L}\big)^{-1} v.$$

Therefore $h^{(0)}$ satisfies the heat equation

$$\begin{cases} \partial_t\, h^{(0)} - D\Delta_x h^{(0)} = 0, \\ h^{(0)}(x, 0) = \varrho_0(x). \end{cases} \tag{3.1.47}$$

In particular $h^{(0)}(t) \in L^{\infty}(\mathbb{R}^2 \times S_1)$ for any $t \geq 0$.

Let us consider equation (ii). By integrating with respect to the uniform measure on S_1 the left hand side vanishes, due to equation (3.1.47). Therefore we can invert the operator $\mathcal{L}$ to obtain

$$\begin{aligned} h^{(2)}(x, v, t) &= \mathcal{L}^{-1}\big(\partial_t h^{(0)}(x, t) + v \cdot \nabla_x (\mathcal{L}^{-1}(v \cdot \nabla_x) h^{(0)}(x, t))\big) + \\ &+ \mathcal{L}^{-1}\big(v \cdot \nabla_x \xi^{(1)}(x, t)\big) + \xi^{(2)}(x, t), \end{aligned} \tag{3.1.48}$$

where $\xi^{(2)}(x,t)$ belongs to the kernel of the operator $\mathcal{L}$.

The equation for $h^{(1)}$ reads

$$\partial_t\, h^{(1)} + v\cdot\nabla_x h^{(2)}(x,v,t) = \mathcal{L}h^{(3)}. \tag{3.1.49}$$

Therefore, integrating with respect to the uniform measure on S_1, using (3.1.48), we get the following closed equation for $\xi^{(1)}(x,t)$

$$\partial_t\, \xi^{(1)} - D\Delta_x \xi^{(1)} = 0. \tag{3.1.50}$$

Since there are no restrictions on the initial condition, we make the simplest choice $\xi^{(1)}(x,0)=0$. Therefore $\xi^{(1)}(x,t)=0$ for any $t\geq 0$ and hence we have the following expression for $h^{(1)}$

$$h^{(1)}(x,v,t) = \mathcal{L}^{-1}(v\cdot\nabla_x h^{(0)}(x,t)).$$

Thanks to Lemma 3.1.10 and the smoothness of $h^{(0)}$ we have

$$\sup_{t\in[0,T]}\|h^{(1)}(t)\|_\infty \leq \frac{1}{2\mu(1-\beta)}\sup_{t\in[0,T]}\|\nabla_x h^{(0)}(t)\|_\infty < +\infty.$$

The expression for the second order coefficient $h^{(2)}$ now reads

$$h^{(2)}(x,v,t) = h^{(2)}_\perp(x,v,t) + \xi^{(2)}(x,t),$$

where we set

$$h^{(2)}_\perp(x,v,t) = \mathcal{L}^{-1}\big(\partial_t h^{(0)}(x,t) + v\cdot\nabla_x(\mathcal{L}^{-1}(v\cdot\nabla_x)h^{(0)}(x,t))\big).$$

We observe that, since $h^{(0)}$ solves the heat equation (3.1.47), using Lemma 3.1.10 it follows that $h^{(2)}_\perp\in L^\infty\big([0,T];L^\infty(\mathbb{R}^2\times S_1)\big)$. Moreover any spatial derivative of $h^{(2)}_\perp$ belongs to $L^\infty\big([0,T];L^\infty(\mathbb{R}^2\times S_1)\big)$ as well.

By using (3.1.50), the left hand side of (3.1.49) belongs to $(Ker\mathcal{L})^\perp$. Therefore we can invert operator $\mathcal{L}$ obtaining

$$\begin{aligned} h^{(3)}(x,v,t) &= \mathcal{L}^{-1}\big(\partial_t\, h^{(1)} + v\cdot\nabla_x h^{(2)}(x,v,t)\big) + \xi^{(3)}(x,t),\\ &= \mathcal{L}^{-1}\big(\partial_t\, \mathcal{L}^{-1}(v\cdot\nabla_x h^{(0)}(x,t)) + v\cdot\nabla_x h^{(2)}(x,v,t)\big) + \xi^{(3)}(x,t), \end{aligned}$$

where $\xi^{(3)}(x,t)\in \operatorname{Ker}\mathcal{L}$. The equation for $h^{(2)}$ reads

$$\partial_t\, h^{(2)} + v\cdot\nabla_x h^{(3)} = \mathcal{L}h^{(4)}.$$

Integrating with respect to the uniform measure on S_1 and using the above expressions for $h^{(3)}$ and $h^{(2)}$ we find the following equation for $\xi^{(2)}(x,t)$

$$\partial_t\, \xi^{(2)} + D\Delta_x \xi^{(2)} = S(x,t), \tag{3.1.51}$$

where

$$
\begin{aligned}
S(x,t) \quad &= \quad -\frac{1}{2\pi}\int_{S_1} dv\ v\cdot\nabla_x\,\mathcal{L}^{-1}\big(\partial_t\,\mathcal{L}^{-1}(v\cdot\nabla_x h^{(0)}(x,t))\big) \\
&\quad\ -\frac{1}{2\pi}\int_{S_1} dv\ v\cdot\nabla_x\,\mathcal{L}^{-1}\left(v\cdot\nabla_x\,h^{(2)}_{\perp}(x,v,t)\right).
\end{aligned}
$$

We notice that $S\in L^{\infty}\big([0,T];L^{\infty}(\mathbb{R}^2)\big)$. As before we make the assumption $\xi^{(2)}(x,0)=0$, then $\xi^{(2)}\in L^{\infty}\big([0,T];L^{\infty}(\mathbb{R}^2)\big)$ and its spatial derivatives as well.

Now we consider the truncated expression (3.1.42). The first three coefficients are uniformly bounded. The remainder R_{η_ε} satisfies

$$
\big(\partial_t+\eta_\varepsilon\,v\cdot\nabla_x\big)R_{\eta_\varepsilon}=\eta_\varepsilon^2\mathcal{L}\,R_{\eta_\varepsilon}-A_{\eta_\varepsilon}, \tag{3.1.52}
$$

with initial condition

$$
R_{\eta_\varepsilon}(x,v,0)=:\bar{R}_{\eta_\varepsilon}(x,v)=-h^{(1)}(x,v,0)-\frac{1}{\eta_\varepsilon}\,h^{(2)}(x,v,0).
$$

Here $A_{\eta_\varepsilon}=\partial_t h^{(1)}+\frac{1}{\eta_\varepsilon}\partial_t\,h^{(2)}+v\cdot\nabla_x h^{(2)}$, then $A_{\eta_\varepsilon}\in L^{\infty}\big([0,T];L^{\infty}(\mathbb{R}^2\times S_1)\big)$. Note that the smoothness hypothesis on ϱ_0 ensures that $\bar{R}_{\eta_\varepsilon}\in L^{\infty}$.

We denote by $S_{\eta_\varepsilon}(t)$ the semigroup associated to the generator $-\eta_\varepsilon\big(v\cdot\nabla_x-\eta_\varepsilon\mathcal{L}\big)$. By equation (3.1.52) we get

$$
R_{\eta_\varepsilon}(t)=S_{\eta_\varepsilon}(t)R_{\eta_\varepsilon}(0)+\int_0^t ds\ S_{\eta_\varepsilon}(t-s)\,A_{\eta_\varepsilon}(s).
$$

By the usual series expansion for $S_{\eta_\varepsilon}(t)$ we obtain

$$
\begin{aligned}
R_{\eta_\varepsilon}(x,v,t)=&\sum_{N\geq 0}e^{-2\mu{\eta_\varepsilon}^2 t}\,(\mu\eta_\varepsilon)^N\int_0^{\eta_\varepsilon t}dt_1\ldots\int_0^{t_{N-1}}dt_N \\
&\int_{-1}^{1}d\rho_1\ldots\int_{-1}^{1}d\rho_N\,\bar{R}_{\eta_\varepsilon}(\gamma^{-\eta_\varepsilon t}(x,v))+ \\
&+\int_0^t ds\sum_{N\geq 0}e^{-2\mu{\eta_\varepsilon}^2(t-s)}\,(\mu\eta_\varepsilon)^N\int_0^{\eta_\varepsilon(t-s)}dt_1\ldots\int_0^{t_{N-1}}dt_N \\
&\int_{-1}^{1}d\rho_1\ldots\int_{-1}^{1}d\rho_N\,A_{\eta_\varepsilon}(\gamma^{-\eta_\varepsilon(t-s)}(x,v),s).
\end{aligned}
$$

Therefore

$$
\sup_{t\in[0,T]}\|R_{\eta_\varepsilon}(t)\|_{\infty}\leq\|\bar{R}_{\eta_\varepsilon}\|_{\infty}+T\sup_{t\in[0,T]}\|A_{\eta_\varepsilon}(t)\|_{\infty}\leq C<+\infty.
$$

$\square$

3.1.6 Proof of Proposition 3.1.5

The proof makes use of the Hilbert expansion in L^2 (see e.g. [EP] and [CIP]). Indeed we follow the same strategy of the previous subsection. Let h^S_ε be the solution of the following equation

$$\begin{cases} v_1 \partial_{x_1} h^S_\varepsilon(x_1, v) = \eta_\varepsilon \, \mathcal{L} h^S_\varepsilon(x_1, v), \\ h^S_\varepsilon(x_1, v) = \rho_1, \qquad x_1 = 0, \quad v_1 > 0, \\ h^S_\varepsilon(x_1, v) = \rho_2, \qquad x_1 = L, \quad v_1 < 0. \end{cases}$$

We assume that h^S_ε has the following form

$$h^S_\varepsilon(x_1, v) = h^{(0)}(x_1) + \sum_{k=1}^{+\infty} \left(\frac{1}{\eta_\varepsilon} \right)^k h^{(k)}(x_1, v).$$

We require $h^{(0)}$ to satisfy the same boundary conditions as the whole solution h^S_ε, namely

$$\begin{cases} h^{(0)}(x_1) = \rho_1, \qquad x_1 = 0, \\ \\ h^{(0)}(x_1) = \rho_2, \qquad x_1 = L. \end{cases} \tag{3.1.53}$$

Comparing terms of the same order we get

$$v_1 \partial_{x_1} h^{(k)} = \mathcal{L}\, h^{(k+1)}, \qquad k \geq 0.$$

The first two equations read

(i) $v_1 \partial_{x_1} h^{(0)} = \mathcal{L}\, h^{(1)}$,

(ii) $v_1 \partial_{x_1} h^{(1)} = \mathcal{L}\, h^{(2)}$,

which have a solution if and only if the left hand side belongs to $(Ker\mathcal{L})^\perp = \{g \in L^2(S_1) : \int_{S_1} g(v)\, dv = 0\}$. Since $v_1 \partial_{x_1} h^{(0)}$ is an odd function of v we can invert the operator $\mathcal{L}$ and set

$$h^{(1)}(x_1, v) = \mathcal{L}^{-1}(v_1 \partial_{x_1} h^{(0)}) + \xi^{(1)}(x_1), \tag{3.1.54}$$

where $\xi^{(1)} \in Ker\mathcal{L}$. We integrate equation (ii) with respect to the uniform measure on S_1. Observing that $\int_{S_1} dv\, \mathcal{L}\, h^{(2)} = 0$, by (3.1.54) we obtain

$$\left(\int_{S_1} dv\, v_1 \mathcal{L}^{-1} v_1 \right) \partial^2_{x_1} h^{(0)} = 0,$$

with the boundary conditions (3.1.53). Therefore

$$h^{(0)}(x_1) = \frac{\rho_1 (L - x_1) + \rho_2 x_1}{L}.$$

With the same strategy as the previous subsection, one can prove that $\xi^{(1)}(x_1) \equiv 0$. Hence

$$h^{(1)}(x_1, v) = h^{(1)}(v_1) = \left(\frac{\rho_2 - \rho_1}{L}\right) \mathcal{L}^{-1}(v_1). \tag{3.1.55}$$

Moreover by equation (ii) we get

$$\begin{aligned} h^{(2)}(x_1, v) &= \mathcal{L}^{-1}\big(v_1 \partial_{x_1} h^{(1)}(x_1, v)\big) + \xi^{(2)}(x_1) \\ &= \xi^{(2)}(x_1), \end{aligned}$$

where in the last step we used (3.1.55). By iterating the same procedure of the previous subsection, since in this case the source term in (3.1.51) is zero, we have that $\xi^{(2)}$ satisfies $\partial^2_{x_1}\xi^{(2)} = 0$. We choose zero boundary conditions so that $\xi^{(2)}(x_1) \equiv 0$. Then

$$h^{(2)}(x_1, v) \equiv 0.$$

We consider the truncated expansion

$$h^S_\varepsilon = h^{(0)} + \frac{1}{\eta_\varepsilon} h^{(1)} + \frac{1}{\eta_\varepsilon} R_{\eta_\varepsilon}. \tag{3.1.56}$$

The remainder R_{η_ε} satisfies

$$v_1 \partial_{x_1} R_{\eta_\varepsilon} = \eta_\varepsilon \mathcal{L}\, R_{\eta_\varepsilon}. \tag{3.1.57}$$

We required $h^{(0)}$ to satisfy the same boundary conditions as the whole solution h^S_ε, then the boundary conditions for R_{η_ε} read

$$\begin{cases} R_{\eta_\varepsilon}(x_1, v) = -\left(\dfrac{\rho_2 - \rho_1}{L}\right) \mathcal{L}^{-1}(v_1), & x_1 = 0, \quad v_1 > 0, \\[2ex] R_{\eta_\varepsilon}(x_1, v) = -\left(\dfrac{\rho_2 - \rho_1}{L}\right) \mathcal{L}^{-1}(v_1), & x_1 = L, \quad v_1 < 0. \end{cases}$$

The unique solution of the above problem is

$$\begin{aligned} R_{\eta_\varepsilon}(x_1, v) = &- e^{\frac{\eta_\varepsilon}{v_1} x_1 \mathcal{L}} \left(\frac{\rho_2 - \rho_1}{L}\right) \mathcal{L}^{-1}(v_1) \chi(v_1 > 0) \\ &- e^{-\frac{\eta_\varepsilon}{v_1}(L - x_1)\mathcal{L}} \left(\frac{\rho_2 - \rho_1}{L}\right) \mathcal{L}^{-1}(v_1) \chi(v_1 < 0). \end{aligned}$$

By (3.1.57) we get

$$-\eta_\varepsilon \left(R_{\eta_\varepsilon}, -\mathcal{L} R_{\eta_\varepsilon}\right) = -b_{\eta_\varepsilon},$$

where the boundary term b_{η_ε} is given by

$$b_{\eta_\varepsilon} = -\int_{v_1 > 0} dv\, v_1 \left(\frac{\rho_2 - \rho_1}{L}\right)^2 \left(\mathcal{L}^{-1}(v_1)\right)^2 \left[e^{\frac{\eta_\varepsilon}{v_1} L \mathcal{L}} - 1\right].$$

We remark that $(\cdot,\cdot)$ and $\|\cdot\|_2$ denote the scalar product and the norm in $L^2((0,L)\times S_1)$ respectively. Observe that $b_{\eta_\varepsilon} \geq 0$. Using the spectral gap of the operator $\mathcal{L}$ we get

$$-b_{\eta_\varepsilon} = -\eta_\varepsilon \left(R_{\eta_\varepsilon}, -\mathcal{L}R_{\eta_\varepsilon}\right) \leq -\lambda\eta_\varepsilon \|R_{\eta_\varepsilon}\|_2^2, \tag{3.1.58}$$

where λ is the first positive eigenvalue of $-\mathcal{L}$. Therefore we obtain

$$\|R_{\eta_\varepsilon}\|_2 \leq \frac{C}{\sqrt{\eta_\varepsilon}}.$$

Since the coefficients $h^{(1)}$ and $h^{(2)}$ are bounded, we have that h_ε^S converges to $h^{(0)}$ in $L^2((0,L)\times S_1)$ for $\eta_\varepsilon \to \infty$.

□

3.1.7 The kinetic description

3.1.8 The extension argument

We remind that h_ε^{out} is the solution of the Boltzmann equation (3.1.20), therefore it can be expressed as

$$\begin{aligned} h_\varepsilon^{out}(x,v,t) = \sum_{n\geq 0} (\mu_\varepsilon \varepsilon)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \\ \int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_n \, \chi(\tau < t_n)\, \chi(\tau > 0)\, e^{-2\mu_\varepsilon \varepsilon (t-\tau)} f_B(\gamma^{-(t-\tau)}(x,v)), \end{aligned} \tag{3.1.59}$$

with $f_B(x,v)$ defined in (3.1.5) and

$$\gamma^{-(t-\tau)}(x,v) = (x - v(t-\tau-t_1) - v_1(t_1-t_2)\cdots - v_n t_n, v_n). \tag{3.1.60}$$

Lemma 3.1.11. *Let h_ε^{out} be the solution of the Boltzmann equation* (3.1.20) *defined in* (3.1.59). *Then*

$$\begin{aligned} h_\varepsilon^{out}(x,v,t) = \sum_{N\geq 0} e^{-2\mu_\varepsilon \varepsilon t} (\mu_\varepsilon \varepsilon)^N \int_0^t dt_1 \ldots \int_0^{t_{N-1}} dt_N \\ \int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_N \, \chi(\tau > 0)\, f_B(\gamma^{-(t-\tau)}(x,v)). \end{aligned} \tag{3.1.61}$$

The above identity follows from the fact that in the last term we added fictitious jumps, those in the time interval $(0,\tau)$ which do not affect $f_B(\gamma^{-(t-\tau)}(x,v))$ but allows us to remove the indicator function $\chi(t_n > \tau)$ replacing consequently the factor $e^{-2\mu_\varepsilon\varepsilon(t-\tau)}$ by the more handable factor $e^{-2\mu_\varepsilon\varepsilon t}$. In view of the particle interpretation it is convenient to think the trajectory γ^{-s}, $s \in (0,t)$ as extended outside Λ, see Figure 3.2. The dashed part of the trajectory is ininfluent for the evaluation of h_ε^{out}.

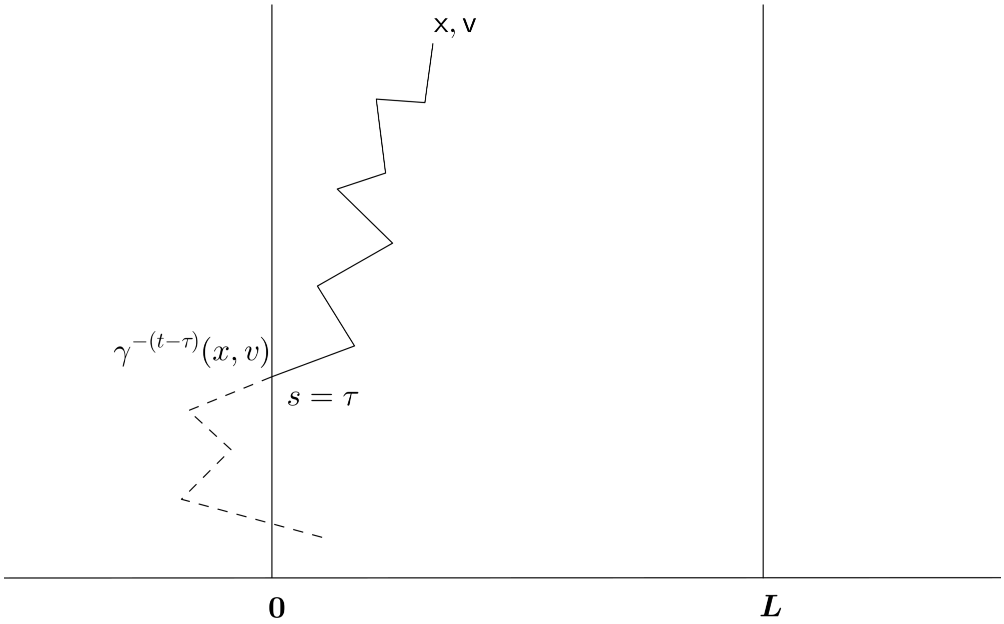

Figure 3.2:

Proof. Observe that for $\tau > 0$, τ given,

$$1 = \sum_{m\geq 0} (\mu_\varepsilon \varepsilon)^m \int_0^t ds_1 \ldots \int_0^{s_{m-1}} ds_m \; \chi(s_1 \leq \tau) \int_{-1}^1 d\xi_1 \ldots \int_{-1}^1 d\xi_m \, e^{-2\mu_\varepsilon \varepsilon \tau}.$$

Using the previous identity we can express h_ε^{out} as

$$h_\varepsilon^{out}(x,v,t) = \sum_{N\geq 0} e^{-2\mu_\varepsilon \varepsilon t} (\mu_\varepsilon \varepsilon)^N \int_0^t dt_1 \ldots \int_0^{t_{N-1}} dt_N \int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_N$$
$$\left(\sum_{n=0}^N \chi(t_n > \tau)\chi(t_{n+1} \leq \tau) \right) \; \chi(\tau > 0) \, f_B(\gamma^{-(t-\tau)}(x,v)),$$

with the convention that $t_0 = t$, $t_{N+1} = 0$. Since

$$\left(\sum_{n=0}^N \chi(t_n > \tau)\chi(t_{n+1} \leq \tau) \right) = 1,$$

we obtain the desired result. $\square$

3.1.9 Proof of Proposition 3.1.6

By (3.1.31) for $(x,v) \in \Lambda \times S_1$, $t > 0$ we have

$$f_\varepsilon^{out}(x,v,t) = e^{-\mu_\varepsilon |B_t(x)\cap\Lambda\setminus B_\varepsilon(x)|} \sum_{q\geq 0} \frac{\mu_\varepsilon^q}{q!} \int_{(B_t(x)\cap\Lambda\setminus B_\varepsilon(x))^q} d\mathbf{c}_q \\ \chi(\tau > 0) f_B(T_{\mathbf{c}_q}^{-(t-\tau)}(x,v)).$$

Here $T_{\mathbf{c}_q}^{-(t-\tau)}(x,v)$ is the flow associated to the initial datum (x,v) for a given scatterers configuration $\mathbf{c}_q$, $B_t(x)$ and $B_\varepsilon(x)$ denote the disks centered in x with radius t and ε respectively.

Since $f_B(T_{\mathbf{c}_q}^{-(t-\tau)}(x,v))$ depends only on the scatterer configurations inside $\Lambda \cap B_t(x)$, we want to add fictitious scatterers outside Λ which do not affect the value $f_B(T_{\mathbf{c}_q}^{-(t-\tau)}(x,v))$ in the same spirit of Lemma 3.1.11. However there is a small difficulty because the scatterers located in the vertical strips $[-\varepsilon,0]\times\mathbb{R}$ and $[L,L+\varepsilon]\times\mathbb{R}$ actually can modify the value of τ. For this reason we introduce

$$\breve{f}_\varepsilon^{out}(x,v,t) = e^{-\mu_\varepsilon |B_t^\varepsilon(x)|} \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{(B_t^\varepsilon(x))^Q} d\mathbf{c}_Q\, \chi(\tau > 0) \\ \big(1 - \chi_{\partial\Lambda}(\mathbf{c}_Q)\big) f_B(T_{\mathbf{c}_Q}^{-(t-\tau)}(x,v)),$$

where

$$\chi_{\partial\Lambda}(\mathbf{c}_Q) = \chi\{\mathbf{c}_Q :\ \exists i = 1,\dots Q \text{ s.t. } c_i \in [-\varepsilon,0]\times\mathbb{R} \cup [L,L+\varepsilon]\times\mathbb{R} \\ \text{and } |x_\varepsilon(-s) - c_i| = \varepsilon,\ s\in[0,t]\}, \tag{3.1.62}$$

allows to have a consistency in the definition of the hitting time for the extended dynamics. Here $B_t^\varepsilon(x) := B_t(x) \setminus B_\varepsilon(x)$. We expect that the contribution due to the obstacles with centers in the vertical strips $[-\varepsilon,0]\times\mathbb{R}$, $[L,L+\varepsilon]\times\mathbb{R}$ influencing the trajectory is indeed negligible in the limit. This fact will be discussed later on (see Section 3.1.10).

Since $|B_t^\varepsilon(x) \setminus \{[-\varepsilon,0]\times\mathbb{R}\cup[L,L+\varepsilon]\times\mathbb{R}\}| \leq |B_t^\varepsilon(x)|$, then $f_\varepsilon^{out} \geq \breve{f}_\varepsilon^{out}$.

We distinguish the obstacles of the configuration $\mathbf{c}_Q = c_1 \dots c_Q$ which, up to the time t, influence the motion, called internal obstacles, and the external ones. More precisely, c_i is internal if

$$\inf_{0\leq s\leq t} |x_\varepsilon(-s) - c_i| = \varepsilon,$$

while c_i is external if

$$\inf_{0\leq s\leq t} |x_\varepsilon(-s) - c_i| > \varepsilon.$$

Here $(x_\varepsilon(-s), v_\varepsilon(-s)) = T^{-s}_{\mathbf{c}_Q}(x,v)$, $s \in [0,t]$. We observe that the characteristic function $\chi_{\partial\Lambda}$ depends only on internal obstacles. Therefore, by integrating over the external obstacles we obtain

$$\breve{f}^{out}_\varepsilon(x,v,t) = \sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B^\varepsilon_t(x)^N} d\mathbf{b}_N \, e^{-\mu_\varepsilon |\mathcal{T}_t(\mathbf{b}_N)|} \chi(\tau > 0)$$
$$\chi(\{\mathbf{b}_N \text{ internal}\}) \big(1 - \chi_{\partial\Lambda}(\mathbf{b}_N)\big) f_B(T^{-(t-\tau)}_{\mathbf{b}_N}(x,v)),$$

where $\mathcal{T}_t(\mathbf{b}_N)$ is the tube

$$\mathcal{T}_t(\mathbf{b}_N) = \{y \in B^\varepsilon_t(x) \text{ s.t. } \exists s \in [0,t] \text{ s.t. } |y - x_\varepsilon(-s)| \leq \varepsilon\}.$$

We define

$$\tilde{f}^{out}_\varepsilon(x,v,t) = e^{-2\mu_\varepsilon \varepsilon t} \sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B^\varepsilon_t(x)^N} d\mathbf{b}_N \, \chi(\{\mathbf{b}_N \text{ internal}\})$$
$$\big(1 - \chi_{\partial\Lambda}(\mathbf{b}_N)\big) \, f_B(T^{-(t-\tau)}_{\mathbf{b}_N}(x,v)) \, \chi(\tau > 0).$$

Since $|\mathcal{T}_t(\mathbf{b}_N)| \leq 2\varepsilon t$, then $f^{out}_\varepsilon \geq \breve{f}^{out}_\varepsilon \geq \tilde{f}^{out}_\varepsilon$.

Note that, according to a classical argument introduced in [G] (see also [DP], [DR]), we remove from $\tilde{f}^{out}_\varepsilon$ all the bad events, namely those untypical with respect to the Markov process described by h^{out}_ε. Then we will show they are unlikely.

For any fixed initial condition (x,v) we order the obstacles $b_1, \dots, b_N$ according to the scattering sequence. Let ρ_i and t_i be the impact parameter and the hitting time of the light particle with $\partial B_\varepsilon(b_i)$ respectively. Then we perform the following change of variables

$$b_1, \dots, b_N \to \rho_1, t_1, \dots, \rho_N, t_N \tag{3.1.63}$$

with

$$0 \leq t_N < t_{N-1} < \dots < t_1 \leq t.$$

Conversely, fixed the impact parameters $\{\rho_i\}$ and the hitting times $\{t_i\}$ we construct the centers of the obstacles $b_i = b(\rho_i, t_i)$. By performing the backward scattering we construct a trajectory $\gamma^{-s}(x,v) := (\xi_\varepsilon(-s), \omega_\varepsilon(-s))$, $s \in [0,t]$, where

$$\begin{cases} \xi_\varepsilon(-t) = x - v(t-t_1) - v_1(t_1 - t_2) \cdots - v_N t_N \\ \omega_\varepsilon(-t) = v_N. \end{cases} \tag{3.1.64}$$

Here $v_1, \dots, v_N$ are the incoming velocities. We remark that ω_ε is an autonomous jump process and ξ_ε is an additive functional of ω_ε.

Observe that the map (3.1.63) is one-to-one, and so $(\xi_\varepsilon(-s), \omega_\varepsilon(-s)) = (x_\varepsilon(-s), v_\varepsilon(-s))$, only outside the following pathological situations.

i) **Recollisions**.
There exists b_i such that for $s \in (t_{j+1}, t_j)$, $j > i$, $\xi_\varepsilon(-s) \in \partial B(b_i, \varepsilon)$.

ii) **Interferences**.
There exists b_j such that $\xi_\varepsilon(-s) \in B(b_j, \varepsilon)$ for $s \in (t_{i+1}, t_i)$, $j > i$.

In order to skip such events we define

$$\begin{aligned}\bar{f}_\varepsilon^{out}(x, v, t) =& e^{-2\mu_\varepsilon \varepsilon t} \sum_{N \geq 0} \mu_\varepsilon^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_N \\ & \chi(\tau > 0)\,(1 - \chi_{\partial\Lambda})(1 - \chi_{rec})(1 - \chi_{int}) f_B(\gamma^{-(t-\tau)}(x, v)),\end{aligned} \tag{3.1.65}$$

where

$$\begin{aligned}\chi_{rec} =& \chi(\{\mathbf{b}_N \text{ s.t. i) is realized}\}) \\ \chi_{int} =& \chi(\{\mathbf{b}_N \text{ s.t. ii) is realized}\}).\end{aligned} \tag{3.1.66}$$

Observe that in (3.1.65) $\gamma^{-(t-\tau)}(x, v) = (x_\varepsilon(-(t-\tau)), v_\varepsilon(-(t-\tau)))$. Moreover

$$\bar{f}_\varepsilon^{out} \leq \tilde{f}_\varepsilon^{out} \leq \breve{f}_\varepsilon^{out} \leq f_\varepsilon^{out}.$$

Next we represent, thanks to Lemma 3.1.11, h_ε^{out}, solution to equation (3.1.20), as

$$\begin{aligned}h_\varepsilon^{out}(x, v, t) =& e^{-2\mu_\varepsilon \varepsilon t} \sum_{N \geq 0} \mu_\varepsilon^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \\ & \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_N \, \chi(\tau > 0)\, f_B(\gamma^{-(t-\tau)}(x, v)).\end{aligned} \tag{3.1.67}$$

Observe that

$$1 - (1 - \chi_{\partial\Lambda})(1 - \chi_{rec})(1 - \chi_{int}) \leq \chi_{\partial\Lambda} + \chi_{rec} + \chi_{int}. \tag{3.1.68}$$

Then by (3.1.65) and (3.1.67) we obtain

$$|h_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t)| \leq \varphi_1(\varepsilon, t), \tag{3.1.69}$$

with

$$\begin{aligned}\varphi_1(\varepsilon, t) :=& \|f_B\|_\infty e^{-2\mu_\varepsilon \varepsilon t} \sum_{N \geq 0} (\mu_\varepsilon)^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \\ & \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_N \, \{\chi_{\partial\Lambda} + \chi_{rec} + \chi_{int}\}.\end{aligned} \tag{3.1.70}$$

We state the following result. The proof is postponed to Section 3.1.10.

Lemma 3.1.12. *Let* $\varphi_1(\varepsilon, t)$ *be defined in* (3.1.70). *For any* $t \in [0, T]$ *we have*

$$\|\varphi_1(\varepsilon, t)\|_{L^\infty} \leq C\varepsilon^{\frac{1}{2}}\, \eta_\varepsilon^3\, t^2. \tag{3.1.71}$$

Let us estimate the difference $\left|f_\varepsilon^{out}(t) - h_\varepsilon^{out}(t)\right|$. By (3.1.69) we have

$$\begin{aligned} \left|f_\varepsilon^{out}(t) - h_\varepsilon^{out}(t)\right| &\leq \left|f_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t)\right| + \left|\bar{f}_\varepsilon^{out}(t) - h_\varepsilon^{out}(t)\right| \\ &\leq \left|f_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t)\right| + \varphi_1(\varepsilon, t). \end{aligned} \tag{3.1.72}$$

Since $\bar{f}_\varepsilon^{out} \leq f_\varepsilon^{out}$, the difference $f_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t)$ is non negative and we can skip the absolute value. Moreover

$$f_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t) \leq \left(f_\varepsilon^{out}(t) - \breve{f}_\varepsilon^{out}(t)\right) + \left(\breve{f}_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t)\right). \tag{3.1.73}$$

Using the fact that the map (3.1.63) is one-to-one outside the pathological sets we can write $\bar{f}_\varepsilon^{out}$ in (3.1.65) as

$$\begin{aligned} \bar{f}_\varepsilon^{out}(t) = e^{-2\mu_\varepsilon \varepsilon t} \sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B_t^\varepsilon(x)^N} d\mathbf{b}_N\, \chi(\{\mathbf{b}_N \text{ internal}\})\, \chi(\tau > 0) \\ (1 - \chi_{\partial\Lambda})(1 - \chi_{rec})(1 - \chi_{int}) f_B(T_{\mathbf{b}_N}^{-(t-\tau)}(x, v)). \end{aligned}$$

Hence

$$\begin{aligned} \breve{f}_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t) = &\sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B_t^\varepsilon(x)^N} d\mathbf{b}_N\, f_B(T_{\mathbf{b}_N}^{-(t-\tau)}(x, v))\ \chi(\{\mathbf{b}_N \text{ internal}\})\, \chi(\tau > 0) \\ &(1 - \chi_{\partial\Lambda})\big(e^{-\mu_\varepsilon |\mathcal{T}_t(\mathbf{b}_N)|} - e^{-2\,\mu_\varepsilon \varepsilon\, t}\,(1 - \chi_{rec})(1 - \chi_{int})\big) \\ \leq &\,||f_B||_\infty \sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B_t^\varepsilon(x)^N} d\mathbf{b}_N\ \chi(\{\mathbf{b}_N \text{ internal}\}) \\ &\big(e^{-\mu_\varepsilon |\mathcal{T}_t(\mathbf{b}_N)|} - e^{-2\,\mu_\varepsilon \varepsilon\, t}\,(1 - \chi_{rec})(1 - \chi_{int})\big). \end{aligned}$$

By observing that

$$\sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B_t^\varepsilon(x)^N} d\mathbf{b}_N\, \chi(\{\mathbf{b}_N \text{ internal}\})\, e^{-\mu_\varepsilon |\mathcal{T}_t(\mathbf{b}_N)|} = 1,$$

we obtain

$$\begin{aligned} \breve{f}_\varepsilon^{out}(t) - \bar{f}_\varepsilon^{out}(t) \leq ||f_B||_\infty \Big(1 - e^{-2\,\mu_\varepsilon \varepsilon\, t} \sum_{N\geq 0} \mu_\varepsilon^N \int_0^t dt_1 \dots \int_0^{t_{N-1}} dt_N \\ \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_N\, (1 - \chi_{rec})(1 - \chi_{int})\Big). \end{aligned}$$

Observe that

$$1-(1-\chi_{rec})(1-\chi_{int}) \leq \chi_{rec}+\chi_{int}.$$

Hence we get

$$\breve{f}_\varepsilon^{out}(t)-\bar{f}_\varepsilon^{out}(t) \leq \varphi_1(\varepsilon,t), \tag{3.1.74}$$

with φ_1 defined in (3.1.70).

Now we consider $f_\varepsilon^{out}(t)-\breve{f}_\varepsilon^{out}(t)$. We observe that

$$\begin{aligned} f_\varepsilon^{out}(x,v,t) = & e^{-\mu_\varepsilon |B_t^\varepsilon(x)\setminus\partial\Lambda^\varepsilon|}\sum_{Q\geq 0}\frac{\mu_\varepsilon^Q}{Q!}\int_{(B_t^\varepsilon(x))^Q} d\mathbf{c}_Q\,\chi(\tau>0) \\ & \big(1-\chi_{\partial\Lambda}(\mathbf{c}_Q)\big) f_B(T_{\mathbf{c}_Q}^{-(t-\tau)}(x,v)), \end{aligned}$$

where $\partial\Lambda^\varepsilon := \big([-\varepsilon,0]\cup[L,L+\varepsilon]\big)\times\mathbb{R}$. By using the previous strategy one can prove

$$f_\varepsilon^{out}(t)-\breve{f}_\varepsilon^{out}(t) \leq \varphi_1(\varepsilon,t). \tag{3.1.75}$$

Therefore (3.1.72), (3.1.74), (3.1.75) and (3.1.71) imply

$$\|f_\varepsilon^{out}(t)-h_\varepsilon^{out}(t)\|_\infty \leq C\varepsilon^{\frac{1}{2}}\,\eta_\varepsilon^3\,t^2.$$

3.1.10 Proof of Lemma 3.1.12 (the control of the pathological sets)

For any measurable function u of the process $(\xi_\varepsilon,\omega_\varepsilon)$ defined in (3.1.64) we set

$$\begin{aligned} \mathbb{E}_{x,v}(u) = & e^{-2\mu_\varepsilon\varepsilon t}\sum_{N\geq 0}(\mu_\varepsilon)^N\int_0^t dt_1\ldots\int_0^{t_{N-1}} dt_N \\ & \int_{-\varepsilon}^{\varepsilon} d\rho_1\ldots\int_{-\varepsilon}^{\varepsilon} d\rho_N\,u(\xi_\varepsilon,\omega_\varepsilon). \end{aligned}$$

Then we realize that

$$\varphi_1(\varepsilon,t) = \|f_B\|_\infty\,\mathbb{E}_{x,v}[\chi_{\partial\Lambda}+\chi_{rec}+\chi_{int}]$$

and we estimate separately the events in (3.1.62) and (3.1.66).

We consider the interference event. Let t_i the first time the light particle hits the i-th scatterer, v_i^- the incoming velocity and v_i^+ the outgoing velocity (for the backward trajectory). Moreover we fix the axis in such a way that v_i^+ is parallel to the x axis. We have

$$\chi_{int} \leq \sum_{i=1}^N\sum_{j>i}\chi_{int}^{i,j},$$

where $\chi_{int}^{i,j}=1$ if the obstacle with center b_j belongs to the tube spanned by $\xi_\varepsilon(-s)$ for $s\in(t_{i+1},t_i)$. We denote by α the angle between v_i^+ and v_{j-1}^+. We have two situations, when the velocity v_{j-1}^+ is transverse to v_i^+ (i.e. $\alpha > \varepsilon^\gamma$ for a suitable positive γ) or when the velocity v_{j-1}^+ is almost parallel to v_i^+ (i.e. $\alpha \le \varepsilon^\gamma$). Then

$$\mathbb{E}_{x,v}[\chi_{int}] \le \mathbb{E}_{x,v}\big[\sum_{i=1}^N \sum_{j>i} \chi_{int}^{i,j} \chi(\alpha > \varepsilon^\gamma)\big] + \mathbb{E}_{x,v}\big[\sum_{i=1}^N \sum_{j>i} \chi_{int}^{i,j} \chi(\alpha \le \varepsilon^\gamma)\big]. \tag{3.1.76}$$

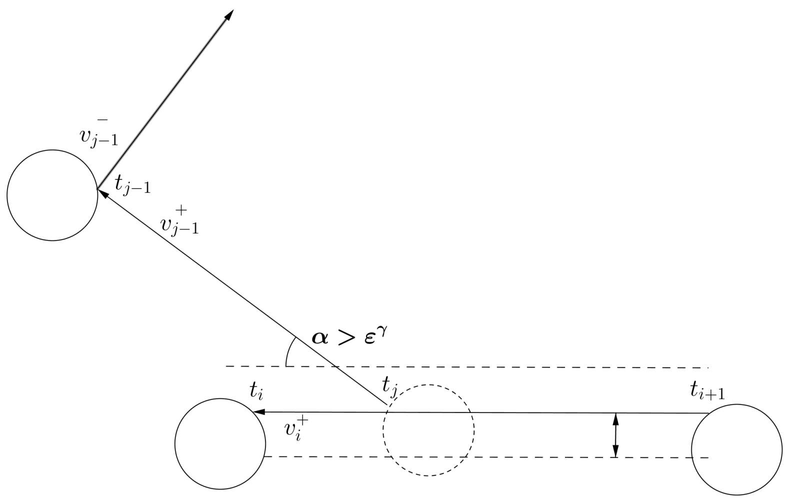

Figure 3.3: Backward Interference-First case

We estimate separately the two contributions. To estimate the first term we fix all the variables $\{t_h\}_{h=1}^N$, $\{\rho_h\}_{h=1}^N$ except t_j. By a simple geometrical argument we argue that the integral over t_j is restricted over an interval of measure at most $C\varepsilon^{1-\gamma}$. Hence we get

$$\begin{aligned} &\mathbb{E}_{x,v}\big[\sum_{i=1}^N \sum_{j>i} \chi_{int}^{i,j} \chi(\alpha > \varepsilon^\gamma)\big] \\ &\le e^{-2\mu_\varepsilon \varepsilon t} \sum_{N\ge 1} (N)^2 (2\mu_\varepsilon \varepsilon)^N \frac{t^{N-1}}{(N-1)!} C\varepsilon^{1-\gamma} \\ &\le e^{-2\mu_\varepsilon \varepsilon t} (2\mu_\varepsilon \varepsilon)^3 t^2 \sum_{N\ge 3} (2\mu_\varepsilon \varepsilon)^{N-3} \frac{t^{N-3}}{(N-3)!} C\varepsilon^{1-\gamma} \\ &\le C\varepsilon^{1-\gamma} \eta_\varepsilon^3 t^2. \end{aligned} \tag{3.1.77}$$

Concerning the second term in (3.1.76), the condition $\alpha \le \varepsilon^\gamma$ implies that the $(j-1)$-th scattering angle θ_{j-1} can varies at most ε^γ (see Figure

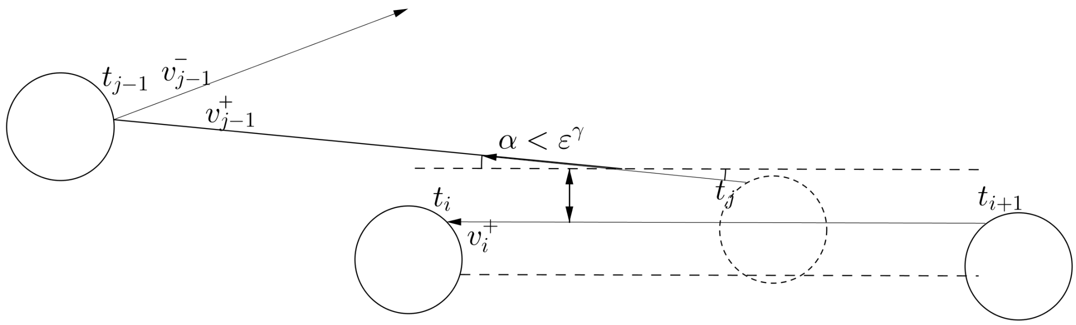

Figure 3.4: Backward Interference-Second case

3.4). Then, fixing all the variables $\{t_h\}_{h=1}^N$, $\{\rho_h\}_{h=1}^N$ except ρ_{j-1}, performing the change of variable $\rho_{j-1} \to \theta_{j-1}$ and recalling that the scattering cross section for a disk of unitary radius is given by $\frac{d\rho}{d\theta} = \frac{1}{2}\sin\frac{\theta}{2}$ we obtain

$$\begin{aligned} \mathbb{E}_{x,v}\big[\sum_{i=1}^{N}\sum_{j>i}\chi_{int}^{i,j}\chi(\alpha \le \varepsilon^\gamma)\big] \le& e^{-2\mu_\varepsilon \varepsilon t}\sum_{N\ge 1}(N)^2(2\mu_\varepsilon\varepsilon)^N\frac{t^N}{(N)!}C\varepsilon^\gamma \\ \le& C\varepsilon^\gamma\eta_\varepsilon^2 t^2. \end{aligned} \tag{3.1.78}$$

By choosing $\gamma = 1/2$, from (3.1.77) and (3.1.78) we obtain

$$\mathbb{E}_{x,v}[\chi_{int}] \le C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^3 t^2. \tag{3.1.79}$$

Finally we consider the recollision event. We have

$$\chi_{rec} \le \sum_{i=1}^{N}\sum_{j>i}\chi_{rec}^{i,j},$$

where $\chi_{rec}^{i,j} = 1$ if the i-th obstacle is recollided in the time interval (t_j, t_{j-1}). Also in this case we have to take into account two possible situations, when $|b_i - b_{j-1}| > \varepsilon^\gamma$ for a suitable positive γ or when $|b_i - b_{j-1}| \le \varepsilon^\gamma$. Then

$$\begin{aligned} \mathbb{E}_{x,v}[\chi_{rec}] \le& \mathbb{E}_{x,v}\Big[\sum_{i=1}^{N}\sum_{j>i}\chi_{rec}^{i,j}\chi\big(|b_i - b_{j-1}| > \varepsilon^\gamma\big)\Big] \\ &+ \mathbb{E}_{x,v}\Big[\sum_{i=1}^{N}\sum_{j>i}\chi_{rec}^{i,j}\chi\big(|b_i - b_{j-1}| \le \varepsilon^\gamma\big)\Big]. \end{aligned} \tag{3.1.80}$$

We look at the first term. Using geometric arguments the condition $|b_i - b_{j-1}| > \varepsilon^\gamma$ gives a bound for the $(j-1)$-th scattering angle θ_{j-1} (see

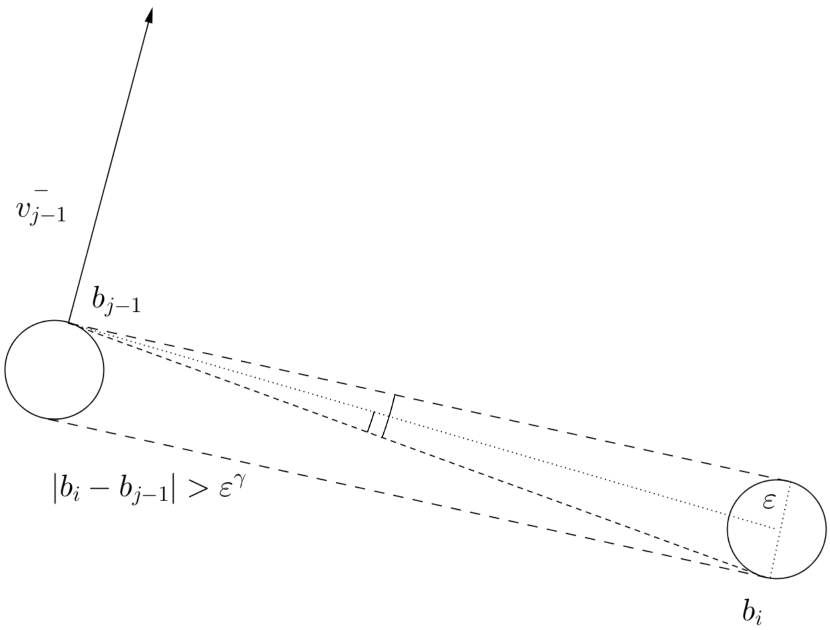

Figure 3.5: Backward Recollision-First case

Figure 3.5). In particular it can varies at most $\varepsilon/\varepsilon^\gamma = \varepsilon^{1-\gamma}$. Therefore, performing the change of variable $\rho_{j-1} \to \theta_{j-1}$ as before, we get

$$\begin{aligned} &\mathbb{E}_{x,v}\Big[\sum_{i=1}^{N}\sum_{j>i}\chi_{rec}^{i,j}\chi\big(|b_i - b_{j-1}| > \varepsilon^\gamma\big)\Big] \\ &\le e^{-2\mu_\varepsilon \varepsilon t}\sum_{N\ge 1}(N)^2(2\mu_\varepsilon\varepsilon)^N\frac{t^N}{(N)!}C\varepsilon^{1-\gamma} \\ &\le C\varepsilon^{1-\gamma}\eta_\varepsilon^2 t^2. \end{aligned} \tag{3.1.81}$$

If $|b_i - b_{j-1}| \le \varepsilon^\gamma$ a simple geometrical argument shows that the time interval $|t_{j-1} - t_j|$ is bounded by ε^γ (see Figure 3.6). Hence, following the same strategy as in (3.1.77), we obtain

$$\begin{aligned} &\mathbb{E}_{x,v}\Big[\sum_{i=1}^{N}\sum_{j>i}\chi_{rec}^{i,j}\chi\big(|b_i - b_{j-1}| \le \varepsilon^\gamma\big)\Big] \\ &\le e^{-2\mu_\varepsilon \varepsilon t}\sum_{N\ge 1}(N)^2(2\mu_\varepsilon\varepsilon)^N\frac{t^{N-1}}{(N-1)!}C\varepsilon^{\gamma} \\ &\le C\varepsilon^{\gamma}\eta_\varepsilon^3 t^2. \end{aligned} \tag{3.1.82}$$

As before we choose $\gamma = 1/2$. Then from (3.1.81) and (3.1.82) we obtain

$$\mathbb{E}_{x,v}[\chi_{rec}] \le C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^3 t^2. \tag{3.1.83}$$

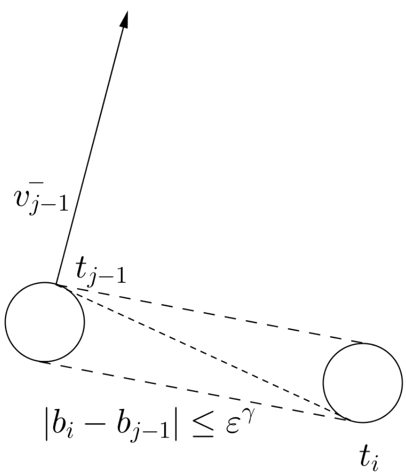

Figure 3.6: Backward Recollision-Second case

We now consider the expectation value for $(1 - \chi_{\partial\Lambda})$, with $\chi_{\partial\Lambda}$ defined in (3.1.62). Observe that $\chi_{\partial\Lambda} = 1$ implies that $\xi_\varepsilon(-(t - t_j)) \in \Lambda^c$ and $d(\xi_\varepsilon(-(t-t_j)), \partial\Lambda) \leq \varepsilon$ for some $j = 1, \dots, N$. As we can see in Figure 3.7, by the same argument used to estimate the interference events in (3.1.77) and (3.1.78) we obtain

$$\mathbb{E}_{x,v}[\chi_{\partial\Lambda}] \leq C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^3 t^2. \tag{3.1.84}$$

By estimates (3.1.79), (3.1.83) and (3.1.84) we obtain

$$\|\varphi_1(\varepsilon, t)\|_\infty \leq C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^3 t^2,$$

for some $C > 0$.

3.1.11 Proof of Proposition 3.1.7

The proof follows the same strategy of the proof of Proposition 3.1.6. Actually it is easier since it does not require the extension trick, but it follows directly by the recollision and interference estimates.

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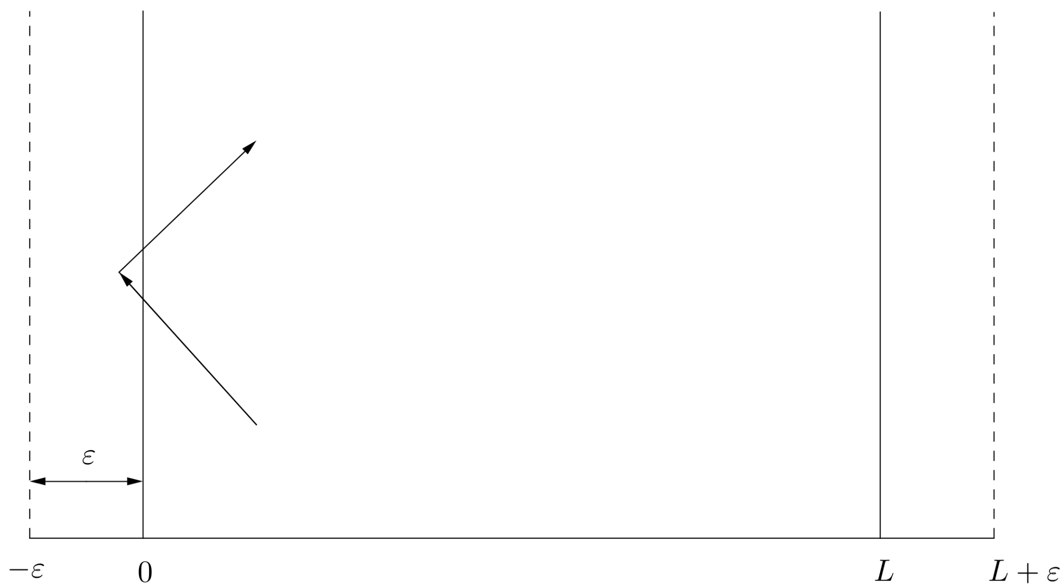

Figure 3.7: $\Lambda \cup \{[-\varepsilon, 0] \times \mathbb{R} \cup [L, L+\varepsilon] \times \mathbb{R}\}$

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3.2

In the present Chapter we present [N3].

Fick's Law for the Lorentz Model in a weak coupling regime

Abstract. In this paper we deal with further recent developments strictly connected to the results obtained in [BNPP]. We consider the Lorentz gas out of equilibrium in a weak coupling regime. Each obstacle of the Lorentz gas generates a smooth radially symmetric potential with compact support. We prove that the macroscopic current in the stationary state is given by the Fick's law of diffusion. The diffusion coefficient is given by the Green-Kubo formula associated to the generator of the diffusion process dictated by the linear Landau equation.

3.2.1 Introduction

The understanding of transport phenomena of nonequilibrium thermodynamics starting form the microscopic dynamics is one of the most challenging problem in statistical mechanics.

Nonequilibrium stationary states describe the state of a mechanical system driven and maintained out of equilibrium. The main characteristic of nonequilibrium stationary states is that they commonly exhibit transport phenomena. They sustain steady flows, for example energy flow, particles flow or momentum flows. The usually conserved quantities, mass, momentum and energy, flow in response to a gradient. For instance the heat flow and the mass flow appears in response to a temperature gradient and a concentration gradient respectively. These processes are well described by phenomenological linear laws, the Fourier's and Fick's law respectively.

In the current literature there are very few rigorous results concerning the derivation of the these phenomenological laws from a microscopic model (see for instance [LS], [LS1], [LS2]). A contribution in this direction is the validation of the Fick's law for the Lorentz model in a low density situation which has been recently proven in [BNPP]. To consider the system out of equilibrium, in [BNPP], they consider the Lorentz gas in a bounded region in the plane and couple the system with two mass reservoirs at the boundaries. More precisely they consider the slice $\Lambda = (0, L) \times \mathbb{R}$ in the plane. In the left half plane there is a free gas of light particles at density ρ_1, in the right half plane there is a free gas of light particles at density ρ_2 which play the role of mass reservoirs. The light particles are not interacting among themselves. Inside Λ there is a Poisson distribution of intensity μ of hard core scatterers Λ. The light particles flow through the boundaries and are elastically reflected by the scatterers. For this model they prove the existence of a stationary state for which

$$J \approx -D\nabla\rho \tag{3.2.1}$$

where J is the mass current, ρ is the mass density and $D > 0$ is the diffusion coefficient. Formula (3.2.1) is the well known Fick's law whose validity has been proven in [BNPP]. We remind that according to the low-density regime considered they can use the linear Boltzmann equation as a bridge between the original mechanical system and the diffusion equation. This strategy works since they provide an explicit control of the error in the kinetic limit which suggests the scale of times for which the diffusive limit can be achieved. The result is presented in a two dimensional setting but it holds in dimension higher than two. The two dimensional case is the most interesting to analyze since the pathologic configurations preventing the Markovianity on a kinetic scale are harder to estimate in this case.

We can wonder if the same result can be achieved if we slightly modify the model. We consider the same geometry described above but inside Λ now we have a Poisson distribution of scatterers which are no longer hard cores. We assume that each obstacle generates a smooth, radial, short-range potential. In the same spirit as in [BNP], [ESY], we scale the range of the interaction and the density of the scatterers according to

$$\begin{aligned} &\phi_\varepsilon(x) = \varepsilon^\alpha \phi(\tfrac{x}{\varepsilon}) \\ &\mu_\varepsilon = \varepsilon^{-(2\alpha+\lambda+1)}\mu. \end{aligned} \tag{3.2.2}$$

with $\alpha \in (0, \frac{1}{2})$ and $\lambda > 0$.

The scaling (3.2.2) means that the kinetic regime describes the system for kinetic times $O(1)$ (i.e. $\lambda = 0$). Observe that when $\lambda = 0$ the limiting cases $\alpha = 0$ and $\alpha = 1/2$ correspond respectively to the low density limit and the weak-coupling limit. In this intermediate scale between the low density and the weak-coupling regime the kinetic equation that appears in the limit is the linear Landau equation. One can go further to diffusive times provided that $\lambda > 0$ is not too large. The intermediate level of description between the mechanical system and the diffusion equation is given by the linear Landau equation with a divergent factor in front of the collision operator. Since the scale of time for which the system diffuses should not prevent the Markov property, there is a constraint on λ. More precisely there exists a threshold $\lambda_0 = \lambda(\alpha)$, emerging from the explicit estimate of the set of pathological configurations producing memory effects, s.t. for $\lambda < \lambda(\alpha)$, the microscopic solution of the time dependent problem converges to the solution of the heat equation in the limit $\varepsilon \to 0$. We refer to [BNP], Section 6, for further details.

This result concerns the time dependent problem. We are now interested in the stationary situation. In this paper we provide a rigorous derivation of Fick's law of diffusion for this model. We prove that there exists a unique stationary solution for the microscopic dynamics which converges to the stationary solution of the heat equation, namely to the linear profile of the density. We underline that in order to obtain the stationary solution of the microscopic dynamics we need to characterize the stationary solution of the linear Landau equation. To handle this problem we will use the analysis of the time dependent problem and the explicit solution of the heat equation.

3.2.2 The model and main results

Let $\Lambda \subset \mathbb{R}^2$ be the strip $(0, L) \times \mathbb{R}$. We consider a Poisson distribution of fixed disks (scatterers) of radius ε in Λ and denote by $c_1, \dots, c_N \in \Lambda$ their centers. This means that, given $\mu > 0$, the probability density of finding N obstacles in a bounded measurable set $A \subset \Lambda$ is

$$\mathbb{P}(\, d\mathbf{c}_N) = e^{-\mu|A|} \frac{\mu^N}{N!} \, dc_1 \dots dc_N \tag{3.2.3}$$

where $|A| = \text{meas}A$ and $\mathbf{c}_N = (c_1, \dots, c_N)$. Since the modulus of the velocity of the test particle is constant, we assume it to be equal to one, so that the phase space of our system is $\Lambda \times S_1$.

We rescale the intensity μ of the obstacles as

$$\mu_\varepsilon = \varepsilon^{-2\alpha-1}\varepsilon^{-\lambda}\mu, \quad \alpha \in (0, 1/8),\ \lambda > 0$$

where, from now on, $\mu > 0$ is fixed. More precisely we make the following assumption.

Assumption 2. We set $\gamma = 1 - 8(\alpha + \lambda/2)$, the parameter λ is such that as $\varepsilon \to 0$,

$$\varepsilon^{\gamma - 4\lambda} \to 0, \tag{3.2.4}$$

namely $\lambda < \frac{1-8\alpha}{8}$.

Accordingly, we denote by $\mathbb{P}_\varepsilon$ the probability density (3.2.3) with μ replaced by μ_ε. $\mathbb{E}_\varepsilon$ will be the expectation with respect to the measure $\mathbb{P}_\varepsilon$.

We now introduce a radial potential $\phi(r)$ such that

- $\phi \in C^2([0, 1])$,
- $\phi(0) > 0$ and $r \to \phi(r)$ is strictly decreasing in $[0, 1]$.

We rescale the intensity of the interaction potential as

$$\phi \to \varepsilon^\alpha \phi.$$

Then the Equations of motion are

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\varepsilon^{\alpha-1} \sum_i \nabla\phi(\frac{|x-c_i|}{\varepsilon}) \end{cases} . \tag{3.2.5}$$

For a given configuration of obstacles $\mathbf{c}_N$, we denote by $T^{-t}_{\mathbf{c}_N}(x,v)$ the (backward) flow, solution of (3.2.5), with initial datum $(x,v) \in \Lambda \times S_1$ and define $t-\tau$, $\tau = \tau(x,v,t,\mathbf{c}_N)$, as the first (backward) hitting time with the boundary. We use the notation $\tau = 0$ to indicate the event such that the trajectory $T^{-s}_{\mathbf{c}_N}(x,v)$, $s \in [0,t]$, never hits the boundary. For any $t \geq 0$ the one-particle correlation function reads

$$f_\varepsilon(x,v,t) = \mathbb{E}_\varepsilon[f_B(T^{-(t-\tau)}_{\mathbf{c}_N}(x,v))\chi(\tau > 0)] + \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_N}(x,v))\chi(\tau = 0)], \tag{3.2.6}$$

where $f_0 \in L^\infty(\Lambda \times S_1)$ and the boundary value f_B is defined by

$$f_B(x,v) := \begin{cases} \rho_1 M(v) & \text{if} \quad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \\ \\ \rho_2 M(v) & \text{if} \quad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \end{cases}$$

with $M(v)$ the density of the uniform distribution on S_1 and $\rho_1, \rho_2 > 0$. Here v_1 denotes the horizontal component of the velocity v. Without loss of generality we assume $\rho_2 > \rho_1$. Since $M(v) = \frac{1}{2\pi}$, from now on we will absorb it in the definition of the boundary values ρ_1, ρ_2. Therefore we set

$$f_B(x,v) := \begin{cases} \rho_1 & \text{if} \quad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \\ \\ \rho_2 & \text{if} \quad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0. \end{cases} \tag{3.2.7}$$

We are interested in the stationary solutions f^S_ε of the above problem. More precisely $f^S_\varepsilon(x,v)$ solves

$$f^S_\varepsilon(x,v) = \mathbb{E}_\varepsilon[f_B(T^{-(t-\tau)}_{\mathbf{c}_N}(x,v))\chi(\tau > 0)] + \mathbb{E}_\varepsilon[f^S_\varepsilon(T^{-t}_{\mathbf{c}_N}(x,v))\chi(\tau = 0)]. \tag{3.2.8}$$

The main result of the present paper can be summarized in the following theorem.

Theorem 3.2.1. *For ε sufficiently small there exists a unique L^∞ stationary solution f^S_ε for the microscopic dynamics (i.e. satisfying* (3.2.8)*). Moreover, as $\varepsilon \to 0$*

$$f^S_\varepsilon \to \varrho^S, \tag{3.2.9}$$

where ϱ^S is the stationary solution of the heat equation with the following boundary conditions

$$\begin{cases} \varrho^S(x) = \rho_1, & x \in \{0\} \times \mathbb{R}, \\ \\ \varrho^S(x) = \rho_2, & x \in \{L\} \times \mathbb{R}. \end{cases} \tag{3.2.10}$$

The convergence is in $L^2((0,L) \times S_1)$.

Some remarks on the above Theorem are in order. The boundary conditions of the problem depend on the space variable only through the horizontal component. As a consequence, the stationary solution f_ε^S of the microscopic problem, as well as the stationary solution ϱ^S of the heat equation, inherits the same feature. This justifies the convergence in $L^2((0,L)\times S_1)$ instead of in $L^2(\Lambda\times S_1)$. The explicit expression for the stationary solution ϱ^S reads

$$\varrho^S(x)=\frac{\rho_1(L-x_1)+\rho_2 x_1}{L}, \tag{3.2.11}$$

where x_1 is the horizontal component of the space variable x. We note that in order to prove Theorem 3.2.1 it is enough to assume that $\varepsilon^{\gamma-3\lambda}\to 0$, i.e. $\lambda<\frac{1-8\alpha}{7}$. The stronger Assumption 2 is needed to prove Theorem 3.2.2 below.

Next, to discuss the Fick's law, we introduce the stationary mass flux

$$J_\varepsilon^S(x)=\varepsilon^{-\lambda}\int_{S_1} v\, f_\varepsilon^S(x,v)\,dv, \tag{3.2.12}$$

and the stationary mass density

$$\varrho_\varepsilon^S(x)=\int_{S_1} f_\varepsilon^S(x,v)\,dv. \tag{3.2.13}$$

Note that J_ε^S is the total amount of mass flowing through a unit area in a unit time interval. Although in a stationary problem there is no typical time scale, the factor $\varepsilon^{-\lambda}$ appearing in the definition of J_ε^S, is reminiscent of the time scaling necessary to obtain a diffusive limit.

Theorem 3.2.2 (Fick's law). *We have*

$$J_\varepsilon^S + D\nabla_x\varrho_\varepsilon^S\to 0 \tag{3.2.14}$$

as $\varepsilon\to 0$. The convergence is in $\mathcal{D}'(0,L)$ and $D>0$ is given by the Green-Kubo formula

$$D=\frac{2}{\mu}\int_{S_1} v\cdot\big(-\Delta_{|v|}^{-1}\big)v\,dv. \tag{3.2.15}$$

Moreover

$$J^S=\lim_{\varepsilon\to 0} J_\varepsilon^S(x), \tag{3.2.16}$$

where the convergence is in $L^2(0,L)$ and

$$J^S=-D\,\nabla\varrho^S=-D\,\frac{\rho_2-\rho_1}{L}, \tag{3.2.17}$$

where ϱ^S is the linear profile (3.2.11).

Observe that, as expected by physical arguments, the stationary flux J^S does not depend on the space variable. Furthermore the diffusion coefficient D is determined by the behavior of the system at equilibrium and in particular it is equal to the diffusion coefficient for the time dependent problem.

3.2.3 Proofs

In order to prove Theorem 3.2.1 our strategy is the following. We introduce the stationary linear Landau equation

$$\begin{cases} (v\cdot\nabla_x)g_\varepsilon^S(x,v)=\varepsilon^{-\lambda}\,\mathcal{L}g_\varepsilon^S(x,v), \\ g_\varepsilon^S(x,v)=\rho_1, \qquad x\in\{0\}\times\mathbb{R}, \quad v_1>0, \\ g_\varepsilon^S(x,v)=\rho_2, \qquad x\in\{L\}\times\mathbb{R}, \quad v_1<0, \end{cases} \tag{3.2.18}$$

where $\mathcal{L}=\frac{\mu}{2}\Delta_{|v|}$ and $\Delta_{|v|}$ is the Laplace Beltrami operator on the circle of radius $|v|=1$, namely S_1. Moreover we introduce the stationary linear Boltzmann equation

$$\begin{cases} (v\cdot\nabla_x)h_\varepsilon^S(x,v)=\varepsilon^{-\lambda}\,\mathrm{L}_\varepsilon h_\varepsilon^S(x,v), \\ h_\varepsilon^S(x,v)=\rho_1, \qquad x\in\{0\}\times\mathbb{R}, \quad v_1>0, \\ h_\varepsilon^S(x,v)=\rho_2, \qquad x\in\{L\}\times\mathbb{R}, \quad v_1<0, \end{cases} \tag{3.2.19}$$

where $\mathrm{L}_\varepsilon:=\varepsilon^{-2\alpha}\mathrm{L}$ and L is the linear Boltzmann operator defined as

$$\mathrm{L}f(v)=\mu\int_{-1}^{1}d\rho\big[f(v')-f(v)\big], \qquad f\in L^1(S_1) \tag{3.2.20}$$

with

$$v'=v-2(\omega\cdot v)\omega \tag{3.2.21}$$

and ω is the unit vector bisecting the angle between the incoming velocity v and the outgoing velocity v' as specified in Figure 3.8.

Since the boundary conditions depend on the space variable only trough the horizontal component, the stationary solution h_ε^S and g_ε^S inherit the same feature, as well as f_ε^S and ϱ^S.

The strategy of the proof consists of two steps. First we prove that there exists a unique g_ε^S which converges, as $\varepsilon\to 0$, to ϱ^S given by (3.2.11). See Proposition 3.2.6 below. Secondly we show that there exists a unique f_ε^S asymptotically equivalent to g_ε^S. See Proposition 3.2.9 below. This result is achieved using two steps. The first one concerns the convergence of f_ε^S towards h_ε^S, the stationary solution of the linear Boltzmann equation, by showing that the memory effects of the mechanical system, preventing the

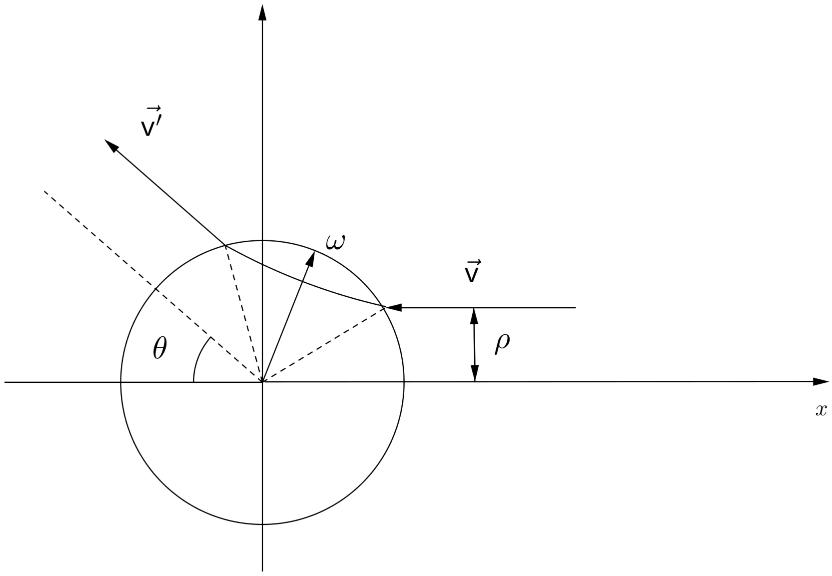

Figure 3.8: The scattering problem

Markovianity, are indeed negligible. The second one concerns the grazing collision limit which guarantees the asymptotic equivalence of h_ε^S and g_ε^S.

Let g_ε be the solution of the problem

$$\begin{cases} \big(\partial_t + v \cdot \nabla_x\big) g_\varepsilon(x,v,t) = \varepsilon^{-\lambda}\, \mathcal{L} g_\varepsilon(x,v,t), \\ g_\varepsilon(x,v,0) = f_0(x,v), \qquad f_0 \in L^\infty(\Lambda \times S_1), \\ g_\varepsilon(x,v,t) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \quad t \geq 0, \\ g_\varepsilon(x,v,t) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \quad t \geq 0. \end{cases} \tag{3.2.22}$$

We can write $g_\varepsilon(t)$ as the sum of two contributions, one due to the backward trajectories hitting the boundary and the other one due to the trajectories which never leave Λ. Therefore we set

$$g_\varepsilon(x,v,t) = g_\varepsilon^{out}(x,v,t) + g_\varepsilon^{in}(x,v,t).$$

Observe that g_ε^{out} solves

$$\begin{cases} \big(\partial_t + v \cdot \nabla_x\big) g_\varepsilon^{out}(x,v,t) = \varepsilon^{-\lambda}\, \mathcal{L} g_\varepsilon^{out}(x,v,t), \\ g_\varepsilon^{out}(x,v,0) = 0, \qquad x \in \Lambda, \\ g_\varepsilon^{out}(x,v,t) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \quad t \geq 0, \\ g_\varepsilon^{out}(x,v,t) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \quad t \geq 0. \end{cases} \tag{3.2.23}$$

We set $\tilde{\mathcal{L}} := \varepsilon^{-\lambda}\, \mathcal{L} - v \cdot \nabla_x$. Let $G_\varepsilon^0(t)$ be the semigroup whose generator is the operator $\tilde{\mathcal{L}}$, i.e. $G^0_\varepsilon(t) = e^{t\tilde{\mathcal{L}}}$. Hence

$$g^{in}_\varepsilon(t) = G^0_\varepsilon(t) f_0.$$

We observe that g^S_ε, solution of (3.2.18), satysfies, for $t_0 > 0$

$$g^S_\varepsilon = g^{out}_\varepsilon(t_0) + G^0_\varepsilon(t_0) g^S_\varepsilon,$$

so that we can formally express g^S_ε as the Neumann series

$$g^S_\varepsilon = \sum_{n\geq 0} (G^0_\varepsilon(t_0))^n g^{out}_\varepsilon(t_0). \tag{3.2.24}$$

We now establish existence and uniqueness of g^S_ε by showing that the Neumann series (3.2.28) converges. In order to do it we extend the action of the semigroup $G^0_\varepsilon(t)$ to the space $L^\infty(\mathbb{R}^2 \times S_1)$, namely

$$G^0_\varepsilon(t)\ell_0(x,v) = \chi_\Lambda(x)\tilde{G}^0_\varepsilon(t)\ell_0(x,v)$$

for any $\ell_0(x,v) \in L^\infty(\mathbb{R}^2 \times S_1)$. Here χ_Λ is the characteristic function of Λ and $\tilde{G}^0_\varepsilon$ is the extension of the semigroup to the whole space $\mathbb{R}^2 \times S_1$. For the sake of simplicity from now on we set $\tilde{G}^0_\varepsilon := G^0_\varepsilon$.

As we proved in [BNPP], the same technique works for h_ε, solution of the following Boltzmann equation

$$\begin{cases} \big(\partial_t + v\cdot\nabla_x\big) h_\varepsilon(x,v,t) = \varepsilon^{-\lambda}\, \mathrm{L}_\varepsilon h_\varepsilon(x,v,t), \\ h_\varepsilon(x,v,0) = f_0(x,v), \qquad f_0 \in L^\infty(\Lambda\times S_1), \\ h_\varepsilon(x,v,t) = \rho_1, \qquad x \in \{0\}\times\mathbb{R}, \quad v_1 > 0, \quad t\geq 0, \\ h_\varepsilon(x,v,t) = \rho_2, \qquad x\in\{L\}\times\mathbb{R}, \quad v_1 < 0, \quad t \geq 0. \end{cases} \tag{3.2.25}$$

The solution h_ε of the problem (3.2.25) has the following explicit representation

$$\begin{aligned} h_\varepsilon(x,v,t) = &\sum_{N\geq 0} (\mu_\varepsilon\varepsilon)^N \int_0^t dt_1 \ldots \int_0^{t_{N-1}} dt_N \\ &\int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_N\, \chi(\tau < t_N)\chi(\tau > 0)\, e^{-2\mu_\varepsilon\varepsilon\,(t-\tau)}\, f_B(\gamma^{-(t-\tau)}(x,v)) + \\ &+ \sum_{N\geq 0} e^{-2\mu_\varepsilon\varepsilon\, t}\, (\mu_\varepsilon\varepsilon)^N \int_0^t dt_1 \ldots \int_0^{t_{N-1}} dt_N \\ &\int_{-1}^1 d\rho_1 \ldots \int_{-1}^1 d\rho_N\, \chi(\tau = 0)\, f_0(\gamma^{-t}(x,v)), \end{aligned} \tag{3.2.26}$$

with f_B defined in (3.2.7). Given x, v, $t_1 \dots t_N$, $\rho_1 \dots \rho_N$, $\gamma^{-t}(x,v)$ denotes the trajectory whose position and velocity are

$$(x - v(t-t_1) - v_1(t_1-t_2) \cdots - v_N t_N, v_N).$$

The transitions $v \to v_1 \to v_2 \cdots \to v_N$ are obtained by means of a scattering with an hard disk with impact parameter ρ_i via (3.2.21). As before $t - \tau$, $\tau = \tau(x, v, t_1 \dots, t_N, \rho_1 \dots \rho_N)$, is the first (backward) hitting time with the boundary. We remind that $\mu_\varepsilon \varepsilon = \mu \varepsilon^{-2\alpha-\lambda}$.

We set

$$h_\varepsilon(x,v,t) = h_\varepsilon^{out}(x,v,t) + h_\varepsilon^{in}(x,v,t).$$

Observe that h_ε^{out} solves

$$\begin{cases} \big(\partial_t + v \cdot \nabla_x\big) h_\varepsilon^{out}(x,v,t) = \varepsilon^{-\lambda}\, \mathrm{L}_\varepsilon h_\varepsilon^{out}(x,v,t), \\ h_\varepsilon^{out}(x,v,0) = 0, \qquad x \in \Lambda, \\ h_\varepsilon^{out}(x,v,t) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \quad t \geq 0, \\ h_\varepsilon^{out}(x,v,t) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \quad t \geq 0. \end{cases} \tag{3.2.27}$$

Let $S_\varepsilon^0(t)$ be the Markov semigroup associated to the second sum in (3.2.26), hence $h_\varepsilon^{in}(t) = S_\varepsilon^0(t) f_0$. Moreover h_ε^S, solution of (3.2.19), satysfies, for $t_0 > 0$

$$h_\varepsilon^S = h_\varepsilon^{out}(t_0) + S_\varepsilon^0(t_0) h_\varepsilon^S,$$

so that we can formally express h_ε^S as the Neumann series

$$h_\varepsilon^S = \sum_{n \geq 0} (S_\varepsilon^0(t_0))^n h_\varepsilon^{out}(t_0). \tag{3.2.28}$$

Proposition 3.2.3. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and for any $\ell_0 \in L^\infty(\mathbb{R}^2 \times S_1)$ we have*

$$||G_\varepsilon^0(\varepsilon^{-\lambda})\ell_0||_\infty \leq \tilde\beta\, ||\ell_0||_\infty, \quad \tilde\beta < 1. \tag{3.2.29}$$

As a consequence there exists a unique stationary solution $g_\varepsilon^S \in L^\infty(\Lambda \times S_1)$ satisfying (3.2.18).

To prove Proposition 3.2.3 we also need the following result

Proposition 3.2.4. *For every $\ell_0 \in L^\infty(\mathbb{R}^2 \times S_1)$*

$$||\Big(G_\varepsilon^0(\varepsilon^{-\lambda} t) - S_\varepsilon^0(\varepsilon^{-\lambda} t)\Big)\ell_0||_\infty \leq C\varepsilon^{2(\alpha-\lambda)}. \tag{3.2.30}$$

Proof. We look at the evolution of $h_\varepsilon^{in}(\varepsilon^{-\lambda}t) - g_\varepsilon^{in}(\varepsilon^{-\lambda}t)$, namely

$$\big(\partial_t + \varepsilon^{-\lambda} v \cdot \nabla_x\big)\big(h_\varepsilon^{in} - g_\varepsilon^{in}\big) = \varepsilon^{-2\lambda}\Big(\tilde{\mathrm{L}}_\varepsilon h_\varepsilon^{in} - \mathcal{L} g_\varepsilon^{in}\Big), \tag{3.2.31}$$

where $\mathcal{L} := \frac{\mu}{2}\Delta_{|v|}$. We observe that we can write (3.2.31) as

$$\big(\partial_t + \varepsilon^{-\lambda} v \cdot \nabla_x\big)\big(h_\varepsilon^{in} - g_\varepsilon^{in}\big) = \varepsilon^{-2\lambda}\Big[\tilde{\mathrm{L}}_\varepsilon\big(h_\varepsilon^{in} - g_\varepsilon^{in}\big) + \big(\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big)g_\varepsilon^{in}\Big]. \quad (3.2.32)$$

Hence we can consider $\big(\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big)g_\varepsilon^{in}$, in (3.2.32), as a source term. Recalling that

$$\tilde{\mathrm{L}}_\varepsilon g_\varepsilon^{in} = \mu\varepsilon^{-2\alpha}\int_{-1}^{1} d\rho\,\big[g_\varepsilon^{in}(v') - g_\varepsilon^{in}(v)\big],$$

we set

$$\begin{aligned} &g_\varepsilon^{in}(v') - g_\varepsilon^{in}(v) \\ &= (v' - v)\cdot\nabla_{|S_1} g_\varepsilon^{in}(v) \\ &\quad + \frac{1}{2}(v'-v)\otimes(v'-v)\nabla_{|S_1}\nabla_{|S_1} g_\varepsilon^{in}(v) \\ &\quad + \frac{1}{6}(v'-v)\otimes(v'-v)\otimes(v'-v)\nabla_{|S_1}\nabla_{|S_1}\nabla_{|S_1} g_\varepsilon^{in}(v) + R_\varepsilon, \end{aligned}$$

with $R_\varepsilon = \mathcal{O}(|v - v'|^4)$. Integrating with respect to v and using symmetry arguments we obtain

$$\tilde{\mathrm{L}}_\varepsilon g_\varepsilon^{in} = \mu\varepsilon^{-2\alpha}\Big\{\frac{1}{2}\Delta_{|v|} g_\varepsilon^{in}\int_{-1}^{1} d\rho\,|v' - v|^2 + \int_{-1}^{1} d\rho\,R_\varepsilon\Big\}.$$

Observe that $|v' - v|^2 = 4\sin^2\frac{\theta_\varepsilon(\rho)}{2}$. (See Figure 3.8). We remind that the scattering angle

$$\theta_\varepsilon(\rho) \le \pi\varepsilon^\alpha \sup_{r\in[0,1]} |r\,\phi'(r)| + \tilde{C}\varepsilon^{2\alpha}$$

and $\max_{\rho\in[0,1]}\theta_\varepsilon(\rho) \le C\varepsilon^\alpha$ (see [DR], Section 3, for further details). Moreover

$$B := \lim_{\varepsilon\to 0}\frac{\mu}{2}\varepsilon^{-2\alpha}\int_{-1}^{1}\theta_\varepsilon(\rho)^2 d\rho$$

is the diffusion coefficient of the Landau equation, $B < \infty$, hence

$$\tilde{\mathrm{L}}_\varepsilon g_\varepsilon^{in} = B\,\Delta_{|v|} g_\varepsilon^{in} + \frac{\mu}{2}\varepsilon^{-2\alpha}\int_{-1}^{1} d\rho\,R_\varepsilon.$$

Therefore

$$\big\|\big(\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big)g_\varepsilon^{in}\big\|_\infty \le C\varepsilon^{2\alpha}, \quad (3.2.33)$$

which vanishes for $\varepsilon \to 0$.

For a smooth reading we set $w_\varepsilon := h_\varepsilon^{in} - g_\varepsilon^{in}$ and $A_\varepsilon := \varepsilon^{-2\lambda}\big(\tilde{\mathrm{L}}_\varepsilon - \mathcal{L}\big)g_\varepsilon^{in}$. Hence (3.2.32) becomes

$$\big(\partial_t + \varepsilon^{-\lambda} v\cdot\nabla_x\big)w_\varepsilon = \varepsilon^{-2\lambda}\tilde{\mathrm{L}}_\varepsilon w_\varepsilon + A_\varepsilon.$$

Let $\tilde{S}_\varepsilon(t) := S^0_\varepsilon(\varepsilon^{-\lambda} t)$ be the semigroup associated to the generator $-\varepsilon^{-\lambda}\big(v \cdot \nabla_x - \varepsilon^{-\lambda}\,\tilde{\mathrm{L}}_\varepsilon\big)$. By equation (3.2.32) we get

$$w_\varepsilon(t) = \tilde{S}_\varepsilon(t) w_\varepsilon(0) + \int_0^t ds\ \tilde{S}_\varepsilon(t-s)\, A_\varepsilon(s).$$

Since $w_\varepsilon(0) = 0$ we get

$$w_\varepsilon(t) = \int_0^t ds\ \tilde{S}_\varepsilon(t-s)\, A_\varepsilon(s).$$

By the usual series expansion for $\tilde{S}_\varepsilon(t)$ we obtain

$$w_\varepsilon(x,v,t) = \int_0^t ds \sum_{N\geq 0} e^{-2\mu\varepsilon^{-2\alpha-2\lambda}(t-s)}\, (\mu_\varepsilon \varepsilon)^N \int_0^{\varepsilon^{-\lambda}(t-s)} dt_1 \ldots \int_0^{t_{N-1}} dt_N$$
$$\int_{-1}^{1} d\rho_1 \ldots \int_{-1}^{1} d\rho_N\, \chi(\tau = 0)\, A_\varepsilon(\gamma^{-\varepsilon^{-\lambda}(t-s)}(x,v), s).$$

Thanks to (3.2.33) we have that A_ε vanishes in the limit, therefore

$$\|w_\varepsilon(t)\|_\infty \leq T \|A_\varepsilon(t)\|_\infty \leq C\varepsilon^{2\alpha - 2\lambda}$$

Hence h^{in}_ε and g^{in}_ε are asymptotically equivalent in L^∞. $\square$

Proposition 3.2.5. *Let $T > 0$. For any $t \in (0,T]$*

$$\|h^{out}_\varepsilon(\varepsilon^{-\lambda} t) - g^{out}_\varepsilon(\varepsilon^{-\lambda} t)\|_\infty \leq C\varepsilon^{2(\alpha-\lambda)} \tag{3.2.34}$$

The proof is essentially the same of Proposition (3.2.4), and to let it work we observe that we need the extension procedure discussed in [BNPP], Section 5, for h^{out}_ε.

Proof of Proposition 3.2.3. From Proposition 2.1 in [BNPP], for any $\ell_0 \in L^\infty(\mathbb{R}^2 \times S_1)$, we have

$$||S^0_\varepsilon(\varepsilon^{-\lambda})\ell_0||_\infty \leq \beta\, ||\ell_0||_\infty, \quad \beta < 1. \tag{3.2.35}$$

Therefore for ε small enough

$$\begin{aligned} ||G^0_\varepsilon(\varepsilon^{-\lambda})\ell_0||_\infty &\leq ||(G^0_\varepsilon(\varepsilon^{-\lambda}) - S^0_\varepsilon(\varepsilon^{-\lambda}))\ell_0||_\infty + ||S^0_\varepsilon(t)\ell_0||_\infty \\ &\leq ||(G^0_\varepsilon(\varepsilon^{-\lambda}) - S^0_\varepsilon(\varepsilon^{-\lambda}))\ell_0||_\infty + \beta\, ||\ell_0||_\infty \end{aligned} \tag{3.2.36}$$

Hence, using (3.2.30) in (3.2.36), we get

$$\begin{aligned} ||G^0_\varepsilon(\varepsilon^{-\lambda})\ell_0||_\infty &\leq ||(G^0_\varepsilon(\varepsilon^{-\lambda}) - S^0_\varepsilon(\varepsilon^{-\lambda}))\ell_0||_\infty + \beta\, ||\ell_0||_\infty|| \\ &\leq\ \tilde{\omega}(\varepsilon) + \beta\, ||\ell_0||_\infty\ < \tilde{\beta}\, ||\ell_0||_\infty, \quad \tilde{\beta} < 1. \end{aligned}$$

Here $\tilde{\omega}(\varepsilon) = C\,\varepsilon^{2(\alpha-\lambda)}$.

Finally, since $\tilde{\beta} < 1$, by (3.2.28) we get

$$||g_\varepsilon^S||_\infty \leq \frac{1}{(1-\tilde{\beta})}\,||g_\varepsilon^{out}(\varepsilon^{-\lambda})||_\infty \leq \frac{1}{(1-\tilde{\beta})}\,\rho_2.$$

$\square$

The last step is the proof of the convergence of g_ε^S to the stationary solution of the diffusion problem

$$\begin{cases} \partial_t \varrho - D\Delta\varrho = 0 \\ \varrho(x,t) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad t \geq 0 \\ \varrho(x,t) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad t \geq 0, \end{cases} \tag{3.2.37}$$

with the diffusion coefficient D given by the Green-Kubo formula (3.2.15). We remind that the stationary solution ϱ^S to the problem (3.2.37) has the following explicit expression

$$\varrho^S(x) = \frac{\rho_1(L-x_1) + \rho_2 x_1}{L}, \tag{3.2.38}$$

where $x = (x_1, x_2)$.

By using the Hilbert expansion technique in L^2 we can prove

Proposition 3.2.6. *Let $g_\varepsilon^S \in L^\infty((0,L) \times S_1)$ be the solution to the problem* (3.2.18)*. Then*

$$g_\varepsilon^S \to \varrho^S \tag{3.2.39}$$

as $\varepsilon \to 0$, where $\varrho^S(x)$ is given by (3.2.38)*. The convergence is in $L^2((0,L)\times S_1)$.*

For the proof we refer to [BNPP], Section 4.2. This concludes our analysis of the Markov part of the proof.

Recalling the expression (3.2.6) for the one-particle correlation function f_ε, we introduce a decomposition analogous to those ones used for $g_\varepsilon(t)$ and $h_\varepsilon(t)$, namely

$$f_\varepsilon^{out}(x,v,t) := \mathbb{E}_\varepsilon[f_B(T_{\mathbf{c}_N}^{-(t-\tau)}(x,v))\chi(\tau > 0)] \tag{3.2.40}$$

and

$$f_\varepsilon^{in}(x,v,t) := \mathbb{E}_\varepsilon[f_0(T_{\mathbf{c}_N}^{-t}(x,v))\chi(\tau = 0)], \tag{3.2.41}$$

so that

$$f_\varepsilon(x,v,t) = f_\varepsilon^{out}(x,v,t) + f_\varepsilon^{in}(x,v,t).$$

Here f_ε^{out} is the contribution due to the trajectories that do leave Λ at times smaller than t, while f_ε^{in} is the contribution due to the trajectories that stay internal to Λ. We introduce the flow $F_\varepsilon^0(t)$ such that

$$(F_\varepsilon^0(t)\ell)(x,v) = \mathbb{E}_\varepsilon[\ell(T_{\mathbf{c}_N}^{-t}(x,v))\chi(\tau=0)], \quad \ell \in L^\infty(\Lambda \times S_1)$$

and remark that F_ε^0 is just the dynamics "inside" Λ. In particular $f_\varepsilon^{in}(t) = F_\varepsilon^0(t)f_0$.

To detect the stationary solution f_ε^S for the microscopic dynamics we proceed as for the Boltzmann evolution (see (3.2.8)) by setting, for $t_0 > 0$,

$$f_\varepsilon^S = f_\varepsilon^{out}(t_0) + F_\varepsilon^0(t_0)f_\varepsilon^S$$

and we can formally express the stationary solution as the Neumann series

$$f_\varepsilon^S = \sum_{n\geq 0}(F_\varepsilon^0(t_0))^n f_\varepsilon^{out}(t_0). \tag{3.2.42}$$

To show the convergence of the series (3.2.42) and hence existence of f_ε^S we first need the following Propositions.

Proposition 3.2.7. *Let $T > 0$. For any $t \in (0,T]$*

$$\|f_\varepsilon^{out}(t) - h_\varepsilon^{out}(t)\|_{L^\infty(\Lambda\times S_1)} \leq C\varepsilon^\gamma t^3, \tag{3.2.43}$$

where h_ε^{out} solves (3.2.27) and $\gamma = 1 - 8(\alpha - \frac{\lambda}{2})$.

Proposition 3.2.8. *For every $\ell_0 \in L^\infty(\Lambda \times S_1)$*

$$||\left(F_\varepsilon^0(t) - S_\varepsilon^0(t)\right)\ell_0||_\infty \leq C||\ell_0||_\infty\,\varepsilon^\gamma\, t^3, \quad \forall t \in [0,T], \tag{3.2.44}$$

where $\gamma = 1 - 8(\alpha - \frac{\lambda}{2})$.

See Section 5 and Section 6 in [BNP], and Section 5 in [BNPP] for the proof. As a corollary we can prove

Proposition 3.2.9. *For ε sufficiently small there exists a unique stationary solution $f_\varepsilon^S \in L^\infty(\Lambda \times S_1)$ satisfying (3.2.8). Moreover*

$$\|f_\varepsilon^S - g_\varepsilon^S\|_\infty \leq C\varepsilon^{\gamma-3\lambda}, \tag{3.2.45}$$

where $\gamma = 1 - 8(\alpha - \frac{\lambda}{2})$.

Proof. We prove the existence and uniqueness of the stationary solution by showing that the Neumann series (3.2.42) converges, namely

$$||F_\varepsilon^0(\varepsilon^{-\lambda})f_0||_\infty \leq \beta'\,||f_0||_\infty, \quad \beta' < 1. \tag{3.2.46}$$

This implies

$$||f_\varepsilon^S||_\infty \le \frac{1}{(1-\beta')}\,||f_\varepsilon^{out}(\varepsilon^{-\lambda})||_\infty \le \frac{1}{(1-\beta')}\,\rho_2, \qquad \beta' < 1.$$

In fact, since

$$||F_\varepsilon^0(\varepsilon^{-\lambda})f_0||_\infty \le ||\left(F_\varepsilon^0(\varepsilon^{-\lambda}) - S_\varepsilon^0(\varepsilon^{-\lambda})\right) f_0||_\infty + ||S_\varepsilon^0(\varepsilon^{-\lambda})f_0||_\infty,$$

thanks to 3.2.8 and Propositions 2.1 in [BNPP] we get

$$\begin{aligned} ||F_\varepsilon^0(\varepsilon^{-\lambda})f_0||_\infty &\le ||f_0||_\infty C\varepsilon^{\gamma-3\lambda} + ||S_\varepsilon^0(\varepsilon^{-\lambda})f_0||_\infty \\ &\le (C\varepsilon^{\gamma-3\lambda} + \beta)||f_0||_\infty \le \beta'||f_0||_\infty, \end{aligned} \tag{3.2.47}$$

with $\beta' < 1$, for ε sufficiently small (remind that $\varepsilon^{\gamma-3\lambda} \to 0$ as $\varepsilon \to 0$). This guarantees the existence and uniqueness of the microscopic stationary solution f_ε^S.

In order to prove (3.2.45) we observe that

$$\|f_\varepsilon^S - g_\varepsilon^S\|_\infty \le \|f_\varepsilon^S - h_\varepsilon^S\|_\infty + \|h_\varepsilon^S - g_\varepsilon^S\|_\infty.$$

We compare the two Neumann series representing f_ε^S and h_ε^S,

$$\begin{aligned} \|f_\varepsilon^S - h_\varepsilon^S\|_\infty &= \|\sum_{n\ge 0}\big((F_\varepsilon^0(\varepsilon^{-\lambda}))^n f_\varepsilon^{out}(\varepsilon^{-\lambda}) - (S_\varepsilon^0(\varepsilon^{-\lambda}))^n h_\varepsilon^{out}(\varepsilon^{-\lambda})\big)\|_\infty \\ &\le \sum_{n\ge 0}\|(F_\varepsilon^0(\varepsilon^{-\lambda}))^n(f_\varepsilon^{out}(\varepsilon^{-\lambda}) - h_\varepsilon^{out}(\varepsilon^{-\lambda}))\|_\infty \\ &\quad + \sum_{n\ge 0}\|\big((F_\varepsilon^0(\varepsilon^{-\lambda}))^n - (S_\varepsilon^0(\varepsilon^{-\lambda}))^n\big)h_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty. \end{aligned} \tag{3.2.48}$$

By (3.2.47), using Proposition 3.2.7, the first sum on the right hand side of (3.2.48) is bounded by

$$\frac{1}{1-\beta'}\|f_\varepsilon^{out}(\varepsilon^{-\lambda}) - h_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty \le C\varepsilon^{\gamma-3\lambda}.$$

As regard to the second sum on the right hand side of (3.2.48) we have

$$\begin{aligned} &\sum_{n\ge 0}\|\big((F_\varepsilon^0(\varepsilon^{-\lambda})^n - (S_\varepsilon^0(\varepsilon^{-\lambda}))^n\big)h_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty \\ &\le \sum_{n\ge 0}\sum_{k=0}^{n-1}\|(F_\varepsilon^0(\varepsilon^{-\lambda}))^{n-k-1}\big(F_\varepsilon^0(\varepsilon^{-\lambda}) - S_\varepsilon^0(\varepsilon^{-\lambda})\big)(S_\varepsilon^0(\varepsilon^{-\lambda}))^k h_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty \\ &\le \sum_{k,\ell\ge 0}\|(F_\varepsilon^0(\varepsilon^{-\lambda}))^\ell\big(F_\varepsilon^0(\varepsilon^{-\lambda}) - S_\varepsilon^0(\varepsilon^{-\lambda})\big)(S_\varepsilon^0(\varepsilon^{-\lambda}))^k h_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty \\ &\le C\,\|h_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty\,\varepsilon^{\gamma-3\lambda}, \end{aligned} \tag{3.2.49}$$

by virtue of (3.2.29), (3.2.47) and (3.2.44).

We compare the two Neumann series representing h_ε^S and g_ε^S,

$$\begin{aligned}\|h_\varepsilon^S - g_\varepsilon^S\|_\infty &= \|\sum_{n\geq 0}\big((S_\varepsilon^0(\varepsilon^{-\lambda}))^n h_\varepsilon^{out}(\varepsilon^{-\lambda}) - (G_\varepsilon^0(\varepsilon^{-\lambda}))^n g_\varepsilon^{out}(\varepsilon^{-\lambda})\big)\|_\infty \\ &\leq \sum_{n\geq 0}\|(S_\varepsilon^0(\varepsilon^{-\lambda}))^n(h_\varepsilon^{out}(\varepsilon^{-\lambda}) - g_\varepsilon^{out}(\varepsilon^{-\lambda}))\|_\infty \\ &+ \sum_{n\geq 0}\|\big((S_\varepsilon^0(\varepsilon^{-\lambda}))^n - (G_\varepsilon^0(\varepsilon^{-\lambda}))^n\big) g_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty.\end{aligned} \tag{3.2.50}$$

By using Proposition 3.2.5 the first sum on the right hand side of (3.2.50) is bounded by

$$\frac{1}{1-\beta'}\|h_\varepsilon^{out}(\varepsilon^{-\lambda}) - g_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty \leq C\varepsilon^{2(\alpha-\lambda)}.$$

As regard to the second sum on the right hand side of (3.2.50) by means of the same trick used in (3.2.49) we get

$$\sum_{n\geq 0}\|\big((S_\varepsilon^0(\varepsilon^{-\lambda}))^n - (G_\varepsilon^0(\varepsilon^{-\lambda}))^n\big) g_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty \leq C\,\|g_\varepsilon^{out}(\varepsilon^{-\lambda})\|_\infty\,\varepsilon^{2(\alpha-\lambda)}.$$

This concludes the proof of Proposition 3.2.9. □

Hence the proof of Theorem 3.2.1 follows from Proposition 3.2.6 and Proposition 3.2.9. We conclude by proving Theorem 3.2.2 which actually is a Corollary of the previous analysis.

Proof of Theorem 3.2.2. By standard computations (see e.g. Section [BNPP], Section 4.2) we have

$$g_\varepsilon^S = \varrho^S + \frac{1}{\varepsilon^{-\lambda}}g^{(1)} + \frac{1}{\varepsilon^{-\lambda}}R_\varepsilon,$$

where

$$g^{(1)}(v) = \mathcal{L}^{-1}(v\cdot\nabla_x\varrho^S) = \frac{\rho_2-\rho_1}{L}\mathcal{L}^{-1}(v_1)$$

and, as we see in [BNPP], Section 4.2, $R_\varepsilon = O(\varepsilon^{\frac{\lambda}{2}})$ in $L^2((0,L)\times S_1)$. Therefore, since $\int_{S_1} v\varrho^S dv = 0$,

$$\varepsilon^{-\lambda}\int_{S_1} v g_\varepsilon^S(x,v)dv = D\nabla_x\varrho^S + O(\varepsilon^{\frac{\lambda}{2}}), \tag{3.2.51}$$

where D is given by (3.2.15). By Theorem 3.2.1 the right hand side of (3.2.51) is close to $D\nabla_x\varrho_\varepsilon^S$ in $\mathcal{D}'((0,L)\times S_1)$, where ϱ_ε^S is given by (3.2.13). On the other hand, by Proposition 3.2.9 and Assumption 2, the left hand side of (3.2.51) is close in $L^\infty((0,L)\times S_1)$ to $J_\varepsilon^S(x)$ defined in (3.2.12). This concludes the proof of (3.2.14). Moreover (3.2.16) and (3.2.17) follow by (3.2.51). □

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Chapter 4

Propagation of Chaos in the wind tree model

J-particle correlation functions

We consider a system with two kinds of particles. The light particles (or wind particles) do not interact between themselves but they interact with the heavy particles (or tree particles) which are supposed to be infinitely heavy compared to the tree particles and are supposed to be at rest and randomly distributed in the plane. We assume that the heavy particles or scatterers are distributed as the space distribution of a perfect gas, i.e. a Poisson distribution of intensity μ. Moreover we suppose that the scatterers are, with respect to the light particles, hard disks of radius ε reflecting the light particles on their surface. Let $(c_1, \dots, c_N)$ in $\mathbb{R}^2$ be the centers of the hard disks. This means that, given $\mu > 0$, the probability density of finding N obstacles in a bounded measurable set $A \subset \mathbb{R}^2$ is

$$\mathbb{P}(\,d\mathbf{c}_N) = e^{-\mu|A|}\frac{\mu^N}{N!}\,dc_1 \dots\, dc_N \tag{4.0.1}$$

where $|A| = \text{meas}(A)$ and $\mathbf{c}_N = (c_1, \dots, c_N)$.
A particle in $\mathbb{R}^2$ moves freely up to the first instant of contact with an obstacle. Then it is elastically reflected and so on. Since the modulus of the velocity of the test particle is constant, we assume it to be equal to one, so that the phase space of our system is $(\mathbb{R}^2 \times S_1)^j$.

To outline a kinetic behavior we rescale the intensity as $\mu_\varepsilon = \varepsilon^{-1}\mu$, with $\mu > 0$ since we are dealing with a low density regime. Accordingly, we denote by $\mathbb{P}_\varepsilon$ the probability density (4.0.1) with μ replaced by μ_ε. $\mathbb{E}_\varepsilon$ will be the expectation with respect to the measure $\mathbb{P}_\varepsilon$ restricted on those configurations of the obstacles whose centers do not belong to the disks of center x_i and radius ε for every $i = 1, \dots, j$. For a given configuration of obstacles $\mathbf{c}_Q$, we denote by $T^t_{\mathbf{c}_Q}(x_1, v_1, \dots, x_j, v_j)$ the configuration into which the initial datum $(x_1, v_1, \dots, x_j, v_j)$ evolves in presence of the hard disks $\mathbf{c}_Q$ in the time

t. For $t \geq 0$ we look at the j particle correlation functions

$$f_{\varepsilon,j}(x_1, v_1, \ldots, x_j, v_j, t) = \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_Q}(x_1, v_1)) \ldots\ldots f_0(T^{-t}_{\mathbf{c}_Q}(x_j, v_j))]. \tag{4.0.2}$$

Here we assumed that the initial distribution $f_{0,j} := f_{0,j}(x_1, v_1, \ldots, x_j, v_j)$ factorize, namely

$$f_{0,j} = f_0^{\otimes j},$$

where the one particle initial distribution f_0 is a continuous, compactly supported function, i.e. $f_0 \in C_0(\mathbb{R}^2 \times S_1)$, with bounded partial derivatives.

The main result can be summarized in the following

Theorem 4.0.10. *Let $f_{\varepsilon,j}$ be defined in* (4.0.2). *Let $T > 0$. For any $t \in (0, T]$, $j \in \mathbb{N}$ we have*

$$f_{\varepsilon,j}(t) \underset{\varepsilon \to 0}{\longrightarrow} h^{\otimes j}(t) \quad \text{in } L^1((\mathbb{R}^2 \times S_1)^j) \tag{4.0.3}$$

where h solves

$$\begin{cases} \big(\partial_t + v \cdot \nabla_x\big) h(x, v, t) = \mathcal{L} h(x, v, t), \\ h(x, v, 0) = f_0(x, v), \end{cases} \tag{4.0.4}$$

with $f_0 \in C_0(\mathbb{R}^2 \times S_1)$, with bounded partial derivatives, and

$$\mathcal{L} h(v) = \mu \int_{-1}^{1} d\rho \{h(v') - h(v)\}.$$

Proof of Theorem 4.0.10. We follow the direct approach proposed by Gallavotti in [G]. We consider

$$f_{\varepsilon,j}(x_1, v_1, \ldots, x_j, v_j, t) = e^{-\mu_\varepsilon |B(\tilde{x})|} \sum_{Q \geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{(B(\tilde{x}))^Q} d\mathbf{c}_Q \, f_0(T^{-t}_{\mathbf{c}_Q}(x_1, v_1)) \ldots f_0(T^{-t}_{\mathbf{c}_Q}(x_j, v_j)). \tag{4.0.5}$$

Here $T^{-t}_{\mathbf{c}_Q}(x_k, v_k)$ is the flow associated to the initial datum (x_k, v_k) for the $k-$th light particle, $k = 1, \ldots, j$, and for a given scatterers configuration $\mathbf{c}_Q$. $B(\tilde{x})$ for $\tilde{x} \in \mathbb{R}^2$ is the smallest open disk such that $\cup_{k=1}^j B_t^\varepsilon(x_k) \subset B(\tilde{x})$ and $B_t^\varepsilon(x_k) := B_t(x_k) \setminus B_\varepsilon(x_k)$ where $B_t(x_k)$ and $B_\varepsilon(x_k)$, for any $k = 1, \ldots, j$, denote the disks centered in x_k with radius t and ε respectively.

We distinguish the obstacles of the configuration $\mathbf{c}_Q = c_1 \dots c_Q$ which, up to the time t, influence the motion, called internal obstacles, and the external ones. More precisely we call c_i internal if

$$\inf_{0 \le s \le t} |x_\varepsilon^k(-s) - c_i| = \varepsilon, \quad k = 1, \dots, j$$

while c_i is external if

$$\inf_{0 \le s \le t} |x_\varepsilon^k(-s) - c_i| > \varepsilon, \quad \forall k = 1, \dots, j.$$

Here $(x_\varepsilon^k(-s), v_\varepsilon^k(-s)) = T_{\mathbf{c}_Q}^{-s}(x_k, v_k)$, $s \in [0,t]$. Therefore, by integrating over the external obstacles we obtain

$$f_{\varepsilon,j}(x_1, v_1, \dots, x_j, v_j, t) = \sum_{N \ge 0} \frac{\mu_\varepsilon^N}{N!} \int_{B(\tilde{x})^N} d\mathbf{b}_N \, e^{-\mu_\varepsilon |\tilde{\mathcal{T}}_t(\mathbf{b}_N)|} \, \chi(\{\mathbf{b}_N \text{ internal}\}) \\ f_0(T_{\mathbf{b}_N}^{-t}(x_1, v_1)) \dots f_0(T_{\mathbf{b}_N}^{-t}(x_j, v_j)). \tag{4.0.6}$$

Here $\tilde{\mathcal{T}}_t(\mathbf{b}_N) := \cup_{k=1}^j \mathcal{T}_t^k(\mathbf{b}_N)$ where

$$\mathcal{T}_t^k(\mathbf{b}_N) = \{y \in B_t^\varepsilon(x_k) \text{ s.t. } \exists s \in [0,t] \text{ s.t. } |y - x_\varepsilon^k(-s)| \le \varepsilon\}.$$

Observe that

$$\chi(\{\mathbf{b}_N \text{ internal}\}) = \chi(\{\mathbf{b}_N \subset \cup_{k=1}^j \mathcal{T}_t^k(\mathbf{b}_N)\}).$$

We introduce

$$\tilde{f}_{\varepsilon,j}(x_1, v_1, \dots, x_j, v_j, t) = e^{-2\mu_\varepsilon \varepsilon j t} \sum_{N \ge 0} \frac{\mu_\varepsilon^N}{N!} \int_{B(\tilde{x})^N} d\mathbf{b}_N \, \chi(\{\mathbf{b}_N \text{ internal}) \\ f_0(T_{\mathbf{b}_N}^{-t}(x_1, v_1)) \dots f_0(T_{\mathbf{b}_N}^{-t}(x_j, v_j)). \tag{4.0.7}$$

Since

$$|\tilde{\mathcal{T}}_t(\mathbf{b}_N)| \le \sum_{k=1}^j |\mathcal{T}_t^k(\mathbf{b}_N)| \le 2\varepsilon j t, \tag{4.0.8}$$

it follows that

$$f_{\varepsilon,j} \ge \tilde{f}_{\varepsilon,j}. \tag{4.0.9}$$

We set

$$\mathbf{b}_N = \mathbf{b}_{n_1} \cup \mathbf{b}_{n_2} \dots \cup \mathbf{b}_{n_j - 1} \cup \mathbf{b}_{n_j},$$

where n_k is the number of obstacles influencing the trajectory of the particle k up to time t. Let $\rho_i^{(k)}$ and $t_i^{(k)}$ be the impact parameter and the hitting time of the light particle with $\partial B_\varepsilon(b_i^{(k)})$ respectively. Then we perform the following change of variables

$$b_1^{(k)} \dots b_{n_k}^{(k)} \to \rho_1^{(k)}, \dots, \rho_{n_k}^{(k)}, t_1^{(k)}, \dots, t_{n_k}^{(k)} \tag{4.0.10}$$

with

$$0 \leq t_{n_k}^{(k)} < t_{n_k-1}^{(k)} < \cdots < t_1^{(k)} \leq t$$

for each wind particle $k = 1, \dots j$. Conversely, for each wind particle $k = 1, \dots j$, fixed the impact parameters $\{\rho_i^{(k)}\}$ and the hitting times $\{t_i^{(k)}\}$ we construct the centers of the obstacles $b_i^{(k)} = b(\rho_i^{(k)}, t_i^{(k)})$. By performing the backward scattering we construct a trajectory $\gamma^s(x_k, v_k) = (\xi_\varepsilon^k(s), \omega_\varepsilon^k(s)) \quad s \in (-t, 0)$, where

$$\begin{cases} \xi_\varepsilon^k(-t) = x_k - v_k(t - t_1^{(k)}) - v_1^{(k)}(t_1^{(k)} - t_2^{(k)}) \cdots - v_{n_k}^{(k)} t_{n_k}^{(k)} \\ \omega_\varepsilon^k(-t) = v_{n_k}^{(k)}. \end{cases} \tag{4.0.11}$$

Here $v_1^{(k)}, \dots, v_{n_k}^{(k)}$ are the incoming velocities.

Also in this case $(\xi_\varepsilon^k(s), \omega_\varepsilon^k(s)) = (x_\varepsilon^k(s), v_\varepsilon^k(s))$ (therefore the mapping (4.0.10) is one-to-one) only outside the following pathological situations

i) **Recollisions**
There exists $b_u^{(k)}$ $u = 1, \dots, n_k$, $k = 1, \dots, j$, s.t. for $s \in (t_v^{(l)}, t_{v+1}^{(l)})$, $t_v^{(l)} > t_u^{(k)}$, $l = 1, \dots, j$, $\xi_\varepsilon^{(l)}(-s) \in \partial B(b_u^{(k)}, \varepsilon)$.

ii) **Interferences**
There exists $b_u^{(k)}$, $u = 1, \dots, n_k$, $k = 1, \dots, j$, such that $\xi_\varepsilon^{(l)}(-s) \in B(b_u^{(k)}, \varepsilon)$ for $s \in (t_v^{(l)}, t_{v+1}^{(l)})$, $t_v^{(l)} < t_u^{(k)}$.

iii) **Same obstacles collided by different particles**
There exists $b_u^{(k)}$, $i = 1, \dots, n_k$, $k = 1, \dots, j$, such that $\xi_\varepsilon^{(l)}(t_v^{(l)}) \in \partial B(b_u^{(k)}, \varepsilon)$, $l \neq k$, $v = 1, \dots, n_l$.

We observe that when $l = k$ in i), ii), we recover the pathological events for the trajectory of one test particle, see [BNPP] Section 5.2. We simply skip such events by setting

$$\begin{aligned} \chi_{rec} &= \chi(\{\mathbf{b}_N \text{ s.t. i) is realized}\}), \\ \chi_{int} &= \chi(\{\mathbf{b}_N \text{ s.t. ii) is realized}\}), \\ \chi_{so} &= \chi(\{\mathbf{b}_N \text{ s.t. iii) is realized}\}), \end{aligned} \tag{4.0.12}$$

and defining

$$\bar{f}_{\varepsilon,j}(x_1,v_1,\dots,x_j,v_j,t) = e^{-2\mu_\varepsilon \varepsilon j t} \sum_{n_1\geq 0}\cdots\sum_{n_j\geq 0} \mu_\varepsilon^{n_1}\dots\mu_\varepsilon^{n_j} \int_0^t dt_1^1\dots dt_{n_1}^1 \dots$$
$$\int_0^{t^j_{n_j-1}} dt^j_{n_j} \int_{-\varepsilon}^{\varepsilon} d\rho_1^1\dots d\rho_{n_1}^1\dots\int_{-\varepsilon}^{\varepsilon} d\rho^j_{n_j}$$
$$(1-\chi_{rec})(1-\chi_{int})(1-\chi_{so})$$
$$f_0(\gamma^{-t}(x_1,v_1))\ \dots f_0(\gamma^{-t}(x_j,v_j))). \tag{4.0.13}$$

Note that $f_{\varepsilon,j} \geq \tilde{f}_{\varepsilon,j} \geq \bar{f}_{\varepsilon,j}$.

Next we remove $(1-\chi_{rec})(1-\chi_{int})(1-\chi_{so})$ by setting

$$h_j(x_1,v_1,\dots,x_j,v_j,t) = e^{-2\mu_\varepsilon \varepsilon j t} \sum_{n_1\geq 0}\cdots\sum_{n_j\geq 0} \mu_\varepsilon^{n_1}\dots\mu_\varepsilon^{n_j} \int_0^t dt_1^1\dots dt_{n_1}^1 \dots$$
$$\int_0^{t^j_{n_j-1}} dt^j_{n_j} \int_{-\varepsilon}^{\varepsilon} d\rho_1^1\dots d\rho_{n_1}^1\dots\int_{-\varepsilon}^{\varepsilon} d\rho^j_{n_j}$$
$$f_0(\gamma^{-t}(x_1,v_1)))\dots f_0(\gamma^{-t}(x_j,v_j))). \tag{4.0.14}$$

We observe that

$$h_j(t) = h(t)^{\otimes j}, \tag{4.0.15}$$

where $h(x,v,t)$ is the solution of (4.0.4), i.e. of the following Boltzmann equation

$$\begin{cases} (\partial_t + v\cdot\nabla_x)h(x,v,t) = \mathcal{L}h(x,v,t), \\ f_0(x,v) = 0, \end{cases}$$

with

$$\mathcal{L}h(v) = \mu_\varepsilon\varepsilon\int_{-1}^{1} d\rho\{h(v')-h(v)\} = \mu\int_{-1}^{1} d\rho\{h(v')-h(v)\}.$$

Observe that

$$1-(1-\chi_{rec})(1-\chi_{int})(1-\chi_{so}) \leq \chi_{rec}+\chi_{int}+\chi_{so}. \tag{4.0.16}$$

Then by (4.0.13) and (4.0.14) we obtain

$$\|h_j(t)-\bar{f}_{\varepsilon,j}(t)\|_{L^1} \leq \|\varphi_1(\varepsilon,t)\|_{L^1}, \tag{4.0.17}$$

with

$$\varphi_1(\varepsilon,t) := e^{-2\mu_\varepsilon \varepsilon j t} \sum_{n_1 \geq 0} \cdots \sum_{n_j \geq 0} \mu_\varepsilon^{n_1} \dots \mu_\varepsilon^{n_j} \int_0^t dt_1^1 \dots dt_{n_1}^1 \cdots$$
$$\int_0^{t^j_{n_j-1}} dt^j_{n_j} \int_{-\varepsilon}^{\varepsilon} d\rho_1^1 \dots d\rho^1_{n_1} \cdots \int_{-\varepsilon}^{\varepsilon} d\rho^j_{n_j} \, (\chi_{rec} + \chi_{int} + \chi_{so})$$
$$f_0(\gamma^{-t}(x_1,v_1))) \dots f_0(\gamma^{-t}(x_j,v_j))). \tag{4.0.18}$$

We can state the following result

Proposition 4.0.11. *Let* $\varphi_1(\varepsilon,t)$ *be defined in* (4.0.18)*. For any* $t \in [0,T]$ *we have*

$$\|\varphi_1(\varepsilon,t)\|_{L^1} \to 0 \quad as \quad \varepsilon \to 0.$$

Hence, thanks to Proposition 4.0.11, we have reduced the problem to the analysis of a Markov process which is an easier task.

□

To conclude the proof of Theorem 4.0.10 let us estimate $\|f_{\varepsilon,j}(t) - h_j(t)\|_{L^1}$. By (4.0.17) we have

$$\begin{aligned}\|f_{\varepsilon,j}(t) - h_j(t)\|_{L^1} &\leq \|f_{\varepsilon,j}(t) - \bar{f}_{\varepsilon,j}(t)\|_{L^1} + \|\bar{f}_{\varepsilon,j}(t) - h_j(t)\|_{L^1} \\ &\leq \|f_{\varepsilon,j}(t) - \bar{f}_{\varepsilon,j}(t)\|_{L^1} + \|\varphi_1(\varepsilon,t)\|_{L^1}.\end{aligned}$$

Since $\bar{f}_{\varepsilon,j} \leq f_{\varepsilon,j}$, the difference $f_{\varepsilon,j}(t) - \bar{f}_{\varepsilon,j}(t)$ is non negative and we can skip the absolute value. Moreover, by using mass conservation for the linear Boltzmann equation, i.e.

$$\int f_{0,j}\, dx_1 dv_1 \dots dx_j dv_j = \int h_j(t)\, dx_1 dv_1 \dots dx_j dv_j$$

and

$$\int f_{\varepsilon,j}(t)\, dx_1 dv_1 \dots dx_j dv_j = \int f_{0,j}\, dx_1 dv_1 \dots dx_j dv_j$$

and Proposition 4.0.11 we obtain that

$$\|f_{\varepsilon,j}(t) - h_j(t)\|_{L^1} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof of Proposition 4.0.11 (Pathological configurations). For any measurable function u of the process $(\xi_\varepsilon^k(s), \omega_\varepsilon^k(s))$ defined in (4.0.11) we set

$$\mathbb{E}(u) = e^{-2\mu_\varepsilon \varepsilon j t} \sum_{n_1 \geq 0} \cdots \sum_{n_j \geq 0} \mu_\varepsilon^{n_1} \dots \mu_\varepsilon^{n_j} \int_0^t dt_1^1 \dots dt_{n_1}^1 \cdots \int_0^{t^j_{n_j-1}} dt^j_{n_j} \int_{-\varepsilon}^{\varepsilon} d\rho_1^1 \dots d\rho_{n_1}^1 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho^j_{n_j} \, u(\xi_\varepsilon^k, \omega_\varepsilon^k).$$

Here $\mathbb{E} = \mathbb{E}_{x_1, v_1, \dots, x_j, v_j}$ and we skip this dependence for notational simplicity. We realize that

$$\varphi_1(\varepsilon, t) \leq \|f_0\|^j_{L^\infty} \, \mathbb{E}[\chi_{rec} + \chi_{int} + \chi_{so}]$$

We estimate separately the events in (4.0.12). We consider the contribution due to the pathological event iii) and we notice that

$$\chi_{so} \leq \chi(\{d(\xi_\varepsilon^{(k)}(t_u^{(k)}), \xi_\varepsilon^{(l)}(t_v^{(l)})) \leq \varepsilon, \quad l \neq k = 1, \dots j, \quad u = 1, \dots, n_k, \; v = 1, \dots n_l\}). \tag{4.0.19}$$

Since the estimates for $\mathbb{E}[\chi_{rec}]$ and $\mathbb{E}[\chi_{int}]$ follow directly, suitably modified, from those ones obtained for a single light particle we obtain

$$\mathbb{E}[\chi_{rec}] \leq C \, j^2 \varepsilon^{\frac{1}{2}} t^2 \tag{4.0.20}$$

and

$$\mathbb{E}[\chi_{int}] \leq C \, j^2 \varepsilon^{\frac{1}{2}} t^2. \tag{4.0.21}$$

See [BNPP], Section 5.3, for the detailed computation. We focus instead on $\mathbb{E}[\chi_{so}]$ and we observe that

$$\chi_{so} \leq \sum_{k=1}^{j} \sum_{l=1}^{j} \chi^{k,l}, \tag{4.0.22}$$

$$\chi^{k,l} \leq \sum_{u=1}^{n_k} \sum_{v=1}^{n_l} \chi^{k,l}_{u,v} \tag{4.0.23}$$

where $\chi^{k,l}_{u,v} = 1$ if the light particles k and l collide the same obstacle at times t_u and t_v respectively. Hence we get

$$\chi_{so} \leq \sum_{k=1}^{j} \sum_{l=1}^{j} \sum_{u=1}^{n_k} \sum_{v=1}^{n_l} \chi^{k,l}_{u,v}. \tag{4.0.24}$$

Moreover, the condition $d(\xi_\varepsilon^{(k)}(t_u^{(k)}), \xi_\varepsilon^{(l)}(t_v^{(l)})) \leq \varepsilon$ implies that $|t_u^{(k)} - t_v^{(l)}| \leq \varepsilon$. Hence the time integration with respect to t_u^k is restricted to a time interval proportional to ε. Performing all the other integrations and summing over k, l, u, v we obtain

$$
\begin{aligned}
\mathbb{E}[\chi_{so}] \ &\leq e^{-2\mu_\varepsilon \varepsilon j t} \sum_{n_1 \geq 0} \cdots \sum_{n_j \geq 0} \mu_\varepsilon^{n_1} \ldots \mu_\varepsilon^{n_j} \int_0^t dt_1^1 \ldots dt_{n_1}^1 \ldots \int_0^{t_{n_j-1}^j} dt_{n_j}^j \\
&\quad \int_{-\varepsilon}^{\varepsilon} d\rho_1^1 \ldots d\rho_{n_1}^1 \ldots \int_{-\varepsilon}^{\varepsilon} d\rho_{n_j}^j \sum_{k=1}^{j} \sum_{l=1}^{j} \sum_{u=1}^{n_k} \sum_{v=1}^{n_l} \chi_{u,v}^{k,l} \\
&\leq j^2 \, e^{-2\mu_\varepsilon \varepsilon j t} \sum_{n_1} \frac{t^{n_1}}{n_1!} (2\mu_\varepsilon \varepsilon)^{n_1} \cdots \sum_{n_{k-1}} \frac{t^{n_{k-1}}}{n_{k-1}!} (2\mu_\varepsilon \varepsilon)^{n_{k-1}} \\
&\quad \sum_{n_k} n_k \frac{t^{n_k - 1}}{n_k - 1!} (2\mu_\varepsilon \varepsilon)^{n_k} C\varepsilon \sum_{n_l} n_l \frac{t^{n_l}}{n_l!} (2\mu_\varepsilon \varepsilon)^{n_l} \cdots \sum_{n_j} \frac{t^{n_j}}{n_j!} (2\mu_\varepsilon \varepsilon)^{n_j} \\
&\leq C \, j^2 \, \varepsilon \, (2\mu_\varepsilon \varepsilon)^3 t^2 .
\end{aligned}
\tag{4.0.25}
$$

We remind that $\mu_\varepsilon \varepsilon = \mu$. Hence, the above quantity is vanishing in the limit $\varepsilon \to 0$. Therefore, by estimates (4.0.20), (4.0.21) and (4.0.25) it results that

$$
\varphi_1(\varepsilon, t) \leq C \|f_0\|_{L^\infty}^j \, j^2 \, \varepsilon^{\frac{1}{2}} t^2 \leq C^j \, j^2 \, \varepsilon^{\frac{1}{2}} t^2
$$

for some $C > 0$. $\square$

Chapter 5

Linear kinetic equations with magnetic field: a rigorous derivation from the Lorentz model [In preparation]

In the present Chapter we present [N1].

Linear kinetic equation with magnetic field: a rigorous derivation from the Lorentz model

Abstract. We consider a test particle moving in random distribution of obstacles in the plane, under the action of a uniform magnetic field, orthogonal to the plane. We show that, in a weak coupling limit, the particle distribution behaves according to the linear Landau equation with the magnetic field. Moreover we show that, in the Boltzmann Grad limit, when each obstacles generates an inverse power law potential, the particle distribution behaves according to the linear Boltzmann equation with the magnetic field. This is in contrast with a recent result which shows that memory effects are not negligible in the Boltzmann-Grad limit.

5.1 Introduction

Consider a point particle of mass one in $\mathbb{R}^d$, $d = 2, 3$ moving in a random distribution of fixed scatterers, whose centers are denoted by $(c_1, \dots, c_N)$. The equations of motion are

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\sum_i \nabla\phi(|x - c_i|) \end{cases} , \tag{5.1.1}$$

here (x, v) denote position and velocity of the test particle, t the time and $\dot{A} = \frac{dA}{dt}$ for any time dependent variable A.

To outline a kinetic behavior it is usually introduced a scaling of the space-time variables and the density of the scatterer distribution. More precisely let $\varepsilon > 0$ be a parameter indicating the ratio between the macroscopic and microscopic variables, then rescale according to the law

$$x \to \varepsilon x, \quad t \to \varepsilon t, \quad \phi \to \varepsilon^\alpha \phi$$

where $\alpha \in [0, \frac{1}{2}]$ is a suitable parameter. In the new variables Eq.n (5.1.1) becomes

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\varepsilon^{\alpha-1} \sum_i \nabla\phi(\frac{|x-c_i|}{\varepsilon}) \end{cases} . \tag{5.1.2}$$

We assume that the scatterers are distributed according to a Poisson distribution of parameter $\mu_\varepsilon = \mu\varepsilon^{-\delta}$, where $\delta = d - 1 + 2\alpha$ and $d = 2, 3$ is the dimension of the physical space. This means that the probability of finding N obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^d$ is given by

$$\mathbb{P}_\varepsilon(\, d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|} \frac{\mu_\varepsilon^N}{N!} \, dc_1, \dots, \, dc_N \tag{5.1.3}$$

where $\mathbf{c}_N = c_1, \dots, c_N$ and $|\Lambda| = \text{meas}(\Lambda)$. Now let $T^t_{\mathbf{c}_N}(x, v)$ be the Hamiltonian flow solution to Eq.n (5.1.2) with initial datum (x, v) in a given sample of obstacles (skipping the ε dependence for notational simplicity) and, for a given probability distribution $f_0 = f_0(x, v)$, consider the quantity

$$f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_N}(x, v))] \tag{5.1.4}$$

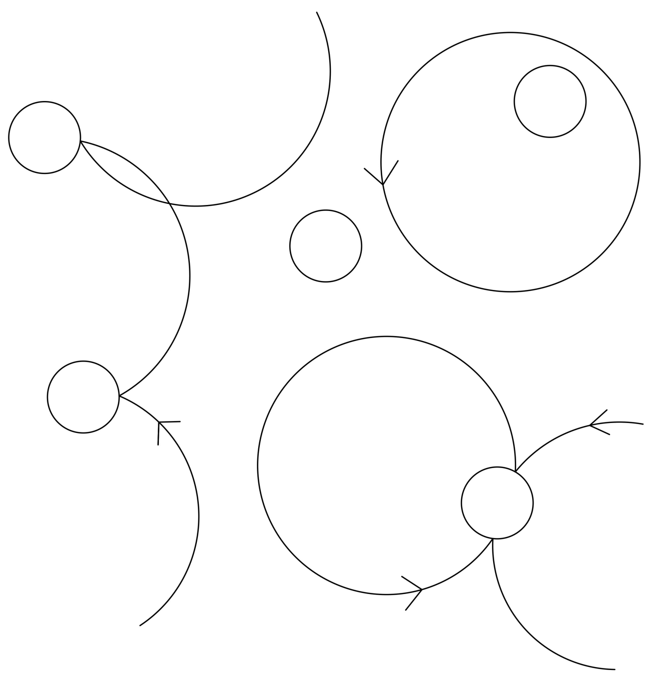
where $\mathbb{E}_\varepsilon$ is the expectation with respect to the measure $\mathbb{P}_\varepsilon$ given by (5.1.3).

In the limit $\varepsilon \to 0$ we expect that the probability distribution (5.1.4) solves a linear kinetic equation depending on the value of α. If $\alpha = 0$ the limit corresponding to such a scaling is called low-density (or Boltzmann-Grad) limit. Then f_ε converges to the solution of a linear Boltzmann equation. See [G], [S], [BBS], [DP]. On the other hand, if $\alpha = \frac{1}{2}$, the corresponding limit, called weak-coupling limit, yields the linear Landau equation as proven in [KP], [DGL], [K]. The intermediate scaling, namely $\alpha \in (0, \frac{1}{2})$, although refer to a low-density situation, leads to the linear Landau equation again. We note that the first group of references make use of abstract techniques while the second ones follow the original constructive idea, due to Gallavotti (see [G]), for the Boltzmann-Grad limit based on a suitable change of variables which can be implemented outside a pathological set of events, such as the recollisions, which, however, can be proven to be a set of negligible $\mathbb{P}_\varepsilon$ measure.

To be more precise consider, as in [G], the hard-sphere potential. Since the Boltzmann equation is the law of a Markov jump process in the velocity variable, events like recollisions, must be negligible in the limit (otherwise the fourth jump cannot be independent of the first three). For an explicit control of the error in the kinetic limit see for instance [BNPP].

In [DR] and [K] it was proven that even if $\alpha > 0$, but sufficiently small, the recollisions are still negligible. Incidentally we note that if α is close to $\frac{1}{2}$, this is not true anymore and it would be interesting to derive the Landau equation in this regime, by means of an explicit constructive approach.

Recently it has been observed that the presence of a given external field is not innocent in the derivation of the linear Boltzmann equation in the Boltzmann-Grad limit. Bobylev et al, in [BMHH1] (see also [BMHH2]), showed that when the test particle moves in a plane, in a Poisson distribution of hard disks, in presence of an external fixed magnetic field (hence performing arcs of circle between two consecutive collisions) the set of pathological configuration is no longer negligible. The situation is described in Figure 5.1. Moreover the probability $\mathbb{P}_R$ of performing an entire Larmor



circle without hitting an obstacle is not vanishing in the limit $\varepsilon \to 0$, since

$$\mathbb{P}_R \simeq e^{-\mu \frac{2\pi R}{|v|}}.$$

Here v is the velocity of the particle and R is the Larmor radius. Therefore, in [BMHH1] and [BMHH2], the authors derive a kinetic equation with memory, i.e. a generalized Boltzmann equation, taking into account these effects. Let $f(x, v, t)$ be the probability density of finding the moving particle at time t at position x with velocity v. This non-markovian kinetic equation which describes the evolution of $f(x,v,t)$ reads as

$$\begin{aligned} \frac{D}{Dt} f^G(x,v,t) =& \mu_\varepsilon \varepsilon \sum_{k=0}^{[t/T_L]} e^{-\nu k T_L} \int_{S_1} dn\,(v\cdot n) \\ & [\chi(v\cdot n) b_n + \chi(-v\cdot n)] f^G(x, S_0^{-k} v, t-kT_L), \end{aligned} \tag{5.1.5}$$

where

$$f^G(x,v,t) = \begin{cases} f(x,v,t) & \text{if } 0 < t < T_L \\ (1-e^{-\nu T_L}) f(x,v,t) & \text{if } t > T_L. \end{cases} \tag{5.1.6}$$

Here $\nu = 2|v|\mu_\varepsilon\varepsilon$ is the collision frequency and T_L the cyclotron period. Furthermore

$$\frac{D}{Dt} = (\partial_t + v\cdot\nabla_x + (v\times B)\cdot\nabla_v)$$

is the generator of free cyclotron motion with frequency $\omega = \frac{qB}{m}$, and $[t/T_L]$ the number of cyclotron periods $T_L = \frac{2\pi}{\omega}$ completed before time t. The angular integration over the unit vector n in (5.1.5) is over the entire unit sphere S_1 centered at the origin. In the gain term (positive contribution)the operator b_n is defined by

$$b_n \phi(v) = \phi(v - 2(v\cdot n)n)$$

where $\phi(v)$ is an arbitrary function of v. The precollisional velocity $v' = v - 2(v\cdot n)n$ becomes v after the elastic collision with the hard disk. Note that $v'\cdot n < 0$. In the loss term (negative contribution), the precollisional velocity, v, is also from the hemisphere $v\cdot n < 0$. Finally, the shift operator S_0^{-k}, when acting on v, rotates the velocity through the angle $-k\theta$, where θ is the scattering angle (from v' to v).

For further readings in this direction we refer to [DR1], [DR2], where a suitable stochastic Lorentz model with an external force field is considered. They show that certain fields prevent the limit process from being Markovian and they prove that the markovianity of the limit can be recovered by introducing an additional stochasticity in the velocity distribution of the obstacles.

In this paper we consider the case of a random distribution of scatterers where each obstacle generates a smooth positive and short-range potential ϕ, with $\alpha > 0$ and sufficiently small. We show that, in this case, the solution of the microscopic dynamics converges, in the intermediate limit (when $\alpha \in (0,1/8)$), to the solution of the linear Landau equation with the additional term due to the magnetic field. Roughly speaking, the heuristic motivation is that, in this case,

$$\mathbb{P}_R \simeq e^{-\mu\varepsilon^{2\alpha}\frac{2\pi R}{|v|}}$$

which vanishes as $\varepsilon \to 0$. Therefore we recover the Markovianity in the limit. In Section 5.2 we establish the model and formulate the result; in Section 5.3 we present the proof.

Furthermore, we observe that if we consider a long range inverse power law interaction potential, in a low density regime (when $\alpha = 0$), we can prove that the memory is lost in the limit. More precisely in Section 5.4 we prove that the microscopic solution converges to the solution of the linear Boltzmann equation with the additional term due to the magnetic field. This comes out from the fact that the probability $\mathbb{P}_R$ of performing an intere Larmor circle without hitting an obstacle is approximatively

$$\mathbb{P}_R \simeq e^{-\mu_\varepsilon \frac{2\pi\varepsilon^\gamma R}{|v|}} = e^{-\mu\varepsilon^{-1}\frac{2\pi\varepsilon^\gamma R}{|v|}} = e^{-\mu\varepsilon^{-1+\gamma}\frac{2\pi R}{|v|}}$$

which vanishes as $\varepsilon \to 0$, for $\gamma < 1$. This shows how the non-markovian behaviour of the limit process, discussed in [BMHH1], [BMHH2], disappears as soon as we slightly modify the microscopic model given by the two dimensional Lorentz Gas.

Along this paper our purpose is to provide a rigorous validation of these linear kinetic equations with magnetic field by using the constructive strategy due to Gallavotti. We remark that, as in [DP], [DR], [BNP], [BNPP] we need explicit estimates of the error in the kinetic limit and this is the crucial part. Moreover, as a future target, it could be interesting to understand if a rigorous derivation of the Generalized Boltzmann equation proposed in [BMHH1], [BMHH2], can be achieved by using the same constructive techniques.

5.2 The Model and the result

We consider the system (5.1.2) in the plane ($d = 2$) under the action of an additional, constant magnetic field, orthogonal to the plane. The equations of motion are

$$\begin{cases} \dot{x} = v \\ \dot{v} = Bv^\perp - \varepsilon^{\alpha-1}\sum_i \nabla\phi(\frac{|x-c_i|}{\varepsilon}) \end{cases}, \tag{5.2.1}$$

where $B > 0$ is the magnitude of the magnetic field and $v^\perp = (v_2, -v_1)$. We assume that the potential $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth and of range 1 i.e. $\phi(r) = 0$ if $r > 1$. Therefore the particle is influenced by the scatterer c_i if $|x - c_i| < \varepsilon$.

Starting at $t = 0$ from the point x with velocity v, the particle moves under the action of the Lorentz force $Bv^\perp$. Suppose that the particle has unitary mass and unitary charge, namely $m, q = 1$, hence between two consecutive scatterers, the particle moves with constant angular velocity $\omega = \frac{qB}{m} = B$ and performs an arc of circle of radius $R = \frac{|v|}{B}$. R is the radius of the cyclotron orbit whose center is situated at the point

$$x_c = x + \frac{\mathcal{R}(\frac{\pi}{2}) \cdot v}{\omega},$$

where the tensor $\mathcal{R}(\psi)$ denotes the rotation of angle ψ.

The precise assumptions on the potential are the following:

*A*1) $\phi \in C^2([0, \infty))$;

*A*2) $\phi \geq 0$, $\phi' \leq 0$ in $(0, 1)$;

*A*3) $\operatorname{supp} \phi \subset [0, 1]$.

On f_0 we assume that

*A*4) $f_0 \in L^1 \cap L^\infty \cap C(\mathbb{R}^2 \times \mathbb{R}^2)$, $f_0 \geq 0$, $\displaystyle\int f_0 \, dx \, dv = 1$.

Moreover we assume

*A*5) The scatterers are distributed according to a Poisson distribution (5.1.3) of intensity $\mu_\varepsilon = \mu \varepsilon^{-\delta}$ with $\delta = 1 + 2\alpha$, $\alpha \in (0, \frac{1}{8})$.

Next we define the Hamiltonian flow $T^t_{\mathbf{c}_N}(x, v)$, associated to the initial datum (x, v), solution of (5.2.1) for a given configuration $\mathbf{c}_N$ of scatterers, and we set

$$f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_N}(x, v))]$$

where $\mathbb{E}_\varepsilon$ denotes the expectation with respect to the Poisson distribution. The first result of the present paper is summarized in the following theorem.

Theorem 5.2.1. *Under assumption A1-A5, for all* $t \in [0, T]$*,*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\cdot; t) = f(\cdot; t) \qquad \in C([0, T]; \mathcal{D}')$$

where f *is the unique weak solution to the Landau equation with magnetic field*

$$\begin{cases} (\partial_t + v \cdot \nabla_x + B \, v^\perp \cdot \nabla_v) f(x, v, t) = \xi \, \Delta_{|v|} f(x, v, t) \\ f(x, v, 0) = f_0(x, v) \end{cases}, \tag{5.2.2}$$

where $\Delta_{|v|}$ *is the Laplace-Beltrami operator on the circle* $S_{|v|}$ *of radius* $|v|$*and* $\xi > 0$*.*

We shall give later explicit expressions of ξ.

5.3 Proof

Following [DP], [DR] and [BNP] we are led to compare f_ε with the solution h_ε of the following Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x + B\, v^\perp \cdot \nabla_v) h_\varepsilon(x,v,t) = \mathrm{L}_\varepsilon h_\varepsilon(x,v,t), \\ h_\varepsilon(x,v,0) = f_0(x,v) \end{cases}, \tag{5.3.1}$$

where

$$\mathrm{L}_\varepsilon h_\varepsilon(v) = \mu_\varepsilon |v| \int_{-\varepsilon}^{\varepsilon} d\rho [h_\varepsilon(v') - h_\varepsilon(v)], \tag{5.3.2}$$

where $v' = v - 2(\omega \cdot v)\omega = \mathcal{R}(\theta_\varepsilon)v$ is the outgoing velocity after a scattering with incoming velocity v and impact parameter $\rho \in [-\varepsilon, \varepsilon]$ generated by the potential $\varepsilon^\alpha \phi(\frac{r}{\varepsilon})$. $\omega = \omega(\rho)$ is the versor bisecting the angle between the incoming and outgoing velocity, θ is the scattering angle and $\mathcal{R}(\theta_\varepsilon)$ is the rotation of angle θ_ε.
By using the invariance of the scattering angle with respect to the space scale, we rewrite the collision operator in the right hand side of (5.3.1) as

$$\mathrm{L}_\varepsilon h_\varepsilon(v) = \mu_\varepsilon \varepsilon |v| \int_{-1}^{1} d\rho [h_\varepsilon(v') - h_\varepsilon(v)]. \tag{5.3.3}$$

We set

$$\begin{aligned} h_\varepsilon(v') - h_\varepsilon(v) = \; & (v' - v) \cdot \nabla_{|S_{|v|}} h_\varepsilon(v) \\ & + \frac{1}{2}(v' - v) \otimes (v' - v) \nabla_{|S_{|v|}} \nabla_{|S_{|v|}} h_\varepsilon(v) \\ & + \frac{1}{6}(v' - v) \otimes (v' - v) \otimes (v' - v) \nabla_{|S_{|v|}} \nabla_{|S_{|v|}} \nabla_{|S_{|v|}} h_\varepsilon(v) + R_{\eta_\varepsilon}, \end{aligned}$$

with $R_{\eta_\varepsilon} = \mathcal{O}(|v - v'|^4)$. Integrating with respect to v and using symmetry arguments we obtain

$$\mathrm{L}_\varepsilon h_\varepsilon = \mu |v| \varepsilon^{-2\alpha} \left\{ \frac{1}{2} \Delta_{|v|} h_\varepsilon \int_{-1}^{1} d\rho\, |v' - v|^2 + \int_{-1}^{1} d\rho\, R_{\eta_\varepsilon} \right\}.$$

Observe that $|v' - v|^2 = 4 \sin^2 \frac{\theta_\varepsilon(\rho)}{2}$, then by direct computation

$$\lim_{\varepsilon \to 0} \frac{\mu \varepsilon^{-2\alpha} |v|}{2} \int_{-1}^{1} d\rho\, |v' - v|^2 = \lim_{\varepsilon \to 0} \frac{\mu \varepsilon^{-2\alpha} |v|}{2} \int_{-1}^{1} \theta_\varepsilon^2(\rho)\, d\rho.$$

Therefore it is rather straightforward to show the following Proposition.

Proposition 5.3.1. *Under the assumptions A1-A4, $h_\varepsilon \to f$ in $C([0,T]; \mathcal{D}')$ where f is the unique weak solution to the Landau equation with magnetic field*

$$\begin{cases} (\partial_t + v \cdot \nabla_x + B\, v^\perp \cdot \nabla_v) f(x,v,t) = \xi \Delta_{|v|} f(x,v,t) \\ f(x,v,0) = f_0(x,v) \end{cases}, \tag{5.3.4}$$

where

$$\xi = \lim_{\varepsilon \to 0} \frac{\mu \varepsilon^{-2\alpha} |v|}{2} \int_{-1}^{1} \theta_\varepsilon^2(\rho)\, d\rho \tag{5.3.5}$$

is the diffusion coefficient.

Remark 5.3.2. *We have still to show that the limit* (5.3.5) *does exist. An explicit expression of ξ has been obtained in [DR]. It is*

$$\xi = \frac{\mu |v|}{2} \int_{-1}^{1} \left(\int_{\rho}^{1} \frac{\rho}{u} \phi'(\frac{\rho}{u}) \frac{du}{\sqrt{1-u^2}} \right)^2 d\rho \tag{5.3.6}$$

An equivalent expression can be found in [K]:

$$\xi = \frac{\pi \mu}{|v|} \int_0^\infty r^2 \hat{\phi}(r)^2\, dr \tag{5.3.7}$$

where $\hat{\phi}$ denotes the Fourier transform of the potential, i.e.

$$\hat{\phi}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ik\cdot x} \phi(|x|)\, dx.$$

Since $\hat{\phi}$ is real and spherically symmetric, we used the usual notational abuse $\hat{\phi}(r) = \hat{\phi}(k)$ if $r = |k|$.

Remark 5.3.3. *Following [DR] we have split the original problem into two parts, one concerning the grazing collision limit discussed up to now, the other concerning the asymptotic equivalence of the h_ε and f_ε and this is the crucial part. Note that the presence of the magnetic field is completely irrelevant as regards the first part.*

Remark 5.3.4. *We avoided to introduce the cross-section $\psi(\theta_\varepsilon) = \dfrac{d\rho}{d\theta_\varepsilon}$ of the problem because the map $\rho \to \theta_\varepsilon(\rho)$ is not monotonic in general. Indeed if ϕ is bounded and ε sufficiently small, $\frac{1}{2}v^2 > \varepsilon^\alpha \phi(0)$ so that $\theta = 0$ for $\rho = 0$ and $\rho = \pm 1$. As a consequence ψ_ε is neither single valued nor bounded.*

Before proving Theorem 5.2.1 we need the following preliminary result concerning the asymptotic behavior of the scattering angle as a function of the impact parameter in the limit when the potential is rescaled as $\phi \to \varepsilon^\alpha \phi$, with $\varepsilon \to 0$, $\alpha > 0$.

Lemma 5.3.5. *The deflection angle $\theta_\varepsilon(\rho)$ of a particle with impact parameter ρ due to a scatterer generating a radial potential $\varepsilon^\alpha \phi$ under the action of the Lorentz force $Bv^\perp$ satisfies*

$$|\theta_\varepsilon(\rho)| \le C \varepsilon^\alpha. \tag{5.3.8}$$

Proof. As established in [DR] (Section 3) estimate (5.3.5) holds when the test particle moves freely with no external field. Hence we just need to compare the dynamics of the particle in presence of the constant magnetic field with the free dynamics. Let $(\underline{x}(t), \underline{v}(t))$ be the solution of the following

$$\begin{cases} \dot{\underline{x}} = v \\ \dot{\underline{v}} = -\varepsilon^{\alpha-1} \sum_i \nabla\phi(\frac{|\underline{x}-c_i|}{\varepsilon}) \end{cases} . \tag{5.3.9}$$

Let τ be the collision time. Since $\tau \le C\varepsilon$, $C > 0$, we get

$$\begin{aligned} |v(\tau) - \underline{v}(\tau)| &= \left| \varepsilon^{\alpha-1} \int_0^\tau ds\, \big(F\big(x(s)/\varepsilon\big) - F\big(\underline{x}(s)/\varepsilon\big)\big) + \int_0^\tau ds\, v^\perp B \right| \\ &\le \varepsilon^{\alpha-1} \int_0^\tau ds\, |F\big(x(s)/\varepsilon\big) - F\big(\underline{x}(s)/\varepsilon\big)| + C_1\varepsilon \\ &\le \varepsilon^{\alpha-2} C_2 \int_0^\tau ds\, |x(s) - \underline{x}(s)| + C_1\varepsilon \\ &= \varepsilon^{\alpha-2} C_2 \int_0^\tau ds \int_0^s dt\, |v(t) - \underline{v}(t)| + C_1\varepsilon \\ &\le \varepsilon^{\alpha-2} C_2 \int_0^\tau ds \int_0^\tau dt\, |v(t) - \underline{v}(t)| + C_1\varepsilon. \end{aligned}$$

By using Gronwall's inequality we obtain

$$|v(\tau) - \underline{v}(\tau)| \le C_1\, \varepsilon\, e^{C_3 \varepsilon^{\alpha-1}\tau} \le C_1\, \varepsilon\, e^{C_3\varepsilon^\alpha},$$

for $\alpha > 0$ and ε sufficiently small. Hence the velocities v and $\underline{v}$ are asymptotically equivalent up to an error term of order ε. This implies the validity of Eq.n (5.3.8) and concludes the proof. □

5.3.1 Strategy

Following the explicit approach in [G], [DR], [DP] we will show the asymptotic equivalence of f_ε, defined by (5.1.4), and h_ε solution of the following Boltzmann equation

$$(\partial_t + v \cdot \nabla_x + B\, v^\perp \cdot \nabla_v) h_\varepsilon(x, v, t) = \mathrm{L}_\varepsilon h_\varepsilon(x, v, t), \tag{5.3.10}$$

where

$$\mathrm{L}_\varepsilon h(v) = \mu \varepsilon^{-2\alpha} |v| \int_{-1}^{1} d\rho \{h(v') - h(v)\}. \tag{5.3.11}$$

This allows to reduce the problem to the analysis of a Markov process which is an easier task. Indeed, the series expansion defining h_ε (obtained perturbing around the loss term) reads as

$$\begin{aligned} h_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q \ge 0} \mu_\varepsilon^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1 \\ \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q\, f_0(\bar{\xi}_\varepsilon(-t), \bar{\eta}_\varepsilon(-t)). \end{aligned} \tag{5.3.12}$$

Eq.n (5.3.12) is an evolution equation for the probability density associated to a particle performing random jumps in the velocity variable at random Markov times.

We consider the microscopic solution f_ε defined by (5.1.4). For $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $t > 0$, we have

$$f_\varepsilon(x,v,t) = e^{-\mu_\varepsilon |B_t(x,v)|} \sum_{N \geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B_t(x,v)^N} d\mathbf{c}_N \, f_0(T^{-t}_{\mathbf{c}_N}(x,v)), \tag{5.3.13}$$

where $T^t_{\mathbf{c}_N}(x,v)$ is the Hamiltonian flow with initial datum (x,v). Finally $B_t(x,v) = B(x, |v|t)$, where here and in the following, $B(x,R)$ denotes the disk of center x and radius R.

Coming back to Eq.n (5.3.13), we distinguish the obstacles of the configuration $\mathbf{c}_N = c_1 \dots c_N$ which, up to the time t, influence the motion, called internal obstacles, and the external ones. More precisely c_i is internal if

$$\inf_{0 \leq s \leq t} |x_\varepsilon(-s) - c_i| < \varepsilon, \tag{5.3.14}$$

while c_i is external if

$$\inf_{0 \leq s \leq t} |x_\varepsilon(-s) - c_i| \geq \varepsilon. \tag{5.3.15}$$

Here $(x_\varepsilon(-s), v_\varepsilon(-s)) = T^{-s}_{\mathbf{c}}(x,v)$, $s \in [0,t]$.

Note that the integration over the external obstacles can be performed so that

$$f_\varepsilon(x,v,t) = \sum_{Q \geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B_t(x,v)^Q} d\mathbf{b}_Q \, e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi(\{\mathbf{b}_Q \text{ internal}\}) f_0(T^{-t}_{\mathbf{b}_Q}(x,v)), \tag{5.3.16}$$

where $\mathcal{T}(\mathbf{b}_Q)$ is the tube

$$\mathcal{T}(\mathbf{b}_Q) = \{y \in B_t(x,v) \text{ s.t. } \exists s \in (0,t) \text{ s.t. } |y - x_\varepsilon(-s)| < \varepsilon\}. \tag{5.3.17}$$

Here and in the sequel $\chi(\{\cdot\})$ is the characteristic function of the event $\{\cdot\}$. Here we are not considering possible overlappings of obstacles. The scattering process can be solved in this case as well. However, as we shall see, this event is negligible because of the moderate densities we are considering. With this purpose we introduce

$$\chi_1(\mathbf{b}_Q) = \chi\{\mathbf{b}_Q \text{ s.t. } b_i \notin B(x,\varepsilon) \text{ and } b_i \notin B(x(-t),\varepsilon) \text{ for all } i = 1,\dots,Q\}. \tag{5.3.18}$$

Moreover, to avoid a first cyclotron orbit completed without suffering collisions and to avoid a repeated collision with the same scatterer without suffering any collision in the meantime (see Figure 5.1), we introduce

$$\chi_{circ}(\mathbf{b}_Q) = \chi\left(\{\mathbf{b}_Q \text{ s.t. } |\mathcal{T}(\mathbf{b}_Q)| \geq 4\pi\varepsilon R\}\right) \tag{5.3.19}$$

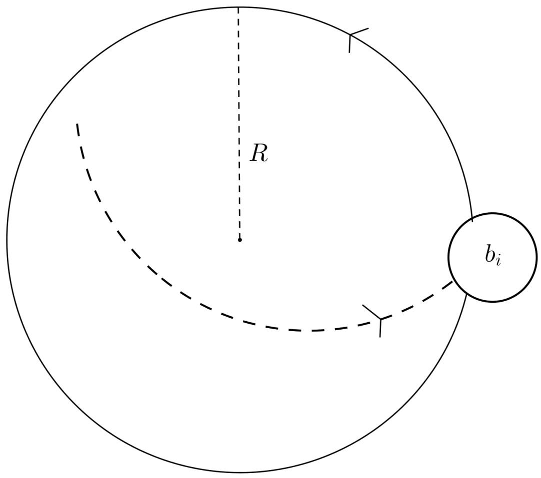


Figure 5.1: Periodic orbit

and we define

$$\breve{f}_\varepsilon(x,v,t) = \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B_t(x,v)^Q} d\mathbf{b}_Q e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi(\{\mathbf{b}_Q \text{ internal}\}) \\ (1-\chi_{circ}(\mathbf{b}_Q))\,\chi_1(\mathbf{b}_Q) f_0(T^{-t}_{\mathbf{b}_Q}(x,v)). \tag{5.3.20}$$

We can show that the contribution due to χ_{circ} is vanishing in the limit as $\varepsilon \to 0$ (see Section 5.3.2 for the explicit computation). Therefore

$$f_\varepsilon \geq \breve{f}_\varepsilon.$$

Observe that for times smaller than the Larmor period T_L we expected to be true the approximation with the dynamics of the test particle in absence of the external field. The unexpected fact is that for times comparable with the Larmor period this still holds due to the smallness of the error term produced by (5.3.19). Hence, for a given configuration $\mathbf{b}_Q$ such that $\chi_1[1-\chi_{circ}](\mathbf{b}_Q) = 1$, we have that the measure of the tube can be estimated by

$$|\mathcal{T}(\mathbf{b}_Q)| \leq 2\varepsilon|v|t. \tag{5.3.21}$$

At this point we define

$$\tilde{f}_\varepsilon(x,v,t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B_t(x,v)^Q} d\mathbf{b}_Q \chi(\{\mathbf{b}_Q \text{ internal}\}) \\ (1-\chi_{circ}(\mathbf{b}_Q))\,\chi_1(\mathbf{b}_Q) f_0(T^{-t}_{\mathbf{b}_Q}(x,v)). \tag{5.3.22}$$

Thanks to (5.3.21) we get

$$f_\varepsilon \geq \breve{f}_\varepsilon \geq \tilde{f}_\varepsilon. \tag{5.3.23}$$

Note that, according to a classical argument introduced in [G] (see also [DP], [DR], [BNP]), we remove from $\tilde{f}_\varepsilon$ all the bad events, namely those untypical with respect to the Markov process described by h_ε^{out}. Then we will show they are unlikely.

For any fixed initial conditions (x, v) we order the obstacles $b_1, \dots, b_N$ according to the scattering sequence. Let ρ_i and t_i be the impact parameter and the entrance time of the light particle in the protection disk around b_i, namely $B_\varepsilon(b_i)$. Then we perform the following change of variables

$$b_1, \dots, b_N \to \rho_1, t_1, \dots, \rho_N, t_N \tag{5.3.24}$$

with

$$0 \le t_N < t_{N-1} < \dots < t_1 \le t.$$

Conversely, fixed the impact parameters $\{\rho_i\}$ and the hitting times $\{t_i\}$ we construct the centers of the obstacles $b_i = b(\rho_i, t_i)$. By performing the backward scattering we construct a trajectory $\gamma^{-s}(x, v) := (\xi_\varepsilon(-s), \eta_\varepsilon(-s)),\ s \in [0, t]$,

However $\gamma^{-s}(x, v) = (x_\varepsilon(-s), v_\varepsilon(-s))$ (therefore the mapping (5.3.24) is one-to-one) only outside the following pathological situations.

i) **Overlapping**.
If b_i and b_j are both internal then $B(b_i, \varepsilon) \cap B(b_j, \varepsilon) \neq \emptyset$.

ii) **Recollisions**.
There exists b_i such that for $s \in (t_{j+1}, t_j)$, $j > i$, $\xi_\varepsilon(-s) \in B(b_i, \varepsilon)$.

iii) **Interferences**.
There exists b_i such that $\xi_\varepsilon(-s) \in B(b_j, \varepsilon)$ for $s \in (t_{i+1}, t_i)$, $j > i$.

We simply skip such events by setting

$$\chi_{ov} = \chi(\{\mathbf{b}_Q \text{ s.t. i) is realized}\}),$$

$$\chi_{rec} = \chi(\{\mathbf{b}_Q \text{ s.t. iii) is realized}\}),$$

$$\chi_{int} = \chi(\{\mathbf{b}_Q \text{ s.t. iv) is realized}\}),$$

and defining

$$\bar{f}_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q \ge 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q$$
$$\chi_1(1 - \chi_{circ})(1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int}) f_0(\gamma^{-t}(x, v)). \tag{5.3.25}$$

Note that

$$\bar{f}_\varepsilon \le \tilde{f}_\varepsilon \le \breve{f}_\varepsilon \le f_\varepsilon.$$

Note also that in (5.3.25) we have used the change of variables (5.3.24) for which, outside the pathological sets i), ii), iii), iv) $T^{-t}_{\mathbf{b}_Q}(x,v) = (x_\varepsilon(-t), v_\varepsilon(-t))$.

Next we remove $\chi_1(1-\chi_{circ})(1-\chi_{ov})(1-\chi_{rec})(1-\chi_{int})$ by setting

$$\begin{aligned}\bar{h}_\varepsilon(x,v,t) = & e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q\geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \\ & \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \, f_0(\gamma^{-t}(x,v)).\end{aligned} \tag{5.3.26}$$

We observe that

$$1-\chi_1(1-\chi_{ov})(1-\chi_{cir})(1-\chi_{rec})(1-\chi_{int}) \leq (1-\chi_1)+\chi_{ov}+\chi_{cir}+\chi_{rec}+\chi_{int}. \tag{5.3.27}$$

Then by (5.3.25) and (5.3.26) we obtain

$$|\bar{h}_\varepsilon(t) - \bar{f}_\varepsilon(t)| \leq \varphi_1(\varepsilon,t)$$

with

$$\begin{aligned}\varphi_1(\varepsilon,t) = \|f_0\|_\infty \, e^{-2\varepsilon^{-2\alpha}|v|t} \sum_{Q\geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \\ (1-\chi_1)+\chi_{ov}+\chi_{cir}+\chi_{rec}+\chi_{int}.\end{aligned} \tag{5.3.28}$$

We state the following result. The proof is postponed to Section 5.3.2.

Proposition 5.3.6. *Let $\varphi_1(\varepsilon,t)$ be defined as in* (5.3.28)*. For any $t \in [0,T]$*

$$\|\varphi_1(\varepsilon,t)\|_{L^1} \to 0$$

as $\varepsilon \to 0$.

Since we are working to achieve the asymptotic equivalence of f_ε and h_ε, we need to compare $\bar{h}_\varepsilon$ with h_ε. This is fulfilled once we consider the collision as instantaneous. More precisely, for the sequence $t_1,\dots,t_Q$ $\rho_1,\dots\rho_Q$ consider the sequence $v_1,\dots,v_Q$ of incoming velocities before the Q collisions. This allows to construct the limiting trajectory $\bar{\gamma}^{-s}(x,v) = (\bar{\xi}_\varepsilon(-s), \bar{\eta}_\varepsilon(-s)),\ s \in [0,t]$, which approximates the trajectory $\gamma^{-s}(x,v)$ up to an error vanishing in the limit. Indeed, since

$$|\xi_\varepsilon(-t) - \bar{\xi}_\varepsilon(-t)| \leq Q\,\varepsilon$$

and Eq.n (5.3.8) holds, due to the Lipschitz continuity of f_0, we can assert that

$$\bar{h}_\varepsilon(x,v,t) = h_\varepsilon(x,v,t) + \varphi_2(x,v,t), \tag{5.3.29}$$

where

$$\sup_{x,v,t\in[0,T]} |\varphi_2(x,v,t)| \leq C\varepsilon^{1-2\alpha}\,T. \tag{5.3.30}$$

For more details see [DP], Section 3.

5.3.2 The control of the pathological sets

In this section we prove Proposition 5.3.6. For any measurable function u of the backward Markov process $(\xi_\varepsilon, \eta_\varepsilon)$ we set

$$\mathbb{E}_{x,v}[u] = e^{-2|v|\mu_\varepsilon \varepsilon t} \sum_{Q\geq 0} (2|v|\mu_\varepsilon)^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q$$
$$\int_{-\varepsilon}^{\varepsilon} d\rho_1 \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \, u(\xi_\varepsilon, \eta_\varepsilon).$$

Then we realize that

$$\varphi_1(\varepsilon, t) = \|f_0\|_\infty \mathbb{E}_{x,v}[(1-\chi_1) + \chi_{ov} + \chi_{cir} + \chi_{rec} + \chi_{int}]$$

and we estimate separately all the events in the right hand side of (5.3.27).

We can skip the estimates of the first two contributions, i.e. $\mathbb{E}_{x,v}[(1-\chi_1)]$ and $\mathbb{E}_{x,v}[\chi_{ov}]$, since the presence of the external field does not affect the classical arguments which can be found in [BNP], [DR], [DP]. The presence of the magnetic field and consequently the circular motion of the test particle strongly affect the explicit estimates of the pathological events ii), iii). Therefore we need a detailed analysis for χ_{cir}, χ_{rec} and χ_{int}.

For what concerns the pathological event due to a recollision with the same scatterer (see Figure 5.2) we observe that $\chi_{cir} = 1$ if there exists an entrance time t_i such that $|t_i - t_{i+1}| \geq T_L$ for some $i = 0, \dots Q-1$ (assuming $t_0 = 0$). Therefore it results

$$\begin{aligned}
\mathbb{E}^M_{x,v}[\chi_{cir}] &= e^{-2t\mu_\varepsilon \varepsilon |v|} \sum_{Q\geq 1} (2|v|\mu_\varepsilon)^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \\
&\quad \dots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \sum_{i=0}^{Q-1} \chi(\{|t_i - t_{i+1}| > T_L\}) \\
&\leq e^{-2t\mu_\varepsilon \varepsilon |v|} \sum_{Q\geq 1} Q \, \frac{(2|v|\mu_\varepsilon \varepsilon)^Q}{(Q-1)!} (t - T_L)^{Q-1} \\
&\leq 2\,(t - T_L) e^{-2\varepsilon^{-2\alpha} T_L} (2|v|\mu_\varepsilon \varepsilon)^2 \\
&\leq C|v| e^{-2\varepsilon^{-2\alpha} T_L} \varepsilon^{-4\alpha} t,
\end{aligned} \tag{5.3.31}$$

for $\alpha > 0$ and ε sufficiently small.

Next we pass to the control of the recollision event. Let t_i the first time the light particle hits the i-th scattering, v_i^- the incoming velocity, v_i^+ the outgoing velocity and t_i^+ the exit time. Moreover we fix the axis in such a way that v_i^+ is parallel to the x axis . We have

$$\chi_{rec} \leq \sum_{i=1}^{Q} \sum_{j>1} \chi^{i,j}_{rec}, \tag{5.3.32}$$

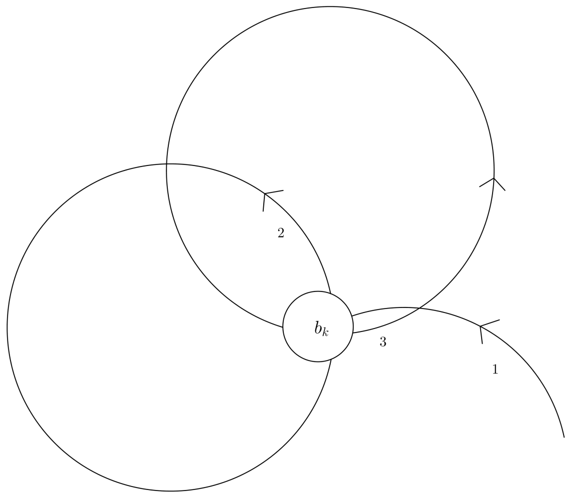


Figure 5.2: Recollision with the same scatter.

where $\chi_{rec}^{i,j} = 1$ if and only if b_i (constructed via the sequence $t_1, \rho_1, \dots, t_i, \rho_i$) is recollided in the time interval (t_j, t_{j-1}).
Note that, since $|\theta_i| \le C\varepsilon^\alpha$, where θ_i is the i-th scattering angle, in order to have a recollision it must be an intermediate velocity v_k, $k = i+1, \dots, j-1$ such that

$$|v_k^+ \cdot v_j^+| \le C\varepsilon^\alpha |v|^2, \tag{5.3.33}$$

namely v_k^+ is almost orthogonal to v_j^+. Then

$$\chi_{rec} \le \sum_{i=1}^{Q} \sum_{j=1}^{Q} \sum_{k=i+1}^{j-1} \chi_{rec}^{i,j,k}, \tag{5.3.34}$$

where $\chi_{rec}^{i,j,k} = 1$ if and only if $\chi_{rec}^{i,j} = 1$ and (5.3.33) is fulfilled.

Moreover, due to the presence of the magnetic field B we need to take into account two different cases. Since the test particle follows a circular trajectory, it could happen that v_k^+ is almost orthogonal to v_j^+ despite there are no scatterers between b_i and b_k. It means that the orthogonality is due to a rotation of $\frac{\pi}{2}$ in a time interval proportional to $\frac{T_L}{4}$. We have the following decomposition

$$\chi_{rec} \le \sum_{i=1}^{Q} \sum_{j=1}^{Q} \sum_{k=i+1}^{j-1} \chi_{rec}^{i,j,k} \le \sum_{i=1}^{Q} \sum_{j=1}^{Q} \sum_{k=i+1}^{j-1} \left(\tilde{\chi}_{rec}^{i,j,k} + \bar{\chi}_{rec}^{i,j,k} \right), \tag{5.3.35}$$

where

$$\tilde{\chi}_{rec}^{i,j,k} := \chi \left(\left\{ |t_{k-1} - t_k| > \frac{T_L}{4} \text{ for some } k = i+1, \dots j-1 \right\} \right).$$

Therefore

$$\mathbb{E}_{x,v}[\chi_{rec}] \leq \quad \mathbb{E}_{x,v}\left[\sum_{i=1}^{Q}\sum_{j=1}^{Q}\sum_{k=i+1}^{j-1}\tilde{\chi}_{rec}^{i,j,k}\right] + \mathbb{E}_{x,v}\left[\sum_{i=1}^{Q}\sum_{j=1}^{Q}\sum_{k=i+1}^{j-1}\bar{\chi}_{rec}^{i,j,k}\right] \tag{5.3.36}$$

and we estimate the two contributions separately. We look at the first one

$$\begin{aligned} &\mathbb{E}_{x,v}^{M}\left[\sum_{i=1}^{Q}\sum_{j=1}^{Q}\sum_{k=i+1}^{j-1}\tilde{\chi}_{rec}^{i,j,k}\right] \\ &\leq e^{-2|v|\varepsilon^{-2\alpha}t}\sum_{Q\geq 3}(Q-1)(Q-2)(Q-3)\frac{(2|v|\varepsilon^{-2\alpha})^{Q}}{Q!}\Big(t-\frac{T_L}{4}\Big)^{Q} \qquad (5.3.37) \\ &\leq 2\,C\,|v|^{3}\Big(t-\frac{T_L}{4}\Big)^{3}e^{-2\varepsilon^{-2\alpha}T_L}\varepsilon^{-6\alpha} \leq C\,|v|^{3}t^{3}e^{-2\varepsilon^{-2\alpha}\frac{T_L}{4}}\varepsilon^{-6\alpha}, \end{aligned}$$

for $\alpha > 0$ and ε sufficiently small. We now consider the second contribution in (5.3.36). Once we fix all the parameters $\rho_1, \dots, \rho_Q$, $t_1, \dots, t_Q$ but t_{k+1} we perform such a time integration. The two branches of the trajectory l_1, l_2 are rigid so that, if the recollision happen the time integration with respect to t_{k+1} is restricted to a time interval proportional to AB. More precisely it is bounded by $C\varepsilon$ (see for instance [BNP], section 5). Performing all the other integrations and summing over i, j, k we obtain

$$\begin{aligned} &\mathbb{E}_{x,v}\left[\sum_{i=1}^{Q}\sum_{j=1}^{Q}\sum_{k=i+1}^{j-1}\bar{\chi}_{rec}^{i,j,k}\right] \\ &\leq C\varepsilon\, e^{-2|v|\varepsilon^{-2\alpha}t}\sum_{Q\geq 3}(Q-1)(Q-2)(Q-3)\frac{(2|v|\varepsilon^{-2\alpha})^{Q}}{(Q-1)!}t^{Q-1} \qquad (5.3.38) \\ &\leq C|v|^{3}t^{3}\varepsilon^{1-8\alpha}, \end{aligned}$$

for $\alpha < 1/8$ and ε sufficiently small.

Following the strategy used in [BNP], since a backward interference is a forward recollision, the estimate for the interference event can be performed by using the Liouville Theorem.

5.4 Concluding Remarks

In this paper we provided a rigorous derivation of the the Linear Landau equation with magnetic field starting from the microscopic dynamics of the Lorentz gas in the intermediate scaling ($\alpha \in (0, \frac{1}{2})$).

We can argue that the same techniques, suitably modified, can be used to derive the Linear Boltzmann equation from the Lorentz model with long-range forces and magnetic field in a low density limit (i.e. the endpoint case $\alpha = 0$).

More precisely we can assume that each obstacle of radius ε generates an inverse power law potential $\phi_\varepsilon(\frac{|x-c|}{\varepsilon})$ where the unrescaled ϕ is an inverse power law potential cutoffed at large distances, i.e.

$$\begin{cases} \phi(x) = \frac{1}{|x|^s} & |x| < \varepsilon^{\gamma-1} \\ \phi(x) = \varepsilon^{-s(\gamma-1)} & |x| \geq \varepsilon^{\gamma-1} \end{cases} , \tag{5.4.1}$$

with $\gamma \in (0,1)$ and $s > 2$. The distribution of the scatterers is a Poisson law of intensity $\mu_\varepsilon = \varepsilon^{-1}\mu$, $\mu > 0$.

The equation of motion, in macroscopic variables, are

$$\begin{cases} \dot{x} = v \\ \dot{v} = Bv^\perp - \varepsilon^{-1} \sum_i \nabla\phi(\frac{|x-c_i|}{\varepsilon}) \end{cases} , \tag{5.4.2}$$

with ϕ given by (5.4.1).

Let $T^t_{\mathbf{c}_N}(x,v)$ be the Hamiltonian flow solution to Eq.n (5.4.2) with initial datum (x,v) in a given sample of obstacles and, for a given probability distribution $f_0 = f_0(x,v)$, consider the quantity

$$f_\varepsilon(x,v,t) = \mathbb{E}_\varepsilon[f_0(T^{-t}_{\mathbf{c}_N}(x,v))] \tag{5.4.3}$$

where $\mathbb{E}_\varepsilon$ is the expectation with respect to the measure $\mathbb{P}_\varepsilon$ given by (5.1.3).

Theorem 5.4.1. *Let f_ε be defined in* (5.4.3). *Then, for any $T > 0$,*

$$\lim_{\varepsilon\to 0} f_\varepsilon(\cdot;t) = f(\cdot;t) \qquad \in C([0,T];\mathcal{D}')$$

where f is the unique weak solution to the linear Boltzmann equation with magnetic field

$$\begin{cases} (\partial_t + v\cdot\nabla_x + B\,v^\perp\cdot\nabla_v)f(t,x,v) = Lf(t,x,v) \\ f(x,v,0) = f_0(x,v), \end{cases} \tag{5.4.4}$$

with

$$Lf(v) = \mu|v| \int_{-\pi}^{\pi} B(\Theta)\left\{f(R_\Theta(v)) - f(v)\right\} d\Theta.$$

To prove the transition from the particle system we are considering to the uncutoffed linear Boltzmann equation (5.4.4) we need the following preliminary result

Proposition 5.4.2. *Let f_ε be defined in* (5.4.3). *Then, for any $T > 0$,*

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - \tilde{h}_\varepsilon\|_{L^\infty([0,T];L^1(\mathbb{R}^2\times S_1))} = 0 \tag{5.4.5}$$

where $\tilde{h}_\varepsilon$ is the unique weak solution of the cutoffed linear Boltzmann equation with magnetic field

$$\begin{cases} (\partial_t + v\cdot\nabla_x + B\,v^\perp\cdot\nabla_v)\tilde{h}_\varepsilon(t,x,v) = \tilde{L}\tilde{h}_\varepsilon(t,x,v) \\ \tilde{h}_\varepsilon(x,v,0) = f_0(x,v) \end{cases} , \tag{5.4.6}$$

with

$$\tilde{L}f(v) = \mu|v| \int_{-\pi}^{\pi} B_{\varepsilon,\gamma}(\theta) \{f(R_\Theta(v)) - f(v)\}\, d\theta.$$

This allows to reduce the problem of the transition from the solution of the cutoffed linear Boltzmann equation to the solution of the uncutoffed linear Boltzmann equation to a partial differential equation problem. Indeed, as in [DP], we can prove that

Proposition 5.4.3. *Let $\tilde{h}_\varepsilon$ solution of* (5.4.6)*. Then, for any $T > 0$,*

$$\tilde{h}_\varepsilon \to f \quad in \quad C([0,T];\mathcal{D}') \tag{5.4.7}$$

where f is the unique weak solution of (5.4.4)*.*

Some comments are in order. The proof of Proposition 5.4.3 is exactly the same performed in [DP]. We remark that due to the presence of the magnetic field B, we obtain a complicate expression for the scattering angle $\theta(M)$, being M the angular momentum. The explicit expression is given in (A.4), see Appendix 5.5.1 for the detailed computation. Moreover it is possible to compute the scattering cross section by using Eq.n (A.9) which gives a complicate explicit expression for $d\theta/dM$. To overcome this difficulty we observe that the scattering cross section in a constant magnetic field is asymptotically close to the free cross section $B_{\varepsilon,\gamma}$, since the error vanishes in the limit $\varepsilon \to 0$.

To prove Proposition 5.4.2 we follow the approach proposed in Section 5.3.1. Roughly speaking, it works since we can show that the probability of finding a closed orbit is vanishing in the limit, namely

$$e^{-\mu_\varepsilon \frac{2\pi\varepsilon^\gamma R}{|v|}} = e^{-\mu\varepsilon^{-1} \frac{2\pi\varepsilon^\gamma R}{|v|}} = e^{-\mu\varepsilon^{-1+\gamma} \frac{2\pi R}{|v|}} \to 0 \quad \text{as} \quad \gamma < 1. \tag{5.4.8}$$

The estimates of the pathological events obtained in Section 5.3.2, modify according to the low density regime and the interaction potential we are considering in this setting. For instance we consider the pathological event due to a circling trajectory and

$$\chi_{circ}(\mathbf{b}_Q) = \chi\left(\{\mathbf{b}_Q \text{ s.t. } |\mathcal{T}(\mathbf{b}_Q)| \geq 4\pi\varepsilon^\gamma R\}\right).$$

By estimating the contribution due to χ_{circ} we get

$$\begin{aligned}
\mathbb{E}^M_{x,v}[\chi_{cir}] &= e^{-2t\mu_\varepsilon \varepsilon^\gamma} \sum_{Q\geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \\
&\quad \dots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \sum_{i=0}^{Q-1} \chi(\{|t_i - t_{i+1}| > T_L\}) \\
&\leq e^{-2t\mu_\varepsilon \varepsilon |v|} \sum_{Q\geq 1} Q \frac{(2|v|\mu_\varepsilon \varepsilon)^Q}{(Q-1)!} (t - T_L)^{Q-1} \\
&\leq 2\, C\, (t - T_L) e^{-2t\mu_\varepsilon \varepsilon^\gamma T_L} (2t\mu_\varepsilon \varepsilon^\gamma)^2 \\
&\leq C e^{-2t\mu_\varepsilon \varepsilon^\gamma T_L} \varepsilon^{2(\gamma-1)} t,
\end{aligned} \tag{5.4.9}$$

for $\gamma \in (0,1)$ and ε sufficiently small.

To control the recollision event, the analogous of *ii*) in Section 5.3.1, we can follow the same strategy used in [DP] (Proof of Proposition 3.1). Also in this case, as we observed in Section 5.3.2, the presence of the magnetic field slightly modify the estimate and we obtain the analogous of Eq.n (5.3.35).

5.5 Appendix

5.5.1 The cross-section

We remind the classical formula giving the scattering angle Θ in terms of the impact parameter ρ for a particle moving in a potential $\Phi(r) = r^{-s}$, $s > 2$.

$$\Theta(\rho) = \pi - 2\int_{r_*}^{+\infty} \frac{\rho\, dr}{r^2\sqrt{1 - 2\Phi_{eff}(r)}}, \tag{A.1}$$

where r_* is a zero of the radicand and

$$\Phi_{eff}(r) = \Phi(r) + \frac{\rho^2}{2r^2} = r^{-s} + \frac{\rho^2}{2r^2}.$$

See [L], Eq.n (18.2). Furthermore, since we deal with a cutoffed inverse power law potential given by (5.4.1), i.e.

$$\begin{cases} \phi(x) = \frac{1}{|x|^s}, & |x| < \varepsilon^{\gamma-1} \\ \phi(x) = \varepsilon^{-s(\gamma-1)} & |x| \geq \varepsilon^{\gamma-1} \end{cases}.$$

with $\gamma \in (0,1)$ and $s > 2$, we need to solve the corresponding scattering problem.

Let $\tilde{\theta} = \pi - 2\tilde{\phi_0}$ be the scattering angle. We set $A := \varepsilon^{\gamma-1}$, it results that

$$\tilde{\phi_0}(\rho) = \arcsin\left(\frac{\rho}{A}\right) + \int_{r_*}^{A} \frac{\rho\, dr}{r^2\sqrt{1 - 2\phi_{eff}(r)}} \tag{A.2}$$

and

$$\frac{d\tilde{\phi_0}}{d\rho} = \frac{1}{\sqrt{1-\rho^2}}\left(1 - \frac{1}{1 - \frac{\phi'(A^-)}{\rho^2}}\right) + \int_{\arcsin\left(\frac{\rho}{A}\right)}^{\pi/2} d\varphi \frac{\sin\varphi}{\left(y - \frac{\rho}{y^2}\phi'(\frac{\rho}{y})\right)^3}\left[\frac{\rho}{y^2}\phi''\left(\frac{\rho}{y}\right) + \frac{2}{y}\phi'\left(\frac{\rho}{y}\right) + \frac{\rho}{y^4}\left(\phi'\left(\frac{\rho}{y}\right)\right)^2\right], \tag{A.3}$$

for $0 < \rho < A$. Since ϕ has a discontinuity of the first derivative in $|x| = A$, here $\phi'(A^-)$ indicates the limit of the derivative as $|x| \to A$ from below. See [PSS], Appendix, for further details. We are now interested in computing the scattering angle θ for a particle moving in a cutoffed inverse power law potential $\phi(r)$, given by (5.4.1), in presence of a uniform magnetic field B. For the sake of simplicity, it is more convenient to express the scattering angle θ in terms of the conserved momentum M, namely $\theta = \theta(M)$. We notice that, as soon as the light particle hits the obstacle, $M = \rho + \varepsilon\frac{B}{2}$. Hence it follows that

$$\theta(M) = \arcsin\left(\frac{M}{A} - \frac{\varepsilon B}{2}A\right) + \int_{r_*}^{A} \frac{(M - \varepsilon\frac{B}{2}r^2)\,dr}{r^2\sqrt{1-2\phi_{eff}(r)}} \tag{A.4}$$

where

$$\phi_{eff}(r) = \phi(r) + \frac{1}{2}\left(\frac{M}{r} - \frac{\varepsilon}{2}Br\right)^2 = \phi(r) + \frac{M^2}{2r^2} + \frac{\varepsilon^2}{8}B^2r^2 - \frac{\varepsilon}{2}BM,$$

and r_* is defined as the solution of the following

$$2\phi_{eff}(r_*) = 1.$$

Let $u = \frac{M}{r}$, $du = -\frac{M}{r^2}\,dr$, than

$$\begin{aligned}\theta(M) = \arcsin&\left(\frac{M}{A} - \frac{\varepsilon B}{2}A\right) + \\ &+ \int_{\frac{M}{A}}^{\frac{M}{r_*}} \frac{du}{\sqrt{1-2\phi_{eff}(\frac{M}{u})}} + \int_{\frac{M}{A}}^{\frac{M}{r_*}} \frac{\varepsilon\frac{B}{2}(\frac{M}{u^2})\,du}{\sqrt{1-2\phi_{eff}(\frac{M}{u})}}.\end{aligned} \tag{A.5}$$

To validate the previous computation we observe that

$$\phi_{eff}\left(\frac{M}{u}\right) = \phi\left(\frac{M}{u}\right) + \frac{u^2}{2} + \frac{\varepsilon^2}{8}B^2\left(\frac{M}{u}\right)^2 - \frac{\varepsilon}{2}BM, \tag{A.6}$$

$$\phi'_{eff}\left(\frac{M}{u}\right) = -\phi'\left(\frac{M}{u}\right)\frac{M}{u^2} + u - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right), \tag{A.7}$$

hence $\phi_{eff}(\frac{M}{u})$ is non decreasing for $u \in \left[\frac{M}{A}, \frac{M}{r_*}\right]$. In fact, since

$$\phi'_{eff}(A) = -\phi'(A)\left(\frac{A^2}{M}\right) + \frac{M}{A} - \frac{\varepsilon^2}{4}\left(\frac{B^2A^3}{M}\right), \tag{A.8}$$

it follows that $\phi'_{eff}(A) \to 0$ for $A \to +\infty$ as soon as $\gamma \in (1/2, 1)$.

We change variables again and we set

$$2\phi_{eff}\left(\frac{M}{u}\right) = \sin^2\varphi,$$

then

$$du = \frac{\sin\varphi\cos\varphi\,d\varphi}{\left(u - \phi'\left(\frac{M}{u}\right)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right)\right)}.$$

Substituting this expression into (A.5) we get

$$\theta(M) = \arcsin\left(\frac{M}{A} - \frac{\varepsilon B}{2}A\right) + \int_{\arcsin\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)}^{\frac{\pi}{2}} \frac{\sin\varphi\, d\varphi}{\left(u - \phi'\left(\frac{M}{u}\right)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right)\right)}$$
$$+ \int_{\arcsin\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)}^{\frac{\pi}{2}} \frac{\varepsilon}{2}\frac{BM}{u^2} \frac{\sin\varphi\, d\varphi}{\left(u - \phi'\left(\frac{M}{u}\right)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right)\right)}.$$

Thanks to a straightforward computation we obtain

$$\begin{aligned}
\frac{d\theta}{dM} =& \frac{1}{A\sqrt{1-\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)^2}} \\
&+ \int_{\arcsin\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)}^{\frac{\pi}{2}} \frac{\sin\varphi}{(u-\phi'(M/u)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right))^3} \\
&\times \left(2\phi'(M/u)\frac{1}{u} + \phi''(M/u)\frac{M}{u^2} + \phi'(M/u)^2\frac{M}{u^4}\right) d\varphi \\
&+ \int_{\arcsin\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)}^{\frac{\pi}{2}} \frac{\sin\varphi}{(u-\phi'(M/u)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right))^3} \\
&\times \left[\phi'(M/u)\frac{\varepsilon^2B^2M^2}{u^5} - \phi''(M/u)\frac{\varepsilon BM^2}{2u^4} + \frac{\varepsilon^4B^4}{16}\frac{M^3}{u^6} - \frac{3M^2\varepsilon^3B^3}{8u^4} - \frac{\varepsilon B}{2}\right] d\varphi \\
&+ \int_{\arcsin\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)}^{\frac{\pi}{2}} \frac{\left(\varepsilon\frac{B}{2}\frac{M}{u^2}\right)\sin\varphi}{(u-\phi'(M/u)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right))^3} \\
&\times \left[\phi'(M/u)\frac{1}{u}\left(2+\frac{\varepsilon^2B^2M^2}{u^4}\right) + \phi''(M/u)\frac{M}{u^2} + \phi'(M/u)^2\frac{M}{u^4}\right. \\
&\left. - \phi''(M/u)\frac{\varepsilon BM^2}{2u^4} + \left(\frac{\varepsilon^2B^2}{4}\right)^2\frac{M^3}{u^6} - \frac{3M^2\varepsilon^3B^3}{8u^4} - \frac{\varepsilon B}{2}\right] \\
&+ \frac{\left(\varepsilon\frac{B}{2}\left(\frac{M}{u^2}\right)\right)\sin\varphi}{(u-\phi'(M/u)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right))^2} \\
&\times \left(\frac{1}{u^2} + \frac{2M}{u^3}\frac{\phi'(M/u)\frac{1}{u} + \frac{\varepsilon^2B^2M}{4u^2} - \frac{\varepsilon B}{2}}{u-\phi'(M/u)\frac{M}{u^2} - \frac{\varepsilon^2}{4}\left(\frac{B^2M^2}{u^3}\right)}\right) d\varphi \\
&- \left(1+\frac{\varepsilon B}{2}\frac{A^2}{M}\right)\frac{\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)}{M\sqrt{1-\left(\frac{M}{A}-\frac{\varepsilon B}{2}A\right)^2}}\frac{1}{\left(1-\phi'(A)\frac{A^3}{M^2} - \frac{\varepsilon^2B^2A^4}{4M^2}\right)}.
\end{aligned} \tag{A.9}$$

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Chapter 6

Appendix

6.1 On the solution of the Linear Boltzmann equation

With the linear Boltzmann equation we enter the world of nonreversible equations. It combines free transport with scattering off of a medium and it is used to model many systems, including neutronic dynamics, radiation transfer, cometary flow and dust particles. The linear Boltzmann equation reads as

$$\begin{cases} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) = \mathrm{L}f(x,v,t) \\ f(x,v,0) = f_0(x,v) \end{cases} . \tag{A.1}$$

with $x \in \mathbb{R}^2$ or in $\mathbb{T}^2$, $v \in S^1$, $t \in \mathbb{R}^+$. Here $\mathrm{L}f = (K - I)f$ where

$$Kf(v') = \int_{S^1} dv\, k(v|v')f(v)$$

and $k(v|v')$ is a transition probability density $k : S^1 \times S^1 \to \mathbb{R}^+$ and $\int_{S^1} k(v|v')\, dv' = 1$ for all v belonging to S^1. (Here we are considering the two-dimensional case for simplicity, however all our considerations can be extended easily to the three dimensional case.)

Observe that the linear Boltzmann equation involves "exchange phenomena" in velocity, thanks to the integral kernel k. It is thus not possible to only consider v as a parameter. As a consequence of this, there is no longer an explicit solution formula based on characteristics. Characteristic lines are now replaced by stochastic functions from a Markov process. This is because the right hand side of equation (A.1) can not be interpreted as something producing characteristics. Instead, we will often regard the right hand side as a source term. We assume that the initial probability distribution f_0 is a bounded, continuous function expressing the initial distribution of our test particle. Hence we can rewrite the equation as

$$f(x,v,t) = f_0(x - vt, v) + \int_0^t (\mathrm{L}f)(x - v(t-s), v, s)\, ds$$

where L denotes the linear operator on the right hand side of the equation. This is just Duhamel's principle, and can be checked by differentiation. Note that the operator $\partial_t + v \cdot \nabla_x$ applied to the first term and the term inside the integral gives zero, so the only nonzero term is ∂_t applied to the integral, which gives Lf, the right hand side of the equation, as desired. Observe that this yields a nice integral equation for f, but not an explicit formula.

Remark 6.1.1. *Recall that Duhamel's principle is based on the idea of solving a linear PDE with a source term and initial data, i.e. a linear inhomogeneous PDE like*

$$\begin{cases} \partial_t f + \underbrace{Tf}_{lin.op} = \underbrace{U}_{source} \\ f|_{t=0} = f_0 \end{cases} .$$

by solving separately the PDE without the source term but with the initial data, and the PDE with the source term but zero initial data. Then, by a linear combination, we obtain the complete solution. Applying this to the linear Boltzmann equation, we can choose $T = v \cdot \nabla_x$ and $Uf = Lf = (K - I)f$. In this case, we can easily write

$$e^{-tT} g(x, v) = g(x - vt, v)$$

so we have

$$f(x, v, t) = f_0(x - vt, v) + \int_0^t (Lf)(x - v(t - s), v, s)\, ds.$$

Alternatively, we could let $T = v \cdot \nabla_x + I$. Hence

$$e^{-tT} g(x, v) = g(x - vt, v)e^{-t}$$

and we obtain

$$f(x, v, t) = f_0(x - vt, v)e^{-t} + \int_0^t e^{-(t-s)}(Kf)(x - v(t - s), v, s)\, ds. \quad \text{(A.2)}$$

The process described by (A.1) is that of a particle with velocity of modulus one, having random transitions (collisions) which preserve the energy. We introduce a physical mechanism for such transitions. Consider a particle, with initial velocity $v \in S^1$, hitting a circular obstacle of unit diameter, whose center c is random. If we denote by ρ the impact parameter and n the outward normal in the collision point, we can compute the outgoing velocity v' (in terms of energy and angular momentum conservation) finding

$$v' = v - 2(v \cdot n)n.$$

We want to associate a probability $k(v|v')\, dv'$ related to the transition $v \to v'$. A reasonable choice is to assume ρ as a random variable uniformly distributed in $[-1/2, 1/2]$, so that the probability to have the transition $v \to v'$ is $d\rho = \frac{d\rho}{dv'} dv'$. However, instead of computing $k(v|v')$ explicitly, we find more convenient to express the operator K, by means of an integration with respect to the angle α. Using

$$d\rho = \left| \frac{d\rho}{d\alpha} \right| d\alpha = |n \cdot v| \, dn$$

we get

$$\begin{aligned} Kf(v') &= \int_{S^1} dn \, |n \cdot v| \, f(v) \chi(n \cdot v \leq 0) \\ &= \int_{S^1} dn \, |n \cdot v| \, f(v) \chi(n \cdot v') \geq 0, \end{aligned}$$

Note that what distinguishes if v' is incoming or outgoing is the scalar product $n \cdot v$ which is positive or negative if v is outgoing or incoming, respectively. If we consider instead the collision of the particle by an obstacle generating a smooth potential ϕ we obtain different transition probabilities. In this case we have an analogous expression for the operator K, i.e.

$$Kf(v') = \int_{S^1} dn \, B(n, |v|) \, f(v),$$

with $B(n, |v|)$ proportional to the cross section associated to the potential ϕ, namely $\left| \frac{d\rho}{d\alpha} \right|$. Moreover in this case we have that $v' = v - 2(\omega \cdot v)\omega$, where ω is the unit vector bisecting the angle between the incoming and the outgoing velocity.

It is obvious that the Cauchy problem associated to the Linear Boltzmann equation has a unique global solution in any reasonable space. Indeed, we can give the explicit solution. We consider (A.1) in the integral form (A.2). If we set $S(t) f_0(x, v) := f_0(x - vt, v) e^{-t}$, from (A.2) we get

$$f(x, v, t) = S(t) f_0(x, v) + \int_0^t S(t - s)(Kf)(x, v, s) \, ds.$$

By iterating we find the following formal series expansion

$$\begin{aligned} f(x, v, t) = &S(t) f_0(x, v) \\ &+ \sum_{m>0} \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \ldots \int_0^{t_m - 1} dt_m \\ &S(t - t - 1) K S(t_1 - t - 2) \ldots K S(t_m) f_0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} f(x,v,t) = & e^{-t} f_0(x - vt, v) \\ & + e^{-t} \sum_{m>0} \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \ldots \int_0^{t_m-1} dt_m \int dv_1 \ldots \int dv_m \\ & k(v|v_1)k(v_1|v_2)\ldots k(v_{m-1}|v_m) \\ & f_0(x - v(t-t_1) - v_1(t_1 - t_2) - \ldots v_m t_m, v_m). \end{aligned} \tag{A.3}$$

Hence, if $f_0 \in L^\infty(\mathbb{R}^2 \times S^1)$ the series converges uniformly in $\mathbb{R}^2 \times S^1 \times [0,T]$, for any $T > 0$. We are interested in the physical interpretation of (A.3). The

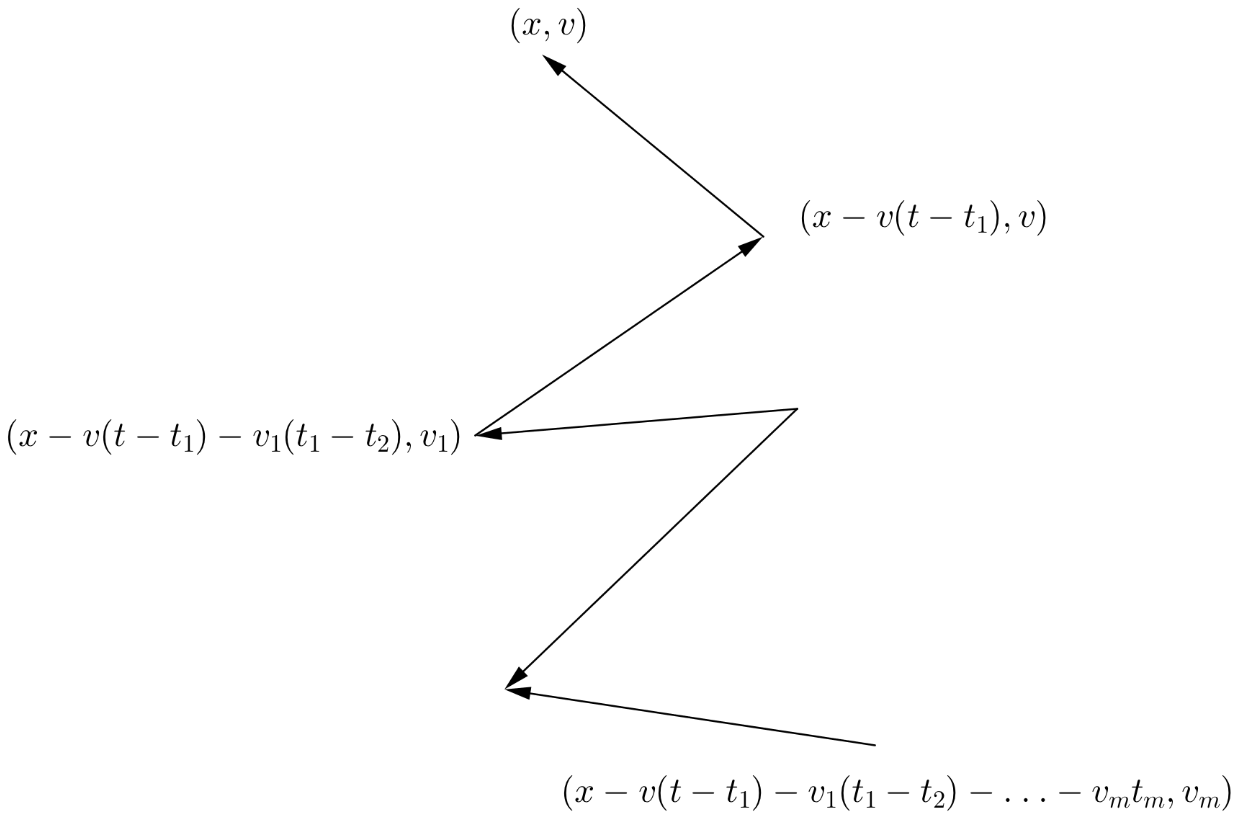

Figure 6.1: Markov jump process

value of f in (x, v) at time t is a sum of infinitely many contributions. The term $e^{-t} f_0(x - vt, v)$ is the value of the initial datum in the backward trajectory (the weight e^{-t} takes into account the mass loss) and is interpreted as the zero collision contribution. The first term of the series in the right hand side of (A.3), i.e.

$$e^{-t}\sum_{m>0}\int_0^{t_1} dt_1 \int dv_1\, k(v|v_1)\, f_0(x - v(t-t_1) - v_1 t_1, v_1),$$

is the contribution to the probability density to be in (x, v) at time t after a collision and is called the one-collision term. The generic m-collision term in (A.3) can be interpreted analogously. Fixed (x, v) we consider the backward free trajectory in the time interval $(t - t_1)$. At this time, being the unit vector n_1 fixed, the particle collides having the transition $v \to v_1 = v - 2n_1(n_1 \cdot v)$. Now the particle moves with the new velocity v_1 in the time interval $(t_1 - t_2)$ and a new collision happens with the same rules and so on. (See Figure 6.1.) We compute now the value of the initial datum at the point $(x - v(t-t_1) - v_1(t_1 - t_2) - \dots v_m t_m, v_m)$. Hence we integrate with respect to the times $t_1, \dots, t_m$ and to the velocities $v_1, \dots, v_n$. This is the contribution to the probability density at time t, due to the fact that the particle performed m collisions in this time.

Remark. A straightforward consequence of (A.3) is that the solution propagates with velocity which can be at maximum 1. This means that if f_0 is supported, with respect to the x variable, in $B(0, R)$ i.e. the disk of radius $R > 0$ and centre 0, it follows that $f(\cdot, t)$ is supported in $B(0, R+t)$. Indeed, it is enough to prove that

$$|v(t-t_1) + v_1(t_1 - t_2) + \cdots + v_m t_m| \le 1$$

for $v_1, \dots, v_m \in S^1$ and $0 \le t_m \le \cdots \le t_1 \le t$.

On the other hand

$$\|f(\cdot, t)\|_\infty \le e^{-t}\left(1 + \sum_{m>0}\frac{t^m}{m!}\right)\|f_0\|_\infty \le \|f_0\|_\infty.$$

Therefore the maximum principle holds.

6.2 On the solution of the linear Landau equation

We consider here the linear Landau equation

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = B\Delta_{|v|} f(x, v, t), \tag{A.1}$$

where $\Delta_{|v|}$ is the Laplace-Beltrami operator on the d-dimensional sphere of radius $|v|$. Due to its linearity, equation (A.1) can be solved explicitly. The direct method to solve this equation is reminiscent of the analogous one valid for the linear diffusion equation, and consists in resorting to Fourier transform. Let $f_0(x, v)$ be an initial distribution function, which we assume to be integrable in $\mathbb{R}^d \times \mathbb{R}^d$. Let us consider the Fourier transform of the distribution f, with respect to both x and v

$$\hat{f}(\eta, \xi, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, v, t) e^{-i(\eta \cdot x + \xi \cdot v)} dx dv. \tag{A.2}$$

Hence Equation (A.1) becomes

$$(\partial_t - \eta \cdot \nabla_\xi) \hat{f}(\eta, \xi, t) + B|\xi|^2 \hat{f}(\eta, \xi, t) = 0. \tag{A.3}$$

It can be easily verified that the solution of equation (A.3) takes the form

$$\hat{f}(\eta, \xi, t) = \hat{f}_0(\eta, \xi + \eta t) e^{-B\left(|\xi|^2 t - \xi \cdot \eta t^2 - \frac{|\eta|^2 t^3}{3}\right)}. \tag{A.4}$$

More precisely, to obtain (A.4), we need to introduce the characteristic differential system associated to

$$(\partial_t - \eta \cdot \nabla_\xi) \hat{f}(\eta, \xi, t) = 0$$

which reads as

$$\begin{cases} \dot{\eta} = 0 \\ \dot{\xi} = -\eta \end{cases} . \tag{A.5}$$

Let $T^t(\eta, \xi) = (T_1^t(\eta, \xi), T_2^t(\eta, \xi)) = (\eta, \xi - \eta t)$ be the solution of (A.5). Hence, since the solution of (A.3) is given by

$$\hat{f}(\eta, \xi, t) = \hat{f}_0(T^{-t}(\eta, \xi)) e^{\int_0^t |T_2^{s-t}(\eta, \xi)|^2 \, ds},$$

by using the explicit expression for $T^t(\eta, \xi)$ we get (A.4).

Now, by applying the inverse Fourier transform $\mathcal{F}^{-1}$, we obtain

$$f(x, v, t) = \mathcal{F}^{-1}\left(\hat{f}_0(\eta, \xi + \eta t)\right) \star \mathcal{F}^{-1}\left(e^{-B\left(|\xi|^2 t - \xi \cdot \eta t^2 - \frac{|\eta|^2 t^3}{3}\right)}\right). \tag{A.6}$$

Hence

$$f(x, v, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x, v, t; x', v') f_0(x', v'). \tag{A.7}$$

where G indicates the Green function corresponding to initial datum

$$f_0(x, v) = \delta(x - x')\delta(v - v').$$

Observe that

$$G(s, v, t) = \mathcal{F}^{-1}\left(e^{-B\left(|\xi|^2 t - \xi \cdot \eta t^2 - \frac{|\eta|^2 t^3}{3}\right)}\right).$$

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