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On the structure of Borel stable abelian subalgebras in $\mathbb{Z}_2$-graded Lie algebras

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Introduction

Let $\mathfrak{g}$ be a complex, finite dimensional, semisimple Lie algebra and let σ be an indecomposable involution of $\mathfrak{g}$, with corresponding eigenspace decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Since $\mathfrak{g}_0$ is reductive, we can fix a Borel subalgebra $\mathfrak{b}_0$. The main problem of this thesis is the following: *describe the maximal abelian $\mathfrak{b}_0$-stable subalgebra of $\mathfrak{g}_1$.*

The origins of this kind of problem go back to a paper of Malcev [15], published in 1945, where the author found the maximal dimension of abelian subalgebras of any simple Lie algebras by explicit case-by-case calculations. In 1965 Kostant gave new motivations to the subject: in fact in [11] he showed a deep connection between the eigenvalues of a Casimir operator if $\mathfrak{g}$ and commutative subalgebras. But only after 35 years from Kostant's paper Peterson [12] introduced a striking and powerful tool, a bijection between abelian ideals of a Borel subalgebra and a certain subset of the affine Weyl group associated to $\mathfrak{g}$. At this point all the main ingredients of the theory were present, and after few years Panyushev [17] and Suter [20] Theorems gave a complete description of the structure of maximal abelian Borel stable subalgebras.

In this work we deal with a generalization of this problem to the $\mathbb{Z}_2$-graded setting introduced by Panyushev in [16], together with an extension of Kostant's theorems: in fact he found a relation between the eigenvalues of a Casimir operator of $\mathfrak{g}_0$ and abelian $\mathfrak{b}_0$-stable subalgebras fo $\mathfrak{g}_1$. The combinatorial translation of the problem is still valid: thanks to Cellini, Möseneder Frajria and Papi [1] we have a bijection between these special subalgebras and the set of the so called σ-minuscule elements, a subset of the affine Weyl group associated to the pair $(\mathfrak{g}, \sigma)$.

Our aim is a generalization of Panyushev's and Suter's Theorems to this setting. The thesis is structured as follows:

Chapter 1: In the first part we state and describe more precisely all the theorems previously cited. In particular we give a complete proof of the bijection between σ-minuscule element and abelian $\mathfrak{b}_0$-stable subalgebras.
The second part is a collection of useful (and well known) combinatorial results on Weyl groups.

Chapter 2: This is the core of the work. The study of maximal σ-minuscule elements is done according to the following steps:

1. We find a necessary condition for the maximality of an element. This let us reduce to the study of only some subset $\mathcal{I}_{\alpha,\mu}$, indexed by simple roots α of the affine Weyl group associated to $(\mathfrak{g}, \sigma)$, and by elements μ of a special subset $\mathcal{M}_\sigma$ of the walls of a polytope.

2. We provide a criterion for $\mathcal{I}_{\alpha,\mu}$ to be non empty. Moreover we show that $\mathcal{I}_{\alpha,\mu}$, if non empty, has minimum (Theorem 2.2.11).
3. We determine the poset structure of $\mathcal{I}_{\alpha,\mu}$, by relating it to a quotient of the subgroup $\widehat{W}_\alpha$ of $\widehat{W}$ generated by the simple reflections orthogonal to the simple root α by a reflection subgroup $\widehat{W}'_\alpha$ (Theorem 2.3.6).
4. We look at intersections among the posets $\mathcal{I}_{\alpha,\mu}$, and we find necessary and sufficient conditions in order that the intersection of two such posets is nonvoid.
5. We study maximal elements in $\mathcal{I}_{\alpha,\mu}$. We show that when $\widehat{W}'_\alpha$ is not standard parabolic, maximal elements appear in pairs of $\mathcal{I}_{\alpha,\mu}$'s: if w is maximal in $\mathcal{I}_{\alpha,\mu}$, then there exist a unique simple root β and a unique wall $\mu' \in \mathcal{M}_\sigma$ such that w is also maximal in $\mathcal{I}_{\beta,\mu'}$ (Lemma 2.5.4).
6. We determine which maximal elements in $\mathcal{I}_{\alpha,\mu}$ are indeed maximal in the set of the σ-minuscule elements (Propositions 2.5.1, 2.5.2).
7. We finally provide a complete parametrization of maximal abelian $\mathfrak{b}_0$-stable subalgebras (Theorem 2.5.3) and uniform formulas for their dimension (Corollary 2.5.6). Our results specialize nicely to Panyushev's and Suter's Theorems quoted above.

Chapter 1

Preliminaries

1.1 Eigenvalues of the Casimir operator and abelian ideals of Borel subalgebras

1.1.1 The Casimir operator

Let $\mathfrak{g}$ be a complex, finite-dimensional, semisimple Lie algebra with Killing form $\langle \cdot, \cdot \rangle$. Choose a basis $\{x_1, \ldots, x_N\}$ of $\mathfrak{g}$, and let $\{x'_1, \ldots, x'_N\}$ be the associated dual basis relative to the Killing form, i.e. $\langle x_i, x'_j \rangle = \delta_{ij}$.
We define the *Casimir operator* $\mathcal{C}$ in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ as

$$\mathcal{C} = \sum_{i,j=1}^{n} x_i \cdot x'_j.$$

Remark 1.1.1. A straightforward calculation shows that the definition of $\mathcal{C}$ doesn't depend from the choice of the basis.

The Casimir operator acts on $\mathfrak{g}$ through the adjoint representation $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$. In particular, since we can extend ad to the exterior algebra $\bigwedge \mathfrak{g}$ setting

$$\mathrm{ad}_x(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \ldots \wedge [x, v_i] \ldots \wedge v_k \quad \forall x, v_1, \ldots, v_k \in \mathfrak{g},$$

we obtain an action $\mathrm{ad}(\mathcal{C})$ of $\mathcal{C}$ on $\bigwedge \mathfrak{g}$

$$\mathrm{ad}(\mathcal{C})(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{n} \mathrm{ad}_{x_i}(\mathrm{ad}_{x'_i}(v_1 \wedge \ldots \wedge v_k)) \quad \forall v_1, \ldots, v_k \in \mathfrak{g}.$$

Remark 1.1.2. It is well known that $\mathcal{C}$ acts as a scalar operator on every irreducible representation. More precisely, fix a Cartan subalgebra $\mathfrak{h}$ and choose a Borel subalgebra of $\mathfrak{g}$ with corresponding positive root system Δ^+: if $\pi : \mathfrak{g} \to \mathrm{End}(V_\lambda)$ is an irreducible representation on a vector space V_λ with highest weight $\lambda \in \mathfrak{h}^*$, then

$$\pi(\mathcal{C}) = \langle \lambda, \lambda + 2\rho \rangle \mathrm{id}_{V_\lambda}$$

where $\rho = \frac{1}{2} \sum_{\gamma \in \Delta^+} \gamma$.

One reason motivating the study of eigenvalues of the Casimir operator on the exterior algebra of $\mathfrak{g}$ is given by the following interpretation of $\mathcal{C}$ in terms of differential operators on $\bigwedge \mathfrak{g}$. First define a coboundary operator d on $\bigwedge \mathfrak{g}$ as

$$d = \frac{1}{2} \sum_{i=1}^{n} \epsilon(x_i') \operatorname{ad}_{x_i} .$$

Herre ϵ denotes the (left) exterior product

$$\epsilon(x)(v_1 \wedge \ldots \wedge v_k) = x \wedge v_1 \wedge \ldots \wedge v_k.$$

Next let $\mathfrak{g}_c$ be a compact form of $\mathfrak{g}$: we can define a conjugation in $\mathfrak{g}$ relative to $\mathfrak{g}_c$ setting $x^* = -u + iv$ whenever $x = u + iv$ with $u, v \in \mathfrak{g}_c$. This let us introduce a hermitian form $\{\cdot, \cdot\}$ on $\mathfrak{g}$, setting $\{x, y\} = \langle x, y^* \rangle$ for every $x, y \in \mathfrak{g}$. Moreover we can extend it to $\bigwedge^k \mathfrak{g}$ via determinants, taking

$$\{v_1 \wedge \ldots \wedge v_k, u_1 \wedge \ldots \wedge u_k\} = \det |\{v_i, u_j\}| \quad \forall v_i, u_j \in \mathfrak{g}$$

and $\{u, v\} = 0$ if $u \in \bigwedge^i \mathfrak{g}_1$ and $v \in \bigwedge^j \mathfrak{g}_1$ with $i \neq j$. From this we obtain the formal adjoint operator ∂ of d: in particular for all $v_1 \wedge \ldots \wedge v_k \in \bigwedge \mathfrak{g}$ we have

$$\partial(v_1 \wedge \ldots \wedge v_p) = \sum_{i<j} (-1)^{i+j+1} [v_i, v_j] \wedge v_1 \ldots \wedge \widehat{v_i} \ldots \wedge \widehat{v_j} \ldots \wedge v_p.$$

which is the Chevalley boundary operator for the Lie algebra homology with trivial coefficients.
Finally we define the *Laplace operator*

$$L = d\partial + \partial d.$$

The statement that justifies the interest arose around the eigenvalues of $\mathcal{C}_{\text{ad}}$ is the following:

Proposition 1.1.1. $L = \frac{1}{2} \operatorname{ad}(\mathcal{C})$

Proof. We recall the following well-known relations (see [13]):

1. $[d, \operatorname{ad}_x] = [\partial, \operatorname{ad}_x] = 0$;
2. $\epsilon(x)\partial + \partial\epsilon(x) = \operatorname{ad}_x$.

Let v be in $\bigwedge \mathfrak{g}$: using the previous identities and the definition of d we obtain

$$\begin{aligned} d\partial(v) &= \frac{1}{2} \sum_{i=1}^{n} \epsilon(x_i) \operatorname{ad}_{x_i'}(\partial(v)) \\ &= \frac{1}{2} \sum_{i=1}^{n} \epsilon(x_i) \partial(\operatorname{ad}_{x_i'}(v)) \\ &= -\partial d(v) + \frac{1}{2} \sum_{i=1}^{n} \operatorname{ad}_{x_i}(\operatorname{ad}_{x_i'}(v)) \end{aligned}$$

and hence the thesis. □

In the next section we will present the first key result of this theory, which is due to Kostant [11].

1.1.2 The link with abelian ideals of Borel subalgebras

From now on we will improperly write $\mathcal{C}$ instead of $\mathrm{ad}(\mathcal{C})$.
Let A be the set of all the commutative subalgebras of $\mathfrak{g}$: for every $\mathfrak{a} \in A$ of dimension k we define the one-dimensional subspace $[\mathfrak{a}] = \bigwedge^k \mathfrak{a}$ of $\bigwedge^k \mathfrak{g}$. Finally let A_k be the subspace generated by all these elements with fixed dimension k.

Theorem 1.1.2 (Kostant [11])**.** *If m_k is the maximal eigenvalue of $\mathcal{C}$ on $\bigwedge^k \mathfrak{g}$, then*

$$m_k \leq k.$$

Moreover the equality holds if and only if *there is a commutative subalgebra of* $\mathfrak{g}$ *of dimension k. In this case the eigenspace associated to m_k is A_k, and every decomposable element of A_k corresponds to a commutative subalgebra of* $\mathfrak{g}$.

This is one of the results that motivate the study of commutative subalgebras of semisimple Lie algebras. In particular Kostant's Theorem gave new motivations to the problem of finding the maximal dimension of these subalgebras.

Continuing to follow [11], we can restrict our study to a smaller class of subspaces. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with associated root system Δ and Weyl group W. Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that Δ^+ is the set of positive roots. Let V be a $\mathfrak{g}$-module: we have

Proposition 1.1.3 (Kostant)**.**

1. *Suppose that $W \subset \bigwedge^k V$ is irreducible. Then W is generated by decomposable vectors if and only if it has a decomposable highest weight vector.*
2. *Let M_k be the eigenspace corresponding to the maximal eigenvalue of $\mathcal{C}$ on $\bigwedge^k V$. Then M_k is a sum of irreducible $\mathfrak{g}$-submodules generated by decomposable vectors.*

This proposition, together with Theorem 1.1.2, implies that, in order to determine the maximal eigenvalue of $\mathcal{C}$ on an exterior power of $\mathfrak{g}$, it suffices to consider the action of $\mathcal{C}$ only on decomposable elements. Furthermore observe that if $v \in \bigwedge^k \mathfrak{g}$ is a decomposable highest weight vector for a submodule of M_k, then the associated k-subalgebra (via Theorem 1.1.2) must be $\mathfrak{b}$-stable.

In conclusion the main problem becomes: *describe, enumerate and find the maximal dimension of the abelian ideals of a Borel subalgebra.*

1.1.3 The connection with the affine Weyl group

After some decades, Konstant returned on the study of abelian subalgebras, presenting in [12] a surprising result of Peterson.

Theorem 1.1.4 (Peterson)**.** *The cardinality of the set of abelian ideals of a Borel subalgebra is 2^n, where $n = \mathrm{rank}(\mathfrak{g})$.*

To prove this theorem Peterson sets up a one-to-one correspondence between the set I_{ab} of all abelian ideals of $\mathfrak{b}$ and a particular subset of the affine Weyl group $\widehat{W}$. To define this subset, let $\widehat{\Delta}$ be the set of affine roots and let $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_{\mathfrak{n}}\}$ be the set of affine simple roots. Denote by $a_0, \dots, a_n$ the labels of the affine Dynkin diagram of $\mathfrak{g}$ and let $\delta = \sum_{i=0}^{n} a_i \alpha_i$ denote the fundamental immaginary root. Moreover let $\widehat{\Delta}^+$ be the set of positive affine roots corresponding to the choice of $\mathfrak{b}$. For any $w \in \widehat{W}$ define the *inversion set*

$$N(w) = \{\gamma \in \widehat{\Delta}^+ \mid w^{-1}(\gamma) \in -\widehat{\Delta}^+\}.$$

Now an element $w \in \widehat{W}$ is said to be *minuscule* if $N(w)$ is of the form $\{\delta - \gamma \mid \gamma \in S\}$, where $S \subset \Delta$. Peterson found the following result:

Proposition 1.1.5. *There exists a one-to-one correspondence between the set of minuscule elements of $\widehat{W}$ and I_{ab}.*

Proof. We sketch the proof following the approach of Cellini and Papi in [4].
Let $\mathfrak{i}$ be an abelian ideal of $\mathfrak{b}$. Write $\mathfrak{i} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and define the set

$$L_{\mathfrak{i}} = \bigcup_{k\geq 1} (-\Phi^k + k\delta),$$

where $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta$. Since $\mathfrak{i}$ is abelian, clearly $\Phi^k = 0$ for every $k \geq 2$. Knowing this, it is easy to verify that $L_{\mathfrak{i}}$ is closed. Furthermore $\widehat{\Delta}^+ \setminus L_{\mathfrak{i}}$ must also be closed: if not, we could find $\alpha, \beta \in \Delta^+ \setminus \Phi$ such that $\alpha - \beta \in \Phi$, clearly a contradiction since $\mathfrak{i}$ is an ideal. This implies, by Proposition 1.3.1(3), that there exists a $w \in \widehat{W}$ such that $N(w) = L_{\mathfrak{i}}$. Thus we have established a bijection between abelian ideals of $\mathfrak{b}$ and the set of minuscule elements. □

Remark 1.1.1. It is possible to prove that $L_{\mathfrak{i}}$ is *biconvex* for all the *ad*-nilpotent ideals $\mathfrak{i}$ of $\mathfrak{b}$, i.e. ideals included in the nilpotent radical of $\mathfrak{b}$. See [2], [3] for further details.

This proposition gives us a new interpretation of the main problem, that is *describe and find the maximal lenght of the minuscule elements of $\widehat{W}$.*

Peterson's approach was the point of departure for a serie of papers regarding abelian ideals and related problems in combinatorics and representation theory. For example:

- in [18], Panyushev and Röhrle, while studying the relationship between spherical nilpotent orbits and abelian ideals of b, observed a bijection between maximal abelian ideals and the set of long simple roots of $\widehat{\mathfrak{g}}$. In [17] Panyushev gave a conceptual explanation of that empirical observation;

- in [20] Suter found a uniform formula for the dimension of maximal abelian ideals in terms of combinatorial invariants related to the associated long simple root: this gave a conceptual explanation of an old resilt of Malcev [15], which was a Lie algebra generalization of a classical result of Schur [19] on the maximal number of linearly independent commuting matrices;

- in [4] Cellini and Papi found independently the same formula of Suter and obtained a subtler description of the poset structure of the set of minuscule elements.

In the next sections we will introduce the main topic of this thesis, that is a generalization to $\mathbb{Z}_2$-graded Lie algebras of the theory previously developed.

1.2 Generalization to $\mathbb{Z}_2$-graded Lie algebras

Let σ be an indecomposable involution acting on a semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding $\mathbb{Z}_2$-gradation, i.e. $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid \sigma(x) = e^{ik\pi}x\}$. We recall that

Proposition 1.2.1. *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-graded semisimple Lie algebra then $\mathfrak{g}_0$ is a reductive Lie subalgebra.*

In view of this we can fix a Cartan subalgebra $\mathfrak{h}_0$ and a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$. Moreover let $\{x_1, \ldots, x_N\}$ be a basis of $\mathfrak{g}$ compatible with the $\mathbb{Z}_2$-grading, that is, a basis consisting of eigenvector for the action of σ. We have the following decompositions:

$$d = d_0 + d_1, \qquad d_i = \frac{1}{2} \sum_{j : x_j \in \mathfrak{g}_i} \epsilon(x_j)\, \mathrm{ad}_{x_j}$$

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1, \qquad \mathcal{C}_i = \sum_{j : x_j \in \mathfrak{g}_i} x_j \cdot x_j'$$

Remark 1.2.1. Observe that $\mathcal{C}_0$ is the Casimir operator of $\mathfrak{g}_0$ with respect to the form $\langle \cdot, \cdot \rangle_{|\mathfrak{g}_0}$. In particular by Theorem 1.1.1 we have $\mathrm{ad}(\mathcal{C}_i) = 2(d_i\partial + \partial d_i)$.

In [16] Panyushev found a generalization of Theorem 1.1.2 to this setting.

Theorem 1.2.2 (Panyushev)**.** *If l_k is the maximal eigenvalue of $\mathcal{C}_0$ on $\bigwedge^k \mathfrak{g}_1$ then*

$$l_k \leq \frac{k}{2}.$$

Moreover the equality holds if and only if *$\mathfrak{g}_1$ contains a k-dimensional commutative subalgebra. In this case the eigenspace associated to l_k is generated by $\bigwedge^k \mathfrak{a}$ where $\mathfrak{a}$ runs through all k-dimensional commutative subalgebras of $\mathfrak{g}_1$.*

Proof. Let $v = v_1 \wedge \ldots \wedge v_k$ be a decomposable element of $\bigwedge^k \mathfrak{g}_1$: we can assume that $v_1, \ldots, v_k$ are orthonormal vectors in $\mathfrak{g}_1$ with respect to the hermitian form $\{\cdot, \cdot\} = \langle \cdot, \cdot^* \rangle$. We recall [16, Proposition 4.1]:

$$d_1([y, z]) = - \sum_{j : x_j \in \mathfrak{g}_0} [x_i, y] \wedge [x_i', z].$$

Using this, it's not difficult to prove that

$$\mathcal{C}_0(v) = \frac{k}{2} v - 2 \sum_{i<j} (-1)^{i+j-1} d_1([v_i, v_j]) \wedge v_1 \ldots \hat{v}_i \ldots \hat{v}_j \ldots \wedge v_k.$$

Define u_{ij} as the generic summand in the right side of the previous equality. Since $v_1, \ldots, v_k$ are orthonormal we obtain that $\{v, v\} = 1$. Moreover

$$\{u_{ij}, v\} = \{d_1([v_i, v_j]), v_i \wedge v_j\} = \{[v_i, v_j], [v_i, v_j]\} \geq 0$$

hence

$$\{\mathcal{C}_0(v), v\} = \frac{k}{2} - 2 \sum_{i<j} \{[v_i, v_j], [v_i, v_j]\} \leq \frac{k}{2}.$$

In conclusion if v is an eigenvector for the action of $\mathcal{C}_0$ on $\bigwedge^k \mathfrak{g}_1$, then the corresponding eigenvalue is less or equal to $\frac{k}{2}$. Observe that the equality holds if and only if all $[v_i, v_j]$ vanish, i.e. the algebra generated by $\{v_1, \ldots, v_k\}$ is commutative.
By Theorem 1.1.3(2), the eigenspace of the maximal eigenvalue is generated by decomposable highest weight vectors, and this suffices to prove the thesis. □

Remark 1.2.2. It is possible to recover Theorem 1.1.2 from the previous result. Let $\widetilde{\mathfrak{g}}$ be a semisimple lie algebra, with decomposition into simple ideals $\widetilde{\mathfrak{g}} = \bigoplus_i \mathfrak{k}_i$. The indecomposability of σ implies that $\mathfrak{k}_i \simeq \mathfrak{k}_j$ and that σ permutes ciclically the factors. Since σ is an involution, up to conjugation we may assume that $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ and that $\sigma(x, y) = \sigma(y, x)$. Observe that in this case $\widetilde{\mathfrak{g}}_0$ is the diagonal, therefore $\widetilde{\mathfrak{g}}_0 \simeq \mathfrak{g}$ as algebras and $\widetilde{\mathfrak{g}}_1 \simeq \mathfrak{g}$ as $\mathfrak{g}$-modules. Let $\widetilde{\mathcal{C}}$ be the Casimir operator of $\widetilde{\mathfrak{g}}$, with σ-decomposition $\widetilde{\mathcal{C}} = \mathcal{C}_0 + \mathcal{C}_1$: observe that $2\widetilde{\mathcal{C}}_0 = \mathcal{C}$, with $\mathcal{C}$ the Casimir operator of $\mathfrak{g}$. Now, by Theorem 1.2.2, we obtain that the maximal eigenvalue of $\widetilde{\mathcal{C}}_0$ on $\bigwedge^k \widetilde{\mathfrak{g}}_1 \simeq \bigwedge^k \mathfrak{g}$ is at most $\frac{k}{2}$. Since the isomorphism $\widetilde{\mathfrak{g}}_1 \simeq \mathfrak{g}$ preserves commutative subalgebras, we have the thesis.

As in [11], we can refine Panyushev's Theorem: first observe that a decomposable element $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \mathfrak{g}_1$ is a highest weight vector *if and only if* the corresponding subspace generated by $\{v_1, \ldots, v_k\}$ is $\mathfrak{b}_0$-stable. Moreover let A_k be the subspace of $\bigwedge^k \mathfrak{g}_1$ generated by the lines $\bigwedge^k \mathfrak{a}$, where $\mathfrak{a}$ ranges over all k-dimensional commutative subalgebras of $\mathfrak{g}_1$: it is known that

Proposition 1.2.3 (Panyushev)**.** *The subspace $A = \bigoplus_k A_k$ of $\bigwedge \mathfrak{g}_1$ is generated by decomposable highest weight vectors.*

This clearly implies that in order to study the maximal eigenvalue of $\mathcal{C}_0$ on $\bigwedge^k \mathfrak{g}_1$, *it suffices to consider the set of all $\mathfrak{b}_0$-stable k-dimensional commutative subspaces of $\mathfrak{g}_1$.*

Even in this case it is possible to translate the problem in combinatorial terms as in Theorem 1.1.5. In particular in [1] Cellini, Möseneder Frajria and Papi found a bijection between the set of $\mathfrak{b}_0$-stable commutative subspaces of $\mathfrak{g}_1$ and a subset of the affine Kac-Moody Lie algebra $\widehat{L}(\mathfrak{g}, \sigma)$.
This time we will explain in detail this relation, giving a complete proof of the related theorems. To make this we need some results on affine Kac-Moody algebras and their homology.

1.2.1 Realization of affine Kac-Moody algebras

The main reference of this section will be [9, Chapter 8].
Let $\mathfrak{g}$ be of type X_N and rank n, σ an automorphism of $\mathfrak{g}$ of order m and k the least positive integer such that σ^k is inner. Denote with $\widehat{\mathfrak{g}}$ the affine Kac-Moody Lie algebra associated to a generalized Cartan matrix of type $X_N^{(k)}$. Define the Chevalley generator $\{E_i, F_i\}_i$ of $\widehat{\mathfrak{g}}$ as in [9]. We recall a theorem of Kac about automorphisms of Lie algebras.

Theorem 1.2.4 (Kac [9])**.** *1. To each $(n+1)$-tuple $\mathbf{s} = (s_0, \ldots, s_n)$ of non-negative coprime integers corresponds an automorphism $\sigma_{\mathbf{s};k}$ of $\mathfrak{g}$ of order $m = k\sum_{i=o}^{n} \alpha_i s_i$, where a_i are the labels of the diagram $X_N^{(k)}$. These automorphisms are defined (uniquely) by $\sigma_{\mathbf{s};k}(E_j) = e^{2\pi i s_j/m} E_j$, $j = 0, \ldots, n$.*

2. *Up to conjugation by an automorphism of* $\mathfrak{g}$*, the automorphisms* $\sigma_{\mathbf{s};k}$ *exhaust all automorphisms of order* m *of* $\mathfrak{g}$*.*

3. *Two automorphisms* $\sigma_{\mathbf{s};k}$ *and* $\sigma_{\mathbf{s}';k'}$ *are conjugate by an automorphism of* $\mathfrak{g}$ *if and only if* $k = k'$ *and the sequence* $\mathbf{s}$ *can be transformed in the sequence* $\mathbf{s}'$ *by an automorphism of the diagram* $X_N^{(k)}$*.*

Observe that the numbers $s_0, \ldots, s_m$ define a $\mathbb{Z}$-grading on $\widehat{\mathfrak{g}}$. In fact, fix a Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ and let $\widehat{\Delta}$ be the associated root system with simple roots $\widehat{\Pi} = \{\alpha_o, \ldots, \alpha_n\}$: for every $\gamma \in \widehat{\Delta}$ with $\gamma = \sum_{i=0}^n m_i\alpha_i$ define the *σ-height*

$$ht_\sigma(\gamma) = \sum_{i=0}^{n} s_i m_i.$$

Let $\widehat{\mathfrak{g}}_\gamma$ be root space corresponding to γ: if $x \in \widehat{\mathfrak{g}}_\gamma$ we set $deg(x) = ht_\sigma(\gamma)$; moreover for every $h \in \widehat{\mathfrak{h}}$ we set $deg(h) = 0$. We denote by $\mathfrak{g}_i$ the subspace generated by all $x \in \widehat{\mathfrak{g}}$ of degree i, and we set $\widehat{\Delta}_i = \{\gamma \in \widehat{\Delta} \mid ht_\sigma(\gamma) = i\}$.

By Theorem 1.2.4 we can assume that σ is an automorphism of type $(s_0, \ldots, s_n; k)$. In particular, since we are interessed in automorphism of order 2, we have to consider only three kind of possibilities:

1. $k = 1$ and there exist two indices p, q such that $\alpha_p = \alpha_q = s_p = s_q = 1$ and $s_i = 0$ for $i \neq p, q$;

2. $k = 1$ and there exists an index p such that $s_p = 1$,$a_p = 2$ and $s_i = 0$ for $i \neq p$;

3. $k = 2$ and there exists an index p such that $s_p = 1$, $a_p = 1$ and $s_i = 0$ for $i \neq p$.

In the rest of the work we will refer to the first case as the *hermitian symmetric case*, and to the remaining two as the *semisimple case.* To explain these definitions we recall the following result.

Theorem 1.2.5 (Kac)**.** *Let* $i_1, \ldots, i_r$ *be all the indices such that* $s_{i_1} = \ldots = s_{i_r} = 0$*. Then the Lie algebra* $\mathfrak{g}_0$ *is isomorphic to a direct sum of the* $(n-r)$*-dimensional center and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the affine diagram* $X_N^{(k)}$ *consisting of the vertices* $i_1, \ldots, i_r$*.*

From the first part of this theorem we obtain that in the semisimple case the subalgebra $\mathfrak{g}_0$ has no center, hence it is semisimple. On the contrary, in the hermitian symmetric case $\mathfrak{g}_0$ is reductive with a 1-dimensional center, therefore we can regard $\mathfrak{g}/\mathfrak{g}_0$ as an infinitesimal hermitian symmetric space.

Remark 1.2.3. By the above theorem it is clear that $\widehat{\Delta}_0$ can be seen as a root system for $\mathfrak{g}_0$.

We can now show a useful realization of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated to the pair $(\mathfrak{g}, \sigma)$. Let $L(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ be the loop algebra of $\mathfrak{g}$ and $\widetilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}c$ its universal central extension, with Lie bracket defined by

$$[t^m \otimes x, t^n \otimes y] = t^{m+n} \otimes [x, y] + m\delta_{m,-n}\langle x, y\rangle c, \quad \forall x, y \in \mathfrak{g}, \forall m, n \in \mathbb{Z}$$

and clearly $[c, t^m \otimes x] = 0$ for all $x \in \mathfrak{g}$, $m \in \mathbb{Z}$. Define a derivation $[d, \cdot]$ of $\widetilde{L}(\mathfrak{g})$ setting

$$[d, p(t) \otimes x] = t\frac{d}{dt}p(t) \otimes x, \quad \forall p(t) \otimes x \in L(\mathfrak{g})$$

and $[d,c]=0$. We obtain the algebra $\widehat{L}(\mathfrak{g})=\widetilde{L}(\mathfrak{g})\oplus\mathbb{C}d$ and its subalgebra

$$\widehat{L}(\mathfrak{g},\sigma)=\sum_{j\in\mathbb{Z}}\mathfrak{g}_{\bar{j}}\otimes t^j+\mathbb{C}c+\mathbb{C}d,$$

where $\bar{j}\in\{0,1\}$ is defined by $j\cong\bar{j}$ mod 2. Let $\mathfrak{h}_0$ be a fixed Cartan subalgebra of $\mathfrak{g}_0$. We have the follwing theorem.

Theorem 1.2.6 (Kac)**.** *There exists an isomorphism* $\Phi:\widehat{\mathfrak{g}}\longrightarrow\widehat{L}(\mathfrak{g},\sigma)$ *such that*

- Φ *maps* $\widehat{\mathfrak{g}}_i$ *onto* $t^i\otimes\mathfrak{g}_{\bar{i}}$ *for* $i\neq 0$*;*
- $\Phi(\widehat{\mathfrak{h}})=1\otimes\mathfrak{h}_0+\mathbb{C}c+\mathbb{C}d$*;*
- $\Phi(\widehat{\mathfrak{g}}_0)=1\otimes\mathfrak{g}_0+\mathbb{C}c+\mathbb{C}d$*.*

Observe that this result implies a refinement of Remark 1.2.3: in fact we have that $\widehat{\Delta}_0$ is isomorphic to the root system of $\mathfrak{g}_0$ corresponding to $\mathfrak{h}_0$. More specifically, define $\delta'\in\widehat{\mathfrak{h}}^*$ setting $\delta'(d)=1$ and $\delta'(c)=\delta'(\mathfrak{h}_0)=0$ (from now on we will omit the isomorphism Φ): following [9] it is possible to prove that

$$\alpha_i=\overline{\alpha_i}+s_i\delta',$$

where $\lambda\to\overline{\lambda}$ is the restriction map from $\widehat{\mathfrak{h}}$ to $\mathfrak{h}_0$.

Remark 1.2.4. Since $\delta=\sum_{i=0}^n a_i\alpha_i$, by the previous equality we obtain that $\delta'=\frac{k}{2}\delta$.

Remark 1.2.5. The assumption of $\mathfrak{g}$ simple can be dropped safely if we assume σ indecomposable: in fact, as explained in [10], most arguments given in [9] can be extended to the case of $\mathfrak{g}$ semisimple but not simple. This is the case of Remark 1.2.2, where we consider $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{g}$, with $\mathfrak{g}$ simple, and σ the flip. In the sequel this case will be referred to as the *adjoint case*.

1.2.2 Homology and representation theory

In this section we will state some results about homology of Kac-Moody algebras. First we recall an extended version of the Garland-Lepowsky Theorem (see [6]). We will follow the presentation of Kumar [14].

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra of rank n, with Cartan decomposition $\mathfrak{g}=\mathfrak{h}\oplus(\bigoplus_{\gamma\in\Delta}\mathfrak{g}_\gamma)$ and let $\Pi=\{\alpha_1,\ldots,\alpha_n\}$ be the set of simple roots. Choose a set of positive roots Δ^+ and a subset $Y\subset\{1,\ldots,n\}$: we define the subalgebras

$$\mathfrak{g}_Y=\mathfrak{h}\oplus\big(\bigoplus_{\gamma\in\Delta_Y}\mathfrak{g}_\gamma\big),\qquad \mathfrak{u}_Y^{\pm}=\bigoplus_{\gamma\in\Delta^{\pm}\setminus\Delta_Y^{\pm}}\mathfrak{g}_\gamma$$

where $\Delta_Y=\Delta\cap(\bigoplus_{i\in Y}\mathbb{Z}\alpha_i)$ is the *parabolic subsystem* associated to Y, and $\Delta_Y^{\pm}=\Delta_Y\cap\Delta^{\pm}$.

Moreover, let $\langle\cdot,\cdot\rangle$ be the Killing form of $\mathfrak{g}$ and $\nu:\mathfrak{h}\to\mathfrak{h}^*$ the induced natural isomorphism: as usual we set $\alpha_i^\vee=\frac{2}{\langle\alpha_i,\alpha_i\rangle}\nu^{-1}(\alpha_i)$. We can define an action on $\mathfrak{h}^*$ of the Weyl group W of $\mathfrak{g}$ in the following way: choose $\rho\in\mathfrak{h}^*$ such that $\rho(\alpha_i^\vee)=1$ for all $i=1,\ldots,n$, and set

$$w*\lambda=w(\lambda+\rho)-\rho,\quad \forall w\in W,\forall\lambda\in\mathfrak{h}^*.$$

Clearly the definition does not depend on the choice of ρ.

We now consider the exterior algebra $\bigwedge\mathfrak{g}$ of $\mathfrak{g}$. First of all we have the following lemma.

Lemma 1.2.7. *Let w be an element of the Weyl group W_Y of Δ_Y. Then the weight space of $\bigwedge \mathfrak{u}_Y^-$ corresponding to the weight $w * 0$ is one dimensional and is spanned by*

$$e_{-\beta_1} \wedge \ldots \wedge e_{-\beta_p}, \quad p = \ell(w)$$

where $N(w) = \{\beta_1, \ldots, \beta_p\}$ and $e_{-\beta_i}$ is a nonzero root vector corresponding to the root $-\beta_i$.

Proof. Fix a basis of $\mathfrak{u}_Y^-$ of root vectors: by the definition of the action of ad on $\bigwedge \mathfrak{g}$, we have that, for every decomposable vector $v = e_{-\gamma_1} \wedge \ldots \wedge e_{-\gamma_p} \in \bigwedge^p \mathfrak{u}_Y^-$,

$$\mathrm{ad}_h(v) = \Big(- \sum_{i=1}^{p} \gamma_i \Big)(h) \cdot v, \quad \forall h \in \mathfrak{h}.$$

In particular v is a root vector for $w * 0$ if and only if $\sum_{i=1}^{p} \gamma_i = \rho - w(\rho)$. Now observe that $s_{\alpha_i}(\rho) = \rho - 2\frac{\langle \rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = \rho - \alpha_i$: this, together with Proposition 1.3.1(2), implies that $w^{-1}(\rho) = \rho - \sum_{\beta_i \in N(w)} \beta_i$, and hence, if v is a root vector,

$$- \sum_{\beta_i \in N(w)} \beta_i = w^{-1}\Big(\sum_{i=1}^{p} \gamma_i \Big). \tag{1.2.1}$$

We claim that $\{\gamma_1, \ldots, \gamma_p\} = N(w)$: observe that, since $w^{-1}(\sum_{i=1}^{p} \gamma_i)$ is negative, there exists an index i_1 such that $w^{-1}(\gamma_{i_1}) < 0$, and then $\gamma_{i_1} \in N(w)$. Now (1.2.1) becomes $- \sum_{\beta_i \in N(w) \setminus \{\gamma_{i_1}\}} \beta_i = w^{-1}(\sum_{i \neq i_1} \gamma_i)$, therefore repeating the previous argument we obtain the claim and the lemma. □

Now let V be a $\mathfrak{g}$-module and set $\Lambda_p = \bigwedge^p \mathfrak{g} \otimes_{\mathbb{C}} V$: it is possible to define a family of operators $\partial_p : \Lambda_p \to \Lambda_{p-1}$ setting

$$\begin{aligned} \partial_p(v_1 \wedge \ldots \wedge v_p \otimes x) = & \sum_{i=1}^{p} (-1)^{i+1} v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p \otimes (v_i \cdot x) + \\ & + \sum_{i<j} (-1)^{i+j+1} [v_i, v_j] \wedge v_1 \ldots \wedge \widehat{v_i} \ldots \wedge \widehat{v_j} \ldots \wedge v_p \otimes x. \end{aligned}$$

when $p > 1$, and $\partial_1(v \otimes x) = 0$ when $p = 1$. Furthermore, we can extend the internal product ϵ and the adjoint action ad to Λ_p, setting

$$\epsilon(w)(v_1 \wedge \ldots \wedge v_p \otimes x) = w \wedge v_1 \wedge \ldots \wedge v_p \otimes x$$
$$\mathrm{ad}_w(v_1 \wedge \ldots \wedge v_p \otimes x) = v_1 \wedge \ldots \wedge v_p \otimes w \cdot x + \sum_{i=1}^{p} v_1 \wedge \ldots \wedge [w, v_i] \wedge \ldots \wedge v_p$$

for every $w \in \mathfrak{g}, v_1 \wedge \ldots \wedge v_p \otimes x \in \Lambda_p$. Observe that the relations stated in the proof of Proposition 1.1.1 are still true, in particular we have

$$[\partial_p, \mathrm{ad}_w] = 0 \tag{1.2.2}$$
$$\mathrm{ad}_w = \epsilon(w)\partial_{p-1} + \partial_p \epsilon(w). \tag{1.2.3}$$

Replacing (1.2.3) in (1.2.2) we obtain that

$$\partial_p \partial_{p+1} \epsilon(w) = \epsilon(w) \partial_{p-1} \partial_p.$$

Moreover, by a recursive argument, this clearly implies that $\partial_p\partial_{p+1} = 0$ and hence that $(\Lambda_p, \partial_p)_{p\geq 1}$ is a chain complex. In conclusion we can define the *Lie algebra homology with coefficients in* V as

$$H_p(\mathfrak{g}, V) = \frac{\ker \partial_p}{\operatorname{Im} \partial_{p+1}}.$$

Denote by W'_Y the (left) quotient of W by Y, that is, the set of minimal lenght elements in the cosets $W_Y w$, $w \in W$. Furthermore, we denote by $V(\lambda)$ (resp. $V_Y(\lambda)$) the irriducible $\mathfrak{g}$-module (resp. $\mathfrak{g}_Y$-module) of highest weight $\lambda \in \mathfrak{h}^*$. We have the following generalization of Garland-Lepowsky Theorem (we refer to [14, Theorem 3.2.7] for the proof).

Theorem 1.2.8. *For any subset $Y \subset \{1, \ldots, n\}$ and any integrable highest weight $\mathfrak{g}$-module $V(\lambda)$, we have*

$$H_p(\mathfrak{u}_Y^-, V(\lambda)) = \bigoplus_{\substack{w\in W'_Y \\ \ell(w)=p}} V_Y(w * \lambda).$$

Now we translate Lemma 1.2.7 and Theorem 1.2.8 to our setting: the Kac-Moody algebra will be the affine algebra $\widehat{\mathfrak{g}}$ associated to the pair $(\mathfrak{g}, \sigma)$, Y will be the set $\{i \mid s_i = 0\}$, and we choose the trivial representation $V(0)$ as $V(\lambda)$. Finally set $\widehat{\mathfrak{u}}_\sigma^- = \mathfrak{u}_Y^-$ and $W'_\sigma = W'_Y$.

Theorem 1.2.9. *We have the following decomposition in irreducible $\widehat{\mathfrak{g}}_0$-modules:*

$$H_p(u_\sigma^-) = \bigoplus_{\substack{w\in W'_\sigma \\ \ell(w)=p}} V(w(\rho) - \rho).$$

Moreover $V(w(\rho) - \rho)$ is one-dimensional and a representative of highest weight vector is $e_{-\beta_1} \wedge \ldots \wedge e_{-\beta_p}$, where $N(w) = \{\beta_1, \ldots, \beta_p\}$ and the $e_{-\beta_i}$ are root vectors.

We conclude this section recalling some results on Lie algebra homology and Hodge theory.
We need to define a Laplacian operator on $\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-$: set

$$\widehat{\mathfrak{h}}_\mathbb{R} = \operatorname{span}_\mathbb{R}(\alpha_1^\vee, \ldots, \alpha_n^\vee) + \mathbb{R}c + \mathbb{R}d$$

and define the antihomomorphism $y \mapsto y^*$ of $\widehat{\mathfrak{g}}$ setting $E_i^* = F_i$, $E_1^* = F_i$ on Chevalley generators and $H^* = H$ when $H \in \widehat{\mathfrak{h}}_\mathbb{R}$. Let $(\cdot, \cdot)$ be the normalized standard form on $\widehat{\mathfrak{g}}$ ([9, Section 6.2]) and define the hermitian form $\{x, y\} = (x, y^*)$ for every $x, y \in \widehat{\mathfrak{g}}$. By [14, Theorem 2.3.13] this hermitian form is positive definite on $\widehat{\mathfrak{u}}_\sigma^-$, therefore, extending it via determinants on $\bigwedge \widehat{\mathfrak{u}}_\sigma^-$, we obtain the adjoint $\partial_p^* : \bigwedge^{p-1} \widehat{\mathfrak{u}}_\sigma^- \to \bigwedge^p \widehat{\mathfrak{u}}_\sigma^-$ of ∂_p, and the associated Laplacian

$$L_p = \partial_{p+1}\partial_{p+1}^* + \partial_p^*\partial_p.$$

Define the set of harmonic p-form $\mathcal{H}_p = \ker(L_p)$: it is known that

Theorem 1.2.10. *1. $\mathcal{H}_p \subseteq \ker(\partial_p)$.*

2. *The natural map* $\mathcal{H}_p \to (\mathcal{H}_p \oplus Im(\partial_{p+1}))/Im(\partial_{p+1})$ *induces an isomorphism* $\mathcal{H}_p \cong H_p(\widehat{\mathfrak{u}}_\sigma^-)$.

Remark 1.2.6. Observe that the grading on $\widehat{\mathfrak{g}}$ defines a grading on $\widehat{\mathfrak{u}}_\sigma^\pm$ and hence on $\bigwedge \widehat{\mathfrak{u}}_\sigma^-$. We denote with $(\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_q$ the subspace of $\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-$ of degree q. Notice that $(\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_q = 0$ if $p > -q$

Remark 1.2.7. It is clear that $\partial_p((\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_q) \subseteq (\bigwedge^{p-1} \widehat{\mathfrak{u}}_\sigma^-)_q$. Moreover, since $\bigwedge^p \widehat{\mathfrak{u}}_\sigma^- = \bigoplus_{q\in\mathbb{Z}}(\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_q$ is an orthogonal sum, we have $\partial_p^*((\bigwedge^{p-1} \widehat{\mathfrak{u}}_\sigma^-)_q) \subseteq (\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_q$. In particular since $(\bigwedge^{p+1} \widehat{\mathfrak{u}}_\sigma^-)_{-p} = 0$ we obtain

$$L_{p_{|(\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_{-p}}} = \partial_p^* \partial_{p_{|(\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-)_{-p}}}. \tag{1.2.4}$$

1.2.3 The link with σ-minuscule elements

Definition 1.2.1. An element $w \in \widehat{W}$ is said to be σ-*minuscule* if $N(w) \subseteq \{\alpha \in \widehat{\Delta}^+ \mid ht_\sigma(\alpha) = 1\}$. Denote by $\mathcal{W}_\sigma^{ab}$ the set of σ-minuscule elements of $\widehat{W}$.

Remark 1.2.1. Note that in the adjoint case $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{g}$ simple, w is σ-minuscule if and only if $N(w) \subset -\Delta_\mathfrak{g}^+ + \delta$, $\Delta_\mathfrak{g}^+$ being the set of positive roots of $\mathfrak{g}$. So we recover Peterson's notion of minuscule elements quoted in the first section.

As preoviously stated, abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ are related to $\mathcal{W}_\sigma^{ab}$, in fact

Theorem 1.2.11. *There is a bijection between* $\mathcal{W}_\sigma^{ab}$ *and the set of abelian* $\mathfrak{b}_0$*-stable subalgebras of* $\mathfrak{g}_1$.

Proof. Let $\mathfrak{a} \subset \mathfrak{g}_1$ be a $\mathfrak{b}_0$-stable commutative subalgebra, and let $\{x_1, \dots, x_p\}$ be a basis of $\mathfrak{a}$. Set

$$v_\mathfrak{a} = t^{-1} \otimes x_1 \wedge \dots \wedge t^{-1} \otimes x_p \in \bigwedge^p \widehat{\mathfrak{g}}_{-1} = \Big(\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-\Big)_{-p}$$

By (1.2.4) and since $\mathfrak{a}$ is abelian we have $\partial_p^* \partial_p(v_\mathfrak{a}) = 0$. Therefore $v_\mathfrak{a}$ is a cycle in $\bigwedge^p \widehat{\mathfrak{u}}_\sigma^-$, and since $\mathfrak{a}$ is $\mathfrak{b}_0$-stable and $v_\mathfrak{a}$ is $\widehat{\mathfrak{h}}$-stable, its homology class is a highest vector for an irreducible component $V_\mathfrak{a}$ of $H_p(\widehat{\mathfrak{u}}_\sigma^-)$. By Theorem 1.2.9 there exists $w \in \widehat{W}$ such that $\ell(w) = p$ and $V_\mathfrak{a} = V(w(\rho) - \rho)$. We need to check that w is σ-minuscule: set $N(w) = \{\beta_1, \dots, \beta_p\}$ and observe that there exists a $c \in \mathbb{C}$ such that

$$e_{-\beta_1} \wedge \dots \wedge e_{-\beta_p} = c \cdot t^{-1} \otimes x_1 \wedge \dots \wedge t^{-1} \otimes x_p,$$

for fixed root vectors $e_{-\beta_i}$. So each $e_{-\beta_i}$ lies in the subspace generated by $t^{-1} \otimes x_1 \dots t^{-1} \otimes x_p$, and this implies that $ht_\sigma(\beta_i) = 1$.
Suppose now conversely that $w \in \mathcal{W}_\sigma^{ab}$ and set $N(w) = \{\beta_1, \dots, \beta_p\}$. Since $ht_\sigma(\beta_i) = 1$, we have that $e_{-\beta_i} \in (\widehat{\mathfrak{u}}_\sigma^-)_{-1}$ and hence $e_{-\beta_i} = t^{-1} \otimes x_i$ with $x_i \in \mathfrak{g}_1$. It is well known that $W'_\sigma = \{w \in \widehat{W} \mid N(w) \cap \Delta_o^+ = \emptyset\}$, in particular if w is σ-minuscule then $w \in W'_\sigma$. Again by Theorem 1.2.9 the element $v = e_{-\beta_1} \wedge \dots \wedge e_{-\beta_p}$ represents a highest weight vector for $V(w(\widehat{\rho}) - \widehat{\rho})$ in $H_p(\widehat{\mathfrak{u}}_\sigma^-)$. By (1.2.4) it follows that

$$L_p(v) = \partial_p^* \partial_p(v) = 0.$$

It is a standard fact that $\partial_p^* \partial(v) = 0$ implies $\partial_p(v) = 0$. It easily follows that the space $\mathfrak{a}$ generated by $\{x_1, \dots, x_p\}$ is abelian. Since v is $\mathfrak{b}_0$-stable, then $\mathfrak{a}$ is also $\mathfrak{b}_0$-stable. □

Remark 1.2.2. The natural isomorphism of $\mathfrak{g}_0$-modules $\mathfrak{g}_1 \cong t^{-1} \otimes \mathfrak{g}_1$ maps the $\mathfrak{b}_0$-stable abelian subspaces of $\mathfrak{g}_1$ to $\mathfrak{b}_0$-stable abelian subspaces of $\widehat{L}(\mathfrak{g}, \sigma)$. Through this isomorphism, the map of the above proposition associates to $w \in \mathcal{W}_\sigma^{ab}$ the $\mathfrak{b}_0$-stable abelian subalgebra $\bigoplus_{i=1}^k \widehat{L}(\mathfrak{g}, \sigma)_{-\beta_i}$.

1.2.4 Maximal eigenvalues of the Casimir operator $\mathcal{C}_0$

In this section we present an interesting result on the maximal eigenvalues l_k of $\mathcal{C}_0$ on $\bigwedge^k \mathfrak{g}_1$. First we have to recall a well known property of the Casimir operator (already stated in Remark 1.1.2).

Lemma 1.2.12. *Let $V(\lambda)$ be an irreducible $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. For every $v \in V(\lambda)$ we have*

$$\mathcal{C}(v) = \langle \lambda, \lambda + 2\rho \rangle v.$$

Moreover if $\mu \in \mathfrak{h}^*$ is a weight of $V(\lambda)$ then $\langle \mu, \mu + 2\rho \rangle \leq \langle \lambda, \lambda + 2\rho \rangle$, and equality holds if and only if $\mu = \lambda$.

The following theorem is a straightforward generalization to the graded setting of a result of Han [7].

Theorem 1.2.13. *Let Δ_1 be the set of weights of $\mathfrak{g}_1$ under the action of $\mathfrak{g}_0$. Set $r = |\Delta_1|$. For $0 \leq k < r$ we have*

$$l_k < l_{k+1}.$$

Proof. The claim clearly holds when $k = 0$. Assume then $1 \leq k < r$: fix a basis $B = \{h_1, \dots, h_m\} \cup \{x_\beta \mid \beta \in \Delta_1\}$ of $\mathfrak{g}_1$, where $h_i \in \mathfrak{h} \cap \mathfrak{g}_1$ and x_β is a weight vector for $\beta \in \Delta_1$. Let $v = v_1 \wedge \dots \wedge v_k$ such that $v_i \in B$ and $\mathcal{C}_0(v) = l_k$: by Theorem 1.1.3 we can assume that v is a highest weight vector whose weight is $\mu = \sum_{\beta \in S} \beta$, where S is the set of weights of the subalgebra generated by $\{v_1, \dots, v_k\}$. Now choose a weight $\alpha \in \Delta_1 \setminus S$ with maximal height (observe that $\Delta_1 \setminus S$ is nonempty since $k < r$). Let x_α be the associated weight vector and define $u = v \wedge e_\alpha$: by the choice of α we have that u is a highest weight vector of weight $\lambda = \mu + \alpha$. Since μ is dominant we have

$$\langle \lambda, \alpha \rangle = \langle \mu, \alpha \rangle + \langle \alpha, \alpha \rangle > 0$$

therefore $\lambda - \alpha = \mu$ is a weight of $V(\lambda)$. This, together with Lemma 1.2.12, implies that

$$l_{k+1} \geq \langle \lambda, \lambda + 2\rho_0 \rangle > \langle \mu, \mu - 2\rho_0 \rangle = l_k.$$

□

1.3 Some results on Weyl groups and root systems

1.3.1 Conventions on root systems

- We assume that K is the *canonical central element* [9, Section 6.2], that is,

$$K = \sum_{i=0}^{n} a_i^\vee \alpha_i^\vee.$$

If we number the Dynkin diagrams as in [9, Tables Aff1, Aff2, Aff 3] then, by Sections 6.1, 6.2, 6.4 of [9],

$$K = \frac{2a_0}{\|\delta - a_0\alpha_0\|}\nu^{-1}(\delta). \tag{1.3.1}$$

- Define $(\mathfrak{h}_0)_\mathbb{R} = \operatorname{span}_\mathbb{R}(\alpha_1^\vee, \dots, \alpha_n^\vee)$. We let

$$C_1 = \{h \in (\mathfrak{h}_0)_\mathbb{R} \mid \overline{\alpha}_i(h) \geq -s_i,\ i = 0, \dots, n\} \tag{1.3.2}$$

  be the *fundamental alcove* of $\widehat{W}$.

- If $v \in \widehat{\mathfrak{h}}^*$, we set $v^\perp = \{x \in \widehat{\mathfrak{h}}^* \mid (x, v) = 0\}$.

- If $S \subseteq \widehat{\Pi}$, we denote by $\Delta(S)$ (resp. $\Delta^+(S)$) the root system generated by S (resp. the set of positive roots corresponding to S). If $A \subseteq \widehat{\Delta}^+$ we denote by $W(A)$ the Weyl group generated (inside $\widehat{W}$) by the reflections in the elements of A.

  We will often identify subsets of the set of simple roots with their Dynkin diagram.

- If R is a finite or affine root system and Π_R is a basis of simple roots, we write the expansion of a root $\gamma \in R$ w.r.t. Π_R as

$$\gamma = \sum_{\alpha \in \Pi_R} c_\alpha(\gamma)\gamma. \tag{1.3.3}$$

  We also set, for $\alpha \in R$,

$$Supp(\alpha) = \{\beta \in \Pi_R \mid c_\beta(\alpha) \neq 0\}.$$

- If R is a finite irreducible root system and Π is a set of simple roots for R, we denote by θ_R (or by θ_Π) its highest root. Recall that the highest root and the highest short root are the only dominant weights belonging to R^+. We will use this remark in the following form:

$$\alpha \in R^+,\, \alpha \text{ long },\, (\alpha, \beta) \geq 0\, \forall\, \beta \in R^+ \implies \alpha = \theta_R.$$

- We recall the definition of dual Coxeter number g_R of a finite irreducible root system R. Write $\theta_R^\vee = \sum_{\alpha \in \Pi_R} c_{\alpha^\vee}(\theta^\vee)\alpha^\vee$ and set

$$g_R = 1 + \sum_{\alpha \in \Pi_R} c_{\alpha^\vee}(\theta^\vee). \tag{1.3.4}$$

  Set finally $\mathbf{g} = \sum_{i=0}^n c_{\alpha_i^\vee}(K)$. This number is called the dual Coxeter number of $\widehat{L}(\mathfrak{g}, \sigma)$.

1.3.2 Combinatorics of inversion sets

If α is a real root in $\widehat{\Delta}^+$, we let s_α denote the reflection in α. If α_i is a simple root we set $s_i = s_{\alpha_i}$.

The following facts are well-known. More details and references can be found in [4].

Proposition 1.3.1. *1. $N(w_1) = N(w_2) \implies w_1 = w_2$.*

2. *If $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w, then*

$$N(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}.$$

   If moreover $\beta_h = s_{i_1} \cdots s_{i_{h-1}}(\alpha_{i_h})$, $1 \le h \le k$, then

$$w = s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}. \tag{1.3.5}$$

3. *$N(w)$ is* biconvex, *i.e. both $N(w)$ and $\widehat{\Delta}^+ \setminus N(w)$ are closed under root addition. Conversely, if $\widehat{\Delta}^+$ has no irreducible components of type $A_1^{(1)}$ and L is a finite subset of real roots which is biconvex, then there exists $w \in \widehat{W}$ such that $L = N(w)$.*

4. *Denote by $\le$ the weak left Bruhat order, that is, $w_1 \le w_2$ if there exists a reduced expression for w_1 which is an initial segment of a reduced expression for w_2). Then*
$$w_1 < w_2 \iff N(w_1) \subset N(w_2).$$

5. *Set $N^{\pm}(w) = N(w) \cup -N(w)$. Then $N^{\pm}(w_1 w_2) = N^{\pm}(w_1) \dot{+} w_1(N^{\pm}(w_2))$, where $\dot{+}$ denotes the symmetric difference. In particular, the following properties are equivalent:*

   (a) $N(w_1 w_2) = N(w_1) \cup w_1(N(w_2))$;

   (b) $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$;

   (c) $w_1(N(w_2)) \subset \widehat{\Delta}^+$.

We also introduce the sets of left and right descents for $w \in \widehat{W}$:

$$L(w) = \{\alpha \in \widehat{\Pi} \mid \ell(s_\alpha w) < \ell(w)\},$$
$$R(w) = \{\alpha \in \widehat{\Pi} \mid \ell(w s_\alpha) < \ell(w)\}.$$

We have that $L(w) = \widehat{\Pi} \cap N(w)$, $R(w) = \widehat{\Pi} \cap N(w^{-1})$.

1.3.3 Reflection subgroups and coset representatives

Let G be a finite or affine reflection group and let ℓ be the length function with respect to a fixed set of Coxeter generators S. Let R be the set of roots of G in the geometric representation, Π_R a system of simple roots for R, and R^+ the corresponding set of positive roots. Let G' be a subgroup of G generated by reflections, and R' be the set of roots $\alpha \in R$ such that $s_\alpha \in G'$, which is easily shown to be a root system.

We recall the main Theorem of [5]:

Theorem 1.3.2 (Deodhar)**.** $\Pi_{R'} = \{\alpha \in R^+ \mid N(s_\alpha) \cap R' = \{\alpha\}\}$ *is a set of simple roots for* R'*, whose associated set of positive roots is* $R'^+ = R' \cap R^+$.

Given $g \in G$, we say that an element $w \in G'g$ is *a minimal right coset representative* if $\ell(w)$ is minimal among the lengths of elements of $G'g$. From Theorem 1.3.2 follows that

Proposition 1.3.3. *Every coset* $G'g$ *has a unique minimal right coset representative* w*. Moreover this element is characterized by the following property:*

$$w^{-1}(\alpha) \in R^+ \quad \text{for all } \alpha \in R'^+. \tag{1.3.6}$$

Proof. First observe that if $\ell(s_\alpha g) > \ell(g)$ for all $\alpha \in \Pi_{R'}$ then clearly $g = w$. Suppose now $\ell(s_{\alpha_1} g) < \ell(g)$ for $\alpha_1 \in \Pi_{R'}$ and continue chosing $\alpha_i \in \Pi_{R'}$ so that $\ell(s_{\alpha_1} \cdots s_{\alpha_i} g) < \ell(s_{\alpha_1} \cdots s_{\alpha_{i-1}} g)$ as long as such α_i can be found. This process clearly must end after at most $\ell(g)$ steps. If it ends with $g_k = s_{\alpha_1} \cdots s_{\alpha_k} g$, then $g_k = w$ since $\ell(s_\alpha g_k) > \ell(g_k)$ for every $\alpha \in \Pi_{R'}$.
To prove the uniqueness of such an element, suppose there exist two minimal coset representative, namely $w_1 = u_1 g, w_2 = u_2 g$, $u_1, u_2 \in G'$. Choose a reduced expression $w_2 = s_{i_1} \cdots s_{i_k}$ with $\alpha_{i_j} \in \Pi_R$, and set $u_1 u_2^{-1} = s_{\gamma_1} \cdots s_{\gamma_q}$ with $\gamma_i \in \Pi_{R'}$: we have

$$w_1 = u_1 u_2^{-1} w_2 = s_{\gamma_1} \cdots s_{\gamma_q} s_{i_1} \cdots s_{i_k}.$$

From this expression we can extract a reduced subword of w_1: since it can not start with some s_{γ_i}, it must be a subword of $s_{i_1} \cdots s_{i_k}$, hence $w_1 \leq w_2$. By simmetry we obtain $w_2 \leq w_1$ and then $w_1 = w_2$. ☐

We will always choose as a coset representative for $G'g$ the minimal right coset representative and (with a slight abuse of notation) we denote by $G' \backslash G$ the set of all minimal right coset representatives. Thus the restriction of the weak order of G on $G' \backslash G$ induces a partial ordering on $G' \backslash G$. When saying *the poset* $G' \backslash G$, we shall always refer to this ordering.

If $\alpha \in R$ and G' is the stabilizer of α in G, then, for each $g \in G$, the minimal length representative of $G'g$ is the unique minimal length element that maps $g^{-1}\alpha$ to α. By formula 1.3.6, this element is characterized by the property

$$w^{-1}(\beta) \in R^+ \quad \text{for all } \beta \in R^+ \text{ orthogonal to } \alpha. \tag{1.3.7}$$

A reflection subgroup G' of G is standard parabolic when $\Pi_{R'} \subseteq \Pi_R$. In this case, if $g \in G$ and w is the minimal right coset representative of $G'g$, then $g = g'w$ with $g' \in G'$ and $\ell(g) = \ell(g') + \ell(w)$. In particular $N(g) \cap R' = N(g')$. Moreover, it is well known that g itself is the minimal representative of $G'g$ if and only if $L(g) \subseteq \Pi_R \setminus \Pi_{R'}$. Therefore $G' \backslash G = \{w \in G \mid L(w) \subseteq \Pi_R \setminus \Pi_{R'}\}$.

If G is finite, the poset $G' \backslash G$ has a unique minimal and a unique maximal element. The identity of G clearly corresponds to the minimum of $G' \backslash G$. If w_0 is the longest element of G and w_0' is the longest element of G', then we have that $N(w_0' w_0) = R \setminus R'$. If $w \in G' \backslash G$, then $N(w) \subseteq R \setminus R'$; therefore $w_0' w_0$ is the unique maximal element of $G' \backslash G$. Note that

$$\ell(w_0' w_0) = |\Delta^+(R)| - |\Delta^+(R')|. \tag{1.3.8}$$

1.3.4 Special elements in finite Weyl groups

We sum up in the following statement the content of Propositions 7.1 and 7.2 from [4], which are based on earlier works of Panyushev and Sommers. Precise attributions of the individual results are done in [4]. The properties below will be used many times in the sequel.

Proposition 1.3.4. *Let R be a finite irreducible root system, W_R its Weyl group. Fix a positive system R^+ and let Π_R, θ_R be the corresponding set of simple root and highest root, respectively.*

1. *For any long root α there exists a unique element $y_\alpha \in W_R$ of minimal length such that $y(\alpha) = \theta_R$.*
2. *$L(y_\alpha) \subset \{\beta \in \Pi_R \mid (\beta, \theta_R) \neq 0\}$. If conversely $v \in W_R$ is such that $v(\alpha) = \theta_R$ and $L(v) \subset \{\beta \in \Pi_R \mid (\beta, \theta_R) \neq 0\}$, then $v = y_\alpha$.*
3. *If $\alpha \in \Pi_R$, then $\ell(y_\alpha) = g_R - 2$, g_R being the dual Coxeter number of R.*
4. *If $\alpha \in \Pi_R$, and $\beta_1 + \beta_2 = \theta_R$, $\beta_1, \beta_2 \in R^+$, then exactly one element among β_1, β_2 belongs to $N(y_\alpha)$, and any element of $N(y_\alpha)$ arises in this way. Conversely, if $y \in W_R$ is such that for any pair $\beta_1, \beta_2 \in R^+$ such that $\beta_1 + \beta_2 = \theta_R$ exactly one of β_1, β_2 belongs to $N(y)$ and $\theta_R \notin N(y)$, then there exists a long simple root β such that $y(\beta) = \theta_R$.*
5. *$N(y_\alpha^{-1}) = \{\beta \in R^+ \mid (\beta, \alpha^\vee) = -1\}$.*
6. *$\gamma \in R^+$, $(\gamma, \theta_R) = 0 \implies \gamma \notin N(y_\alpha)$.*

Proof.

1. We prove that there exists a unique element of minimal lenght in $A = \{v \in W_R \mid v(\theta_R) = \alpha\}$. This clearly is equivalent to our thesis. Let then $m = \min_{v \in A} \ell(v)$ and let $v \in A$ be an element such that $\ell(v) = m$. If $v' \in A$, then $v' = vx$ with $x \in \mathrm{Stab}_{W_R}(\theta_R)$. We shall prove that $\ell(v') = \ell(v) + \ell(x)$. Let $v = s_{i_1} \cdots s_{i_k}$, $x = s_{j_1} \cdots s_{j_h}$ be reduced expressions: if by contradiction $s_{i_1} \cdots s_{i_k} s_{j_1} \cdots s_{j_h}$ is not reduced, then there exist $u <_B v$ and $y <_B x$, $<_B$ being the Bruhat order, such that $uy = vx$. But

$$\mathrm{Stab}_{W_R}(\theta_R) = \langle s_\beta \mid \beta \in \Pi_R \cap \theta_R^\perp \rangle,$$

therefore $y \in \mathrm{Stab}_{W_R}(\theta_R)$ too. It follows that $u(\alpha) = \theta_R$, which is a contradiction since $\ell(w)$ is minimal.

2. For all $y \in W_R$, if $y(\alpha) = \theta_R$ then $y = xy_\alpha$ with $x \in \mathrm{Stab}_{W_R}(\theta_R)$ and $\ell(y) = \ell(x) + \ell(y_\alpha)$. In particular, $\ell(s_\beta y_\alpha) > \ell(y_\alpha)$ for all $\beta \in \Pi_R \cap \theta_R^\perp$, hence $L_{y_\alpha} \subseteq \Pi_R \setminus \theta_R^\perp$. Moreover $L_x \subseteq \theta_R^\perp$ and $N(x) \subseteq N(Y)$, therefore if $y \neq y_\alpha$ we must have $L_y \not\subseteq \Pi_R \setminus \theta_R^\perp$.

3. Let $\alpha \in \Pi_R$ and $v \in W$ be such that $v(\alpha) = \theta_R$. We first prove that $\ell(v) \geq g_R - 2$. Let $v = s_{\gamma_1} \cdots s_{\gamma_k}$ be a reduced expression. Set $v_0 = 1$, and $v_h = \prod_{i=1}^h s_{\gamma_i}$, for $1 \leq h \leq k$. Since $N(v_h) \subseteq N(v)$, $v_h^{-1}(\theta_R)$ is a long positive root for $0 \leq h \leq k$: in particular $v_{h-1}^{-1}(\theta_R) \neq \gamma_h$, for $0 < h \leq k$. For any positive root β and any $\gamma \in \Pi_R$, we have $s_\gamma(\beta) = \beta - (\beta, \gamma^\vee)\gamma$. Moreover, if β is long and $\beta \neq \gamma$, then $|(\beta^\vee, \gamma)| \leq 1$, hence $|(\beta, \gamma^\vee)| = |(\beta^\vee, \gamma)\frac{(\beta,\beta)}{(\gamma,\gamma)}| \leq |\frac{(\beta,\beta)}{(\gamma,\gamma)}| = |\frac{(\theta_R,\theta_R)}{(\gamma,\gamma)}|$. Therefore $v_{h-1}^{-1}(\theta_R) - v_h^{-1}(\theta_R) \leq$

$\frac{(\theta_R,\theta_R)}{(\gamma_h,\gamma_h)}\gamma_h$, for $1 \leq h \leq k$, hence $\theta_R - \alpha = v_0^{-1}(\theta_R) - v_k^{-1}(\theta_R) \leq \sum_{i=1}^{k} \frac{(\theta_R,\theta_R)}{(\gamma_i,\gamma_i)}\gamma_i$. Since $\theta_R = \sum_{\gamma\in\Pi_R} m_\gamma \gamma = \sum_{\gamma\in\Pi_R} m_\gamma^\vee \frac{(\theta_R,\theta_R)}{(\gamma,\gamma)}\gamma$, we obtain that each $\gamma \neq \alpha$ occurs in the sequence $(\gamma_1,\dots,\gamma_k)$ at least $m_\gamma^\vee$ times, and α at least $m_\alpha^\vee - 1$ times. It follows that $\ell(v) = k \geq \sum_{\gamma\in\Pi_R} m_\gamma^\vee - 1 = g_R - 2$.

Next, we show that there exists a $w \in W_R$ such that $w(\alpha) = \theta_R$ and $\ell(w) \leq g_R - 2$. If β is a long root and $\beta \neq \theta_R$, then there exists a simple root γ such that $(\beta,\gamma^\vee) < 0$ and hence $s_\gamma(\beta) = \beta + \frac{(\theta_R,\theta_R)}{(\gamma,\gamma)}\gamma$. Therefore, if α is a simple long root, we can find a sequence of simple roots $(\gamma_1,\dots,\gamma_k)$ such that $(s_{\gamma_{i-1}}\cdots s_{\gamma_1}(\alpha),\gamma_i^\vee) < 0$, $s_{\gamma_i}\cdots s_{\gamma_1}(\alpha) = s_{\gamma_{i-1}}\cdots s_{\gamma_1}(\alpha) + \frac{(\theta_R,\theta_R)}{(\gamma_i,\gamma_i)}\gamma_i$, for $i \leq k$, and $s_{\gamma_k}\cdots s_{\gamma_1}(\alpha) = \theta_R$. Then clearly $k = \sum_{\gamma\in\Pi_R} m_\gamma^\vee - 1 = g_R - 2$ and therefore $\ell(s_{\gamma_k}\cdots s_{\gamma_1}) \leq g_R - 2$. This concludes the proof.

4. We have $\theta_R \notin N(y_\alpha)$, therefore, since $N(y_\alpha)$ is biconvex, for any pair β_1,β_2 in Δ_R^+ such that $\theta_R = \beta_1 + \beta_2$, *at most* one of β_1,β_2 lies in $N(y_\alpha)$. If $\alpha \in \Pi_R$, since $y_\alpha(\alpha) = \theta_R$, we have $N(y_\alpha s_\alpha) = N(y_\alpha) \cup \{\theta_R\}$: this implies that *at least* one of β_1,β_2 lies in $N(y_\alpha s_\alpha)$ and hence in $N(y_\alpha)$.

Now let $y \in W_R$ be such that for any pair β_1,β_2 in Δ_R^+ such that $\theta_R = \beta_1 + \beta_2$, exactly one of β_1,β_2 lies in $N(y)$, and, moreover, $\theta_R \notin N(y)$. Then $N(y) \cup \{\theta_R\}$ is still biconvex, therefore, by Proposition 1.3.1(3), there exists $y' \in W$ such that $N(y') = N(y) \cup \{\theta_R\}$. By Proposition 1.3.1(1), $y' = yx$ with $\ell(y') = \ell(y) + \ell(x)$, therefore $\ell(x) = 1$ or equivalently $x = s_\alpha$ for some $\alpha \in \Pi_R$. Then by Proposition 1.3.1(2) we obtain that $\theta_R = y(\alpha)$, hence that α is a long simple root.

5. Assume $y_\alpha(\beta) < 0$. Then $\alpha \neq \beta$ and $-y_\alpha(\beta) \in N(y_\alpha)$. By part 2, $\mathit{Supp}\,(-y_\alpha(\beta)) \setminus \theta_R^\perp \neq \emptyset$, hence $(\beta,\alpha^\vee) = -(-y_\alpha(\beta),\theta_R^\vee) = -1$. Conversely, assume $(\beta,\alpha^\vee) = -1$. Then $(y_\alpha(\beta),\theta_R^\vee) = -1$, hence $y_\alpha(\beta) < 0$.

6. The claim follows from part 2 and Proposition 1.3.1(3). □

Chapter 2

The structure of Borel stable abelian subalgebras

Recall that Π_0 denotes the set of simple roots of $\mathfrak{g}_0$ corresponding to Δ_0^+. In general Π_0 is disconnected and we write $\Sigma|\Pi_0$ to mean that Σ is a connected component of Π_0. Clearly, the Weyl group W_0 of $\mathfrak{g}_0$ is the direct product of the $W(\Sigma)$, $\Sigma|\Pi_0$. If θ_Σ is the highest root of $\Delta(\Sigma)$, set

$$\begin{aligned}
\widehat{\Delta}_0 &= \{\alpha + \mathbb{Z}k\delta \mid \alpha \in \Delta_0\} \cup \pm\mathbb{N}k\delta,\\
\widehat{\Pi}_0 &= \Pi_0 \cup \{k\delta - \theta_\Sigma \mid \Sigma|\Pi_0\},\\
\widehat{\Delta}_0^+ &= \Delta_0^+ \cup \{\alpha \in \widehat{\Delta}_0 \mid \alpha(d) > 0\}.
\end{aligned}$$

Denote by $\widehat{W_0}$ the Weyl group of $\widehat{\Delta}_0$. Let $\widehat{\Delta}_{re} = \widehat{W}\widehat{\Pi}$ be the set of real roots of $\widehat{L}(\mathfrak{g}, \sigma)$. If $\lambda \in \mathfrak{h}_0^*$, then we let $\mathfrak{g}_\lambda \subset \mathfrak{g}$ be the corresponding weight space.

Definition 2.0.1. We say that a real root α is *noncompact* if $\mathfrak{g}_{\overline{\alpha}} \subset \mathfrak{g}_1$, *compact* if $\mathfrak{g}_{\overline{\alpha}} \subset \mathfrak{g}_0$, and *complex* if it is neither compact nor noncompact.

If α is a complex root then a corresponding eigenvector $x_{\overline{\alpha}}$ in $\mathfrak{g}_{\overline{\alpha}}$ decomposes as

$$x_{\overline{\alpha}} = u_{\overline{\alpha}} + v_{\overline{\alpha}}$$

with $u_{\overline{\alpha}} \in \mathfrak{g}_0$ and $v_{\overline{\alpha}} \in \mathfrak{g}_1$. Then $u_{\overline{\alpha}}$ is a root vector in $\mathfrak{g}_0$ for $\overline{\alpha}$ and $v_{\overline{\alpha}}$ is a weight vector in $\mathfrak{g}_1$ for the weight $\overline{\alpha}$. From this follows that α is a compact root if and only if $\overline{\alpha} \in \widehat{\Delta}$ and $\delta' + \overline{\alpha} \notin \widehat{\Delta}$, α is a noncompact root if and only if $\overline{\alpha} \notin \widehat{\Delta}$ and $\delta' + \overline{\alpha} \in \widehat{\Delta}$, and α is a complex root if and only if $\overline{\alpha} \in \widehat{\Delta}$ and $\delta' + \overline{\alpha} \in \widehat{\Delta}$. More precisely if $k = 1$, since $\delta' \notin \widehat{\Delta}$, then a real root is either compact or noncompact. If $k = 2$ and $\mathfrak{g}$ is simple then $\delta' = \delta$, hence, by the very definition of $\widehat{L}(\mathfrak{g}, \sigma)$, $\alpha \in \widehat{\Delta}_{re}$ is either compact or noncompact if and only if α is a long root (i.e., $\|\alpha\|$ is largest among the possible root lengths). If $\mathfrak{g}$ is not simple, since σ is indecomposable, all the real roots are complex.

Recall that for every $\alpha \in \widehat{\Delta}$

$$ht_\sigma(\alpha) = \sum_{i=0}^{n} s_i c_{\alpha_i}(\alpha)$$

and, for $i \in \mathbb{Z}$, $\widehat{\Delta}_i = \{\alpha \in \widehat{\Delta} \mid ht_\sigma(\alpha) = i\}$.

Remark 2.0.1. Since $\alpha_i = s_i\delta' + \overline{\alpha}_i$ (Section 1.2.1), for any $\alpha \in \widehat{\Delta}$, we have that $\alpha = ht_\sigma(\alpha)\delta' + \overline{\alpha}$. In particular, since $k\delta = 2\delta'$, $ht_\sigma(k\delta) = 2$. By definition, the roots θ_Σ, $\Sigma|\Pi_0$, are the maximal roots having σ-height equal to 0, with respect to the usual order $\leq$ on roots: $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a sum of positive roots or zero. It follows that the roots $k\delta - \theta_\Sigma$ are the minimal roots having σ-height equal to 2. More generally, if $s \in \mathbb{Z}$, $\{sk\delta - \theta_\Sigma \mid \Sigma|\Pi_0\}$ is the set of minimal roots in $\widehat{\Delta}_{2s}$. Similarly, $\Pi_1 + sk\delta$ is the set of minimal roots in $\widehat{\Delta}_{2s+1}$.

Remark 2.0.1. It will be useful, from a notational point of view, to introduce the following generalization of the σ-height. Given $A \subseteq \widehat{\Pi}$ and $\gamma \in \widehat{\Delta}$, set

$$ht_A(\gamma) = \sum_{\alpha \in A} c_\alpha(\gamma).$$

In particular, the σ-height equals ht_{Π_1} and the usual height equals $ht_{\widehat{\Pi}}$. In these two cases we will keep using ht_σ and ht.

As previously stated we are interested in the set of the σ-minuscule elements

$$\mathcal{W}_\sigma^{ab} = \{w \in W \mid N(w) \subseteq \widehat{\Delta}_1\}.$$

In this chapter we will study the structure of some particular subset of $\mathcal{W}_\sigma^{ab}$, regarded as posets under the weak Bruhat order. In the next section we present the main result of [1] about the cardinality of $\mathcal{W}_\sigma^{ab}$.

2.1 The cardinality of the set of σ-minuscule elements

Consider the set

$$D_\sigma = \bigcup_{w \in \mathcal{W}_\sigma^{ab}} wC_1.$$

Observe that in the adjoint case D_σ is twice the fundamental alcove of the affine Weyl group of $\mathfrak{g}$.

The main idea of [1] is similar to the approach of Peterson in Theorem 1.1.4: in fact the authors find the formula for the cardinality of the set $\mathcal{W}_\sigma^{ab}$ calculating the volume of the polytope D_σ. More precisely, in the semisimple case they calculate the ratio $\mathrm{Vol}(P_\sigma)/\mathrm{Vol}(C_1)$ where P_σ is a fundamental domain for $W(\widehat{\Pi}_0)$. Since, by [1, Proposition 5.8, Lemma 5.11], there exist $w_\sigma \in \widehat{W}$ such that

$$P_\sigma = \begin{cases} D_\sigma & \text{if } \alpha \in \Pi_1 \text{ is short} \\ D_\sigma \cup w_\sigma(C_1) & \text{otherwise} \end{cases}$$

then the following theorem holds.

Theorem 2.1.1 (Cellini, Möseneder, Papi). *If $\mathfrak{g}_0$ is semisimple, $\Pi_1 = \{\alpha\}$ and $\chi_l(\alpha)$ is the truth function which is 1 if α is long and noncomplex and 0 otherwise, then*

$$|\mathcal{W}_\sigma^{ab}| = a_0(\chi_l(\alpha) + 1)k^{n-L}\frac{|W|}{|W(\widehat{\Pi}_0)|} - \chi_l(\alpha),$$

where L is the number of long roots in $\{\alpha_1, \ldots, \alpha_n\}$.
If $\mathfrak{g}_0$ is not semisimple then

$$|\mathcal{W}_\sigma^{ab}| = \frac{|W|}{|W(\widehat{\Pi}_0)|}\left(1 + \frac{l_\sigma}{l_f}\right),$$

where l_σ, l_f are the connection indexes of, respectively, $\widehat{\Delta}_0$ and Δ.

We conclude this section with a result that refines [1, Proposition 4.1]. Let a be the squared length of a long root in $\widehat{\Delta}^+$. Define

$$\widehat{\Pi}_0^* = \Pi_0 \cup \left\{ k\delta - \theta_\Sigma \mid a \leq 2\|\theta_\Sigma\|^2 \right\}, \tag{2.1.1}$$

$$\Phi_\sigma = \widehat{\Pi}_0^* \cup \{\alpha + k\delta \mid \alpha \in \Pi_1, \alpha \text{ long and noncomplex}\} \tag{2.1.2}$$

Remark 2.1.1.

(1) It is immediate to see that $\widehat{\Pi}_0^* = \widehat{\Pi}_0$, unless $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $G_2^{(1)}$ or $A_2^{(2)}$. Indeed, in the latter cases there exists $\Sigma|\Pi_0$ such that $\frac{a}{\|\theta_\Sigma\|^2} = 3, 4$, respectively.

(2) When $|\Pi_1| = 2$, then both roots in Π_1 are long; moreover, for any $\Sigma|\Pi_0$, both roots in Π_1 are not orthogonal to Σ. This is most easily seen by a brief inspection of the untwisted Dynkin diagrams, recalling that, by Section 1.2.1, $k = 1$ and the labels of the roots in Π_1 in the Dynkin diagram of $\widehat{\Pi}$ are equal to 1. Anyway, we provide a uniform argument. Let $\Pi_1 = \{\alpha, \beta\}$: since $k = 1$ and $c_\alpha(\delta) = 1$, $\delta - \alpha$ is a root and belongs to $\Delta(\widehat{\Pi} \setminus \{\alpha\})$. Since the support of $\delta - \alpha$ is $\widehat{\Pi} \setminus \{\alpha\}$, we see that $\widehat{\Pi} \setminus \{\alpha\}$ is connected. We claim that $\delta - \alpha$ is the highest root $\Delta(\widehat{\Pi} \setminus \{\alpha\})$. Otherwise, if $\beta > \delta - \alpha$ and $\beta \in \Delta(\widehat{\Pi} \setminus \{\alpha\})$, then $\beta - \delta$ would be a root with positive coefficients in some simple root in $\widehat{\Pi} \setminus \{\alpha\}$ and coefficient -1 in α. In particular, we obtain that $\delta - \alpha$ is long with respect to $\Delta(\widehat{\Pi} \setminus \{\alpha\})$ and, since it has the same length as α, that both $\delta - \alpha$ and α are long. For proving the second claim, observe that $\Sigma \cup \{\beta\} \subseteq Supp(\delta - \alpha) = \widehat{\Pi} \setminus \{\alpha\}$ and the latter is connected. Hence β has to be nonorthogonal to Σ. Switching the role of α and β we get the second claim.

If $\alpha \in \widehat{\Delta}$ then we let $H_\alpha^+ = \{h \in (\mathfrak{h}_0)_\mathbb{R} \mid \alpha(d + h) \geq 0\}$. This is the previously annunced result.

Proposition 2.1.2.

$$D_\sigma = \bigcap_{\alpha \in \Phi_\sigma} H_\alpha^+.$$

Proof. By [1, Propositions 4.1 and 5.8] and by Remark 2.1.1 (2), we have that $D_\sigma = \bigcap_{\alpha \in \Phi'_\sigma} H_\alpha^+$, where

$$\Phi'_\sigma = \widehat{\Pi}_0 \cup \{\alpha + k\delta \mid \alpha \in \Pi_1, \alpha \text{ long and noncomplex}\}.$$

(Actually Propositions 4.1 and 5.8 of [1] cover only the cases when $\mathfrak{g}$ is simple, but the argument is easily extended to the adjoint case.) Therefore, we have only to prove that we can restrict from $\widehat{\Pi}_0$ to $\widehat{\Pi}_0^*$, i.e. that if Σ is a component of Π_0 such that $a > 2\|\theta_\Sigma\|^2$, then $\theta_\Sigma(x) \leq k$ for all $x \in D_\sigma$. By Remark 2.1.1 (1), $\widehat{\Pi}$ is of type $G_2^{(1)}$ or $A_2^{(2)}$, in particular Π_1 has a single element: set $\Pi_1 = \{\tilde{\alpha}\}$. Note that $\tilde{\alpha}$ is long. We proceed in steps.

1. $\tilde{\alpha} + 3\theta_\Sigma \in \widehat{\Delta}^+$: this follows from $(\tilde{\alpha}, \theta_\Sigma^\vee) < -2$.

2. $2\tilde{\alpha} + 3\theta_\Sigma \in \widehat{\Delta}^+_{re}$: indeed $(\tilde{\alpha}, \tilde{\alpha} + 3\theta_\Sigma) < 0$ and $\|2\tilde{\alpha} + 3\theta_\Sigma\| > 0$.

3. $k\delta - 2\tilde{\alpha} - 3\theta_\Sigma \in \Delta_0^+$: relation $k\delta - 2\tilde{\alpha} - 3\theta_\Sigma \in \widehat{\Delta}$ follows from (2); it is also clear that it belongs to Δ_0. So it remains to show that it is positive. Indeed (1) implies $k\delta - \tilde{\alpha} - 3\theta_\Sigma \in \widehat{\Delta}$, and this root is positive since $c_{\tilde{\alpha}}(k\delta - \tilde{\alpha} - 3\theta_\Sigma) = 1$, hence $(k\delta - \tilde{\alpha} - 3\theta_\Sigma) - \tilde{\alpha} \in \widehat{\Delta}^+$.

Now we can conclude, since $(k\delta - 2\tilde{\alpha} - 3\theta_\Sigma)(x) \geq 0$ implies $\theta_\Sigma(x) \leq \frac{k}{3} - \frac{2}{3}(\tilde{\alpha}, x) \leq k$. □

2.2 The poset $\mathcal{I}_{\alpha,\mu}$ and its minimal elements

We now begin the study of maximal elements of $\mathcal{W}_\sigma^{ab}$. Set

$$\mathcal{M}_\sigma = \Phi_\sigma \backslash (\widehat{\Pi} \cap \Phi_\sigma). \tag{2.2.1}$$

The following proposition gives us a necessary condition for the maximality of an element.

Proposition 2.2.1. *If $w \in \mathcal{W}_\sigma^{ab}$ is maximal, then there is $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ such that $w(\alpha) = \mu$.*

Proof. By Proposition 2.1.2, we have that, if $\alpha \in \widehat{\Pi}$, $w(\alpha) \in \widehat{\Delta}^+$, then $ws_\alpha(C_1) \not\subset D_\sigma$, hence there exists $\mu \in \Phi_\sigma$ such that $ws_\alpha(C_1) \not\subset H_\mu^+$. It follows that $\mu \in N(ws_\alpha)$. Since $N(ws_\alpha) = N(w) \cup \{w(\alpha)\}$, we see that $w(\alpha) = \mu$. We need therefore to prove that there is a simple root α such that $w(\alpha) \in \widehat{\Delta}^+$ and $w(\alpha) \notin \Pi_0$.

Assume on the contrary that, if $\alpha \in \widehat{\Pi}$ and $w(\alpha) \in \widehat{\Delta}^+$, then $w(\alpha) \in \Pi_0$. Then, for all $\alpha \in \widehat{\Pi}$, $ht_\sigma(w(\alpha)) \leq 0$ and, hence, for all $\beta \in \widehat{\Delta}^+$, we have that $ht_\sigma(w(\beta)) \leq 0$. It follows that, for all $\beta \in \widehat{\Delta}^+$, if $w(\beta)$ is positive, then $w(\beta) \in \Delta_0$. Equivalently, $w(\widehat{\Delta}^+) \cap \widehat{\Delta}^+ \subseteq \Delta_0$. Hence, in particular, $w(\widehat{\Delta}^+) \backslash \widehat{\Delta}^+$ is infinite, but this is impossible, since $w(\widehat{\Delta}^+) \setminus \widehat{\Delta}^+ = -N(w)$. □

Remark 2.2.1. In the adjoint case observe that $\mathcal{M}_\sigma = \{-\theta + 2\delta\}$ and $D_\sigma = 2C_1$, hence Proposition 2.2.1 becomes geometrically evident.

The previous proposition suggests the following definition. Let $\alpha \in \widehat{\Pi}$, $\mu \in \mathcal{M}_\sigma$: we define

$$\mathcal{I}_{\alpha,\mu} = \{w \in \mathcal{W}_\sigma^{ab} \mid w(\alpha) = \mu\}.$$

In this section we find necessary and sufficient conditions for the poset $\mathcal{I}_{\alpha,\mu}$ to be nonempty, and in such a case we show that it has minimum.

2.2.1 The case $\mu = k\delta - \theta_\Sigma$, with $\Sigma|\Pi_0$

Definition 2.2.1. Let $\Sigma|\Pi_0$, and consider the subgraph of $\widehat{\Pi}$ with $\{\alpha \in \widehat{\Pi} \mid (\alpha, \theta_\Sigma) \leq 0\}$ as set of vertices. We call $A(\Sigma)$ the union of the connected components of this subgraph which contain at least one root of Π_1. Moreover, we set

$$\Gamma(\Sigma) = A(\Sigma) \cap \Sigma.$$

Remark 2.2.2. If $|\Pi_1| = 1$ then, obviously, $A(\Sigma)$ is connected. If $|\Pi_1| = 2$ then a brief inspection shows that there is only one case when $A(\Sigma)$ is disconnected, namely when $\widehat{\Pi}$ is of type $C_n^{(1)}$. Note that in such a case Π_0 is connected and θ_{Π_0} is a short root.

Example 2.2.1. We number affine Dynkin diagrams as in [9, Tables Aff1 and Aff2].

1. Let $\widehat{L}(\mathfrak{g},\sigma)$ be of type $B_n^{(1)}$ $(n \geq 5)$ and $\Pi_1 = \{\alpha_p\}$, $4 \leq p \leq n-1$. Then Π_0 has two components, say Σ_1, of type D_p, with simple roots $\{\alpha_i, |\ 0 \leq i \leq p-1\}$, and Σ_2 of type B_{n-p} and simple roots $\{\alpha_i, |\ p+1 \leq i \leq n\}$. We have $A(\Sigma_1) = \{\alpha_{p-1}, \dots, \alpha_n\}$, $\Gamma(\Sigma_1) = \{\alpha_{p-1}\}$, and $A(\Sigma_2) = \{\alpha_0, \dots, \alpha_{p+1}\}$, $\Gamma(\Sigma_2) = \{\alpha_{p+1}\}$.

2. Let $\widehat{L}(\mathfrak{g},\sigma)$ be of type $E_6^{(1)}$ and $\Pi_1 = \{\alpha_6\}$. Then Π_0 has two components: Σ_1, of type A_5, with simple roots $\{\alpha_1, \dots, \alpha_5\}$, and $\Sigma_2 = \{\alpha_0\}$, of type A_1. We have $A(\Sigma_1) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_0\}$, $\Gamma(\Sigma_1) = \{\alpha_2, \alpha_3, \alpha_4\}$ and $A(\Sigma_2) = \widehat{\Pi} \setminus \{\alpha_0\}$, $\Gamma(\Sigma_2) = \emptyset$.

3. Let $\widehat{L}(\mathfrak{g},\sigma)$ be of type $A_n^{(1)}$, $(n > 2)$, and $\Pi_1 = \{\alpha_0, \alpha_p\}$, $1 < p < n$. Then Π_0 has two components: Σ_1, of type A_{p-1}, with simple roots $\{\alpha_i, |\ 1 \leq i \leq p-1\}$, and Σ_2 of type A_{n-p} and simple roots $\{\alpha_i, |\ p+1 \leq i \leq n\}$. We have and $A(\Sigma_1) = \Sigma_2 \cup \Pi_1$, $A(\Sigma_2) = \Sigma_1 \cup \Pi_1$, and $\Gamma(\Sigma_i) = \emptyset$ for $i = 1, 2$.

Remark 2.2.3. Assume that $\Sigma|\Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \Pi_1$, and set

$$r_\Sigma = -(\alpha, \theta_\Sigma^\vee).$$

By Remark 2.1.1 (2), r_Σ is independent from the choice of $\alpha \in \Pi_1$. Moreover, we see that $r_\Sigma = 1$ if and only if θ_Σ is long and non complex while, in the remaining cases, since we are assuming that $k\delta - \theta_\Sigma \in \widehat{\Pi}_0^*$, we have that $r_\Sigma = 2$. If $r_\Sigma = 2$, then, for $\alpha \in \Pi_1$, either $\|\alpha\| = 2\|\theta_\Sigma\|$, or $\overline{\alpha} = -\overline{\theta}_\Sigma$. The latter instance occurs in the adjoint case, so that $k = 2$ and θ_Σ is long and complex. In the first case, θ_Σ is a short root, and k may be 1 or 2. In fact, $k = 2$ and θ_Σ is complex, except in the following two cases: $\mathfrak{g}$ is of type B_n, $\Pi_1 = \{\alpha_{n-1}\}$ and $\theta_\Sigma = \alpha_n$ or $\mathfrak{g}$ is of type C_n, $\Pi_1 = \{\alpha_0, \alpha_n\}$, $\Sigma = \{\alpha_1, \dots, \alpha_{n-1}\}$. (Dynkin diagrams are numbered as in [9, Tables Aff1 and Aff2]).

From now on we will distinguish roots in two types, according to the following definition.

Definition 2.2.2. We say that $\alpha \in \widehat{\Delta}^+_{re}$ is of *type 1* if it is long and non complex and of *type 2* otherwise.

By the above remark, if $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, its type is r_Σ.

Lemma 2.2.2. *Assume* $\Sigma|\Pi_0$ *and* $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$. *If* $\frac{k}{r_\Sigma} \in \mathbb{Z}$, *then* $A(\Sigma)$ *is connected,* $\frac{k}{r_\Sigma}\delta - \theta_\Sigma$ *is a root, and*

$$Supp\left(\frac{k}{r_\Sigma}\delta - \theta_\Sigma\right) \subseteq A(\Sigma). \tag{2.2.2}$$

Proof. Note that $\frac{k}{r_\Sigma} \in \mathbb{Z}$ if and only if $r_\Sigma = 1$ or $k = r_\Sigma = 2$, in any case $\frac{k}{r_\Sigma} \in \{1, 2\}$. If $\frac{k}{r_\Sigma} = 2$, then $\frac{k}{r_\Sigma}\delta - \theta_\Sigma \in \Delta$ and, if $\frac{k}{r_\Sigma} = 1$ then, either $k = 1$ or $k = 2$ and θ_Σ is complex. In both cases, $\frac{k}{r_\Sigma}\delta - \theta_\Sigma \in \Delta$.

We now prove that $Supp\,(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) \subset A(\Sigma)$. Note that $\Pi_1 \subset Supp\,(\frac{k}{r_\Sigma}\delta - \theta_\Sigma)$, hence we need only to prove that $\alpha \notin Supp\,(\frac{k}{r_\Sigma}\delta - \theta_\Sigma)$ for any $\alpha \in \Sigma$ such that

$(\alpha, \theta_\Sigma) > 0$. We next show that, for such an α, we have $c_\alpha(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) = 0$. We have:

$$2\frac{r_\Sigma}{k} = -\sum_{\beta \in \Pi_1} c_\beta(\delta)(\beta, \theta_\Sigma^\vee) = \sum_{\substack{\beta \in \Sigma \\ (\beta, \theta_\Sigma) > 0}} c_\beta(\delta)(\beta, \theta_\Sigma^\vee).$$

The first equality follows by the definition of r_Σ, and the second by the relation $(\delta, \theta_\Sigma) = 0$. If there is only one root $\alpha \in \Sigma$ such that $(\alpha, \theta_\Sigma) > 0$, we obtain that

$$\frac{k}{r_\Sigma} c_\alpha(\delta)(\alpha, \theta_\Sigma^\vee) = 2 = c_\alpha(\theta_\Sigma)(\alpha, \theta_\Sigma^\vee),$$

hence $c_\alpha(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) = 0$. If there is more than one root in Σ not orthogonal to θ_Σ then $\sum_{\alpha \in \Sigma, (\alpha, \theta_\Sigma) > 0} c_\alpha(\theta_\Sigma)(\alpha, \theta_\Sigma^\vee) = 2$, hence $(\alpha, \theta_\Sigma) = c_\alpha(\theta_\Sigma) = 1$ for all $\alpha \in \Sigma$ not orthogonal to θ_Σ.

Since $\frac{k}{r_\Sigma} \sum_{\alpha \in \Sigma, (\alpha, \theta_\Sigma) > 0} c_\alpha(\delta) = 2$, $\frac{k}{r_\Sigma} \in \mathbb{Z}$, and $c_\alpha(\delta) > 0$ for all $\alpha \in \widehat{\Pi}$, we obtain $\frac{k}{r_\Sigma} c_\alpha(\delta) = 1$ and again we have $c_\alpha(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) = 0$, as desired. □

Note that, if θ_Σ is of type 1 or $k = 2$, then $\frac{k}{r_\Sigma} \in \mathbb{Z}$. In particular $A(\Sigma)$ is connected.

Proposition 2.2.3. *Assume $\Sigma | \Pi_0$ and $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$. If θ_Σ is of type 1, then $k\delta - \theta_\Sigma$ is the highest root of $\Delta(A(\Sigma))$. If $k = 2$ and θ_Σ is of type 2, then $\delta - \theta_\Sigma$ is either the highest root of $\Delta(A(\Sigma))$, or its highest short root.*

Proof. Our assumptions imply in any case that $\frac{k}{r_\Sigma} \in \mathbb{Z}$. By (2.2.2) we have that $\frac{k}{r_\Sigma}\delta - \theta_\Sigma \in \Delta(A(\Sigma))$. By the definition of $A(\Sigma)$, $\frac{k}{r_\Sigma}\delta - \theta_\Sigma$ is a dominant root in $\Delta(A(\Sigma))$, therefore, since $\Delta(A(\Sigma))$ is a finite root system, we obtain that it is either the highest root of $\Delta(A(\Sigma))$ or its highest short root. If θ_Σ is of type 1, then it is a long root, so, since $r_\Sigma = 1$, $k\delta - \theta_\Sigma$ is the highest root of $\Delta(A(\Sigma))$. If θ_Σ is of type 2, then $r_\Sigma = 2$, hence $\frac{k}{r_\Sigma} = 1$. In this case, θ_Σ may be short or long, and $\delta - \theta_\Sigma$ is the highest short or long root of $\Delta(A(\Sigma))$, according to its length. □

Lemma 2.2.4. *Assume $\Sigma | \Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$ and θ_Σ of type 2. Let s be the element of minimal length in $\widehat{W}$ such that $s(\theta_\Sigma) = k\delta - \theta_\Sigma$. Then $s \in W(A(\Sigma))$ and is an involution. Moreover,*

$$N(s) = \{\beta \in \widehat{\Delta}_1^+ \mid (\beta, \theta_\Sigma^\vee) = -2\},$$

in particular, $s \in \mathcal{W}_\sigma^{ab}$.

Proof. First we assume $k = 2$. We claim that in this case $s = s_{\delta - \theta_\Sigma}$, which directly implies that it is an involution and, by Proposition 2.2.3, that it belongs to $W(A(\Sigma))$. It is immediate that $s_{\delta - \theta_\Sigma}(\theta_\Sigma) = 2\delta - \theta_\Sigma$. Moreover, for each $\alpha \in \widehat{\Delta}^+$ which is orthogonal to θ_Σ we have $s_{\delta - \theta_\Sigma}(\alpha) = \alpha \in \widehat{\Delta}^+$, therefore s is the unique element of minimal length that maps $\delta - \theta_\Sigma$ to $2\delta - \theta_\Sigma$. We study $N(s)$. For each $\beta \in \Delta^+(A(\Sigma))$,

$$s(\beta) = \beta + (\beta, \theta_\Sigma^\vee)(\delta - \theta_\Sigma)$$

hence $s(\beta) < 0$ if and only if $(\beta, \theta_\Sigma^\vee) < 0$. Thus if $(\beta, \theta_\Sigma^\vee) = -2$, then $\beta \in N(s)$. It remains to prove the converse. Assume $s(\beta) < 0$, hence $(\beta, \theta_\Sigma^\vee) < 0$: since $(\alpha, \theta_\Sigma) \geq 0$

for all $\alpha \in \widehat{\Pi} \setminus \Pi_1$, this implies that $ht_\sigma(\beta) \geq 1$. Now we observe that, if $\beta \in N(s)$, then also $-s(\beta) \in N(s)$, therefore $ht_\sigma(-s(\beta)) \geq 1$ as well. Since

$$ht_\sigma(s(\beta)) = ht_\sigma(\beta) + (\beta, \theta_\Sigma^\vee) ht_\sigma(\delta - \theta_\Sigma) = ht_\sigma(\beta) + (\beta, \theta_\Sigma^\vee),$$

we obtain that $-(\beta, \theta_\Sigma^\vee) = ht_\sigma(\beta) + ht_\sigma(-s(\beta)) \geq 2$. But $k\delta - \theta_\Sigma$ belongs to $\mathcal{M}_\sigma \subset \Pi_0^*$, therefore, by (2.1.1), we have $-(\beta, \theta_\Sigma^\vee) \leq \frac{2\|\beta\|}{\|\theta_\Sigma\|} \leq 2\sqrt{2}$, so we can conclude that $-(\beta, \theta_\Sigma^\vee) = 2$ and $ht_\sigma(\beta) = ht_\sigma(-s(\beta)) = 1$.

Now we assume $k = 1$. By Remark 2.2.3, then either $\mathfrak{g}$ is of type B_n, $\Pi_1 = \{\alpha_{n-1}\}$ and $\theta_\Sigma = \alpha_n$, or $\mathfrak{g}$ is of type C_n, $\Pi_1 = \{\alpha_0, \alpha_n\}$, $\Sigma = \{\alpha_1, \dots, \alpha_{n-1}\}$. In the first case a straightforward check shows that $s = s_{n-1} \cdots s_2 s_0 s_1 s_2 \cdots s_{n-1} = s_{\alpha_0+\alpha_2+\ldots+\alpha_{n-1}} s_{\alpha_1+\alpha_2+\ldots+\alpha_{n-1}}$ maps α_n to $\delta - \alpha_n$, α_{n-1} to $\alpha_{n-1} + 2\alpha_n - \delta$, fixes α_i, $i = 2, \dots, n-2$ and switches α_0 and α_1. A positive root γ is orthogonal to α_n if and only if $c_{\alpha_{n-1}}(\gamma) = c_{\alpha_n}(\gamma)$. Therefore s keeps positive any positive root orthogonal to α_n, as required. It is clear that s is an involution, being conjugated to $s_0 s_1$. A direct computation shows that $N(s) = \{\beta \in \Delta^+(\widehat{\Pi} \setminus \{\alpha_n\}) \mid c_{\alpha_{n-1}}(\beta) = 1\} = \{\beta \in \widehat{\Delta}_1^+ \mid (\beta, \theta_\Sigma^\vee) = -2\}$.

For $\mathfrak{g}$ of type C_n, $s = s_0 s_n$ maps $\theta_\Sigma = \alpha_1 + \cdots + \alpha_{n-1}$ to $\delta - \theta_\Sigma = \alpha_0 + \cdots + \alpha_n$. Moreover, a root in $\widehat{\Delta}^+$ is orthogonal to θ_Σ if and only if it is of the form $A \cup (\mathbb{N}\delta \pm A)$ where A is formed by the roots in the subsystem generated by $\alpha_2, \dots, \alpha_{n-2}$ and by the roots $2\alpha_i + \ldots + 2\alpha_{n-1} + \alpha_n, 2 \leq i \leq n-1$ and $\alpha_1 + \ldots + \alpha_n$. A direct check shows that these roots are kept positive by s, which is therefore minimal. It is immediate to see that $N(s) = \{\alpha_0, \alpha_n\} = \{\beta \in \widehat{\Delta}_1^+ \mid (\beta, \theta_S^\vee) = -2\}$. □

Lemma 2.2.5. *Assume* $\Sigma | \Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, *and* $\|\alpha\| = \|\theta_\Sigma\|$.

1. *If* θ_Σ *is of type 1,* $\alpha \in A(\Sigma)$, *and* w_α *is the element of minimal length such that* $w_\alpha(\alpha) = k\delta - \theta_\Sigma$, *then* $w_\alpha \in \mathcal{W}_\sigma^{ab}$.

2. *If* θ_Σ *is of type 2,* $\alpha \in \Sigma$, v_α *is the element of minimal length in* $W(\Sigma)$ *such that* $v_\alpha(\alpha) = \theta_\Sigma$, *and* s *is the element of minimal length in* $\widehat{W}$ *such that* $s(\theta_\Sigma) = k\delta - \theta_\Sigma$, *then* $sv_\alpha \in \mathcal{W}_\sigma^{ab}$. *Moreover,* $\ell(sv_\alpha) = \ell(s) + \ell(v_\alpha)$ *and* sv_α *is the element of minimal length in* $\widehat{W}$ *that maps* α *to* $k\delta - \theta_\Sigma$.

Proof.

1. By Proposition 2.2.3 (1) and Proposition 1.3.4 (5), if $\beta \in N(w_\alpha)$, then there exists $\beta' \in \widehat{\Delta}^+$ such that $\beta + \beta' = k\delta - \theta_\Sigma$. By Remark 2.0.1, each root less than μ in the usual root order has σ-height strictly less than 2, hence $ht_\sigma(\beta) = ht_\sigma(\beta') = 1$.

2. Assume first $k = 2$, so that $s = s_{\delta - \theta_\Sigma}$. We first show that $s_{\delta-\theta_\Sigma}(\beta) = \beta + \delta - \theta_\Sigma$ for each $\beta \in N(v_\alpha)$. This amounts to prove that $(\theta_\Sigma^\vee, \beta) = 1$ for each $\beta \in N(v_\alpha)$, which follows again from Proposition 1.3.4, (2). Thus we obtain that the σ-height of the roots in $s_{\delta-\theta_\Sigma}(N(v_\alpha))$ is 1; moreover,

$$N(sv_\alpha) = N(s_{\delta-\theta_\Sigma}) \cup s_{\delta-\theta_\Sigma}(N(v_\alpha))$$

and $\ell(sv_\alpha) = \ell(s) + \ell(v_\alpha)$. Since by Lemma 2.2.4, for each $\beta \in N(s)$, $ht_\sigma(\beta) = 1$, we conclude that $sv_\alpha \in \mathcal{W}_\sigma^{ab}$. It remains to prove the assertion about the minimal length. Notice that the above considerations show in particular that, for each $\beta \in N(sv_\alpha)$, we have that $(\beta, k\delta - \theta_\Sigma) \neq 0$. By subsection 1.3.3, it follows that sv_α is the unique element of minimal length that maps α to $k\delta - \theta_\Sigma$.

In the case of B_n, one has $N(sv_\alpha) = N(s) = \{\beta \in \Delta^+(\widehat{\Pi} \setminus \{\alpha_n\}) \mid c_{\alpha_{n-1}}(\beta) = 1\}$. This follows noting that $L(s) = \{\alpha_{n-1}\}, \ell(s) = 2n-2 = |\Delta^+(\widehat{\Pi} \setminus \{\alpha_n\})| - |\Delta^+(\widehat{\Pi} \setminus \{\alpha_{n-1}, \alpha_n\})|$.

In the case C_n, we first remark that $sv_{\alpha_i} = s_0 \cdots s_{i-1} s_n \cdots s_{i+1}$, $1 \leq i \leq n-1$. Thus,

$$\begin{aligned} N(sv_{\alpha_i}) &= N(s_0 \cdots s_{i-1} s_n \cdots s_{i+1}) \\ &= \{\alpha_0 + \ldots + \alpha_k \mid 0 \leq k \leq i-1\} \cup \{\alpha_h + \ldots + \alpha_n \mid i+1 \leq h \leq n\}, \end{aligned}$$

whose elements have clearly σ-height 1. The same argument used in case $k = 2$ proves that also in this case sv_α is the unique element of minimal length that maps α to $k\delta - \theta_\Sigma$. □

Lemma 2.2.6. *Assume $\mu \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, and $w \in \mathcal{I}_{\alpha,\mu}$. Then*

1. *for each $\beta \in N(w)$, $\mu + \beta \notin N(w)$;*
2. *for each $\beta, \beta' \in \widehat{\Delta}^+$ such that $\beta + \beta' = \mu$, exactly one of β, β' belongs to $N(w)$.*

Proof.

1. We have

$$N(ws_\alpha) = N(w) \cup \{\mu\}. \tag{2.2.3}$$

If, for some $\beta \in N(w)$, $\beta + \mu \in \widehat{\Delta}^+$, then by the convexity properties, we would obtain $\beta + \mu \in N(w)$: this cannot happen since $ht_\sigma(\beta + \mu) \geq 3$, while w is σ-minuscule.

2. By the convexity properties, relation (2.2.3) implies that $N(ws_\alpha)$ contains at least one summand of each decomposition $\mu = \beta + \beta'$, hence $N(w)$ does. Since $\mu \notin N(w)$, it contains exactly one summand. □

Lemma 2.2.7. *Assume $\mu = k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, and $w \in \mathcal{I}_{\alpha,\mu}$. Then there exists $u \in \widehat{W}$ such that*

$$\{\beta \in N(w) \mid \mu - \beta \in \widehat{\Delta}^+\} = N(u).$$

Moreover, u belongs to $\mathcal{I}_{\alpha,\mu}$ and it is its minimal element.

Proof. Set $U = \{\beta \in N(w) \mid \mu - \beta \in \widehat{\Delta}^+\}$. We first prove the existence of u: we have only to check that U is biconvex. We observe that, if $\beta, \beta' \in U$, then $\beta + \beta'$ is not a root, otherwise it would belong to $N(w)$, which impossible since $ht_\sigma(\beta + \beta') = 2$ and w is σ-minuscule. Thus we have only to check that, if $\beta \in U$ and $\beta = \gamma + \gamma'$, then at least one of γ, γ' belongs to U. Clearly, at least (in fact exactly) one of γ, γ', say γ, belongs to $N(w)$. We have to prove that $\mu - \gamma$ is a positive root. Set $\beta' = \mu - \beta$: by definition, β' is a positive root and it is immediate that $ht_\sigma(\beta') = 1$. Since $\gamma + \gamma' + \beta' = \mu$, at least one of $\gamma + \beta'$, $\gamma' + \beta'$, is a root, otherwise, by the Jacobi identity, $\gamma + \gamma' + \beta'$ would not be a root. But $\gamma + \beta'$ cannot be a root, otherwise it would have σ-height equal to 2, while being less than μ. Therefore $\mu - \gamma = \gamma' + \beta'$ is a root, as required.

It remains to prove that $u \in \mathcal{I}_{\alpha,\mu}$. It is clear that $u \in \mathcal{W}_\sigma^{ab}$, we have only to check that $u(\alpha) = \mu$. By Lemma 2.2.6 (2), $N(w)$ contains exactly one summand of any decomposition of μ as a sum of two positive roots and, by the definition of u, $N(u)$

has the same property. From this fact, we easily deduce that $N(u) \cup \{\mu\}$ is biconvex, hence that there exist a simple root $\beta \in \widehat{\Pi}$ such that $N(us_\beta) = N(u) \cup \{\mu\}$. But $N(us_\beta) = N(u) \cup \{u(\beta)\}$, hence $u(\beta) = \mu$. We must prove that $\beta = \alpha$. Since $u \leq w$, there exists $z \in \widehat{W}$ such that $w = uz$ and $N(w) = N(u) \cup uN(z)$. If $\beta \neq \alpha$, since $w(\beta) = uz(\beta) \neq \mu$, we obtain that $z(\beta) \neq \beta$, hence, by formula (1.3.5), that $N(z)$ contains at least one root γ such that $\gamma \not\perp \beta$. Then $u(\gamma) \not\perp \mu$ and $u(\gamma) \in N(w) \setminus N(u)$: we show that this is a contradiction. In fact, $u(\gamma) \not\perp \mu$ implies that either $\mu + u(\gamma)$ or $\mu - u(\gamma)$ is a positive root: the first instance is impossible by Lemma 2.2.6 (1); the second one is impossible because it would imply that $u(\gamma) \in N(u)$. □

In Lemma 2.2.5 we have constructed elements w_α and sw_α belonging to $\mathcal{I}_{\alpha,\mu}$, under certain restrictions on α and μ, so proving in particular that, under such restrictions, $\mathcal{I}_{\alpha,\mu}$ is not empty. It is quite clear that, by the the minimality assertions of both Lemmas 2.2.5 and 2.2.7, the element u built in Lemma 2.2.7 must be equal to w_α or sw_α, when these are defined. In the next proposition we prove that u is in any case equal to some w_α or sw_α, with the restriction on α stated il Lemma 2.2.5. We have therefore determined necessary and sufficient conditions under which $\mathcal{I}_{\alpha,\mu}$ is not empty.

Proposition 2.2.8. *Assume $\Sigma|\Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, and $w \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$.*

1. *If θ_Σ is of type 1, then $\alpha \in A(\Sigma)$ and $w \geq w_\alpha$;*
2. *If θ_Σ is of type 2, then $\alpha \in \Sigma$ and $w \geq sv_\alpha$.*

Proof.

1. Set $\mu = k\delta - \theta_\Sigma$ and consider the element u built in Lemma 2.2.7. By Lemma 2.2.6 (2) and by the definition of u, $N(u)$ contains exactly one summand of any decomposition of μ as a sum of two positive roots, and each element of $N(u)$ is one of the summands of such a decomposition. By Proposition 2.2.3 and by Proposition 1.3.4 (6), there exists a simple root $\beta \in A(\Sigma)$ such that $u(\beta) = \mu$, and u is the minimal length element with this property. But $u(\alpha) = \mu$, hence $\alpha = \beta \in A(\Sigma)$, and $u = w_\alpha$.

2. As above, we set $\mu = k\delta - \theta_\Sigma$ and consider the element u built in Lemma 2.2.7. We claim that in this case $\alpha \in \Sigma$ and $u = sv_\alpha$, which clearly implies the thesis.

We start proving that $s < u$, which, by Lemma 2.2.4, consists in proving that all $\beta \in \widehat{\Delta}_1^+$ such that $(\beta, \mu^\vee) = 2$ belong to $N(u)$. Assume $\beta \in \widehat{\Delta}_1^+$ and $(\beta, \mu^\vee) = 2$: this imply that $\mu - \beta$ and $2\mu - \beta$ are roots, and positive, having positive σ-height. By Lemma 2.2.6 (1), $\mu - \beta \notin N(u)$, since $\mu - \beta + \mu$ is a root, hence $\beta \in N(u)$. So $s < u$, i.e. there exists $v \in \widehat{W}$ such that $u = sv$ and $N(u) = N(s) \cup sN(v)$. It remains to prove that $\alpha \in \Sigma$ and $v = v_\alpha$.

First, we prove that for all $\beta \in N(u)$, we have that $(\beta, \mu^\vee) > 0$. Assume by contradiction that $\beta \in N(u)$ and $(\beta, \mu^\vee) = 0$, and set $\beta' = \mu - \beta$. Then $ht_\sigma(\beta') = 1$ and $(\beta', \mu^\vee) = 2$: by the previous part, this implies $\beta' \in N(u)$, which is impossible. Therefore we have $(\beta, \mu^\vee) > 0$, hence $(\beta, \mu^\vee) \in \{1, 2\}$, since $\mu \in \mathcal{M}_\sigma$. It follows that $sN(v) \subseteq \{\beta \in \widehat{\Delta}_1^+ \mid (\beta, \mu^\vee) = 1\}$.

Now, we claim that $N(v) \subseteq \Delta(\Sigma)$ and that, for each $\beta \in \Delta^+(\Sigma)$ such that $\theta_\Sigma - \beta$ is positive, exactly one amog β and $\theta_\Sigma - \beta$ belongs to $N(v)$. Assume $\beta \in N(v)$ and set $\beta' = s(\beta)$. Then $ht_\sigma(\beta') = 1$ and $(\beta', \mu^\vee) = 1$, so that $\mu - \beta'$ is a positive root, $(\mu-\beta',\mu^\vee)=1$, and $ht_\sigma(\mu-\beta')=1$. By the explicit description of $N(s)$, $\theta_\Sigma-\beta=s(\mu-\beta')$ is positive, hence $\beta\in\Delta(\Sigma)$. Now let $\beta\in\Delta(\Sigma)^+$ be such that $\theta_\Sigma-\beta\in\Delta(\Sigma)^+$ and set $\beta'=s(\beta)$. Then, θ_Σ being long with respect to $\Delta(\Sigma)$, we obtain that $(\beta,\theta_\Sigma^\vee)=(\theta_\Sigma-\beta,\theta_\Sigma^\vee)=1$, hence $(\beta',\mu^\vee)=(\mu-\beta',\mu^\vee)=1$. Moreover, by the explict description of $N(s)$, both β' and $\mu-\beta'$ are positive, therefore both have σ-height equal to 1. By Lemma 2.2.6 (2), it follows that exactly one among them belongs to $N(u)$, hence to $sN(v)$, therefore, exactly one among β and $\theta_\Sigma-\beta$ belongs to $N(v)$. Thus v has the property of Proposition 1.3.4 (6), whence there exists $\beta\in\Sigma$ such that $v=v_\beta$. But $sv_\beta(\beta)=\mu$, hence $\beta=\alpha$. □

2.2.2 The case $\mu=k\delta+\beta$, with $\beta\in\Pi_1$

According to our definitions, (2.1.2) and (2.2.1), the assumption that $\beta+k\delta\in\mathcal{M}_\sigma$ implies that β is long.

We start refining the analysis done in [1, Lemma 5.10].

Proposition 2.2.9. *If $\mathfrak{g}_0$ is semisimple, then $\mathfrak{g}_1$ is irreducible as a $\mathfrak{g}_0$-module. If $\mathfrak{g}_0$ is not semisimple, then $\mathfrak{g}_1$ has two irreducible components as a $\mathfrak{g}_0$-module. As a consequence,*

1. *if $\Pi_1=\{\alpha\}$ and w_0 is the longest element of W_0, then $w_0(\alpha)=k\delta-\alpha$;*
2. *if $\Pi_1=\{\alpha,\beta\}$, then $w_0(\alpha)=\delta-\beta$ and $w_0(\beta)=\delta-\alpha$.*

Proof. As shown in the proof of [1, Lemma 5.10], $t^{-1}\otimes\mathfrak{g}_1$ occurs as a submodule of the homology $H_1(\mathfrak{u}^-)$ where $\mathfrak{u}^-=\sum_{\alpha\in(-\widehat{\Delta}^+)\setminus\Delta_0}\widehat{L}(\mathfrak{g},\sigma)_\alpha$. By Garland-Lepowsky theorem, this homology decomposes as $\oplus_{\alpha\in\Pi_1}V(-\alpha)$, as a sum of irreducible $(\mathfrak{g}_0+\mathbb{C}K+\mathbb{C}d)$-modules, which stay irreducible as $\mathfrak{g}_0$-modules. It follows that

$$t^{-1}\otimes\mathfrak{g}_1=V(-\alpha),\ \text{if }\Pi_1=\{\alpha\}. \tag{2.2.4}$$

Moreover, it is clear that $-\alpha$ occurs as a highest weights of $t^{-1}\otimes\mathfrak{g}_1$, for any $\alpha\in\Pi_1$, hence,

$$t^{-1}\otimes\mathfrak{g}_1=V(-\alpha)\oplus V(-\beta),\ \text{if }\Pi_1=\{\alpha,\beta\}\text{ with }\alpha\neq\beta. \tag{2.2.5}$$

Since $\mathfrak{g}_1$ is self-dual as a $\mathfrak{g}_0$-module, if $\Pi_1=\{\alpha\}$ we obtain that $w_0(\bar\alpha)=-\bar\alpha$, hence $w_0(\alpha)=w_0(\delta'+\bar\alpha)=\delta'-\bar\alpha=2\delta'-\alpha=k\delta-\alpha$ (cfr. Section 1.2.1), as claimed.

Assume $\Pi_1=\{\alpha,\beta\}$. Notice that in this case $k=1$, so that $\delta=2\delta'$ and that $c_\alpha(\delta)=c_\beta(\delta)=1$ (see Section 1.2.1). We have two cases:

1. $V(-\alpha)^*=V(-\alpha)$ and $V(-\beta)^*=V(-\beta)$,
2. $V(-\alpha)^*=V(-\beta)$ and $V(-\beta)^*=V(-\alpha)$.

In the first case we have $w_0(\bar\alpha)=-\bar\alpha$, which forces $w_0(\alpha)=w_0(\delta'+\bar\alpha)=\delta'-\bar\alpha=\delta-\alpha$ and this is not possible since $c_\alpha(w_0(\alpha))=c_\alpha(\alpha)=1$, while $c_\alpha(\delta-\alpha)=0$. Hence (2) holds. It follows that $w_0(\bar\alpha)=-\bar\beta$ and $w_0(\bar\beta)=-\bar\alpha$. Therefore, $w_0(\alpha)=\delta'-\bar\beta=2\delta'-\beta=\delta-\beta$ and $w_0(\beta)=\delta'-\bar\alpha=2\delta'-\alpha=\delta-\alpha$. □

Proposition 2.2.10. *Assume $\mu=\alpha+k\delta\in\mathcal{M}_\sigma$, with $\alpha\in\Pi_1$. Set $\Pi_{0,\alpha}=\Pi_0\cap\alpha^\perp$, $W_{0,\alpha}=W(\Pi_{0,\alpha})$, and denote by $w_{0,\alpha}$ the longest element of $W_{0,\alpha}$.*

1. *If* $\Pi_1 = \{\alpha\}$*, then* $\mathcal{I}_{\gamma,\mu} \neq \emptyset$ *if and only if* $\gamma = \alpha$*. Moreover,*
$$\mathcal{I}_{\alpha,\mu} = \{s_\alpha w_{0,\alpha} w_0\}.$$

2. *If* $\Pi_1 = \{\alpha, \beta\}$*, then* $\mathcal{I}_{\gamma,\alpha+k\delta} \neq \emptyset$ *if and only if* $\gamma = \beta$*. Moreover,*
$$\min \mathcal{I}_{\beta,\alpha+k\delta} = s_\alpha w_{0,\alpha} w_0.$$

Proof. Set $x = s_\alpha w_{0,\alpha} w_0$. By Proposition 2.2.9, we have that:

1. if $\Pi_1 = \{\alpha\}$, then $x(\alpha) = s_\alpha w_{0,\alpha} w_0(\alpha) = s_\alpha w_{0,\alpha}(k\delta - \alpha) = s_\alpha(k\delta - \alpha) = k\delta + \alpha$;
2. if $\Pi_1 = \{\alpha, \beta\}$, then $x(\beta) = s_\alpha w_{0,\alpha} w_0(\beta) = s_\alpha w_{0,\alpha}(k\delta - \alpha) = s_\alpha(k\delta - \alpha) = k\delta + \alpha$.

We prove that x is σ-minuscule. Since $w_{0,\alpha} w_0 \in W_0$, it is clear that $N(w_{0,\alpha} w_0) \subseteq \Delta_0^+$. In fact, we have $N(w_{0,\alpha} w_0) = \Delta_0^+ \setminus \Delta(\Pi_{0,\alpha})$. Since α is long, for each $\gamma \in \Delta_0^+ \setminus \Delta(\Pi_{0,\alpha})$, we have $s_\alpha(\gamma) = \gamma + \alpha$, hence $N(x) = \{\alpha\} \cup s_\alpha N(w_{0,\alpha} w_0) \subseteq \widehat{\Delta}_1$, as claimed.

So we have proved that $x \in \mathcal{I}_{\alpha,\alpha+k\delta}$, if $\Pi_1 = \{\alpha\}$, and $x \in \mathcal{I}_{\beta,\alpha+k\delta}$, if $\Pi_1 = \{\alpha, \beta\}$.

Now we treat separately the two cases. First, let $\Pi_1 = \{\alpha\}$ and assume that $w \in \mathcal{I}_{\gamma,\alpha+k\delta}$, with $\gamma \in \widehat{\Pi}$. Then $N(ws_\gamma) = N(w) \cup \{\alpha + k\delta\}$, hence, since w is σ-minuscule,
$$w(C_1) \subseteq \bigcap_{\eta \in \widehat{\Pi}_0} H_\eta^+ \setminus \bigcap_{\eta \in \Phi_\sigma} H_\eta^+ = P_\sigma \setminus D_\sigma,$$
where we denote by P_σ the polytope $\bigcap_{\eta \in \widehat{\Pi}_0} H_\eta^+$. But by [1, Lemma 5.11], there is exactly one $w \in \widehat{W}$ such that $w(C_1) \subseteq P_\sigma \setminus D_\sigma$, hence $w = x$, $\gamma = \alpha$, and $\mathcal{I}_{\alpha,\alpha+k\delta} = \{x\}$.

Now we assume $\Pi_1 = \{\alpha, \beta\}$, $\gamma \in \widehat{\Pi}$ and $w \in \mathcal{I}_{\gamma,\delta+\alpha}$. We will show that $\gamma = \beta$ and $x \leq w$. By Remark 2.1.1 (2) both roots in Π_1 are long; moreover, $\delta - \alpha$ is the highest root of $\Delta(\widehat{\Pi} \setminus \{\alpha\})$. For any $\gamma \in \widehat{\Pi} \setminus \{\alpha\}$, let v_γ be the element of minimal length that maps γ to $\delta - \alpha$. We start proving that $w_{0,\alpha} w_0 = v_\beta$. In fact, it is clear that $w_{0,\alpha} w_0(\beta) = \delta - \alpha$, so it suffices then to check that $(w_{0,\alpha} w_0)^{-1}(\gamma) > 0$ for all $\gamma \in \widehat{\Pi}$ such that $(\alpha, \gamma) = 0$. If $\gamma \in \Pi_{0,\alpha}$ then $(w_{0,\alpha} w_0)^{-1}(\gamma) = w_0 w_{0,\alpha}(\gamma) > 0$. Moreover, in any case $(w_{0,\alpha} w_0)^{-1}(\beta) > 0$, since $N(w_{0,\alpha} w_0) \subset \Delta_0$. Thus we obtain $w_{0,\alpha} w_0 = v_\beta$, $x = s_\alpha v_\beta$, and $N(x) = \{\alpha\} \cup s_\alpha(N(v_\beta))$.

Now we consider w. Since $w(\gamma) = \delta + \alpha$, we have $w^{-1}(\alpha) = -\delta + \gamma$ hence $\alpha \in N(w)$. It follows that $w = s_\alpha z$ with $\ell(w) = 1 + \ell(z)$. In particular, $N(w) = \{\alpha\} \cup s_\alpha(N(z))$. Since $z(\gamma) = \delta - \alpha$, we have that $N(zs_\gamma) = N(z) \cup \{\delta - \alpha\}$, so the biconvexity of $N(zs_\gamma)$ implies that for any pair $\eta_1, \eta_2 \in \widehat{\Delta}^+$ such that $\eta_1 + \eta_2 = \delta - \alpha$ exactly one of η_1, η_2 belong to $N(z)$. Moreover, $\delta - \alpha$ being a long root, for any such pair of roots we have $(\eta_i, (\delta - \alpha)^\vee) = 1$, for $i = 1, 2$, since $(\eta_1 + \eta_2, (\delta - \alpha)^\vee) = 2$ and $(\eta_i, (\delta - \alpha)^\vee) \leq 1$, for $i = 1, 2$. It follows that $s_\alpha(\eta_i) = \eta_i + \alpha$ and therefore, that $ht_\sigma(s_\alpha(\eta_i)) = ht_\sigma(\eta_i) + 1$, for $i = 1, 2$. Now, if $\eta_i \in N(z)$, $s_\alpha(\eta_i) \in N(w)$, and we obtain that $ht_\sigma(\eta_i) = 0$. But $ht_\sigma(\delta - \alpha) = 1$, so that one of the η_i has σ-height equal to 1 and the other has σ-height equal to 0. This implies that for any pair $\eta_1, \eta_2 \in \widehat{\Delta}^+$ such that $\eta_1 + \eta_2 = \delta - \alpha$, $N(z)$ contains exactly the summand η_i having σ-height equal to 0. This must hold in particular when we take $w = x$ and so $z = v_\beta$. In this case we clearly obtain that $N(v_\beta)$ is exactly the set of the summands of σ-height equal to 0 of all the decomposition of $\delta-\alpha$ as a sum of two positive roots. So, for a general w, we obtain that $N(v_\beta)\subseteq N(z)$, whence $w_{\beta,\alpha+\delta}=s_\alpha v_\beta\le s_\alpha z=w$ as desired.

It remains to prove that $\gamma=\beta$. We have $z=v_\beta y$ with $N(z)=N(v_\beta)\cup v_\beta N(y)$, and $y(\gamma)=\beta$. If $\gamma\neq\beta$, then $N(y)$ would contain some roots not orthogonal to β, whence $v_\beta N(y)$ contains some root η not orthogonal to $\delta-\alpha$, hence to α. It follows that $s_\alpha(\eta)=\eta\pm\alpha\in N(w)$. But $\eta-\alpha\notin N(w)$, being summable to α that belongs to $N(w)$, hence $s_\alpha(\eta)=\eta+\alpha\in N(w)$. In particular, $ht_\sigma(\eta)=0$, and $\delta-\alpha-\eta\in\widehat{\Delta}^+$: this implies that $\eta\in N(v_\beta)$, a contradiction. □

We sum up the results we have obtained in the following theorem. If S is a connected subset of the set of simple roots, we denote by $S_{\overline{\ell}}$ the set of elements of S of the same length of θ_S. It is clear that, with respect to $\Delta(S)$, θ_S is a long root, therefore $S_{\overline{\ell}}$, is the set of the long roots of S, with respect to the subsystem $\Delta(S)$. With notation as in Lemma 2.2.5 and Proposition 2.2.10, we set

$$w_{\alpha,\mu}=\begin{cases} w_\alpha & \text{if } \mu=k\delta-\theta_\Sigma,\ \theta_\Sigma \text{ is of type 1, and } \alpha\in A(\Sigma)_{\overline{\ell}}\\ sv_\alpha & \text{if } \mu=k\delta-\theta_\Sigma,\ \theta_\Sigma \text{ is of type 2, and } \alpha\in \Sigma_{\overline{\ell}}\\ s_\beta w_{0,\beta}w_0 & \text{if } \mu=\beta+k\delta,\ \beta\in\Pi_1\end{cases}\tag{2.2.6}$$

and

$$\widehat{\Pi}_\mu=\begin{cases} A(\Sigma)_{\overline{\ell}} & \text{if } \mu=k\delta-\theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 1}\\ \Sigma_{\overline{\ell}} & \text{if } \mu=k\delta-\theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 2}\\ \Pi_1 & \text{if } \mu=\beta+k\delta \text{ and } \{\beta\}=\Pi_1\\ \Pi_1\setminus\{\beta\} & \text{if } \mu=\beta+k\delta,\ \beta\in\Pi_1, \text{ and } |\Pi_1|=2\end{cases}\tag{2.2.7}$$

Theorem 2.2.11. *Assume $\mu\in\mathcal{M}_\sigma$ and $\alpha\in\widehat{\Pi}$. Then $\mathcal{I}_{\alpha,\mu}\neq\emptyset$ if and only if $\alpha\in\widehat{\Pi}_\mu$. Moreover,*

$$w_{\alpha,\mu}=\min\,\mathcal{I}_{\alpha,\mu}.$$

Proof. The claim follows directly from Lemma 2.2.5, Proposition 2.2.8, and Proposition 2.2.10. □

2.2.3 Reduced expressions of minimal elements

The aim of this section is to find reduced expressions for the minimal elements $w_{\alpha,\mu}$ defined in (2.2.6). To make this, we use a decomposition formula for $k\delta$ (see Corollary 2.2.16).

We start from the following formula, which is a variation of e.g. [9, Exercise 3.12].

Lemma 2.2.12. *Let $w\in W,\mu,\gamma\in\Delta$, $w=s_{i_1}\cdots s_{i_k}$ a reduced expression and $\beta_j=s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j=1,\dots,k$. Then*

$$w^{-1}(\gamma)=\gamma-\sum_{j=1}^{k}\frac{2(\beta_j,\gamma)}{|\alpha_{i_j}|^2}\alpha_{i_j}$$

or equivalently

$$w(\mu) = \mu + \sum_{j=1}^{k} \frac{2(w^{-1}(\beta_j), \mu)}{|\alpha_{i_j}|^2} \alpha_{i_j}$$

Proof. We proceed by induction on $\ell(w)$. We have

$$\begin{aligned} w^{-1}(\gamma) &= s_{i_k} \cdots s_{i_1}(\gamma) = s_{i_{k-1}} \cdots s_{i_1}(\gamma) - (s_{i_{k-1}} \cdots s_{i_1}(\gamma), \alpha_{i_k})\, \alpha_{i_k}^{\vee} = \\ &= s_{i_{k-1}} \cdots s_{i_1}(\gamma) - (\gamma, \beta_k)\, \alpha_{i_k}^{\vee} \end{aligned}$$

The first claim follows by inductive hypothesis. The second claim follows from the first setting $\mu = w(\gamma)$. □

Now we generalize Definition 2.2.1.

Definition 2.2.1. Let Σ be a component of Π_0 and set $\Gamma_0(\Sigma) = \Pi_\Sigma, \theta_0(\Sigma) = \theta_\Sigma$. We define recursively $\Gamma_i(\Sigma) \subset \Gamma_{i-1}(\Sigma)$ as the the component not orthogonal to Π_1 in $\{\gamma \in \Gamma_{i-1}(\Sigma) \mid (\gamma, \theta_{\Gamma_{i-1}(\Sigma)}) = 0\}$ (if it exists), and we denote by $\theta_i(\Sigma)$ the highest root $\theta_{\Gamma_i(\Sigma)}$ of $\Gamma_i(\Sigma)$.

Remark 2.2.4. Clearly $\Gamma_1(\Sigma) = \Gamma(\Sigma)$.

We need some technical lemmas on the sets Γ_i and their highest roots. For the sake of simplicity we fix a component Σ of Π_0 and we omit Σ in $\theta_i(\Sigma)$ and $\Gamma_i(\Sigma)$. Recall that α_Σ is the unique simple root in Π_Σ not orthogonal to $\alpha \in \Pi_1$.

Definition 2.2.3. For any component Σ we define the elements $\delta_\Sigma \in \widehat{\mathfrak{h}}^*$ as

$$k\delta = \sum_{\alpha \in \Pi_1} k c_\alpha(\delta)\alpha + \sum_{\Sigma|\Pi_0} \delta_\Sigma.$$

Lemma 2.2.13. *Let w_Σ be the maximum of $W(\Pi_\Sigma)$. We have*

$$w_\Sigma(x) = -x, \ \forall x \in \Gamma_i.$$

In particular $w_\Sigma(\theta_i) = -\theta_i$.

Proof. First we observe that for every $\alpha \in \Pi_\Sigma$ we have $(\alpha, \theta_\Sigma) = 0$ if and only if $(w_\Sigma(\alpha), \theta_\Sigma) = 0$. In the semisimple case, by Proposition 2.2.9, we have $w_\Sigma(\alpha) = w_\Sigma^{-1}(\alpha) = \alpha + \delta_\Sigma$, therefore $(\alpha_\Sigma, w_\Sigma^{-1}(\alpha)) = (\alpha_\Sigma, \alpha + \delta_\Sigma) = -(\alpha_\Sigma, \alpha)$ and then $w_\Sigma(\alpha_\Sigma) = -\alpha_\Sigma$. Moreover, if $\Pi_1 = \{\alpha, \beta\}$, with similar calculations it is possible to show that $w_\Sigma(\alpha_\Sigma) = -\beta_\Sigma$. This implies that $w_\Sigma(\Gamma_1) = -\Gamma_1$ and hence $w_\Sigma(\theta_1) = -\theta_1$. The general case follows arguing by induction. □

Lemma 2.2.14. *Let $\alpha \in \Pi_1$ and $i \in \mathbb{N}$ such that Γ_i is not empty. We have $(\alpha, \theta_\Sigma^\vee) = (\alpha, \theta_i^\vee)$ and $|\theta_0| = |\theta_i|$.*

Proof. First we suppose $\widehat{\mathfrak{g}}$ not of type $A_{2l}^{(2)}$. We observe that since $\theta_0 \geq \theta_i$ then $(\alpha^\vee, \theta_0) \leq (\alpha^\vee, \theta_i) \leq -1$: this implies that

$$(\alpha, \theta_0^\vee) \leq \frac{|\theta_i|^2}{|\theta_0|^2}(\alpha, \theta_0^\vee) \leq -\frac{|\alpha|^2}{|\theta_0|^2} \tag{2.2.8}$$

If $|\alpha| > |\theta_0|$ then, by our assumption, θ_i must be a short root for every i: since in this case $(\alpha, \theta_0^\vee) = -|\alpha|^2|\theta_0|^{-2}$ we have the thesis. On the contrary let $|\alpha| < |\theta_0|$: if

we suppose $|\alpha|^2|\theta_0|^{-2} = |\theta_i|^2|\theta_0|^{-2} < 1$ we would have $(\alpha, \theta_i^\vee) < -1$, hence $|\alpha| \geq |\theta_0|$ and a contradiction. Therefore $|\theta_i| = |\theta_0|$ and $(\alpha, \theta_i) = (\alpha, \theta_0) = -1$. The remaining case is $|\alpha| = |\theta|$: clearly we can assume θ_0 long. Moreover suppose there exists i such that $|\theta_i| < |\theta_0|$ and let j be the smallest index with such property: we have $\Gamma_j \subseteq \widehat{\Pi}_{\text{short}}$ and hence α_Σ short. Since α is long and Π_Σ contains at least a long root, we have that the diagram of $\{\alpha\} \cup \Sigma$ contains a subdiagram of type $C_l^{(1)}$ for some l. This clearly implies that $\widehat{\mathfrak{g}}$ is of type $C_l^{(1)}$ and hence a contradiction since θ_0 would be short. In conclusion $|\theta_i| = |\theta_0|$ for every i and therefore by (2.2.8) we get $(\alpha, \theta_\Sigma^\vee) = (\alpha, \theta_i^\vee) = -1$.
The case $\mathfrak{g} \simeq A_{2l}^{(2)}$ is dealt with a brief inspection. □

Proposition 2.2.15. *Let N be the maximal integer such that Γ_N is not empty. We have*

$$\delta_\Sigma = r_\Sigma \sum_{i=0}^{N} \theta_i.$$

Proof. We define $w = w_\Sigma \prod_{i=0}^{N} s_{\theta_i}$: by Lemma 2.2.14 and Lemma 2.2.13 we have that $w(\alpha) = \alpha + \delta_\Sigma - r_\Sigma \sum_{i=0}^{N} \theta_i$. From Lemma 2.2.12 follows that if $(w^{-1}(\gamma), \alpha) = 0$ for every $\gamma \in N(w)$ then $w(\alpha) = \alpha$: hence we need to describe explicitly $N(w)$ or, equivalently, the set

$$A = \Big\{\gamma \in \Delta_\Sigma^+ \mid \prod_{i=0}^{N} s_{\theta_i}(\gamma) > 0\Big\}.$$

Let γ be in A and suppose that exists $j \leq N$ such that $(\theta_j^\vee, \gamma) \neq 0$ and $(\theta_i, \gamma) = 0$ for all $i < j$: we observe that $\theta_i \notin A$ for every $i \leq N$, therefore

$$\prod_{i=0}^{N} s_{\theta_i}(\gamma) = \gamma - \theta_j - \sum_{i=j+1}^{N} (\theta_i^\vee, \gamma)\theta_i$$

with $|(\theta_i^\vee, \gamma)| < 2$. Since $\gamma \in \Delta^+(\Gamma_j)$ and $\Gamma_i \subset \Gamma_j$, we have that $\gamma - \sum_{i=j+1}^{N} (\theta_i^\vee, \gamma)\theta_i \in \Delta^+(\Gamma_j)$: this implies $\prod_{i=0}^{N} s_{\theta_i}(\gamma) < 0$ and hence a contradiction.
So we have proved that $N(w) \subset \bigcap_i \theta_i^\perp$: by Lemma 2.2.13 we also have $w^{-1}(N(w)) \subset \bigcap_i \theta_i^\perp$ and in particular $w^{-1}(N(w)) \in \Delta(\Gamma_N) \cap \theta_N^\perp$. This implies that $w^{-1}(N(w)) \in \Delta_\Sigma \cap \Pi_1^\perp$, hence the thesis. □

Corollary 2.2.16. *For every component Σ, denote by $N(\Sigma)$ the maximal integer such that $\Gamma_{N(\Sigma)}(\Sigma) \neq \emptyset$. We have*

$$k\delta = \sum_{\alpha \in \Pi_1} c_\alpha(k\delta)\alpha + \sum_{\Sigma | \Pi_0} \sum_{i=0}^{N(\Sigma)} \theta_i(\Sigma).$$

Now we have all the tools to prove the main result of this section.
We start with some remarks on the components of Π_0 when $|\Pi_1| = 1$.

Lemma 2.2.17. *1. Suppose that $\alpha \in \Pi_1$ is a short root. Then Π_0 has at most two component. In particular if $\Pi_0 = \Pi_\Sigma$ then $N(\Sigma) \leq 1$, and if $\Pi_0 = \Pi_\Sigma \cup \Pi_{\Sigma'}$ then $N(\Sigma) = N(\Sigma') = 0$.*

2. *Suppose that $\alpha \in \Pi_1$ is a long root. Then when $\Pi_0 = \Pi_\Sigma$ and θ_Σ is of type 1 we have $N(\Sigma) > 2$, while if θ_Σ is long and of type 2 we have $N(\Sigma) = 1$. Finally if $\Pi_0 = \Pi_\Sigma \cup \Pi_{\Sigma'}$ and $\theta_\Sigma, \theta_{\Sigma'}$ are of type 1, then $N(\Sigma) + N(\Sigma') > 0$.*

Proof. By Corollary 2.2.16 we have

$$0 = (\alpha^\vee, k\delta) = 2(\alpha^\vee, \alpha) + \sum_{\Sigma|\Pi_0} \sum_{i=0}^{N(\Sigma)} r_\Sigma(\alpha^\vee, \theta_i(\Sigma)).$$

Since, by Lemma 2.2.14, $|\theta_i(\Sigma)| = |\theta_0(\Sigma)|$ and $(\alpha, \theta_i(\Sigma)^\vee) = -r_\Sigma$ for every $i \le N(\Sigma)$, we obtain

$$4 = \sum_{\Sigma|\Pi_0} r_\Sigma^2 (N(\Sigma)+1) \frac{|\theta_0(\Sigma)|^2}{|\alpha|^2} \tag{2.2.9}$$

Now let $\alpha \in \Pi_1$ be a short root. Since we are interested in the case $k\delta - \theta_\Sigma \in \widehat{\Pi}_0^*$ (see Section 2.1), we have $|\theta_0(\Sigma)|^2|\alpha|^{-2} = 2$. Therefore (2.2.9) becomes

$$2 = \sum_{\Sigma \text{ of type } 1} (N(\Sigma)+1) + 2 \sum_{\substack{\theta_\Sigma \text{ short} \\ \Sigma \text{ of type } 2}} (N(\Sigma)+1) + 4 \sum_{\substack{\theta_\Sigma \text{ long} \\ \Sigma \text{ of type } 2}} (N(\Sigma)+1),$$

from which we easily obtain part 1.
Finally, let $\alpha \in \Pi_1$ be a long root. With the same assumption of the previous case, from (2.2.9) we obtain

$$4 = \sum_{\Sigma \text{ of type } 1} (N(\Sigma)+1) + 2 \sum_{\substack{\theta_\Sigma \text{ short} \\ \Sigma \text{ of type } 2}} (N(\Sigma)+1) + 4 \sum_{\substack{\theta_\Sigma \text{ long} \\ \Sigma \text{ of type } 2}} (N(\Sigma)+1),$$

from which the second part of the lemma follows. □

Definition 2.2.2. Given $\alpha \in \Pi_1$ and $\Sigma|\Pi_0$, we set $W_\Sigma = W(\Pi_\Sigma)$, $w_{\Sigma,\alpha} = \max(W(\Pi_\Sigma \cap \widehat{\Pi}_\alpha))$. Recall that $w_\Sigma = \max(W_\Sigma)$. Moreover fix $\alpha \in \Pi_1$: we denote by $v_{i,\alpha,\Sigma}$ the minimal lenght element of the set $\{v \in W(\Pi_\Sigma) \mid v(\alpha) = \alpha + r_\Sigma \theta_i(\Sigma)\}$.

Remark 2.2.5. Observe that when $\alpha \in \Pi_1$ is long we have $v_{i,\alpha,\Sigma} = u_{i,\alpha,\Sigma} s_\Sigma$, where $u_{i,\alpha,\Sigma}$ is the minimal lenght element such that $u_{i,\alpha,\Sigma}(\alpha_\Sigma) = \theta_\Sigma$. In fact, as we have observed in the proof of Lemma 2.2.14, when α and θ_Σ are long roots, α_Σ must be also long, therefore we have $(\alpha, \alpha_\Sigma^\vee) = (\alpha, \theta_\Sigma^\vee) = -r_\Sigma$.
Finally observe that $v_{i,\alpha,\Sigma} \in W_\Sigma / W(\Pi_\Sigma \setminus \{\alpha_\Sigma\})$, and that

$$\ell(v_{i,\alpha,\Sigma}) = g_{\Gamma_i(\Sigma)} - 1. \tag{2.2.10}$$

The following theorem is the announced result on reduced expressions of the elements $w_{\alpha,\mu}$. To make the formulas a bit clearer, we set $\Pi_1 = \{\alpha_p, \alpha_q\}$ in the hermitian case, and $\Pi_1 = \{\alpha_p\}$ in the semisimple one. Furthermore, we omit α_p in $v_{i,\alpha_p,\Sigma}$.

Theorem 2.2.18. *If θ_Σ is of type 1 and $\alpha \in \Gamma_1(\Sigma)_{\overline{\ell}}$, then*

$$w_{\alpha,\mu} = \begin{cases} (\prod_{\gamma \in \Pi_1} s_\gamma) v_\alpha^1 & \text{if } N(\Sigma) = 1 \text{ and } \Pi_0 = \Pi_\Sigma \\ s_q w_0 w_{0,\alpha_p} v_{0,\Sigma}^{-1} v_{1,\Sigma}^{-1} s_p v_\alpha^1 & \text{otherwise,} \end{cases} \tag{2.2.11}$$

where v_α^1 *is the element of minimal lenght such that* $v_\alpha^1(\alpha) = \theta_1(\Sigma)$.
If $\alpha \in \Sigma'_{\overline{\ell}}$ *with* $\Sigma \neq \Sigma'$ *and* θ_Σ, $\theta_{\Sigma'}$ *of type 1, then*

$$w_{\alpha,\mu} = \begin{cases} (\prod_{\gamma\in\Pi_1} s_\gamma)v'_\alpha & \text{if } N(\Sigma) = N(\Sigma') = 0 \text{ and } \Pi_0 = \Pi_\Sigma \cup \Pi_{\Sigma'} \\ s_q w_0 w_{0,\alpha_p} v_{0,\Sigma}^{-1} v_{0,\Sigma'}^{-1} s_p v'_\alpha & \text{otherwise,} \end{cases} \tag{2.2.12}$$

where v'_α *is the element of minimal lenght such that* $v'_\alpha(\alpha) = \theta_{\Sigma'}$.
If $\alpha \in \Sigma_{\overline{\ell}}$ *with* θ_Σ *of type 2, then*

$$w_{\alpha,\mu} = \begin{cases} (\prod_{\gamma\in\Pi_1} s_\gamma)v_\alpha & \text{if } N(\Sigma) = 0 \text{ and } \Pi_0 = \Pi_\Sigma \\ s_q w_0 w_{0,\alpha_p} v_{0,\Sigma}^{-1} s_p v_\alpha & \text{otherwise,} \end{cases} \tag{2.2.13}$$

where v_α *is the element of minimal lenght such that* $v_\alpha(\alpha) = \theta_\Sigma$.

Proof. Denote by $\widetilde{w}$ the second member of the equalities: by Lemma 2.2.5, it's enough to check that $\widetilde{w}(\alpha) = \mu$ and that the lenght of $\widetilde{w}$ is minimal.

We begin by considering the first cases of our equalities. Suppose then θ_Σ of type 1, $N(\Sigma) = 1, \Pi_0 = \Pi_\Sigma$ and $\alpha \in \Gamma_1(\Sigma)_{\overline{\ell}}$: in the hermitian symmetric case, by Corollary 2.2.16, we have

$$\Big(\prod_{\gamma\in\Pi_1} s_\gamma\Big)v_\alpha^1(\alpha) = \theta_1(\Sigma) - (\alpha_p^\vee, \theta_1(\Sigma))\alpha_p - (\alpha_q^\vee, \theta_1(\Sigma))\alpha_q = \delta - \theta_\Sigma.$$

In the semisimple case, by Lemma 2.2.17(2), we can assume α_p short. This implies that

$$\Big(\prod_{\gamma\in\Pi_1} s_\gamma\Big)v_\alpha^1(\alpha) = \theta_1(\Sigma) + 2\alpha_p = \delta - \theta_\Sigma.$$

Clearly $(\prod_{\gamma\in\Pi_1} s_\gamma)v_\alpha^1$ is of minimal lenght, therefore we have the thesis in the first case of (2.2.11). The remaining cases are similar to the previous one. The only difference occurs in (2.2.13) when $N(\Sigma) = 0, \Pi_0 = \Pi_\Sigma, \Pi_1 = \{\alpha_p\}$ and α_p is *long*: by Lemma 2.2.17(2) this implies that θ_Σ is long and complex, therefore, since $\overline{\alpha}_p = -\overline{\theta}_\Sigma$, again we have the claim.

Finally, we deal with the most generic cases of our equalities. First observe that, by Lemma 2.2.17(1), we can assume that Π_1 is composed of long roots. This implies that, by Remark 2.2.5, we have $v_{i,\Sigma} = u_{i,\Sigma} s_\Sigma$. The first step consists in proving that

$$w_\Sigma w_{\Sigma,\alpha_p} = v_{N(\Sigma),\Sigma} \cdots v_{0,\Sigma}. \tag{2.2.14}$$

To make this, observe that $v_{N(\Sigma),\Sigma} \cdots v_{0,\Sigma}(\alpha_p) = \alpha_p + r_\Sigma \sum_i \theta_i(\Sigma) = \alpha_p + \delta_\Sigma$, by Proposition 2.2.15. Moreover it is not difficult to prove that $v_{N(\Sigma),\Sigma} \cdots v_{0,\Sigma}$ must be of minimal lenght. From this we obtain that $v_{N(\Sigma),\Sigma} \cdots v_{0,\Sigma} \in W_\Sigma / W(\Pi_\Sigma \setminus \{\alpha_\Sigma\})$, hence, since $w_\Sigma w_{\Sigma,\alpha_p}$ is the maximum of this quotient with respect to the weak order, we have $w_\Sigma w_{\Sigma,\alpha_p} \geq v_{N(\Sigma),\Sigma} \cdots v_{0,\Sigma}$ and hence $s_q \prod_{\Sigma'\neq\Sigma} w_{\Sigma'} w_{\Sigma',\alpha_p} v_{N(\Sigma),\Sigma} \cdots v_{0,\Sigma} \in \mathcal{W}_\sigma^{ab}$. By Lemma 2.2.9 we have that $I_{\alpha_p,k\delta-\alpha_q} = \{s_p w_0 w_{0,\alpha_p}\}$, therefore we obtain the desired equality. Once established (2.2.14), it's easy to check that $\widetilde{w}(\alpha) = \mu$ in all three cases.

It remains to prove that $\ell(w_{\alpha,\mu}) = \ell(\widetilde{w})$: from Remark 2.2.5, and in particular Equation (2.2.10), we obtain that

$$\ell(v_{i,\Sigma}^{-1} w_0 w_{0,\alpha_p}) = \ell(w_0 w_{0,\alpha_p}) - \ell(v_{i,\Sigma}) = g - g_{\Gamma_i(\Sigma)}.$$

The claim on the lenghts of $w_{\alpha,\mu}$ and $\widetilde{w}$ is now an easy calculation. □

Example 2.2.2. As usually, we number affine Dynkin diagrams as in [9, Tables Aff1 and Aff2].

1. Let $\widehat{L}(\mathfrak{g},\sigma)$ be of type $E_7^{(1)}$ and $\Pi_1 = \{\alpha_7\}$. We set $\alpha = \alpha_3$ and Σ as the only component of Π_0. Observe that $N(\Sigma) = 3$. Following formula 2.2.11, we have:
$$v_\alpha^1 = s_1s_2s_5s_4, \quad v_{1,\Sigma} = s_1s_2s_5s_4s_3, \quad v_{0,\Sigma} = s_0s_1s_2s_6s_5s_4s_3$$
$$w_0w_{0,\alpha_7} = s_3s_2s_1s_0s_4s_3s_2s_1s_5s_4s_3s_2s_6s_5s_4s_3$$
therefore $w_{\alpha,\mu} = s_7s_3s_2s_4s_3s_7s_1s_2s_5s_4$.

2. Let $\widehat{L}(\mathfrak{g},\sigma)$ be of type $B_7^{(1)}$ and $\Pi_1 = \{\alpha_4\}$. We choose Σ as the D_4-type component and $\alpha = \alpha_5 \in \Sigma'$. This time, since $N(\Sigma) = 1$, we have:
$$v'_\alpha = s_6s_7s_6, \quad v_{0,\Sigma} = s_0s_1s_2s_3, \quad v_{0,\Sigma'} = s_6s_7s_6s_5$$
$$w_0w_{0,\alpha_4} = s_3s_0s_1s_2s_3s_5s_6s_7s_6s_5,$$
therefore, by formula 2.2.12, we obtain $w_{\alpha,\mu} = s_4s_3s_5s_4s_6s_7s_6$.

3. Let $\widehat{L}(\mathfrak{g},\sigma)$ be of type $A_9^{(2)}$ and $\Pi_1 = \{\alpha_5\}$. Choose $\alpha = \alpha_4$. By formula 2.2.13 we have:
$$v_\alpha = s_3s_4s_2s_1s_0s_2, \quad v_{0,\Sigma} = s_3s_2s_0s_1s_2s_3s_5s_4$$
$$w_0w_{0,\alpha_5} = s_4s_3s_2s_0s_1s_2s_3s_4,$$
hence $w_{\alpha,\mu} = s_5s_4s_5s_3s_4s_2s_1s_0s_2$.

2.3 The poset structure of $\mathcal{I}_{\alpha,\mu}$

We now study the poset structure of the sets $\mathcal{I}_{\alpha,\mu}$. This study is motivated by the following result, that shows that, in most cases, the maximal elements of the sets $\mathcal{I}_{\alpha,\mu}$ are maximal in the whole poset $\mathcal{W}_\sigma^{ab}$.

Proposition 2.3.1. *Suppose $w \in \mathcal{I}_{\alpha,\mu}$ and that $v \geq w$ with $v \in \mathcal{W}_\sigma^{ab}$. If $v \notin \mathcal{I}_{\alpha,\mu}$ then $\alpha \in \Pi_1$ and $v \in \mathcal{I}_{\alpha,k\delta+\beta}$ where $\beta = \alpha$ if $|\Pi_1| = 1$ and $\Pi_1 = \{\alpha, \beta\}$ otherwise.*

Proof. If $v \notin \mathcal{I}_{\alpha,\mu}$, write $v = wxs_\gamma y$ with $wx \in \mathcal{I}_{\alpha,\mu}$, $wxs_\gamma \notin \mathcal{I}_{\alpha,\mu}$ and $\ell(v) = \ell(w) + \ell(x) + \ell(y) + 1$. Then $(\gamma, \alpha) < 0$. Set $(\alpha, \gamma^\vee) = -r$ and consider $wxs_\gamma s_\alpha$. We have
$$N(wxs_\gamma s_\alpha) = N(wxs_\gamma) \cup \{wx(\alpha + r\gamma)\} = N(wxs_\gamma) \cup \{\mu + rwx(\gamma)\}.$$
Note that $ht_\sigma(\mu + rwx(\gamma)) = ht_\sigma(\mu) + r$. Since the latter root is not simple, there exists $\eta \in \widehat{\Pi}$ such that $\mu + wx(\gamma) - \eta \in \widehat{\Delta}^+$. Since $N(wxs_\gamma) \subset \widehat{\Delta}_1$ and $N(wxs_\gamma s_\alpha)$ is convex, we have that $\eta \notin \Pi_0$. Hence $\mu + rwx(\gamma)$ is minimal in $\widehat{\Delta}_{ht_\sigma(\mu)+r}$. Now we use Remark 2.0.1 about minimal roots. If $ht_\sigma(\mu) + r = 2s$ with $s > 1$ then $\mu + rwx(\gamma) = ks\delta - \theta_\Sigma$ for some $\Sigma|\Pi_0$. But then, by convexity, $k\delta - \theta_\Sigma \in N(wxs_\gamma)$ which is absurd. If $ht_\sigma(\mu) + r = 2s + 1$ with $s > 1$ then $\mu + rwx(\gamma) = ks\delta + \beta$ for some $\beta \in \Pi_1$. But then, by convexity, $k\delta + \beta \in N(wxs_\gamma)$ which is absurd. Therefore $ht_\sigma(\mu) = 2$ and $r = 1$. It follows that there exists $\beta \in \Pi_1$ such that $\mu + wx(\gamma) = \beta + k\delta$. In turn, we deduce that $wxs_\gamma \in \mathcal{I}_{\alpha,k\delta+\beta}$. By Proposition 2.2.10, (1), we have $\alpha \in \Pi_1$ as claimed, and $wxs_\gamma \in \mathcal{I}_{\alpha,k\delta+\beta}$ with $\beta = \alpha$ if $|\Pi_1| = 1$ and $\Pi_1 = \{\alpha, \beta\}$ otherwise. Since $v \geq wxs_\gamma \in \mathcal{I}_{\alpha,k\delta+\beta}$ and $ht_\sigma(k\delta + \beta) = 3$, by the first part of the proof, we have that $v \in \mathcal{I}_{\alpha,k\delta+\beta}$, as claimed. $\square$

We now turn to the description of the poset structure of $\mathcal{I}_{\alpha,\mu}$: we will show that it is isomorphic to a poset $G'\backslash G$ for suitable reflection subgroups G, G' of $\widehat{W}$.

Definition 2.3.1. For $\alpha \in \widehat{\Pi}$, and $\Sigma|\Pi_0$, we set

$$\widehat{\Pi}_\alpha = \widehat{\Pi} \cap \alpha^\perp, \quad \widehat{W}_\alpha = W(\widehat{\Pi}_\alpha)$$

Lemma 2.3.2. *Let $\mu \in \mathcal{M}_\sigma$, $u, v \in \mathcal{I}_{\alpha,\mu}$, and $u < v$. Then $v = ux$ with $x \in \widehat{W}_\alpha$. In particular,*

$$\mathcal{I}_{\alpha,\mu} \subseteq w_{\alpha,\mu}\widehat{W}_\alpha.$$

Proof. By assumption, there exists $x \in \widehat{W}$ such that $N(v) = N(u) \cup uN(x)$: suppose by contradiction that $x \notin \widehat{W}_\alpha$. Then we may assume $x = x_1 s_\beta x_2$ with $\ell(x) = \ell(x_1) + \ell(x_2) + 1$, $x_1 \in \widehat{W}_\alpha$, and $\beta \in \widehat{\Pi}$, $\beta \not\perp \alpha$. Then $N(ux_1) \cup ux_1(\beta) \subseteq N(v)$. But $(\beta, \alpha) < 0$, hence $(ux_1(\beta), ux_1(\alpha)) = (ux_1(\beta), \mu) < 0$, therefore $ux_1(\beta) + \mu$ is a root: this cannot happen by Lemma 2.2.6 (1). □

By Lemma 2.3.2, $\mathcal{I}_{\alpha,\mu}$ is in bijection, in a natural way, with a subset of $\widehat{W}_\alpha$, namely, the subset of all $u \in \widehat{W}_\alpha$ such that $w_{\alpha,\mu}u \in \mathcal{I}_{\alpha,\mu}$. We will show that this subset is a system of minimal coset representatives of $\widehat{W}_\alpha$ modulo a certain subgroup $\widehat{W}_{\alpha,\mu}$. This will take the rest of the section.

We start with giving a combinatorial characterization of the elements u such that $w_{\alpha,\mu}u \in \mathcal{I}_{\alpha,\mu}$.

Definition 2.3.2. We set

$$B_\mu = \begin{cases} \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 1,} \\ \Pi_1 & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 2,} \\ \{\beta\} & \text{if } \mu = k\delta + \beta,\ \beta \in \Pi_1, \end{cases}$$

$$V_{\alpha,\mu} = \{w \in \widehat{W}_\alpha \mid ht_{B_\mu}(\gamma) = 1 \ \forall\, \gamma \in N(w)\},$$

if $B_\mu \neq \emptyset$. If $B_\mu = \emptyset$, we set $V_{\alpha,\mu} = \{1\}$.

Lemma 2.3.3. *Assume $\Sigma|\Pi_0$, $\mu = k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, and set*

$$B_\Sigma = \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma) > 0\}.$$

Then

1. *For all $\eta \in \widehat{\Delta}$,*
$$(\eta, \mu^\vee) = ht_\sigma(\eta) r_\Sigma - ht_{B_\Sigma}(\eta)\varepsilon_\Sigma, \tag{2.3.1}$$
where $\varepsilon_\Sigma = 2$, if $|\Sigma| = 1$, $\varepsilon_\Sigma = 1$, otherwise.

2. *$B_\Sigma = \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\}$, unless $|\Sigma| = 1$.*

Proof. It is clear that for $\gamma \in \widehat{\Pi}$, we have $(\gamma, \theta_\Sigma) < 0$ if and only if $\gamma \in \Pi_1$; moreover, recall that $r_\Sigma = -(\gamma, \theta_\Sigma^\vee) = (\gamma, \mu^\vee)$ for all $\gamma \in \Pi_1$.

On the other hand, by definition, we have $(\gamma, \theta_\Sigma) > 0$ if and only if $\gamma \in B_\Sigma$. Clearly, $B_\Sigma \subseteq \Sigma$, and since θ_Σ is long with respect to $\Delta(\Sigma)$, it follows that, if $\gamma \in B_\Sigma$, then $(\gamma, \theta_\Sigma^\vee) = 1$ unless $\Sigma = B_\Sigma = \{\theta_\Sigma\}$, in which case $(\gamma, \theta_\Sigma^\vee) = 2$. Therefore, for any $\eta \in \widehat{\Delta}$,

$$(\eta, \mu^\vee) = \sum_{\gamma \in \Pi_1} c_\gamma(\eta)(\gamma, \mu^\vee) + \sum_{\gamma \in B_\Sigma} c_\gamma(\eta)(\gamma, \mu^\vee) = ht_\sigma(\eta) r_\Sigma - ht_{B_\Sigma}(\eta)\varepsilon_\Sigma$$

as wished. □

Lemma 2.3.4. *Assume $\mathcal{I}_{\alpha,\mu} \neq \emptyset$. For any $u \in \widehat{W}$, $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$ if and only if $u \in V_{\alpha,\mu}$.*

Proof. We deal with the three cases that occur in the definition of B_μ one by one. We shall use several times relation (2.3.1) from Lemma 2.3.3.

1. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 1. Then $\alpha \in A(\Sigma)$, μ is the highest root of $A(\Sigma)$, and $w_{\alpha,\mu} \in W(A(\Sigma))$. It is clear that $B_\Sigma \cap A(\Sigma) = \emptyset$, in fact, by Definition 2.2.1, $A(\Sigma)$ is a connected component of $\widehat{\Pi} \setminus B_\Sigma$. In particular, for all $\eta \in \widehat{\Delta}$, $ht_{B_\Sigma}(w_{\alpha,\mu}(\eta)) = ht_{B_\Sigma}(\eta)$. Recall that r_Σ is the type of θ_Σ. By (2.3.1),

$$\begin{aligned}(w_{\alpha,\mu}(\eta), \mu^\vee) &= ht_\sigma(w_{\alpha,\mu}(\beta)) - ht_{B_\Sigma}(w_{\alpha,\mu}(\eta))\varepsilon_\Sigma = \\ &= ht_\sigma(w_{\alpha,\mu}(\beta)) - ht_{B_\Sigma}(\eta)\varepsilon_\Sigma.\end{aligned}$$

Now, assume $u \in V_{\alpha,\mu}$. If $u = 1$, obviously $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$. So we may assume $u \neq \mathbf{1}$ and $|\Sigma| > 1$. If $\eta \in N(u)$, then $(\eta, \alpha) = 0$, so that $(w_{\alpha,\mu}(\eta), \mu) = 0$; moreover, $ht_{B_\Sigma}(\eta) = \varepsilon_\Sigma = 1$. Therefore, by the above identities we obtain that $ht_\sigma(w_{\alpha,\mu}(\eta)) = 1$. Thus $N(w_{\alpha,\mu} u) = N(w_{\alpha,\mu}) \cup w_{\alpha,\mu} N(u) \subseteq \widehat{\Delta}_1$, hence $w_{\alpha,\mu} u \in \mathcal{W}_\sigma^{ab}$. Since $u(\alpha) = \alpha$, we conclude that $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$.

Conversely, if $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$ with $u \neq 1$, then, by Lemma 2.3.2, $u \in \widehat{W}_\alpha$, so that, if $\eta \in N(u)$, then $(\eta, \alpha) = 0$, hence $(w_{\alpha,\mu}(\eta), \mu) = 0$. Moreover, $ht_\sigma(w_{\alpha,\mu}(\eta)) = 1$. It follows that $\varepsilon_\Sigma = 1$ and $ht_{B_\Sigma}(\eta) = 1$, so $ht_{B_\Sigma}(\eta) = ht_{B_\mu}(\eta) = 1$, hence $u \in V_{\alpha,\mu}$.

2. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 2. Then $\alpha \in \Sigma$, and $w_{\alpha,\mu} = s v_\alpha$, where v_α is the minimal element that maps α to θ_Σ and s is the minimal element that maps θ_Σ to μ. We also know that s is an involution. In this case, $B_\mu = \Pi_1$, hence $B_\mu \cap \Sigma = \emptyset$. Thus the B_μ-height is the σ-height and, since $v_\alpha \in W(\Sigma)$, we have that v_α preserves the σ-height. Similarly, since $s \in W(A(\Sigma))$, s preserves the B_Σ-height. Therefore, for all $\eta \in \widehat{\Delta}$, we obtain that

$$\begin{aligned}(w_{\alpha,\mu}(\eta), \mu^\vee) = (v_\alpha(\eta), s\mu^\vee) &= (v_\alpha(\eta), \theta_\Sigma^\vee) = -(v_\alpha(\eta), \mu^\vee) \\ &= -2ht_\sigma(v_\alpha(\eta)) + ht_{B_\Sigma}(v_\alpha(\eta))\varepsilon_\Sigma \\ &= -2ht_\sigma(\eta) + ht_{B_\Sigma}(v_\alpha(\eta))\varepsilon_\Sigma,\end{aligned}$$

and also that

$$\begin{aligned}(w_{\alpha,\mu}(\eta), \mu^\vee) &= 2ht_\sigma(w_{\alpha,\mu}(\eta)) - ht_{B_\Sigma}(w_{\alpha,\mu}(\eta))\varepsilon_\Sigma \\ &= 2ht_\sigma(w_{\alpha,\mu}(\eta)) - ht_{B_\Sigma}(v_\alpha(\eta))\varepsilon_\Sigma.\end{aligned}$$

In particular, if $(\mu^\vee, w_{\alpha,\mu}(\eta)) = 0$, then $ht_\sigma(w_{\alpha,\mu}(\eta)) = ht_\sigma(\eta) = ht_{B_\mu}(\eta)$. By Lemma 2.3.2, this directly implies that $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$ if and only if $u \in V_{\alpha,\mu}$.

3. $\mu = k\delta + \beta$, $\beta \in \Pi_1$. If $|\Pi_1| = 1$, then $V_{\alpha,\mu} = \{1\}$ and, by Proposition 2.2.10, $\mathcal{I}_{\alpha,\mu} = \{w_{\alpha,\mu}\}$. So we may assume $|\Pi_1| = 2$, $\Pi_1 = \{\alpha, \beta\}$. Then, with notation as in Proposition 2.2.10, we have that $w_{\alpha,\mu} = s_\beta v_\beta$. Since $v_\beta(\alpha) = \delta - \beta$, we deduce that $v_\beta^{-1}(\beta) = \delta - \alpha$, hence, if $(\gamma, \alpha) = 0$, then

$$s_\beta v_\beta(\gamma) = v_\beta(\gamma) - (v_\beta(\gamma), \beta^\vee)\beta = v_\beta(\gamma) - (\gamma, \delta - \alpha^\vee)\beta = v_\beta(\gamma).$$

It follows that, if $\gamma \in \widehat{\Delta}_\alpha^+$, then $ht_\sigma(w_{\alpha,\mu}(\gamma)) = ht_\sigma(v_\beta(\gamma)) = c_\beta(\gamma) = ht_{B_\mu}(\gamma)$ and we can argue as in case 2. □

Lemma 2.3.5. *Assume $\alpha \in \widehat{\Pi}$, $\mu \in \mathcal{M}_\sigma$, $\mathcal{I}_{\alpha,\mu} \neq \emptyset$, $B_\mu \neq \emptyset$, and set*

$$\Delta^2_{\alpha,\mu} = \{\gamma \in \Delta(\widehat{\Pi}_\alpha) \mid ht_{B_\mu}(\gamma) \geq 2\}.$$

Then $\Delta^2_{\alpha,\mu} \neq \emptyset$ if and only if $\mu = k\delta - \theta_\Sigma$ with θ_Σ of type 1 and $|\Sigma| > 1$, and $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$. In this case, θ_Σ is the minimal element in $\Delta^2_{\alpha,\mu}$, with respect to the usual root order. Moreover, $ht_{B_\mu}(\theta_\Sigma) = ht_{B_\Sigma}(\theta_\Sigma) = 2$.

Proof. We deal with the three cases that occur in the definition of B_μ one by one.

1. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 1. Then $\alpha \in A(\Sigma)$ and $|\Sigma| > 1$, since we are assuming $B_\mu \neq \emptyset$. In particular $B_\Sigma = B_\mu = \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\}$.

We first prove that if $\gamma \in \Delta(\widehat{\Pi}_\alpha)$ and $ht_{B_\mu}(\gamma) \geq 2$ then $\gamma \geq \theta_\Sigma$. We notice $(\beta, \theta_\Sigma^\vee) \in \{0, 1\}$ for any $\beta \in \Delta^+(\Sigma) \setminus \{\theta_\Sigma\}$, therefore, since $(\theta_\Sigma, \theta_\Sigma^\vee) = 2$, $ht_{B_\mu}(\theta_\Sigma) = 2$ and θ_Σ is the unique root in $\Delta(\Sigma)$ with this property. It follows that we can assume $\gamma \notin \Delta(\Sigma)$, so that $ht_\sigma(\gamma) > 0$. Since $c_\alpha(k\delta - \gamma) > 0$, we have that $k\delta - \gamma$ is a positive root, hence $ht_\sigma(\gamma) \leq 2$. If $ht_\sigma(\gamma) = 1$, then $(\gamma, \theta_\Sigma^\vee) = 1$, hence $\gamma - \theta_\Sigma$ is a root, which can't be negative, since γ is supported also outside Σ. So it is positive, hence $\gamma \geq \theta_\Sigma$. Suppose now $ht_\sigma(\gamma) = 2$. Then $k\delta - \gamma \in \Delta_0$, hence it should belong to the component Σ' of Π_0 to which α belongs, since $c_\alpha(k\delta - \gamma) > 0$. Thus $\gamma = k\delta - \beta$ with $\beta \in \Sigma'$. If $\Sigma = \Sigma'$, then $\alpha \in \Gamma(\Sigma)$. Let Z be the component of $\Gamma(\Sigma)$ containing α. Let $\eta \in \Pi_1$ be a root such that $(\eta, \theta_Z) < 0$. We have that $\eta + \theta_Z + \theta_\Sigma$ is a root, so $k\delta - \eta - \theta_\Sigma - \theta_Z$ is a root, which is positive since its σ-height is 1. It follows that $k\delta \geq \theta_\Sigma + \theta_Z$, hence $\gamma \geq \theta_\Sigma - \beta + \theta_Z$. But then $c_\alpha(\gamma) > 0$, which is impossible. We have therefore $\Sigma' \neq \Sigma$. But then $\gamma = k\delta - \beta$ with $\beta \notin \Sigma$, so, clearly, $\gamma \geq \theta_\Sigma$.

It remains only to check that $\Delta^2_{\alpha,\mu} \neq \emptyset$ if and only if $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$. If $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$ then $\theta_\Sigma \in \Delta(\widehat{\Pi}_\alpha)$, hence $\theta_\Sigma \in \Delta^2_{\alpha,\mu}$. Assume now $\gamma \in \Delta^2_{\alpha,\mu}$. If $\alpha \in \Pi_1 \cup \Sigma$, then $\theta_\Sigma \notin \Delta(\widehat{\Pi}_\alpha)$, hence $\gamma > \theta_\Sigma$. This is absurd since it implies $c_\alpha(\gamma) > 0$.

2. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 2. Then $\alpha \in \Sigma$ and $B_\mu = \Pi_1$, so that the B_μ-height is the σ-height. We shall prove that, if $\gamma \in \Delta(\widehat{\Pi}_\alpha)$, then $ht_\sigma(\gamma) \leq 1$.

Consider first the case $k = 2$. Assume $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. Notice that, if $\delta - \gamma$ is a root, then it is positive, since then $c_\alpha(\delta - \gamma) > 0$. Since $ht_\sigma(\delta) = 1$, this implies that $ht_\sigma(\gamma) \leq 1$. Now, assume by contradiction that $ht_\sigma(\gamma) > 1$. Since, in any case, $2\delta - \gamma \in \widehat{\Delta}^+$, we obtain that $ht_\sigma(\gamma) = 2$ and $2\delta - \gamma \in \Delta_0^+$. In turn, this implies that $2\delta - \gamma \in \Delta(\Sigma)$, since $c_\alpha(2\delta - \gamma) > 0$, and $\alpha \in \Sigma$. Thus, since θ_Σ is of type 2, also $2\delta - \gamma$ is of type 2. But this implies that $\delta - \gamma$ is a root, hence that $ht_\sigma(\gamma) \leq 1$: a contradiction.

Next, consider the case $k = 1$. In case B_n, we have $\Sigma = \{\alpha_n\}$ and $\Pi_1 = \{\alpha_{n-1}\}$, so $\alpha = \alpha_n$ and and $ht_\sigma(\gamma) = 0$ for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. In case C_n, we have $\Sigma =$

$\{\alpha_1, \dots, \alpha_{n-1}\}$ and $\Pi_1 = \{\alpha_0, \alpha_n\}$, so it is clear that for all $\alpha \in \Sigma$, and for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$, $ht_\sigma(\gamma) \leq 1$.

3. $\mu = k\delta + \beta$, $\beta \in \Pi_1$. In this case $\alpha \in \Pi_1$ and $B_\mu \subseteq \Pi_1$, so it is clear that, if $\Pi_1 = \{\alpha\}$, then $ht_\sigma(\gamma) = 0$ for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. If $\Pi_1 = \{\alpha, \beta\}$, we obtain in any case that $ht_\sigma(\gamma) \leq 1$ for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. □

Definition 2.3.3. Given $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ such that $\mathcal{I}_{\alpha,\mu} \neq \emptyset$, we set

$$\widehat{\Pi}_{\alpha,\mu} = \widehat{\Pi}_\alpha \setminus B_\mu,$$

$$\widehat{\Pi}^*_{\alpha,\mu} = \begin{cases} \widehat{\Pi}_{\alpha,\mu} \cup \{\theta_\Sigma\} & \text{if } \mu = k\delta - \theta_\Sigma, \theta_\Sigma \text{ of type 1}, |\Sigma| > 1, \\ & \alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1), \\ \widehat{\Pi}_{\alpha,\mu} & \text{in all other cases;} \end{cases}$$

$$\widehat{W}_{\alpha,\mu} = W(\widehat{\Pi}^*_{\alpha,\mu}).$$

The main results of this section is the following statement. Recall that we identify a coset space with the set of minimal length coset representatives.

Theorem 2.3.6. *Let $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ be such that $\mathcal{I}_{\alpha,\mu} \neq \emptyset$. Then the map $u \mapsto w_{\alpha,\mu} u$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$.*

Proof. By Lemma 2.3.4, we have only to prove that $\widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha = V_{\alpha,\mu}$.

Let $u \in V_{\alpha,\mu}$, $u \neq \mathbf{1}$. To prove that $u \in \widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$ we have to show that if $\beta \in \widehat{\Pi}^*_{\alpha,\mu}$, then $u^{-1}(\beta) \in \widehat{\Delta}^+$: this is immediate from the definitions, since $ht_{B_\mu}(\beta) \in \{0, 2\}$, while, for all $\gamma \in N(u)$, $ht_{B_\mu}(\gamma) = 1$.

Conversely, assume $u \in \widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$, $u \neq \mathbf{1}$, and $\gamma \in N(u)$. If, by contradiction, $ht_{B_\mu}(\gamma) = 0$, then, by the biconvexity property of $N(u)$, we obtain that there exists some $\beta \in (\widehat{\Pi}_\alpha \setminus B_\mu) \cap N(u)$: this contradicts the definition of $\widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$. Therefore, $ht_{B_\mu}(\gamma) > 0$. By Lemma 2.3.5, this implies that $ht_{B_\mu}(\gamma) = 1$ in all cases except when $\mu = k\delta - \theta_\Sigma$, with θ_Σ of type 1 and $|\Sigma| > 1$. It remains to prove that also in this case $ht_{B_\mu}(\gamma) = 1$. First, we observe that, by Lemma 2.2.2, $ht_{B_\Sigma}(k\delta - \theta_\Sigma) = 0$: it follows that $ht_{B_\Sigma}(k\delta) = ht_{B_\Sigma}(\theta_\Sigma)$ and, by Lemma 2.3.5, that $ht_{B_\mu}(k\delta) = 2$. Hence, $ht_{B_\mu}(\gamma) \leq 2$, since $k\delta - \gamma$ is a positive root. Now, if we assume, by contradiction, that $ht_{B_\mu}(\gamma) = 2$, then by Lemma 2.3.5, we obtain that γ is equal to θ_Σ plus a, possibly empty, sum of positive roots with null B_μ-height. By the biconvexity of $N(u)$, this implies that some root in $(\widehat{\Pi}_\alpha \setminus B_\mu) \cup \{\theta_\Sigma\}$ belongs to $N(u)$, in contradiction with the definition of $\widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$. □

2.4 Intersections among $\mathcal{I}_{\alpha,\mu}$'s

Our goal in this Section is the proof of the following Theorem.

Theorem 2.4.1. *1. If $\alpha \neq \beta$, then $\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} \neq \emptyset$ if and only if $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta'_\Sigma$ with $\Sigma \neq \Sigma'$, $\alpha \in \Sigma'$, $\beta \in \Sigma$, and $\alpha, \beta, \theta_\Sigma, \theta_{\Sigma'}$ all of type 1.*

2. *If* $\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} \neq \emptyset$, *then*

$$\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} \cong W((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)\backslash W(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta),$$

the isomorphism being $u \mapsto w_{\alpha,\beta} u$, $u \in W((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)\backslash W(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)$, *where*

$$w_{\alpha,\beta} = \sup\{\min \mathcal{I}_{\alpha,\mu}, \min \mathcal{I}_{\beta,\mu'}\}.$$

Statements (1), (2) are proved in Propositions 2.4.7, 2.4.6, respectively.

Definition 2.4.1. Assume that Σ and Σ' are distinct components of Π_0. We define

$$A(\Sigma, \Sigma') = A(\Sigma) \cap A(\Sigma').$$

Moreover, we set

$$W_{\Sigma,\Sigma'} = W(A(\Sigma, \Sigma')), \qquad W^1_{\Sigma,\Sigma'} = W(A(\Sigma, \Sigma') \setminus \Pi_1)$$

and denote by $u_{\Sigma,\Sigma'}$ the maximal element in $W^1_{\Sigma,\Sigma'}\backslash W_{\Sigma,\Sigma'}$.

According to Definition 2.4.1 and Subsection 1.3.3,

$$N(u_{\Sigma,\Sigma'}) = \{\beta \in \Delta(A(\Sigma, \Sigma')) \mid ht_\sigma(\beta) > 0\}. \tag{2.4.1}$$

It is clear from Definition 2.2.1 that $\Sigma' \subseteq A(\Sigma)$; in fact, we have the partition

$$A(\Sigma) = \bigcup_{\substack{\Sigma' | \Pi_0 \\ \Sigma' \neq \Sigma}} \Sigma' \cup \Pi_1 \cup \Gamma(\Sigma). \tag{2.4.2}$$

From this, we obtain

$$A(\Sigma, \Sigma') = \Gamma(\Sigma) \cup \Pi_1 \cup \Gamma(\Sigma') \cup \Sigma'', \tag{2.4.3}$$

where $\Sigma'' = \Pi_0 \setminus (\Sigma \cup \Sigma')$. In particular we obtain the partition

$$A(\Sigma) = A(\Sigma, \Sigma') \cup (\Sigma' \setminus \Gamma(\Sigma')). \tag{2.4.4}$$

Remark 2.4.1. From equation (2.4.3) and Definition 2.2.1, we obtain directly that θ_Σ and $\theta_{\Sigma'}$ are orthogonal to all the roots in $A(\Sigma, \Sigma')$, except the ones in Π_1. This implies that $(\beta, \theta_\Sigma) \leq 0$ and $(\beta, \theta_{\Sigma'}) \leq 0$ for all $\beta \in \Delta(A(\Sigma, \Sigma'))$. Moreover, by equation (2.4.1), for any $\beta \in A(\Sigma, \Sigma')$, we have the following equivalences of conditions:

$$(\beta, \theta_\Sigma) < 0 \iff \beta \in N(u_{\Sigma,\Sigma'}) \iff (\beta, \theta_{\Sigma'}) < 0.$$

Lemma 2.4.2. *Let* Σ *and* Σ' *be distinct components of* Π_0. *Then*

$$u_{\Sigma,\Sigma'} \in \mathcal{W}_\sigma^{ab}.$$

Proof. By formula (2.4.1), for all $\beta \in N(u_{\Sigma,\Sigma'})$, $ht(\beta) > 0$. Therefore, it suffices to prove that, for all $\beta \in \Delta(A(\Sigma, \Sigma'))$, $ht_\sigma(\beta) < 2$. Assume by contradiction that $\beta \in \Delta(A(\Sigma, \Sigma'))$ and $ht_\sigma(\beta) < 2$. Then $ht_\sigma(k\delta - \beta) = 0$, hence $k\delta - \beta$ belongs to some component Σ'' of Π_0. At least one among Σ, Σ', say Σ, is not Σ''. Hence $(k\delta - \beta, \theta_\Sigma) = 0$, which gives $(\beta, \theta_\Sigma) = 0$: this is impossible, by Remark 2.4.1. □

Remark 2.4.2. If Z is a connected component of $A(\Sigma,\Sigma')$, then the sum of the roots in Z is a root and, by the Lemma 2.4.2, it has σ-height at most 1. This implies, in particular, that Z contains at most one root of Π_1.

Though we shall not need this fact, we notice that $A(\Sigma,\Sigma')$ is connected except in type $A_n^{(1)}$, in which case $A(\Sigma,\Sigma')=\Pi_1$, with Π_1 disconnected, since $\Sigma\neq\Sigma'$.

Lemma 2.4.3. *Let Σ and Σ' be distinct components of Π_0. If θ_Σ and $\theta_{\Sigma'}$ are both of type 1, then*

1. *$u_{\Sigma,\Sigma'}(\theta_\Sigma)=\theta_{A(\Sigma')}=k\delta-\theta_{\Sigma'}$, and $u_{\Sigma,\Sigma'}$ is the element of minimal length in $\widehat{W}$, with this property;*
2. *$u^2_{\Sigma,\Sigma'}=1$.*

Proof.

1. Set $u=u_{\Sigma,\Sigma'}$. Since $L(u)\subset\Pi_1$, by Proposition 1.3.4 (3), it suffices to show that $u(\theta_\Sigma)=\theta_{A(\Sigma')}=k\delta-\theta_{\Sigma'}$. This is equivalent to show that $u^{-1}(\theta_{\Sigma'})=k\delta-\theta_\Sigma$. Since $\theta_{\Sigma'}$ is of type 1, hence long, and $u^{-1}(\theta_{\Sigma'})\in A(\Sigma)$, it suffices to show that $(u^{-1}(\theta_{\Sigma'}),\gamma)\geq 0$ for each $\gamma\in A(\Sigma)=A(\Sigma,\Sigma')\cup(\Sigma'\backslash\Gamma(\Sigma'))$. We know that $u=u_{0,\Pi_1}u_0$, where u_0 is the longest element of $W(A(\Sigma,\Sigma'))$ and u_{0,Π_1} is the longest element of $A(\Sigma,\Sigma')\backslash\Pi_1$. Since the only roots in $A(\Sigma,\Sigma')$ not orthogonal to $\theta_{\Sigma'}$ are the roots in Π_1, we see that $u^{-1}(\theta_{\Sigma'})=u_0(\theta_{\Sigma'})$. Thus, since $(\theta_{\Sigma'},\gamma)\leq 0$ when $\gamma\in A(\Sigma,\Sigma')$, we see that $(u^{-1}(\theta_{\Sigma'}),\gamma)=(\theta_{\Sigma'},u_0(\gamma))\geq 0$ for $\gamma\in A(\Sigma,\Sigma')$. Next we deal with the case $\gamma\in\Sigma'\setminus\Gamma(\Sigma')$. If $(\gamma,\theta_{\Sigma'})=0$ then $u(\gamma)=\gamma$, hence $(u^{-1}(\theta_{\Sigma'}),\gamma)=(\theta_{\Sigma'},\gamma)=0$. If instead $(\gamma,\theta_{\Sigma'})\neq 0$ and $ht_\sigma(u(\gamma))=1$, we are done because $(\theta^\vee_{\Sigma'},u(\gamma))=(\theta^\vee_{\Sigma'},\gamma)-1\geq 0$. If $ht_\sigma(u(\gamma))=2$ then $ht_\sigma(k\delta-u(\gamma))=0$, so $k\delta-u(\gamma)$ belongs to some component of Π_0. If this component is Σ', then $0=(k\delta-u(\gamma),\theta_\Sigma)$ gives a contradiction, since $c_\eta(k\delta-u(\gamma))\neq 0$ for all $\eta\in\Sigma$ such that $(\eta,\theta_\Sigma)\neq 0$. In the other case we have $0=(k\delta-u(\gamma),\theta_{\Sigma'})$ hence $0=(u(\gamma),\theta_{\Sigma'})$ and we are done.

2. Set again $u=u_{\Sigma,\Sigma'}$. Since $u_0(\theta_\Sigma)=k\delta-\theta_{\Sigma'}$ we see that, if $\alpha\in\Pi_1$, then $u_0(\alpha)=-\alpha$. In fact, if Z is the component of $A(\Sigma,\Sigma')$ containing α, then, by Remark 2.4.2, α is the only root in Z that is not orthogonal to $\theta_{\Sigma'}$. By [8], it follows that u is an involution which permutes $A(\Sigma,\Sigma')\setminus\Pi_1$ and maps $\alpha\in\Pi_1$ to $-\theta_Z$. □

Lemma 2.4.4. *Let $\Sigma\neq\Sigma'$, $\theta_\Sigma,\theta_{\Sigma'}$ of type 1, $\alpha\in A(\Sigma)_{\overline{\ell}}$, $\beta\in A(\Sigma')_{\overline{\ell}}$, and assume that $w\in\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}\cap\mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$. Then*

1. *$u_{\Sigma,\Sigma'}\leq w$;*
2. *$\alpha\in\Sigma'$ and $\beta\in\Sigma$;*
3. *$uv_\alpha v_\beta\leq w$, where v_α is the element of minimal length in $W(\Sigma')$ that maps α to $\theta_{\Sigma'}$, and v_β is the element of minimal length in $W(\Sigma)$ that maps β to θ_Σ. Moreover, $uv_\alpha=w_{\alpha,k\delta-\theta_\Sigma}$, and $uv_\beta=w_{\beta,k\delta-\theta_{\Sigma'}}$.*

Proof.

1. Let $u=u_{\Sigma,\Sigma'}$. If $u\not\leq w$, then there is $\gamma\in N(u)$ such that $\gamma\notin N(w)$. Note that $(\gamma,\theta^\vee_\Sigma)=(\gamma,\theta^\vee_{\Sigma'})=-1$, hence $\theta_\Sigma+\gamma,\theta_\Sigma+\gamma+\theta_{\Sigma'}\in\widehat{\Delta}$. In particular we have that $k\delta-\theta_\Sigma-\gamma\in N(w)$. But then $k\delta-\theta_{\Sigma'}+k\delta-\gamma-\theta_\Sigma=2k\delta-\theta_\Sigma-\gamma-\theta_{\Sigma'}\in N(w)$, which is absurd. We have therefore $u\leq w$.

2. - 3. From (1) we obtain that $w = uv$ with $v(\alpha) = \theta_{\Sigma'}$. Let $U = \{\beta \in N(v) \mid \theta_{\Sigma'} - \beta \in \widehat{\Delta}^+\}$. Arguing as in the proof of Lemma 2.2.7, we see that U is biconvex, hence there is an element $x \in W(\Sigma')$ such that $N(x) = U$. Since x satisfies the hypothesis of Proposition 1.3.4 (6), we see that there is a root $\gamma \in \Sigma'$ such that $x = v_\gamma$, where v_γ is the element of minimal length that maps γ to $\theta_{\Sigma'}$. We conclude that $v_\gamma \leq v$. We now show that $\ell(uv_\gamma) = \ell(u) + \ell(v_\gamma)$; for this it suffices to prove that $u^{-1}(\eta) = u(\eta) \in \widehat{\Delta}^+$ for $\eta \in N(v_\gamma)$. If not, then $\eta \in N(u)$, hence, by Remark 2.4.1, $(\eta, \theta_{\Sigma'}^\vee) < 0$; but $\eta \in \Sigma'$, hence $(\eta, \theta_{\Sigma'}) \geq 0$. We now prove that $L(uv_\gamma) = \Pi_1$. We have $N(uv_\gamma) = N(u) \cup u(N(v_\gamma))$. Since $L(u) = \Pi_1$, it suffices to prove that $u(\eta) \notin \widehat{\Pi}$ for any $\eta \in N(v_\gamma)$. Since $\eta \in \Sigma'$, we have

$$(u(\eta), \theta_{\Sigma'}) = (\eta, u(\theta_{\Sigma'})) = (\eta, k\delta - \theta_\Sigma) = 0.$$

This implies that if $u(\eta) = \xi \in \widehat{\Pi}$, then $\xi \notin \Pi_1$ and, since $u \in W(A(\Sigma, \Sigma'))$, we see that, for any $\nu \in B_{\Sigma'}$, we have $0 = c_\nu(\xi) = c_\nu(u(\eta)) = c_\nu(\eta)$, hence $(\eta, \theta_{\Sigma'}) = 0$, against Proposition 1.3.4 (8). Since $uv_\gamma(\gamma) = k\delta - \theta_\Sigma = \theta_{A(\Sigma')}$ and $L(uv_\gamma) \subset \Pi_1$, we can apply Proposition 1.3.4 (3), to get $uv_\gamma = w_{\gamma, k\delta - \theta_\Sigma}$. This implies that $w_{\gamma, k\delta - \theta_\Sigma} \leq w$, so, by Proposition 2.3.1, $w \in \mathcal{I}_{\gamma, \mu}$, hence $\alpha = \gamma \in \Sigma'$ and $uv_\alpha \leq w$. Similarly, $\beta = \gamma \in \Sigma$ and $uv_\beta \leq w$. Since

$$N(uv_\alpha v_\beta) = N(u) \cup u(N(v_\alpha)) \cup u(N(v_\beta)), \tag{2.4.5}$$

we get that $uv_\alpha v_\beta \leq w$. □

Proposition 2.4.5. *Assume $\Sigma \neq \Sigma'$, $\theta_\Sigma, \theta_{\Sigma'}$ of type 1, $\alpha \in A(\Sigma)_{\overline{\ell}}$, and $\beta \in A(\Sigma')_{\overline{\ell}}$. Then $\mathcal{I}_{\alpha, k\delta - \theta_\Sigma} \cap \mathcal{I}_{\beta, k\delta - \theta_{\Sigma'}} \neq \emptyset$ if and only if $\alpha \in \Sigma'$ and $\beta \in \Sigma$. In this case,*

$$\min(\mathcal{I}_{\alpha, k\delta - \theta_\Sigma} \cap \mathcal{I}_{\beta, k\delta - \theta_{\Sigma'}}) = uv_\alpha v_\beta,$$

where v_α is the element of minimal length in $W(\Sigma')$ that maps α to $\theta_{\Sigma'}$, and v_β is the element of minimal length in $W(\Sigma)$ that maps β to θ_Σ.

Proof. We first prove that, if $\alpha \in \Sigma'$ and $\beta \in \Sigma$, then $uv_\alpha v_\beta \in \mathcal{I}_{\alpha, k\delta - \theta_\Sigma} \cap \mathcal{I}_{\beta, k\delta - \theta_{\Sigma'}}$. Indeed, it is clear that it suffices to prove that $uv_\alpha v_\beta \in \mathcal{W}_\sigma^{ab}$. As shown above $w_{\alpha, k\delta - \theta_\Sigma} = uv_\alpha$ and $w_{\beta, k\delta - \theta_{\Sigma'}} = uv_\beta$. From (2.4.5) we deduce that $N(uv_\alpha v_\beta) = N(w_{\alpha, k\delta - \theta_\Sigma}) \cup N(w_{\beta, k\delta - \theta_{\Sigma'}})$ hence $uv_\alpha v_\beta$ is a σ-minuscule element.

The remaining statements follow from Lemma 2.4.4. □

Definition 2.4.2. Let $\Sigma \neq \Sigma'$, θ_Σ and $\theta_{\Sigma'}$ of type 1. Consider $\alpha \in \Sigma'_{\overline{\ell}}$, $\beta \in \Sigma_{\overline{\ell}}$ and let v_α be the element of minimal length in $W(\Sigma')$ that maps α to $\theta_{\Sigma'}$ and v_β the element of minimal length in $W(\Sigma)$ that maps β to θ_Σ. Then we set

$$w_{\alpha,\beta} = u_{\Sigma,\Sigma'} v_\alpha v_\beta.$$

Proposition 2.4.6. *Let $\Sigma \neq \Sigma'$, $\theta_\Sigma, \theta_{\Sigma'}$ of type 1, $\alpha \in \Sigma'_{\overline{\ell}}$ and $\beta \in \Sigma_{\overline{\ell}}$. Then*

$$w_{\alpha,\beta} = \sup\{\min \mathcal{I}_{\alpha,\mu}, \min \mathcal{I}_{\beta,\mu'}\}$$

and

$$w_{\alpha,\beta} x \in \mathcal{I}_{\alpha, k\delta - \theta_\Sigma} \cap \mathcal{I}_{\beta, k\delta - \theta_{\Sigma'}}$$

if and only if

$$x \in W((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1) \backslash W(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta).$$

Proof. Since $N(uv_\alpha v_\beta) = N(w_{\alpha,k\delta-\theta_\Sigma}) \cup N(w_{\beta,k\delta-\theta_{\Sigma'}})$, it follows that

$$w_{\alpha,\beta} = \sup\{w_{\alpha,k\delta-\theta_\Sigma}, w_{\beta,k\delta-\theta_{\Sigma'}}\}.$$

Take $x \in W((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)\backslash W(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)$. We now show that $w_{\alpha,\beta}x \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$. We may assume that $x \neq 1$, in particular $|\Sigma| > 1$. It suffices to see that $w_{\alpha,\beta}x$ is σ-minuscule. Writing $w_{\alpha,\beta}x = w_{\alpha,k\delta-\theta_\Sigma}v_\beta x$, by the proof of Theorem 2.3.6, it suffices to prove that $v_\beta x \in V_{\alpha,k\delta-\theta_\Sigma}$. Since we already know that $v_\beta \in V_{\alpha,k\delta-\theta_\Sigma}$, we are left with proving that $ht_{B_\Sigma}(v_\beta(\gamma)) = 1$ for each $\gamma \in N(x)$. We have $(v_\beta(\gamma), \theta_\Sigma) = (\gamma, \beta) = 0$, hence $ht_{B_\Sigma}(v_\beta(\gamma)) = ht_\sigma(v_\beta(\gamma)) = ht_\sigma(\gamma) \geq 1$. Actually, the latter σ-height is 1: if it were 2, then $k\delta - \gamma$ would belong to some component, but this is impossible since both α and β belong to its support.

Viceversa assume $w_{\alpha,\beta}x \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$, with $\ell(w_{\alpha,\beta}x) = \ell(w_{\alpha,\beta}) + \ell(x)$ and $x \neq 1$. By Lemma 2.3.2, we get $v_\beta x \in \widehat{W}_\alpha, v_\alpha x \in \widehat{W}_\beta$; but $v_\beta \in \widehat{W}_\alpha$, hence $x \in \widehat{W}_\alpha$ and similarly $x \in \widehat{W}_\beta$. We are left with proving that $L(x) \subseteq \Pi_1$, so take $\gamma \in N(x) \cap \widehat{\Pi}$. Recall that $v_\beta x \in V_{\alpha,k\delta-\theta_\Sigma}$, hence

$$1 = ht_{B_\Sigma}(v_\beta(\gamma)) = ht_\sigma(v_\beta(\gamma)). \tag{2.4.6}$$

If $\gamma \notin \Pi_1$, then $v_\beta(\gamma) \in \widehat{\Delta}_0$, so $ht_\sigma(v_\beta(\gamma)) = 0$ against (2.4.6). Therefore $\gamma \in \Pi_1$, as desired. □

Proposition 2.4.7. *Assume $\mu, \mu' \in \mathcal{M}_\sigma$, and $\alpha, \beta \in \widehat{\Pi}$. Then $\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} \neq \emptyset$ if and only if either $\alpha = \beta$ and $\mu = \mu'$, or $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta'_\Sigma$, with $\Sigma \neq \Sigma'$ and θ_Σ, $\theta_{\Sigma'}$ of type 1, $\alpha \in \Sigma'_{\overline{\ell}}$, and $\beta \in \Sigma_{\overline{\ell}}$.*

Proof. In Proposition 2.4.5, we settled the cases $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta'_\Sigma$, with $\Sigma \neq \Sigma'$ and θ_Σ, $\theta_{\Sigma'}$ of type 1. It remains to prove that $\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} = \emptyset$ in all other non trivial cases.

We suppose by contradiction that there is $w \in \mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'}$ and treat the possibile cases one by one.

1. Let $\alpha, \beta \in \Pi_1$ and $\mu = k\delta + \beta$, $\mu' = k\delta + \alpha$. Since $N(w_{\beta,k\delta+\alpha}) \subset N(w)$ and $w^{-1}_{\beta,k\delta+\alpha}(\alpha) = -k\delta + \beta$ we see that $\alpha \in N(w)$. If $\Pi_0 = \emptyset$ then $(\alpha, \beta) \neq 0$, so $k\delta + \alpha + \beta \in \widehat{\Delta}$ and this implies that $k\delta + \alpha + \beta \in N(w)$. This is impossible since $ht_\sigma(k\delta + \alpha + \beta) = 4$. If $\Pi_0 \neq \emptyset$ and $\Sigma|\Pi_0$ then $\theta_\Sigma + \alpha \in \widehat{\Delta}$, so $k\delta - \theta_\Sigma - \alpha \in \widehat{\Delta}^+$. Since $k\delta - \alpha = \theta_\Sigma + k\delta - \theta_\Sigma - \alpha$, using the explicit expression for $w_{\beta,k\delta+\alpha}$ given in Proposition 2.2.10, we see that $\theta_\Sigma + \alpha = s_\alpha(\theta_\Sigma) \in N(w)$. Since $\alpha + \beta + \theta_\Sigma \in \widehat{\Delta}$, this implies that $(k\delta + \beta) + (\alpha + \theta_\Sigma) \in N(w)$ and again this gives a contradiction.

2. Let $\alpha, \gamma \in \Pi_1$, $\mu = k\delta + \gamma$, $\mu' = k\delta - \theta_\Sigma$. As above, we see that $\theta_\Sigma + \gamma \in N(w_{\alpha,k\delta+\gamma}) \subset N(w)$. But then $k\delta - \theta_\Sigma + \theta_\Sigma + \gamma = k\delta + \gamma \in N(w)$ and this is impossible.

3. Let $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta_{\Sigma'}$ with θ_Σ of type 2. We have clearly $\Sigma \neq \Sigma'$. Assume first θ_Σ complex. If $\delta - \theta_\Sigma$ is a simple root then $\widehat{\Pi} = \Sigma \cup \Pi_1$ contrary to the assumption that $\Sigma \neq \Sigma'$. Thus $\delta - \theta_\Sigma$ is not simple. We now rely on the explicit description of $w_{\alpha,\mu}$ given in Lemma 2.2.8. If $\gamma \in \Pi_1$, then $\gamma \in N(s_{\delta-\theta_\Sigma})$, hence $2\delta - 2\theta_\Sigma - \gamma \in N(s_{\delta-\theta_\Sigma}) \subset N(w_{\alpha,\mu}) \subset N(w)$. But then $(2\delta - \theta_{\Sigma'}) + (2\delta - 2\theta_\Sigma - \gamma) = 4\delta - \theta_{\Sigma'} - \gamma - \theta_\Sigma \in N(w)$ and this is not possible. It remains to check the case when θ_Σ is short compact. There is only a case when this occurs and Π_0 has more than one component, namely type $B_n^{(1)}$ with $\Pi_1 = \{\alpha_{n-1}\}$. By the explicit description of $w_{\alpha,\mu}$ given for this case in Lemma 2.2.5, we see that $\theta_{\Sigma'} \in N(w_{\alpha,\mu}) \subset N(w)$ and this gives clearly a contradiction. □

2.5 Maximal elements and dimension formulas

In this section we give a parametrization of the maximal ideals in $\mathcal{W}_\sigma^{ab}$ and compute their dimension.

As a first step in our classification of maximal ideals, we determine which $\mathcal{I}_{\alpha,\mu}$ admits maximum. Let Π_1^1 denote the set of roots of type 1 in Π_1.

Proposition 2.5.1.

1. *If θ_Σ is of type 1 (resp. type 2) and $\alpha \in \Gamma(\Sigma)_{\overline{\ell}} \cup \Pi_1^1$ (resp. $\alpha \in \Sigma_{\overline{\ell}}$) then $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$ has maximum.*

2. *If $\theta_\Sigma, \theta_{\Sigma'}, \alpha \in \Sigma', \beta \in \Sigma, \Sigma \neq \Sigma'$, are all roots of type 1, then $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$ has maximum.*

3. *If $\Pi_1^1 = \Pi_1$, then, $\mathcal{I}_{\alpha,\beta+k\delta}$, $\alpha, \beta \in \Pi_1$ has maximum if nonempty.*

Proof. Recall that, by Theorem 2.3.6, $\mathcal{I}_{\alpha,\mu}$ is isomorphic to $\widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$. The subgroup $\widehat{W}_{\alpha,\mu}$ is standard parabolic for any α and μ except when $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 1, $|\Sigma| > 1$, and $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$. The existence of the maximum in cases (1) and (3) follows now from subsection 1.3.3. The same applies to $\mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}} \cap \mathcal{I}_{\beta,k\delta-\theta_\Sigma}$, by Theorem 2.4.1. □

We already saw in Proposition 2.3.1 that, in many cases, the maximal elements of $\mathcal{I}_{\alpha,\mu}$ are maximal in $\mathcal{W}_\sigma^{ab}$. The next result deals with the missing cases.

Proposition 2.5.2. *If $\Pi_1 = \Pi_1^1 = \{\alpha, \beta\}$ (with possibly $\alpha = \beta$), θ_Σ is of type 1, and $w \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$, then $w \leq max(\mathcal{I}_{\alpha,k\delta+\beta})$.*

Proof. By Proposition 2.5.1, $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$ has maximum. From subsection 1.3.3, we see that its maximum is $w_{max} = w_{\alpha,k\delta-\theta_\Sigma} w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}$, where w_{0,B_Σ} is the longest element of $W(\widehat{\Pi}_\alpha \backslash B_\Sigma)$ and $w_{0,\widehat{\Pi}_\alpha}$ is the longest element of $\widehat{W}_\alpha$. Clearly there is a root $\alpha_\Sigma \in \Sigma$ such that $(\alpha_\Sigma, \alpha) \neq 0$, and we note that this root is necessarily unique, for, otherwise, $\Sigma \cup \{\alpha\}$ would contain a loop, and this is only possible in the adjoint case of type A_n. But in this case α is not of type 1.

We now show that $w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}(\alpha_\Sigma) = \theta_\Sigma$. This is clear if $|\Sigma| = 1$ so we assume $|\Sigma| > 1$. Recall that $w_{0,\alpha}$ is the longest element of $W((\Pi_0)_\alpha)$. Let w_{B_Σ} be the longest element of $W((\Pi_0)_\alpha \backslash B_\Sigma)$. Obviously $N(w_{B_\Sigma} w_{0,\alpha}) \subset N(w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha})$ and we know that $w_{B_\Sigma} w_{0,\alpha}(\alpha_\Sigma) = \theta_\Sigma$. We show that $v(\alpha_\Sigma) = \theta_\Sigma$ for any v such that $w_{B_\Sigma} w_{0,\alpha} \leq v \leq w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}$. This is proven by induction on $\ell(v) - \ell(w_{B_\Sigma} w_{0,\alpha})$. Assume that $v(\alpha_\Sigma) = \theta_\Sigma$ and $w_{B_\Sigma} w_{0,\alpha} \leq v < v s_\gamma \leq w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}$ with $\gamma \in \widehat{\Pi}_\alpha$. We need to prove that $v s_\gamma(\alpha_\Sigma) = \theta_\Sigma$. Set $(\alpha_\Sigma, \gamma^\vee) = -r$ with $r \in \mathbb{Z}^+$. Then $v s_\gamma(\alpha_\Sigma) = \theta_\Sigma + r v(\gamma)$. Observe that $v \in V_{\alpha,k\delta-\theta_\Sigma}$, so $ht_{B_\Sigma}(v(\gamma)) = 1$. It follows that $ht_{B_\Sigma}(v s_\gamma(\alpha_\Sigma)) = 2 + r$. We claim that $ht_{B_\Sigma}(\nu) \leq 2$ for any $\nu \in \Delta(\widehat{\Pi} \backslash \{\alpha\})$. Indeed this is obvious if $|\Pi_1| = 1$ and, in the hermitian symmetric case it follows from (2.3.1) and the observation that, in this case, $ht_\sigma(\nu) \le 1$. We conclude that $r = 0$ and $vs_\gamma(\alpha_\Sigma) = \theta_\Sigma$.

We have shown that $w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}(\alpha_\Sigma) = \theta_\Sigma$, therefore we get $w_{max}(\alpha_\Sigma) = w_{\alpha,k\delta-\theta_\Sigma}(\theta_\Sigma)$. Now

$$ht_\sigma(w_{\alpha,k\delta-\theta_\Sigma}(\theta_\Sigma)) = (w_{\alpha,k\delta-\theta_\Sigma}(\theta_\Sigma), k\delta - \theta_\Sigma^\vee) + ht_{B_\Sigma}(\theta_\Sigma) = 1.$$

This proves that $w_{max}s_{\alpha_\Sigma} \in \mathcal{W}_\sigma^{ab}$, so $w_{max}s_{\alpha_\Sigma} \in \mathcal{I}_{\alpha,k\delta+\beta}$. □

Proposition 2.5.1 allows us to give the following definition:

Definition 2.5.1. If θ_Σ is of type 1 (resp. type 2) and $\alpha \in \Gamma(\Sigma)_{\overline{\ell}}$ (resp. $\alpha \in \Sigma_{\overline{\ell}}$), we let $MI(\alpha)$ be the maximum of $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$. If $\Sigma \ne \Sigma'$ and $\theta_\Sigma, \theta_{\Sigma'}, \alpha \in \Sigma', \beta \in \Sigma$ are all roots of type 1, we let $MI(\alpha,\beta)$ be the maximum of $\mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}} \cap \mathcal{I}_{\beta,k\delta-\theta_\Sigma}$. If $\alpha, \beta \in \Pi_1^1$ with $\mathcal{I}_{\alpha,k\delta+\beta} \ne \emptyset$, we let $MI(\alpha)$ be its maximum.

We are finally ready to state the main result of the paper, which gives a complete parametrization of the set of maximal abelian $\mathfrak{b}_0$-stable subspaces in $\mathcal{W}_\sigma^{ab}$. For notational reasons, it is convenient to fix an arbitrary total order $\prec$ on the components of Π_0.

Theorem 2.5.3. *The maximal $\mathfrak{b}_0$-stable abelian subalgebras are parametrized by the set*

$$\mathcal{M} = \left(\bigcup_{\substack{\Sigma|\Pi_0 \\ \Sigma \text{ of type 1}}} \Gamma(\Sigma)_{\overline{\ell}} \right) \cup \left(\bigcup_{\substack{\Sigma|\Pi_0 \\ \Sigma \text{ of type 2}}} \Sigma_{\overline{\ell}} \right) \cup \left(\bigcup_{\substack{\Sigma,\Sigma'|\Pi_0,\, \Sigma\prec\Sigma' \\ \Sigma,\Sigma' \text{ of type 1}}} (\Sigma_{\overline{\ell}} \times \Sigma'_{\overline{\ell}}) \right) \cup \Pi_1^1. \quad (2.5.1)$$

Remark 2.5.1. In the adjoint case, there is just one component Σ in Π_0, which is the set of simple roots of $\mathfrak{g}$. In the r.h.s of (2.5.1) the only surviving term is $\Sigma_{\overline{\ell}}$, so $\mathcal{M}$ is the set of long simple roots of $\mathfrak{g}$. This parametrization has been first discovered by Panyushev [17].

Now we begin to work in view of the proof of Theorem 2.5.3. We need to study the maximal elements of $\mathcal{I}_{\alpha,\mu}$. This is immediate when $\mathcal{I}_{\alpha,\mu}$ has maximum, more delicate in the other cases. We also need to determine when a maximal element of $\mathcal{W}_\sigma^{ab}$ occurs in different $\mathcal{I}_{\alpha,\mu}$'s. The description of the intersections among different $\mathcal{I}_{\alpha,\mu}$'s given in Section 2.4 is the key to solve both problems. We start with the following

Lemma 2.5.4. *Assume $\Sigma \ne \Sigma'$. If $\theta_\Sigma, \theta_{\Sigma'}, \alpha \in \Sigma$, are all roots of type 1 and $w \in \mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}}$ is maximal, then there is $\eta \in \Sigma'$ such that $w(\eta) = k\delta - \theta_\Sigma$.*

Proof. Write $w = w_{\alpha,k\delta-\theta_{\Sigma'}}x$ with x maximal in $V_{\alpha,k\delta-\theta_{\Sigma'}}$. If $\Sigma' = \{\theta_{\Sigma'}\}$, then by Lemma 2.3.5 and Theorem 2.3.6, $x = 1$, so $w(\theta_{\Sigma'}) = u_{\Sigma,\Sigma'}v_\alpha(\theta_{\Sigma'}) = k\delta - \theta_\Sigma$.

If $|\Sigma'| > 1$, then by Definition 2.3.3, we have that $\widehat{W}_{\alpha,k\delta-\theta_{\Sigma'}} \ne \{1\}$. It follows that x cannot be the longest element of $W(\widehat{\Pi}_\alpha)$, hence there is a root γ in $\widehat{\Pi}_\alpha$ such that $x(\gamma) > 0$. Since x is maximal, then $ht_{B_{\Sigma'}}(x(\gamma)) \ne 1$, hence $ht_{B_{\Sigma'}}(x(\gamma)) \in \{0, 2\}$. Next we exclude that $ht_{B_{\Sigma'}}(x(\gamma)) = 0$ for all γ. We start with proving that if $ht_{B_{\Sigma'}}(x(\gamma)) = 0$, then $x(\gamma)$ is simple. Indeed, if $x(\gamma) - \beta \in \widehat{\Delta}_\alpha^+$ with $\beta \notin B_{\Sigma'}$, then, by convexity of $N(x)$, we have that $\beta\in N(x)$, contradicting the fact that $x\in V_{\alpha,k\delta-\theta_{\Sigma'}}$. If, for all roots γ in $\widehat{\Pi}_\alpha$ such that $x(\gamma)>0$ we have that $x(\gamma)\in\widehat{\Pi}\backslash B_{\Sigma'}$, then, arguing as in Proposition 2.2.1, we see that $N(x)$ is the set of roots β in $\widehat{\Delta}^+_\alpha$ such that $ht_{B_{\Sigma'}}(\beta)>0$. Since $(\theta_{\Sigma'},\theta^\vee_{\Sigma'})=2$ and $|\Sigma'|>1$, we see that this contradicts again the fact that $x\in V_{\alpha,k\delta-\theta_{\Sigma'}}$.

Therefore there is γ such that $ht_{B_{\Sigma'}}(x(\gamma))=2$. Then, arguing as above, we see that $x(\gamma)$ is minimal among the roots β such that $ht_{B_{\Sigma'}}(\beta)=2$. By Lemma 2.3.5, we have that $x(\gamma)=\theta_{\Sigma'}$.

Arguing as in the proof of parts (2), (3) of Lemma 2.4.4, one checks that there is $\eta\in\Sigma'$ such that $v_\eta\le x$. It follows that $w_{\alpha,k\delta-\theta_{\Sigma'}}v_\eta\le w$. Since $w_{\alpha,k\delta-\theta_{\Sigma'}}v_\eta=u_{\Sigma,\Sigma'}v_\alpha v_\eta\in\mathcal{I}_{\eta,k\delta-\theta_\Sigma}$, by Proposition 2.3.1 we have $w\in\mathcal{I}_{\eta,k\delta-\theta_\Sigma}$ as desired. □

We are now ready to prove Theorem 2.5.3.

Proof of Theorem 2.5.3. Consider the map $MI:\mathcal{M}\to\mathcal{W}^{ab}_\sigma$ defined in Definition 2.5.1. Let MAX be the set of maximal abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$. By Propositions 2.3.1 and 2.5.2, it is clear that $MI(m)\in MAX$ for any $m\in\mathcal{M}$. We next prove that $MI:\mathcal{M}\to MAX$ is bijective. First we show that $MI(\mathcal{M})=MAX$. Let w be maximal. By Proposition 2.2.1 we have that w is maximal in $\mathcal{I}_{\alpha,\mu}$ for some $\mu\in\mathcal{M}_\sigma$. If $\alpha\in\Pi_1$ and it is of type 2, then μ is of type 2, hence $\mu=k\delta-\theta_\Sigma$ with θ_Σ of type 2, but this case is ruled out by Theorem 2.2.11. We can therefore assume α of type 1. From Proposition 2.5.2 we deduce $\mu=\beta+k\delta$ so that $\alpha,\beta\in\Pi_1^1$. Hence $w=MI(\alpha)$. If $\alpha\notin\Pi_1$ then, by Proposition 2.2.10, we have that $\mu=k\delta-\theta_\Sigma$. If $\alpha\in\Sigma$ and θ_Σ is of type 1 (resp. type 2), then by Theorem 2.2.11, we have $\alpha\in\Gamma(\Sigma)_{\overline{\ell}}$ (resp. $\Sigma_{\overline{\ell}}$) and by Proposition 2.5.1 we have $w=MI(\alpha)$. Finally assume $\alpha\in\Sigma'\neq\Sigma$. In particular, by Theorem 2.2.11, α, θ_Σ, and $\theta_{\Sigma'}$ are of type 1. By Lemma 2.5.4 and Proposition 2.5.1 (2), we see that there is $\beta\in\Sigma'$ such that $w=MI(\alpha,\beta)$.

Finally we prove that MI is injective. Set

$$Y=\bigcup_{\substack{\Sigma|\Pi_0\\ \Sigma\text{ of type }1}}\Gamma(\Sigma)_{\overline{\ell}}\cup\bigcup_{\substack{\Sigma|\Pi_0\\ \Sigma\text{ of type }2}}\Sigma_{\overline{\ell}}\cup\Pi_1^1. \tag{2.5.2}$$

If $\alpha,\beta\in Y$, it follows readily from Theorem 2.4.1 that $MI(\alpha)=MI(\beta)$ implies $\alpha=\beta$. Theorem 2.4.1 also implies that $MI(\alpha)\neq MI(\beta,\gamma)$ for $\alpha\in Y$ and $(\beta,\gamma)\in\Sigma_{\overline{\ell}}\times\Sigma'_{\overline{\ell}}$ with $\beta,\gamma,\Sigma,\Sigma'$ of type 1. Suppose finally that $MI(\alpha,\beta)=MI(\gamma,\eta)$ with $\alpha\in\Sigma_{\overline{\ell}}$, $\beta\in\Sigma'_{\overline{\ell}}$, $\gamma\in\Sigma''_{\overline{\ell}}$, $\eta\in\Sigma'''_{\overline{\ell}}$, and $\Sigma,\Sigma',\Sigma'',\Sigma'''$ all of type 1, and $\Sigma\prec\Sigma'$, $\Sigma''\prec\Sigma'''$. Set $w=MI(\alpha,\beta)=MI(\gamma,\eta)$. We have $w\in\mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}}\cap\mathcal{I}_{\gamma,k\delta-\theta_{\Sigma'''}}\neq\emptyset$. Thus either $\alpha=\gamma$ and $\Sigma'=\Sigma'''$ or $\gamma\in\Sigma'$ and $\alpha\in\Sigma'''$. In the first case we have $w(\eta)=k\delta-\theta_\Sigma$ so $\beta=\eta$. In the second case we have $\Sigma=\Sigma'''$ and $\Sigma''=\Sigma'$ contradicting the fact that $\Sigma''\prec\Sigma'''$. □

We can improve Theorem 2.5.3 by computing the dimension of maximal abelian subspaces.

Recall from (1.3.4) that g_R denotes the dual Coxeter number of a finite irreducible root system R. Suppose $\Sigma|\Pi_0$. To simplify notation, we set $g_\Sigma=g_{\Delta(\Sigma)}$ and, if θ_Σ is type 1, $g_{A(\Sigma)}=g_{\Delta(A(\Sigma))}$ (note that in this case $\Delta(A(\Sigma))$ is irreducible by Remark 2.2.2). Also recall from Section 1.3.1 that K is the canonical central element of $\widehat{L}(\mathfrak{g},\sigma)$ and $\mathbf{g}$ is its dual Coxeter number and from Section 2.1 that we denote by a the squared length of a long root in $\widehat{\Delta}^+$.

Lemma 2.5.5. *Let $\gamma \in \widehat{\Delta}_{re}$. Then*

1. $(k\delta+\gamma)^\vee = \frac{a}{\|\gamma\|^2}K+\gamma^\vee$. *In particular,* $(k\delta-\theta_\Sigma)^\vee = r_\Sigma K - \theta_\Sigma^\vee$.
2. *If θ_Σ is of type 1, then* $g_{A(\Sigma)} = \mathbf{g} - g_\Sigma + 2$. *In particular, if $\alpha \in A(\Sigma)_{\overline{\ell}}$, then* $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = \mathbf{g} - g_\Sigma$.
3. *If θ_Σ is of type 2 and $\alpha \in \Sigma_{\overline{\ell}}$, then* $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = \mathbf{g}-1$.
4. *If $\alpha,\beta \in \Pi_1^1$, and $\beta \neq \alpha$ if $|\Pi_1^1| = 2$, then* $\ell(w_{\beta,k\delta+\alpha}) = \mathbf{g}-1$.

Proof. We compute, using (1.3.1):

$$(k\delta+\gamma)^\vee = \frac{2k}{\|\gamma\|^2}\nu^{-1}(\delta)+\gamma^\vee = \frac{k}{a_0}\frac{\|\delta-a_0\alpha_0\|^2}{\|\gamma\|^2}K+\gamma^\vee.$$

A direct inspection shows that $\frac{k\|\delta-a_0\alpha_0\|^2}{a_0} = a$. This proves (1).

To prove the first part of (2) we observe that $g_{A(\Sigma)} = ht_{\widehat{\Pi}^\vee}((k\delta-\theta_\Sigma)^\vee)+1$. The result then follows readily from (1). By Proposition 1.3.4 (3) and (2.2.6), we see that if θ_Σ is of type 1 and $\alpha \in A(\Sigma)_{\overline{\ell}}$ then $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = g_{A(\Sigma)}-2 = \mathbf{g}-g_\Sigma$. For (3) recall that $w_{\alpha,k\delta-\theta_\Sigma} = sv_\alpha$, s being the element of $\widehat{W}$ described in Lemma 2.2.4 and v_α the element of minimal length in $W(\Sigma)$ mapping α to θ_Σ. It follows that $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = \ell(s)+g_\Sigma-2$. It is therefore enough to show that $\ell(s) = \mathbf{g}-g_\Sigma+1$. Recall that, by Lemma 2.2.12,

$$w^{-1}(\lambda) = \lambda - \sum_{i=1}^{l}(\lambda,\beta_j^\vee)\alpha_{i_j}. \tag{2.5.3}$$

Here $w \in \widehat{W}$, $\lambda \in \widehat{\mathfrak{h}}^*$, $s_{i_1}\cdots s_{i_l}$ is a reduced expression of w and $\beta_j = s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})$ (so that $N(w) = \{\beta_1,\dots,\beta_l\}$ and $l = \ell(w)$). Applying (2.5.3) to $w = s$ and $\lambda = k\delta-\theta_\Sigma$ and using Lemma 2.2.4, we obtain that $s(\lambda) = \lambda - 2\sum_{i=1}^{l} r_j\alpha_{i_j}$, where $r_j = \frac{\|\lambda\|^2}{\|\beta_j\|^2}$. In turn, recalling that $s(\mu) = \theta_\Sigma$ and applying $\frac{2}{(\theta_\Sigma,\theta_\Sigma)}\nu^{-1}$ to the previous equality we get

$$\theta_\Sigma^\vee = (k\delta-\theta_\Sigma)^\vee - 2\sum_{j=1}^{l}\alpha_{i_j}^\vee.$$

In particular, taking $ht_{\widehat{\Pi}^\vee}$ of both sides, we obtain $2\ell(s) = ht_{\widehat{\Pi}^\vee}((k\delta-\theta_\Sigma)^\vee)-g_\Sigma+1$. Now use part (1) (recall that $r_\Sigma = 2$) to finish the proof.

To prove (4), we recall that, by Proposition 2.2.10 $w_{\beta,\alpha+k\delta} = s_\alpha w_{0,\alpha}w_0$, hence $N(w_{\beta,\alpha+k\delta}) = \{\alpha\}\cup s_\alpha N(w_{0,\alpha}w_0)$. By definition, for all $\gamma \in N(w_{0,\alpha}w_0)$, we have $(\gamma,\alpha^\vee) < 0$, hence $(s_\alpha\gamma,\alpha^\vee) > 0$. Now it is clear that $s_\alpha\gamma \neq \alpha$, so that $s_\alpha\gamma-\alpha$ is a root. Since

$$\frac{\|s_\alpha\gamma-\alpha\|^2}{\|\alpha\|^2} = 1+\frac{\|s_\alpha\gamma\|^2}{\|\alpha\|^2}-(s_\alpha\gamma,\alpha^\vee).$$

and α is long, then $(s_\alpha\gamma,\alpha^\vee) = 2$ and $\|s_\alpha\gamma-\alpha\| = 0$ or $(s_\alpha\gamma,\alpha^\vee) = 1$. The first case implies $s_\alpha\gamma = c\delta+\alpha$ for some $c \in \mathbb{R}\setminus\{0\}$. This is not possible, since $ht_\sigma(s_\alpha\gamma) = 1$. Hence $(s_\alpha\gamma,\alpha^\vee) = 1$ for all $\gamma \in N(w_{0,\alpha}w_0)$. Now, formula (2.5.3) with $w = w_{\beta,\alpha+k\delta}$ and $\lambda = \alpha + k\delta$ gives $\beta = \alpha + k\delta - 2\alpha - \sum_{i=2}^{l}(\alpha, \beta_i^\vee)\alpha_{i_j}$, with $\{\beta_2, \dots, \beta_l\} = s_\alpha N(w_{0,\alpha}w_0)$ and, applying $\frac{2}{(\alpha,\alpha)}\nu^{-1}$,

$$\beta^\vee = K - \alpha^\vee - \sum_{i=2}^{l} \alpha_{i_j}^\vee.$$

It follows that $l = \mathbf{g} - 1$, as claimed. □

If $\mathfrak{g}$ is a simple Lie algebra, let $g_\mathfrak{g}$ be the dual Coxeter number of the root system of $\mathfrak{g}$. It is know that $\mathbf{g} = g_\mathfrak{g}$ if $\mathfrak{g}$ is simple and that $\mathbf{g} = g_\mathfrak{g}$ in the adjoint case $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$. The following result gives our dimension formulas.

Theorem 2.5.6. *If θ_Σ is of type 1 and $\alpha \in \Gamma(\Sigma)_{\overline{\ell}}$, then*

$$\dim MI(\alpha) = \mathbf{g} - g_\Sigma + |\widehat{\Delta}_\alpha^+| - |\Delta^+(\widehat{\Pi}_{\alpha,\mu}^*)|. \tag{2.5.4}$$

If $\alpha \in \Pi_1^1$, or $\alpha \in \Sigma_{\overline{\ell}}$ with θ_Σ of type 2, then

$$\dim MI(\alpha) = \mathbf{g} - 1 + |\widehat{\Delta}_\alpha^+| - |\Delta^+(\widehat{\Pi}_{\alpha,\mu}^*)|. \tag{2.5.5}$$

If $\alpha \in \Sigma_{\overline{\ell}}$, $\beta \in \Sigma'_{\overline{\ell}}$, with $\Sigma \neq \Sigma'$ and θ_Σ, $\theta_{\Sigma'}$ of type 1, then

$$\dim MI(\alpha, \beta) = \mathbf{g} - 2 + |\Delta^+(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)| - |\Delta^+((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)|. \tag{2.5.6}$$

Proof. By (1.3.8), Theorems 2.2.11 and 2.3.6 imply that, for $\alpha \in Y$ (cf. (2.5.2))

$$\dim MI(\alpha) = \ell(w_{\alpha,\mu}) + |\widehat{\Delta}_\alpha^+| - |\Delta^+(\widehat{\Pi}_{\alpha,\mu}^*)|.$$

Using part (2) of the previous Lemma we obtain (2.5.4). Likewise, if θ_Σ is of type 2 and $\alpha \in \Sigma_{\overline{\ell}}$, or $\alpha \in \Pi_1^1$ then (2.5.5) follows from (3), (4) in Lemma 2.5.5.

Finally, we have to prove (2.5.6). Theorem 2.4.1 gives

$$\dim MI(\alpha, \beta) = \ell(w_{\alpha,\beta}) + |\Delta^+(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)| - |\Delta^+((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)|.$$

So it remains to show that $\ell(w_{\alpha,\beta}) = \mathbf{g} - 2$. From Lemma 2.4.4 (3), we know that $w_{\alpha,\beta} = u_{\Sigma,\Sigma'}v_\alpha v_\beta$ with $\ell(w_{\alpha,\beta}) = \ell(u_{\Sigma,\Sigma'}) + \ell(v_\alpha) + \ell(v_\beta)$, where v_α, v_β are the elements of minimal length mapping α, β, respectively, to the highest root of their component. By Proposition 1.3.4 (4), the lengths of the latter elements are $g_\Sigma - 2$, $g_{\Sigma'} - 2$, respectively. We know that $u_{\Sigma,\Sigma'}v_\beta$ is the element of minimal length in $W(A(\Sigma))$ mapping β to $k\delta - \theta_\Sigma$. Hence $\ell(u_{\Sigma,\Sigma'}) + \ell(v_\beta) = g_{A(\Sigma)} - 2$. Using Lemma 2.5.5 (1), we have

$$\ell(u_{\Sigma,\Sigma'}) = g_{A(\Sigma)} - g_{\Sigma'} = \mathbf{g} - g_\Sigma - g_{\Sigma'} + 2,$$

hence (2.5.6) is proven. □

Remark 2.5.2. The dimension formula in the adjoint case is a specialization of (2.5.5) and is due to Suter [20]. For a refinement of Suter's formula, see [4, Theorem 8.13].

Proposition 2.5.7. *In the hermitian case, if $\alpha \in \Pi_1$, we have*

$$\dim(MI(\alpha)) = \frac{\dim(\mathfrak{g}_1)}{2}.$$

Proof. Let $\Pi_1 = \{\alpha, \beta\}$. It is clear that a root of $\widehat{L}(\mathfrak{g}, \sigma)$ has σ-height 1 if it is greater or equal than exactly one among α, β. Hence

$$t^{-1} \otimes \mathfrak{g}_1 = \bigoplus_{\substack{\gamma \geq \alpha \\ \beta \notin Supp(\gamma)}} \widehat{L}(\mathfrak{g}, \sigma)_{-\gamma} \oplus \bigoplus_{\substack{\gamma \geq \beta \\ \alpha \notin Supp(\gamma)}} \widehat{L}(\mathfrak{g}, \sigma)_{-\gamma}. \tag{2.5.7}$$

Since there is a automorphism of the Dynkin diagram of $\widehat{L}(\mathfrak{g}, \sigma)$ switching the elements of Π_1, the two summands in the r.h.s. of (2.5.7) have both dimension $\dim(\mathfrak{g}_1)/2$. Set $F_\alpha = \{-\gamma \in \widehat{\Delta} \mid \gamma \geq \beta \textit{ and } \alpha \notin Supp(\gamma)\}$. It is clear that, if $-\gamma', -\gamma'' \in F_\alpha$, then $-\gamma' - \gamma'' \notin \widehat{\Delta}$; moreover, for each $\eta \in \Delta_0^+$ such that $-\gamma + \eta \in \widehat{\Delta}$ we have that $-\gamma + \eta \in F_\alpha$. It follows that $\bigoplus_{\gamma \geq \beta, \alpha \notin Supp(\gamma)} \widehat{L}(\mathfrak{g}, \sigma)_{-\gamma}$ is an abelian $\mathfrak{b}_0$-stable subspace of $t^{-1} \otimes \mathfrak{g}_1$, hence, by Remark 1.2.2, it corresponds to a $\mathfrak{b}_0$-stable abelian subspace of $\mathfrak{g}_1$. In order to conclude the proof, we shall prove that the element of $\mathcal{W}_\sigma^{ab}$ corresponding to the latter subspace is $MI(\alpha)$. Set $z = MI(\alpha)$. By formula (2.2.6), Theorem 2.2.11, and Lemma 2.3.2, $z = s_\beta z'$ with $z' \in W(\widehat{\Pi} \setminus \{\alpha\})$. It follows that $N(z) \subseteq -F_\alpha$ and therefore, by the maximality of $MI(\alpha)$, that $N(z) = -F_\alpha$: this proves the claim. □

Remark 2.5.3. If we take $\Pi_1 = \{\alpha_0, \beta\}$, where α_0 is the extra node of the extended Dynkin diagram associated to $\mathfrak{g}$, then the sum $\mathfrak{i}$ of all root subspaces corresponding to $\{\gamma \geq \beta \mid \alpha_0 \notin Supp(\gamma)\}$ is an ideal of the Borel subalgebra of $\mathfrak{g}$ corresponding to the simple system $\widehat{\Pi} \setminus \{\alpha_0\}$. Moreover, if w is the element associated to this abelian ideal via Peterson's bijection quoted in the Introduction, then $N(w) = \{\gamma \geq \alpha_0 \mid \beta \notin Supp(\gamma)\}$. Now Proposition 2.2.10 implies that $w(\beta) = \delta + \alpha_0$, hence this ideal is included in the maximal ideal associated to β via the Panyushev bijection [17]. By Theorem 2.5.6 and Suter dimension formula, we obtain that $\mathfrak{i}$ is exactly this maximal ideal. Notice that this applies to any simple root β of $\mathfrak{g}$ that occurs with coefficient 1 in the highest root of $\mathfrak{g}$.

Remark 2.5.4. As recalled in the Introduction, Panyushev [16] investigated the maximal eigenvalue of the Casimir element of $\mathfrak{g}_0$ w.r.t. the Killing form of $\mathfrak{g}$. In particular he showed that in the hermitian case $N = \frac{\dim(\mathfrak{g}_1)}{2}$ gives the required maximal eigenvalue. By the previous Proposition, if $v_1, \ldots, v_N$ is any basis of $MI(\alpha)$, then $v_1 \wedge \ldots \wedge v_N$ is an explicit eigenvector of maximal eigenvalue.

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