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Eigenvalues and homogenization of some fully nonlinear partial differential equations

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Introduction

This thesis is a contribution to the theory of viscosity solutions of fully nonlinear equations of first and second order. It is divided in two parts. The first one, which collects the results contained in [97], [96] and [98], studies the generalized principal eigenvalues associated to nonlinear elliptic operators of second order with Neumann boundary conditions. Three different classes of operators are considered respectively in Chapters 1, 2 and 3.

The second part is dedicated to homogenization problems for first order local and non-local Hamilton-Jacobi equations. In Chapter 4 we presents the results of [92], where we study the time-dependent Peierls-Nabarro model which is a phase field model for dislocation dynamics. This model leads to a non-local parabolic PDE with a first order Lévy operator. After a proper rescaling, a macroscopic model describing the evolution of a density of dislocations is obtained. In Chapter 5 we study the rate of convergence in homogenization of local parabolic Hamilton-Jacobi equations, the results of this chapter are contained in [2].

We recall that viscosity solutions were introduced by M. G. Crandall and P. L. Lions [41] in 1981 for first order Hamilton-Jacobi equations. Successively, many authors have contributed to the theory of viscosity solutions which was extended to a large class of first and second order partial differential equations. For a good description of viscosity solutions we refer to Crandall, Ishii and Lions [40], to the CIME course [14] and to the books: Fleming and Soner [54] for applications to the stochastic control theory; Barles [17], Bardi and Capuzzo Dolcetta [13] for first order equations and applications to the deterministic control; Cabré and Caffarelli [32] for the regularity of solutions of uniformly elliptic equations.

Motivated by applications to finance but also by an increasing number of other mathematical models (physical sciences, mechanics, biological models etc.), the theory of viscosity solutions has been extended to the context of partial integro-differential equations, i.e. partial equations involving non-local operators such as the Lévy ones. To the best of our knowledge the first paper devoted to this extension was the one by Soner [107] in the context of stochastic control of jump diffusion processes. Following Soner's work, a quite general class of integro-differential equations nonlinear with respect the non-local operators was developed by Awatif, [12]. Successively, many results were extended to equations involving also second order derivatives, see for instance [84] and [22].

In the framework of viscosity solutions, the homogenization problem for first order Hamilton-Jacobi equations has been extensively studied. First of all, Lions Papanicolaou and Varadhan [88] completely solved the problem in the case of time-independent periodic and coercive Hamiltonians. After this seminal paper, homogenization of Hamilton-Jacobi equations for coercive Hamiltonians has been treated for a wider class of periodic situations, c.f. Ishii [72], for problems set on bounded domains, c.f. Alvarez [3], Horie and Ishii [64], for equations with different structures, c.f. Alvarez and Ishii [8], for deterministic control problems in L^∞, c.f. Alvarez and Barron [6], for almost periodic Hamiltonians, c.f. Ishii [70], and for Hamiltonians with stochastic dependence, c.f. Souganidis [108].

For different structures assumptions, non-local operators and further results on homogenization we refer the reader to e.g. [49], [50], [53], [24], [18], [67], [68], [5], [4], [35] and references therein.

PART I

I.1 An overview of the literature

The eigenvalue problem for linear elliptic operators has a very long history and goes back to Fourier. Here we limit to recall some classical results.

Let L be a general uniformly elliptic linear operator of second order defined in a bounded domain $\Omega \subset \mathbb{R}^N$, of the form

$$Lu = -a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u, \quad \text{in } \Omega \tag{I.1.1}$$

that is uniformly elliptic, i.e.,

$$a|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le A|\xi|^2 \quad \text{for any } x \in \Omega,\ \xi \in \mathbb{R}^N,$$

with a and A positive constants. Let B be the boundary operator defined by

$$Bu = \gamma(x)u + \delta\frac{\partial u}{\partial \overrightarrow{n}} \quad \text{on } \partial\Omega,$$

where $\overrightarrow{n}(x)$ is the exterior normal to Ω at $x \in \partial\Omega$, and either $\delta = 0$ and $\gamma \equiv 1$ (i.e. B is the Dirichlet boundary operator) or $\delta = 1$ and $\gamma \ge 0$ (i.e. B is the Neumann or oblique derivative boundary operator).

It is well known that when $\partial\Omega$ and the coefficients of L and B are sufficiently regular, there is a real number λ_1, called *first* or *principal eigenvalue* for L with associated the boundary operator B, such that

a) (λ_1 is an eigenvalue) there exists a positive function ϕ, called *principal eigenfunction*, solution of

$$\begin{cases} L\phi = \lambda_1\phi & \text{in } \Omega \\ B\phi = 0 & \text{on } \partial\Omega; \end{cases} \tag{I.1.2}$$

b) (λ_1 is the smallest real eigenvalue) for any eigenvalue $\lambda \neq \lambda_1$ of L with boundary operator B then

$$\text{Re}\lambda > \lambda_1.$$

The existence of such a pair (λ_1, ϕ) follows from the Krein Rutman Theorem, see for example [9]. Other properties of the principal eigenvalue are the following: λ_1 is simple, i.e. any solution of (I.1.2) is a multiple of ϕ; if ψ is a positive eigenfunction with eigenvalue λ, then $\lambda = \lambda_1$; λ_1 is an isolated eigenvalue.

When L associated to the boundary operator B is self-adjoint (with respect to $L^2(\Omega)$), i.e. L has the form

$$Lu = -\partial_i(a_{ij}(x)\partial_j u) + c(x)u,$$

$\delta = 0$ and $\gamma \equiv 1$, the first eigenvalue can be characterized as the minimum of the so called Rayleigh quotient:

$$\lambda_1 = \min_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega (a_{ij}(x)\partial_i u \partial_j u + c(x)u^2)dx}{\int_\Omega u^2 dx}, \tag{I.1.3}$$

see e.g. [61]. A variational formulation for the principal eigenvalue of a general linear second-order operator with Dirichlet boundary conditions and with maximum principle was given by Donsker and Varadhan in [45] and [46].

More recently, Berestycki, Nirenberg and Varadhan [26] gave a characterization of the first eigenvalue of a linear uniformly elliptic operator of second order through the maximum principle. They consider the operator (I.1.1) defined in a bounded domain Ω and with associated Dirichlet boundary conditions. In addition to the uniform ellipticity condition, they assume on the coefficients of L:

$$a_{ij} \in C(\Omega), \quad b_i,\, c \in L^\infty(\Omega) \quad \text{for any } i, j.$$

One of the main feature of the paper is that no regularity is required on $\partial\Omega$. In this thesis we will not deal with the question of the regularity of the domain and will always assume $\partial\Omega$ smooth.

Berestycki, Nirenberg and Varadhan define

$$\lambda_1 := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ in } \Omega \text{ supersolution of } Lv = \lambda v \text{ in } \Omega\}. \tag{I.1.4}$$

Here a solution (resp. sub, supersolution) of $Lu = f$ in Ω is a function $u \in W^{2,p}_{\text{loc}}(\Omega)$ for all finite $p > N$, satisfying $Lu = f$ (resp. $\leq$, $\geq$) for a.e. $x \in \Omega$.

In [26], the authors show that λ_1 defined as in (I.1.4) is the principal eigenvalue of L, i.e. there exists a positive function ϕ such that the pair (λ_1, ϕ) have properties (a) and (b) with $Bu = u$. They also prove that the maximum principle for L holds true if and only if $\lambda_1 > 0$. Indeed, they show that if $\lambda_1 > 0$, u is a subsolution of $Lu = 0$ in Ω and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω (i.e. the maximum principle holds). On the other hand, if $\lambda \geq \lambda_1$ then ϕ is a counterexample to the maximum principle.

Several other properties of the first eigenvalue, such as simplicity and stability, are established in [26].

In view of its relation with the maximum and the comparison principles, the concept of principal eigenvalue has been extended to nonlinear operators to study the associated boundary value problems.

A classical example of nonlinear eigenvalue problem is the one associated to the p-Laplacian defined for $1 < p < \infty$ by:

$$\Delta_p u = \text{div}(|Du|^{p-2}Du).$$

The natural way to define the principal eigenvalue for the p-Laplacian is by generalizing definition (I.1.3) to the nonlinear case. Indeed, if λ_1 is the minimum of the associated Rayleigh quotient

$$\frac{\int_\Omega |Du|^p dx}{\int_\Omega |u|^p dx}$$

among all the functions belonging to $W_0^{1,p}(\Omega) \setminus \{0\}$, then any minimizing function ϕ is a weak solution of the corresponding Euler-Lagrange equation

$$\begin{cases} -\Delta_p \phi = \lambda_1 |\phi|^{p-2}\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{I.1.5}$$

Moreover any solution of (I.1.5) does not change sign in Ω and (I.1.5) does not admit non-trivial solutions if we replace λ_1 by $\lambda < \lambda_1$, see e.g. [90]. Because of these properties, the minimum λ_1 of the Rayleigh quotient is called the principal eigenvalue of $-\Delta_p$ and the solutions of (I.1.5) are called principal eigenfunctions.

Remark that the operator $\Delta_p u + \lambda_1 |u|^{p-2}u$ is homogeneous of degree $p-1$. This implies that if ϕ is solution of (I.1.5) then $t\phi$ is solution of the same problem for any $t \in \mathbb{R}$. On the other hand, it is possible to prove that if ϕ and ψ are two principal eigenfunctions, then there exists $t \in \mathbb{R} \setminus \{0\}$ such that $\phi = t\psi$. This property of λ_1 is called simplicity and was proved indipendently by Anane [10] and by Otani and Teshima [95]. The principle eigenvalue of the p-Laplacian has almost all the typical properties of the first eigenvalue of a linear operator, see [90].

The method of the minimization of the Rayleigh quotient uses heavily the variational structure of the operator and cannot be applied to operators which have not this property. An important step in the study of the eigenvalue problem for nonlinear operators in non-divergence form was made by Lions in [87]. In that paper, using probabilistic and analytic methods, he showed the existence of principal eigenvalues for the uniformly elliptic Bellman operator and obtained results for the related Dirichlet problems. Very recently, many authors, inspired by [26], have developed an eigenvalue theory for fully nonlinear operators which are non-variational. The Pucci's extremal operators $\mathcal{M}_{a,A}(D^2u)$ have been treated by Quaas [103] and Busca, Esteban and Quaas [31]. Related results have been obtained by Birindelli and Demengel in [28] for singular or degenerate elliptic operators, like $|Du|^\alpha \mathcal{M}_{a,A}(D^2u)$, $\alpha > -1$, the p-Laplacian and some of its non-variational generalizations. In [104] Quaas and Sirakov have studied the eigenvalue problem for fully nonlinear elliptic operators which are convex and positively homogenous, like the Hamilton-Jacobi-Bellman one. In that paper many properties of the principal eigenvalues, including the fact that they are simple and isolated, have been established. Similar results have been obtained by Ishii and Yoshimura [76] for non-convex operators. The existence of a principal eigenvalue defined as in [26] for the ∞-Laplacian has been proved by Juutinen in [80] together with many other results. A different approach to investigate the eigenvalue problem for ∞-Laplacian consists in studying the asymptotic behavior, as $p \to \infty$, of the p-Laplacian eigenvalue equation, see [82] and [59]. This second method uses the variational formulation of the approximate problems and leads to a different limit eigenvalue problem, see [80].

All the mentioned papers treat Dirichlet boundary conditions.

I.2 Outline of the results of Chapters 1, 2 and 3

In the first part of this thesis we study the eigenvalue problem for some classes of fully nonlinear homogenous operators $G(x,u,Du,D^2u)$ with Neumann boundary conditions. We are thus concerned with the fully nonlinear elliptic problem

$$\begin{cases} G(x,u,Du,D^2u) = g(x) & \text{in } \Omega \\ B(x,u,Du) = 0 & \text{on } \partial\Omega, \end{cases} \tag{I.2.6}$$

where Ω is a bounded domain of class C^2 and $B(x,u,Du)$ is a first order boundary operator. The problem we consider may have weak solutions which are not smooth enough to satisfy the differential equations involved in the classical sense. We adopt the notion of viscosity solution as the weak solution of (I.2.6).

Let us recall the definition of viscosity solution of (I.2.6), for $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times$ *S(N)* $\to \mathbb{R}$ and $B : \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ continuous functions.

Definition I.2.1. *A function $u \in USC(\overline{\Omega})$ (resp. $u \in LSC(\overline{\Omega})$) is called* viscosity subsolution *(resp.* supersolution*) of* (I.2.6) *if the following conditions hold*

(i) For every $x_0 \in \Omega$, for all $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp. minimum) at x_0, one has

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq \ (\textit{resp.} \ \geq)\ g(x_0).$$

(ii) For every $x_0 \in \partial\Omega$, for all $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp. minimum) at x_0, one has

$$(G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - g(x_0)) \wedge B(x_0, u(x_0), D\varphi(x_0)) \leq 0$$

(resp.

$$(G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - g(x_0)) \vee B(x_0, u(x_0), D\varphi(x_0)) \geq 0).$$

A viscosity solution *is a continuous function which is both a subsolution and a supersolution.*

In Chapters 1 and 3 we consider operators G of the form

$$G(x,u,Du,D^2u) = -F(x,Du,D^2u) + b(x) \cdot Du|Du|^\alpha + c(x)|u|^\alpha u, \tag{I.2.7}$$

with $\alpha > -1$ and b, c, g continuous functions on $\overline{\Omega}$. F is a fully nonlinear operator that may be singular or degenerate where the gradient vanishes. Since we cannot test when the test function has zero gradient, we have to modify the previous definition of solution of the Neumann problem associated to these operators. We adopt the notion of viscosity solution given by Birindelli and Demengel in [28] adapted to our case.

Definition I.2.2. *Any function $u \in USC(\overline{\Omega})$ (resp. $u \in LSC(\overline{\Omega})$) is called* viscosity subsolution *(resp.* supersolution*) of* (I.2.6) *with G defined as in* (I.2.7)*, if the following conditions hold* *(i) For every $x_0 \in \Omega$, for all $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp. minimum) at x_0 and $D\varphi(x_0) \neq 0$, one has*

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq \text{ (resp. } \geq \text{) } g(x_0).$$

If $u \equiv k =$const. in a neighborhood of x_0, then

$$c(x_0)|k|^\alpha k \leq \text{ (resp. } \geq \text{) } g(x_0).$$

(ii) For every $x_0 \in \partial\Omega$, for all $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp. minimum) at x_0 and $D\varphi(x_0) \neq 0$, one has

$$(G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - g(x_0)) \wedge B(x_0, u(x_0), D\varphi(x_0)) \leq 0$$

(resp.

$$(G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - g(x_0)) \vee B(x_0, u(x_0), D\varphi(x_0)) \geq 0).$$

If $u \equiv k =$const. in a neighborhood of x_0 in $\overline{\Omega}$, then

$$(c(x_0)|k|^\alpha k - g(x_0)) \wedge B(x_0, k, 0) \leq 0$$

(resp.

$$(c(x_0)|k|^\alpha k - g(x_0)) \vee B(x_0, k, 0) \geq 0).$$

A viscosity solution *is a continuous function which is both a subsolution and a supersolution.*

This is a new definition of viscosity solution when $\alpha = 0$ and $F(x, Du, D^2u)$ is the ∞-Laplacian, studied in Chapter 3, which is equivalent to the standard one.

Singular fully nonlinear operators

In Chapter 1 we present the results contained in [97]. We study the maximum principle, principal eigenvalues, regularity and existence for viscosity solutions of the problem (I.2.6) where G is defined as in (I.2.7) and

$$B(x, u, Du) = B(Du) = \frac{\partial u}{\partial \overrightarrow{n}}$$

is the pure Neumann boundary operator. Here and in what follows $\overrightarrow{n}(x)$ is the exterior normal to the domain Ω at $x \in \partial\Omega$. The fully nonlinear and singular operator F satisfies the following homogeneity and ellipticity conditions

(F1) For all $t \in \mathbb{R}^*, \mu \geq 0$, $(x, p, X) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\} \times S(N)$

$$F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X).$$

(F2) There exist $a, A > 0$ such that for $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}, M, N \in S(N)$, $N \geq 0$

$$a|p|^\alpha \text{tr} N \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \text{tr} N.$$

In addition, we will assume on F some Hölder's continuity hypothesis that will be made precise in Chapter 1. Remark that G is positively homogenous of degree $\alpha+1$:

$$G(x,tu,tDu,tD^2u) = t^{\alpha+1}G(x,u,Du,D^2u) \quad \text{for any } t>0.$$

In this class of operators one can consider for example

$$F(Du,D^2u) = |Du|^{\alpha}\mathcal{M}^{\pm}_{a,A}(D^2u),$$

$\alpha > -1$, where $\mathcal{M}^{\pm}_{a,A}(D^2u)$ are the Pucci's operators defined for $X \in S(N)$ by

$$\mathcal{M}^{+}_{a,A}(X) = A\sum_{e_i>0} e_i + a\sum_{e_i<0} e_i,$$

$$\mathcal{M}^{-}_{a,A}(X) = a\sum_{e_i>0} e_i + A\sum_{e_i<0} e_i,$$

where $e_1, ..., e_N$ are the eigenvalues of X (see e.g. [32]).

Other examples for F are given by the p-Laplacian with $\alpha = p-2$, and non-variational generalizations of the p-Laplacian, depending explicitly on x, like the operator

$$F(x,Du,D^2u) = |Du|^{q-2}\mathrm{tr}(B_1(x)D^2u) + c_0|Du|^{q-4}\langle D^2uB_2(x)Du, B_2(x)Du\rangle,$$

with $\alpha = q-2$, where $q>1$, B_1 and B_2 are θ-Hölderian functions with $\theta > \frac{1}{2}$, which send $\overline{\Omega}$ into $S(N)$, $aI \le B_1 \le AI$, $-\sqrt{a}I \le B_2 \le \sqrt{A}I$ and $c_0 > -1$.

These operators were introduced by Birindelli and Demengel in [28] (see also [27]), where the Dirichlet eigenvalue problem is studied. They assume slightly less general structure conditions on F, but on the other hand, some of their results can be applied to degenerate elliptic equations.

Following the ideas of [26], we define

$$\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v>0 \text{ bounded viscosity supersolution of } G(x,v,Dv,D^2v) = \lambda v^{\alpha+1} \text{ in } \Omega, \frac{\partial v}{\partial \overrightarrow{n}} = 0 \text{ on } \partial\Omega\}.$$

We prove that $\overline{\lambda}$ is a generalized principal eigenvalue of G with the Neumann boundary condition. Indeed, we show the existence of a positive and Lipschitz continuous function ϕ^+ that is a viscosity solution of

$$\begin{cases} G(x,\phi^+,D\phi^+,D^2\phi^+) = \overline{\lambda}(\phi^+)^{\alpha+1} & \text{in} \quad \Omega \\ \frac{\partial \phi^+}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover, as in the linear case, the maximum principle for G with the Neumann condition holds true if and only if $\overline{\lambda} > 0$. This means that if $\overline{\lambda} > 0$, whenever u is a viscosity subsolution of

$$\begin{cases} G(x,u,Du,D^2u) = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{I.2.8}$$

then $u \le 0$ in Ω. On the other hand, if $\overline{\lambda} \le 0$ then the principal eigenfunction ϕ^+ is a counterexample to the maximum principle. In particular, $\overline{\lambda}$ is the smallest real number λ such that there exists a positive solution of

$$\begin{cases} G(x, v, Dv, D^2v) = \lambda |v|^\alpha v & \text{in} \quad \Omega \\ \frac{\partial v}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{I.2.9}$$

Indeed, by the maximum principle if $\lambda < \overline{\lambda}$ any subsolution of (I.2.9) must be non-positive.

It is not hard to prove that a sufficient condition to the maximum principle to hold is that the function $c(x)$ in (I.2.7) satisfies: $c \ge 0$ in Ω and $c \not\equiv 0$. On the other hand, if $c(x) \le 0$ the maximum principle fails, since the positive constants are subsolution of (I.2.8). It is natural to wonder if the condition $\overline{\lambda} > 0$ is weaker than the non-negativity of $c(x)$. We give a positive answer to this question, by constructing an explicit example of a bounded positive viscosity supersolution of $G(x, v, Dv, D^2v) = \lambda v^{\alpha+1}$ in Ω, $\frac{\partial v}{\partial \overrightarrow{n}} = 0$ on $\partial\Omega$, with $\lambda > 0$, for some $c(x)$ changing sign in Ω. The existence of such v implies, by definition, $\overline{\lambda} > 0$.

For fully nonlinear operators it is possible to define another principal eigenvalue

$$\underline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ bounded viscosity subsolution of}$$
$$G(x, u, Du, D^2u) = \lambda |u|^\alpha u \text{ in } \Omega, \frac{\partial u}{\partial \overrightarrow{n}} = 0 \text{ on } \partial\Omega\}.$$

If $F(x, p, X) = -F(x, p, -X)$ then $\overline{\lambda} = \underline{\lambda}$, otherwise $\overline{\lambda}$ may be different from $\underline{\lambda}$.

Symmetrical results can be proved for $\underline{\lambda}$: there exists a negative and Lipschitz continuous eigenfunction ϕ^- associated to $\underline{\lambda}$. Moreover, the minimum principle for G with the Neumann condition is valid if and only if $\underline{\lambda} > 0$. We say that the minimum principle holds true if whenever u is a supersolution of (I.2.8) then $u \ge 0$ in Ω.

The minimum principle implies that $\underline{\lambda}$ is the smallest real number λ such that there exists a negative solution of (I.2.9).

We conclude Chapter 1, by applying the results about the principal eigenvalues to solve the Neumann problem (I.2.6) for (I.2.7). We prove that if $\overline{\lambda} > 0$ and $g \ge 0$, $g \not\equiv 0$, (I.2.6) admits a positive solution which is unique if $g > 0$. Symmetrically, if $\underline{\lambda} > 0$ and $g \le 0$, $g \not\equiv 0$, then there exists a negative solution of (I.2.6) which is unique if $g < 0$. Finally, if both the eigenvalues are positive, the Neumann problem (I.2.6) is solvable for any right-hand side g.

The Isaacs operators

In Chapter 2 we present the results obtained in [96]. We consider uniformly elliptic operators, $G(x, u, Du, D^2u)$, which are positively homogenous of order 1 and with some additional assumptions that will be made precise in Chapter 2. This class includes the uniformly elliptic Hamilton-Jacobi-Bellman operator

$$G(x, u, Du, D^2u) = \sup_{\alpha \in \mathcal{A}} \{-\mathrm{tr}(A_\alpha(x) D^2u) + b_\alpha(x) \cdot Du + c_\alpha(x) u\}, \tag{I.2.10}$$

which arises in stochastic optimal control, see [13] and [54], or more in general the Isaacs operator

$$G(x, u, Du, D^2u) = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-\mathrm{tr}(A_{\alpha,\beta}(x) D^2u) + b_{\alpha,\beta}(x) \cdot Du + c_{\alpha,\beta}(x) u\}, \tag{I.2.11}$$

which arises in stochastic differential games, see [54].

To G we associate the following boundary condition

$$B(x,u,Du) = f(x,u) + \frac{\partial u}{\partial \vec{n}} = 0 \quad x \in \partial\Omega. \tag{I.2.12}$$

Typically f will be

$$f(x,u) = \gamma(x)u,$$

with $\gamma(x)$ continuous and non-negative on $\partial\Omega$.

The main result of Chapter 2 is the strong comparison principle between sub and supersolutions: we show that if u and v are respectively a sub and a supersolution of (I.2.6) for the considered operators G and B, and $u \leq v$, then either $u \equiv v$ or $u < v$ on $\overline{\Omega}$. This result can be applied to develop an eigenvalue theory for the uniformly elliptic homogenous operators we are studying. Again as in [26], we define

$$\begin{aligned}\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of}\\ G(x,v,Dv,D^2v) = \lambda v \text{ in } \Omega,\ B(x,v,Dv) = 0 \text{ on } \partial\Omega\},\end{aligned}$$

$$\begin{aligned}\underline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ on } \overline{\Omega} \text{ bounded viscosity subsolution of}\\ G(x,u,Du,D^2u) = \lambda u \text{ in } \Omega,\ B(x,u,Du) = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

With the aid of the strong comparison principle we are able to show that $\overline{\lambda}$ and $\underline{\lambda}$ are principal eigenvalues, i.e., that there exist a positive and a negative function, ϕ^+ and ϕ^- that are respectively viscosity solutions of

$$\begin{cases} G(x,\phi^+,D\phi^+,D^2\phi^+) = \overline{\lambda}\phi^+ & \text{in} \quad \Omega \\ B(x,\phi^+,D\phi^+) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{I.2.13}$$

$$\begin{cases} G(x,\phi^-,D\phi^-,D^2\phi^-) = \underline{\lambda}\phi^- & \text{in} \quad \Omega \\ B(x,\phi^-,D\phi^-) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{I.2.14}$$

Moreover, the maximum principle (resp. the minimum principle) for G with boundary condition (I.2.12) holds true if and only if $\overline{\lambda}$ (resp. $\underline{\lambda}$) is positive.

The strong comparison principle implies some of the basic properties of principal eigenvalues. The first one is simplicity: if u is a solution of (I.2.13) (resp. solution of (I.2.14)) and u is positive (resp. negative) at some point, then there exists $t > 0$ such that $u \equiv t\phi^+$ (resp. $u \equiv t\phi^-$). If, in addition, F is convex or concave, the same statement holds true for some $t \in \mathbb{R}$, without conditions on the sign of u. The Hamilton-Jacobi-Bellmann operator (I.2.11) is an example of convex operator.

We also prove that the principal eigenvalues are isolated and they are the only eigenvalues to which there correspond eigenfunctions that do not change sign in Ω.

A comparison between the principal eigenvalues for G corresponding to Dirichlet and Neumann boundary conditions is also possible. If we denote by $\overline{\lambda} = \overline{\lambda}_N$, $\underline{\lambda} = \underline{\lambda}_N$ and $\overline{\lambda}_D$, $\underline{\lambda}_D$ respectively the principal eigenvalues corresponding to the Neumann and the Dirichlet problems, then we have: $\overline{\lambda}_N < \overline{\lambda}_D$ and $\underline{\lambda}_N < \underline{\lambda}_D$.

The strong comparison principle is not known for the singular operators that we study in Chapter 1. This is the main reason why some questions about the properties of principal eigenvalues are still open problems, both in the Neumann and

Dirichlet cases. Recently, Birindelli and Demengel [29] have proved simplicity in the Dirichlet case for any smooth bounded domain Ω of $\mathbb{R}^N$ when $\partial\Omega$ has one connected components and when $N = 2$ and $\partial\Omega$ has at most two connected components.

We have already remarked that for nonlinear operators the two principal eigenvalues may not coincide. We exhibit an example in which this eventuality always occurs. Indeed, we show that when

$$G(x, u, Du, D^2u) = -\mathcal{M}(D^2u) + b(x) \cdot Du + c(x)u,$$

where $\mathcal{M}(D^2u)$ is one of the Pucci's operators and $c(x)$ is not constant, then we have $\overline{\lambda} \neq \underline{\lambda}$.

Finally, we apply our results to the study of the Neumann problem (I.2.6). It is well known that for the operators considered in Chapter 2 (I.2.6) is uniquely solvable if there exists $\sigma > 0$ such that for any $(x, p, X) \in \overline{\Omega} \times \mathbb{R}^N \times S(N)$ the function $r \to G(x, r, p, X) - \sigma r$ is non-decreasing on $\mathbb{R}$, see e.g [40]. For the Isaacs operator (I.2.10) this condition is equivalent to $c_{\alpha,\beta}(x) \geq c_0 > 0$ for all $x \in \overline{\Omega}$ and $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$. Without requiring this assumption, we prove that if $\overline{\lambda} > 0$ and $g \geq 0$ (resp. $\underline{\lambda} > 0$ and $g \leq 0$), $g \not\equiv 0$, then (I.2.6) admits a positive (resp. negative) Lipschitz continuous solution which is unique if $g > 0$ (resp. $g < 0$). The boundary value problem (I.2.6) is solvable for any right-hand side if both the two eigenvalues are positive. It is easy to see that the non-decreasing monotonicity of $r \to G(x, r, p, X) - \sigma r$ implies $\overline{\lambda}, \underline{\lambda} > 0$.

The ∞-Laplacian

In Chapter 3 we extend the results obtained for the singular operators considered in Chapter 1 to the operator:

$$G(x, u, Du, D^2u) = -\Delta_\infty u + b(x) \cdot Du + c(x)u \tag{I.2.15}$$

where

$$\Delta_\infty u = \langle D^2u \frac{Du}{|Du|}, \frac{Du}{|Du|} \rangle,$$

is the 1-homogeneous version of the ∞-Laplacian. The results presented are contained in [98].

Because of its strong degeneracy, the ∞-Laplacian does not satisfy the assumption (F2) of Chapter 1, so it is not covered by [28] or [97].

The ∞-Laplacian, which arises from the optimal Lipschitz extension problem, see [11], appears also in the Monge-Kantorovich mass transfer problem, see [51], and recently, some authors have introduced a game theoretic interpretation of it, see [100].

Since the operator (I.2.15) is 1-homogenous, we define the principal eigenvalue as follows

$$\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of}$$
$$G(x, v, Dv, D^2v) = \lambda v \text{ in } \Omega,\ \frac{\partial v}{\partial \overrightarrow{n}} = 0 \text{ on } \partial\Omega\}.$$

Remark that since $\Delta_\infty(-u) = -\Delta_\infty u$, if we define $\underline{\lambda}$ in the usual way, then $\overline{\lambda} = \underline{\lambda}$.

We show that all the results presented in Chapter 1 such as the existence of eigenfunctions and the solvability of the associated Neumann problem are true for the operator (I.2.15).

The question of simplicity of the principal eigenvalue is still open in both the Dirichlet and the Neumann cases. The main difficulty comes from the fact that the strong comparison principle for ∞-Laplacian is not valid as shown in [83].

A further application of the eigenvalue theory is the decay estimate for solutions of the following Neumann evolution problem:

$$\begin{cases} h_t = \Delta_\infty h + c(x)h & \text{in } (0,+\infty)\times\Omega \\ \dfrac{\partial h}{\partial \overrightarrow{n}} = 0 & \text{on } [0,+\infty)\times\partial\Omega \\ h(0,x) = h_0(x) & \text{for } x\in\Omega. \end{cases} \tag{I.2.16}$$

We show that if h is a solution of (I.2.16) and the principal eigenvalue of the stationary operator associated to (I.2.16) is positive, then h decays to zero exponentially and that the rate of the decay depends on it. Precisely, if $\overline{\lambda}$ and v are respectively the principal eigenvalue and a principal eigenfunction, then

$$\sup_{\Omega\times[0,+\infty)} \frac{h(t,x)e^{\overline{\lambda}t}}{v(x)} \le \sup_{\Omega} \frac{h_0^+(x)}{v(x)},$$

where $h_0^+ = \max\{h_0, 0\}$ denotes the positive part of h_0.

PART II

I.3 The Peierls-Nabarro model for dislocation dynamics

Dislocations are defect lines in crystalline solids whose motion is directly responsible for the plastic deformation of these materials. Their typical length is of order of $10^{-6}m$ with thickness of order of $10^{-9}m$.

The theoretical study of dislocations, started in the 30's, had a considerable development in the 90's thanks to the increasing power of computers which made possible to simulate the 3D behavior of dislocations.

Recently, a new approach has emerged: the so called *phase field model of dislocations*, where dislocations are described by variations of continuous fields (see for instance [105], [43] and [57]). This approach has the advantage that the possible topological changes during dislocation movement are automatically taken into account and that the interactions of dislocations movement with other defects or phases can be easily incorporated in the model.

A brief presentation of the theory of dislocations

We are going to present some basic facts of the theory of dislocations in crystals. For a complete presentation of the theory we refer the reader to the book by Hirth, Lothe [63].

Let us start with the geometrical description of a dislocation. Consider a perfect crystal cube, acted on a shear stress as shown in Figure I.3.16-(a). The *edge dislocation* in Figure I.3.16-(b) is producing by slicing the cube from the left to right normal to z and displacing the cube to comply with the stress. The dislocation is the line representing the boundary of the slipped region.

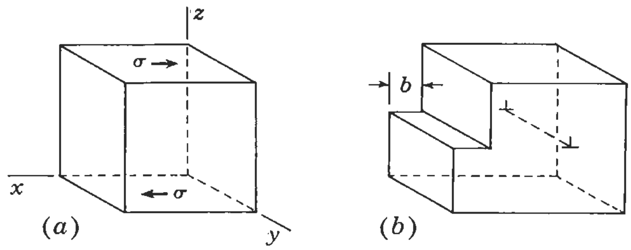

Figure I.3.16. Geometry of an edge dislocation.

Similarly, the cut displacement shown in Figure I.3.16 produces a *screw dislocation* which is again the boundary of the area over which slip has occurred.

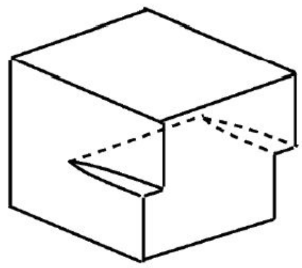

Figure I.3.16. Geometry of a screw dislocation.

There are essentially two invariants to characterize a dislocation. The first one is $\overrightarrow{n}$, the normal vector to the *slip plane*. Indeed, at least at low temperature, a dislocation moves in a well defined crystallographic plane. The second one is the *Burgers vector* $\overrightarrow{b}$. The classical way to define it is the following: in Figure I.3.16-(a) we can see a schematic view of an edge dislocation in the crystal which is divided into a "good" region where the distribution of atoms is close to the distribution of a perfect crystal, and a "bad" region near the dislocation line where the displacement is large. Let us choose a sense of the dislocation line by assigning a unit vector $\overrightarrow{\xi}$ tangent to the dislocation. Let us consider a circuit, called Burgers circuit, right-handed oriented with respect to $\overrightarrow{\xi}$, which lies entirely in good material and enclosing the dislocation. Then we draw the same circuit in the perfect reference lattice, Figure I.3.16-(b). The vector required to close the latter circuit is called the Burgers vector $\overrightarrow{b} = \overrightarrow{BA}$.

For an edge dislocation $\overrightarrow{b}$ is perpendicular to $\overrightarrow{\xi}$, while for a screw dislocation $\overrightarrow{b}$ is parallel to $\overrightarrow{\xi}$. In the general case, the angle between the Burgers vector and the dislocation line is arbitrary and we have a mixed dislocation.

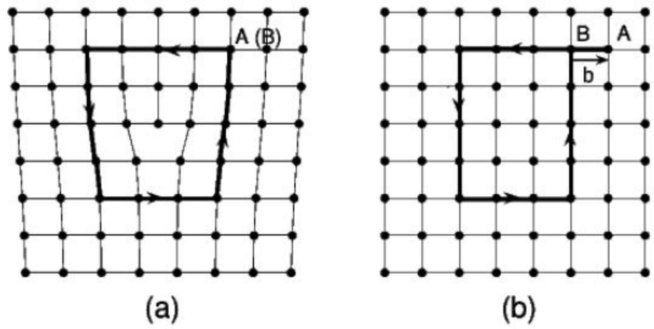

Figure I.3.16. Burgers circuit.

Next, let us define the elastic energy associated to a dislocation line. Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. From now on, we denote the coordinates by $x = (x_1, x_2, x_3)$. Consider a linear elastic material represented by the whole space. Let σ_{ij}, $i, j = 1, 2, 3$ be the stresses in the body. In absence of internal torque

$$\sigma_{ij} = \sigma_{ji} \quad \text{for any } i, j = 1, 2, 3.$$

If there are no body forces, the stresses need to satisfy the equilibrium equation of classical elasticity:

$$\mathrm{div}(\sigma) = 0 \tag{I.3.17}$$

where $(\mathrm{div}(\sigma))_j = \sum_{i=1}^{3} \partial_i \sigma_{ij}$. When acted upon by stresses a body deforms; let $U : \mathbb{R}^3 \to \mathbb{R}^3$ be the displacement vector. The strain is defined as

$$e = e(U) \quad \text{where} \quad e_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right).$$

For small distortions $\partial U_i / \partial x_j$, the stresses depends linearly on deformations (Hooke's law):

$$\sigma = \Lambda : e,$$

where Λ_{ijkl} are the elastic constants of the body satisfying the following symmetry property

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{jilk},$$

and the following coercivity assumption for some $m > 0$

$$\frac{1}{2} e : \Lambda : e \geq m|e|^2$$

for all constant symmetric matrices $e = (e_{ij}) \in$ *S(3)*. Here $(\Lambda : e)_{ij} = \sum_{k,l=1}^{3} \Lambda_{ijkl} e_{kl}$ and $|e|^2 = \sum_{i,j=1}^{3} e_{ij}^2$.

The equation of linear elasticity (I.3.17) is the Euler-Lagrange equation (minimizing with respect to U) associated to the elastic energy

$$\mathcal{E}^{el} = \frac{1}{2}\int_{\mathbb{R}^3} e : \Lambda : e dx. \tag{I.3.18}$$

If the medium is elastically isotropic, i.e. the elastic properties are independent of direction, only two independent elastic constants are required: the Lamé constant λ and the shear modulus μ, with $\mu > 0$ and $3\lambda + 2\mu > 0$. In this case the elastic energy has the form

$$\mathcal{E}^{el} = \int_{\mathbb{R}^3} \mu|e(U)|^2 + \frac{\lambda}{2}|\mathrm{tr}e(U)|^2 dx. \tag{I.3.19}$$

Now, let us assume that there is a dislocation line in the material, represented by the boundary Γ of a smooth domain Ω_0 contained in the plane $\{x_3 = 0\}$ (slip plane). In this case we have to consider a plastic deformation e^{pl} in addition to the elastic one e^{el}:

$$e(U) = e^{el} + e^{pl},$$

where

$$e^{pl} = e^0\rho\delta_0(x_3), \quad \text{with } e^0 = \frac{1}{2}\left(\overrightarrow{b} \otimes \overrightarrow{n} + \overrightarrow{n} \otimes \overrightarrow{b}\right).$$

Here $\delta_0(x_3)$ is the Dirac mass only in the x_3 component, $\overrightarrow{n} = e_3$ and ρ is the characteristic function of Ω_0. The classical theory of dislocations asserts that the dislocation line creates a distortion in the strain e such that

$$\mathrm{div}(\Lambda : e(U)) = \mathrm{div}(\rho\delta_0(x_3)\Lambda : e^0)$$

in the sense of distributions. The dislocation line is surrounded by a region, known as the *dislocation core*, where the atomic positions cannot be described by a small homogenous deformation of the reference crystal. To take into consideration the nonlinear atomic interaction across the slip plane, in many models a tension line energy is added to the elastic energy, see for instance [30] and [25]. The energy is thus formally given by

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{R}^3} e^{el} : \Lambda : e^{el} dx + \int_\Gamma W(\overrightarrow{n})dS,$$

where W is an energy of tension line. In the dislocation core the linear continuum elastic theory is not longer valid, in other words, the elastic energy may be infinite close to the dislocation line. A simple way to overcome this difficulty has been proposed by Alvarez, Hoch, Le Bouar and Monneau [7] and consists in introducing a regularizing core tensor χ and replacing e^{el} by $e^{el}_\chi = \chi\star e^{el}$, where $\star$ is the convolution operation. We refer to [7] for more details.

Description of the Peierls-Nabarro model

The Peierls-Nabarro model is a phase field model incorporating atomic features into continuum framework.

We briefly review the model for a straight dislocation, see [63] for a detailed presentation. As an example, consider an edge dislocation in a crystal with simple cubic lattice. In a Cartesian system of coordinates $x_1x_2x_3$, consider the section of the crystal on the plane x_1x_2. Assume that the dislocation is located along the x_3 axis and the Burgers vector is in the direction of the x_1 axis. Thus the x_1x_3 plane is the slip plane of the dislocation. The length of the Burgers vector is b and corresponds to the magnitude of the lattice. The disregistry of the upper half crystal $\{x_2 > 0\}$ relative to the lower half $\{x_2 < 0\}$ in the direction of the Burgers vector is $\phi(x_1)$, where ϕ is an increasing function such that $\phi(-\infty) = 0$ and $\phi(+\infty) = b$.

In the Peierls-Nabarro model, the total energy is written as

$$\mathcal{E} = \mathcal{E}^{el} + \mathcal{E}^{mis},$$

where $\mathcal{E}^{el}$ is the long-range interaction elastic energy induced by the slip (I.3.18), and $\mathcal{E}^{mis}$ is the so called *misfit energy* due to the nonlinear atomic interaction across the slip plane

$$\mathcal{E}^{mis} = \int_{-\infty}^{+\infty} W(\phi(x_1))dx_1,$$

where $W(\phi)$ is the interplanar potential. In the classical Peierls-Nabarro model, [99]-[93], isotropic elasticity is used for $\mathcal{E}^{el}$ and $W(\phi)$ in $\mathcal{E}^{mis}$ is approximated by the Frenkel sinusoidal potential

$$W(\phi) = \frac{\mu b^2}{4\pi^2 d}\left(1 - \cos\left(\frac{2\pi\phi}{b}\right)\right),$$

where d is the lattice spacing perpendicular to the slip plane. The equilibrium configuration of the dislocation is obtained by minimizing the total energy with respect to ϕ. The solution employed in Nabarro [93], has the explicit form

$$\phi(x_1) = \frac{b}{\pi}\arctan\left(\frac{2(1-\nu)x_1}{d}\right) + \frac{b}{2}, \tag{I.3.20}$$

where $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is the Poisson ratio.

In the general model, one can consider a potential W satisfying

(i) $W(u + b) = W(u)$ for all $u \in \mathbb{R}$;

(ii) $W(b\mathbb{Z}) = 0 < W(a)$ for all $a \in \mathbb{R} \setminus b\mathbb{Z}$.

The periodicity of W reflects the periodicity in the crystal, while the minimum property is consistent with the fact that the perfect crystal is assumed to minimize the energy.

The displacement is represented by a scalar function $U(x_1, x_2)$ which is antisymmetric in x_2 and such that $U(x_1, 0) = \frac{1}{2}\phi(x_1)$. The elastic energy can be written as integral over the slip plane. Suppose that, replacing U by $2U$, the elastic energy has formally the following form

$$\mathcal{E}^{el} = \frac{1}{4}\int_{\mathbb{R}^2} |\nabla U|^2 dx_1dx_2 = \frac{1}{2}\int_{\mathbb{R}\times(0,+\infty)} |\nabla U|^2 dx_1dx_2,$$

with $U(x_1, 0) = \phi(x_1)$. Then, the total energy is

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{R}\times(0,+\infty)} |\nabla U|^2 dx_1dx_2 + \int_{\partial(\mathbb{R}\times(0,+\infty))} W(U)dS.$$

A local minimizer of the energy is a solution of

$$\begin{cases} \Delta U = 0, & \text{in } \Omega \\ \frac{\partial U}{\partial \vec{n}} = -W'(U), & \text{on } \partial\Omega \end{cases} \tag{I.3.21}$$

where $\Omega = \{(x_1, x_2) \,|\, x_2 > 0\}$ and $\frac{\partial U}{\partial \vec{n}} = -\frac{\partial U}{\partial x_2}$ is the exterior normal derivative of U, see Cabré and Solà-Morales [33]. It is well known that if V is a smooth function on $\mathbb{R}^N \times [0, +\infty)$ which is harmonic in $\mathbb{R}^N \times (0, +\infty)$ and $V(x_1, ..., x_N, 0) = v(x_1, ..., x_N)$, then $\frac{\partial V}{\partial x_{N+1}}(x_1, ..., x_N, 0) = -(-\Delta)^{\frac{1}{2}} v(x_1, ..., x_N)$, where the half-Laplacian $(-\Delta)^{\frac{1}{2}}$ is the fractional operator defined on the Schwartz class $S(\mathbb{R}^N)$ by

$$(-\Delta)^{\frac{1}{2}} v = \mathcal{F}^{-1}(|\cdot|\mathcal{F}(v)),$$

see [86]. Here $\mathcal{F}$ is the Fourier transform. For a bounded real smooth function v defined on $\mathbb{R}^N$, the linear operator $-(-\Delta)^{\frac{1}{2}}$ is given by the Lévy-Khintchine formula (see Theorem 1 in [66]):

$$-(-\Delta)^{\frac{1}{2}} v(y) = C_N \int_{\mathbb{R}^N} (v(y+z) - v(y) - \nabla v(y) \cdot z \mathbf{1}_{\{|z| \le 1\}}) \frac{dz}{|z|^{N+1}} \tag{I.3.22}$$

where C_N is a constant depending on the dimension N and $\mathbf{1}_{\{|z| \le 1\}}$ is the characteristic function of the set $\{|z| \le 1\}$. Hence, for smooth solutions, system (I.3.21) can be rewritten for $\phi(x_1) = U(x_1, 0)$ as

$$-(-\Delta)^{\frac{1}{2}} \phi = W'(\phi) \quad \text{on } \mathbb{R},$$

where $-(-\Delta)^{\frac{1}{2}}$ is the non-local operator defined by (I.3.22) with $N = 1$. In [33] the authors show that for smooth potentials W satisfying assumptions (i) and (ii) above, there exists a function ϕ solution of

$$\begin{cases} -(-\Delta)^{\frac{1}{2}} \phi = W'(\phi), & \text{on } \mathbb{R} \\ \phi' > 0, \quad \text{and} \quad \phi(-\infty) = 0 \quad \phi(+\infty) = b. \end{cases} \tag{I.3.23}$$

The function ϕ is called layer solution. If in addition W satisfies

(iii) $W''(b\mathbb{Z}) > 0$,

then the layer solution is unique up to translations. When the elastic energy has the general form (I.3.18), then the phase transition ϕ is a solution of

$$\mathcal{I}_1[\phi] = W'(\phi) \quad \text{on } \mathbb{R}, \tag{I.3.24}$$

where $\mathcal{I}_1$ is the anisotropic Lévy operator of order 1, defined on bounded C^2-functions by

$$\mathcal{I}_1[v](x) = \int_{\mathbb{R}} (v(x+z) - v(x) - zv'(y)\mathbf{1}_{\{|z| \le 1\}}) \frac{1}{|z|^2} g\left(\frac{z}{|z|}\right) dz$$

with g positive and even function. See [7] for the characterization of g. In the special case of isotropic elasticity, we have $g(z) = \frac{\mu}{2\pi(1-\nu)}$ and the function (I.3.20) is a layer solution of (I.3.24).

Outline of the results of Chapter 4

In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane at a velocity of order $10ms^{-1}$. To take into consideration the dynamics effects in motion of dislocations, in Chapter 4 we study the evolutive version in dimension N of the Peierls-Nabarro model (I.3.24) introduced in the previous section. The results presented are contained in [92]. Precisely, we consider the non-local parabolic equation

$$\partial_t u = \mathcal{I}_1[u(t,\cdot)] - W'(u) + \sigma(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \tag{I.3.25}$$

where $\mathcal{I}_1$ is the anisotropic Lévy operator of order 1, defined on bounded C^2-functions $U : \mathbb{R}^N \to \mathbb{R}$ for $r > 0$ by

$$\begin{aligned}\mathcal{I}_1[U](x) = &\int_{|z|\le r} (U(x+z) - U(x) - \nabla U(x)\cdot z)\frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \\ &+ \int_{|z|>r} (U(x+z) - U(x))\frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz.\end{aligned}$$

Here g is a positive and even continuous function and W and σ are periodic functions. See Chapter 4 for the precise assumptions on g, σ and W. In the model σ has been introduced to take into account the possible external applied shear stress on the material.

We suppose that at initial time $t = 0$ u satisfies

$$u(0,x) = \frac{1}{\epsilon} u_0(\epsilon x) \quad \text{on } \mathbb{R}^N, \tag{I.3.26}$$

where u_0 is a regular bounded function and ϵ is a small positive parameter.

Problem (I.3.25)-(I.3.26) for $N = 1$ models the dynamics of a collection of parallel straight edge dislocation lines with the same Burgers vector, all contained in the same slip plane and moving in a landscape with periodic obstacles. The parameter ϵ takes into account the fact that the number of dislocations is increasing of order $1/\epsilon$.

We want to identify at *large scale* an evolution model for the dynamics of a density of dislocations and we do this by a periodic homogenization approach. We consider the following rescaling

$$u^\epsilon(t,x) = \epsilon u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right),$$

the functions u^ϵ are solutions of

$$\begin{cases} \partial_t u^\epsilon = \mathcal{I}_1[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{I.3.27}$$

A good notion of solution for system (I.3.27) is the notion of viscosity solution for non-local equations given for instance in [12]. See also [34], [22] and [65].

We prove that the limit u^0 of u^ϵ as $\epsilon \to 0$ exists and is the unique solution of the homogenized problem

$$\begin{cases} \partial_t u = \overline{H}(\nabla u, \mathcal{I}_1[u(t,\cdot)]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{I.3.28}$$

for some continuous function $\overline{H}$ usually called *effective Hamiltonian.* As usual in periodic homogenization, the limit equation is determined by a *cell problem.* In our case, such a problem is for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the following:

$$\begin{cases} \lambda + \partial_t v = \mathcal{I}_1[v(\tau,\cdot)] + L - W'(v + \lambda t + p\cdot x) + \sigma(t,x) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ v(0,x) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{I.3.29}$$

We show that there is a unique number $\lambda = \lambda(p, L)$ for which there exists a solution of (I.3.29) which is bounded on $\mathbb{R}^+ \times \mathbb{R}^N$. The effective Hamiltonian of (I.3.28) is thus defined as follows: $\overline{H}(p, L) := \lambda(p, L)$. The function $\overline{H}(p, L)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$ and non-decreasing in L.

A specific technical difficulty in this problem is to deal with the case $\lambda = p = 0$. In order to overcome it, following [67] and [68], we consider cell problems in a higher dimensional space. The lack of smooth solutions for these problems, has induced us to construct regular approximated sub and supercorrectors, i.e. regular sub and supersolutions of approximate $N + 1$-dimensional cell problems, and this is enough to conclude. Let us also point out that, differently from the case of equations independent of u^ϵ/ϵ, correctors here are not periodic with respect to the space variable in general. Moreover, correctors are necessarily time dependent.

The homogenized equation (I.3.28) can be interpreted as the plastic flow rule in a model for macroscopic crystal plasticity, i.e. a relationship between the plastic strain velocity and the stress. In the homogenized equation (I.3.28):

- u^0 is the plastic strain;
- $\partial_t u^0$ is the plastic strain velocity;
- ∇u^0 is the dislocation density;
- $\mathcal{I}_1[u^0]$ is the internal stress created by the density of dislocations contained in a slip plane.

In dimension $N = 1$, when $\mathcal{I}_1$ is the half-Laplacian and the periodic stress σ is equal to 0, we get

$$\overline{H}(p, L) \sim \frac{1}{2}\gamma|p|L \tag{I.3.30}$$

for small p and L, where $\gamma = 2(\int_{\mathbb{R}}(\phi')^2)^{-1}$ is the inverse of the so called damping factor and ϕ is a layer solution, i.e. a solution of (I.3.23). This characterization of the effective Hamiltonian is known in physics as Orowan's law.

Remark I.3.1. Fractional reaction-diffusion equations of the form

$$\partial_t u = \mathcal{I}_1[u] + f(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N \tag{I.3.31}$$

where $N \geq 2$ and f is a bistable nonlinearity have been studied by Imbert and Souganidis [69]. In this paper the authors show that solutions of (I.3.31), after properly rescaling them, exhibit a moving interface. Analogous results have been obtained by González and Monneau [62] for the evolutive Peierls-Nabarro model in dimension $N = 1$. In the one dimensional space, the moving interface are points. The dynamics of these particles corresponds to the classical discrete dislocation dynamics, in the particular case of of parallel straight edge dislocation lines in the same slip plane with the same Burgers vector. Considering the motion of these particles they identify at large scale an evolution model for the dynamics of a density of dislocations that corresponds to (I.3.28).

Finally, let us mention that in [57] and [58] Garroni and Muller study a variational model for dislocations that is the variational formulation of the stationary Peierls-Nabarro equation.

I.4 Local Hamilton-Jacobi equations

In Chapter 5 we present the results of [2]. We consider homogenization problems for first order local Hamilton-Jacobi equations with u^ϵ/ϵ periodic dependence, namely

$$\begin{cases} u^\epsilon_t + H\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{u^\epsilon}{\epsilon}, Du^\epsilon\right) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u^\epsilon(0,x) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{I.4.32}$$

with the following assumptions on the Hamiltonian H:

(H1) Periodicity: for any $(t,x,u,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$H(t+1, x+k, u+1, p) = H(t,x,u,p) \quad \text{for any } k \in \mathbb{Z}^N;$$

(H2) Regularity: $H : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and there exists a constant $C_1 > 0$ such that, for almost every $(t,x,u,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$|D_{(t,x)}H(t,x,u,p)| \le C_1(1+|p|), \quad |D_uH(t,x,u,p)| \le C_1, \quad |D_pH(t,x,u,p)| \le C_1;$$

(H3) $H(t,x,u,p) \to +\infty$ as $|p| \to +\infty$ uniformly for $(t,x,u) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$;

(H4) There exists a constant C such that for almost every $(t,x,u,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$|D_pH(t,x,u,p) \cdot p - H(t,x,u,p)| \le C.$$

Problem (I.4.32) with H independent of t was introduced by Imbert and Monneau [67] as a simplified model for dislocation dynamics in material science. The complete model is introduced in [68] and leads to non-local first order equations of the type

$$u^\epsilon_t + \left(c\left(\frac{x}{\epsilon}\right) + M^\epsilon\left(\frac{u^\epsilon}{\epsilon}\right)\right)|Du^\epsilon| + H\left(\frac{u^\epsilon}{\epsilon}, Du^\epsilon\right) = 0$$

where M^ϵ is a non-local jump operator and c is a periodic velocity. In the latter model, the level sets of the solution u^ϵ describe dislocations.

Going back to (I.4.32), it was proved in [67] that, with H independent of t,

- under assumptions (H1) and (H2), there exists a unique bounded continuous viscosity solution of (I.4.32);

- under assumptions (H1)-(H3), the limit u^0 of u^ϵ as $\epsilon \to 0$ exists and it is the unique bounded continuous solution of the homogenized problem

$$\begin{cases} u^0_t + \overline{H}(Du^0) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u^0(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{I.4.33}$$

where the effective Hamiltonian $\overline{H}$ is uniquely defined by the long time behavior of the solution of

$$\begin{cases} \lambda = v_t + H(x, -\lambda t + p\cdot x + v, p + Dv), & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ v(0,x) = 0, & x \in \mathbb{R}^N. \end{cases} \tag{I.4.34}$$

More precisely, we have the following theorem

Theorem I.4.1 (Imbert-Monneau, [67])**.** *Let H be independent of t. Assume (H1)-(H3) and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Then, as $\epsilon \to 0$, the sequence u^ϵ converges locally uniformly in $(0,+\infty)\times\mathbb{R}^N$ to the solution u^0 of* (I.4.33)*, where, for any $p \in \mathbb{R}^N$ $\overline{H}(p)$ is defined as the unique number λ for which there exists a bounded continuous viscosity solution of* (I.4.34)*. Moreover $\overline{H} : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies the coercivity property*

$$\overline{H}(p) \to +\infty \quad \textit{as } |p| \to +\infty.$$

The proof in [67] is rather involved: it uses a *twisted* perturbed test function for a higher dimensional problem posed in $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$.
Under the additional assumption (H4), an easier proof of Theorem I.4.1 was given by Barles, [18], as a byproduct of a general result on the homogenization of Hamilton-Jacobi equations with non-coercive Hamiltonians.

Remark I.4.2. The hypothesis (H4) which was not used in [67] guarantees the existence of a function H_∞ such that

$$H_\infty(t,x,u,p) = \lim_{s\to 0^+} sH(t,x,u,s^{-1}p).$$

Moreover H_∞ satisfies (H1)-(H3).

In [18], thanks to assumption (H4), the equation for u^ϵ is interpreted as an equation for the motion of a graph: indeed, following [18], for $t \in \mathbb{R}$, $(x,y) \in \mathbb{R}^{N+1}$, $(p_x,p_y) \in \mathbb{R}^{N+1}$, let us introduce the non-coercive Hamiltonian F defined by

$$F(t,x,y,p_x,p_y) = \begin{cases} |p_y| H(t,x,y,|p_y|^{-1}p_x), & \text{if } p_y \neq 0, \\ H_\infty(t,x,y,p_x), & \text{otherwise.} \end{cases}$$

The function $U^\epsilon(t,x,y) := u^\epsilon(t,x) - y$ satisfies

$$\begin{cases} U^\epsilon_t + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon + y}{\epsilon}, D_x U^\epsilon, D_y U^\epsilon\right) = 0, & (t,x,y) \in (0,+\infty)\times\mathbb{R}^{N+1}, \\ U^\epsilon(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}. \end{cases} \tag{I.4.35}$$

In [18] Barles proves that the sequence U^ϵ converges to the solution U^0 of the following problem

$$\begin{cases} U^0_t + \overline{F}(D_x U^0, D_y U^0) = 0, & (t,x,y) \in (0,+\infty)\times\mathbb{R}^{N+1}, \\ U^0(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}, \end{cases} \tag{I.4.36}$$

where for $(p_x, p_y) \in \mathbb{R}^{N+1}$, $\overline{F}(p_x, p_y)$ is the unique number λ for which the cell problem

$$V_t + F(t, x, y, p_x + D_x V, p_y + D_y V) = \lambda \quad \text{in } \mathbb{R} \times \mathbb{R}^{N+1}. \tag{I.4.37}$$

admits bounded sub and supersolutions. This result makes it possible to solve the homogenization problem for (I.4.32):

Theorem I.4.3 (Barles, [18])**.** *Assume (H1)-(H4). Then the sequence u^ϵ converges locally uniformly in $(0, +\infty) \times \mathbb{R}^N$ to the solution u^0 of* (I.4.33)*. The function $\overline{H}(p)$ in* (I.4.33) *can be characterized as follows: $\overline{H}(p) = \overline{F}(p, -1)$, where, for any $(p_x, p_y) \in \mathbb{R}^{N+1}$, $\overline{F}(p_x, p_y)$ is the unique number λ for which the equation* (I.4.37) *admits bounded sub and supersolutions in $\mathbb{R} \times \mathbb{R}^{N+1}$.*

An important step in the proof of Theorem I.4.3 consists of homogenizing the non-coercive level-set equation satisfied by $\mathbb{1}_{\{U^\epsilon \geq 0\}}$.

Outline of the results of Chapter 5

In Chapter 5, we tackle two questions:

- Is it possible to estimate the rate of convergence of u^ϵ to u^0 when $\epsilon \to 0$?
- Is is possible to approximate numerically the effective Hamiltonian?

The first question was answered by Capuzzo Dolcetta and Ishii, [36] for a more classical homogenization problem: the estimate $\|u^\epsilon - u^0\|_\infty \leq C\epsilon^{\frac{1}{3}}$ was obtained for Hamilton-Jacobi equations of the type

$$u^\epsilon + H\left(x, \frac{x}{\epsilon}, u^\epsilon\right) = 0,$$

where $(x, y, p) \to H(x, y, p)$ is a coercive Hamiltonian, uniformly Lipschitz continuous for $|p|$ bounded and periodic with respect to y; moreover, if $H(x, y, p)$ does not depend on x, then the convergence is linear in ϵ.

We show that in the present case, it is possible to obtain the same rates of convergence as $\epsilon \to 0$ by adapting the proof in [36] using the arguments contained in [18]. The main idea is to approximate U^ϵ (with an error smaller than ϵ) by a discontinuous function $\widetilde{U}^\epsilon$ which takes integer values where U^ϵ has noninteger values and which is a discontinuous viscosity solution of

$$\widetilde{U}^\epsilon_t + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, D_x\widetilde{U}^\epsilon, D_y\widetilde{U}^\epsilon\right) = 0, \qquad (t, x, y) \in (0, +\infty) \times \mathbb{R}^{N+1}.$$

The latter equation has to be compared with (I.4.35). This approximation $\widetilde{U}^\epsilon$ is obtained as the limit as $\delta \to 0$ of $\phi_\delta(U^\epsilon)$ where $(\phi_\delta)_\delta$ is a sequence of increasing functions. The method of Capuzzo Dolcetta and Ishii [36] can then be applied to $\widetilde{U}^\epsilon$. The second question was studied in [1] for equation

$$u^\epsilon + H\left(\frac{x}{\epsilon}, u^\epsilon\right) = 0,$$

where $(y, p) \to H(y, p)$ is a coercive Hamiltonian, uniformly Lipschitz continuous for $|p|$ bounded and periodic with respect to y; in this article, a complete numerical method for solving the homogenized problem was studied, including as a main step the approximation of the effective Hamiltonian by solving discrete cell problems. Error estimates were proved. Here, we study the approximation of the cell problem (I.4.37) by Eulerian schemes in the discrete torus. We have preferred to study the approximation of the noncoercive $N + 2$ dimensional problem (I.4.37) rather than that of the coercive $N+1$ dimensional problem (I.4.34) because the solution of (I.4.34) may not be periodic. We prove the discrete analogue of the ergodicity Theorems in [18], i.e. that there exists a unique real number $\lambda_h^{\Delta t}$ such that the discrete analogue of (I.4.37) has a solution. The arguments in the proof are the discrete counterparts of those in [18]. We also show that the discrete effective Hamiltonian converges to the effective Hamiltonian when the grid step of the discrete cell problem tends to zero.

Part I

Neumann generalized principal eigenvalues

Chapter 1

Fully nonlinear singular operators

In this chapter we study the maximum principle, principal eigenvalues, regularity and existence for viscosity solutions of the Neumann boundary value problem

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du|Du|^\alpha + (c(x) + \lambda)|u|^\alpha u = g(x) & \text{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x) \rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.0.1}$$

where Ω is a bounded domain of class C^2, $\overrightarrow{n}(x)$ is the exterior normal to the domain Ω at x, $\alpha > -1$, $\lambda \in \mathbb{R}$ and b, c, g are continuous functions on $\overline{\Omega}$. F is a fully nonlinear operator that may be singular at the points where the gradient vanishes. $F : \overline{\Omega} \times \mathbb{R}^N \setminus \{0\} \times S(N) \to \mathbb{R}$ satisfies the following conditions

(F1) For all $t \in \mathbb{R}^*, \mu \geq 0$, $(x, p, X) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\} \times S(N)$

$$F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X).$$

(F2) There exist $a, A > 0$ such that for $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}, M, N \in S(N)$, $N \geq 0$

$$a|p|^\alpha \text{tr} N \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \text{tr} N.$$

(F3) There exist $C_1 > 0$ and $\theta \in (\frac{1}{2}, 1]$ such that for all $x, y \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in S(N)$

$$|F(x, p, X) - F(y, p, X)| \leq C_1|x - y|^\theta |p|^\alpha \|X\|.$$

(F4) There exist $C_2 > 0$ and $\nu \in (\frac{1}{2}, 1]$ such that for all $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $p_0 \in \mathbb{R}^N$, $|p_0| \leq \frac{|p|}{2}$, $X \in S(N)$

$$|F(x, p + p_0, X) - F(x, p, X)| \leq C_2|p|^{\alpha - \nu}|p_0|^\nu \|X\|.$$

The domain Ω is supposed to be bounded and of class C^2. In particular, it satisfies the interior sphere condition and the uniform exterior sphere condition, i.e.,

(Ω1) For each $x \in \partial\Omega$ there exist $R > 0$ and $y \in \Omega$ for which $|x - y| = R$ and $B(y, R) \subset \Omega$.

($\Omega 2$) There exists $r > 0$ such that $B(x + r\overrightarrow{n}(x), r) \cap \Omega = \emptyset$ for any $x \in \partial\Omega$.

From the property ($\Omega 2$) it follows that

$$\langle y - x, \overrightarrow{n}(x) \rangle \leq \frac{1}{2r}|y - x|^2 \quad \text{for } x \in \partial\Omega \text{ and } y \in \overline{\Omega}. \tag{1.0.2}$$

Moreover, the C^2-regularity of Ω implies the existence of a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ on which the distance from the boundary

$$d(x) := \inf\{|x - y|, y \in \partial\Omega\}, \quad x \in \overline{\Omega}$$

is of class C^2. We still denote by d a C^2 extension of the distance function to the whole $\overline{\Omega}$. Without loss of generality we can assume that $|Dd(x)| \leq 1$ in $\overline{\Omega}$.

Here we adopt the notion of viscosity solution given in Definition I.2.2. We call strong viscosity subsolutions (resp., supersolutions) the viscosity subsolutions (resp., supersolutions) that satisfy $B(x, u, Du) \leq$ (resp., $\geq$) 0 in the viscosity sense for all $x \in \partial\Omega$. If $\lambda \to B(x, r, p - \lambda\overrightarrow{n})$ is non-increasing in $\lambda \geq 0$, then classical subsolutions (resp., supersolutions) are strong viscosity subsolutions (resp., supersolutions), see [40] Proposition 7.2.

In the definition of viscosity solution the test functions can be substituted by the elements of the semijets $\overline{J}^{2,+}u(x_0)$ when u is a subsolution and $\overline{J}^{2,-}u(x_0)$ when u is a supersolution, see [40].

For simplicity of notation we denote

$$G(x, u, Du, D^2u) := F(x, Du, D^2u) + b(x) \cdot Du|Du|^\alpha + c(x)|u|^\alpha u.$$

Remark that the function b, c and g correspond respectively to $-b, -c$ and $-g$ of (I.2.7).

Following the ideas of [26], we define the principal eigenvalue as

$$\begin{aligned}\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ bounded viscosity supersolution of}\\ G(x, v, Dv, D^2v) + \lambda v^{\alpha+1} = 0 \text{ in } \Omega, \langle Dv, \overrightarrow{n} \rangle = 0 \text{ on } \partial\Omega\}.\end{aligned} \tag{1.0.3}$$

$\overline{\lambda}$ is well defined since the above set is not empty; indeed, $-|c|_\infty$ belongs to it, being $v(x) \equiv 1$ a corresponding supersolution. Furthermore it is an interval because if λ belongs to it then so does any $\lambda' < \lambda$.

We will prove that $\overline{\lambda}$ is an "eigenvalue" for $-G$ which admits a positive "eigenfunction", in the sense that there exists $\phi > 0$ solution of

$$\begin{cases} G(x, \phi, D\phi, D^2\phi) + \overline{\lambda}\phi^{\alpha+1} = 0 & \text{in} \quad \Omega \\ \langle D\phi, \overrightarrow{n}(x) \rangle = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover, $\overline{\lambda}$ can be characterized as the supremum of those λ for which the operator $G(x, u, Du, D^2u) + \lambda|u|^\alpha u$ with the Neumann boundary condition satisfies the maximum principle. As a consequence $\overline{\lambda}$ is the least "eigenvalue" to which there correspond "eigenfunctions" positive somewhere. These results are applied to obtain existence and uniqueness for the boundary value problem (1.0.1).

For fully nonlinear operators it is possible to define another principal eigenvalue

$$\underline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ bounded viscosity subsolution of } G(x,u,Du,D^2u) + \lambda|u|^\alpha u = 0 \text{ in } \Omega, \langle Du, \overrightarrow{n}\rangle = 0 \text{ on } \partial\Omega\}.$$

If $F(x,p,X) = -F(x,p,-X)$ then $\overline{\lambda} = \underline{\lambda}$, otherwise $\overline{\lambda}$ may be different from $\underline{\lambda}$. Symmetrical results can be obtained for $\underline{\lambda}$.

The classical assumption which guarantees the solvability of the Neumann problem (1.0.1) with $\lambda = 0$ is $c < 0$ in $\overline{\Omega}$. We show that the right hypothesis for any right-hand side is the positivity of the two principal eigenvalues.

In the next we establish a Lipschitz regularity result for viscosity solutions of (1.0.1). Section 1.2 is devoted to the study of the maximum principle for subsolutions of (1.0.1). In Section 1.2.1 we show that it holds (even for more general boundary conditions) for $G(x,u,Du,D^2u)$ if $c(x) \le 0$ and $c \not\equiv 0$, see Theorem 1.2.5. One of the main result of this chapter is that the maximum principle holds for $G(x,u,Du,D^2u)+\lambda|u|^\alpha u$ for any $\lambda < \overline{\lambda}$, as we show in Theorem 1.2.9 of Section 1.2.2. In particular it holds for $G(x,u,Du,D^2u)$ if $\overline{\lambda} > 0$. It is natural to wonder if the result of Theorem 1.2.9 is stronger than that of Theorem 1.2.5; indeed if $c \equiv 0$, one has $\overline{\lambda} = 0$. A positive answer is given in Section 1.2.3, where we construct an explicit example of a bounded positive viscosity supersolution of $G(x,v,Dv,D^2v) + \lambda v^{\alpha+1} = 0$ in Ω, $\langle Dv, \overrightarrow{n}\rangle = 0$ on $\partial\Omega$, $\lambda > 0$, with $c(x)$ changing sign. The existence of such v implies, by definition, $\overline{\lambda} > 0$. Finally, in Section 1.3 we show some existence and comparison theorems.

1.1 Lipschitz continuity of viscosity solutions

Theorem 1.1.1. *Let Ω be a bounded domain of class C^2. Suppose that F satisfies (F2)-(F4) and that b, c, g are bounded in Ω. If $u \in C(\overline{\Omega})$ is a viscosity solution of*

$$\begin{cases} F(x,Du,D^2u) + b(x)\cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) & \textit{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \textit{on} \quad \partial\Omega, \end{cases}$$

then

$$|u(x) - u(y)| \le C_0|x-y| \quad \forall x,y \in \overline{\Omega},$$

where C_0 depends on Ω, N, α, a, A, θ, ν, C_1, C_2, $|b|_\infty$, $|c|_\infty$, $|g|_\infty$, and $|u|_\infty$.

The Theorem is an immediate consequence of the next lemma. To prove the lemma we adopt the technique used in Proposition III.1 of [75] for Dirichlet problems, that we modify taking test functions which depend on $d(x)$.

The lemma plays a key role also in the proof of Theorem 1.2.9 in the next section.

Lemma 1.1.2. *Assume the hypothesis of Theorem 1.1.1 and suppose that g and h are bounded functions. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} F(x,Du,D^2u) + b(x)\cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) & \textit{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \textit{on} \quad \partial\Omega, \end{cases}$$

and $v \in LSC(\overline{\Omega})$ *a viscosity supersolution of*

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha + c(x)|v|^\alpha v = h(x) & in \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega, \end{cases}$$

with u *and* v *bounded, or* $v \geq 0$ *and bounded. If* $m = \max_{\overline{\Omega}}(u - v) \geq 0$*, then there exists* $C_0 > 0$ *such that*

$$u(x) - v(y) \leq m + C_0|x - y| \quad \forall x, y \in \overline{\Omega}, \tag{1.1.4}$$

where C_0 *depends on* Ω, N, α, a, A, θ, ν, C_1, C_2, $|b|_\infty$, $|c|_\infty$, $|g|_\infty$, $|h|_\infty$, $|v|_\infty$, m *and* $|u|_\infty$ *or* $\sup_{\overline{\Omega}} u$.

Proof. We set

$$\Phi(x) = MK|x| - M(K|x|)^2,$$

and

$$\varphi(x, y) = m + e^{-L(d(x)+d(y))}\Phi(x - y),$$

where L is a fixed number greater than $2/(3r)$ with r the radius in the condition $(\Omega 2)$ and K and M are two positive constants to be chosen later. If $K|x| \leq \frac{1}{4}$, then

$$\Phi(x) \geq \frac{3}{4}MK|x|. \tag{1.1.5}$$

We define

$$\Delta_K := \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \leq \frac{1}{4K}\right\}.$$

We fix M such that

$$\max_{\overline{\Omega}^2}(u(x) - v(y)) \leq m + e^{-2Ld_0}\frac{M}{8}, \tag{1.1.6}$$

where $d_0 = \max_{x\in\overline{\Omega}} d(x)$, and we claim that taking K large enough, one has

$$u(x) - v(y) - \varphi(x, y) \leq 0 \quad \text{for } (x, y) \in \Delta_K \cap \overline{\Omega}^2.$$

In this case (1.1.4) is proven. To show the last inequality we suppose by contradiction that for some $(\overline{x}, \overline{y}) \in \Delta_K \cap \overline{\Omega}^2$

$$u(\overline{x}) - v(\overline{y}) - \varphi(\overline{x}, \overline{y}) = \max_{\Delta_K\cap\overline{\Omega}^2}(u(x) - v(y) - \varphi(x, y)) > 0.$$

Here we have dropped the dependence of $\overline{x}$, $\overline{y}$ on K for simplicity of notations.

Observe that if $v \geq 0$, since from (1.1.5) $\Phi(x - y)$ is non-negative in Δ_K and $m \geq 0$, one has $u(\overline{x}) > 0$.

Clearly $\overline{x} \neq \overline{y}$. Moreover the point $(\overline{x}, \overline{y})$ belongs to $\text{int}(\Delta_K) \cap \overline{\Omega}^2$. Indeed, if $|x - y| = \frac{1}{4K}$, by (1.1.6) and (1.1.5) we have

$$u(x) - v(y) \leq m + e^{-2Ld_0}\frac{M}{8} \leq m + e^{-L(d(x)+d(y))}\frac{1}{2}MK|x - y| \leq \varphi(x, y).$$

Since $\overline{x} \neq \overline{y}$ we can compute the derivatives of φ in $(\overline{x}, \overline{y})$ obtaining

$$D_x\varphi(\overline{x},\overline{y}) = e^{-L(d(\overline{x})+d(\overline{y}))}MK\Big\{ - L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{x}) + (1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}\Big\},$$

$$D_y\varphi(\overline{x},\overline{y}) = e^{-L(d(\overline{x})+d(\overline{y}))}MK\Big\{ - L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{y}) - (1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}\Big\}.$$

Observe that for large K

$$e^{-2Ld_0}\frac{MK}{4} \leq e^{-L(d(\overline{x})+d(\overline{y}))}MK\left(\frac{1}{2}-L|\overline{x}-\overline{y}|\right) \leq |D_x\varphi(\overline{x},\overline{y})|, |D_y\varphi(\overline{x},\overline{y})| \leq 2MK. \tag{1.1.7}$$

Using (1.0.2), if $\overline{x} \in \partial\Omega$ we have

$$\begin{aligned}
&\langle D_x\varphi(\overline{x},\overline{y}), \overrightarrow{n}(\overline{x})\rangle \\
&= e^{-Ld(\overline{y})}MK\Big\{L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|) + (1-2K|\overline{x}-\overline{y}|)\langle\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}, \overrightarrow{n}(\overline{x})\rangle\Big\} \\
&\geq e^{-Ld(\overline{y})}MK\Big\{\frac{3}{4}L|\overline{x}-\overline{y}| - (1-2K|\overline{x}-\overline{y}|)\frac{|\overline{x}-\overline{y}|}{2r}\Big\} \\
&\geq \frac{1}{2}e^{-Ld(\overline{y})}MK|\overline{x}-\overline{y}|\left(\frac{3}{2}L-\frac{1}{r}\right) > 0,
\end{aligned}$$

since $\overline{x} \neq \overline{y}$ and $L > 2/(3r)$. Similarly, if $\overline{y} \in \partial\Omega$

$$\langle -D_y\varphi(\overline{x},\overline{y}), \overrightarrow{n}(\overline{y})\rangle \leq \frac{1}{2}e^{-Ld(\overline{x})}MK|\overline{x}-\overline{y}|\left(-\frac{3}{2}L+\frac{1}{r}\right) < 0.$$

In view of definition of sub and supersolution, we conclude that

$$G(\overline{x}, u(\overline{x}), D_x\varphi(\overline{x},\overline{y}), X) \geq g(\overline{x}) \quad \text{if } (D_x\varphi(\overline{x},\overline{y}), X) \in \overline{J}^{2,+}u(\overline{x}),$$

$$G(\overline{y}, v(\overline{y}), -D_y\varphi(\overline{x},\overline{y}), Y) \leq h(\overline{y}) \quad \text{if } (-D_y\varphi(\overline{x},\overline{y}), Y) \in \overline{J}^{2,-}v(\overline{y}).$$

Since $(\overline{x},\overline{y}) \in \text{int}\Delta_K \cap \overline{\Omega}^2$, it is a local maximum point of $u(x) - v(y) - \varphi(x,y)$ in $\overline{\Omega}^2$. Then applying Theorem 3.2 in [40], for every $\epsilon > 0$ there exist $X, Y \in S(N)$ such that $(D_x\varphi(\overline{x},\overline{y}), X) \in \overline{J}^{\,2,+}u(\overline{x})$, $(-D_y\varphi(\overline{x},\overline{y}), Y) \in \overline{J}^{\,2,-}v(\overline{y})$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2(\varphi(\overline{x},\overline{y})) + \epsilon(D^2(\varphi(\overline{x},\overline{y})))^2. \tag{1.1.8}$$

Now we want to estimate the matrix on the right-hand side of the last inequality.

$$\begin{aligned}
D^2\varphi(\overline{x},\overline{y}) &= \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}) + D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) \\
&\quad + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}) + e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).
\end{aligned}$$

We set

$$A_1 := \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_2 := D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_3 := e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).$$

Observe that

$$A_1 \le CK|\overline{x}-\overline{y}| \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{1.1.9}$$

Here and henceforth C denotes various positive constants independent of K.

For A_2 we have the following estimate

$$A_2 \le CK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + CK \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{1.1.10}$$

Indeed for $\xi,\,\eta \in \mathbb{R}^N$ we compute

$$\begin{aligned}\langle A_2(\xi,\eta),(\xi,\eta)\rangle &= 2Le^{-L(d(\overline{x})+d(\overline{y}))}\{\langle Dd(\overline{x})\otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\xi\rangle \\ &\quad + \langle Dd(\overline{y})\otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\eta\rangle\} \le CK(|\xi|+|\eta|)|\eta-\xi| \\ &\quad \le CK(|\xi|^2+|\eta|^2) + CK|\eta-\xi|^2.\end{aligned}$$

Now we consider A_3. The matrix $D^2(\Phi(\overline{x}-\overline{y}))$ has the form

$$D^2(\Phi(\overline{x}-\overline{y})) = \begin{pmatrix} D^2\Phi(\overline{x}-\overline{y}) & -D^2\Phi(\overline{x}-\overline{y}) \\ -D^2\Phi(\overline{x}-\overline{y}) & D^2\Phi(\overline{x}-\overline{y}) \end{pmatrix},$$

and the Hessian matrix of $\Phi(x)$ is

$$D^2\Phi(x) = \frac{MK}{|x|}\left(I - \frac{x\otimes x}{|x|^2}\right) - 2MK^2 I. \tag{1.1.11}$$

If we choose

$$\epsilon = \frac{|\overline{x}-\overline{y}|}{2MKe^{-L(d(\overline{x})+d(\overline{y}))}},$$

then we have the following estimates

$$\begin{gathered}\epsilon A_1^2 \le CK|\overline{x}-\overline{y}|^3 I_{2N}, \quad \epsilon A_2^2 \le CK|\overline{x}-\overline{y}| I_{2N}, \\ \epsilon(A_1A_2+A_2A_1) \le CK|\overline{x}-\overline{y}|^2 I_{2N}, \\ \epsilon(A_1A_3+A_3A_1) \le CK|\overline{x}-\overline{y}| I_{2N}, \quad \epsilon(A_2A_3+A_3A_2) \le CKI_{2N},\end{gathered} \tag{1.1.12}$$

where $I_{2N} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then using (**??**), (1.1.10), (1.1.12) and observing that

$$(D^2(\Phi(\overline{x}-\overline{y})))^2 = \begin{pmatrix} 2(D^2\Phi(\overline{x}-\overline{y}))^2 & -2(D^2\Phi(\overline{x}-\overline{y}))^2 \\ -2(D^2\Phi(\overline{x}-\overline{y}))^2 & 2(D^2\Phi(\overline{x}-\overline{y}))^2 \end{pmatrix},$$

from (1.1.8) we conclude that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le O(K)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where

$$B = CKI + e^{-L(d(\overline{x})+d(\overline{y}))}\left[D^2\Phi(\overline{x}-\overline{y}) + \frac{|\overline{x}-\overline{y}|}{MK}(D^2\Phi(\overline{x}-\overline{y}))^2\right].$$

The last inequality can be rewritten as follows

$$\begin{pmatrix} \widetilde{X} & 0 \\ 0 & -\widetilde{Y} \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with $\widetilde{X} = X - O(K)I$ and $\widetilde{Y} = Y + O(K)I$.

Now we want to get a good estimate for $\mathrm{tr}(\widetilde{X}-\widetilde{Y})$, as in [75]. For that aim let

$$0 \le P := \frac{(\overline{x}-\overline{y})\otimes(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^2} \le I.$$

Since $\widetilde{X}-\widetilde{Y} \le 0$ and $\widetilde{X}-\widetilde{Y} \le 4B$, we have

$$\mathrm{tr}(\widetilde{X}-\widetilde{Y}) \le \mathrm{tr}(P(\widetilde{X}-\widetilde{Y})) \le 4\mathrm{tr}(PB).$$

We have to compute $\mathrm{tr}(PB)$. From (1.1.11), observing that the matrix $(1/|x|^2)x\otimes x$ is idempotent, i.e., $[(1/|x|^2)x\otimes x]^2 = (1/|x|^2)x\otimes x$, we compute

$$(D^2\Phi(x))^2 = \frac{M^2K^2}{|x|^2}(1-4K|x|)\left(I - \frac{x\otimes x}{|x|^2}\right) + 4M^2K^4I.$$

Then, since $\mathrm{tr}P = 1$ and $4K|\overline{x}-\overline{y}| \le 1$, we have

$$\begin{aligned}\mathrm{tr}(PB) &= CK + e^{-L(d(\overline{x})+d(\overline{y}))}(-2MK^2 + 4MK^3|\overline{x}-\overline{y}|)\\ &\le CK - e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 < 0,\end{aligned}$$

for large K. This gives

$$|\mathrm{tr}(\widetilde{X}-\widetilde{Y})| = -\mathrm{tr}(\widetilde{X}-\widetilde{Y}) \ge 4e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 - 4CK \ge CK^2,$$

for large K. Since $\|B\| \le \frac{CK}{|\overline{x}-\overline{y}|}$, we have

$$\|B\|^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac{1}{2}} \le \left(\frac{CK}{|\overline{x}-\overline{y}|}\right)^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac{1}{2}} \le \frac{C}{K^{\frac{1}{2}}|\overline{x}-\overline{y}|^{\frac{1}{2}}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|.$$

The Lemma III.I in [75] ensures the existence of a universal constant C depending only on N such that

$$\|\widetilde{X}\|, \|\widetilde{Y}\| \le C\{|\mathrm{tr}(\widetilde{X}-\widetilde{Y})| + \|B\|^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac{1}{2}}\}.$$

Thanks to the above estimates we can conclude that

$$\|\widetilde{X}\|,\ \|\widetilde{Y}\| \le C|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|\left(1 + \frac{1}{K^{\frac{1}{2}}|\overline{x}-\overline{y}|^{\frac{1}{2}}}\right). \tag{1.1.13}$$

Now, using the assumptions (F2), (F3) and (F4) concerning F, the definition of $\widetilde{X}$ and $\widetilde{Y}$ and the fact that u and v are respectively sub and supersolution we compute

$$\begin{aligned} g(\overline{x}) - c(\overline{x})|u(\overline{x})|^\alpha u(\overline{x}) &\leq F(\overline{x}, D_x\varphi, X) + b(\overline{x}) \cdot D_x\varphi|D_x\varphi|^\alpha \\ &\leq F(\overline{x}, D_x\varphi, \widetilde{X}) + |D_x\varphi|^\alpha O(K) + b(\overline{x}) \cdot D_x\varphi|D_x\varphi|^\alpha \\ &\leq F(\overline{y}, -D_y\varphi, \widetilde{Y}) + C_1|\overline{x} - \overline{y}|^\theta |D_x\varphi|^\alpha \|\widetilde{X}\| \\ &+ CK^\nu|\overline{x} - \overline{y}|^\nu |D_x\varphi|^{\alpha-\nu}\|\widetilde{X}\| + a|D_y\varphi|^\alpha \mathrm{tr}(\widetilde{X} - \widetilde{Y}) \\ &+ |D_x\varphi|^\alpha O(K) + b(\overline{x}) \cdot D_x\varphi|D_x\varphi|^\alpha \\ &\leq b(\overline{y}) \cdot D_y\varphi|D_y\varphi|^\alpha - c(\overline{y})|v(\overline{y})|^\alpha v(\overline{y}) + h(\overline{y}) \\ &+ C_1|\overline{x} - \overline{y}|^\theta |D_x\varphi|^\alpha \|\widetilde{X}\| + CK^\nu|\overline{x} - \overline{y}|^\nu |D_x\varphi|^{\alpha-\nu}\|\widetilde{X}\| \\ &+ a|D_y\varphi|^\alpha \mathrm{tr}(\widetilde{X} - \widetilde{Y}) + |D_y\varphi|^\alpha \vee |D_x\varphi|^\alpha O(K) \\ &+ b(\overline{x}) \cdot D_x\varphi|D_x\varphi|^\alpha. \end{aligned}$$

From this inequalities, using (1.1.7), (1.1.13) and the fact that $\theta, \nu > \frac{1}{2}$ we get

$$\begin{aligned} g(\overline{x}) - h(\overline{y}) - c(\overline{x})|u(\overline{x})|^\alpha u(\overline{x}) + c(\overline{y})|v(\overline{y})|^\alpha v(\overline{y}) \leq |D_y\varphi|^\alpha \vee |D_x\varphi|^\alpha [a\mathrm{tr}(\widetilde{X} - \widetilde{Y}) \\ + C_1|\overline{x} - \overline{y}|^\theta \|\widetilde{X}\| + C|\overline{x} - \overline{y}|^\nu \|\widetilde{X}\| + O(K)] \leq CK^\alpha[a\mathrm{tr}(\widetilde{X} - \widetilde{Y}) + o(|\mathrm{tr}(\widetilde{X} - \widetilde{Y})|)]. \end{aligned}$$

If both u and v are bounded, then the first member in the last inequalities is bounded from below by $-|g|_\infty - |h|_\infty - |c|_\infty(|u|_\infty^{\alpha+1} + |v|_\infty^{\alpha+1})$. Otherwise, if v is non-negative and bounded, then $u(\overline{x}) \geq 0$ and that quantity is greater than $-|g|_\infty - |h|_\infty - |c|_\infty(\sup u)^{\alpha+1} - |c|_\infty|v|_\infty^{\alpha+1}$. On the other hand, the last member goes to $-\infty$ as $K \to +\infty$, hence taking K large enough we obtain a contradiction and this concludes the proof. □

Remark 1.1.3. If F satisfies (F2) and (F3), u is a subsolution of $G(x, u, Du, D^2u) = g$, v is a supersolution of $G(x, v, Dv, D^2v) = h$ in Ω, $u \leq v$ on $\partial\Omega$ and $m > 0$ then the estimate (1.1.4) still holds for any $x, y \in \Omega$. To prove this define $\varphi = m + MK|x| - M(K|x|)^2$ and follow the proof of Lemma 1.1.2.

Since the Lipschitz estimate depends only on the bounds of the solution, of g and on the structural constants, an immediate consequence of Theorem 1.1.1 is the following compactness criterion that will be useful in the last section.

Corollary 1.1.4. *Assume the hypothesis of Theorem 1.1.1 on Ω, F and b. Suppose that $(g_n)_n$ is a sequence of continuous and uniformly bounded functions and $(u_n)_n$ is a sequence of uniformly bounded viscosity solutions of*

$$\begin{cases} F(x, Du_n, D^2u_n) + b(x) \cdot Du_n|Du_n|^\alpha = g_n(x) & \text{in} \quad \Omega \\ \langle Du_n, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then the sequence $(u_n)_n$ is relatively compact in $C(\overline{\Omega})$.

1.2 The Maximum Principle and the principal eigenvalues

We say that the operator $G(x, u, Du, D^2u)$ with the Neumann boundary condition satisfies the maximum principle if whenever $u \in USC(\overline{\Omega})$ is a viscosity subsolution of

$$\begin{cases} G(x, u, Du, D^2u) = 0 & \text{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x) \rangle = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u \leq 0$ in $\overline{\Omega}$.

We first prove that the maximum principle holds under the classical assumption $c \leq 0$, also for domain which are not of class C^2 and with more general boundary conditions. Then we show that the operator $G(x, u, Du, D^2u) + \lambda|u|^\alpha u$ with the Neumann boundary condition satisfies the maximum principle for any $\lambda < \overline{\lambda}$. This is the best result that one can expect, indeed, as we will see in the last section, $\overline{\lambda}$ admits a positive eigenfunction which provides a counterexample to the maximum principle for $\lambda \geq \overline{\lambda}$.

Finally, we give an example of $c(x)$ which changes sign in Ω and such that the associated principal eigenvalue $\overline{\lambda}$ is positive.

1.2.1 The case $c(x) \leq 0$

In this subsection we assume that Ω is of class C^1 and satisfies the interior sphere condition (Ω1). We need the comparison principle between sub and supersolutions of the Dirichlet problem when $c < 0$ in Ω. This result is proven in [28] under different assumptions on F and b; thanks to the estimate (1.1.4), see Remark 1.1.3, we can show it using the same strategy of [28], if F satisfies the conditions (F2) and (F3) and b is continuous and bounded on Ω.

Theorem 1.2.1. *Let Ω be bounded. Assume that (F2) and (F3) hold, that b, c and g are continuous and bounded on Ω and $c < 0$ in Ω. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are respectively sub and supersolution of*

$$F(x, Du, D^2u) + b(x) \cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) \quad \text{in } \Omega,$$

and $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.

For convenience of the reader we postpone the proof of the theorem to the next subsection.

The previous comparison result allows us to establish the strong minimum and maximum principles, for sub and supersolutions of the Neumann problem even with the following more general boundary condition

$$f(x, u) + \langle Du, \overrightarrow{n}(x) \rangle = 0 \quad x \in \partial\Omega,$$

for some $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$. We do not assume any regularity on f.

Proposition 1.2.2. *Let Ω be a C^1 domain satisfying (Ω1). Assume that (F1)-(F3) hold, that b and c are bounded and continuous on Ω and that $f(x,0) \leq 0$ for all $x \in \partial\Omega$. If $v \in LSC(\overline{\Omega})$ is a non-negative viscosity supersolution of*

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha + c(x)|v|^\alpha v = 0 & \text{in} \quad \Omega \\ f(x,v) + \langle Dv, \overrightarrow{n}(x) \rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2.14}$$

then either $v \equiv 0$ or $v > 0$ in $\overline{\Omega}$.

Proof. The assumption $(F2)$ and the fact that $F(x,p,0) = 0$ imply that

$$F(x,p,M) \geq |p|^\alpha \mathcal{M}^-_{a,A} M = |p|^\alpha (atr(M^+) - Atr(M^-)) =: H(p,M),$$

where $M = M^+ - M^-$ is the minimal decomposition of M into positive and negative symmetric matrices. It follows, since v is non-negative, that it suffices to prove the proposition when v is a supersolution of the Neumann problem for the equation

$$H(Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha - |c|_\infty v^{1+\alpha} = 0 \quad \text{in } \Omega. \tag{1.2.15}$$

Moreover we can assume $|c|_\infty > 0$. Following the proof of Theorem 2 in [28] it can be showed that $v > 0$ in Ω. We prove that v cannot vanish on the boundary of Ω. We suppose by contradiction that x_0 is some point in $\partial\Omega$ on which $v(x_0) = 0$. For the interior sphere condition (Ω1) there exist $R > 0$ and $y \in \Omega$ such that the ball centered in y and of radius R, $B(y,R)$, is contained in Ω and $x_0 \in \partial B(y,R)$. Fixed $0 < \rho < R$, let us construct a subsolution of (1.2.15) in the annulus $\rho < |x-y| = r < R$. Let us consider the function $\phi(x) = e^{-kr} - e^{-kR}$, where k is a positive constant to be determined. If we compute the derivatives of ϕ we get

$$D\phi(x) = -ke^{-kr}\frac{(x-y)}{r},\ D^2\phi(x) = \left(k^2e^{-kr} + \frac{k}{r}e^{-kr}\right)\frac{(x-y)\otimes(x-y)}{r^2} - \frac{k}{r}e^{-kr}I.$$

The eigenvalues of $D^2\phi(x)$ are k^2e^{-kr} of multiplicity 1 and $-ke^{-kr}/r$ of multiplicity $N-1$. Then

$$\begin{aligned} &H(D\phi, D^2\phi) + b(x) \cdot D\phi|D\phi|^\alpha - |c|_\infty \phi^{1+\alpha} \\ &\geq e^{-(\alpha+1)kr}\left(ak^{\alpha+2} - \left(A\frac{N-1}{\rho} + |b|_\infty\right)k^{\alpha+1} - |c|_\infty\right). \end{aligned}$$

Take k such that

$$ak^{\alpha+2} - \left(A\frac{N-1}{\rho} + |b|_\infty\right)k^{\alpha+1} - |c|_\infty > \epsilon,$$

for some $\epsilon > 0$, then ϕ is a strict subsolution of the equation (1.2.15). Now choose $m > 0$ such that

$$m(e^{-k\rho} - e^{-kR}) = v_1 := \inf_{|x-y|=\rho} v(x) > 0,$$

and define $w(x) = m(e^{-kr} - e^{-kR})$. By homogeneity w is still a subsolution of (1.2.15) in the annulus $\rho < |x-y| < R$, moreover $w = v_1 \leq v$ if $|x-y| = \rho$ and

$w = 0 \leq v$ if $|x-y| = R$. Then by the comparison principle, Theorem 1.2.1, $w \leq v$ in the entire annulus.

Now let δ be a positive number smaller than $R-\rho$. In $B(x_0,\delta)\cap\overline{\Omega}$ it is again $w \leq v$, in fact where $|x-y| > R$ it is $w < 0 \leq v$; moreover $w(x_0) = v(x_0) = 0$. Then w is a test function for v at x_0. But

$$H(Dw(x_0), D^2w(x_0)) + b(x_0)\cdot Dw(x_0)|Dw(x_0)|^\alpha - |c|_\infty w^{1+\alpha}(x_0) > 0,$$

and

$$f(x_0, w(x_0)) + \langle Dw(x_0), \overrightarrow{n}(x_0)\rangle = f(x_0,0) + \frac{\partial w}{\partial \overrightarrow{n}}(x_0) \leq -kme^{-kR} < 0.$$

This contradicts the definition of v. Finally v cannot be zero in $\overline{\Omega}$. □

Remark 1.2.3. By Proposition 1.2.2 the supersolutions in the definition (1.0.3) are positive in the whole $\overline{\Omega}$.

Proposition 1.2.4. *Let Ω be a C^1 domain satisfying (Ω1). Assume that (F1)-(F3) hold, that b and c are bounded and continuous on Ω and that $f(x,0) \geq 0$ for all $x \in \partial\Omega$. If $u \in USC(\overline{\Omega})$ is a non-positive viscosity subsolution of* (1.2.14) *then either $u \equiv 0$ or $u < 0$ in $\overline{\Omega}$.*

Proof. The proof is similar to the proof of Proposition 1.2.2, observing that (F1) and the fact that $F(x,p,0) = 0$ imply that

$$F(x,p,M) \leq |p|^\alpha(A\mathrm{tr}(M^+) - a\mathrm{tr}(M^-)).$$

□

For $x \in \partial\Omega$, let us introduce $S(x)$, the symmetric operator corresponding to the second fundamental form of $\partial\Omega$ in x oriented with the exterior normal to Ω.

Theorem 1.2.5 (Maximum Principle for $c \leq 0$)**.** *Assume the hypothesis of Proposition 1.2.4. In addition suppose that Ω is bounded, $c \leq 0$, $c \not\equiv 0$ and $r \to f(x,r)$ is non-decreasing on $\mathbb{R}$. If $u \in USC(\overline{\Omega})$ is a viscosity subsolution of* (1.2.14) *then $u \leq 0$ in $\overline{\Omega}$. The same conclusion holds also if $c \equiv 0$ in the following two cases*

(i) Ω is a C^2 domain and there exists $\overline{x} \in \partial\Omega$ such that $S(\overline{x}) \leq 0$, $\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle > 0$ and $f(\overline{x}, r) > 0$ for any $r > 0$;

(ii) There exists $\overline{x} \in \partial\Omega$ such that $f(\overline{x},r) > 0$ for any $r > 0$ and u is a strong subsolution.

Proof. Let u be a subsolution of (1.2.14) and $c \not\equiv 0$. First let us suppose $u \equiv k$ =const. By definition

$$c(x)|k|^\alpha k \geq 0 \qquad \text{in } \Omega,$$

which implies $k \leq 0$.

Now we assume that u is not a constant. We argue by contradiction; suppose that $\max_{\overline{\Omega}} u = u(x_0) > 0$, for some $x_0 \in \overline{\Omega}$. Define $\widetilde{u}(x) := u(x) - u(x_0)$. Since $c \leq 0$ and f is non-decreasing, $\widetilde{u}$ is a non-positive subsolution of (1.2.14). Then, from Proposition 1.2.4, either $u \equiv u(x_0)$ or $u < u(x_0)$ in $\overline{\Omega}$. In both cases we get a contradiction.

Let us turn to the case $c \equiv 0$. Suppose that Ω is a C^2 domain, $S(\overline{x}) \leq 0$, $\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle > 0$ and $f(\overline{x}, r) > 0$ for any $r > 0$ and some point $\overline{x} \in \partial\Omega$. We have to prove that u cannot be a positive constant. Suppose by contradiction that $u \equiv k$. In general, if ϕ is a C^2 function, $\overline{x} \in \partial\Omega$ and $S(\overline{x}) \leq 0$, then $(D\phi(\overline{x}) - \lambda\overrightarrow{n}(\overline{x}), D^2\phi(\overline{x})) \in J^{2,+}\phi(\overline{x})$, for $\lambda \geq 0$ (see [40] Remark 2.7). Hence $(-\lambda\overrightarrow{n}(\overline{x}), 0) \in J^{2,+}u(\overline{x})$. But

$$f(\overline{x}, k) - \lambda\langle\overrightarrow{n}(\overline{x}), \overrightarrow{n}(\overline{x})\rangle = f(\overline{x}, k) - \lambda > 0,$$

for $\lambda > 0$ small enough, and

$$G(\overline{x}, k, -\lambda\overrightarrow{n}(\overline{x}), 0) = -\lambda^{\alpha+1}\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle < 0.$$

This contradicts the definition of u.

Finally, if u is a strong subsolution, $f(\overline{x}, r) > 0$ for $r > 0$ and some $\overline{x} \in \partial\Omega$, $u \equiv k > 0$, then the boundary condition is not satisfied at $\overline{x}$ for $p = 0$. □

Remark 1.2.6. Under the same assumptions of Theorem 1.2.5, but now with f satisfying $f(x, 0) \leq 0$ for all $x \in \partial\Omega$ and with $f(\overline{x}, r) < 0$ for any $r < 0$ and some $\overline{x} \in \partial\Omega$ in (i) and (ii), using Proposition 1.2.2 we can prove the minimum principle, i.e., if $u \in LSC(\overline{\Omega})$ is a viscosity supersolution of (1.2.14) then $u \geq 0$ in $\overline{\Omega}$.

Remark 1.2.7. C^2 convex sets satisfy the condition $S \leq 0$ in every point of the boundary.

Remark 1.2.8. If $c \equiv 0$ and $f \equiv 0$ a counterexample to the maximum principle is given by the positive constants.

1.2.2 The threshold for the Maximum Principle

In this subsection and in the rest of the paper we always assume that Ω is bounded and of class C^2, that F satisfies (F1)-(F4), that b and c are continuous on $\overline{\Omega}$.

Theorem 1.2.9 (Maximum Principle for $\lambda < \overline{\lambda}$)**.** *Let $\lambda < \overline{\lambda}$ and let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} G(x, u, Du, D^2u) + \lambda|u|^\alpha u = 0 & \text{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2.16}$$

then $u \leq 0$ in $\overline{\Omega}$.

Remark 1.2.10. Similarly it is possible to prove that if $\lambda < \underline{\lambda}$ and v is a supersolution of (1.2.16) then $v \geq 0$ in $\overline{\Omega}$.

Corollary 1.2.11. *The quantities $\overline{\lambda}$ and $\underline{\lambda}$ are finite.*

Proof. It suffices to observe that $\overline{\lambda}$, $\underline{\lambda} \leq |c|_\infty$, since when the zero order coefficient is $c(x) + |c|_\infty$ the maximum and the minimum principles do not hold. The theorems fail respectively for the positive and negative constants. □

In the proof of Theorem 1.2.9 the Lemma 1.1.2 is one of the main ingredient. Furthermore, we need the following two results. The first one is an adaptation of Lemma 1 of [28] for supersolutions of the Neumann boundary value problem; the second one is a lemma due to Barles and Ramaswamy, [23].

Lemma 1.2.12. *Let $v \in LSC(\overline{\Omega})$ be a viscosity supersolution of*

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha - \beta(v(x)) = g(x) & \text{in} \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for some functions $g, \beta \in USC(\overline{\Omega})$. Suppose that $\overline{x} \in \overline{\Omega}$ is a strict local minimum of $v(x) + C|x - \overline{x}|^q e^{-kd(x)}$, $k > \frac{q}{2r}$, where r is the radius in the condition (Ω2) and $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$. Moreover suppose that v is not locally constant around $\overline{x}$. Then

$$-\beta(v(\overline{x})) \leq g(\overline{x}).$$

Remark 1.2.13. Similarly, if β, $g \in LSC(\overline{\Omega})$, $u \in USC(\overline{\Omega})$ is a supersolution, $\overline{x}$ is a strict local maximum of $u(x) - C|x - \overline{x}|^q e^{-kd(x)}$, $k > \frac{q}{2r}$, $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$ and u is not locally constant around $\overline{x}$, it can be proved that

$$-\beta(u(\overline{x})) \geq g(\overline{x}).$$

Lemma 1.2.14. *If $X, Y \in S(N)$ satisfy*

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

then we have

$$X - Y \leq -\frac{1}{2\zeta}(tX + (1-t)Y)^2 \quad \textit{for all } t \in [0, 1].$$

Proof of Theorem 1.2.9. Let $\tau \in]\lambda, \overline{\lambda}[$, then by definition there exists $v > 0$ in $\overline{\Omega}$ bounded viscosity supersolution of

$$\begin{cases} G(x, v, Dv, D^2v) + \tau v^{\alpha+1} = 0 & \text{in} \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2.17}$$

We argue by contradiction that u has a positive maximum in $\overline{\Omega}$. As in [28], we define $\gamma' := \sup_{\overline{\Omega}}(u/v) > 0$ and $w = \gamma v$, with $\gamma \in (0, \gamma')$ to be determined. By homogeneity, w is still a supersolution of (1.2.17). Let $\overline{y} \in \overline{\Omega}$ be such that $u(\overline{y})/v(\overline{y}) = \gamma'$. Since $u(\overline{y}) - w(\overline{y}) = (\gamma' - \gamma)v(\overline{y}) > 0$, the supremum of $u - w$ is strictly positive, then by upper semicontinuity there exists $\overline{x} \in \overline{\Omega}$ such that

$$u(\overline{x}) - w(\overline{x}) = \max_{\overline{\Omega}}(u - w) = m > 0.$$

Clearly $u(\overline{x}) > w(\overline{x}) > 0$, moreover $u(\overline{x}) \leq \gamma' v(\overline{x}) = \frac{\gamma'}{\gamma} w(\overline{x})$, from which

$$w(\overline{x}) \geq \frac{\gamma}{\gamma'} u(\overline{x}). \tag{1.2.18}$$

Fix $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$ and $k > q/(2r)$, where r is the radius in the condition (Ω2), and define for $j \in \mathbb{N}$ the functions $\phi \in C^2(\overline{\Omega} \times \overline{\Omega})$ and $\psi \in USC(\overline{\Omega} \times \overline{\Omega})$ by

$$\phi(x,y) = \frac{j}{q}|x-y|^q e^{-k(d(x)+d(y))}, \quad \psi(x,y) = u(x) - w(y) - \phi(x,y).$$

Let $(x_j, y_j) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of ψ, then $m = \psi(\overline{x}, \overline{x}) \leq u(x_j) - w(y_j) - \phi(x_j, y_j)$, from which

$$\frac{j}{q}|x_j - y_j|^q \leq (u(x_j) - w(y_j) - m)e^{k(d(x_j)+d(y_j))} \leq C, \tag{1.2.19}$$

where C is independent of j. The last relation implies that, up to subsequence, x_j and y_j converge to some $\overline{z} \in \overline{\Omega}$ as $j \to +\infty$. Classical arguments show that

$$\lim_{j \to +\infty} \frac{j}{q}|x_j - y_j|^q = 0, \quad \lim_{j \to +\infty} u(x_j) = u(\overline{z}), \quad \lim_{j \to +\infty} w(y_j) = w(\overline{z}),$$

and

$$u(\overline{z}) - w(\overline{z}) = m.$$

Claim 1 *For j large enough, there exist x_j and y_j such that (x_j, y_j) is a maximum point of ψ and $x_j \neq y_j$.*

Indeed if $x_j = y_j$ we have

$$\psi(x_j, x) = u(x_j) - w(x) - \frac{j}{q}|x - x_j|^q e^{-k(d(x_j)+d(x))} \leq \psi(x_j, x_j) = u(x_j) - w(x_j),$$

and

$$\psi(x, x_j) = u(x) - w(x_j) - \frac{j}{q}|x - x_j|^q e^{-k(d(x)+d(x_j))} \leq \psi(x_j, x_j) = u(x_j) - w(x_j).$$

Then x_j is a minimum point for

$$W(x) := w(x) + \frac{j}{q} e^{-kd(x_j)} |x - x_j|^q e^{-kd(x)},$$

and a maximum point for

$$U(x) := u(x) - \frac{j}{q} e^{-kd(x_j)} |x - x_j|^q e^{-kd(x)}.$$

We first exclude that x_j is both a strict local minimum and a strict local maximum. Indeed in that case, if u and w are not locally constant around x_j, by Lemma 1.2.12

$$(c(x_j) + \tau) w(x_j)^{\alpha+1} \leq (c(x_j) + \lambda) u(x_j)^{\alpha+1}.$$

The same result holds if u or w are locally constant by definition of sub and supersolution. The last inequality leads to a contradiction, as we will see at the end of the proof. Hence x_j cannot be both a strict local minimum and a strict local maximum. In the first case there exist $\delta > 0$ and $R > \delta$ such that

$$\begin{aligned} w(x_j) &= \min_{\substack{\delta \le |x-x_j| \le R \\ x \in \overline{\Omega}}} \left(w(x) + \frac{j}{q}|x - x_j|^q e^{-k(d(x_j)+d(x))} \right) \\ &= w(y_j) + \frac{j}{q}|y_j - x_j|^q e^{-k(d(x_j)+d(y_j))}, \end{aligned}$$

for some $y_j \neq x_j$, so that (x_j, y_j) is still a maximum point for ψ. In the other case, similarly, one can replace x_j by a point $y_j \neq x_j$ such that (y_j, x_j) is a maximum for ψ. This concludes the Claim 1.

Now computing the derivatives of ϕ we get

$$D_x\phi(x, y) = j|x - y|^{q-2} e^{-k(d(x)+d(y))}(x - y) - k\frac{j}{q}|x - y|^q e^{-k(d(x)+d(y))} Dd(x),$$

and

$$D_y\phi(x, y) = -j|x - y|^{q-2} e^{-k(d(x)+d(y))}(x - y) - k\frac{j}{q}|x - y|^q e^{-k(d(x)+d(y))} Dd(y).$$

Denote $p_j := D_x\phi(x_j, y_j)$ and $r_j := -D_y\phi(x_j, y_j)$. Since $x_j \neq y_j$, p_j and r_j are different from 0 for j large enough. Indeed

$$|p_j|, |r_j| \geq j|x_j - y_j|^{q-1} e^{-k(d(x_j)+d(y_j))} \left(1 - \frac{k}{q}|x_j - y_j|\right) \geq \frac{j}{2}|x_j - y_j|^{q-1} e^{-2kd_0},$$

where $d_0 = \max_{\overline{\Omega}} d(x)$. Using (1.0.2), if $x_j \in \partial\Omega$ then

$$\langle p_j, \overrightarrow{n}(x_j)\rangle \geq j|x_j - y_j|^q e^{-kd(y_j)} \left(-\frac{1}{2r} + \frac{k}{q}\right) > 0,$$

and if $y_j \in \partial\Omega$ then

$$\langle r_j, \overrightarrow{n}(y_j)\rangle \leq j|x_j - y_j|^q e^{-kd(x_j)} \left(\frac{1}{2r} - \frac{k}{q}\right) < 0,$$

since $k > q/(2r)$ and $x_j \neq y_j$. In view of definition of sub and supersolution we conclude that

$$G(x_j, u(x_j), p_j, X) + \lambda u(x_j)^{\alpha+1} \geq 0 \quad \text{if } (p_j, X) \in \overline{J}^{2,+} u(x_j),$$

$$G(y_j, w(y_j), r_j, Y) + \tau w(y_j)^{\alpha+1} \leq 0 \quad \text{if } (r_j, Y) \in \overline{J}^{2,-} w(y_j).$$

Then the previous relations hold for $(x_j, y_j) \in \overline{\Omega}^2$, provided j is large.

Now, applying Theorem 3.2 of [40] for any $\epsilon > 0$ there exist $X_j, Y_j \in S(N)$ such that $(p_j, X_j) \in \overline{J}^{2,+} u(x_j)$, $(r_j, Y_j) \in \overline{J}^{2,-} w(y_j)$ and

$$-\left(\frac{1}{\epsilon} + \|D^2\phi(x_j, y_j)\|\right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq D^2\phi(x_j, y_j) + \epsilon(D^2\phi(x_j, y_j))^2. \tag{1.2.20}$$

Claim 2 *X_j and Y_j satisfy*

$$-\zeta_j \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_j - \widetilde{X_j} & 0 \\ 0 & -Y_j + \widetilde{Y_j} \end{pmatrix} \leq \zeta_j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \qquad (1.2.21)$$

where $\zeta_j = Cj|x_j - y_j|^{q-2}$, for some positive constant C independent of j and some matrices $\widetilde{X_j}$, $\widetilde{Y_j} = O(j|x_j - y_j|^q)$.

To prove the claim we need to estimate $D^2\phi(x_j, y_j)$.

$$D^2\phi(x_j, y_j) = \frac{j}{q}|x_j - y_j|^q D^2(e^{-k(d(x_j)+d(y_j))}) + D(e^{-k(d(x_j)+d(y_j))}) \otimes \frac{j}{q} D(|x_j - y_j|^q) + \frac{j}{q} D(|x_j - y_j|^q) \otimes D(e^{-k(d(x_j)+d(y_j))}) + e^{-k(d(x_j)+d(y_j))} \frac{j}{q} D^2(|x_j - y_j|^q).$$

We denote

$$A_1 := \frac{j}{q}|x_j - y_j|^q D^2(e^{-k(d(x_j)+d(y_j))}),$$

$$A_2 := De^{-k(d(x_j)+d(y_j))} \otimes \frac{j}{q} D(|x_j - y_j|^q) + \frac{j}{q} D(|x_j - y_j|^q) \otimes D(e^{-k(d(x_j)+d(y_j))}),$$

$$A_3 := e^{-k(d(x_j)+d(y_j))} \frac{j}{q} D^2(|x_j - y_j|^q).$$

For A_1 and A_3 we have

$$A_1 \leq Cj|x_j - y_j|^q \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$$A_3 \leq (q-1)j|x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Here and henceforth, as usual, the letter C denotes various constants independent of j. Now we consider the quantity $\langle A_2(\xi, \eta), (\xi, \eta)\rangle$ for ξ, $\eta \in \mathbb{R}^N$. We have

$$\begin{aligned} \langle A_2(\xi, \eta), (\xi, \eta)\rangle &= 2kj|x_j - y_j|^{q-2} e^{-k(d(x_j)+d(y_j))} [\langle Dd(x_j) \otimes (x_j - y_j)(\eta - \xi), \xi\rangle \\ &\quad + \langle Dd(y_j) \otimes (x_j - y_j)(\eta - \xi), \eta\rangle] \\ &\leq Cj|x_j - y_j|^{q-1}|\xi - \eta|(|\xi| + |\eta|) \\ &\leq Cj|x_j - y_j|^{q-1} \left(\frac{|\xi - \eta|^2}{|x_j - y_j|} + \frac{(|\xi| + |\eta|)^2}{4}|x_j - y_j| \right) \\ &\leq C\left[j|x_j - y_j|^{q-2}|\xi - \eta|^2 + j|x_j - y_j|^q(|\xi|^2 + |\eta|^2) \right]. \end{aligned}$$

The last inequality can be rewritten equivalently in this way

$$A_2 \leq Cj|x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + Cj|x_j - y_j|^q \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Finally if we choose

$$\epsilon = \frac{1}{j|x_j - y_j|^{q-2}},$$

we get the same estimates for the matrix $\epsilon(D^2\phi(x_j,y_j))^2$. In conclusion we have

$$D^2\phi(x_j,y_j)+\epsilon(D^2\phi(x_j,y_j))^2 \le Cj|x_j-y_j|^{q-2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + Cj|x_j-y_j|^{q}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence, since $\|D^2\phi(x_j,y_j)\| \le Cj|x_j-y_j|^{q-2}$, (1.2.20) implies (1.2.21) and the Claim 2 is proved.

Claim 3 $F(x_j,p_j,X_j-\widetilde{X_j})-F(y_j,r_j,Y_j-\widetilde{Y_j}) \le o_j$, *where* $o_j \to 0$ *as* $j \to +\infty$.

First we need to know that the quantity $j|x_j-y_j|^{q-1}$ is bounded uniformly in j. This is a simple consequence of Lemma 1.1.2. Indeed, since $m>0$ and w is positive and bounded, the estimate (1.1.4) holds for u and w; then using it in (1.2.19) and dividing by $|x_j-y_j| \neq 0$ we obtain

$$\frac{j}{q}|x_j-y_j|^{q-1} \le C_0 e^{k(d(x_j)+d(y_j))} \le C.$$

Consequently, there exists $R>0$ such that for large j

$$C\zeta_j|x_j-y_j| \le \frac{j}{2}|x_j-y_j|^{q-1}e^{-k(d(x_j)+d(y_j))} \le |p_j|,\ |r_j| \le 2j|x_j-y_j|^{q-1} \le R. \tag{1.2.22}$$

Denote for simplicity $Z_j := X_j-\widetilde{X_j}$ and $W_j := Y_j-\widetilde{Y_j}$. By (1.2.21) and Lemma 1.2.14 with $t=0$, we have

$$Z_j-W_j \le -\frac{1}{2\zeta_j}W_j^2.$$

As in the appendix of [21] we use the previous relation, the Cauchy-Schwarz's inequality and the properties of F to get the estimate of the claim

$$\begin{aligned} F(x_j,p_j,Z_j)-F(y_j,r_j,W_j) &= F(x_j,p_j,Z_j)-F(x_j,p_j,W_j)+F(x_j,p_j,W_j) \\ &\quad - F(y_j,p_j,W_j)+F(y_j,p_j,W_j)-F(y_j,r_j,W_j) \\ &\le -\frac{a}{2\zeta_j}|p_j|^\alpha \mathrm{tr}W_j^2 + C_1|x_j-y_j|^\theta|p_j|^\alpha\|W_j\| \\ &\quad + C_2|p_j|^{\alpha-\nu}|p_j-r_j|^\nu\|W_j\| \le -\frac{a}{2\zeta_j}|p_j|^\alpha \mathrm{tr}W_j^2 \\ &\quad + \frac{a}{4\zeta_j}|p_j|^\alpha \mathrm{tr}W_j^2 + \frac{C_1^2|x_j-y_j|^{2\theta}|p_j|^{2\alpha}\zeta_j}{a|p_j|^\alpha} \\ &\quad + \frac{a}{4\zeta_j}|p_j|^\alpha \mathrm{tr}W_j^2 + \frac{C_2^2|p_j|^{2(\alpha-\nu)}|p_j-r_j|^{2\nu}\zeta_j}{a|p_j|^\alpha} \\ &= C\zeta_j|x_j-y_j|^{2\theta}|p_j|^\alpha + C\zeta_j|p_j|^{\alpha-2\nu}|p_j-r_j|^{2\nu}. \end{aligned}$$

Now consider the first term of the last quantity. Using (1.2.22) we have

$$C\zeta_j|x_j-y_j|^{2\theta}|p_j|^\alpha \le \frac{C\zeta_j|x_j-y_j|^{2\theta}|p_j|^{\alpha+1}}{\zeta_j|x_j-y_j|} \le CR^{\alpha+1}|x_j-y_j|^{2\theta-1},$$

and the last term goes to 0 as $j \to +\infty$ since $\theta > \frac{1}{2}$. It remains to estimate $C\zeta_j|p_j|^{\alpha-2\nu}|p_j - r_j|^{2\nu}$. Observe that

$$|p_j - r_j| \le 2k\frac{j}{q}|x_j - y_j|^q = C\zeta_j|x_j - y_j|^2,$$

then we have

$$\begin{aligned} C\zeta_j|p_j|^{\alpha-2\nu}|p_j - r_j|^{2\nu} &= C|p_j|^{\alpha+1}\frac{\zeta_j}{|p_j|}\frac{|p_j - r_j|^{2\nu}}{|p_j|^{2\nu}} \le \frac{CR^{\alpha+1}}{|x_j - y_j|}|x_j - y_j|^{2\nu} \\ &= CR^{\alpha+1}|x_j - y_j|^{2\nu-1}. \end{aligned}$$

Also the last quantity goes to 0 as $j \to +\infty$ since $\nu > \frac{1}{2}$ and this concludes the Claim 3.

Now using the properties of F and the fact that u and w are respectively sub and supersolution we compute

$$\begin{aligned} -(\lambda + c(x_j))u(x_j)^{\alpha+1} &\le F(x_j, p_j, X_j) + b(x_j)\cdot p_j|p_j|^\alpha \\ &\le F(x_j, p_j, X_j - \widetilde{X_j}) + b(x_j)\cdot p_j|p_j|^\alpha \ + |p_j|^\alpha O\left(j|x_j - y_j|^q\right) \\ &\le F(y_j, r_j, Y_j - \widetilde{Y_j}) + b(x_j)\cdot p_j|p_j|^\alpha + |p_j|^\alpha O\left(j|x_j - y_j|^q\right) + o_j \\ &\le -(\tau + c(y_j))w(y_j)^{\alpha+1} + b(x_j)\cdot p_j|p_j|^\alpha - b(y_j)\cdot r_j|r_j|^\alpha \\ &+ (|p_j|^\alpha \vee |r_j|^\alpha) O\left(j|x_j - y_j|^q\right) + o_j. \end{aligned}$$

Sending $j \to +\infty$ we obtain

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))w(\overline{z})^{\alpha+1}. \tag{1.2.23}$$

Indeed $o_j \to 0$ as $j \to +\infty$ and

$$(|p_j|^\alpha \vee |r_j|^\alpha)O\left(j|x_j - y_j|^q\right) \le C(j|x_j - y_j|^{q-1})^{\alpha+1}|x_j - y_j| \le CR^{\alpha+1}|x_j - y_j| \to 0$$

as $j \to +\infty$. Moreover, up to subsequence p_j, $r_j \to p_0 \in \mathbb{R}^N$. If $p_0 \ne 0$ then

$$b(x_j)\cdot p_j|p_j|^\alpha,\ b(y_j)\cdot r_j|r_j|^\alpha \to b(\overline{z})\cdot p_0|p_0|^\alpha$$

and so the difference goes to 0, otherwise

$$|b(x_j)\cdot p_j||p_j|^\alpha \le |b(x_j)||p_j|^{\alpha+1} \to 0 \quad \text{as } j \to +\infty.$$

The same result holds for $b(y_j)\cdot r_j|r_j|^\alpha$.

If $\tau + c(\overline{z}) > 0$, from (1.2.18) and (1.2.23) we have

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} u(\overline{z})^{\alpha+1},$$

and taking γ sufficiently close to γ' in order that $\frac{\tau\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} - \lambda}{1 - \left(\frac{\gamma}{\gamma'}\right)^{\alpha+1}} > |c|_\infty$, we get a contradiction. Finally if $\tau + c(\overline{z}) \le 0$ we obtain

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))w(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))u(\overline{z})^{\alpha+1},$$

once more a contradiction since $\lambda < \tau$. □

Proof of Lemma 1.2.12. Without loss of generality we can assume that $\overline{x} = 0$.
Since the minimum is strict there exists a small $\delta > 0$ such that

$$v(0) < v(x) + C|x|^q e^{-kd(x)} \quad \text{for any } x \in \overline{\Omega},\ 0 < |x| \leq \delta.$$

Since v is not locally constant and $q > 1$, for any $n > \delta^{-1}$ there exists $(t_n, z_n) \in B(0, \frac{1}{n})^2 \cap \overline{\Omega}^2$ such that

$$v(t_n) > v(z_n) + C|z_n - t_n|^q e^{-kd(z_n)}.$$

Consequently, for $n > \delta^{-1}$ the minimum of the function $v(x) + C|x - t_n|^q e^{-kd(x)}$ in $\overline{B}(0, \delta) \cap \overline{\Omega}$ is not achieved on t_n. Indeed

$$\min_{|x| \leq \delta,\, x \in \overline{\Omega}} (v(x) + C|x - t_n|^q e^{-kd(x)}) \leq v(z_n) + C|z_n - t_n|^q e^{-kd(z_n)} < v(t_n).$$

Let $y_n \neq t_n$ be some point in $\overline{B}(0, \delta) \cap \overline{\Omega}$ on which the minimum is achieved. Passing to the limit as n goes to infinity, t_n goes to 0 and, up to subsequence, y_n converges to some $y \in \overline{B}(0, \delta) \cap \overline{\Omega}$. By the lower semicontinuity of v and the fact that 0 is a local minimum of $v(x) + C|x|^q e^{-kd(x)}$ we have

$$v(0) \leq v(y) + C|y|^q e^{-kd(y)} \leq \liminf_{n \to +\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}),$$

and using that $v(0) + C|t_n|^q e^{-kd(0)} \geq v(y_n) + C|y_n - t_n|^q e^{-kd(y_n)}$, one has

$$v(0) \geq \limsup_{n \to +\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}).$$

Then

$$v(0) = v(y) + C|y|^q e^{-kd(y)} = \lim_{n \to +\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}).$$

Since 0 is a strict local minimum of $v(x) + C|x|^q e^{-kd(x)}$, the last equalities imply that $y = 0$ and $v(y_n)$ goes to $v(0)$ as $n \to +\infty$. Then for large n, y_n is an interior point of $B(0, \delta)$ so that the function

$$\varphi(x) = v(y_n) + C|y_n - t_n|^q e^{-kd(y_n)} - C|x - t_n|^q e^{-kd(x)}$$

is a test function for v at y_n. Moreover, the gradient of φ

$$D\varphi(x) = -Cq|x - t_n|^{q-2} e^{-kd(x)}(x - t_n) + kC|x - t_n|^q e^{-kd(x)} Dd(x)$$

is different from 0 at $x = y_n$ for small δ, indeed

$$|D\varphi(y_n)| \geq C|y_n - t_n|^{q-1} e^{-kd(y_n)}(q - k|y_n - t_n|) \geq C|y_n - t_n|^{q-1} e^{-kd(y_n)}(q - 2k\delta) > 0.$$

Using (1.0.2), if $y_n \in \partial\Omega$ we have

$$\langle D\varphi(y_n), \overrightarrow{n}(y_n) \rangle \leq C|y_n - t_n|^q \left(\frac{q}{2r} - k \right) < 0,$$

since $k > q/(2r)$. Then we conclude that

$$F(y_n, D\varphi(y_n), D^2\varphi(y_n)) + b(y_n)\cdot D\varphi(y_n)|D\varphi(y_n)|^\alpha - \beta(v(y_n)) \le g(y_n).$$

This inequality together with the condition (F2) implies that

$$-|D\varphi(y_n)|^\alpha A\text{tr}(D^2\varphi(y_n))^- + b(y_n)\cdot D\varphi(y_n)|D\varphi(y_n)|^\alpha - \beta(v(y_n)) \le g(y_n). \quad (1.2.24)$$

Observe that $D^2\varphi(y_n) = |y_n - t_n|^{q-2}M$, where M is a matrix such that $\text{tr}M^+$ and $\text{tr}M^-$ are bounded by a constant independent of δ and n. Hence, from (1.2.24) we get

$$C_0|y_n - t_n|^{\alpha(q-1)+q-2} - \beta(v(y_n)) \le g(y_n),$$

for some constant C_0, where the exponent $\alpha(q-1)+q-2 = q(\alpha+1)-(\alpha+2) > 0$. Passing to the limit, since β and g are upper semicontinuous we get

$$-\beta(v(0)) \le g(0),$$

which is the desired conclusion. □

We conclude sketching the proof of Theorem 1.2.1.

Proof of Theorem 1.2.1. Suppose by contradiction that $\max_{\overline{\Omega}}(u-v) = m > 0$. Since $u \le v$ on the boundary, the supremum is achieved inside Ω. Let us define for $j \in \mathbb{N}$ and some $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$

$$\psi(x,y) = u(x) - v(y) - \frac{j}{q}|x-y|^q.$$

Suppose that (x_j, y_j) is a maximum point for ψ in $\overline{\Omega}^2$. Then $|x_j - y_j| \to 0$ as $j \to +\infty$ and up to subsequence $x_j, y_j \to \overline{x}$, $u(x_j) \to u(\overline{x})$, $v(y_j) \to v(\overline{x})$ and $j|x_j - y_j|^q \to 0$ as $j \to +\infty$. Moreover, $\overline{x}$ is such that $u(\overline{x}) - v(\overline{x}) = m$ and we can choose $x_j \ne y_j$. Recalling by Remark 1.1.3 that the estimate (1.1.4) holds in Ω, we can proceed as in the proof of Theorem 1.2.9 to get

$$-c(\overline{x})|u(\overline{x})|^\alpha u(\overline{x}) \le -c(\overline{x})|v(\overline{x})|^\alpha v(\overline{x}).$$

This is a contradiction since $c(\overline{x}) < 0$. □

1.2.3 The Maximum Principle for $c(x)$ changing sign: an example.

In the previous subsections we have proved that $G(x, u, Du, D^2u)$ with the Neumann boundary condition satisfies the maximum principle if $c(x) \le 0$ or without condition on the sign of $c(x)$ provided $\overline{\lambda} > 0$. In this subsection we want to prove that these two cases do not coincide, i.e., that there exists some $c(x)$ which changes sign in Ω such that the associated principal eigenvalue $\overline{\lambda}$ is positive. To prove this, by definition of $\overline{\lambda}$, it suffices to find a function $c(x)$ changing sign for which there exists a bounded positive supersolution of

$$\begin{cases} F(x, Dv, D^2v) + b(x)\cdot Dv|Dv|^\alpha + c(x)|v|^\alpha v = -m & \text{in} \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \quad (1.2.25)$$

where $m > 0$.

In the rest of this subsection we will construct an explicit example of such function. For simplicity, let us suppose that $b \equiv 0$ and Ω is the ball of center 0 and radius R. We will look for c such that:

$$\begin{cases} c(x) < 0 & \text{if } R-\epsilon < |x| \le R \\ c(x) \le -\beta_1 & \text{if } \rho < |x| \le R-\epsilon \\ c(x) \le \beta_2 & \text{if } |x| \le \rho, \end{cases} \tag{1.2.26}$$

where $0 < \rho < R - \epsilon$ and ϵ, β_1, β_2 are positive constants which satisfy a suitable inequality. Remark that in the ball of radius ρ, $c(x)$ may assume positive values.

In order to construct a supersolution, we define the function

$$v(x) := \begin{cases} D & \text{if } R-\epsilon < |x| \le R \\ E|x|^2 - E(R+\rho-\epsilon)|x| + D + E\rho(R-\epsilon) & \text{if } \rho < |x| \le R-\epsilon \\ D + 1 - e^{k(|x|-\rho)} & \text{if } |x| \le \rho, \end{cases} \tag{1.2.27}$$

where D, E, k are positive constants to be chosen later.

Lemma 1.2.15. *The function v defined in* (1.2.27) *has the following properties*

(i) v is continuous on $\overline{B}(0,R)$ and of class C^2 in the sets $B(0,\rho)\setminus\{0\}$, $B(0,R-\epsilon)\setminus\overline{B}(0,\rho)$, $\overline{B}(0,R)\setminus\overline{B}(0,R-\epsilon)$;

(ii) v is positive provided $D > \frac{E}{4}(R-\rho-\epsilon)^2$;

(iii) $J^{2,-}v(x) = \emptyset$ if $x = 0$, $|x| = R-\epsilon$ and if $|x| = \rho$ provided $E(R-\rho-\epsilon) > k$.

Proof. The proof of (i) is a very simple calculation.

For (ii) we observe that v is positive if $R-\epsilon \le |x| \le R$ and $|x| \le \rho$ since $D, k > 0$. In the region $\{\rho \le |x| \le R-\epsilon\}$ v is positive on the boundary where takes the value D, while in the interior $Dv(x) = 2Ex - E(R+\rho-\epsilon)\frac{x}{|x|} = 0$ if $|x| = \frac{R+\rho-\epsilon}{2}$. In such points $v(x) = -\frac{E}{4}(R-\rho-\epsilon)^2 + D$, then they are global minimums where v takes positive value if $D > \frac{E}{4}(R-\rho-\epsilon)^2$.

Now we turn to (iii). Let $\widehat{x} \in \Omega$ be such that $|\widehat{x}| = \rho$ and let $(p, X) \in J^{2,-}v(\widehat{x})$, then by definition of semi-jet

$$v(x) \ge v(\widehat{x}) + \langle p, x-\widehat{x}\rangle + \frac{1}{2}\langle X(x-\widehat{x}), x-\widehat{x}\rangle + o(|x-\widehat{x}|^2), \tag{1.2.28}$$

as $x \to \widehat{x}$. If we take $x = \widehat{x} + t\overrightarrow{n}(\widehat{x})$, for $t > 0$, where $\overrightarrow{n}(\widehat{x}) = \frac{\widehat{x}}{|\widehat{x}|}$ is the exterior normal to the sphere of radius ρ at $\widehat{x}$, then $|x| > \rho$ and dividing (1.2.28) by t we have

$$\frac{v(\widehat{x} + t\overrightarrow{n}(\widehat{x})) - v(\widehat{x})}{t} \ge p_n + O(t),$$

where $p_n = p \cdot \overrightarrow{n}(\widehat{x})$. Letting $t \to 0^+$ we get

$$p_n \le \langle 2E\widehat{x} - E(R+\rho-\epsilon)\frac{\widehat{x}}{|\widehat{x}|}, \frac{\widehat{x}}{|\widehat{x}|}\rangle = -E(R-\rho-\epsilon).$$

On the other hand, if we take $x = \widehat{x} - t\overrightarrow{n}(\widehat{x})$, $t > 0$, in (1.2.28) and divide by $-t$, letting $t \to 0^+$ we get

$$p_n \geq \langle -ke^{k(|\widehat{x}|-\rho)} \frac{\widehat{x}}{|\widehat{x}|}, \frac{\widehat{x}}{|\widehat{x}|} \rangle = -k.$$

In conclusion

$$E(R-\rho-\epsilon) \leq -p_n \leq k.$$

Assuming the hypothesis in (iii) the previous condition cannot never be satisfied, then $J^{2,-}v(\widehat{x}) = \emptyset$.

In the same way it can be proved that if $\widehat{x} \in \Omega$ is such that $|\widehat{x}| = R - \rho$ and $(p, X) \in J^{2,-}v(\widehat{x})$ then

$$E(R-\rho-\epsilon) \leq p_n \leq 0,$$

and clearly also this condition cannot be satisfied, consequently $J^{2,-}v(\widehat{x}) = \emptyset$.

Finally it is easy to see that $J^{2,-}v(0) = \emptyset$. □

Proposition 1.2.16. *There exist ϵ, β_1, $\beta_2 > 0$ such that for any $c(x)$ satisfying* (1.2.26) *the function v defined in* (1.2.27) *is a positive continuous viscosity solution of* (1.2.25).

Proof. Clearly v satisfies the boundary condition. Since the semi-jet $J^{2,-}v(x)$ is empty if $|x| = \rho$, $|x| = R - \epsilon$ and $x = 0$, in such points we have nothing to test. In $B(0, \rho) \setminus \{0\}$, $B(0, R-\epsilon) \setminus \overline{B}(0, \rho)$, $B(0, R) \setminus \overline{B}(0, R-\epsilon)$ v is of class C^2, then it suffices to prove that v is a classical supersolution of (1.2.25) in these open sets.

Case I: $R - \epsilon < |x| < R$.

Since $c < 0$ and continuous on $\{R - \epsilon \leq |x| \leq R\}$, we have

$$c(x)v^{\alpha+1} = c(x)D^{\alpha+1} \leq -m_1 < 0. \tag{1.2.29}$$

Hence, by definition v is supersolution.

Case II: $\rho < |x| < R - \epsilon$.

In this set

$$Dv(x) = E[2|x| - (R+\rho-\epsilon)]\frac{x}{|x|}, \quad D^2v(x) = 2EI - E(R+\rho-\epsilon)\frac{1}{|x|}\left(I - \frac{x \otimes x}{|x|^2}\right).$$

Since $-(R-\rho-\epsilon) \leq 2|x| - (R+\rho-\epsilon) \leq R-\rho-\epsilon$, using (F2) we compute

$$F(x, Dv, D^2v) \leq E^{\alpha+1}(R-\rho-\epsilon)^\alpha \left[\frac{2AN(R-\epsilon) - a(R+\rho-\epsilon)N + a(R+\rho-\epsilon)}{R-\epsilon}\right]$$
$$= E^{\alpha+1}\frac{(R-\rho-\epsilon)^\alpha}{R-\epsilon}\{N[(A-a)(R-\epsilon) + A(R-\epsilon) - a\rho] + a(R+\rho-\epsilon)\}.$$

Observe that all the factors in the last member are positive. Using the last computation, the fact that in the minimum points v takes the value $D - \frac{E}{4}(R-\rho-\epsilon)^2$ (see the proof of Lemma 1.2.15) and that $c \leq -\beta_1$, we have

$$F(x, Dv, D^2v) + c(x)v^{\alpha+1} \leq E^{\alpha+1}\frac{(R-\rho-\epsilon)^\alpha}{R-\epsilon}\{N[(A-a)(R-\epsilon) + A(R-\epsilon) - a\rho]$$
$$+ a(R+\rho-\epsilon)\} - \beta_1\left[D - \frac{E}{4}(R-\rho-\epsilon)^2\right]^{\alpha+1} =: -m_2. \tag{1.2.30}$$

The above quantity is negative if

$$D > \frac{E}{4}(R-\rho-\epsilon)^2 + EC, \tag{1.2.31}$$

where

$$C := \frac{(R-\rho-\epsilon)^{\frac{\alpha}{\alpha+1}}}{\beta_1^{\frac{1}{\alpha+1}}(R-\epsilon)^{\frac{1}{\alpha+1}}}\{N[(A-a)(R-\epsilon)+A(R-\epsilon)-a\rho]+a(R+\rho-\epsilon)\}^{\frac{1}{\alpha+1}} > 0.$$

Case III: $0 < |x| < \rho$.

Here we have

$$Dv(x) = -ke^{k(|x|-\rho)}\frac{x}{|x|}, \quad D^2v(x) = -k^2e^{k(|x|-\rho)}\frac{x\otimes x}{|x|^2} - ke^{k(|x|-\rho)}\frac{1}{|x|}\left(I - \frac{x\otimes x}{|x|^2}\right).$$

Then

$$F(x, Dv, D^2v) + c(x)v^{\alpha+1} \leq -k^{\alpha+1}e^{(\alpha+1)k(|x|-\rho)}a\left(k + \frac{N-1}{|x|}\right) + \beta_2(D+1$$
$$- e^{k(|x|-\rho)})^{\alpha+1} \leq -k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k+\frac{N-1}{\rho}\right) + \beta_2(D+1-e^{-k\rho})^{\alpha+1} =: -m3. \tag{1.2.32}$$

The last quantity is negative if

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k+\frac{N-1}{\rho}\right)}{(D+1-e^{-k\rho})^{\alpha+1}}. \tag{1.2.33}$$

Since E must satisfy the condition in (iii) of Lemma 1.2.15, we choose

$$E := \frac{k}{R-\rho-\epsilon'}, \tag{1.2.34}$$

for $\epsilon < \epsilon' < R - \rho$. Furthermore we take

$$D := \frac{E}{4}(R-\rho-\epsilon)^2 + EC + \epsilon = \frac{k(R-\rho-\epsilon)^2}{4(R-\rho-\epsilon')} + \frac{kC}{R-\rho-\epsilon'} + \epsilon. \tag{1.2.35}$$

With this choice of D, (1.2.31) is satisfied and v is positive by (ii) of Lemma 1.2.15. Observe that

$$D \to k\left\{\frac{R-\rho}{4} + \left[\frac{2NAR-(N-1)a(R+\rho)}{\beta_1 R(R-\rho)}\right]^{\frac{1}{\alpha+1}}\right\}$$

as $\epsilon, \epsilon' \to 0^+$.

Finally we can write the relation between β_1 and β_2:

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k+\frac{N-1}{\rho}\right)}{\left(k\left\{\frac{R-\rho}{4}+\left[\frac{2NAR-(N-1)a(R+\rho)}{\beta_1R(R-\rho)}\right]^{\frac{1}{\alpha+1}}\right\}+1-e^{-k\rho}\right)^{\alpha+1}}. \tag{1.2.36}$$

Suppose that (1.2.36) holds for some $k > 0$, then we can choose $\epsilon' > \epsilon > 0$ so small that

$$\beta_2 < \frac{k^{\alpha+1} e^{-(\alpha+1)k\rho} a \left(k + \frac{N-1}{\rho}\right)}{(D + 1 - e^{-k\rho})^{\alpha+1}},$$

where D is defined by (1.2.35). Define E as in (1.2.34), then v is a positive supersolution of (1.2.25) with m the minimum between the quantity m_1, m_2 and m_3 defined respectively in (1.2.29), (1.2.30) and (1.2.32). Observe that the size of ϵ is given by (1.2.36). □

Remark 1.2.17. If we call $UB(\beta_2)$ the upper bound of β_2 in (1.2.36), we can see that if we choose $k = \frac{1}{\rho}$ then $UB(\beta_2)$ goes to $+\infty$ as $\rho \to 0^+$, that is, if the set where c is positive becomes smaller then the values of c in this set can be very large. On the contrary, for any value of k, if $\rho \to R^-$ then $UB(\beta_2)$ goes to 0. Finally, for any k, if $\beta_1 \to 0^+$ then again $UB(\beta_2)$ goes to 0. So there is a sort of balance between β_1 and β_2. This behavior can be explained by the following example: consider the equation $\Delta v + c(x)v = 0$ which is a subcase of our equation and suppose that $v > 0$ in $\overline{\Omega}$ is a classical solution of $\Delta v + c(x)v \leq 0$ in Ω, $\frac{\partial v}{\partial \overrightarrow{n}} \geq 0$ on $\partial\Omega$. Then dividing by v and integrating by part we get

$$\int_\Omega c(x)dx \leq -\int_\Omega \frac{|Dv|^2}{v^2}dx - \int_{\partial\Omega} \frac{1}{v}\frac{\partial v}{\partial \overrightarrow{n}}dS \leq 0, \tag{1.2.37}$$

the first inequality being strict if $\Delta v + c(x)v \not\equiv 0$. If the supersolution is C^2 piecewise with $J^{2,-}v = \emptyset$ in the non-regular points, as the one constructed before, then we can repeat this computation in any set where v is C^2 getting again

$$\int_\Omega c(x)dx < 0.$$

Remark 1.2.18. The construction above can be repeated for any C^2 domain. The assumptions on c and the supersolution v can be rewritten respectively as follows

$$\begin{cases} c(x) < 0 & \text{if } d(x) < \epsilon \\ c(x) \leq -\beta_1 & \text{if } \epsilon \leq d(x) < \delta \\ c(x) \leq \beta_2 & \text{if } d(x) \geq \delta, \end{cases}$$

$$v(x) := \begin{cases} D & \text{if } d(x) < \epsilon \\ E(\delta + \epsilon - d(x))^2 + E(\delta + \epsilon)(d(x) - \epsilon - \delta) + D + E\epsilon\delta & \text{if } \epsilon \leq d(x) < \delta \\ D + 1 - e^{k(\delta - d(x))} & \text{if } d(x) \geq \delta, \end{cases}$$

where $0 < \epsilon < \delta$ and $d(x)$ is precisely the distance function, not one of its C^2 extensions. We recall some properties of the distance function:

- There exists $\mu > 0$ such that d is of class C^2 in $\Omega_\mu := \{x \in \overline{\Omega} | \, d(x) < \mu\}$ and the eigenvalues of the hessian matrix of d at x are 0 and $\frac{k_i}{1+d(x)k_i}$, $i = 1, ..., N-1$, where k_i are the principal curvatures of $\partial\Omega$ corresponding to the directions orthogonal to $\overrightarrow{n}$ at the point $y = x - d(x)Dd(x)$;

- d is semi-concave in Ω, i.e., there exists $s_0 > 0$ such that $d(x) - \frac{s_0}{2}|x|^2$ is concave;
- If $J^{2,-}d(x) \neq \emptyset$, d is differentiable at x and $|Dd(x)| = 1$.

We choose δ so small that in $\Omega_{\delta+\delta'}$ d is of class C^2 for some small $\delta' > 0$. Then, as in previous example, where δ was $R-\rho$, it can be proved that v is continuous on $\overline{\Omega}$, positive if $D > \frac{E}{4}(\delta-\epsilon)^2$ and of class C^2 on Ω_ϵ, $\Omega_\delta \setminus \overline{\Omega}_\epsilon$. Furthermore, $J^{2,-}v(x) = \emptyset$ if $d(x) = \epsilon$ and if $d(x) = \delta$ provided $E(\delta - \epsilon) > k$.

Let K be such that $|k_i(x)| \leq K$ for all i and all $x \in \partial\Omega$. Then, if $\epsilon < d(x) < \delta$ we have the following estimate

$$\begin{aligned} F(x, Dv, D^2v) + c(x)v^{\alpha+1} \leq E^{\alpha+1}(\delta-\epsilon)^\alpha \Big\{ & 2A + [A\delta + (A-2a)\epsilon](N-1)K \\ & + [(2A-a)\delta - a\epsilon]\frac{(N-1)K}{1-\delta K}\Big\} - \beta_1 \left[D - \frac{E}{4}(\delta-\epsilon)^2\right]^{\alpha+1}. \end{aligned}$$

Now suppose $d(x) > \delta$, then $v(x) = D + 1 - e^{k(\delta - d(x))}$. Let $\overline{x} \in \Omega$ be such that $d(\overline{x}) > \delta$ and let ψ be a C^2 function such that $(v-\psi)(x) \geq (v-\psi)(\overline{x}) = 0$ for all x in a small neighborhood of $\overline{x}$. Then the function ϕ defined as

$$\phi(x) := -\frac{1}{k}\log(D + 1 - \psi(x)) + \delta$$

is a C^2 function in a neighborhood of $\overline{x}$, such that $(d-\phi)(x) \geq (d-\phi)(\overline{x}) = 0$. This implies that $J^{2,-}d(\overline{x}) \neq \emptyset$. According to some of the properties of d recalled before, on such point d is differentiable, $D\phi(\overline{x}) = Dd(\overline{x})$ and $D^2\phi(\overline{x}) \leq s_0 I$. Then it easy to check that for $k > \frac{s_0 AN}{a}$

$$\begin{aligned} F(\overline{x}, D\psi(\overline{x}), D^2\psi(\overline{x})) + c(\overline{x})v^{\alpha+1} \leq\ & k^{\alpha+1}e^{-(\alpha+1)k(R-\delta)}(s_0AN - ka) \\ & + \beta_2(D + 1 - e^{-k(R-\delta)})^{\alpha+1}, \end{aligned}$$

where $R := \max_{\overline{\Omega}} d(x)$.

We can repeat the argument used before to conclude that v is a positive supersolution of (1.2.25) if ϵ is small enough and β_1 and β_2 satisfy the following inequality for some $k > \frac{s_0 AN}{a}$

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k(R-\delta)}(ka - s_0AN)}{\left\{k\left[\frac{\delta}{4} + \left[\frac{2A+(N-1)\delta K[A+(2A-a)(1-\delta K)^{-1}]}{\beta_1\delta}\right]^{\frac{1}{\alpha+1}}\right] + 1 - e^{-k(R-\delta)}\right\}^{\alpha+1}}.$$

Of course the relation between β_1 and β_2 can be bettered if we have more informations about the domain Ω.

1.3 Some existence results

This section is devoted to the problem of the existence of a solution of

$$\begin{cases} F(x, Du, D^2u) + b(x)\cdot Du|Du|^\alpha + (c(x)+\lambda)|u|^\alpha u = g(x) & \text{in } \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3.38)$$

The first existence result for (1.3.38) is obtained when $\lambda = 0$ and $c < 0$, via Perron's method. Using it, we will prove the existence of a positive solution of (1.3.38) when g is non-positive and $\lambda < \overline{\lambda}$, without condition on the sign of c. Then we will show the existence of a positive principal eigenfunction corresponding to $\overline{\lambda}$, that is a solution of (1.3.38) when $g \equiv 0$ and $\lambda = \overline{\lambda}$. For the last two results we will follow the proof given in [28] for the analogous theorems with the Dirichlet boundary condition.

Symmetrical results can be obtained for the eigenvalue $\underline{\lambda}$.

Finally, we will prove that the Neumann problem (1.3.38) is solvable for any right-hand side if $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$.

Comparison results guarantee for (1.3.38) the uniqueness of the solution when $c < 0$, of the positive solution when $\lambda < \overline{\lambda}$ and $g < 0$ and of the negative solution when $\lambda < \underline{\lambda}$ and $g > 0$.

Theorem 1.3.1. *Suppose that $c < 0$ and g is continuous on $\overline{\Omega}$. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are respectively viscosity sub and supersolution of*

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) & \text{in } \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.39}$$

with u and v bounded or $v \geq 0$ and bounded, then $u \leq v$ in $\overline{\Omega}$. Moreover (1.3.39) *has a unique viscosity solution.*

Proof. We suppose by contradiction that $\max_{\overline{\Omega}}(u - v) = m > 0$. Repeating the proof of Theorem 1.2.9 taking v as w, we arrive to the following inequality

$$-c(\overline{z})|u(\overline{z})|^\alpha u(\overline{z}) \leq -c(\overline{z})|v(\overline{z})|^\alpha v(\overline{z}),$$

where $\overline{z} \in \overline{\Omega}$ is such that $u(\overline{z}) - v(\overline{z}) = m > 0$. This is a contradiction since $c(\overline{z}) < 0$.

The existence of a solution follows from Perron's method of Ishii [73] and the comparison result just proved, provided there is a bounded subsolution and a bounded supersolution of (1.3.39). Since c is negative and continuous on $\overline{\Omega}$, there exists $c_0 > 0$ such that $c(x) \leq -c_0$ for every $x \in \overline{\Omega}$. Then

$$u_1 := -\left(\frac{|g|_\infty}{c_0}\right)^{\frac{1}{\alpha+1}}, \quad u_2 := \left(\frac{|g|_\infty}{c_0}\right)^{\frac{1}{\alpha+1}}$$

are respectively a bounded sub and supersolution of (1.3.39).

Put

$$u(x) := \sup\{\varphi(x) |\, u_1 \leq \varphi \leq u_2 \text{ and } \varphi \text{ is a subsolution of (1.3.39) }\},$$

then u is a solution of (1.3.39). We first show that the upper semicontinuous envelope of u defined as

$$u^*(x) := \lim_{\rho\downarrow 0} \sup\{u(y) : y \in \overline{\Omega} \text{ and } |y - x| \leq \rho\}$$

is a subsolution of (1.3.39). Indeed if $(p, X) \in J^{2,+}u(x_0)$ and $p \neq 0$ then by the standard arguments of the Perron's method it can be proved that $G(x_0, u(x_0), p, X) \geq g(x_0)$ if $x_0 \in \Omega$ and $(-G(x_0, u(x_0), p, X) + g(x_0)) \wedge \langle p, \overrightarrow{n}(x_0)\rangle \leq 0$ if $x_0 \in \partial\Omega$.

Now suppose $u^* \equiv k$ in a neighborhood of $x_0 \in \overline{\Omega}$. If $x_0 \in \partial\Omega$ clearly u^* is subsolution in x_0. Assume that x_0 is an interior point of Ω. We may choose a sequence of subsolutions $(\varphi_n)_n$ and a sequence of points $(x_n)_n$ in Ω such that $x_n \to x_0$ and $\varphi_n(x_n) \to k$. Suppose that $|x_n - x_0| < a_n$ with a_n decreasing to 0 as $n \to +\infty$. If, up to subsequence, φ_n is constant in $B(x_0, a_n)$ for any n, then passing to the limit in the relation $c(x_n)|\varphi_n(x_n)|^\alpha \varphi_n(x_n) \geq g(x_n)$ we get $c(x_0)|k|^\alpha k \geq g(x_0)$ as desired. Otherwise, suppose that for any n φ_n is not constant in $B(x_0, a_n)$. Repeating the argument of Lemma 1.2.12 we find a sequence $\{(t_n, y_n)\}_{n\in\mathbb{N}} \subset \Omega^2$ and a small $\delta > 0$ such that $|t_n - x_0| < a_n$, $|y_n - x_0| \leq \delta$, $t_n \neq y_n$, $\varphi_n(x) - |x - t_n|^q \leq \varphi_n(y_n) - |y_n - t_n|^q$ for any $x \in B(x_0, \delta)$, with $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$ and $u^* \equiv k$ in $\overline{B}(x_0, \delta)$. Up to subsequence $y_n \to y \in \overline{B}(x_0, \delta)$ as $n \to +\infty$. We have

$$\begin{aligned} k &= \lim_{n\to+\infty}(\varphi_n(x_n) - |x_n - t_n|^q) \leq \liminf_{n\to+\infty}(\varphi_n(y_n) - |y_n - t_n|^q) \\ &\leq \limsup_{n\to+\infty}(\varphi_n(y_n) - |y_n - t_n|^q) \leq k - |y - x_0|^q. \end{aligned}$$

The last inequalities imply that $y = x_0$ and $\varphi_n(y_n) \to k$. Then for large n, y_n is an interior point of $B(x_0, \delta)$ and $\phi_n(x) := \varphi_n(y_n) - |y_n - t_n|^q + |x - t_n|^q$ is a test function for φ_n at y_n. Passing to the limit as $n \to +\infty$ in the relation $G(y_n, \varphi_n(y_n), D\phi_n(y_n), D^2\phi_n(y_n)) \geq g(y_n)$, we get again $c(x_0)|k|^\alpha k \geq g(x_0)$. In conclusion u^* is a subsolution of (1.3.39). Since $u_1 \leq u^* \leq u_2$, it follows from the definition of u that $u = u^*$.

Finally the lower semicontinuous envelope of u defined as

$$u_*(x) := \lim_{\rho\downarrow 0} \inf\{u(y) : y \in \overline{\Omega} \text{ and } |y - x| \leq \rho\}$$

is a supersolution. Indeed, if it is not, the Perron's method provides a viscosity subsolution of (1.3.39) greater than u, contradicting the definition of u. If $u_* \equiv k$ in a neighborhood of $x_0 \in \Omega$ and $c(x_0)|k|^\alpha k > g(x_0)$ then for small δ and ρ, the subsolution is

$$u_{\delta,\rho}(x) := \begin{cases} \max\{u(x), k + \frac{\delta\rho^2}{8} - \delta|x - x_0|^2\} & \text{if } |x - x_0| < \rho, \\ u(x) & \text{otherwise.} \end{cases}$$

Hence u_* is a supersolution of (1.3.39) and then, by comparison, $u^* = u \leq u_*$, showing that u is continuous and is a solution.

The uniqueness of the solution is an immediate consequence of the comparison principle just proved. □

Theorem 1.3.2. *Suppose $g \in LSC(\overline{\Omega})$, $h \in USC(\overline{\Omega})$, $h \leq 0$, $h \leq g$ and $g(x) > 0$ if $h(x) = 0$. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of* (1.3.38) *and $v \in LSC(\overline{\Omega})$ be a bounded positive viscosity supersolution of* (1.3.38) *with g replaced by h. Then $u \leq v$ in $\overline{\Omega}$.*

Remark 1.3.3. The existence of a such v implies $\lambda \leq \overline{\lambda}$.

Proof. It suffices to prove the theorem for $h < g$. Indeed, for $l > 1$ the function defined by $v_l := lv$ is a supersolution of (1.3.38) with right-hand side $l^{\alpha+1}h(x)$. By the assumptions on h and g, $l^{\alpha+1}h < g$. If $u \le lv$ for any $l > 1$, passing to the limit as $l \to 1^+$, one obtains $u \le v$ as desired.

Hence we can assume $h < g$. By upper semicontinuity $\max_{\overline{\Omega}}(h-g) = -M < 0$. Suppose by contradiction that $u > v$ somewhere in Ω. Then there exists $\overline{y} \in \overline{\Omega}$ such that

$$\gamma' := \frac{u(\overline{y})}{v(\overline{y})} = \max_{x\in\overline{\Omega}} \frac{u(x)}{v(x)} > 1.$$

Define $w = \gamma v$ for some $1 \le \gamma < \gamma'$. Since $h \le 0$ and $\gamma \ge 1$, $\gamma^{\alpha+1}h \le h$ and then w is still a supersolution of (1.3.38) with right-hand side h. The supremum of $u - w$ is strictly positive then, by upper semicontinuity, there exists $\overline{x} \in \overline{\Omega}$ such that $u(\overline{x}) - w(\overline{x}) = \max_{\overline{\Omega}}(u-w) > 0$. We have $u(\overline{x}) > w(\overline{x})$ and $w(\overline{x}) \ge \frac{\gamma}{\gamma'}u(\overline{x})$. Repeating the proof of Theorem 1.2.9, we get

$$g(\overline{z}) - (\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le h(\overline{z}) - (\lambda + c(\overline{z}))w(\overline{z})^{\alpha+1},$$

where $\overline{z}$ is some point in $\overline{\Omega}$ where the maximum of $u-w$ is attained. If $\lambda + c(\overline{z}) \le 0$, then

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le h(\overline{z}) - g(\overline{z}) - (\lambda + c(\overline{z}))w(\overline{z})^{\alpha+1} < -(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1},$$

which is a contradiction. If $\lambda + c(\overline{z}) > 0$, then

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le h(\overline{z}) - g(\overline{z}) - (\lambda + c(\overline{z}))\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} u(\overline{z})^{\alpha+1}.$$

If we choose γ sufficiently close to γ' in order that

$$|\lambda + c|_\infty \left[\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} - 1\right] (\max_{\overline{\Omega}} u)^{\alpha+1} \ge -\frac{M}{2},$$

we get once more a contradiction. □

Theorem 1.3.4. *Suppose that $\lambda < \overline{\lambda}$, $g \le 0$, $g \not\equiv 0$ and g is continuous on $\overline{\Omega}$, then there exists a positive viscosity solution of* (1.3.38). *If $g < 0$, the positive solution is unique.*

Proof. If $\lambda < -|c|_\infty$ then the existence of the solution is guaranteed by Theorem 1.3.1. Let us suppose $\lambda \ge -|c|_\infty$ and define by induction the sequence $(u_n)_n$ by $u_1 = 0$ and u_{n+1} as the solution of

$$\begin{cases} F(x, Du_{n+1}, D^2u_{n+1}) + b(x)\cdot Du_{n+1}|Du_{n+1}|^\alpha & \\ \quad +(c(x) - |c|_\infty - 1)|u_{n+1}|^\alpha u_{n+1} = g - (\lambda + |c|_\infty + 1)|u_n|^\alpha u_n & \text{in } \Omega \\ \langle Du_{n+1}, \overrightarrow{n}(x)\rangle = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists by Theorem 1.3.1. By the comparison principle, since $g \le 0$ and $g \not\equiv 0$ the sequence is positive and increasing. We use the argument of Theorem 7 of [28] to prove that $(u_n)_n$ is also bounded. Suppose that it is not, then dividing by $|u_{n+1}|_\infty^{\alpha+1}$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_{n+1} is a solution of

$$\begin{cases} F(x, Dv_{n+1}, D^2v_{n+1}) + b(x)\cdot Dv_{n+1}|Dv_{n+1}|^\alpha & \\ \quad +(c(x) - |c|_\infty - 1)v_{n+1}^{\alpha+1} = \frac{g}{|u_{n+1}|_\infty^{\alpha+1}} - (\lambda + |c|_\infty + 1)\frac{u_n^{\alpha+1}}{|u_{n+1}|_\infty^{\alpha+1}} & \text{in } \Omega \\ \langle Dv_{n+1}, \overrightarrow{n}(x)\rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

By Corollary 1.1.4, $(v_n)_n$ converges to a positive function v with $|v|_\infty = 1$ which satisfies

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha + (c(x) + \lambda)v^{\alpha+1} & \\ \quad = (\lambda + |c|_\infty + 1)(1-k)v^{\alpha+1} \geq 0 & \text{in } \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on } \partial\Omega, \end{cases}$$

where $k := \lim_{n\to+\infty} \frac{|u_n|_\infty^{\alpha+1}}{|u_{n+1}|_\infty^{\alpha+1}} \leq 1$. This contradicts the maximum principle, Theorem 1.2.9.

Then $(u_n)_n$ is bounded and letting n go to infinity, by the compactness result, the sequence converges to a function u which is a solution. Moreover, the solution is positive in $\overline{\Omega}$ by the strong minimum principle, Proposition 1.2.2.

If $g < 0$, the uniqueness of the positive solution follows from Theorem 1.3.2.

□

Theorem 1.3.5 (Existence of principal eigenfunctions)**.** *There exists $\phi > 0$ in $\overline{\Omega}$ viscosity solution of*

$$\begin{cases} F(x, D\phi, D^2\phi) + b(x) \cdot D\phi|D\phi|^\alpha + (c(x) + \overline{\lambda})\phi^{\alpha+1} = 0 & \textit{in } \Omega \\ \langle D\phi, \overrightarrow{n}(x)\rangle = 0 & \textit{on } \partial\Omega. \end{cases}$$

Moreover ϕ is Lipschitz continuous on $\overline{\Omega}$.

Proof. Let λ_n be an increasing sequence which converges to $\overline{\lambda}$. Let u_n be the positive solution of (1.3.38) with $\lambda = \lambda_n$ and $g \equiv -1$. By Theorem 1.3.4 the sequence $(u_n)_n$ is well defined. Following the argument of the proof of Theorem 8 of [28], we can prove that it is unbounded, otherwise one would contradict the definition of $\overline{\lambda}$. Then, up to subsequence $|u_n|_\infty \to +\infty$ as $n \to +\infty$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_n satisfies (1.3.38) with $\lambda = \lambda_n$ and $g \equiv -\frac{1}{|u_n|_\infty^{\alpha+1}}$. By Corollary 1.1.4, we can extract a subsequence converging to a positive function ϕ with $|\phi|_\infty = 1$ which is the desired solution. By Theorem 1.1.1 the solution is also Lipschitz continuous on $\overline{\Omega}$. □

Remark 1.3.6. With the same arguments used in the proofs of Theorems 1.3.2, 1.3.4 and 1.3.5 one can prove: the comparison result between $u \in USC(\overline{\Omega})$ bounded and negative viscosity subsolution of (1.3.38) and $v \in LSC(\overline{\Omega})$ supersolution of (1.3.38) with g replaced by h, provided $g \geq 0$, $h \leq g$ and $h(x) < 0$ if $g(x) = 0$; the existence of a negative viscosity solution of (1.3.38), for $\lambda < \underline{\lambda}$ and $g \geq 0$, $g \not\equiv 0$; the existence of a negative Lipschitz first eigenfunction corresponding to $\underline{\lambda}$, i.e., a solution of (1.3.38) with $\lambda = \underline{\lambda}$ and $g \equiv 0$.

Theorem 1.3.7. *Suppose that $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$ and g is continuous on $\overline{\Omega}$, then there exists a viscosity solution of* (1.3.38).

Proof. If $g \equiv 0$, by the maximum and minimum principles the only solution is $u \equiv 0$. Let us suppose $g \not\equiv 0$. Since $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$ by Theorem 1.3.4 and Remark 1.3.6 there exist v_0 positive viscosity solution of (1.3.38) with right-hand side $-|g|_\infty$ and u_0 negative viscosity solution of (1.3.38) with right-hand side $|g|_\infty$.

Let us suppose $\lambda + |c|_\infty \geq 0$. Let $(u_n)_n$ be the sequence defined in the proof of Theorem 1.3.4 with $u_1 = u_0$, then by comparison Theorem 1.3.1 we have $u_0 = u_1 \leq u_2 \leq ... \leq v_0$. Hence, by the compactness Corollary 1.1.4 the sequence converges to a continuous function which is the desired solution. $\square$

Chapter 2

Isaacs operators

In this chapter we want to develop an eigenvalue theory for a class of fully nonlinear operators with Neumann boundary conditions in a bounded C^2 domain Ω. Precisely, we consider a uniformly elliptic operator which is positively homogenous of order 1

$$F[u](x) = F(x, u, Du, D^2u), \tag{2.0.1}$$

for any $u \in C^2(\overline{\Omega})$, with some additional assumptions that will be made precise in the next section. This class includes the non-convex Isaacs operator I.2.10.

To (2.0.1) we associate the following boundary condition

$$B(x, u, Du) = f(x, u) + \frac{\partial u}{\partial \overrightarrow{n}} = 0 \quad x \in \partial\Omega, \tag{2.0.2}$$

where $\overrightarrow{n}(x)$ is the exterior normal to the domain Ω at x.

Following the ideas of [26], we define the principal eigenvalues as

$$\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of}$$
$$F(x, v, Dv, D^2v) = \lambda v \text{ in } \Omega,\ B(x, v, Dv) = 0 \text{ on } \partial\Omega\},$$

$$\underline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ on } \overline{\Omega} \text{ bounded viscosity subsolution of}$$
$$F(x, u, Du, D^2u) = \lambda u \text{ in } \Omega,\ B(x, u, Du) = 0 \text{ on } \partial\Omega\}.$$

We will prove that $\overline{\lambda}$ and $\underline{\lambda}$ are "eigenvalues" for F which admit respectively a positive and a negative "eigenfunction". Moreover, we show that $\overline{\lambda}$ (resp., $\underline{\lambda}$) can be characterized as the supremum of those λ for which the operator $F - \lambda I$ with boundary condition (2.0.2) satisfies the maximum (resp., minimum) principle. As a consequence, $\overline{\lambda}$ (resp., $\underline{\lambda}$) is the least "eigenvalue" to which there correspond "eigenfunctions" positive (resp., negative) somewhere.

Other properties of the principal eigenvalues are established: we show that they are simple, isolated and the only "eigenvalues" to which there correspond "eigenfunctions" which do not change sign in Ω. Finally, we obtain Lipschitz regularity, uniqueness and existence results for viscosity solutions of

$$\begin{cases} F(x, u, Du, D^2u) = g(x) & \text{in} \quad \Omega \\ B(x, u, Du) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.0.3}$$

In particular, we prove that (2.0.3) is solvable for any continuous right-hand side if the two principal eigenvalues are positive.

In the next section we give assumptions. In Section 2.2 we prove the strong comparison principle between sub and supersolutions of (2.0.3). This allows us to prove the maximum principle for subsolutions of the Neumann boundary value problem. We first show it under the classical assumption that F be proper, see Theorem 2.3.1; then we prove in Theorem 2.3.5 that the operator $F - \lambda I$ with boundary condition (2.0.2) satisfies the maximum principle for any $\lambda < \overline{\lambda}$. Using the example given in [97] we show that the result of Theorem 2.3.5 is stronger than that of Theorem 2.3.1, i.e., that there exist non-proper operators which have positive principal eigenvalue $\overline{\lambda}$, and then for which the maximum principle holds.

In Section 2.4 we establish a Lipschitz regularity result for viscosity solutions of (2.0.3). In Section 2.5 we show some existence and comparison theorems. In Section 2.6 we establish some of the basic properties of the principal eigenvalues. Finally, in Section 2.7 we show, through an example, that $\overline{\lambda}$ and $\underline{\lambda}$ may be different.

2.1 Assumptions

We recall that $\mathcal{M}^+_{a,A}$, $\mathcal{M}^-_{a,A}$: $S(N) \to \mathbb{R}$ are the Pucci's extremal operators defined by

$$\mathcal{M}^+_{a,A}(X) = A \sum_{e_i>0} e_i + a \sum_{e_i<0} e_i,$$

$$\mathcal{M}^-_{a,A}(X) = a \sum_{e_i>0} e_i + A \sum_{e_i<0} e_i,$$

where $e_1, ..., e_N$ are the eigenvalues of X (see e.g. [32]).

The operator F is supposed to be continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N)$, moreover we shall make the following assumptions:

(F1) For all $(x, r, p, X) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N)$ and $t \geq 0$

$$F(x, tr, tp, tX) = tF(x, r, p, X).$$

(F2) There exist $b, c > 0$ such that for $x \in \overline{\Omega}$, $r, s \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $X, Y \in S(N)$

$$\begin{aligned}\mathcal{M}^-_{a,A}(Y - X) - b|p - q| - c|r - s| &\leq F(x, r, p, X) - F(x, s, q, Y) \\ &\leq \mathcal{M}^+_{a,A}(Y - X) + b|p - q| + c|r - s|.\end{aligned}$$

(F3) For each $T > 0$ there exists a continuous function ω_T with $\omega_T(0) = 0$, such that if $X, Y \in S(N)$ and $\zeta > 0$ satisfy

$$-3\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $x, y \in \overline{\Omega}$, $r \in [-T, T]$, $p \in \mathbb{R}^N$

$$F(y, r, p, Y) - F(x, r, p, X) \leq \omega_T(\zeta|x - y|^2 + |x - y|(|p| + 1)).$$

(F4) There exists $C_1 > 0$ such that for all $x, y \in \overline{\Omega}$ and $X \in S(N)$

$$|F(x,0,0,X) - F(y,0,0,X)| \leq C_1|x-y|^{\frac{1}{2}}\|X\|.$$

Remark that (F1) implies that $F(x,0,0,0) \equiv 0$.

The Isaacs operator (I.2.10) is continuous and satisfies (F1) and (F2) if $aI \leq A_{\alpha,\beta}(x) \leq AI$ for any $x \in \overline{\Omega}$, $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$ and the functions $A_{\alpha,\beta}$, $b_{\alpha,\beta}$, $c_{\alpha,\beta}$ are continuous on $\overline{\Omega}$ uniformly in α and β, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary index sets. If the matrices $A_{\alpha,\beta}$ are equi-Hölderian of exponent $\frac{1}{2}$, i.e., for some constant $C > 0$

$$\|A_{\alpha,\beta}(x) - A_{\alpha,\beta}(y)\| \leq C|x-y|^{\frac{1}{2}} \quad \text{for all } x, y \in \overline{\Omega} \text{ and } (\alpha,\beta) \in \mathcal{A} \times \mathcal{B},$$

then F satisfies (F4). Finally, (F3) is satisfied by F if, in addition to the uniform elliptic condition $A_{\alpha,\beta}(x) \geq aI$ and the equi-continuity of $c_{\alpha,\beta}$, the functions $A_{\alpha,\beta}$ and $b_{\alpha,\beta}$ are equi-Lipschitz continuous, i.e., there exists $L > 0$ such that for all $x, y \in \overline{\Omega}$ and $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$

$$\|A_{\alpha,\beta}(x) - A_{\alpha,\beta}(y)\| \leq L|x-y|, \quad |b_{\alpha,\beta}(x) - b_{\alpha,\beta}(y)| \leq L|x-y|.$$

We assume throughout the paper that Ω is a bounded domain of $\mathbb{R}^N$ of class C^2. In particular it satisfies the interior sphere condition and the uniform exterior sphere condition, i.e.,

(Ω1) For each $x \in \partial\Omega$ there exist $R > 0$ and $y \in \Omega$ for which $|x-y| = R$ and $B(y,R) \subset \Omega$.

(Ω2) There exists $r > 0$ such that $B(x + r\overrightarrow{n}(x), r) \cap \Omega = \emptyset$ for any $x \in \partial\Omega$.

From property (Ω2) it follows that

$$\langle \overrightarrow{n}(x), y - x \rangle \leq \frac{1}{2r}|y-x|^2 \quad \text{for } x \in \partial\Omega \text{ and } y \in \overline{\Omega}. \tag{2.1.4}$$

Moreover, the C^2-regularity of Ω implies the existence of a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ on which the distance from the boundary

$$d(x) := \inf\{|x-y|, y \in \partial\Omega\}, \quad x \in \overline{\Omega}$$

is of class C^2. We still denote by d a C^2 extension of the distance function to the whole $\overline{\Omega}$. Without loss of generality we can assume that $|Dd(x)| \leq 1$ on $\overline{\Omega}$.

On the function f in (2.0.2) we shall suppose

(f1) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

(f2) For all $(x,r) \in \partial\Omega \times \mathbb{R}$ and $t \geq 0$

$$f(x,tr) = tf(x,r).$$

For the existence results we will assume in addition

(f3) For all $x \in \partial\Omega$ $r \to f(x,r)$ is non-decreasing on $\mathbb{R}$.

Clearly, $f(x,u) = \gamma(x)u$ with $\gamma(x) \geq 0$ and continuous on $\partial\Omega$, satisfies all the three hypothesis.

In this chapter, for (2.0.3) we adopt the classical notion of viscosity solution Definition I.2.1 given in the Introduction. In the definition the test functions can be substituted by the elements of the semi-jets $\overline{J}^{2,+}u(x_0)$ when u is a subsolution and $\overline{J}^{2,-}u(x_0)$ when u is a supersolution, see [40].

One of the motivation for the relaxed boundary conditions required in Definition I.2.1 is the stability under uniform convergence. Actually, if the operator F satisfies (F2) and the domain Ω the exterior sphere condition, viscosity subsolutions (resp., supersolutions) satisfy in the viscosity sense $B(x, u(x), Du(x)) \leq$ (resp. $\geq$) 0 for any $x \in \partial\Omega$, as shown in the following proposition due to Hitoshi Ishii, [74], whose proof is given for the reader's convenience.

Proposition 2.1.1. *Suppose that Ω satisfies the exterior sphere condition. If there exists $b > 0$ such that for $x \in \overline{\Omega}$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $X, Y \in$ S(N)*

$$F(x,r,p,X) - F(x,r,q,Y) \geq \mathcal{M}^-_{a,A}(Y - X) - b|p - q|,$$

and u is a viscosity subsolution of (2.0.3) *then u satisfies in the viscosity sense*

$$B(x_0, u(x_0), Du(x_0)) \leq 0,$$

for any $x_0 \in \partial\Omega$. If

$$F(x,r,p,X) - F(x,r,q,Y) \leq \mathcal{M}^+_{a,A}(Y - X) + b|p - q|,$$

and u is a viscosity supersolution of (2.0.3) *then u satisfies in the viscosity sense*

$$B(x_0, u(x_0), Du(x_0)) \geq 0,$$

for any $x_0 \in \partial\Omega$.

Proof. We show the proposition for subsolutions. Set

$$g(t) = -Kt^2 + \epsilon t \quad \forall t \in \mathbb{R},$$

where $K \gg 1$ and $0 < \epsilon \ll 1$. Observe that $g(0) = 0$, $g'(0) = \epsilon$, $g''(0) = -2K$, and

$$0 < t < \frac{\epsilon}{K} \quad \Longrightarrow \quad g(t) > 0.$$

Let $\varphi \in C^2(\overline{\Omega})$ and $x_0 \in \partial\Omega$. Assume that $u - \varphi$ attains a maximum at x_0. We need to prove that $f(x_0, u(x_0)) + \langle \overrightarrow{n}(x_0), D\varphi(x_0)\rangle \leq 0$.

Let $y_0 \in \mathbb{R}^N$ and $R > 0$ satisfy

$$B(y_0, R) \cap \overline{\Omega} = \{x_0\}.$$

We may assume by translation that $y_0 = 0$. We set

$$\psi(x) = g(|x| - R) \quad \forall x \in \mathbb{R}^N.$$

Note that $\psi(x_0) = g(0) = 0$,

$$D\psi(x_0) = g'(0)\frac{x_0}{|x_0|} = \epsilon e_0, \text{ where } e_0 = \frac{x_0}{|x_0|},$$

$$\overrightarrow{n}(x_0) \cdot D\psi(x_0) = -e_0 \cdot \epsilon e_0 = -\epsilon,$$

$$\begin{aligned} D^2\psi(x_0) &= g''(0) e_0 \otimes e_0 + \frac{g'(0)}{|x_0|}(I - e_0 \otimes e_0) \\ &= -2K e_0 \otimes e_0 + \frac{\epsilon}{R}(I - e_0 \otimes e_0), \end{aligned}$$

$$\mathcal{M}^-_{a,A}(-D^2\psi(x_0)) = -\frac{\epsilon(N-1)A}{R} + 2Ka,$$

$$R < |x| < R + \frac{\epsilon}{K} \quad \Longrightarrow \quad \psi(x) > 0.$$

Moreover we observe that $u - \varphi - \psi$ attains a local maximum at x_0. Remark that

$$\begin{aligned} &F(x_0, u(x_0), D\varphi(x_0) + D\psi(x_0), D^2\varphi(x_0) + D^2\psi(x_0)) \\ &\geq F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - b|D\psi(x_0)| + \mathcal{M}^-_{a,A}(-D^2\psi(x_0)) \\ &\geq F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - b\epsilon - \frac{\epsilon(N-1)A}{R} + 2Ka. \end{aligned}$$

We fix $K \gg 1$ so that for any $0 < \epsilon < 1$

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - b\epsilon - \frac{\epsilon(N-1)A}{R} + 2Ka > g(x_0).$$

Then, by definition of subsolution we get that

$$0 \geq f(x_0, u(x_0)) + \overrightarrow{n}(x_0) \cdot (D\varphi(x_0) + D\psi(x_0)) = f(x_0, u(x_0)) + \overrightarrow{n}(x_0) \cdot D\varphi(x_0) - \epsilon,$$

from which we obtain

$$f(x_0, u(x_0)) + \overrightarrow{n}(x_0) \cdot D\varphi(x_0) \leq 0,$$

as desired. □

2.2 The Strong Comparison Principle

The strong comparison principle is the key ingredient in the development of our theory.

Theorem 2.2.1. *Assume that (F2), (F3), (f1) hold and that g is continuous on $\overline{\Omega}$. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be respectively a sub and a supersolution of*

$$\begin{cases} F(x, u, Du, D^2u) = g(x) & \text{in} \quad \Omega \\ B(x, u, Du) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

If $u \leq v$ on $\overline{\Omega}$ then either $u < v$ on $\overline{\Omega}$ or $u \equiv v$ on $\overline{\Omega}$.

Let us recall that for the sub and the supersolutions of the Dirichlet problem the following theorem holds, see [76].

Theorem 2.2.2. *Assume that (F2), (F3) hold and that g is continuous on Ω. Let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ be respectively a sub and a supersolution of*

$$F(x, u, Du, D^2u) = g(x).$$

If $u \le v$ in Ω then either $u < v$ in Ω or $u \equiv v$ in Ω.

Proof of Theorem 2.2.1. Assume $u \not\equiv v$, then by Theorem 2.2.2 $u < v$ in Ω. Suppose by contradiction that there exists a point $x_0 \in \partial\Omega$ on which $u(x_0) = v(x_0)$.

The interior sphere condition (Ω1) implies that there exist $R > 0$ and $y_0 \in \Omega$ such that the ball centered in y_0 and of radius R, B_1, is contained in Ω and $x_0 \in \partial B_1$. Let for $k > 2/R^2$ and $x \in \overline{\Omega}$

$$w(x) := e^{-kR^2} - e^{-k|x-y_0|^2}.$$

This function has the following properties

$$\begin{aligned} w(x) &< 0 \quad \text{in } B_1, \\ w(x) &= 0 \quad \text{on } \partial B_1, \\ w(x) &> 0 \quad \text{outside } \overline{B}_1. \end{aligned}$$

Let B_2 be the ball of center y_0 and radius $\frac{R}{2}$ and $-m := \max_{\overline{B}_2}(u - v) < 0$. Choose $\sigma > 0$ so small that

$$\sigma \inf_{\overline{B}_2} w \ge -\frac{m}{2}. \tag{2.2.5}$$

Let us define for $j \in \mathbb{N}$ the functions

$$\phi(x, y) := \frac{j}{2}|x - y|^2 + \frac{\sigma}{2}(w(x) + w(y)) - f(x_0, u(x_0))\langle \overrightarrow{n}(x_0), x - y\rangle,$$

and

$$\psi(x, y) := u(x) - v(y) - \phi(x, y).$$

Let $(x_j, y_j) \in \overline{\Omega}^2$ be a maximum point of ψ in $\overline{\Omega}^2$. We have

$$\begin{aligned} 0 = u(x_0) - v(x_0) - \sigma w(x_0) \le u(x_j) - v(y_j) - \frac{j}{2}|x_j - y_j|^2 - \frac{\sigma}{2}(w(x_j) + w(y_j)) \\ + f(x_0, u(x_0))\langle \overrightarrow{n}(x_0), x_j - y_j\rangle, \end{aligned} \tag{2.2.6}$$

from which we can see that $|x_j - y_j| \to 0$ as $j \to +\infty$. Up to subsequence, x_j and y_j converge to some $\overline{z} \in \overline{\Omega}$. Standard arguments show that

$$\lim_{j\to+\infty} \frac{j}{2}|x_j - y_j|^2 = 0, \quad \lim_{j\to+\infty} u(x_j) \to u(\overline{z}) \text{ and } \lim_{j\to+\infty} v(y_j) \to v(\overline{z}).$$

Passing to the limit in (2.2.6) we get

$$\sigma w(\overline{z}) \le u(\overline{z}) - v(\overline{z}) \le 0, \tag{2.2.7}$$

which implies that the limit point $\overline{z}$ belongs to $\overline{B}_1$. Furthermore, since $u(\overline{z}) - v(\overline{z}) - \sigma w(\overline{z}) \geq 0$, it cannot belong to $\overline{B}_2$, indeed by (2.2.5) we have $u(x) - v(x) - \sigma w(x) \leq -\frac{m}{2} < 0$, for any $x \in \overline{B}_2$. In conclusion

$$\frac{R}{2} < |\overline{z} - y_0| \leq R.$$

Computing the derivatives of ϕ we get

$$D_x\phi(x, y) = j(x - y) + \sigma k e^{-k|x-y_0|^2}(x - y_0) - f(x_0, u(x_0))\overrightarrow{n}(x_0),$$

$$D_y\phi(x, y) = -j(x - y) + \sigma k e^{-k|y-y_0|^2}(y - y_0) + f(x_0, u(x_0))\overrightarrow{n}(x_0).$$

If $x_j \in \partial\Omega$ then $\overline{z} = x_0$ and using (2.1.4) we have

$$\begin{aligned} B(x_j, u(x_j), D_x\phi(x_j, y_j)) &\geq f(x_j, u(x_j)) - f(x_0, u(x_0))\langle\overrightarrow{n}(x_0), \overrightarrow{n}(x_j)\rangle \\ &\quad - \frac{j}{2r}|x_j - y_j|^2 + \sigma k e^{-k|x_j-y_0|^2}\langle x_j - y_0, \overrightarrow{n}(x_j)\rangle > 0 \end{aligned}$$

for large j, since the last term goes to $\sigma k e^{-kR^2}R$ as $j \to +\infty$, being $\overrightarrow{n}(x_0) = \frac{x_0 - y_0}{R}$. Similarly if $y_j \in \partial\Omega$ then $\overline{z} = x_0$ and $u(x_0) = v(x_0)$ so that

$$\begin{aligned} B(y_j, v(y_j), -D_y\phi(x_j, y_j)) &\leq f(y_j, v(y_j)) - f(x_0, u(x_0))\langle\overrightarrow{n}(x_0), \overrightarrow{n}(y_j)\rangle \\ &\quad + \frac{j}{2r}|x_j - y_j|^2 - \sigma k e^{-k|y_j-y_0|^2}\langle y_j - y_0, \overrightarrow{n}(y_j)\rangle < 0 \end{aligned}$$

for large j. Then x_j and y_j are internal points and

$$F(x_j, u(x_j), D_x\phi(x_j, y_j), X) \leq g(x_j) \quad \text{if } (D_x\phi(x_j, y_j), X) \in \overline{J}^{2,+}u(x_j),$$

$$F(y_j, v(y_j), -D_y\phi(x_j, y_j), Y) \geq g(y_j) \quad \text{if } (-D_y\phi(x_j, y_j), Y) \in \overline{J}^{2,-}v(y_j).$$

Then the previous relations hold for $(x_j, y_j) \in \overline{\Omega}^2$, provided j is large.

Since (x_j, y_j) is a local maximum point of $\psi(x, y) = (u(x) - \frac{\sigma}{2}w(x)) - (v(y) + \frac{\sigma}{2}w(y)) - \frac{j}{2}|x - y|^2 + f(x_0, u(x_0))\langle\overrightarrow{n}(x_0), x - y\rangle$ in $\overline{\Omega}^2$, applying Theorem 3.2 of [40] there exist $X_j, Y_j \in S(N)$ such that $(D_x\phi(x_j, y_j), X_j) \in \overline{J}^{2,+}u(x_j)$, $(-D_y\phi(x_j, y_j), Y_j) \in \overline{J}^{2,-}v(y_j)$ and

$$-3j\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_j - \frac{\sigma}{2}D^2w(x_j) & 0 \\ 0 & -(Y_j + \frac{\sigma}{2}D^2w(y_j)) \end{pmatrix} \leq 3j\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The hessian matrix of $w(x)$ is

$$D^2w(x) = 2ke^{-k|x-y_0|^2}I - 4k^2e^{-k|x-y_0|^2}(x - y_0) \otimes (x - y_0).$$

Its eigenvalues are $2ke^{-k|x-y_0|^2}$ with multiplicity $N-1$ and $2ke^{-k|x-y_0|^2}(1-2k|x-y_0|^2)$ with multiplicity 1. In the annulus $B_1 \setminus \overline{B}_2$ we have $2ke^{-k|x-y_0|^2}(1 - 2k|x - y_0|^2) \leq 2ke^{-k|x-y_0|^2}\left(1 - k\frac{R^2}{2}\right) < 0$ since $k > \frac{2}{R^2}$.

Using the fact that u and v are respectively sub and supersolution and the properties of the operator F we have

$$\begin{aligned}
g(y_j) &\le F(y_j, v(y_j), -D_y\phi, Y_j) \\
&\le F(y_j, v(y_j), -D_y\phi, Y_j + \frac{\sigma}{2}D^2w(y_j)) + \frac{\sigma}{2}\mathcal{M}^+_{a,A}(D^2w(y_j)) \\
&\le F(x_j, v(y_j), -D_y\phi, X_j - \frac{\sigma}{2}D^2w(x_j)) + \omega_T(o_j) + \frac{\sigma}{2}\mathcal{M}^+_{a,A}(D^2w(y_j)) \\
&\le F(x_j, u(x_j), D_x\phi, X_j) + \omega_T(o_j) + \frac{\sigma}{2}\mathcal{M}^+_{a,A}(D^2w(x_j)) + \frac{\sigma}{2}\mathcal{M}^+_{a,A}(D^2w(y_j)) \\
&+ b\frac{\sigma}{2}|Dw(x_j)| + b\frac{\sigma}{2}|Dw(y_j)| + c|u(x_j) - v(y_j)| \\
&\le g(x_j) + \omega_T(o_j) + \frac{\sigma}{2}\mathcal{M}^+_{a,A}(D^2w(x_j)) + \frac{\sigma}{2}\mathcal{M}^+_{a,A}(D^2w(y_j)) + b\frac{\sigma}{2}|Dw(x_j)| \\
&+ b\frac{\sigma}{2}|Dw(y_j)| + c|u(x_j) - v(y_j)|,
\end{aligned}$$

where $o_j = j|x_j - y_j|^2 + |x_j - y_j|(|D_y\phi| + 1) \to 0$ as $j \to +\infty$. Then

$$\begin{aligned}
g(y_j) \le\; & g(x_j) + A(N-1)\sigma k e^{-k|x_j - y_0|^2} + a\sigma k e^{-k|x_j-y_0|^2}(1 - 2k|x_j - y_0|^2) \\
&+ A(N-1)\sigma k e^{-k|y_j-y_0|^2} + a\sigma k e^{-k|y_j-y_0|^2}(1 - 2k|y_j - y_0|^2) \\
&+ k\sigma e^{-k|x_j-y_0|^2} b|x_j - y_0| + k\sigma e^{-k|y_j-y_0|^2} b|y_j - y_0| + c|u(x_j) - v(y_j)| + \omega_T(o_j).
\end{aligned}$$

Passing to the limit as $j \to +\infty$ we get

$$2\sigma e^{-k|\overline{z}-y_0|^2}\{-2ak^2|\overline{z} - y_0|^2 + [A(N-1) + a + b|\overline{z} - y_0|]k\} + c|u(\overline{z}) - v(\overline{z})| \ge 0.$$

Using (2.2.7) and the fact that $\frac{R}{2} < |\overline{z} - y_0| \le R$, we have

$$\begin{aligned}
0 &\le 2\sigma e^{-k|\overline{z}-y_0|^2}\{-2ak^2|\overline{z} - y_0|^2 + [A(N-1) + a + b|\overline{z} - y_0|]k\} + c|u(\overline{z}) - v(\overline{z})| \\
&\le 2\sigma e^{-k|\overline{z}-y_0|^2}\left\{-ak^2\frac{R^2}{2} + [A(N-1) + a + bR]k\right\} + \sigma c(e^{-k|\overline{z}-y_0|^2} - e^{-kR^2}) \\
&\le \sigma e^{-k|\overline{z}-y_0|^2}\{-ak^2R^2 + 2[A(N-1) + a + bR]k + c\}.
\end{aligned}$$

If we fix $k > 2/R^2$ so large that

$$-ak^2R^2 + 2[A(N-1) + a + bR]k + c < 0,$$

we obtain a contradiction, then $u < v$ on $\overline{\Omega}$. □

Remark 2.2.3. In Theorem 2.2.1 the domain Ω may be unbounded. In that case, in the proof of the theorem it suffices to maximize $\psi(x, y)$ on the compact set $(\overline{B}(y_0, 2R) \cap \overline{\Omega})^2$, instead of the whole $\overline{\Omega}$.

A consequence of Theorem 2.2.1 are the following strong maximum and minimum principles.

Corollary 2.2.4. *Assume the hypothesis of Theorem 2.2.1. If $f(x, 0) \le 0$ for any $x \in \partial\Omega$ and $v \in LSC(\overline{\Omega})$ is a non-negative viscosity supersolution of*

$$\begin{cases} F(x, v, Dv, D^2v) = 0 & \text{in} \quad \Omega \\ B(x, v, Dv) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.8}$$

then either $v \equiv 0$ *or* $v > 0$ *on* $\overline{\Omega}$*. If* $f(x,0) \geq 0$ *for any* $x \in \partial\Omega$ *and* $u \in USC(\overline{\Omega})$ *is a non-positive viscosity subsolution of* (2.2.8) *then either* $u \equiv 0$ *or* $u < 0$ *on* $\overline{\Omega}$*.*

Proof. If $f(x,0) \leq 0$ for any $x \in \partial\Omega$ then $u \equiv 0$ is a subsolution of (2.2.8). The thesis follows applying Theorem 2.2.1. □

2.3 The Maximum Principle and the principal eigenvalues

We say that F with boundary condition (2.0.2) satisfies the maximum principle, if whenever $u \in USC(\overline{\Omega})$ is a viscosity subsolution of

$$\begin{cases} F(x,u,Du,D^2u) = 0 & \text{in} \quad \Omega \\ B(x,u,Du) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u \leq 0$ on $\overline{\Omega}$. We first prove that the maximum principle holds if F is proper, i.e., if $r \to F(x,r,p,M)$ is non-decreasing. Observe that we do not require the stronger condition $F(x,r,p,X) - \sigma r$ non-decreasing in r for some $\sigma > 0$, in which case the comparison principle holds (see [40] Theorem 7.5) and implies the maximum principle if $u \equiv 0$ is a supersolution.

Successively, we show that the operator $F - \lambda I$ with boundary condition (2.0.2) satisfies the maximum principle for any $\lambda < \overline{\lambda}$. To prove that the two results do not coincide, we construct a class of operators which are not proper but that have positive principal eigenvalue $\overline{\lambda}$, hence for which the maximum principle holds.

2.3.1 The case F proper

Theorem 2.3.1. *Assume that (F2), (F3), (f1) and (f3) hold, that* $r \to F(x,r,p,M)$ *is non-decreasing on* $\mathbb{R}$ *for all* $(x,p,M) \in \overline{\Omega} \times \mathbb{R}^N \times S(N)$*,* $F(x,0,0,0) \geq 0$ *for all* $x \in \Omega$*,* $f(x,0) \geq 0$ *for all* $x \in \partial\Omega$ *and*

$$\max_{x \in \partial\Omega} f(x,r) \vee \max_{x \in \overline{\Omega}} F(x,r,0,0) > 0 \textit{ for any } r > 0. \tag{2.3.9}$$

If $u \in USC(\overline{\Omega})$ *is a viscosity subsolution of*

$$\begin{cases} F(x,u,Du,D^2u) = 0 & \textit{in} \quad \Omega \\ B(x,u,Du) = 0 & \textit{on} \quad \partial\Omega, \end{cases} \tag{2.3.10}$$

then $u \leq 0$ *on* $\overline{\Omega}$*.*

Proof. Let u be a subsolution of (2.3.10). First let us suppose $u \equiv k$ =const. By definition of subsolution and Proposition 2.1.1

$$F(x,k,0,0) \leq 0, \text{ for any } x \in \Omega$$

and

$$B(x,k,0) = f(x,k) \leq 0 \text{ for any } x \in \partial\Omega.$$

Then the hypothesis (2.3.9) implies $k \le 0$.

Now we assume that u is not a constant. We argue by contradiction; suppose that $\max_{\overline{\Omega}} u = u(x_0) > 0$, for some $x_0 \in \overline{\Omega}$. Define $\widetilde{u}(x) := u(x) - u(x_0)$. Since $r \to F(x,r,p,M)$ and $r \to f(x,r)$ are non-decreasing, $\widetilde{u}$ is a non-positive subsolution of (2.3.10). The properties $F(x,0,0,0) \ge 0$ and $f(x,0) \ge 0$ imply that $v \equiv 0$ is a supersolution of (2.3.10). Then it follows from Theorem 2.2.1 that either $u \equiv u(x_0)$ or $u < u(x_0)$ on $\overline{\Omega}$. In both cases we get a contradiction.

□

Remark 2.3.2. Under the assumptions of Theorem 2.3.1, but now with $F(x,0,0,0) \le 0$ for all $x \in \Omega$, $f(x,0) \le 0$ for all $x \in \partial\Omega$ and $\min_{x\in\partial\Omega} f(x,r) \vee \min_{x\in\overline{\Omega}} F(x,r,0,0) < 0$ for any $r < 0$, we can prove the minimum principle, i.e., if $u \in LSC(\overline{\Omega})$ is a viscosity supersolution of (2.3.10) then $u \ge 0$ on $\overline{\Omega}$.

Remark 2.3.3. If F does not depend on r and $f \equiv 0$ a counterexample to the validity of the maximum principle is given by the positive constants.

2.3.2 The Maximum Principle for $\lambda < \overline{\lambda}$

We set

$$\begin{aligned} \overline{E} := \{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of} \\ F(x,v,Dv,D^2v) = \lambda v \text{ in } \Omega,\ B(x,v,Dv) = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

$$\begin{aligned} \underline{E} := \{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ on } \overline{\Omega} \text{ bounded viscosity subsolution of} \\ F(x,u,Du,D^2u) = \lambda u \text{ in } \Omega,\ B(x,u,Du) = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

The set $\overline{E}$ is not empty, indeed the function $v(x) = e^{-|f(\cdot,1)|_\infty d(x)}$ satisfies

$$\begin{aligned} F(x,v,Dv,D^2v) - \lambda v \ge e^{-|f(\cdot,1)|_\infty d(x)}\big\{ -\mathcal{M}^+_{a,A}\big(|f(\cdot,1)|^2_\infty Dd(x) \otimes Dd(x) \\ - |f(\cdot,1)|_\infty D^2 d(x)\big) - b|f(\cdot,1)|_\infty - c - \lambda\big\} \ge 0, \end{aligned}$$

in Ω, for λ small enough, and

$$B(x,v,Dv) = f(x,1) + |f(x,1)|_\infty \ge 0,$$

on $\partial\Omega$. As a consequence $\overline{\lambda} = \sup \overline{E}$ is well defined. Similarly we can prove that $\underline{E}$ is not empty. We shall show that $\overline{\lambda}$ and $\underline{\lambda}$ are finite.

We want to remark that since in the sequel we will assume (f2), which implies $f(x,0) = 0$ for any $x \in \partial\Omega$, by Corollary 2.2.4 any non-negative supersolution (resp., non-positive subsolution) of $F(x,v,Dv,D^2v) = 0$ in Ω, $B(x,v,Dv) = 0$ on $\partial\Omega$ which is non-zero will be positive (resp., negative) in all $\overline{\Omega}$.

Theorem 2.3.4. *Assume that (F1)-(F3), (f1) and (f2) hold. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be respectively sub and supersolution of*

$$\begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega \\ B(x,u,Du) = 0 & \text{on } \partial\Omega. \end{cases}$$

If v is bounded, $v > 0$ on $\overline{\Omega}$ and $u(x_0) > 0$ for some $x_0 \in \overline{\Omega}$, then there exists $t > 0$ such that $v \equiv tu$. The same conclusion holds if u is bounded, $u < 0$ on $\overline{\Omega}$ and $v(x_0) < 0$.

Proof. Suppose that $v > 0$ on $\overline{\Omega}$ and $u(x_0) > 0$. We prove the theorem trough a typical argument which is used in [26] for the linear case and Dirichlet boundary condition. Set $w_t = u - tv$. If t is large enough $w_t < 0$ on $\overline{\Omega}$. We define

$$\tau = \inf\{t \,|\, w_t < 0 \text{ on } \overline{\Omega}\}.$$

Clearly $w_\tau \leq 0$. If $\max_{\overline{\Omega}} w_\tau = m < 0$, then for any $x \in \overline{\Omega}$

$$w_{\tau-\epsilon}(x) = u(x) - (\tau - \epsilon)v(x) \leq m + \epsilon|v|_\infty < 0,$$

for ϵ small enough. This contradicts the definition of τ. Then w_τ vanishes somewhere on $\overline{\Omega}$ and $\tau > 0$ since $u(x_0) > 0$. In conclusion $u \leq \tau v$ and $u(x) = \tau v(x)$ for some $x \in \overline{\Omega}$. Since τv is again a supersolution, by Theorem 2.2.1 we have $u \equiv \tau v$.

If the inequalities satisfied by u and v are reversed, that is $u < 0$ and $v(x_0) < 0$, we consider the function $w_t = tu - v$ and use the same argument. □

Theorem 2.3.5 (Maximum Principle for $\lambda < \overline{\lambda}$)**.** *Assume that (F1)-(F3), (f1) and (f2) hold and $\lambda < \overline{\lambda}$. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} F(x,u,Du,D^2u) = \lambda u & \text{in} \quad \Omega \\ B(x,u,Du) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3.11}$$

then $u \leq 0$ on $\overline{\Omega}$.

Proof. Let $\tau \in]\lambda, \overline{\lambda}[$, then by definition there exists $v > 0$ on $\overline{\Omega}$ bounded viscosity supersolution of

$$\begin{cases} F(x,v,Dv,D^2v) = \tau v & \text{in} \quad \Omega \\ B(x,v,Dv) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then v satisfies

$$\begin{cases} F(x,v,Dv,D^2v) - \lambda v \geq (\tau - \lambda)v > 0 & \text{in} \quad \Omega \\ B(x,v,Dv) \geq 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3.12}$$

in the viscosity sense. Suppose by contradiction that $u(x_0) > 0$ for some $x_0 \in \overline{\Omega}$. Applying Theorem 2.3.4 to the operator $F - \lambda I$, there exists $t > 0$ such that $u \equiv tv$. Then u is positive on $\overline{\Omega}$ and by homogeneity satisfies (2.3.12) in the viscosity sense. Since in addition u is a viscosity subsolution of (2.3.11), using Lemma 7.3 of [76] we get

$$(\tau - \lambda)u \leq 0 \quad \text{in } \Omega,$$

which is a contradiction. □

Remark 2.3.6. Similarly, we can prove the minimum principle for $\lambda < \underline{\lambda}$, i.e., if $u \in LSC(\overline{\Omega})$ is a viscosity supersolution of (2.3.11) and $\lambda < \underline{\lambda}$ then $u \geq 0$ on $\overline{\Omega}$.

Corollary 2.3.7. *Under the assumptions of Theorem 2.3.5, the quantities $\overline{\lambda}$ and $\underline{\lambda}$ are finite.*

Proof. By Theorem 2.3.5 it suffices to find $\lambda \in \mathbb{R}$ and a function w which is a positive subsolution of

$$\begin{cases} F(x, w, Dw, D^2 w) = \lambda w & \text{in} \quad \Omega \\ B(x, w, Dw) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

For

$$\lambda \geq -\mathcal{M}^{-}_{a,A}\left(|f(\cdot,1)|^2_\infty Dd(x) \otimes Dd(x) + |f(\cdot,1)|_\infty D^2 d(x)\right) + b|f(\cdot,1)|_\infty + c,$$

a subsolution is $w(x) = e^{|f(\cdot,1)|_\infty d(x)}$. □

2.3.3 An example

We want to show there exist some operators which are not proper but whose first eigenvalue $\overline{\lambda}$ is positive.

For simplicity, let us suppose that F is independent of the gradient variable and that Ω is the ball of center 0 and radius R. We assume in addition that for all $(x, X) \in \overline{\Omega} \times S(N)$ and any $r > 0$

$$F(x, r, X) \geq -\mathcal{M}^{+}_{a,A}(X) + c_0(x) r, \tag{2.3.13}$$

for some functions $c_0(x)$. The Isaacs operator (I.2.10) satisfies (2.3.13) if

$$c_{\alpha,\beta}(x) \geq c_0(x) \quad \text{for all } x \in \overline{\Omega} \text{ and } (\alpha, \beta) \in \mathcal{A} \times \mathcal{B}.$$

In this case the operator is proper if $c_0(x) \geq 0$. Since we are interested in non-proper F, we are looking for functions $c_0(x)$ in (2.3.13) that may be negative somewhere. We suppose that

$$\begin{cases} c_0(x) > 0 & \text{if } R - \epsilon < |x| \leq R \\ c_0(x) \geq \beta_1 & \text{if } \rho < |x| \leq R - \epsilon \\ c_0(x) \geq -\beta_2 & \text{if } |x| \leq \rho, \end{cases}$$

where $0 < \rho < R$, $\epsilon > 0$ is small enough and β_1, $\beta_2 > 0$. Remark that in the ball of radius ρ, $c_0(x)$ may assume negative values. To prove that $\overline{\lambda} > 0$ it suffices to find $v > 0$ bounded supersolution of

$$\begin{cases} -\mathcal{M}^{+}_{a,A}(D^2 v) + c_0(x) v = \lambda v & \text{in} \quad \Omega \\ B(x, v, Dv) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for some $\lambda > 0$. Assume $f(x, r) \geq 0$ for any $x \in \partial\Omega$ and $r \geq 0$, then, as shown in [97], such supersolution v exists if β_1 and β_2 satisfy the following inequality for some $k > 0$

$$\beta_2 < \frac{k e^{-k\rho} a \left(k + \frac{N-1}{\rho}\right)}{k \frac{R-\rho}{4} + k \frac{2NAR-(N-1)a(R+\rho)}{\beta_1 R(R-\rho)} + 1 - e^{-k\rho}}.$$

As observed in [97], from the last relation we can see that choosing $k = \frac{1}{\rho}$ the term on the right-hand side goes to $+\infty$ as $\rho \to 0^+$, that is, if the set where $c_0(x)$ is negative becomes smaller then the values of $c_0(x)$ in this set can be very negative. On the contrary, for any value of k, if $\rho \to R^-$ then β_2 goes to 0. Finally, for any k, if $\beta_1 \to 0^+$ then again β_2 goes to 0. So there is a sort of balance between β_1 and β_2. In [97] we present an example to explain this behavior. For operators which satisfy (2.3.13), the property $\overline{\lambda} > 0$ can be proved in any C^2 domain, under similar assumptions on $c_0(x)$, see [97].

2.4 Lipschitz regularity

In this section we shall prove that viscosity solutions are Lipschitz continuous on $\overline{\Omega}$. We want to mention the works of Barles and Da Lio [20] and Milakis and Silvestre [91] about Hölder estimates of viscosity solutions of fully nonlinear elliptic equations associated to Neumann type boundary conditions.

Theorem 2.4.1. *Assume that (F1), (F2), (F4), (f1) and (f2) hold. Let g be a bounded function and $u \in C(\overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} F(x,u,Du,D^2u) = g(x) & in \quad \Omega \\ B(x,u,Du) = 0 & on \quad \partial\Omega, \end{cases}$$

then there exists $C_0 > 0$ such that

$$|u(x) - u(y)| \leq C_0|x-y| \quad \forall x, y \in \overline{\Omega}, \tag{2.4.14}$$

where C_0 depends on N, a, A, b, c, C_1, Ω and $|f(\cdot, u(\cdot))|_\infty$.

Proof. We follow the proof of Proposition III.1 of [75], that we modify taking test functions which depend on the distance function and that are suitable for the Neumann boundary conditions.

We set

$$\Phi(x) = MK|x| - M(K|x|)^2,$$

and

$$\varphi(x,y) = e^{-L(d(x)+d(y))}\Phi(x-y),$$

where L is a fixed number greater than $\frac{2}{3r}$ with r the radius in the condition (Ω2) and K and M are two positive constants to be chosen later. If $K|x| \leq \frac{1}{4}$, then

$$\Phi(x) \geq \frac{3}{4}MK|x|. \tag{2.4.15}$$

We define

$$\Delta_K := \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \, |x-y| \leq \frac{1}{4K}\right\}.$$

We fix M such that

$$\max_{\overline{\Omega}^2} |u(x) - u(y)| \leq e^{-2Ld_0}\frac{M}{8}, \tag{2.4.16}$$

where $d_0 = \max_{x\in\overline{\Omega}} d(x)$, and we claim that taking δ small enough and K large enough, one has

$$\delta(u(x) - u(y)) - \varphi(x,y) \leq 0 \quad \text{for } (x,y) \in \Delta_K \cap \overline{\Omega}^2. \tag{2.4.17}$$

In this case (2.4.14) is proven. To show (2.4.17) we suppose by contradiction that for some $(\overline{x}, \overline{y}) \in \Delta_K \cap \overline{\Omega}^2$

$$\delta u(\overline{x}) - \delta u(\overline{y}) - \varphi(\overline{x}, \overline{y}) = \max_{\Delta_K \cap \overline{\Omega}^2} (\delta u(x) - \delta u(y) - \varphi(x, y)) > 0. \tag{2.4.18}$$

Observe that δu is again a solution since both F and B are positively homogeneous. Here we have dropped the dependence of $\overline{x}$, $\overline{y}$ on K and δ for simplicity of notations.

Clearly $\overline{x} \neq \overline{y}$. Moreover the point $(\overline{x}, \overline{y})$ belongs to $\mathrm{int}(\Delta_K) \cap \overline{\Omega}^2$. Indeed, if $|x - y| = \frac{1}{4K}$, by (2.4.16) and (2.4.15) for $\delta \leq 1$ we have

$$\delta u(x) - \delta u(y) \leq |u(x) - u(y)| \leq e^{-2Ld_0} \frac{M}{8} \leq e^{-L(d(x)+d(y))} \frac{1}{2} MK|x - y| \leq \varphi(x, y).$$

Since $\overline{x} \neq \overline{y}$ we can compute the derivatives of φ in $(\overline{x}, \overline{y})$ obtaining

$$\begin{aligned} D_x\varphi(\overline{x}, \overline{y}) = & - Le^{-L(d(\overline{x})+d(\overline{y}))} MK|\overline{x} - \overline{y}|(1 - K|\overline{x} - \overline{y}|)Dd(\overline{x}) \\ & + e^{-L(d(\overline{x})+d(\overline{y}))} MK(1 - 2K|\overline{x} - \overline{y}|) \frac{(\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|}, \\ D_y\varphi(\overline{x}, \overline{y}) = & - Le^{-L(d(\overline{x})+d(\overline{y}))} MK|\overline{x} - \overline{y}|(1 - K|\overline{x} - \overline{y}|)Dd(\overline{y}) \\ & - e^{-L(d(\overline{x})+d(\overline{y}))} MK(1 - 2K|\overline{x} - \overline{y}|) \frac{(\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|}. \end{aligned}$$

Observe that for $K \geq \frac{L}{4}$

$$|D_x\varphi(\overline{x}, \overline{y})|, |D_y\varphi(\overline{x}, \overline{y})| \leq 2MK. \tag{2.4.19}$$

Using (2.1.4), if $\overline{x} \in \partial\Omega$ we have

$$\begin{aligned} B(\overline{x}, \delta u(\overline{x}), D_x\varphi(\overline{x}, \overline{y})) = & f(\overline{x}, \delta u(\overline{x})) + Le^{-Ld(\overline{y})} MK|\overline{x} - \overline{y}|(1 - K|\overline{x} - \overline{y}|) \\ & + e^{-Ld(\overline{y})} MK(1 - 2K|\overline{x} - \overline{y}|)\langle \overrightarrow{n}(\overline{x}), \frac{(\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|}\rangle \\ \geq & \frac{1}{2} e^{-Ld(\overline{y})} MK|\overline{x} - \overline{y}| \left(\frac{3}{2}L - \frac{1}{r} \right) - \delta |f(\cdot, u(\cdot))|_\infty > 0, \end{aligned} \tag{2.4.20}$$

since $\overline{x} \neq \overline{y}$, $L > \frac{2}{3r}$, for δ small enough. Similarly, if $\overline{y} \in \partial\Omega$ then

$$B(\overline{y}, \delta u(\overline{y}), -D_y\varphi(\overline{x}, \overline{y})) \leq \frac{1}{2} e^{-Ld(\overline{x})} MK|\overline{x} - \overline{y}| \left(-\frac{3}{2}L + \frac{1}{r} \right) + \delta |f(\cdot, u(\cdot))|_\infty < 0.$$

Then $\overline{x}, \overline{y} \in \Omega$ and

$$F(\overline{x}, \delta u(\overline{x}), D_x\varphi(\overline{x}, \overline{y}), X) \leq \delta g(\overline{x}), \quad \text{if } (D_x\varphi(\overline{x}, \overline{y}), X) \in \overline{J}^{2,+} \delta u(\overline{x}),$$

$$F(\overline{y}, \delta u(\overline{y}), -D_y\varphi(\overline{x}, \overline{y}), Y) \geq \delta g(\overline{y}) \quad \text{if } (-D_y\varphi(\overline{x}, \overline{y}), Y) \in \overline{J}^{2,-} \delta u(\overline{y}).$$

Since $(\overline{x}, \overline{y}) \in \mathrm{int}\Delta_K \cap \overline{\Omega}^2$, it is a local maximum point of $\delta u(x) - \delta u(y) - \varphi(x, y)$ in $\overline{\Omega}^2$. Then applying Theorem 3.2 in [40], for every $\epsilon > 0$ there exist $X, Y \in S(N)$ such that $(D_x\varphi(\overline{x}, \overline{y}), X) \in \overline{J}^{2,+} \delta u(\overline{x})$, $(-D_y\varphi(\overline{x}, \overline{y}), Y) \in \overline{J}^{2,-} \delta u(\overline{y})$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2(\varphi(\overline{x}, \overline{y})) + \epsilon (D^2(\varphi(\overline{x}, \overline{y})))^2. \tag{2.4.21}$$

Now we want to estimate the matrix on the right-hand side of the last inequality.

$$D^2\varphi(\overline{x},\overline{y}) = \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}) + D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y}))$$
$$+ D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}) + e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).$$

We set

$$A_1 := \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_2 := D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_3 := e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).$$

Observe that

$$A_1 \le CK|\overline{x}-\overline{y}| \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.4.22}$$

Here and henceforth C denotes various positive constants independent of K and δ.

For A_2 we have the following estimate

$$A_2 \le CK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + CK \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{2.4.23}$$

Indeed for ξ, $\eta \in \mathbb{R}^N$ we compute

$$\langle A_2(\xi,\eta),(\xi,\eta)\rangle = 2Le^{-L(d(\overline{x})+d(\overline{y}))}\{\langle Dd(\overline{x}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\xi\rangle$$
$$+ \langle Dd(\overline{y}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\eta\rangle\} \le CK(|\xi|+|\eta|)|\eta-\xi|$$
$$\le CK(|\xi|^2+|\eta|^2) + CK|\eta-\xi|^2.$$

Now we consider A_3. The matrix $D^2(\Phi(\overline{x}-\overline{y}))$ has the form

$$D^2(\Phi(\overline{x}-\overline{y})) = \begin{pmatrix} D^2\Phi(\overline{x}-\overline{y}) & -D^2\Phi(\overline{x}-\overline{y}) \\ -D^2\Phi(\overline{x}-\overline{y}) & D^2\Phi(\overline{x}-\overline{y}) \end{pmatrix},$$

and the Hessian matrix of $\Phi(x)$ is

$$D^2\Phi(x) = \frac{MK}{|x|}\left(I - \frac{x\otimes x}{|x|^2}\right) - 2MK^2 I. \tag{2.4.24}$$

If we choose

$$\epsilon = \frac{|\overline{x}-\overline{y}|}{2MKe^{-L(d(\overline{x})+d(\overline{y}))}}, \tag{2.4.25}$$

then we have the following estimates

$$\epsilon A_1^2 \le CK|\overline{x}-\overline{y}|^3 I_{2N}, \quad \epsilon A_2^2 \le CK|\overline{x}-\overline{y}|I_{2N},$$
$$\epsilon(A_1A_2 + A_2A_1) \le CK|\overline{x}-\overline{y}|^2 I_{2N}, \tag{2.4.26}$$
$$\epsilon(A_1A_3 + A_3A_1) \le CK|\overline{x}-\overline{y}|I_{2N}, \quad \epsilon(A_2A_3 + A_3A_2) \le CKI_{2N},$$

where $I_{2N} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then using (2.4.22), (2.4.23), (2.4.26) and observing that

$$(D^2(\Phi(\overline{x}-\overline{y})))^2 = \begin{pmatrix} 2(D^2\Phi(\overline{x}-\overline{y}))^2 & -2(D^2\Phi(\overline{x}-\overline{y}))^2 \\ -2(D^2\Phi(\overline{x}-\overline{y}))^2 & 2(D^2\Phi(\overline{x}-\overline{y}))^2 \end{pmatrix},$$

from (2.4.21) we can conclude that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le O(K) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where

$$B = CKI + e^{-L(d(\overline{x})+d(\overline{y}))}\left[D^2\Phi(\overline{x}-\overline{y}) + \frac{|\overline{x}-\overline{y}|}{MK}(D^2\Phi(\overline{x}-\overline{y}))^2\right]. \tag{2.4.27}$$

The last inequality can be rewritten as follows

$$\begin{pmatrix} \widetilde{X} & 0 \\ 0 & -\widetilde{Y} \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with $\widetilde{X} = X - O(K)I$ and $\widetilde{Y} = Y + O(K)I$.

Now we want to get a good estimate for $\mathrm{tr}(\widetilde{X}-\widetilde{Y})$, as in [75]. For that aim let

$$0 \le P := \frac{(\overline{x}-\overline{y}) \otimes (\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^2} \le I.$$

Since $\widetilde{X}-\widetilde{Y} \le 0$ and $\widetilde{X}-\widetilde{Y} \le 4B$, we have

$$\mathrm{tr}(\widetilde{X}-\widetilde{Y}) \le \mathrm{tr}(P(\widetilde{X}-\widetilde{Y})) \le 4\mathrm{tr}(PB).$$

We have to compute $\mathrm{tr}(PB)$. From (2.4.24), observing that the matrix $(1/|x|^2)x \otimes x$ is idempotent, i.e., $[(1/|x|^2)x \otimes x]^2 = (1/|x|^2)x \otimes x$, we compute

$$(D^2\Phi(x))^2 = \frac{M^2K^2}{|x|^2}(1-4K|x|)\left(I - \frac{x \otimes x}{|x|^2}\right) + 4M^2K^4 I.$$

Then, since $\mathrm{tr}P = 1$ and $4K|\overline{x}-\overline{y}| \le 1$, we have

$$\begin{aligned} \mathrm{tr}(PB) &= CK + e^{-L(d(\overline{x})+d(\overline{y}))}(-2MK^2 + 4MK^3|\overline{x}-\overline{y}|) \\ &\le CK - e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 < 0, \end{aligned}$$

for large K. This gives

$$|\mathrm{tr}(\widetilde{X}-\widetilde{Y})| = -\mathrm{tr}(\widetilde{X}-\widetilde{Y}) \ge 4e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 - 4CK \ge CK^2,$$

for large K. Since $\|B\| \le \frac{CK}{|\overline{x}-\overline{y}|}$, we have

$$\|B\|^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac{1}{2}} \le \left(\frac{CK}{|\overline{x}-\overline{y}|}\right)^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac{1}{2}} \le \frac{C}{K^{\frac{1}{2}}|\overline{x}-\overline{y}|^{\frac{1}{2}}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|.$$

The Lemma III.I in [75] ensures the existence of a universal constant C depending only on N such that

$$\|\widetilde{X}\|, \|\widetilde{Y}\| \le C\{|\mathrm{tr}(\widetilde{X}-\widetilde{Y})| + \|B\|^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac{1}{2}}\}.$$

Thanks to the above estimates we can conclude that

$$\|\widetilde{X}\|,\ \|\widetilde{Y}\| \le C|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|\left(1+\frac{1}{K^{\frac{1}{2}}|\overline{x}-\overline{y}|^{\frac{1}{2}}}\right). \tag{2.4.28}$$

Now, using assumptions (F2) and (F4) concerning F, the definition of $\widetilde{X}$ and $\widetilde{Y}$ and the fact that δu is sub and supersolution we compute

$$\begin{aligned}\delta g(\overline{y}) &\le F(\overline{y},\delta u(\overline{y}),-D_y\varphi,Y) \le F(\overline{y},\delta u(\overline{y}),-D_y\varphi,\widetilde{Y})+O(K)\\ &\le F(\overline{y},\delta u(\overline{x}),D_x\varphi,\widetilde{X})+c\delta|u(\overline{x})-u(\overline{y})|+b|D_x\varphi+D_y\varphi|\\ &+a\mathrm{tr}(\widetilde{X}-\widetilde{Y})+O(K)\\ &\le F(\overline{x},\delta u(\overline{x}),D_x\varphi,\widetilde{X})+2c\delta|u(\overline{x})|+2b|D_x\varphi|+C_1|\overline{x}-\overline{y}|^{\frac{1}{2}}\|\widetilde{X}\|\\ &+c\delta|u(\overline{x})-u(\overline{y})|+b|D_x\varphi+D_y\varphi|+a\mathrm{tr}(\widetilde{X}-\widetilde{Y})+O(K)\\ &\le \delta g(\overline{x})+2c\delta|u(\overline{x})|+2b|D_x\varphi|+C_1|\overline{x}-\overline{y}|^{\frac{1}{2}}\|\widetilde{X}\|+c\delta|u(\overline{x})-u(\overline{y})|\\ &+b|D_x\varphi+D_y\varphi|+a\mathrm{tr}(\widetilde{X}-\widetilde{Y})+O(K).\end{aligned}$$

From this inequalities, using (2.4.19) and (2.4.28), we get

$$\begin{aligned}&\delta g(\overline{y})-\delta g(\overline{x})-2c\delta|u(\overline{x})|-c\delta|u(\overline{x})-u(\overline{y})|\\ &\le O(K)+C|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|(|\overline{x}-\overline{y}|^{\frac{1}{2}}+K^{-\frac{1}{2}})+a\mathrm{tr}(\widetilde{X}-\widetilde{Y})\\ &= a\mathrm{tr}(\widetilde{X}-\widetilde{Y})+o(|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|),\end{aligned} \tag{2.4.29}$$

as $K\to+\infty$. Since g and u are bounded, the first member in (2.4.29) is bounded from below by the quantity $-2|g|_\infty-4c|u|_\infty$ which is independent of δ. But the last term in (2.4.29) goes to $-\infty$ as $K\to+\infty$, hence taking K so large that

$$a\mathrm{tr}(\widetilde{X}-\widetilde{Y})+o(|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|)<-2|g|_\infty-4c|u|_\infty,$$

and then δ so small that the last member in (2.4.20) is positive, we obtain a contradiction and this concludes the proof. □

Remark 2.4.2. The regularity theorem can be shown also for solutions of the Neumann problem for the operator

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\{-\mathrm{tr}(A_{\alpha,\beta}(x)D^2u)+b_{\alpha,\beta}(x)\cdot Du+c_{\alpha,\beta}(x)u-g_{\alpha,\beta}(x)\},$$

if the functions $g_{\alpha,\beta}$ are bounded uniformly in α and β.

Since the Lipschitz estimate depends only on the bounds of the solution of g and on the structural constants, an immediate consequence of the previous theorem is the following compactness criterion that will be useful in the next sections.

Corollary 2.4.3. *Assume the same hypothesis of Theorem 2.4.1. Suppose that $(g_n)_n$ is a sequence of continuous and uniformly bounded functions and $(u_n)_n$ is a sequence of uniformly bounded viscosity solutions of*

$$\begin{cases}F(x,u_n,Du_n,D^2u_n)=g_n(x) & \text{in } \Omega\\ B(x,u_n,Du_n)=0 & \text{on } \partial\Omega.\end{cases}$$

Then the sequence $(u_n)_n$ is relatively compact in $C(\overline{\Omega})$.

2.5 Existence results

This section is devoted to the problem of the existence of a solution of

$$\begin{cases} F(x,u,Du,D^2u) = \lambda u + g(x) & \text{in} \quad \Omega \\ B(x,u,Du) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5.30}$$

Using the well known result which guarantees that (2.5.30) with $\lambda = 0$ is uniquely solvable if F satisfies

(F5) There exists $\sigma > 0$ such that for any $(x,p,X) \in \overline{\Omega} \times \mathbb{R}^N \times S(N)$ the function $r \to F(x,r,p,X) - \sigma r$ is non-decreasing on $\mathbb{R}$,

see [40] Theorem 7.5, we will prove the existence of a positive solution of (2.5.30) when g is non-negative and $\lambda < \overline{\lambda}$, without requiring (F5). The solution is unique if $g > 0$. Then we will show the existence of a positive principal eigenfunction corresponding to $\overline{\lambda}$, that is a solution of (2.5.30) when $g \equiv 0$ and $\lambda = \overline{\lambda}$. For the last two results we will follow the proof given in [28] for the analogous theorems with the Dirichlet boundary condition.

Symmetrical results can be obtained for the eigenvalue $\underline{\lambda}$.

Finally, we will prove that the Neumann problem (2.5.30) is solvable for any right-hand side if $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$.

The following is a well known result, see [40] Theorem 7.5.

Theorem 2.5.1. *Suppose that (F2), (F3), (F5), (f1) and (f3) hold and that g is continuous on $\overline{\Omega}$. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are respectively sub and supersolution of*

$$\begin{cases} F(x,u,Du,D^2u) = g(x) & \text{in} \quad \Omega \\ B(x,u,Du) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.5.31}$$

then $u \le v$ on $\overline{\Omega}$. Moreover (2.5.31) *has a unique viscosity solution.*

Theorem 2.5.2. *Assume that (F1)-(F3), (f1) and (f2) hold. Suppose $h \ge 0$, $g \le h$ and $g(x) < 0$ if $h(x) = 0$. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} F(x,u,Du,D^2u) = \lambda u + g(x) & \text{in} \quad \Omega \\ B(x,u,Du) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and let $v \in LSC(\overline{\Omega})$ be a bounded positive viscosity supersolution of

$$\begin{cases} F(x,v,Dv,D^2v) = \lambda v + h(x) & \text{in} \quad \Omega \\ B(x,v,Dv) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5.32}$$

Then $u \le v$ on $\overline{\Omega}$.

Remark 2.5.3. The existence of such a v implies $\lambda \le \overline{\lambda}$.

Remark 2.5.4. Similarly, we can prove the comparison result between u and v if u is negative and bounded, $g \le 0$, $g \le h$ and $h(x) > 0$ if $g(x) = 0$.

Proof. Suppose by contradiction that $\max_{\overline{\Omega}}(u - v) = u(\overline{x}) - v(\overline{x}) > 0$ for some $\overline{x} \in \overline{\Omega}$. Set $w_t = u - tv$. If t is large enough $w_t < 0$ on $\overline{\Omega}$. We define

$$\tau = \inf\{t \,|\, w_t < 0 \text{ on } \overline{\Omega}\}.$$

As in the proof of Theorem 2.3.4, $w_\tau \leq 0$ and vanishes in some point, i.e., $u \leq \tau v$ and $u(x) = \tau v(x)$ for some $x \in \overline{\Omega}$. Moreover, since $u(\overline{x}) > v(\overline{x})$ we know that $\tau > 1$, which implies that $h \leq \tau h$, being h non-negative. Then τv is still a supersolution of (2.5.32) and $u \equiv \tau v$ by Theorem 2.2.1. Hence, applying Lemma 7.3 of [76] we get

$$\tau h \leq g,$$

which contradicts the assumptions on g and h. □

Theorem 2.5.5. *Suppose that (F1)-(F4), (f1)-(f3) hold, that $\lambda < \overline{\lambda}$, $g \geq 0$, $g \not\equiv 0$ and g is continuous on $\overline{\Omega}$, then there exists a positive viscosity solution of* (2.5.30). *The positive solution is unique if $g > 0$.*

Proof. The condition (F2) implies that $r \to F(x, r, p, X) + cr$ is non-decreasing. Hence the operator $F + (2c + |\lambda|)I$ satisfies (F5) with $\sigma = c + |\lambda|$, so that by Theorem 2.5.1 the sequence $(u_n)_n$ defined by $u_1 = 0$ and u_{n+1} as the solution of

$$\begin{cases} F(x, u_{n+1}, Du_{n+1}, D^2u_{n+1}) + (2c + |\lambda|)u_{n+1} = g + (2c + |\lambda| + \lambda)u_n & \text{in } \Omega \\ B(x, u_{n+1}, Du_{n+1}) = 0 & \text{on } \partial\Omega, \end{cases}$$

is well defined. By the comparison Theorems 2.5.1 and 2.2.1, since $g \geq 0$ and $g \not\equiv 0$ the sequence is positive and increasing.

We use the argument of Theorem 7 of [28] to prove that $(u_n)_n$ is also bounded. Suppose that it is not, then dividing by $|u_{n+1}|_\infty$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_{n+1} is a solution of

$$\begin{cases} F(x, v_{n+1}, Dv_{n+1}, D^2v_{n+1}) + (2c + |\lambda|)v_{n+1} & \\ \quad = \frac{g}{|u_{n+1}|_\infty} + (2c + |\lambda| + \lambda)\frac{u_n}{|u_{n+1}|_\infty} & \text{in } \Omega \\ B(x, v_{n+1}, Dv_{n+1}) = 0 & \text{on } \partial\Omega. \end{cases}$$

By Corollary 2.4.3, $(v_n)_n$ converges along a subsequence to a positive function v which satisfies

$$\begin{cases} F(x, v, Dv, D^2v) - \lambda v = (2c + |\lambda| + \lambda)(k - 1)v \leq 0 & \text{in } \Omega \\ B(x, v, Dv) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $k := \limsup_{n\to+\infty} \frac{|u_n|_\infty}{|u_{n+1}|_\infty} \leq 1$. This contradicts the maximum principle, Theorem 2.3.5. Then $(u_n)_n$ is bounded and letting n go to infinity, by the compactness result, the sequence converges uniformly to a function u which is a solution. Moreover the solution is positive on $\overline{\Omega}$ by Corollary 2.2.4.

The uniqueness of the positive solution follows from Theorem 2.5.2. □

Theorem 2.5.6 (Existence of principal eigenfunctions)**.** *Suppose that (F1)-(F4), (f1)-(f3) hold. Then there exists $\phi > 0$ on $\overline{\Omega}$ viscosity solution of*

$$\begin{cases} F(x,\phi,D\phi,D^2\phi) = \overline{\lambda}\phi & in \quad \Omega \\ B(x,\phi,D\phi) = 0 & on \quad \partial\Omega. \end{cases} \tag{2.5.33}$$

Moreover ϕ is Lipschitz continuous on $\overline{\Omega}$.

Proof. Let λ_n be an increasing sequence which converges to $\overline{\lambda}$. Let u_n be a positive solution of

$$\begin{cases} F(x,u_n,Du_n,D^2u_n) = \lambda_n u_n + 1 & \text{in} \quad \Omega \\ B(x,u_n,Du_n) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By Theorem 2.5.5 the sequence $(u_n)_n$ is well defined. Following the argument of the proof of Theorem 8 of [28] we can prove that it is unbounded, otherwise one would contradict the definition of $\overline{\lambda}$. Then, up to subsequence, $|u_n|_\infty \to +\infty$ as $n \to +\infty$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_n satisfies

$$\begin{cases} F(x,v_n,Dv_n,D^2v_n) = \lambda_n v_n + \frac{1}{|u_n|_\infty} & \text{in} \quad \Omega \\ B(x,v_n,Dv_n) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then, by Corollary 2.4.3, we can extract a subsequence converging to a function ϕ with $|\phi|_\infty = 1$ which is positive on $\overline{\Omega}$ by Corollary 2.2.4 and is the desired solution. By Theorem 2.4.1 the solution is also Lipschitz continuous on $\overline{\Omega}$. □

Remark 2.5.7. With the same arguments used in the proofs of Theorems 2.5.5 and 2.5.6 one can prove: the existence of a negative viscosity solution of (2.5.30), for $\lambda < \underline{\lambda}$ and $g \leq 0$, $g \not\equiv 0$, which is unique if $g < 0$ by Remark 2.5.4; the existence of a negative Lipschitz principal eigenfunction corresponding to $\underline{\lambda}$, i.e., a solution of

$$\begin{cases} F(x,\phi,D\phi,D^2\phi) = \underline{\lambda}\phi & \text{in} \quad \Omega \\ B(x,\phi,D\phi) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5.34}$$

Theorem 2.5.8. *Suppose that (F1)-(F4), (f1)-(f3) hold. Suppose that $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$ and g is continuous on $\overline{\Omega}$, then there exists a viscosity solution of* (2.5.30).

Proof. If $g \equiv 0$, by the maximum and minimum principles the only solution is $u \equiv 0$. Let us suppose $g \not\equiv 0$. Since $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$ by Theorem 2.5.5 and Remark 2.5.7 there exist $v_0 \in C(\overline{\Omega})$ positive viscosity solution of (2.5.30) with right-hand side $|g|_\infty$ and $u_0 \in C(\overline{\Omega})$ negative viscosity solution of (2.5.30) with right-hand side $-|g|_\infty$.

Let $(u_n)_n$ be the sequence defined in the proof of Theorem 2.5.5 with $u_1 = u_0$. By comparison Theorem 2.5.1 we have $u_0 = u_1 \leq u_2 \leq ... \leq v_0$. Hence, by the compactness Corollary 2.4.3 the sequence converges to a continuous function which is the desired solution. □

Remark 2.5.9. The existence results can be shown also for the operator

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\{-\mathrm{tr}(A_{\alpha,\beta}(x)D^2u)+b_{\alpha,\beta}(x)\cdot Du+c_{\alpha,\beta}(x)u-g_{\alpha,\beta}(x)\},$$

if the functions $g_{\alpha,\beta}$ are continuous uniformly in α and β. In particular, in that case, if $\overline{\lambda}$ and $\underline{\lambda}$ are positive there exists a viscosity solution of

$$\begin{cases}\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\{-\mathrm{tr}(A_{\alpha,\beta}(x)D^2u)+b_{\alpha,\beta}(x)\cdot Du+c_{\alpha,\beta}(x)u-g_{\alpha,\beta}(x)\}=0 & \text{in } \Omega\\ B(x,u,Du)=0 & \text{on } \partial\Omega.\end{cases}$$

2.6 Properties of the principal eigenvalues

In this section we establish some of the basic properties of the principal eigenvalues. We denote by ϕ^+ a positive eigenfunction corresponding to $\overline{\lambda}$ and by ϕ^- a negative eigenfunction corresponding to $\underline{\lambda}$. Throughout this section we assume (F1)-(F4) and (f1)-(f3).

The next result states that the principal eigenfunctions are simple, in the sense that they are equal up to a multiplicative constant.

Proposition 2.6.1. *If $u\in USC(\overline{\Omega})$ is a viscosity subsolution of*

$$\begin{cases}F(x,u,Du,D^2u)=\overline{\lambda}u & \text{in } \Omega\\ B(x,u,Du)=0 & \text{on } \partial\Omega,\end{cases}\tag{2.6.35}$$

and $u(x_0)>0$ for some $x_0\in\overline{\Omega}$ then there exists $t>0$ such that $u\equiv t\phi^+$. If $u\in LSC(\overline{\Omega})$ is a viscosity supersolution of (2.6.35) *with $\overline{\lambda}$ replaced by $\underline{\lambda}$ and $u(x_0)<0$, then there exists $t>0$ such that $u\equiv t\phi^-$.*

Assume in addition

$$-F(x,-r,-p,-X)\le F(x,r,p,X) \text{ for any } (x,r,p,X)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\times \mathrm{S(N)}\tag{2.6.36}$$

and

$$-f(x,-r)\le f(x,r) \text{ for any } (x,r)\in\partial\Omega\times\mathbb{R}.\tag{2.6.37}$$

If $u\in C(\overline{\Omega})$, $u\not\equiv 0$, is a viscosity subsolution of (2.6.35) *then there exists $t\in\mathbb{R}$ such that $u\equiv t\phi^+$. If $u\in C(\overline{\Omega})$, $u\not\equiv 0$ is a viscosity solution of* (2.6.35) *with $\overline{\lambda}$ replaced by $\underline{\lambda}$, there exists $t\in\mathbb{R}$ such that $u\equiv t\phi^-$.*

Proof. If u is a subsolution (resp., supersolution) of (2.6.35) (resp., of (2.6.35) with $\underline{\lambda}$ instead of $\overline{\lambda}$) and $u(x_0)>0$ (resp., $u(x_0)<0$), then by Theorem 2.3.4 we have $u\equiv t\phi^+$ (resp., $u\equiv t\phi^-$) for some $t>0$.

Now assume (2.6.36)-(2.6.37) and let $u\not\equiv 0$ be a subsolution of (2.6.35). If u is positive somewhere we are in the previous case. If u is negative on $\overline{\Omega}$ then the function $w:=-u$ is a positive continuous supersolution of

$$\begin{cases}F(x,w,Dw,D^2w)-\overline{\lambda}w\ge -F(x,-w,-Dw,-D^2w)+\overline{\lambda}(-w)\ge 0 & \text{in } \Omega\\ B(x,w,Dw)\ge -B(x,-w,-Dw)\ge 0 & \text{on } \partial\Omega.\end{cases}$$

Hence, again from Theorem 2.3.4 it follows that $u\equiv t\phi^+$, for some $t<0$.

Finally, let $u \not\equiv 0$ be a solution of (2.6.35) with $\underline{\lambda}$ instead of $\overline{\lambda}$. Remark that conditions (2.6.36)-(2.6.37) imply $\underline{\lambda} \leq \overline{\lambda}$. If $\underline{\lambda} < \overline{\lambda}$, then by the maximum principle, Theorem 2.3.5, $u < 0$ on $\overline{\Omega}$ and we are in the first case. If $\underline{\lambda} = \overline{\lambda}$, by the simplicity of $\overline{\lambda}$ just proved, $u \equiv t\phi^-$ for some $t < 0$. □

Remark 2.6.2. If F and f satisfy

$$-F(x,-r,-p,-X) \geq F(x,r,p,X) \text{ for any } (x,r,p,X) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N)$$

and

$$-f(x,-r) \geq f(x,r) \text{ for any } (x,r) \in \partial\Omega \times \mathbb{R},$$

then, applying Proposition 2.6.1 to the operator $G(x,r,p,X) = -F(x,-r,-p,-X)$ with $B(x,r,p) = \widetilde{f}(x,r) + \langle p, \overrightarrow{n}(x)\rangle$, where $\widetilde{f}(x,r) = -f(x,-r)$, we get again simplicity of principal eigenvalues.

Remark 2.6.3. Convex and 1-homogeneous operators satisfy the assumption (2.6.36).

Proposition 2.6.4. $\overline{\lambda}$ *(resp.,* $\underline{\lambda}$*) is the only eigenvalue corresponding to a positive (resp., negative) eigenfunction.*

Proof. Let u be a positive eigenfunction corresponding to μ. By the definition of $\overline{\lambda}$, we have $\mu \leq \overline{\lambda}$. If $\mu < \overline{\lambda}$, we must have $u \leq 0$ by Theorem 2.3.5, which is a contradiction. Thus $\mu = \lambda$. □

The following proposition states that the principal eigenvalues are isolated.

Proposition 2.6.5. *There exists* $\epsilon > 0$ *such that the problem*

$$\begin{cases} F(x,u,Du,D^2u) = \lambda u & in \quad \Omega \\ B(x,u,Du) = 0 & on \quad \partial\Omega, \end{cases} \tag{2.6.38}$$

has no solutions $u \not\equiv 0$*, for* $\lambda \in (-\infty, \max\{\underline{\lambda}, \overline{\lambda}\} + \epsilon) \setminus \{\overline{\lambda}, \underline{\lambda}\}$.

Proof. We may suppose without loss of generality that $\overline{\lambda} \leq \underline{\lambda}$. If $\lambda < \overline{\lambda} \leq \underline{\lambda}$ then it follows from the maximum and minimum principles that $u \equiv 0$ is the only solution of (2.6.38).

If $\lambda < \underline{\lambda}$ and $u \not\equiv 0$ is a solution of (2.6.38), by the minimum principle we have $u > 0$ on $\overline{\Omega}$. Then Proposition 2.6.4 implies $\lambda = \overline{\lambda}$.

Finally suppose that there exists a sequence $\lambda_n \downarrow \underline{\lambda}$ such that the problem (2.6.38) with $\lambda = \lambda_n$ has a solution $\phi_n \not\equiv 0$. We can assume that $|\phi_n|_\infty = 1$ for any n. Then by the compactness criterion, Corollary 2.4.3, the sequence $(\phi_n)_n$ converges uniformly on $\overline{\Omega}$ to a function $\phi \not\equiv 0$ which is a solution of (2.6.38) with $\lambda = \underline{\lambda}$. By Proposition 2.6.4 the functions ϕ_n change sign in Ω while by Proposition 2.6.1 and Theorem 2.2.1 either $\phi > 0$ or $\phi < 0$ on $\overline{\Omega}$. This contradicts the uniform convergence of $(\phi_n)_n$ to ϕ. □

We want to conclude this section with the following comparison, suggested by Hitoshi Ishii [74], between $\overline{\lambda} = \overline{\lambda}_N$ and $\overline{\lambda}_D$ respectively the principal eigenvalues corresponding to the Neumann and the Dirichlet problems.

Proposition 2.6.6. $\overline{\lambda}_N < \overline{\lambda}_D$.

Proof. Let v and w be respectively the eigenfunctions corresponding to $\overline{\lambda}_N$ and $\overline{\lambda}_D$. That is

$$F(x, v, Dv, D^2v) = \overline{\lambda}_N v \text{ in } \Omega, \quad B(x, v, Dv) = 0 \text{ on } \partial\Omega, \quad v > 0 \text{ on } \overline{\Omega},$$

$$F(x, w, Dw, D^2w) = \overline{\lambda}_D w \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega, \quad w > 0 \text{ in } \Omega.$$

Since $f(x, 0) = 0$, we see that w satisfies

$$F(x, w, Dw, D^2w) = \overline{\lambda}_D w \text{ in } \Omega, \quad B(x, w, Dw) \leq 0 \text{ on } \partial\Omega.$$

Let us suppose $\overline{\lambda}_N \geq \overline{\lambda}_D$. Then

$$F(x, w, Dw, D^2w) \leq \overline{\lambda}_N w \text{ in } \Omega, \quad B(x, w, Dw) \leq 0 \text{ on } \partial\Omega.$$

Replacing w by its constant multiple tw with $t > 0$, we may assume that $w \leq v$ on $\overline{\Omega}$ and $w(x_0) = v(x_0)$ for some $x_0 \in \Omega$. Note that $w(x) = 0 < v(x)$ for all $x \in \partial\Omega$. By Theorem 2.2.2 we must have $w \equiv v$ or $w < v$ on $\overline{\Omega}$. This is a contradiction.

□

2.7 The Pucci's operators

In this section we want to show that the two principal eigenvalues of the following operator

$$F(x, u, Du, D^2u) = -\mathcal{M}^+_{a,A}(D^2u) + b(x) \cdot Du + c(x)u,$$

with the pure Neumann boundary condition may be different. Suppose $b \in C^{0,1}(\overline{\Omega})$, $c \in C^{0,\beta}(\overline{\Omega})$ for some $\beta > 0$ and Ω of class $C^{2,\beta}$.

If $c(x) \equiv c_0$ is constant then it is easy to see that $\overline{\lambda} = \underline{\lambda} = c_0$ and by Proposition 2.6.1 the only eigenfunctions are the constants. Nevertheless, if $c(x)$ is not constant the two principal eigenvalues never coincide, unless $\mathcal{M}^+_{a,A}$ is the Laplacian. To prove this we need the following lemma, whose proof is given for the sake of completness.

Lemma 2.7.1. *Suppose that Ω is a $C^{2,\beta}$ domain, $b \in C^{0,\beta}(\overline{\Omega})$ and $c \in C^{0,\beta}(\overline{\Omega})$, for some $0 < \beta \leq 1$. Then the viscosity solutions of*

$$\begin{cases} -\Delta u + b(x) \cdot Du + c(x)u = 0 & \text{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x) \rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.7.39}$$

are in $C^2(\overline{\Omega})$.

Proof. Consider the problem

$$\begin{cases} -\Delta v + b(x) \cdot Dv + v = f(x) & \text{in } \Omega \\ \langle Dv, \overrightarrow{n}(x) \rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7.40}$$

where $f(x) = (1 - c(x))u(x)$. By Theorem 2.4.1, u is Lipschitz continuous on $\overline{\Omega}$ and then the function f is Hölder continuous on $\overline{\Omega}$. Moreover, it is clear that u is a solution of (2.7.40). The classical theory says that (2.7.40) has a solution $v \in C^2(\overline{\Omega})$. By uniqueness of viscosity solutions of (2.7.40), we find that $u = v$. □

Proposition 2.7.2. *Assume the hypothesis of Lemma 2.7.1 and let* $b \in C^{0,1}(\overline{\Omega})$*. If* $A \neq a$ *and* $\overline{\lambda} = \underline{\lambda}$ *then* $c(x)$ *is constant.*

Proof. Let ϕ be a positive eigenfunction of $\overline{\lambda}$, i.e.

$$\begin{cases} -\mathcal{M}^+_{a,A}(D^2\phi) + b(x)\cdot D\phi + (c(x) - \overline{\lambda})\phi = 0 & \text{in} \quad \Omega \\ \langle D\phi, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.7.41}$$

and let $-\psi$ be a negative eigenfunction corresponding to $\underline{\lambda}$. Since $\mathcal{M}^+_{a,A}(-D^2\psi) = -\mathcal{M}^-_{a,A}(D^2\psi)$, ψ satisfies

$$\begin{cases} -\mathcal{M}^-_{a,A}(D^2\psi) + b(x)\cdot D\psi + (c(x) - \underline{\lambda})\psi = 0 & \text{in} \quad \Omega \\ \langle D\psi, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.7.42}$$

If $\overline{\lambda} = \underline{\lambda}$ then by Proposition 2.6.1 $\psi = t\phi$ for some $t > 0$. We can assume $\psi = \phi$. By summing the first equations in (2.7.41) and (2.7.42), we can see that ϕ is a positive viscosity solution of

$$\begin{cases} -(A + a)\Delta\phi + 2b(x)\cdot D\phi + 2(c(x) - \overline{\lambda})\phi = 0 & \text{in} \quad \Omega \\ \langle D\phi, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then by Lemma 2.7.1, $\phi \in C^2(\overline{\Omega})$. Subtracting the first equations in (2.7.41) and (2.7.42), we can see that ϕ is a classical solution of

$$(A - a)\sum_{i=1}^{N} |e_i(x)| = 0 \quad \text{in} \quad \Omega,$$

where $e_1(x), ..., e_N(x)$ are the eigenvalues of $D^2\phi(x)$. Since $A \neq a$, the last equation implies that $e_i(x) = 0$ in Ω for any $i = 1...N$. In particular, taking into consideration the boundary condition, ϕ is a classical solution of

$$\begin{cases} \Delta\phi = 0 & \text{in} \quad \Omega \\ \langle D\phi, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and then has to be constant. This implies

$$c(x) - \overline{\lambda} = 0 \quad \text{in } \Omega,$$

i.e., $c \equiv \overline{\lambda}$ is constant. □

Chapter 3

The infinity-Laplacian

In this chapter we study the maximum principle, the principal eigenvalue, regularity, existence and uniqueness for viscosity solutions of the Neumann boundary value problem

$$\begin{cases} \Delta_\infty u + b(x) \cdot Du + (c(x) + \lambda)u = g(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.0.1}$$

where Ω is a bounded smooth domain, $\overrightarrow{n}(x)$ is the exterior normal to the domain Ω at x, b, c and g are continuous functions on $\overline{\Omega}$, $\lambda \in \mathbb{R}$ and

$$\Delta_\infty u = \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \rangle, \tag{3.0.2}$$

for $u \in C^2(\Omega)$, is the 1-homogeneous version of the ∞-Laplacian.

We define and investigate the properties of the principal eigenvalue of the operator

$$-(\Delta_\infty + b(x) \cdot D + c(x)),$$

with the Neumann boundary condition and as an application, we get existence and uniqueness results for (3.0.1) and a decay estimate for the solution of the associated evolution problem.

Following the ideas of [26], we define the principal eigenvalue as

$$\begin{aligned} \overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of} \\ \Delta_\infty v + b(x) \cdot Dv + (c(x) + \lambda)v = 0 \text{ in } \Omega,\ \frac{\partial v}{\partial \overrightarrow{n}} = 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{3.0.3}$$

The quantity $\overline{\lambda}$ is well defined since the above set is not empty; indeed, $-|c|_\infty$ belongs to it, being $v(x) \equiv 1$ a corresponding supersolution. Furthermore it is an interval because if λ belongs to it then so does any $\lambda' < \lambda$.

We will prove that $\overline{\lambda}$ is an "eigenvalue" for $-(\Delta_\infty + b(x) \cdot D + c(x))$ which admits a positive "eigenfunction". As in the linear case it can be characterized as the supremum of those λ for which $\Delta_\infty + b(x) \cdot D + c(x) + \lambda$ with the Neumann boundary condition satisfies the maximum principle. As a consequence, $\overline{\lambda}$ is the least "eigenvalue", i.e., the least number λ for which there exists a non-zero solution of

$$\begin{cases} \Delta_\infty u + b(x)\cdot Du + (c(x)+\lambda)u = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

These results are applied to obtain existence and uniqueness for the boundary value problem (3.0.1).

Remark that since $\Delta_\infty(-u) = -\Delta_\infty u$, $\overline{\lambda}$ can be defined also in the following way

$$\begin{aligned} \overline{\lambda} = \sup\{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ on } \overline{\Omega} \text{ bounded viscosity subsolution of} \\ \Delta_\infty u + b(x)\cdot Du + (c(x)+\lambda)u = 0 \text{ in } \Omega,\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{3.0.4}$$

For a fully nonlinear operator, $\overline{\lambda}$ defined as in (3.0.3) may be different from the quantity defined as in (3.0.4), see Chapter 2.

In the next section we give assumptions. In Section 3.2 we establish a Lipschitz regularity result for viscosity solutions of (3.0.1). Section 3.3 is devoted to the maximum principle for subsolutions of (3.0.1). In Section 3.3.1 we show that it holds (even for more general boundary conditions) for $\Delta_\infty + b(x)\cdot D + c(x)$ if $c(x) \le 0$ and $c \not\equiv 0$, see Theorem 3.3.4. One of the main result of this chapter is that the maximum principle holds for $\Delta_\infty + b(x)\cdot D + c(x) + \lambda$ for any $\lambda < \overline{\lambda}$, as we show in Theorem 3.3.8 of Section 3.3.2. In particular it holds for $\Delta_\infty + b(x)\cdot D + c(x)$ if $\overline{\lambda} > 0$. Following the example given in [97] we show that the result of Theorem 3.3.8 is stronger than that of Theorem 3.3.4, i.e., that there exist some functions $c(x)$ changing sign in Ω for which the principal eigenvalue of $\Delta_\infty + b(x)\cdot D + c(x)$ is positive and then for which the maximum principle holds.

In Section 3.4 we show some existence and comparison theorems. In particular, we prove that the Neumann problem (3.0.1) is solvable for any right-hand side if $\lambda < \overline{\lambda}$.

Finally, in Section 3.5 we prove a decay estimate for solutions of the Neumann evolution problem.

3.1 Assumptions and definitions

Let $\sigma : \mathbb{R}^N \to S(N)$ be the function defined by

$$\sigma(p) := \frac{p \otimes p}{|p|^2}.$$

The ∞-Laplacian can be written as

$$\Delta_\infty u = \operatorname{tr}(\sigma(Du)D^2u),$$

for any $u \in C^2(\Omega)$.

It easy to check that σ has the following properties:

- $\sigma(p)$ is homogeneous of order 0, i.e., for any $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}^N$

$$\sigma(\alpha p) = \sigma(p);$$

- For all $p \in \mathbb{R}^N$
$$0 \le \sigma(p) \le I,$$
where I is the identity matrix in $\mathbb{R}^N$;

- $\sigma(p)$ is idempotent, i.e.,
$$(\sigma(p))^2 = \sigma(p);$$

- For any $p \in \mathbb{R}^N \setminus \{0\}$ and $p_0 \in \mathbb{R}^n$ with $|p_0| \le \frac{|p|}{2}$
$$\operatorname{tr}\left[(\sigma(p+p_0) - \sigma(p))^2\right] \le 8\frac{|p_0|^2}{|p|^2}. \tag{3.1.5}$$

The domain Ω is supposed to be bounded and of class C^2. In particular, it satisfies the interior sphere condition and the uniform exterior sphere condition, i.e.

(Ω1) For each $x \in \partial\Omega$ there exist $R > 0$ and $y \in \Omega$ for which $|x - y| = R$ and $B(y, R) \subset \Omega$.

(Ω2) There exists $r > 0$ such that $B(x + r\overrightarrow{n}(x), r) \cap \Omega = \emptyset$ for any $x \in \partial\Omega$.

From the property (Ω2) it follows that

$$\langle y - x, \overrightarrow{n}(x)\rangle \le \frac{1}{2r}|y - x|^2 \quad \text{for } x \in \partial\Omega \text{ and } y \in \overline{\Omega}. \tag{3.1.6}$$

Moreover, the C^2-regularity of Ω implies the existence of a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ on which the distance from the boundary

$$d(x) := \inf\{|x - y|, y \in \partial\Omega\}, \quad x \in \overline{\Omega}$$

is of class C^2. We still denote by d a C^2 extension of the distance function to the whole $\overline{\Omega}$. Without loss of generality we can assume that $|Dd(x)| \le 1$ on $\overline{\Omega}$.

For (3.0.1) we adopt the notion of viscosity solution Definition I.2.2 given in the Introduction.

It is possible to define sub and supersolutions of the ∞-Laplace equation also using the semicontinous extensions of the function $(p, X) \to \operatorname{tr}(\sigma(p)X)$ as done in [80] and [81]. In Definition I.2.2 it is remarkable that nothing is required in the case $D\varphi(x_0) = 0$ if u is not constant.

We call strong viscosity subsolutions (resp., supersolutions) the viscosity subsolutions (resp., supersolutions) that satisfy $B(x, u, Du) \le$ (resp., $\ge$) 0 in the viscosity sense for all $x \in \partial\Omega$. If $\lambda \to B(x, r, p - \lambda\overrightarrow{n})$ is non-increasing in $\lambda \ge 0$, then classical subsolutions (resp., supersolutions) are strong viscosity subsolutions (resp., supersolutions), see [40] Proposition 7.2.

In the definitions the test functions can be substituted by the elements of the semijets $\overline{J}^{2,+}u(x_0)$ when u is a subsolution and $\overline{J}^{2,-}u(x_0)$ when u is a supersolution, see [40].

3.2 Lipschitz continuity of viscosity solutions

It is known that the ∞-harmonic functions, i.e., the solution of $\Delta_\infty u = 0$ are locally Lipschitz continuous, see e.g. [11]. We now show the Lipschitz regularity in the whole $\overline{\Omega}$ of the solutions of the Neumann problem associated to the ∞-Laplacian plus lower order terms.

Theorem 3.2.1. *Assume that Ω is a bounded domain of class C^2 and that b, c, g are bounded in Ω. If $u \in C(\overline{\Omega})$ is a viscosity solution of*

$$\begin{cases} \Delta_\infty u + b(x)\cdot Du + c(x)u = g(x) & in \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & on \quad \partial\Omega, \end{cases}$$

then

$$|u(x) - u(y)| \leq C_0|x-y| \quad \forall x, y \in \overline{\Omega},$$

where C_0 depends on Ω, N, $|b|_\infty$, $|c|_\infty$, $|g|_\infty$, and $|u|_\infty$.

The Theorem is an immediate consequence of the next lemma, the proof of which, though following the line of Proposition III.1 of [75], introduces new test functions that, in particular, depend on the distance function $d(x)$.

The lemma will be used also in the proof of Theorem 3.3.8 in the next section.

Lemma 3.2.2. *Assume the hypothesis of Theorem 3.2.1 and suppose that g and h are bounded functions. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} \Delta_\infty u + b(x)\cdot Du + c(x)u = g(x) & in \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & on \quad \partial\Omega, \end{cases}$$

and $v \in LSC(\overline{\Omega})$ a viscosity supersolution of

$$\begin{cases} \Delta_\infty v + b(x)\cdot Dv + c(x)v = h(x) & in \quad \Omega \\ \frac{\partial v}{\partial \overrightarrow{n}} = 0 & on \quad \partial\Omega, \end{cases}$$

with u and v bounded, or $v \geq 0$ and bounded. If $m = \max_{\overline{\Omega}}(u-v) \geq 0$, then there exists $C_0 > 0$ such that

$$u(x) - v(y) \leq m + C_0|x-y| \quad \forall x, y \in \overline{\Omega}, \tag{3.2.7}$$

where C_0 depends on Ω, N, $|b|_\infty$, $|c|_\infty$, $|g|_\infty$, $|h|_\infty$, $|v|_\infty$, m and $|u|_\infty$ or $\sup_{\overline{\Omega}} u$.

Proof. We set

$$\Phi(x) = MK|x| - M(K|x|)^2,$$

and

$$\varphi(x,y) = m + e^{-L(d(x)+d(y))}\Phi(x-y),$$

where L is a fixed number greater than $2/(3r)$ with r the radius in the condition (Ω2) and where K and M are two positive constants to be chosen later. If $K|x| \leq \frac{1}{4}$, then

$$\Phi(x) \geq \frac{3}{4}MK|x|. \tag{3.2.8}$$

We define

$$\Delta_K := \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N |\, |x - y| \leq \frac{1}{4K}\right\}.$$

We fix M such that

$$\max_{\overline{\Omega}^2}(u(x) - v(y)) \leq m + e^{-2Ld_0}\frac{M}{8}, \tag{3.2.9}$$

where $d_0 = \max_{x\in\overline{\Omega}} d(x)$. To prove (3.2.7) it is enough to show that taking K large enough, one has

$$u(x) - v(y) - \varphi(x, y) \leq 0 \quad \text{for } (x, y) \in \Delta_K \cap \overline{\Omega}^2.$$

Suppose by contradiction that for each K there is some point $(\overline{x}, \overline{y}) \in \Delta_K \cap \overline{\Omega}^2$ such that

$$u(\overline{x}) - v(\overline{y}) - \varphi(\overline{x}, \overline{y}) = \max_{\Delta_K\cap\overline{\Omega}^2}(u(x) - v(y) - \varphi(x, y)) > 0.$$

Here we have dropped the dependence of $\overline{x}$, $\overline{y}$ on K for simplicity of notations.

Observe that if $v \geq 0$, since from (3.2.8) $\Phi(x - y)$ is non-negative in Δ_K and $m \geq 0$, one has $u(\overline{x}) > 0$.

Clearly $\overline{x} \neq \overline{y}$. Moreover the point $(\overline{x}, \overline{y})$ belongs to $\text{int}(\Delta_K) \cap \overline{\Omega}^2$. Indeed, if $|x - y| = \frac{1}{4K}$, by (3.2.9) and (3.2.8) we have

$$u(x) - v(y) \leq m + e^{-2Ld_0}\frac{M}{8} \leq m + e^{-L(d(x)+d(y))}\frac{1}{2}MK|x - y| \leq \varphi(x, y).$$

Since $\overline{x} \neq \overline{y}$ we can compute the derivatives of φ at $(\overline{x}, \overline{y})$ obtaining

$$\begin{aligned} D_x\varphi(\overline{x}, \overline{y}) = e^{-L(d(\overline{x})+d(\overline{y}))}MK\Big\{ &- L|\overline{x} - \overline{y}|(1 - K|\overline{x} - \overline{y}|)Dd(\overline{x}) \\ &+ (1 - 2K|\overline{x} - \overline{y}|)\frac{(\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|}\Big\}, \end{aligned}$$

$$\begin{aligned} D_y\varphi(\overline{x}, \overline{y}) = e^{-L(d(\overline{x})+d(\overline{y}))}MK\Big\{ &- L|\overline{x} - \overline{y}|(1 - K|\overline{x} - \overline{y}|)Dd(\overline{y}) \\ &- (1 - 2K|\overline{x} - \overline{y}|)\frac{(\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|}\Big\}. \end{aligned}$$

Observe that for large K

$$0 < e^{-L(d(\overline{x})+d(\overline{y}))}MK\left(\frac{1}{2} - L|\overline{x} - \overline{y}|\right) \leq |D_x\varphi(\overline{x}, \overline{y})|, |D_y\varphi(\overline{x}, \overline{y})| \leq 2MK. \tag{3.2.10}$$

Using (3.1.6), if $\overline{x} \in \partial\Omega$ we have

$$\begin{aligned} &\langle D_x\varphi(\overline{x}, \overline{y}), \overrightarrow{n}(\overline{x})\rangle \\ &= e^{-Ld(\overline{y})}MK\Big\{L|\overline{x} - \overline{y}|(1 - K|\overline{x} - \overline{y}|) + (1 - 2K|\overline{x} - \overline{y}|)\langle\frac{(\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|}, \overrightarrow{n}(\overline{x})\rangle\Big\} \\ &\geq e^{-Ld(\overline{y})}MK\Big\{\frac{3}{4}L|\overline{x} - \overline{y}| - (1 - 2K|\overline{x} - \overline{y}|)\frac{|\overline{x} - \overline{y}|}{2r}\Big\} \\ &\geq \frac{1}{2}e^{-Ld(\overline{y})}MK|\overline{x} - \overline{y}|\left(\frac{3}{2}L - \frac{1}{r}\right) > 0, \end{aligned}$$

since $\overline{x} \neq \overline{y}$ and $L > 2/(3r)$. Similarly, if $\overline{y} \in \partial\Omega$

$$\langle -D_y\varphi(\overline{x},\overline{y}), \overrightarrow{n}(\overline{y})\rangle \leq \frac{1}{2}e^{-Ld(\overline{x})}MK|\overline{x}-\overline{y}|\left(-\frac{3}{2}L+\frac{1}{r}\right) < 0.$$

In view of definition of sub and supersolution, we conclude that

$$\mathrm{tr}(\sigma(D_x\varphi(\overline{x},\overline{y}))X)+b(\overline{x})\cdot D_x\varphi(\overline{x},\overline{y})+c(\overline{x})u(\overline{x}) \geq g(\overline{x}) \text{ if } (D_x\varphi(\overline{x},\overline{y}),X) \in \overline{J}^{2,+}u(\overline{x}),$$

$$\mathrm{tr}(\sigma(D_y\varphi(\overline{x},\overline{y}))Y)-b(\overline{y})\cdot D_y\varphi(\overline{x},\overline{y})+c(\overline{y})v(\overline{y}) \leq h(\overline{y}) \text{ if } (-D_y\varphi(\overline{x},\overline{y}),Y) \in \overline{J}^{2,-}v(\overline{y}).$$

Then the previous inequalities holds for any maximum point $(\overline{x},\overline{y}) \in \Delta_K \cap \overline{\Omega}^2$, provided K is large enough.

Since $(\overline{x},\overline{y}) \in \mathrm{int}\Delta_K \cap \overline{\Omega}^2$, it is a local maximum of $u(x)-v(y)-\varphi(x,y)$ in $\overline{\Omega}^2$. Applying Theorem 3.2 in [40], for every $\epsilon > 0$ there exist $X, Y \in$ *S(N)* such that $(D_x\varphi(\overline{x},\overline{y}),X) \in \overline{J}^{2,+}u(\overline{x})$, $(-D_y\varphi(\overline{x},\overline{y}),Y) \in \overline{J}^{2,-}v(\overline{y})$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2(\varphi(\overline{x},\overline{y})) + \epsilon(D^2(\varphi(\overline{x},\overline{y})))^2. \tag{3.2.11}$$

Now we want to estimate the matrix on the right-hand side of the last inequality.

$$\begin{aligned} D^2\varphi(\overline{x},\overline{y}) &= \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}) + D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) \\ &\quad + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}) + e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})). \end{aligned}$$

We set

$$A_1 := \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_2 := D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_3 := e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).$$

Observe that

$$A_1 \leq CK|\overline{x}-\overline{y}|\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.2.12}$$

Here and henceforth C denotes various positive constants independent of K.

For A_2 we have the following estimate

$$A_2 \leq CK\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + CK\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.2.13}$$

Indeed for ξ, $\eta \in \mathbb{R}^N$ we compute

$$\begin{aligned} \langle A_2(\xi,\eta),(\xi,\eta)\rangle &= 2Le^{-L(d(\overline{x})+d(\overline{y}))}\{\langle Dd(\overline{x}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\xi\rangle \\ &\quad + \langle Dd(\overline{y}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\eta\rangle\} \leq CK(|\xi|+|\eta|)|\eta-\xi| \\ &\quad \leq CK(|\xi|^2+|\eta|^2) + CK|\eta-\xi|^2. \end{aligned}$$

Now we consider A_3. The matrix $D^2(\Phi(\overline{x}-\overline{y}))$ has the form

$$D^2(\Phi(\overline{x}-\overline{y})) = \begin{pmatrix} D^2\Phi(\overline{x}-\overline{y}) & -D^2\Phi(\overline{x}-\overline{y}) \\ -D^2\Phi(\overline{x}-\overline{y}) & D^2\Phi(\overline{x}-\overline{y}) \end{pmatrix},$$

and the Hessian matrix of $\Phi(x)$ is

$$D^2\Phi(x) = \frac{MK}{|x|}\left(I - \frac{x\otimes x}{|x|^2}\right) - 2MK^2 I. \tag{3.2.14}$$

If we choose

$$\epsilon = \frac{|\overline{x}-\overline{y}|}{2MKe^{-L(d(\overline{x})+d(\overline{y}))}}, \tag{3.2.15}$$

then we have the following estimates

$$\begin{gathered}\epsilon A_1^2 \le CK|\overline{x}-\overline{y}|^3 I_{2N}, \quad \epsilon A_2^2 \le CK|\overline{x}-\overline{y}| I_{2N},\\ \epsilon(A_1A_2 + A_2A_1) \le CK|\overline{x}-\overline{y}|^2 I_{2N}, \\ \epsilon(A_1A_3 + A_3A_1) \le CK|\overline{x}-\overline{y}| I_{2N}, \quad \epsilon(A_2A_3 + A_3A_2) \le CKI_{2N},\end{gathered} \tag{3.2.16}$$

where $I_{2N} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then using (**??**), (3.2.13), (3.2.16) and observing that

$$(D^2(\Phi(\overline{x}-\overline{y})))^2 = \begin{pmatrix} 2(D^2\Phi(\overline{x}-\overline{y}))^2 & -2(D^2\Phi(\overline{x}-\overline{y}))^2 \\ -2(D^2\Phi(\overline{x}-\overline{y}))^2 & 2(D^2\Phi(\overline{x}-\overline{y}))^2 \end{pmatrix},$$

from (**??**) we conclude that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le O(K)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where

$$B = CKI + e^{-L(d(\overline{x})+d(\overline{y}))}\left[D^2\Phi(\overline{x}-\overline{y}) + \frac{|\overline{x}-\overline{y}|}{MK}(D^2\Phi(\overline{x}-\overline{y}))^2\right].$$

The last inequality can be rewritten as follows

$$\begin{pmatrix} \widetilde{X} & 0 \\ 0 & -\widetilde{Y} \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with $\widetilde{X} = X - O(K)I$ and $\widetilde{Y} = Y + O(K)I$. Multiplying on the left the previous inequality by the non-negative symmetric matrix

$$\begin{pmatrix} \sigma(D_x\varphi(\overline{x},\overline{y})) & 0 \\ 0 & \sigma(D_y\varphi(\overline{x},\overline{y})) \end{pmatrix},$$

and taking traces we get

$$\mathrm{tr}(\sigma(D_x\varphi(\overline{x},\overline{y}))\widetilde{X}) - \mathrm{tr}(\sigma(D_y\varphi(\overline{x},\overline{y}))\widetilde{Y}) \le \mathrm{tr}(\sigma(D_x\varphi(\overline{x},\overline{y}))B) + \mathrm{tr}(\sigma(D_y\varphi(\overline{x},\overline{y}))B). \tag{3.2.17}$$

We want to get a good estimate for the matrix on the right-hand side above. For that aim let

$$0 \le P := \frac{(\overline{x}-\overline{y})\otimes(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^2} \le I,$$

and let us compute $\mathrm{tr}(PB)$. From (3.2.14), since the matrix $(1/|x|^2)x\otimes x$ is idempotent, we get

$$(D^2\Phi(x))^2 = \frac{M^2K^2}{|x|^2}(1-4K|x|)\left(I-\frac{x\otimes x}{|x|^2}\right)+4M^2K^4I.$$

Then, using that $\mathrm{tr}P=1$ and $4K|\overline{x}-\overline{y}|\le 1$, we have

$$\begin{aligned}\mathrm{tr}(PB) &= CK + e^{-L(d(\overline{x})+d(\overline{y}))}(-2MK^2+4MK^3|\overline{x}-\overline{y}|)\\ &\le CK - e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 \le -CK^2,\end{aligned}$$

for large K. The vector $D_x\varphi(\overline{x},\overline{y})$ can be written in the following way

$$D_x\varphi(\overline{x},\overline{y}) = e^{-L(d(\overline{x})+d(\overline{y}))}MK(v_1+v_2),$$

where

$$v_1 = -L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{x}),\quad v_2=(1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|},$$

and so

$$\sigma(D_x\varphi(\overline{x},\overline{y})) = \frac{v_1\otimes v_1}{|v_1+v_2|^2}+\frac{v_1\otimes v_2+v_2\otimes v_1}{|v_1+v_2|^2}+\frac{v_2\otimes v_2}{|v_1+v_2|^2}.$$

Since $K|\overline{x}-\overline{y}|\le\frac{1}{4}$, for large K we have

$$\frac{1}{4}=\frac{1}{2}-\frac{1}{4}\le|v_2|-|v_1|\le|v_1+v_2|\le|v_1|+|v_2|\le 2,$$

and

$$\|B\|\le\frac{CK}{|\overline{x}-\overline{y}|}.$$

Then

$$\begin{aligned}\left|\mathrm{tr}\left(\frac{v_1\otimes v_1}{|v_1+v_2|^2}B\right)\right| &\le C|\overline{x}-\overline{y}|^2\|B\|\le CK|\overline{x}-\overline{y}|,\\ \left|\mathrm{tr}\left(\frac{v_1\otimes v_2+v_2\otimes v_1}{|v_1+v_2|^2}B\right)\right| &\le C|\overline{x}-\overline{y}|\|B\|\le CK\end{aligned}$$

and

$$\mathrm{tr}\left(\frac{v_2\otimes v_2}{|v_1+v_2|^2}B\right)=\frac{1}{|v_1+v_2|^2}\mathrm{tr}(PB)\le -CK^2.$$

In conclusion

$$\mathrm{tr}(\sigma(D_x\varphi(\overline{x},\overline{y})B))\le O(K)-CK^2.$$

The same estimate holds for $\mathrm{tr}(\sigma(D_y\varphi(\overline{x},\overline{y}))B)$. Hence, from (3.2.17) we conclude that

$$\mathrm{tr}(\sigma(D_x\varphi(\overline{x},\overline{y})\widetilde{X})-\mathrm{tr}(\sigma(D_y\varphi(\overline{x},\overline{y})\widetilde{Y})\le O(K)-CK^2.$$

Now, using the previous estimate, the definition of $\widetilde{X}$ and $\widetilde{Y}$ and the fact that u and v are respectively sub and supersolution we compute

$$\begin{aligned}g(\overline{x})-c(\overline{x})u(\overline{x}) &\le \mathrm{tr}(\sigma(D_x\varphi)X)+b(\overline{x})\cdot D_x\varphi\\ &\le \mathrm{tr}(\sigma(D_x\varphi)\widetilde{X})+O(K)+b(\overline{x})\cdot D_x\varphi\\ &\le \mathrm{tr}(\sigma(D_y\varphi)Y)+O(K)-CK^2+b(\overline{x})\cdot D_x\varphi\\ &\le b(\overline{y})\cdot D_y\varphi-c(\overline{y})v(\overline{y})+h(\overline{y})+O(K)-CK^2+b(\overline{x})\cdot D_x\varphi.\end{aligned}$$

From this inequalities, using (3.2.10) we get

$$g(\overline{x}) - h(\overline{y}) - c(\overline{x})u(\overline{x}) + c(\overline{y})v(\overline{y}) \leq O(K) - CK^2.$$

If both u and v are bounded, then the member on the left-hand side of the last inequality is bounded from below by $-|g|_\infty - |h|_\infty - |c|_\infty(|u|_\infty + |v|_\infty)$. Otherwise, if v is non-negative and bounded, then $u(\overline{x}) \geq 0$ and that quantity is greater than $-|g|_\infty - |h|_\infty - |c|_\infty(\sup u + |v|_\infty)$. On the other hand, the member on the right-hand side goes to $-\infty$ as $K \to +\infty$, hence taking K large enough we obtain a contradiction and this concludes the proof. □

Remark 3.2.3. If u is a subsolution of $\Delta_\infty u + b(x) \cdot Du + c(x)u = g$, v is a supersolution of $\Delta_\infty v + b(x) \cdot Dv + c(x)v = h$ in Ω, $u \leq v$ on $\partial\Omega$ and $m > 0$ then the estimate (3.2.7) still holds for any $x, y \in \Omega$. To prove this define $\varphi = m + MK|x| - M(K|x|)^2$ and follow the proof of Lemma 3.2.2.

Since the Lipschitz estimate depends only on the bounds of the solution of g and on the structural constants, an immediate consequence of Theorem 3.2.1 is the following compactness criterion that will be useful in the next sections.

Corollary 3.2.4. *Assume the hypothesis of Theorem 3.2.1 on Ω, F and b. Suppose that $(g_n)_n$ is a sequence of continuous and uniformly bounded functions and $(u_n)_n$ is a sequence of uniformly bounded viscosity solutions of*

$$\begin{cases} \Delta_\infty u_n + b(x) \cdot Du_n = g_n(x) & \text{in} \quad \Omega \\ \frac{\partial u_n}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then the sequence $(u_n)_n$ is relatively compact in $C(\overline{\Omega})$.

3.3 The Maximum Principle and the principal eigenvalues

We say that the operator $\Delta_\infty + b(x) \cdot D + c(x)$ with the Neumann boundary condition satisfies the maximum principle if whenever $u \in USC(\overline{\Omega})$ is a viscosity subsolution of

$$\begin{cases} \Delta_\infty u + b(x) \cdot Du + c(x)u = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u \leq 0$ on $\overline{\Omega}$.

We first prove that the maximum principle holds under the classical assumption $c \leq 0$, also for domain which are not of class C^2 and with more general boundary conditions. Then we show that the operator $\Delta_\infty + b(x) \cdot D + c(x) + \lambda$ with the Neumann boundary condition satisfies the maximum principle for any $\lambda < \overline{\lambda}$. This is the best result that one can expect, indeed, as we will see, $\overline{\lambda}$ admits a positive eigenfunction which provides a counterexample to the maximum principle for $\lambda \geq \overline{\lambda}$.

Finally, we give an example of class of functions $c(x)$ which change sign in Ω and such that the associated principal eigenvalue $\overline{\lambda}$ is positive.

3.3.1 The case $c(x) \leq 0$

In this subsection we assume that Ω is of class C^1 and satisfies the interior sphere condition ($\Omega 1$). We need the comparison principle between sub and supersolutions of the Dirichlet problem when $c < 0$ in Ω. This result is known for the operator $\Delta_\infty u + b(x)\cdot Du + c(x)u$ when b is Lipschitz continuous or b satisfies $\langle b(x) - b(y), x - y\rangle \leq 0$, see e.g. [40]. Actually, we can remove these conditions.

Theorem 3.3.1. *Let Ω be bounded. Assume that b, c and g are continuous and bounded in Ω and $c < 0$ on $\overline{\Omega}$. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are respectively sub and supersolution of*

$$\Delta_\infty u + b(x)\cdot Du + c(x)u = g(x) \quad \text{in } \Omega,$$

and $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.

For convenience of the reader the proof of the theorem will be sketched at the end of the next subsection.

The previous comparison result allows us to establish the strong minimum and maximum principles, for sub and supersolutions of the Neumann problem even with the following more general boundary condition

$$f(x, u) + \frac{\partial u}{\partial \overrightarrow{n}} = 0 \quad x \in \partial\Omega,$$

for some $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

Proposition 3.3.2. *Let Ω be a C^1 domain satisfying ($\Omega 1$). Suppose that b and c are bounded and continuous in Ω and that $f(x, 0) \leq 0$ for all $x \in \partial\Omega$. If $v \in LSC(\overline{\Omega})$ is a non-negative viscosity supersolution of*

$$\begin{cases} \Delta_\infty v + b(x)\cdot Dv + c(x)v = 0 & \text{in } \quad \Omega \\ f(x, v) + \frac{\partial v}{\partial \overrightarrow{n}} = 0 & \text{on } \quad \partial\Omega, \end{cases} \tag{3.3.18}$$

then either $v \equiv 0$ or $v > 0$ on $\overline{\Omega}$.

Proof. Since v is non-negative, it is supersolution in Ω of the equation

$$\Delta_\infty v + b(x)\cdot Dv - |c|_\infty v = 0. \tag{3.3.19}$$

Without loss of generality we can assume $|c|_\infty > 0$. Suppose by contradiction that $v \not\equiv 0$ vanishes somewhere in Ω. Then we can find $x_1, x_0 \in \Omega$ and $R > 0$ such that $B(x_1, \frac{3}{2}R) \subset \Omega$, $v > 0$ in $B(x_1, R)$, $|x_1 - x_0| = R$ and $v(x_0) = 0$. Let us construct a subsolution of (3.3.19) in the annulus $\frac{R}{2} < |x - x_1| = r < \frac{3}{2}R$.

Let us consider the function $\phi(x) = e^{-kr} - e^{-kR}$, where k is a positive constant to be determined. It easy to see that for radial functions $g(x) = \varphi(r)$, $\Delta_\infty g(x) = \varphi''(r)$. Then

$$\begin{aligned} \Delta_\infty \phi + b(x)\cdot D\phi - |c|_\infty \phi &= k^2 e^{-kr} - k e^{-kr} b(x)\cdot \frac{(x-y)}{r} - |c|_\infty (e^{-kr} - e^{-kR}) \\ &\geq e^{-kr}\left(k^2 - |b|_\infty k - |c|_\infty\right). \end{aligned}$$

Take k such that
$$k^2 - |b|_\infty k - |c|_\infty > 0,$$
then ϕ is a strict subsolution of the equation (3.3.19). Now choose $m > 0$ such that
$$m(e^{-k\frac{R}{2}} - e^{-kR}) = v_1 := \inf_{|x-x_1|=\frac{R}{2}} v(x) > 0,$$
and define $w(x) = m(e^{-kr} - e^{-kR})$. By homogeneity w is still a subsolution of (3.3.19) in the annulus $\frac{R}{2} < |x - x_1| < \frac{3}{2}R$, moreover $w = v_1 \le v$ if $|x - x_1| = \frac{R}{2}$ and $w < 0 \le v$ if $|x - x_1| = \frac{3}{2}R$. Then by the comparison principle, Theorem 3.3.1, $w \le v$ in the entire annulus.

Since $v(x_0) = w(x_0) = 0$, w is a test function for v at x_0 with $Dw(x_0) \neq 0$. But
$$\Delta_\infty w(x_0) + b(x_0) \cdot Dw(x_0) - |c|_\infty v(x_0) > 0,$$
and this contradicts the definition of v. Then $v > 0$ in Ω.

Now suppose by contradiction that x_0 is some point in $\partial\Omega$ on which $v(x_0) = 0$. The interior sphere condition (Ω1) implies that there exist $R > 0$ and $y \in \Omega$ such that the ball centered in y and of radius R, $B(y, R)$, is contained in Ω and $x_0 \in \partial B(y, R)$. Fixed $0 < \rho < R$, as before the function $w(x) = m(e^{-kr} - e^{-kR})$ is a strict subsolution of (3.3.19) in the annulus $\rho < |x - y| = r < R$, where m is such that $m(e^{-k\rho} - e^{-kR}) = v_1 := \inf_{|x-y|=\rho} v(x) > 0$. Since $w \le v$ on the boundary of the annulus then again by the comparison principle, Theorem 3.3.1, $w \le v$ in the entire annulus.

Now let δ be a positive number smaller than $R - \rho$. In $B(x_0, \delta) \cap \overline{\Omega}$ still $w \le v$, since for $|x - y| > R$, $w < 0 \le v$; moreover $w(x_0) = v(x_0) = 0$. Then w is a test function for v at x_0. But
$$\Delta_\infty w(x_0) + b(x_0) \cdot Dw(x_0) - |c|_\infty v(x_0) > 0,$$
and
$$f(x_0, v(x_0)) + \frac{\partial w}{\partial \vec{n}}(x_0) = f(x_0, 0) - kme^{-kR} < 0.$$
This contradicts the definition of v. Finally v cannot be zero on $\overline{\Omega}$. □

Similarly we can prove

Proposition 3.3.3. *Let Ω be a C^1 domain satisfying (Ω1). Assume that b and c are bounded and continuous in Ω and that $f(x, 0) \ge 0$ for all $x \in \partial\Omega$. If $u \in USC(\overline{\Omega})$ is a non-positive viscosity subsolution of* (3.3.18) *then either $u \equiv 0$ or $u < 0$ on $\overline{\Omega}$.*

For $x \in \partial\Omega$, let us introduce $S(x)$, the symmetric operator corresponding to the second fundamental form of $\partial\Omega$ in x oriented with the exterior normal to Ω.

Theorem 3.3.4 (Maximum Principle for $c \le 0$)**.** *Assume the hypothesis of Proposition 3.3.3. In addition suppose that Ω is bounded, $c \le 0$, $c \not\equiv 0$ and $r \to f(x, r)$ is non-decreasing on $\mathbb{R}$. If $u \in USC(\overline{\Omega})$ is a viscosity subsolution of* (3.3.18) *then $u \le 0$ on $\overline{\Omega}$. The same conclusion holds also if $c \equiv 0$ in the following two cases*

(i) Ω is a C^2 domain and for any $r > 0$ there exists $\overline{x} \in \partial\Omega$ such that $f(\overline{x}, r) > 0$, $S(\overline{x}) \le 0$ and $\langle b(\overline{x}), \vec{n}(\overline{x})\rangle > 0$;

(ii) $\max_{x\in\partial\Omega} f(x,r) > 0$ *for any* $r > 0$ *and* u *is a strong subsolution.*

Proof. Let u be a subsolution of (3.3.18) and $c \not\equiv 0$. First let us suppose $u \equiv k$ =const. By definition

$$c(x)k \geq 0 \qquad \text{in } \Omega,$$

which implies $k \leq 0$.

Now we assume that u is not a constant. We argue by contradiction; suppose that $\max_{\overline{\Omega}} u = u(x_0) > 0$, for some $x_0 \in \overline{\Omega}$. Define $\widetilde{u}(x) := u(x) - u(x_0)$. Since $c \leq 0$ and f is non-decreasing, $\widetilde{u}$ is a non-positive subsolution of (3.3.18). Then, from Proposition 3.3.3, either $u \equiv u(x_0)$ or $u < u(x_0)$ on $\overline{\Omega}$. In both cases we get a contradiction.

Let us turn to the case $c \equiv 0$. We have to prove that u cannot be a positive constant. Suppose by contradiction that $u \equiv k$. Suppose that Ω is a C^2 domain and let $\overline{x} \in \partial\Omega$ be such that $S(\overline{x}) \leq 0$, $\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle > 0$ and $f(\overline{x}, k) > 0$. In general, if ϕ is a C^2 function, $\overline{x} \in \partial\Omega$ and $S(\overline{x}) \leq 0$, then $(D\phi(\overline{x}) - \lambda\overrightarrow{n}(\overline{x}), D^2\phi(\overline{x})) \in J^{2,+}\phi(\overline{x})$, for $\lambda \geq 0$ (see [40] Remark 2.7). Hence $(-\lambda\overrightarrow{n}(\overline{x}), 0) \in J^{2,+}u(\overline{x})$. But

$$f(\overline{x}, k) - \lambda\langle\overrightarrow{n}(\overline{x}), \overrightarrow{n}(\overline{x})\rangle = f(\overline{x}, k) - \lambda > 0,$$

for $\lambda > 0$ small enough, and

$$-\lambda\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle < 0.$$

This contradicts the definition of u.

Finally if u is a strong subsolution, $u \equiv k > 0$ and $f(\overline{x}, k) > 0$ for some $\overline{x} \in \partial\Omega$, then the boundary condition is not satisfied at $\overline{x}$ for $p = 0$. □

Remark 3.3.5. Under the same assumptions of Theorem 3.3.4, but now with f satisfying $f(x, 0) \leq 0$ for all $x \in \partial\Omega$ and with $f(\overline{x}, r) < 0$ for $r < 0$ in (i) and $\min_{x\in\partial\Omega} f(x, r) < 0$ for $r < 0$ in (ii), using Proposition 3.3.2 we can prove the minimum principle, i.e., if $u \in LSC(\overline{\Omega})$ is a viscosity supersolution of (3.3.18) then $u \geq 0$ on $\overline{\Omega}$.

Remark 3.3.6. C^2 convex sets satisfy the condition $S \leq 0$ in every point of the boundary.

Remark 3.3.7. If $c \equiv 0$ and $f \equiv 0$ a counterexample to the maximum principle is given by the positive constants.

3.3.2 The threshold for the Maximum Principle

In this subsection and in the rest of the paper we always assume that Ω is bounded and of class C^2 and that b and c are continuous on $\overline{\Omega}$.

Theorem 3.3.8 (Maximum Principle for $\lambda < \overline{\lambda}$)**.** *Let* $\lambda < \overline{\lambda}$ *and let* $u \in USC(\overline{\Omega})$ *be a viscosity subsolution of*

$$\begin{cases} \Delta_\infty u + b(x)\cdot Du + (c(x)+\lambda)u = 0 & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3.20}$$

then $u \leq 0$ *on* $\overline{\Omega}$.

Corollary 3.3.9. *The quantity* $\overline{\lambda}$ *is finite.*

Proof. It suffices to observe that $\overline{\lambda} \leq |c|_\infty$, since when the zero order coefficient is $c(x) + |c|_\infty$ the maximum principle does not hold. A counterexample is given by the positive constants. □

In the proof of Theorem 3.3.8 we need the following result which is an adaptation of Lemma 1 of [28] for supersolutions of the Neumann boundary value problem.

Lemma 3.3.10. *Let* $v \in LSC(\overline{\Omega})$ *be a viscosity supersolution of*

$$\begin{cases} \Delta_\infty v + b(x) \cdot Dv - \beta(v(x)) = g(x) & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for some functions $g, \beta \in USC(\overline{\Omega})$. *Suppose that* $\overline{x} \in \overline{\Omega}$ *is a strict local minimum of* $v(x) + C|x - \overline{x}|^q e^{-kd(x)}$, $k > \frac{q}{2r}$, *where* r *is the radius in the condition* ($\Omega 2$) *and* $q > 2$. *Moreover suppose that* v *is not locally constant around* $\overline{x}$. *Then*

$$-\beta(v(\overline{x})) \leq g(\overline{x}).$$

Remark 3.3.11. Similarly, if β, $g \in LSC(\overline{\Omega})$, $u \in USC(\overline{\Omega})$ is a supersolution, $\overline{x}$ is a strict local maximum of $u(x) - C|x - \overline{x}|^q e^{-kd(x)}$, $k > \frac{q}{2r}$, $q > 2$ and u is not locally constant around $\overline{x}$, it can be proved that

$$-\beta(u(\overline{x})) \geq g(\overline{x}).$$

Proof of Theorem 3.3.8. Let $\tau \in]\lambda, \overline{\lambda}[$, then by definition there exists $v > 0$ on $\overline{\Omega}$ bounded viscosity supersolution of

$$\begin{cases} \Delta_\infty v + b(x) \cdot Dv + (c(x) + \tau)v = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3.21}$$

We argue by contradiction and suppose that u has a positive maximum in $\overline{\Omega}$. As in [28], we define $\gamma' := \sup_{\overline{\Omega}}(u/v) > 0$ and $w = \gamma v$, with $\gamma \in (0, \gamma')$ to be determined. By homogeneity, w is still a supersolution of (3.3.21). Let $\overline{y} \in \overline{\Omega}$ be such that $u(\overline{y})/v(\overline{y}) = \gamma'$. Since $u(\overline{y}) - w(\overline{y}) = (\gamma' - \gamma)v(\overline{y}) > 0$, the supremum of $u - w$ is strictly positive, then by upper semicontinuity there exists $\overline{x} \in \overline{\Omega}$ such that

$$u(\overline{x}) - w(\overline{x}) = \max_{\overline{\Omega}}(u - w) = m > 0.$$

Clearly $u(\overline{x}) > w(\overline{x}) > 0$, moreover $u(\overline{x}) \leq \gamma' v(\overline{x}) = \frac{\gamma'}{\gamma} w(\overline{x})$, from which

$$w(\overline{x}) \geq \frac{\gamma}{\gamma'} u(\overline{x}). \tag{3.3.22}$$

Fix $q > 2$ and $k > q/(2r)$, where r is the radius in the condition ($\Omega 2$), and define for $j \in \mathbb{N}$ the functions $\phi \in C^2(\overline{\Omega} \times \overline{\Omega})$ and $\psi \in USC(\overline{\Omega} \times \overline{\Omega})$ by

$$\phi(x, y) = \frac{j}{q}|x - y|^q e^{-k(d(x)+d(y))}, \quad \psi(x, y) = u(x) - w(y) - \phi(x, y).$$

Let $(x_j, y_j) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of ψ, then $m = \psi(\overline{x}, \overline{x}) \le u(x_j) - w(y_j) - \phi(x_j, y_j)$, from which

$$\frac{j}{q}|x_j - y_j|^q \le (u(x_j) - w(y_j) - m)e^{k(d(x_j)+d(y_j))} \le C, \tag{3.3.23}$$

where C is independent of j. The last relation implies that, up to subsequence, x_j and y_j converge to some $\overline{z} \in \overline{\Omega}$ as $j \to +\infty$. Classical arguments show that

$$\lim_{j\to+\infty} \frac{j}{q}|x_j - y_j|^q = 0, \quad \lim_{j\to+\infty} u(x_j) = u(\overline{z}), \quad \lim_{j\to+\infty} w(y_j) = w(\overline{z}),$$

and

$$u(\overline{z}) - w(\overline{z}) = m.$$

Claim 1 *For j large enough, there exist x_j and y_j such that (x_j, y_j) is a maximum point of ψ and $x_j \neq y_j$.*

Indeed if $x_j = y_j$ we have

$$\psi(x_j, x) = u(x_j) - w(x) - \frac{j}{q}|x - x_j|^q e^{-k(d(x_j)+d(x))} \le \psi(x_j, x_j) = u(x_j) - w(x_j),$$

and

$$\psi(x, x_j) = u(x) - w(x_j) - \frac{j}{q}|x - x_j|^q e^{-k(d(x)+d(x_j))} \le \psi(x_j, x_j) = u(x_j) - w(x_j).$$

Then x_j is a minimum point for

$$W(x) := w(x) + \frac{j}{q}e^{-kd(x_j)}|x - x_j|^q e^{-kd(x)},$$

and a maximum point for

$$U(x) := u(x) - \frac{j}{q}e^{-kd(x_j)}|x - x_j|^q e^{-kd(x)}.$$

We first exclude that x_j is both a strict local minimum and a strict local maximum. Indeed in that case, if u and w are not locally constant around x_j, by Lemma 3.3.10

$$(c(x_j) + \tau)w(x_j) \le (c(x_j) + \lambda)u(x_j).$$

The same result holds if u or w are locally constant by definition of sub and supersolution. The last inequality leads to a contradiction, as we will see at the end of the proof. Hence x_j cannot be both a strict local minimum and a strict local maximum. In the first case there exist $\delta > 0$ and $R > \delta$ such that

$$\begin{aligned} w(x_j) &= \min_{\substack{\delta \le |x - x_j| \le R \\ x \in \overline{\Omega}}} \left(w(x) + \frac{j}{q}|x - x_j|^q e^{-k(d(x_j)+d(x))} \right) \\ &= w(y_j) + \frac{j}{q}|y_j - x_j|^q e^{-k(d(x_j)+d(y_j))}, \end{aligned}$$

for some $y_j \neq x_j$, so that (x_j, y_j) is still a maximum point for ψ. In the other case, similarly, one can replace x_j by a point $y_j \neq x_j$ such that (y_j, x_j) is a maximum for ψ. This concludes the Claim 1.

Now computing the derivatives of ϕ we get

$$D_x\phi(x,y) = j|x-y|^{q-2}e^{-k(d(x)+d(y))}(x-y) - k\frac{j}{q}|x-y|^q e^{-k(d(x)+d(y))}Dd(x),$$

and

$$D_y\phi(x,y) = -j|x-y|^{q-2}e^{-k(d(x)+d(y))}(x-y) - k\frac{j}{q}|x-y|^q e^{-k(d(x)+d(y))}Dd(y).$$

Denote $p_j := D_x\phi(x_j,y_j)$ and $r_j := -D_y\phi(x_j,y_j)$. Since $x_j \neq y_j$, p_j and r_j are different from 0 for j large enough. Indeed

$$0 < \frac{j}{2}|x_j-y_j|^{q-1}e^{-2kd_0} \le |p_j|, |r_j| \le 2j|x_j-y_j|^{q-1}, \tag{3.3.24}$$

for large j, where $d_0 = \max_{\overline{\Omega}} d(x)$. Using (3.1.6), if $x_j \in \partial\Omega$ then

$$\langle p_j, \overrightarrow{n}(x_j)\rangle \ge j|x_j-y_j|^q e^{-kd(y_j)}\left(-\frac{1}{2r}+\frac{k}{q}\right) > 0,$$

and if $y_j \in \partial\Omega$ then

$$\langle r_j, \overrightarrow{n}(y_j)\rangle \le j|x_j-y_j|^q e^{-kd(x_j)}\left(\frac{1}{2r}-\frac{k}{q}\right) < 0,$$

since $k > q/(2r)$ and $x_j \neq y_j$. In view of definition of sub and supersolution we conclude that

$$\mathrm{tr}(\sigma(p_j)X) + b(x_j)\cdot p_j + (c(x_j)+\lambda)u(x_j) \ge 0 \quad \text{if } (p_j,X) \in \overline{J}^{2,+}u(x_j),$$

$$\mathrm{tr}(\sigma(r_j)Y) + b(y_j)\cdot r_j + (c(y_j)+\tau)w(y_j) \le 0 \quad \text{if } (r_j,Y) \in \overline{J}^{2,-}w(y_j).$$

Applying Theorem 3.2 of [40] for any $\epsilon > 0$ there exist $X_j, Y_j \in$ *S(N)* such that $(p_j,X_j) \in \overline{J}^{2,+}u(x_j)$, $(r_j,Y_j) \in \overline{J}^{2,-}w(y_j)$ and

$$-\left(\frac{1}{\epsilon}+\|D^2\phi(x_j,y_j)\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le D^2\phi(x_j,y_j)+\epsilon(D^2\phi(x_j,y_j))^2. \tag{3.3.25}$$

Claim 2 *X_j and Y_j satisfy*

$$\begin{pmatrix} X_j-\widetilde{X_j} & 0 \\ 0 & -Y_j+\widetilde{Y_j} \end{pmatrix} \le \zeta_j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{3.3.26}$$

where $\zeta_j = Cj|x_j-y_j|^{q-2}$, for some positive constant C independent of j and some matrices $\widetilde{X_j}$, $\widetilde{Y_j} = O(j|x_j-y_j|^q)$.

To prove the claim we need to estimate $D^2\phi(x_j,y_j)$.

$$D^2\phi(x_j,y_j) = \frac{j}{q}|x_j-y_j|^q D^2(e^{-k(d(x_j)+d(y_j))}) + D(e^{-k(d(x_j)+d(y_j))})\otimes\frac{j}{q}D(|x_j-y_j|^q)$$
$$+\frac{j}{q}D(|x_j-y_j|^q)\otimes D(e^{-k(d(x_j)+d(y_j))}) + e^{-k(d(x_j)+d(y_j))}\frac{j}{q}D^2(|x_j-y_j|^q).$$

We denote

$$A_1 := \frac{j}{q}|x_j - y_j|^q D^2(e^{-k(d(x_j)+d(y_j))}),$$

$$A_2 := De^{-k(d(x_j)+d(y_j))} \otimes \frac{j}{q} D(|x_j - y_j|^q) + \frac{j}{q} D(|x_j - y_j|^q) \otimes D(e^{-k(d(x_j)+d(y_j))}),$$

$$A_3 := e^{-k(d(x_j)+d(y_j))} \frac{j}{q} D^2(|x_j - y_j|^q).$$

For A_1 and A_3 we have

$$A_1 \leq Cj|x_j - y_j|^q \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$$A_3 \leq (q-1)j|x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Here and henceforth, as usual, the letter C denotes various constants independent of j. Now we consider the quantity $\langle A_2(\xi, \eta), (\xi, \eta) \rangle$ for ξ, $\eta \in \mathbb{R}^N$. We have

$$\begin{aligned} \langle A_2(\xi,\eta),(\xi,\eta)\rangle &= 2kj|x_j - y_j|^{q-2} e^{-k(d(x_j)+d(y_j))} [\langle Dd(x_j) \otimes (x_j - y_j)(\eta - \xi), \xi \rangle \\ &\quad + \langle Dd(y_j) \otimes (x_j - y_j)(\eta - \xi), \eta \rangle] \\ &\leq Cj|x_j - y_j|^{q-1} |\xi - \eta|(|\xi| + |\eta|) \\ &\leq Cj|x_j - y_j|^{q-1} \left(\frac{|\xi - \eta|^2}{|x_j - y_j|} + \frac{(|\xi| + |\eta|)^2}{4} |x_j - y_j| \right) \\ &\leq C \left[j|x_j - y_j|^{q-2} |\xi - \eta|^2 + j|x_j - y_j|^q (|\xi|^2 + |\eta|^2) \right]. \end{aligned}$$

The last inequality can be rewritten equivalently in this way

$$A_2 \leq Cj|x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + Cj|x_j - y_j|^q \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Finally if we choose

$$\epsilon = \frac{1}{j|x_j - y_j|^{q-2}},$$

we get the same estimates for the matrix $\epsilon(D^2\phi(x_j, y_j))^2$. In conclusion we have

$$\begin{aligned} D^2\phi(x_j, y_j) + \epsilon(D^2\phi(x_j, y_j))^2 &\leq Cj|x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\ &\quad + Cj|x_j - y_j|^q \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \end{aligned}$$

and (3.3.25) implies (3.3.26). The Claim 2 is proved.

Now, multiplying the inequality (3.3.26) on the left for the non-negative symmetric matrix

$$\begin{pmatrix} \sigma(p_j)\sigma(p_j) & \sigma(p_j)\sigma(r_j) \\ \sigma(r_j)\sigma(p_j) & \sigma(r_j)\sigma(r_j) \end{pmatrix} = \begin{pmatrix} \sigma(p_j) & \sigma(p_j)\sigma(r_j) \\ \sigma(r_j)\sigma(p_j) & \sigma(r_j) \end{pmatrix},$$

taking traces and using (3.1.5) and (3.3.24), we get

$$\begin{aligned}\operatorname{tr}(\sigma(p_j)(X_j-\widetilde{X}_j))-\operatorname{tr}(\sigma(r_j)(Y_j-\widetilde{Y}_j)) &\le \zeta_j \operatorname{tr}[(\sigma(p_j)-\sigma(r_j))^2] \le \frac{8\zeta_j}{|p_j|^2}|p_j-r_j|^2\\ &\le C\frac{j|x_j-y_j|^{q-2}j^2|x_j-y_j|^{2q}}{j^2|x_j-y_j|^{2(q-1)}}\\ &= Cj|x_j-y_j|^q.\end{aligned}$$

Now using that u and w are respectively sub and supersolution we compute

$$\begin{aligned}-(\lambda+c(x_j))u(x_j) &\le \operatorname{tr}(\sigma(p_j)X_j)+b(x_j)\cdot p_j\\ &\le \operatorname{tr}(\sigma(p_j)(X_j-\widetilde{X}_j))+b(x_j)\cdot p_j+O\left(j|x_j-y_j|^q\right)\\ &\le \operatorname{tr}(\sigma(r_j)(Y_j-\widetilde{Y}_j))+b(x_j)\cdot p_j+O\left(j|x_j-y_j|^q\right)\\ &\le -(\tau+c(y_j))w(y_j)+b(x_j)\cdot p_j-b(y_j)\cdot r_j+O\left(j|x_j-y_j|^q\right).\end{aligned}$$

The quantity $b(x_j)\cdot p_j-b(y_j)\cdot r_j$ goes to 0 as $j\to+\infty$. Indeed, since $m>0$ and w is positive and bounded, the estimate (3.2.7) of Lemma 3.2.2 holds for u and w; using it in (3.3.23) and dividing by $|x_j-y_j|\neq 0$ we obtain

$$\frac{j}{q}|x_j-y_j|^{q-1}\le C_0 e^{k(d(x_j)+d(y_j))}\le C.$$

Then by (3.3.24) we conclude that the sequences $\{p_j\}$ and $\{r_j\}$ are bounded, so that, since in addition $|p_j-r_j|\le Cj|x_j-y_j|^q\to 0$ as $j\to+\infty$, up to subsequence $p_j, r_j\to p_0$ as $j\to+\infty$.

Hence, sending $j\to+\infty$ we obtain

$$-(\lambda+c(\overline{z}))u(\overline{z})\le-(\tau+c(\overline{z}))w(\overline{z}).$$

If $\tau+c(\overline{z})>0$, using (3.3.22) we get

$$-(\lambda+c(\overline{z}))u(\overline{z})\le-(\tau+c(\overline{z}))\frac{\gamma}{\gamma'}u(\overline{z}),$$

and taking γ sufficiently close to γ' in order that $\frac{\tau\frac{\gamma}{\gamma'}-\lambda}{1-\frac{\gamma}{\gamma'}}>|c|_\infty$, we obtain a contradiction. Finally if $\tau+c(\overline{z})\le 0$ we have

$$-(\lambda+c(\overline{z}))u(\overline{z})\le-(\tau+c(\overline{z}))w(\overline{z})\le-(\tau+c(\overline{z}))u(\overline{z}),$$

once more a contradiction since $\lambda<\tau$. □

Proof of Lemma 3.3.10. Without loss of generality we can assume that $\overline{x}=0$.

Since the minimum is strict there exists a small $\delta>0$ such that

$$v(0)<v(x)+C|x|^q e^{-kd(x)}\quad \text{for any } x\in\overline{\Omega},\ 0<|x|\le\delta.$$

Since v is not locally constant and $q>1$ for any $n>\delta^{-1}$ there exists $(t_n,z_n)\in B(0,\frac{1}{n})^2\cap\overline{\Omega}^2$ such that

$$v(t_n)>v(z_n)+C|z_n-t_n|^q e^{-kd(z_n)}.$$

Consequently, for $n > \delta^{-1}$ the minimum of the function $v(x) + C|x - t_n|^q e^{-kd(x)}$ in $\overline{B}(0,\delta) \cap \overline{\Omega}$ is not achieved on t_n. Indeed

$$\min_{|x|\le\delta,\, x\in\overline{\Omega}} (v(x) + C|x - t_n|^q e^{-kd(x)}) \le v(z_n) + C|z_n - t_n|^q e^{-kd(z_n)} < v(t_n).$$

Let $y_n \neq t_n$ be some point in $\overline{B}(0,\delta) \cap \overline{\Omega}$ on which the minimum is achieved. Passing to the limit as n goes to infinity, t_n goes to 0 and, up to subsequence, y_n converges to some $y \in \overline{B}(0,\delta) \cap \overline{\Omega}$. By the lower semicontinuity of v and the fact that 0 is a local minimum of $v(x) + C|x|^q e^{-kd(x)}$ we have

$$v(0) \le v(y) + C|y|^q e^{-kd(y)} \le \liminf_{n\to+\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}),$$

and using that $v(0) + C|t_n|^q e^{-kd(0)} \ge v(y_n) + C|y_n - t_n|^q e^{-kd(y_n)}$, one has

$$v(0) \ge \limsup_{n\to+\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}).$$

Then

$$v(0) = v(y) + C|y|^q e^{-kd(y)} = \lim_{n\to+\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}).$$

Since 0 is a strict local minimum of $v(x) + C|x|^q e^{-kd(x)}$, the last equalities imply that $y = 0$ and $v(y_n)$ goes to $v(0)$ as $n \to +\infty$. Then for large n, y_n is an interior point of $B(0,\delta)$ so that the function

$$\varphi(x) = v(y_n) + C|y_n - t_n|^q e^{-kd(y_n)} - C|x - t_n|^q e^{-kd(x)}$$

is a test function for v at y_n. Moreover, the gradient of φ

$$D\varphi(x) = -Cq|x - t_n|^{q-2} e^{-kd(x)}(x - t_n) + kC|x - t_n|^q e^{-kd(x)} Dd(x)$$

is different from 0 at $x = y_n$ for small δ, indeed

$$|D\varphi(y_n)| \ge C|y_n - t_n|^{q-1} e^{-kd(y_n)}(q - k|y_n - t_n|) \ge C|y_n - t_n|^{q-1} e^{-kd(y_n)}(q - 2k\delta) > 0.$$

Using (3.1.6), if $y_n \in \partial\Omega$ we have

$$\langle D\varphi(y_n), \overrightarrow{n}(y_n)\rangle \le C|y_n - t_n|^q \left(\frac{q}{2r} - k\right) < 0,$$

since $k > q/(2r)$. Then we conclude that

$$\operatorname{tr}\Big(\sigma(D\varphi(y_n))D^2\varphi(y_n)\Big) + b(y_n)\cdot D\varphi(y_n) - \beta(v(y_n)) \le g(y_n).$$

Observe that $D^2\varphi(y_n) = |y_n - t_n|^{q-2}M$, where M is a bounded matrix. Hence, from the last inequality we get

$$C_0|y_n - t_n|^{q-2} - \beta(v(y_n)) \le g(y_n),$$

for some constant C_0. Passing to the limit, since β and g are upper semicontinuous we obtain

$$-\beta(v(0)) \le g(0),$$

which is the desired conclusion. □

We conclude sketching the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. Suppose by contradiction that $\max_{\overline{\Omega}}(u-v) = m > 0$. Since $u \leq v$ on the boundary, the supremum is achieved inside Ω. Let us define for $j \in \mathbb{N}$ and some $q > 2$

$$\psi(x,y) = u(x) - v(y) - \frac{j}{q}|x-y|^q.$$

Suppose that (x_j, y_j) is a maximum point for ψ in $\overline{\Omega}^2$. Then $|x_j - y_j| \to 0$ as $j \to +\infty$ and up to subsequence $x_j, y_j \to \overline{x}$, $u(x_j) \to u(\overline{x})$, $v(y_j) \to v(\overline{x})$ and $j|x_j - y_j|^q \to 0$ as $j \to +\infty$. Moreover, $\overline{x}$ is such that $u(\overline{x}) - v(\overline{x}) = m$ and we can choose $x_j \neq y_j$. Recalling by Remark 3.2.3 that the estimate (3.2.7) holds in Ω, we can proceed as in the proof of Theorem 3.3.8 to get

$$-c(\overline{x})u(\overline{x}) \leq -c(\overline{x})v(\overline{x}).$$

This is a contradiction since $c(\overline{x}) < 0$. □

3.3.3 The Maximum Principle for $c(x)$ changing sign: an example.

In the previous subsections we have proved that $\Delta_\infty + b(x)\cdot D + c(x)$ with the Neumann boundary condition satisfies the maximum principle if $c(x) \leq 0$ or without condition on the sign of $c(x)$ provided $\overline{\lambda} > 0$. In this subsection we want to show that these two cases do not coincide, i.e., that there exists some $c(x)$ which changes sign in Ω such that the associated principal eigenvalue $\overline{\lambda}$ is positive. To prove this, by definition of $\overline{\lambda}$, it suffices to find a function $c(x)$ changing sign for which there exists a bounded positive supersolution of

$$\begin{cases} \Delta_\infty v + b(x)\cdot Dv + (c(x)+\lambda)v = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3.27}$$

for some $\lambda > 0$. For simplicity, let us suppose that $b \equiv 0$ and Ω is the ball of center 0 and radius R. We will look for c such that:

$$\begin{cases} c(x) < 0 & \text{if } R-\epsilon < |x| \leq R \\ c(x) \leq -\beta_1 & \text{if } \rho < |x| \leq R-\epsilon \\ c(x) \leq \beta_2 & \text{if } |x| \leq \rho, \end{cases} \tag{3.3.28}$$

where $0 < \rho < R-\epsilon$ and ϵ, β_1, β_2 are positive constants. Remark that in the ball of radius ρ, $c(x)$ may assume positive values. Following [97], it is possible to construct a supersolution of (3.3.27) if ϵ is small enough and

$$\beta_2 < \frac{k^2 e^{-k\rho}}{\frac{k}{4}(R-\rho) + \frac{2k}{\beta_1(R-\rho)} + 1 - e^{-k\rho}},$$

for some $k > 0$. From the last relation we can see that choosing $k = \frac{1}{\rho}$ the term on the right-hand side goes to $+\infty$ as $\rho \to 0^+$, that is, if the set where $c_0(x)$ is positive becomes smaller then the values of $c_0(x)$ in this set can be very large. On the contrary, for any value of k, if $\rho \to R^-$ then β_2 goes to 0. Finally for any k if $\beta_1 \to 0^+$, then again β_2 goes to 0.

3.4 Some existence results

This section is devoted to the problem of the existence of a solution of

$$\begin{cases} \Delta_\infty u + b(x) \cdot Du + (c(x) + \lambda)u = g(x) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.4.29}$$

The first existence result for (3.4.29) is obtained when $\lambda = 0$ and $c < 0$, via Perron's method. Then, we will prove the existence of a positive solution of (3.4.29) when g is non-positive and $\lambda < \overline{\lambda}$ (without condition on the sign of c). These two results will allow us to prove that the Neumann problem (3.4.29) is solvable for any right-hand side if $\lambda < \overline{\lambda}$. Finally, we will prove the existence of a positive principal eigenfunction corresponding to $\overline{\lambda}$, that is a solution of (3.4.29) when $g \equiv 0$ and $\lambda = \overline{\lambda}$.

Comparison results guarantee for (3.4.29) the uniqueness of the solution when $c < 0$ and when $\lambda < \overline{\lambda}$ and $g < 0$ or $g > 0$.

Theorem 3.4.1. *Suppose that $c < 0$ and g is continuous on $\overline{\Omega}$. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are respectively viscosity sub and supersolution of*

$$\begin{cases} \Delta_\infty u + b(x) \cdot Du + c(x)u = g(x) & \textit{in} \quad \Omega \\ \dfrac{\partial u}{\partial \overrightarrow{n}} = 0 & \textit{on} \quad \partial\Omega, \end{cases} \tag{3.4.30}$$

with u and v bounded or $v \geq 0$ and bounded, then $u \leq v$ on $\overline{\Omega}$. Moreover (3.4.30) *has a unique viscosity solution.*

Proof. We suppose by contradiction that $\max_{\overline{\Omega}}(u - v) = m > 0$. Repeating the proof of Theorem 3.3.8 taking v as w, we arrive to the following inequality

$$-c(\overline{z})u(\overline{z}) \leq -c(\overline{z})v(\overline{z}),$$

where $\overline{z} \in \overline{\Omega}$ is such that $u(\overline{z}) - v(\overline{z}) = m > 0$. This is a contradiction since $c(\overline{z}) < 0$.

The existence of a solution follows from Perron's method of Ishii, see e.g. [40], and the comparison result just proved, provided there is a bounded subsolution and a bounded supersolution of (3.4.30). Since c is negative and continuous on $\overline{\Omega}$, there exists $c_0 > 0$ such that $c(x) \leq -c_0$ for every $x \in \overline{\Omega}$. Then

$$u_1 := -\frac{|g|_\infty}{c_0}, \quad u_2 := \frac{|g|_\infty}{c_0}$$

are respectively a bounded sub and supersolution of (3.4.30).

Define

$$u(x) := \sup\{\varphi(x) |\, u_1 \leq \varphi \leq u_2 \text{ and } \varphi \text{ is a subsolution of (3.4.30) }\},$$

we claim that u is a solution of (3.4.30). We first show that the upper semicontinuous envelope of u defined as

$$u^*(x) := \lim_{\rho \downarrow 0} \sup\{u(y) : y \in \overline{\Omega} \text{ and } |y - x| \leq \rho\}$$

is a subsolution of (3.4.30). Indeed if $(p, X) \in J^{2,+}u(x_0)$ and $p \neq 0$ then by the standard arguments of the Perron's method it can be proved that $\mathrm{tr}(\sigma(p)X) + b(x_0) \cdot p + c(x_0)u(x_0) \geq g(x_0)$ if $x_0 \in \Omega$ and $(-\mathrm{tr}(\sigma(p)X) - b(x_0) \cdot p - c(x_0)u(x_0) + g(x_0)) \wedge \langle p, \overrightarrow{n}(x_0)\rangle \leq 0$ if $x_0 \in \partial\Omega$.

Now suppose $u^* \equiv k$ in a neighborhood of $x_0 \in \overline{\Omega}$. If $x_0 \in \partial\Omega$ clearly u^* is subsolution at x_0. Assume that x_0 is an interior point of Ω. We may choose a sequence of subsolutions $(\varphi_n)_n$ and a sequence of points $(x_n)_n$ in Ω such that $x_n \to x_0$ and $\varphi_n(x_n) \to k$. Suppose that $|x_n - x_0| < a_n$ with a_n decreasing to 0 as $n \to +\infty$. If, up to subsequence, φ_n is constant in $B(x_0, a_n)$ for any n, then passing to the limit in the relation $c(x_n)\varphi_n(x_n) \geq g(x_n)$ we get $c(x_0)k \geq g(x_0)$ as desired. Otherwise, suppose that for any n φ_n is not constant in $B(x_0, a_n)$. Repeating the argument of Lemma 3.3.10 we find a sequence $\{(t_n, y_n)\}_{n\in\mathbb{N}} \subset \Omega^2$ and a small $\delta > 0$ such that $|t_n - x_0| < a_n$, $|y_n - x_0| \leq \delta$, $t_n \neq y_n$, $\varphi_n(x) - |x - t_n|^q \leq \varphi_n(y_n) - |y_n - t_n|^q$ for any $x \in B(x_0, \delta)$, with $q > 2$ and $u^* \equiv k$ in $\overline{B}(x_0, \delta)$. Up to subsequence $y_n \to y \in \overline{B}(x_0, \delta)$ as $n \to +\infty$. We have

$$
\begin{aligned}
k &= \lim_{n\to+\infty}(\varphi_n(x_n) - |x_n - t_n|^q) \leq \liminf_{n\to+\infty}(\varphi_n(y_n) - |y_n - t_n|^q) \\
&\leq \limsup_{n\to+\infty}(\varphi_n(y_n) - |y_n - t_n|^q) \leq k - |y - x_0|^q.
\end{aligned}
$$

The last inequalities imply that $y = x_0$ and $\varphi_n(y_n) \to k$. Then, for large n, y_n is an interior point of $B(x_0, \delta)$ and $\phi_n(x) := \varphi_n(y_n) - |y_n - t_n|^q + |x - t_n|^q$ is a test function for φ_n at y_n. Passing to the limit as $n \to +\infty$ in the relation $\Delta_\infty\phi_n(y_n) + b(y_n) \cdot D\phi_n(y_n) + c(y_n)\varphi_n(y_n)) \geq g(y_n)$, we get again $c(x_0)k \geq g(x_0)$. In conclusion u^* is a subsolution of (3.4.30). Since $u^* \leq u_2$, it follows from the definition of u that $u = u^*$.

Finally the lower semicontinuous envelope of u defined as

$$
u_*(x) := \lim_{\rho\downarrow 0} \inf\{u(y) : y \in \overline{\Omega} \text{ and } |y - x| \leq \rho\}
$$

is a supersolution. Indeed, if it is not, the Perron's method provides a viscosity subsolution of (3.4.30) greater than u, contradicting the definition of u. If $u_* \equiv k$ in a neighborhood of $x_0 \in \Omega$ and $c(x_0)k > g(x_0)$ then for small δ and ρ, the subsolution is

$$
u_{\delta,\rho}(x) := \begin{cases} \max\{u(x), k + \frac{\delta\rho^2}{8} - \delta|x - x_0|^2\} & \text{if } |x - x_0| < \rho, \\ u(x) & \text{otherwise.} \end{cases}
$$

Hence u_* is a supersolution of (3.4.30) and then, by comparison, $u^* = u \leq u_*$, showing that u is continuous and is a solution.

The uniqueness of the solution is an immediate consequence of the comparison principle just proved. □

Theorem 3.4.2. *Suppose $g \in LSC(\overline{\Omega})$, $h \in USC(\overline{\Omega})$, $h \leq 0$, $h \leq g$ and $g(x) > 0$ if $h(x) = 0$. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of* (3.4.29) *and $v \in LSC(\overline{\Omega})$ be a bounded positive viscosity supersolution of* (3.4.29) *with g replaced by h. Then $u \leq v$ on $\overline{\Omega}$.*

Remark 3.4.3. The existence of a such v implies $\lambda \leq \overline{\lambda}$.

Proof. It suffices to prove the theorem for $h < g$. Indeed, for $l > 1$ the function lv is a supersolution of (3.4.29) with right-hand side $lh(x)$ and by the assumptions on h and g, $lh < g$. If $u \le lv$ for any $l > 1$, passing to the limit as $l \to 1^+$, one obtains $u \le v$ as desired.

Hence we can assume $h < g$. By upper semicontinuity $\max_{\overline{\Omega}}(h-g) = -M < 0$. Suppose by contradiction that $u > v$ somewhere in $\overline{\Omega}$. Then there exists $\overline{y} \in \overline{\Omega}$ such that

$$\gamma' := \frac{u(\overline{y})}{v(\overline{y})} = \max_{x\in\overline{\Omega}} \frac{u(x)}{v(x)} > 1.$$

Define $w = \gamma v$ for some $1 \le \gamma < \gamma'$. Since $h \le 0$ and $\gamma \ge 1$, $\gamma h \le h$ and then w is still a supersolution of (3.4.29) with right-hand side h. The supremum of $u - w$ is strictly positive then, by upper semicontinuity, there exists $\overline{x} \in \overline{\Omega}$ such that $u(\overline{x}) - w(\overline{x}) = \max_{\overline{\Omega}}(u-w) > 0$. We have $u(\overline{x}) > w(\overline{x})$ and $w(\overline{x}) \ge \frac{\gamma}{\gamma'}u(\overline{x})$. Repeating the proof of Theorem 3.3.8, we get

$$g(\overline{z}) - (\lambda + c(\overline{z}))u(\overline{z}) \le h(\overline{z}) - (\lambda + c(\overline{z}))w(\overline{z}),$$

where $\overline{z}$ is some point in $\overline{\Omega}$ where the maximum of $u - w$ is attained. If $\lambda + c(\overline{z}) \le 0$, then

$$-(\lambda + c(\overline{z}))u(\overline{z}) \le h(\overline{z}) - g(\overline{z}) - (\lambda + c(\overline{z}))w(\overline{z}) < -(\lambda + c(\overline{z}))u(\overline{z}),$$

which is a contradiction. If $\lambda + c(\overline{z}) > 0$, then

$$-(\lambda + c(\overline{z}))u(\overline{z}) \le h(\overline{z}) - g(\overline{z}) - (\lambda + c(\overline{z}))\frac{\gamma}{\gamma'}u(\overline{z}).$$

If we choose γ sufficiently close to γ' in order that

$$|\lambda + c|_\infty \left(\frac{\gamma}{\gamma'} - 1\right) \max_{\overline{\Omega}} u \ge -\frac{M}{2},$$

we get once more a contradiction. □

Theorem 3.4.4. *Suppose that $\lambda < \overline{\lambda}$, $g \le 0$, $g \not\equiv 0$ and g is continuous on $\overline{\Omega}$, then there exists a positive viscosity solution of* (3.4.29)*. If $g < 0$, the solution is unique.*

Proof. We follow the proof of Theorem 7 of [28].

If $\lambda < -|c|_\infty$ then the existence of the solution is guaranteed by Theorem 3.4.1. Let us suppose $\lambda \ge -|c|_\infty$ and define by induction the sequence $(u_n)_n$ by $u_1 = 0$ and u_{n+1} as the solution of

$$\begin{cases} \Delta_\infty u_{n+1} + b(x)\cdot Du_{n+1} + (c(x) - |c|_\infty - 1)u_{n+1} = g - (\lambda + |c|_\infty + 1)u_n & \text{in } \Omega \\ \dfrac{\partial u_{n+1}}{\partial \overrightarrow{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists by Theorem 3.4.1. By the comparison principle, since $g \le 0$ and $g \not\equiv 0$ the sequence is positive and increasing.

We claim that $(u_n)_n$ is also bounded. Suppose that it is not, then dividing by $|u_{n+1}|_\infty$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_{n+1} is a solution of

$$\begin{cases} \Delta_\infty v_{n+1} + b(x)\cdot Dv_{n+1} + (c(x) - |c|_\infty - 1)v_{n+1} & \\ \quad = \frac{g}{|u_{n+1}|_\infty} - (\lambda + |c|_\infty + 1)\frac{u_n}{|u_{n+1}|_\infty} & \text{in} \quad \Omega \\ \dfrac{\partial v_{n+1}}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By Corollary 3.2.4, $(v_n)_n$ converges to a positive function v with $|v|_\infty = 1$, which satisfies

$$\begin{cases} \Delta_\infty v + b(x)\cdot Dv + (c(x) + \lambda)v & \\ \quad = (\lambda + |c|_\infty + 1)(1-k)v \geq 0 & \text{in} \quad \Omega \\ \dfrac{\partial v_{n+1}}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $k := \lim_{n\to+\infty} \frac{|u_n|_\infty}{|u_{n+1}|_\infty} \leq 1$. This contradicts the maximum principle, Theorem 3.3.8.

Then $(u_n)_n$ is bounded and letting n go to infinity, by the compactness result, the sequence converges to a function u which is a solution. Moreover, the solution is positive on $\overline{\Omega}$ by the strong minimum principle, Proposition 3.3.2.

If $g < 0$, the uniqueness of the solution follows from Theorem 3.4.2. □

Remark 3.4.5. Clearly, since the operator Δ_∞ is odd, by Theorem 3.4.4, there exists a negative solution of (3.4.29) for $\lambda < \underline{\lambda}$ and $g \geq 0$, $g \not\equiv 0$, which is unique if $g > 0$.

Theorem 3.4.6. *Suppose that $\lambda < \overline{\lambda}$ and g is continuous on $\overline{\Omega}$, then there exists a viscosity solution of* (3.4.29)*.*

Proof. If $g \equiv 0$, by the maximum principle the only solution is $u \equiv 0$. Let us suppose $g \not\equiv 0$. Since $\lambda < \overline{\lambda}$ by Theorem 3.4.4 there exist v_0 positive viscosity solution of (3.4.29) with right-hand side $-|g|_\infty$ and u_0 negative viscosity solution of (3.4.29) with right-hand side $|g|_\infty$.

Let us suppose $\lambda + |c|_\infty \geq 0$. Let $(u_n)_n$ be the sequence defined in the proof of Theorem 3.4.4 with $u_1 = u_0$, then by comparison Theorem 3.4.1 we have $u_0 = u_1 \leq u_2 \leq ... \leq v_0$. Hence, by the compactness Corollary 3.2.4 the sequence converges to a continuous function which is the desired solution. □

Theorem 3.4.7 (Existence of principal eigenfunctions)**.** *There exists $\phi > 0$ on $\overline{\Omega}$ viscosity solution of*

$$\begin{cases} \Delta_\infty \phi + b(x)\cdot D\phi + (c(x) + \overline{\lambda})\phi = 0 & \textit{in} \quad \Omega \\ \dfrac{\partial \phi}{\partial \overrightarrow{n}} = 0 & \textit{on} \quad \partial\Omega. \end{cases}$$

Moreover ϕ is Lipschitz continuous on $\overline{\Omega}$.

Proof. Let λ_n be an increasing sequence which converges to $\overline{\lambda}$. Let u_n be the positive solution of (3.4.29) with $\lambda = \lambda_n$ and $g \equiv -1$. By Theorem 3.4.4 the sequence $(u_n)_n$ is well defined. Following the argument of the proof of Theorem 8 of [28], it can proved that it is unbounded, otherwise one would contradict the definition of $\overline{\lambda}$. Then, up to subsequence $|u_n|_\infty \to +\infty$ as $n \to +\infty$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_n satisfies (3.4.29) with $\lambda = \lambda_n$ and $g \equiv -\frac{1}{|u_n|_\infty}$. Then by Corollary 3.2.4, we can extract a subsequence converging to a positive function ϕ with $|\phi|_\infty = 1$ which is the desired solution. By Theorem 3.2.1 the solution is also Lipschitz continuous on $\overline{\Omega}$. □

3.5 A decay estimate for solutions of the evolution problem

In this section we want to study the asymptotic behavior as $t \to +\infty$ of the solution $h(t,x)$ of the evolution problem

$$\begin{cases} h_t = \Delta_\infty h + c(x)h & \text{in } (0,+\infty)\times\Omega \\ \dfrac{\partial h}{\partial \overrightarrow{n}} = 0 & \text{on } [0,+\infty)\times\partial\Omega \\ h(0,x) = h_0(x) & \text{for } x\in\Omega, \end{cases} \tag{3.5.31}$$

where h_0 is a continuous function on $\overline{\Omega}$. As in [80] and in [81] we use the semi-continuous extensions of the function $(p,X) \to \operatorname{tr}(\sigma(p)X)$ to define the viscosity solutions of (3.5.31). For $X \in S(N)$, let us denote its smaller and larger eigenvalue respectively by $m(X)$ and $M(X)$, that is

$$m(X) := \min_{|\xi|=1}\langle X\xi,\xi\rangle,$$

$$M(X) := \max_{|\xi|=1}\langle X\xi,\xi\rangle.$$

Definition 3.5.1. *Any function $u \in USC([0,+\infty)\times\overline{\Omega})$ (resp., $u \in LSC([0,+\infty)\times\overline{\Omega})$) is called* viscosity subsolution *(resp.,* supersolution*) of* (3.5.31) *if for any $x \in \overline{\Omega}$, $u(0,x) \le h_0(x)$ (resp., $u(0,x) \ge h_0(x)$) and if the following conditions hold*

(i) For every $(t_0,x_0) \in (0,+\infty)\times\Omega$, for all $\varphi \in C^2([0,+\infty)\times\overline{\Omega})$, such that $u-\varphi$ has a local maximum (resp., minimum) at (t_0,x_0), one has

$$\begin{cases} \varphi_t(t_0,x_0) \le \Delta_\infty\varphi(t_0,x_0) + c(x_0)u(t_0,x_0)\ (\textit{resp.},\ \ge) & \textit{if } D\varphi(t_0,x_0)\ne 0, \\ \varphi_t(t_0,x_0) \le M(D^2\varphi(t_0,x_0)) + c(x_0)u(t_0,x_0) & \textit{if } D\varphi(t_0,x_0) = 0 \\ (\textit{resp.},\ \varphi_t(t_0,x_0) \ge m(D^2\varphi(t_0,x_0)) + c(x_0)u(t_0,x_0)). \end{cases}$$

(ii) For every $(t_0,x_0) \in (0,+\infty)\times\partial\Omega$, for all $\varphi \in C^2([0,+\infty)\times\overline{\Omega})$, such that $u-\varphi$ has a local maximum (resp., minimum) at (t_0,x_0) and $D\varphi(t_0,x_0)\ne 0$, one has

$$(\varphi_t(t_0,x_0) - \Delta_\infty\varphi(t_0,x_0) - c(x_0)u(t_0,x_0)) \wedge \langle D\varphi(t_0,x_0), \overrightarrow{n}(x_0)\rangle \le 0.$$

(resp.,

$$(\varphi_t(t_0,x_0) - \Delta_\infty\varphi(t_0,x_0) - c(x_0)u(t_0,x_0)) \vee \langle D\varphi(t_0,x_0), \overrightarrow{n}(x_0)\rangle \ge 0.)$$

Remark that if $(t_0, x_0) \in (0, +\infty) \times \partial\Omega$ and $D\varphi(t_0, x_0) = 0$, then the boundary condition is satisfied.

We will show that if the principal eigenvalue of the stationary operator associated to (3.5.31) is positive, then h decays to zero exponentially and that the rate of the decay depends on it. Let $\overline{\lambda}$ and v be respectively the principal eigenvalue and a principal eigenfunction, i.e., v is a positive solution of

$$\begin{cases} \Delta_\infty v + (c(x) + \overline{\lambda})v = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \overrightarrow{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 3.5.1. *Let $h \in C(\overline{\Omega} \times [0, +\infty))$ be a solution of* (3.5.31) *then*

$$\sup_{\Omega \times [0,+\infty)} \frac{h(t,x)e^{\overline{\lambda}t}}{v(x)} \leq \sup_{\Omega} \frac{h_0^+(x)}{v(x)}, \tag{3.5.32}$$

where $h_0^+ = \max\{h_0, 0\}$ denotes the positive part of h_0.

Proof. It suffices to prove that, fixed $\lambda < \overline{\lambda}$

$$\sup_{[0,T)\times\Omega} \frac{h(t,x)e^{\lambda t}}{v(x)} \leq \sup_{\Omega} \frac{h_0^+(x)}{v(x)},$$

for any $T > 0$. This implies that

$$\sup_{[0,T)\times\Omega} \frac{h(t,x)e^{\overline{\lambda}t}}{v(x)} \leq \sup_{\Omega} \frac{h_0^+(x)}{v(x)},$$

for any $T > 0$ and consequently (3.5.32). Let us denote $H(t,x) = h(t,x)e^{\lambda t}$, it is easy to see that $H(t,x)$ satisfies

$$\begin{cases} H_t = \Delta_\infty H + (c(x) + \lambda)H & \text{in } [0, +\infty) \times \Omega \\ \dfrac{\partial H}{\partial \overrightarrow{n}} = 0 & \text{on } [0, +\infty) \times \partial\Omega \\ H(0,x) = h_0(x) & \text{for } x \in \Omega. \end{cases} \tag{3.5.33}$$

Suppose by contradiction that there exists $T > 0$ such that

$$\gamma' := \sup_{[0,T)\times\Omega} \frac{h(t,x)e^{\lambda t}}{v(x)} > \sup_{\Omega} \frac{h_0^+(x)}{v(x)} =: \overline{h} \geq 0. \tag{3.5.34}$$

Let us denote $w = \gamma v$, where

$$\overline{h} < \gamma < \gamma'$$

and γ is sufficiently close to γ' in order that

$$\frac{\overline{\lambda}\frac{\gamma}{\gamma'} - \lambda}{1 - \frac{\gamma}{\gamma'}} > |c|_\infty. \tag{3.5.35}$$

Since $\gamma < \gamma'$, the function $H - w$ has a positive maximum on $[0, T] \times \overline{\Omega}$.

Fix $q > 2$, $k > \frac{q}{2r}$ and $\epsilon > 0$ small, for $j \in \mathbb{N}$ we define the function

$$\phi(t,x,s,y) = \left(\frac{j}{q}|x-y|^q + \frac{j}{2}|t-s|^2\right) e^{-k(d(x)+d(y))} + \frac{\epsilon}{T-t},$$

and we consider the supremum of

$$H(t,x) - w(y) - \phi(t,x,s,y)$$

over $([0,T) \times \overline{\Omega})^2$. Let (t_j, x_j, s_j, y_j) be a point in $(\overline{\Omega} \times [0,T))^2$ where the maximum is attained. From

$$H(t_j,x_j) - w(y_j) - \phi(t_j,x_j,t_j,y_j) \le H(t_j,x_j) - w(y_j) - \phi(t_j,x_j,s_j,y_j)$$

we deduce that

$$t_j = s_j.$$

Let $(\widehat{t},\widehat{x}) \in [0,T[\times\overline{\Omega}$ be such that $H(\widehat{t},\widehat{x}) - w(\widehat{x}) = l > 0$, then for ϵ small enough we have

$$\frac{l}{2} \le H(\widehat{t},\widehat{x}) - w(\widehat{x}) - \frac{\epsilon}{T-\widehat{t}} \le H(t_j,x_j) - w(y_j) - \frac{\epsilon}{T-t_j} - \frac{j}{q}|x_j-y_j|^q e^{-k(d(x_j)+d(y_j))}.$$

Since $\frac{\epsilon}{T-t} \to +\infty$ as $t \uparrow T$, the previous inequality implies that, up to subsequence $(t_j,x_j,y_j) \to (\overline{t},\overline{x},\overline{x})$ as $j \to +\infty$ with $\overline{t} < T$ and that

$$H(\overline{t},\overline{x}) - w(\overline{x}) > 0. \tag{3.5.36}$$

Moreover

$$\lim_{j\to+\infty} \frac{j}{q}|x_j-y_j|^q = 0,$$

and from (3.5.34) we deduce that

$$w(\overline{x}) \ge \frac{\gamma}{\gamma'} H(\overline{t},\overline{x}). \tag{3.5.37}$$

Finally, since $\gamma > \overline{h}$, it is $\overline{t} > 0$. Hence for j large enough, $0 < t_j < T$.

As in Theorem 3.3.8 the following holds true.

Claim *For j large enough, we can choose $x_j \neq y_j$.*

Indeed, suppose that $x_j = y_j$, then (t_j, x_j) is a maximum point for

$$U(t,x) := H(t,x) - \frac{\epsilon}{T-t} - e^{-kd(x_j)}\left(\frac{j}{q}|x-x_j|^q + \frac{j}{2}|t-t_j|^2\right) e^{-kd(x)},$$

and a minimum point for

$$W(t,x) := w(x) + e^{-kd(x_j)}\left(\frac{j}{q}|x-x_j|^q + \frac{j}{2}|t-t_j|^2\right) e^{-kd(x)}.$$

We prove that (t_j, x_j) is not both a strict local maximum and a strict local minimum. Indeed, in that case, if $H(t,x) - \frac{\epsilon}{T-t}$ is not locally constant around

(t_j, x_j), following the proof of Lemma 3.3.10, we can construct sequences $(t_n, x_n)_n$, $(s_n, y_n)_n$ converging to (t_j, x_j) as $n \to +\infty$, such that $(t_n, x_n) \neq (s_n, y_n)$ and

$$\varphi(t,x) := C\left(\frac{|x-x_n|^q}{q} + \frac{|t-t_n|^2}{2}\right)e^{-kd(x)} + \frac{\epsilon}{T-t} + H(s_n, y_n)$$
$$- \frac{\epsilon}{T-s_n} - C\left(\frac{|y_n-x_n|^q}{q} + \frac{|s_n-t_n|^2}{2}\right)e^{-kd(y_n)}$$

is a test function for $H(t,x)$ at (s_n, y_n), where $C = je^{-kd(x_j)}$. If $y_n \in \partial\Omega$, then

$$\langle D\varphi(s_n, y_n), \overrightarrow{n}(y_n)\rangle \geq C\left[\left(\frac{k}{q} - \frac{1}{2r}\right)|x_n - y_n|^q + \frac{k}{2}|s_n - t_n|^2\right] > 0.$$

Then $D\varphi(s_n, y_n) \neq 0$ and by definition of subsolution

$$\frac{\epsilon}{(T-s_n)^2} + Ce^{-kd(y_n)}(s_n - t_n) \leq \Delta_\infty(\varphi(s_n, y_n)) + (c(y_n) + \lambda)H(s_n, y_n).$$

If y_n is an interior point and $D\varphi(s_n, y_n) \neq 0$, then again the previous inequality holds true, otherwise if $D\varphi(s_n, y_n) = 0$, we have

$$\frac{\epsilon}{(T-s_n)^2} + Ce^{-kd(y_n)}(s_n - t_n) \leq M(D^2\varphi(s_n, y_n)) + (c(y_n) + \lambda)H(s_n, y_n).$$

Passing to the limit as $n \to +\infty$, from both the previous relations we get

$$\frac{\epsilon}{(T-t_j)^2} \leq (c(x_j) + \lambda)H(t_j, x_j).$$

By definition of subsolution, we get the same inequality if $H(t,x) - \frac{\epsilon}{T-t}$ is locally constant around (t_j, x_j).

Proceeding in the same way, if either w is locally constant around x_j or not, since (t_j, x_j) is a strict local minimum of $W(t,x)$, we get

$$(c(x_j) + \overline{\lambda})w(x_j) \leq 0.$$

Then, passing to the limit as $j \to +\infty$, we finally obtain

$$(c(\overline{x}) + \overline{\lambda})w(\overline{x}) < \frac{\epsilon}{(T-\overline{t})^2} \leq (c(\overline{x}) + \lambda)H(\overline{t}, \overline{x}), \tag{3.5.38}$$

which contradicts (3.5.35), (3.5.36) and (3.5.37).

Hence (t_j, x_j) cannot be both a strict local maximum and a strict local minimum. In the first case, there exists $(s_j, y_j) \neq (t_j, x_j)$ such that

$$H(s_j, y_j) - w(x_j) - \frac{\epsilon}{T-s_j} - \left(\frac{j}{q}|x_j - y_j|^q + \frac{j}{2}|t_j - s_j|^2\right)e^{-k(d(x_j)+d(y_j))}$$
$$= H(t_j, x_j) - w(x_j) - \frac{\epsilon}{T-t_j} = \sup_{([0,T)\times\Omega)^2}(H(t,x) - w(y) - \phi(t,x,s,y)).$$

As before we get that $s_j = t_j$, then $x_j \neq y_j$ and this concludes the claim.

From the claim we deduce that $D_x\phi(t_j, x_j, t_j, y_j)$ and $D_y\phi(t_j, x_j, t_j, y_j)$ are different from 0. Moreover there exist $X_j, Y_j \in S(N)$ satisfying (3.3.26) such that $\left(\frac{\epsilon}{(T-t_j)^2}, D_x\phi(t_j, x_j, t_j, y_j), X_j\right) \in \mathcal{P}^{2,+}H(t_j, x_j)$ and $(-D_y\phi(t_j, x_j, t_j, y_j), Y_j) \in J^{2,-}w(y_j)$. Now we can proceed as in the proof of Theorem 3.3.8 to obtain (3.5.38) and hence to reach a contradiction.

□

Part II

Homogenization of first order Hamilton-Jacobi equations

Chapter 4

The Peierls-Nabarro model for dislocation dynamics

In this chapter we are concerned with a non-local Hamilton-Jacobi equation describing dislocation dynamics. We investigate the limit as $\epsilon \to 0$ of the solution u^ϵ of:

$$\begin{cases} \partial_t u^\epsilon = \mathcal{I}_1[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.0.1}$$

where $\mathcal{I}_1$ is an anisotropic Lévy operator of order 1, defined on bounded C^2- functions for $r > 0$ by

$$\begin{aligned} \mathcal{I}_1[U](x) = & \int_{|z|\le r} (U(x+z) - U(x) - \nabla U(x)\cdot z) \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \\ & + \int_{|z|>r} (U(x+z) - U(x)) \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz, \end{aligned}$$

where the function g satisfies

(H1) $g \in C(\mathbf{S}^{N-1})$, $g > 0$;

(H2) $\exists r_0 > 0$ such that

$$c_0 := \inf_{e\in[0,1)^N} \int_{\{|z|>r_0\}\cap\{|z+e|>r_0\}} \min\left\{\frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right), \frac{1}{|z+e|^{N+1}} g\left(\frac{z+e}{|z+e|}\right)\right\} dz > 0.$$

On the functions W, σ and u_0 we assume:

(H3) $W \in C^{1,1}(\mathbb{R})$ and $W(v+1) = W(v)$ for any $v \in \mathbb{R}$;

(H4) $\sigma \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\sigma(t+1,x) = \sigma(t,x)$, $\sigma(t,x+k) = \sigma(t,x)$ for any $k \in \mathbb{Z}^N$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$;

(H5) $u_0 \in W^{2,\infty}(\mathbb{R}^N)$.

We prove that the limit u^0 of u^ϵ as $\epsilon \to 0$ exists and is the unique solution of the homogenized problem

$$\begin{cases} \partial_t u = \overline{H}(\nabla_x u, \mathcal{I}_1[u(t,\cdot)]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.0.2}$$

for some continuous function $\overline{H}$ usually called *effective Hamiltonian*.

Notation.

It is convenient to introduce the singular measure defined on $\mathbb{R}^N \setminus \{0\}$ by

$$\mu(dz) = \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz = \mu_0(z)dz,$$

and to denote

$$\mathcal{I}_1^{1,r}[U,x] = \int_{|z|\le r} (U(x+z) - U(x) - \nabla U(x)\cdot z)\mu(dz),$$

$$\mathcal{I}_1^{2,r}[U,x] = \int_{|z|>r} (U(x+z) - U(x))\mu(dz),$$

$$\mathcal{I}_1[U,x] = \mathcal{I}_1^{1,r}[U,x] + \mathcal{I}_1^{2,r}[U,x].$$

Sometimes when $r = 1$ we will omit r and we will write simply $\mathcal{I}_1^1$ and $\mathcal{I}_1^2$.

See the Appendix for further notations used in this chapter.

4.1 Main results

As usual in periodic homogenization, the limit equation is determined by a *cell problem*. In our case, such a problem is for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the following:

$$\begin{cases} \lambda + \partial_\tau v = \mathcal{I}_1[v(\tau,\cdot)] + L - W'(v + \lambda\tau + p\cdot y) + \sigma(\tau,y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ v(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.1.3}$$

where $\lambda = \lambda(p,L)$ is the unique number for which there exists a solution of (4.1.3) which is bounded on $\mathbb{R}^+ \times \mathbb{R}^N$. In order to solve (4.1.3), we show for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the existence of a unique solution of

$$\begin{cases} \partial_\tau w = \mathcal{I}_1[w(\tau,\cdot)] + L - W'(w + p\cdot y) + \sigma(\tau,y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ w(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.1.4}$$

and we look for some $\lambda \in \mathbb{R}$ for which $w - \lambda\tau$ is bounded. Precisely we have:

Theorem 4.1.1 (Ergodicity)**.** *Assume (H1)-(H5). For $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique viscosity solution $w \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ of* (4.1.4) *and there exists a unique $\lambda \in \mathbb{R}$ such that w satisfies: $\frac{w(\tau,y)}{\tau}$ converges towards λ as $\tau \to +\infty$, locally uniformly in y. The real number λ is denoted by $\overline{H}(p,L)$. The function $\overline{H}(p,L)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$ and non-decreasing in L.*

With the aid of the bounded solution of (4.1.3), usually called *corrector*, we prove the convergence of the sequence u^ϵ to the solution of (4.0.2).

Theorem 4.1.2 (Convergence)**.** *Assume (H1)-(H5). The solution u^ϵ of* (4.0.1) *converges towards the solution u^0 of* (4.0.2) *locally uniformly in (t,x), where $\overline{H}$ is defined in Theorem 4.1.1.*

4.2 Results about viscosity solutions for non-local equations

The classical notion of viscosity solution can be adapted for Hamilton-Jacobi equations involving non-local operators, see for instance [12]. In this section we state comparison principles, existence and regularity results for viscosity solutions of (4.0.1) and (4.0.2), that will be used later in the proofs.

4.2.1 Definition of viscosity solution

We first recall the definition of viscosity solution for a general first order non-local equation with associated an initial condition:

$$\begin{cases} u_t = F(t,x,u,Du,\mathcal{I}_1[u]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.2.5}$$

where $F(t,x,u,p,L)$ is continuous and non-decreasing in L.

Definition 4.2.1 (r-viscosity solution)**.** *A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is a r-viscosity subsolution (resp., supersolution) of* (4.2.5) *if $u(0,x) \le (u_0)^*(x)$ (resp., $u(0,x) \ge (u_0)_*(x)$) and for any $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$, any $\tau \in (0,t_0)$ and any test function $\phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^N)$ such that $u-\phi$ attains a local maximum (resp., minimum) at the point (t_0,x_0) on $Q_{(\tau,r)}(t_0,x_0)$, then we have*

$$\partial_t\phi(t_0,x_0) - F(t_0,x_0,u(t_0,x_0),\nabla_x\phi(t_0,x_0),\mathcal{I}_1^{1,r}[\phi(t_0,\cdot),x_0] + \mathcal{I}_1^{2,r}[u(t_0,\cdot),x_0]) \le 0$$
(resp., ≥ 0).

A function $u \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ is a r-viscosity solution of (4.2.5) *if it is a r-viscosity sub and supersolution of* (4.2.5).

It is classical that the maximum in the above definition can be supposed to be global and this will be used later. We have also the following property, see e.g. [12]:

Proposition 4.2.1 (Equivalence of the definitions)**.** *Assume $F(t,x,u,p,L)$ continuous and non-decreasing in L. Let $r > 0$ and $r' > 0$. A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is a r-viscosity subsolution (resp., supersolution) of* (4.2.5) *if and only if it is a r'-viscosity subsolution (resp., supersolution) of* (4.2.5).

Because of this proposition, if we do not need to emphasize r, we will omit it when calling viscosity sub and supersolutions.

4.2.2 Comparison principle and existence results

In this subsection, we successively give comparison principles and existence results for (4.0.1) and (4.0.2). The following comparison theorem is shown in [81] for more general parabolic integro-PDEs.

Proposition 4.2.2 (Comparison Principle for (4.0.1))**.** *Consider $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ subsolution and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ supersolution of* (4.0.1)*, then $u \le v$ on $\mathbb{R}^+ \times \mathbb{R}^N$.*

Following [81] it can also be proved the comparison principle for (4.0.1) in bounded domains. Since we deal with a non-local equation, we need to compare the sub and the supersolution everywhere outside the domain.

Proposition 4.2.3 (Comparison Principle on bounded domains for (4.0.1))**.** *Let Ω be a bounded domain of $\mathbb{R}^+\times\mathbb{R}^N$ and let $u\in USC_b(\mathbb{R}^+\times\mathbb{R}^N)$ and $v\in LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$ be respectively a sub and a supersolution of*

$$\partial_\tau u^\epsilon = \mathcal{I}_1[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon},\frac{x}{\epsilon}\right)$$

in Ω. If $u\le v$ outside Ω, then $u\le v$ in Ω.

Proposition 4.2.4 (Existence for (4.0.1))**.** *For $\epsilon>0$ there exists $u^\epsilon\in C_b(\mathbb{R}^+\times\mathbb{R}^N)$ (unique) viscosity solution of* (4.0.1)*. Moreover, there exists a constant $C>0$ independent of ϵ such that*

$$|u^\epsilon(t,x)-u_0(x)|\le Ct. \tag{4.2.6}$$

Proof. Adapting the argument of [65], we can construct a solution by Perron's method if we construct sub and supersolutions of (4.0.1). Since $u_0\in W^{2,\infty}$, the two functions $u^{\pm}(t,x):=u_0(x)\pm Ct$ are respectively a super and a subsolution of (4.0.1) for any $\epsilon>0$, if

$$C\ge D_N\|u_0\|_{2,\infty}+\|W'\|_\infty+\|\sigma\|_\infty,$$

with D_N depending on the dimension N. By comparison we also get the estimate (4.2.6). □

We next recall the comparison and the existence results for (4.0.2).

Proposition 4.2.5 ([68], Proposition 3)**.** *Let $\overline{H}:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be continuous with $\overline{H}(p,\cdot)$ non-decreasing on $\mathbb{R}$ for any $p\in\mathbb{R}^N$. If $u\in USC_b(\mathbb{R}^+\times\mathbb{R}^N)$ and $v\in LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$ are respectively a sub and a supersolution of* (4.0.2)*, then $u\le v$ on $\mathbb{R}^+\times\mathbb{R}^N$. Moreover there exists a (unique) viscosity solution of* (4.0.2)*.*

In the next sections, we will embed the problem in the higher dimensional space $\mathbb{R}^+\times\mathbb{R}^{N+1}$ by adding a new variable x_{N+1} in the equations. We will need the following proposition showing that sub and supersolutions of the higher dimensional problem are also sub and supersolutions of the lower dimensional one. This in particular implies that the comparison principle between sub and supersolutions remains true increasing the dimension.

Proposition 4.2.6. *Assume $F(t,x,x_{N+1},U,p,L)$ continuous and non-decreasing in L. Suppose that $U\in LSC_b(\mathbb{R}^+\times\mathbb{R}^{N+1})$ (resp., $U\in USC_b(\mathbb{R}^+\times\mathbb{R}^{N+1})$ is a viscosity supersolution (resp., subsolution) of*

$$U_t = F(t,x,x_{N+1},U,D_xU,\mathcal{I}_1[U(t,\cdot,x_{N+1})]) \quad in \quad \mathbb{R}^+\times\mathbb{R}^{N+1},$$

then, for any $x_{N+1}\in\mathbb{R}$ U is a viscosity supersolution (resp., subsolution) of

$$U_t = F(t,x,x_{N+1},U,D_xU,\mathcal{I}_1[U(t,\cdot,x_{N+1})]) \quad in \quad \mathbb{R}^+\times\mathbb{R}^{N}.$$

Proof. We show the result for supersolutions. Fix $x^0_{N+1} \in \mathbb{R}$. Let us consider a point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ and a smooth function $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ such that

$$U(t, x, x^0_{N+1}) - \varphi(t, x) \geq U(t_0, x_0, x^0_{N+1}) - \varphi(t_0, x_0) = 0 \quad \text{for } (t, x) \in Q_{\tau,r}(t_0, x_0).$$

We have to show that

$$\partial_t \varphi(t_0, x_0) \geq F(t_0, x_0, x^0_{N+1}, U(t_0, x_0, x^0_{N+1}), D_x\varphi(t_0, x_0), \mathcal{I}^1_1[\varphi(t_0, \cdot), x_0] + \mathcal{I}^2_1[U(t_0, \cdot, x^0_{N+1}), x_0]).$$

Without loss of generality, we can assume that the minimum is strict. For $\epsilon > 0$ let $\varphi_\epsilon : \mathbb{R}^+ \times \mathbb{R}^{N+1} \to \mathbb{R}$ be defined by

$$\varphi_\epsilon(t, x, x_{N+1}) = \varphi(t, x) - \frac{1}{\epsilon}|x_{N+1} - x^0_{N+1}|^2.$$

Let $(t_\epsilon, x_\epsilon, x^\epsilon_{N+1})$ be a minimum point of $U - \varphi_\epsilon$ in $Q_{\tau,r}(t_0, x_0, x^0_{N+1})$. Standard arguments show that $(t_\epsilon, x_\epsilon, x^\epsilon_{N+1}) \to (t_0, x_0, x^0_{N+1})$ as $\epsilon \to 0$ and that $\lim_{\epsilon\to 0} U(t_\epsilon, x_\epsilon, x^\epsilon_{N+1}) = U(t_0, x_0, x^0_{N+1})$. In particular, $(t_\epsilon, x_\epsilon, x^\epsilon_{N+1})$ is internal to $Q_{\tau,r}(t_0, x_0, x^0_{N+1})$ for ϵ small enough, then we get

$$\partial_t \varphi(t_\epsilon, x_\epsilon) \geq F(t_\epsilon, x_\epsilon, U(t_\epsilon, x_\epsilon, x^\epsilon_{N+1}), D_x\varphi(t_\epsilon, x_\epsilon), \mathcal{I}^1_1[\varphi(t_\epsilon, \cdot), x_\epsilon] + \mathcal{I}^2_1[U(t_\epsilon, \cdot, x^\epsilon_{N+1}), x_\epsilon]). \tag{4.2.7}$$

By the Dominate Convergence Theorem $\lim_{\epsilon\to 0} \mathcal{I}^1_1[\varphi(t_\epsilon, \cdot), x_\epsilon] = \mathcal{I}^1_1[\varphi(t_0, \cdot), x_0]$; by the Fatou's Lemma and the convergence of $U(t_\epsilon, x_\epsilon, x^\epsilon_{N+1})$ to $U(t_0, x_0, x^0_{N+1})$, we deduce that

$$\mathcal{I}^2_1[U(t_0, \cdot, x^0_{N+1}), x_0] \leq \liminf_{\epsilon\to 0} \mathcal{I}^2_1[U(t_\epsilon, \cdot, x^\epsilon_{N+1}), x_\epsilon].$$

Then, passing to the limit in (4.2.7) and using the continuity and monotonicity of F, we get the desired inequality. □

4.2.3 Hölder regularity

In this subsection we state and prove a regularity result for sub and supersolutions of semilinear non-local equations.

Proposition 4.2.7 (Hölder regularity)**.** *Assume (H1) and let g_1, $g_2 : \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be bounded functions. Suppose that $u \in C(\mathbb{R}^+ \times \mathbb{R}^N)$ and bounded on $\mathbb{R}^+ \times \mathbb{R}^N$ is a viscosity subsolution of*

$$\begin{cases} \partial_t u = \mathcal{I}_1[u(t, \cdot)] + g_1(t, x, u) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0, x) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases}$$

and a viscosity supersolution of

$$\begin{cases} \partial_t u = \mathcal{I}_1[u(t, \cdot)] + g_2(t, x, u) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0, x) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

Then, for any $0 < \alpha < 1$, $u \in C^\alpha_x(\mathbb{R}^+ \times \mathbb{R}^N)$ with $< u >^\alpha_x \leq C$, where C depends on $\|u\|_\infty$, $\|g_1\|_\infty$ and $\|g_2\|_\infty$.

Proof.

Suppose by contradiction that u does not belong to $C^\alpha_x(\mathbb{R}^+\times\mathbb{R}^N)$. Let $u^{\epsilon,\epsilon'}$ and $u_{\epsilon,\epsilon'}$ be respectively the double-parameters sup and inf convolution of u in $\mathbb{R}^+\times\mathbb{R}^N$, i.e.

$$u^{\epsilon,\epsilon'}(t,x)=\sup_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^N}\left(u(s,y)-\frac{1}{2\epsilon}|x-y|^2-\frac{1}{2\epsilon'}(t-s)^2\right),$$

$$u_{\epsilon,\epsilon'}(t,x)=\inf_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^N}\left(u(s,y)+\frac{1}{2\epsilon}|x-y|^2+\frac{1}{2\epsilon'}(t-s)^2\right).$$

Then $u^{\epsilon,\epsilon'}$ is semiconvex and is a subsolution of

$$\partial_t u^{\epsilon,\epsilon'}=\mathcal{I}_1[u^{\epsilon,\epsilon'}(t,\cdot)]+g_1^{\epsilon,\epsilon'}(t,x)\quad\text{in}\quad(t_{\epsilon'},+\infty)\times\mathbb{R}^N$$

and $u_{\epsilon,\epsilon'}$ is semiconcave and is a supersolution of

$$\partial_t u_{\epsilon,\epsilon'}=\mathcal{I}_1[u_{\epsilon,\epsilon'}(t,\cdot)]+g_2^{\epsilon,\epsilon'}(t,x)\quad\text{in}\quad(t_{\epsilon'},+\infty)\times\mathbb{R}^N,$$

where $t_{\epsilon'}\to 0$ as $\epsilon'\to 0$ and $g_1^{\epsilon,\epsilon'}$ and $g_2^{\epsilon,\epsilon'}$ are bounded functions such that $g_i^{\epsilon,\epsilon'}(t,x)\to g_i(t,x,u(t,x))$ as $\epsilon,\epsilon'\to 0$, $\|g_i^{\epsilon,\epsilon'}\|_\infty=\|g_i\|_\infty$, $i=1,2$, see e.g. [12]. Let us consider smooth functions $\psi_1(t)$ and $\psi_2(x)$ with bounded first and second derivatives such that $\psi_1(t)\to+\infty$ as $t\to+\infty$, $\psi_2(x)\to+\infty$ as $|x|\to+\infty$ and there exists $K_0>0$ such that $|\psi_2(x)|\le K_0(1+\sqrt{x})$. Then, for any $K>0$ and ϵ' and β small enough, the supremum of the function $u^{\epsilon,\epsilon'}(t,x_1)-u_{\epsilon,\epsilon'}(t,x_2)-\phi(t,x_1,x_2)$ on $\mathbb{R}^+\times\mathbb{R}^{2N}$, where $\phi(t,x_1,x_2)=K|x_1-x_2|^\alpha+\beta\psi_1(t)+\beta\psi_2(x_1)$, is positive and is attained at some point $(\bar{t},\bar{x}_1,\bar{x}_2)\in(t_{\epsilon'},+\infty)\times\mathbb{R}^{2N}$, with $\bar{x}_1\neq\bar{x}_2$. Remark that

$$|\bar{x}_1-\bar{x}_2|\le\left(\frac{2\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^N}|u(t,x)|}{K}\right)^{\frac{1}{\alpha}}.$$

In order to apply the Jensen's Lemma, see e.g. Lemma A.3 of [40], we have to transform $(\bar{t},\bar{x}_1,\bar{x}_2)$ into a strict maximum point. To do so, we consider a smooth bounded function $h:\mathbb{R}^+\to\mathbb{R}$, with bounded derivatives, such that $h(0)=0$ and $h(s)>0$ for $s>0$ and we set $\theta(t,x_1,x_2)=h((t-\bar{t})^2)+h(|x_1-\bar{x}_1|^2)+h(|x_2-\bar{x}_2|^2)$. Next we consider a smooth function $\chi:\mathbb{R}^N\to\mathbb{R}$ such that $\chi(x)=1$ if $|x|\le 1/4$ and $\chi(x)=0$ for $|x|\ge 1/2$. Clearly $(\bar{t},\bar{x}_1,\bar{x}_2)$ is a strict maximum point of $u^{\epsilon,\epsilon'}(t,x_1)-u_{\epsilon,\epsilon'}(t,x_2)-\phi(t,x_1,x_2)-\theta(t,x_1,x_2)$ and by Jensen's Lemma, for every small and positive δ there exist $t^\delta\in\mathbb{R}$, q_1^δ, $q_2^\delta\in\mathbb{R}^N$ with $|t^\delta|,|q_1^\delta|,|q_2^\delta|\le\delta$ such that the function

$$u^{\epsilon,\epsilon'}(t,x_1)-u_{\epsilon,\epsilon'}(t,x_2)-K|x_1-x_2|^\alpha-\varphi_1(t,x_1)-\varphi_2(x_2),\tag{4.2.8}$$

where

$$\varphi_1(t,x_1)=\beta\psi_1(t)+\beta\psi_2(x_1)+h((t-\bar{t})^2)+h(|x_1-\bar{x}_1|^2)+t^\delta t+\chi(x_1-\bar{x}_1)q_1^\delta\cdot x_1,$$

$$\varphi_2(x_2)=h(|x_2-\bar{x}_2|^2)+\chi(x_2-\bar{x}_2)q_2^\delta\cdot x_2,$$

has a maximum at $(t^\delta,x_1^\delta,x_2^\delta)$ with $|t_\delta-\bar{t}|$, $|x_1^\delta-\bar{x}_1|$, $|x_2^\delta-\bar{x}_2|\le\delta$ and $u^{\epsilon,\epsilon'}(t,x_1)-u_{\epsilon,\epsilon'}(t,x_2)$ twice differentiable at $(t^\delta,x_1^\delta,x_2^\delta)$. In particular $u^{\epsilon,\epsilon'}$ is twice differentiable w.r.t. x_1 at (t^δ, x_1^δ) and $u_{\epsilon,\epsilon'}$ is twice differentiable with respect to x_2 at (t^δ, x_2^δ). Moreover, the fact that $(t^\delta, x_1^\delta, x_2^\delta)$ is a maximum point implies that

$$\nabla_{x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) = \nabla_{x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) + \nabla_{x_1}\varphi_1(t^\delta, x_1^\delta) + \nabla_{x_2}\varphi_2(x_2^\delta),$$

and for any $z \in \mathbb{R}^N$

$$\begin{aligned} &u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta + z) - u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) - \nabla_{x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) \cdot z\mathbf{1}_B(z) \\ &\le u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta + z) - u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) - \nabla_{x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) \cdot z\mathbf{1}_B(z) \\ &+ \varphi_1(t^\delta, x_1^\delta + z) - \varphi_1(t^\delta, x_1^\delta) - \nabla_{x_1}\varphi_1(t^\delta, x_1^\delta) \cdot z\mathbf{1}_B(z) \\ &+ \varphi_2(x_2^\delta + z) - \varphi_2(x_2^\delta) - \nabla_{x_2}\varphi_2(x_2^\delta) \cdot z\mathbf{1}_B(z), \end{aligned} \tag{4.2.9}$$

where we denote by B the unit ball of $\mathbb{R}^N$ and by $\mathbf{1}_B(z)$ the indicator function of B. The last inequality, in particular implies that

$$\mathcal{I}_1^2[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] \le \mathcal{I}_1^2[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta] + \mathcal{I}_1^2[\varphi_1(t^\delta, \cdot), x_1^\delta] + \mathcal{I}_1^2[\varphi_2, x_2^\delta].$$

By doubling the variables and passing to the limit, we can obtain the following viscosity inequalities

$$a \le \mathcal{I}_1[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] + g_1^{\epsilon,\epsilon'}(t^\delta, x_1^\delta),$$

$$b \ge \mathcal{I}_1[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta] + g_2^{\epsilon,\epsilon'}(t^\delta, x_2^\delta),$$

with $a - b = \partial_t\varphi_1(t^\delta, x_1^\delta)$. Then, using (4.2.9), we get

$$\begin{aligned} a &\le \mathcal{I}_1[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] + g_1^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) \\ &\le \mathcal{I}_1[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta] + g_2^{\epsilon,\epsilon'}(t^\delta, x_2^\delta) + \|g_1\|_\infty + \|g_2\|_\infty \\ &+ \mathcal{I}_1^2[\varphi_1(t^\delta, \cdot), x_1^\delta] + \mathcal{I}_1^2[\varphi_2, x_2^\delta] + (\mathcal{I}_1^1[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] - \mathcal{I}_1^1[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta]) \\ &\le b + \|g_1\|_\infty + \|g_2\|_\infty + \mathcal{I}_1^2[\varphi_1(t^\delta, \cdot), x_1^\delta] + \mathcal{I}_1^2[\varphi_2, x_2^\delta] \\ &+ (\mathcal{I}_1^1[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] - \mathcal{I}_1^1[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta]). \end{aligned} \tag{4.2.10}$$

Now, let us estimate the term $\mathcal{I}_1^1[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] - \mathcal{I}_1^1[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta]$. For δ small enough, we can assume $x_1^\delta \neq x_2^\delta$. Since $(t^\delta, x_1^\delta, x_2^\delta)$ is a maximum point for the function (4.2.8) we have

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

where

$$X = D^2_{x_1x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) - D^2_{x_1x_1}\varphi_1(t^\delta, x_1^\delta), \quad Y = -D^2_{x_2x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) - D^2_{x_2x_2}\varphi(x_2^\delta),$$

$$B = \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(I + (\alpha - 2)P) \quad \text{and} \quad P = \frac{(x_1^\delta - x_2^\delta) \otimes (x_1^\delta - x_2^\delta)}{|x_1^\delta - x_2^\delta|^2},$$

where $\otimes$ is the tensor product of matrices. This implies that $X + Y \le 0$ and $X + Y \le 4B$. We need a more precise estimate as in [75]. Since $0 \le P \le I$, using the properties of symmetric matrices one has

$$X + Y \le (X + Y)P \le 4PB,$$

that is

$$D^2_{x_1x_1}u^{\epsilon,\epsilon'}(t^\delta,x_1^\delta)-D^2_{x_2x_2}u_{\epsilon,\epsilon'}(t^\delta,x_2^\delta)\le 4PB+D^2_{x_1x_1}\varphi_1(t^\delta,x_1^\delta)+D^2_{x_2x_2}\varphi(x_2^\delta). \tag{4.2.11}$$

Now, we are ready to estimate $\mathcal{I}_1^1[u^{\epsilon,\epsilon'}(t^\delta,\cdot),x_1^\delta]-\mathcal{I}_1^1[u_{\epsilon,\epsilon'}(t^\delta,\cdot),x_2^\delta]$. We first remark that from (4.2.9) we have

$$\mathcal{I}_1^1[u^{\epsilon,\epsilon'}(t^\delta,\cdot),x_1^\delta]-\mathcal{I}_1^1[u_{\epsilon,\epsilon'}(t^\delta,\cdot),x_2^\delta]=\frac{1}{2}\int_{|z|\le 1}\left\{z^t\left[D^2_{x_1x_1}u^{\epsilon,\epsilon'}(t^\delta,x_1^\delta)-D^2_{x_2x_2}u_{\epsilon,\epsilon'}(t^\delta,x_2^\delta)\right]z\right.$$
$$\left.+\,o_{\epsilon,\epsilon',\delta}(|z|^2)\right\}\mu(dz)\le C$$

where C is independent of the parameters. Moreover, by semiconvexity we have

$$\frac{1}{2}\int_{|z|\le 1}\left\{z^t\left[D^2_{x_1x_1}u^{\epsilon,\epsilon'}(t^\delta,x_1^\delta)-D^2_{x_2x_2}u_{\epsilon,\epsilon'}(t^\delta,x_2^\delta)\right]z\right\}\mu(dz)\ge-\frac{1}{\epsilon}\int_{|z|\le 1}|z|^2\mu(dz).$$

The two previous inequalities imply that the quantity $\int_{|z|\le 1}o_{\epsilon,\epsilon',\delta}(|z|^2)\mu(dz)$ is bounded from above by $C+\frac{1}{\epsilon}\int_{|z|\le 1}|z|^2\mu(dz)$. Using this estimate, (4.2.11) and that P is idempotent, i.e., $P^2=P$, we get

$$\begin{aligned}\mathcal{I}_1^1[u^{\epsilon,\epsilon'}(t^\delta,\cdot),x_1^\delta]-\mathcal{I}_1^1[u_{\epsilon,\epsilon'}(t^\delta,\cdot),x_2^\delta]&\le 2\int_{|z|\le 1}z^tBPz\mu(dz)+\widetilde{C}+\frac{1}{\epsilon}\int_{|z|\le 1}|z|^2\mu(dz)\\ =-2\alpha K|x_1^\delta-x_2^\delta|^{\alpha-2}(1-\alpha)\int_{|z|\le 1}z^tPz\mu(dz)&+\widetilde{C}+\frac{1}{\epsilon}\int_{|z|\le 1}|z|^2\mu(dz).\end{aligned} \tag{4.2.12}$$

Remark that by (H1)

$$\int_{|z|\le 1}z^tPz\mu(dz)=C_N>0.$$

Finally, from (4.2.10) and (4.2.12), we obtain

$$\begin{aligned}2\alpha K|x_1^\delta-x_2^\delta|^{\alpha-2}(1-\alpha)C_N&\le\widetilde{C}+\frac{\widetilde{C}_N}{\epsilon}-\partial_t\varphi_1(t^\delta,x_1^\delta)+\|g_1\|_\infty+\|g_2\|_\infty\\&\quad+\mathcal{I}_1^2[\varphi_1(t^\delta,\cdot),x_1^\delta]+\mathcal{I}_1^2[\varphi_2,x_2^\delta]\\&\le\widetilde{C}+\frac{\widetilde{C}_N}{\epsilon}+\beta\|\psi_1'\|_\infty+\|h'\|_\infty+C\delta+\|g_1\|_\infty+\|g_2\|_\infty\\&\quad+C\beta K_0+C\|h\|_\infty,\end{aligned}$$

which is a contradiction for K large enough and fixed ϵ. Hence $u\in C_x^\alpha(\mathbb{R}^+\times\mathbb{R}^N)$. □

4.3 The proof of convergence

This section is dedicated to the proof of Theorem 4.1.2. Before presenting it, we first imbed the problem in a higher dimensional one. Precisely, we consider U^ϵ solution of

$$\begin{cases}\partial_t U^\epsilon=\mathcal{I}_1[U^\epsilon(t,\cdot,x_{N+1})]-W'\left(\frac{U^\epsilon}{\epsilon}\right)+\sigma\left(\frac{t}{\epsilon},\frac{x}{\epsilon}\right) & \text{in}\quad\mathbb{R}^+\times\mathbb{R}^{N+1}\\ U^\epsilon(0,x,x_{N+1})=u_0(x)+x_{N+1} & \text{on}\quad\mathbb{R}^{N+1}.\end{cases} \tag{4.3.13}$$

By Proposition 4.2.6 and Proposition 4.2.2, the comparison principle holds true for (4.3.13). Then, as in the proof of Proposition 4.2.4, by Perron's method we have:

Proposition 4.3.1 (Existence for (4.3.13))**.** *For $\epsilon > 0$ there exists $U^\epsilon \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ (unique) viscosity solution of* (4.3.13)*. Moreover, there exists a constant $C > 0$ independent of ϵ such that*

$$|U^\epsilon(t, x, x_{N+1}) - u_0(x) - x_{N+1}| \leq Ct. \tag{4.3.14}$$

Let us exhibit the link between the problem in $\mathbb{R}^N$ and the problem in $\mathbb{R}^{N+1}$.

Lemma 4.3.2 (Link between the problems on $\mathbb{R}^N$ and on $\mathbb{R}^{N+1}$)**.** *If u^ϵ and U^ϵ denote respectively the solution of* (4.0.1) *and* (4.3.13)*, then we have*

$$\left| U^\epsilon(t, x, x_{N+1}) - u^\epsilon(t, x) - \epsilon \left\lfloor \frac{x_{N+1}}{\epsilon} \right\rfloor \right| \leq \epsilon,$$

$$U^\epsilon \left(t, x, x_{N+1} + \epsilon \left\lfloor \frac{a}{\epsilon} \right\rfloor \right) = U^\epsilon(t, x, x_{N+1}) + \epsilon \left\lfloor \frac{a}{\epsilon} \right\rfloor \quad \text{for any } a \in \mathbb{R}. \tag{4.3.15}$$

This lemma is a consequence of comparison principle for (4.3.13) and of invariance by ϵ-translations w.r.t. x_{N+1}.

We need to make more precise the dependence of the real number λ given by Theorem 4.1.1 on its variables. The following properties will be shown in the next section.

Proposition 4.3.3 (Properties of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$. Let $\overline{H}(p, L)$ be the constant defined by Theorem 4.1.1, then $\overline{H} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the following properties:*

(i) $\overline{H}(p, L) \to \pm\infty$ *as* $L \to \pm\infty$ *for any* $p \in \mathbb{R}^N$*;*

(ii) $\overline{H}(p, \cdot)$ *is non-decreasing on* $\mathbb{R}$ *for any* $p \in \mathbb{R}^N$*;*

(iii) If $\sigma(\tau, y) = \sigma(\tau, -y)$ *then*

$$\overline{H}(p, L) = \overline{H}(-p, L);$$

(iv) If $W'(-s) = -W'(s)$ *and* $\sigma(\tau, -y) = -\sigma(\tau, y)$ *then*

$$\overline{H}(p, -L) = -\overline{H}(p, L).$$

In the proof of convergence, we will use smooth approximate sub and super-correctors on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$. More precisely, we consider for $P = (p, 1) \in \mathbb{R}^{N+1}$ and $L \in \mathbb{R}$:

$$\begin{cases} \lambda + \partial_\tau V = L + \mathcal{I}_1[U(\tau, \cdot, y_{N+1})] - W'(V + P \cdot Y + \lambda\tau) + \sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{4.3.16}$$

Here and in what follows, we denote $Y = (y, y_{N+1})$. We will use also the notation $X = (x, x_{N+1})$.

Then, we have the following proposition.

Proposition 4.3.4 (Smooth approximate correctors)**.** *Let λ be the constant defined by Theorem 4.1.1. For any fixed $p \in \mathbb{R}^N$, $P = (p, 1)$, $L \in \mathbb{R}$ and $\eta > 0$ small enough, there exist real numbers $\lambda^+_\eta(p, L)$, $\lambda^-_\eta(p, L)$, a constant $C > 0$ (independent of η, p and L) and bounded super and subcorrectors V^+_η, V^-_η i.e. respectively a super and a subsolution of*

$$\begin{cases} \lambda_\eta^{\overset{+}{-}} + \partial_\tau V_\eta^{\overset{+}{-}} = L + \mathcal{I}_1[V_\eta^{\overset{+}{-}}(\tau, \cdot, y_{N+1})] \\ \qquad\qquad -W'(V_\eta^{\overset{+}{-}} + P \cdot Y + \lambda_\eta^{\overset{+}{-}}\tau) + \sigma(\tau, y) \overset{-}{+} o_\eta(1) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V_\eta^{\overset{+}{-}}(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{4.3.17}$$

where $0 \le o_\eta(1) \to 0$ as $\eta \to 0^+$, such that

$$\lim_{\eta \to 0^+} \lambda^+_\eta(p, L) = \lim_{\eta \to 0^+} \lambda^-_\eta(p, L) = \lambda(p, L), \tag{4.3.18}$$

$\lambda_\eta^{\overset{+}{-}}$ satisfy (i) and (ii) of Proposition 4.3.3 and for any $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$

$$|V_\eta^{\overset{+}{-}}(\tau, Y)| \le C. \tag{4.3.19}$$

Moreover $V_\eta^{\overset{+}{-}}$ are of class C^2 w.r.t. y_{N+1}, and for any $0 < \alpha < 1$

$$-1 \le \partial_{y_{N+1}} V_\eta^{\overset{+}{-}} \le \frac{\|W''\|_\infty}{\eta}, \tag{4.3.20}$$

$$< \partial_{y_{N+1}} V_\eta^{\overset{+}{-}} >^\alpha_y, \ \|\partial^2_{y_{N+1}y_{N+1}} V_\eta^{\overset{+}{-}}\|_\infty \le C_\eta. \tag{4.3.21}$$

4.3.1 Proof of Theorem 4.1.2

By (4.3.14), we know that the family of functions $\{U^\epsilon\}_{\epsilon > 0}$ is locally bounded, then $U^+ := \limsup^*_{\epsilon \to 0} U^\epsilon$ is everywhere finite. Classically we prove that U^+ is a subsolution of

$$\begin{cases} \partial_t U = \overline{H}(\nabla_x U, \mathcal{I}_1[U(t, \cdot, x_{N+1})]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ U(0, x, x_{N+1}) = u_0(x) + x_{N+1} & \text{on} \quad \mathbb{R}^{N+1}. \end{cases} \tag{4.3.22}$$

Similarly, we can prove that $U^- = \liminf_{*\epsilon \to 0} U^\epsilon$ is a supersolution of (4.3.22). Moreover $U^+(0, x, x_{N+1}) = U^-(0, x, x_{N+1}) = u_0(x) + x_{N+1}$. The comparison principle for (4.3.22), which is an immediate consequence of Propositions 4.2.5 and 4.2.6, then implies that $U^+ \le U^-$. Since the reverse inequality $U^- \le U^+$ always holds true, we conclude that the two functions coincide with U^0, the unique viscosity solution of (4.3.22).

The link between problems (4.0.2) and (4.3.22) is given by the following lemma.

Lemma 4.3.5. *Let u^0 and U^0 be respectively the solutions of* (4.0.2) *and* (4.3.22)*. Then, we have*

$$U^0(t, x, x_{N+1}) = u^0(t, x) + x_{N+1},$$

$$U^0(t, x, x_{N+1} + a) = U^0(t, x, x_{N+1}) + a.$$

Lemma 4.3.5 is a consequence of comparison principle for (4.3.22) and the invariance by translations w.r.t. y_{N+1}.

By Lemmata 4.3.2 and 4.3.5, the convergence of U^ϵ to U^0 proves in particular that u^ϵ converges towards u^0 viscosity solution of (4.0.2).

We argue by contradiction. We consider a test function ϕ such that $U^+ - \phi$ attains a strict global zero maximum at (t_0, X_0) with $t_0 > 0$ and $X_0 = (x_0, x^0_{N+1})$, and we suppose that there exists $\theta > 0$ such that

$$\partial_t \phi(t_0, X_0) = \overline{H}(\nabla_x \phi(t_0, X_0), L_0) + \theta,$$

where

$$\begin{aligned} L_0 = &\int_{|x|\le 1} (\phi(t_0, x_0 + x, x^0_{N+1}) - \phi(t_0, X_0) - \nabla_x \phi(t_0, X_0) \cdot x)\mu(dx) \\ &+ \int_{|x|>1} (U^+(t_0, x_0 + x, x^0_{N+1}) - U^+(t_0, X_0))\mu(dx). \end{aligned} \tag{4.3.23}$$

By Proposition 4.3.3, we know that there exists $L_1 > 0$ such that

$$\overline{H}(\nabla_x \phi(t_0, X_0), L_0) + \theta = \overline{H}(\nabla_x \phi(t_0, X_0), L_0 + L_1).$$

By Proposition 4.3.4, we can consider a sequence $L_\eta \to L_1$ as $\eta \to 0^+$, such that $\lambda^+_\eta(\nabla_x \phi(t_0, X_0), L_0 + L_\eta) = \lambda(\nabla_x \phi(t_0, X_0), L_0 + L_1)$. We choose η so small that $L_\eta - o_\eta(1) \ge L_1/2 > 0$, where $o_\eta(1)$ is defined in Proposition 4.3.4. Let V^+_η be the approximate supercorrector given by Proposition 4.3.4 with

$$p = \nabla_x \phi(t_0, X_0), \quad L = L_0 + L_\eta$$

and

$$\lambda^+_\eta = \lambda^+_\eta(p, L_0 + L_\eta) = \partial_t \phi(t_0, X_0).$$

For simplicity of notations, in the following we denote $V = V^+_\eta$. We consider the function $F(t, X) = \phi(t, X) - p \cdot x - \lambda t$, and as in [67] and [68] we introduce the "x_{N+1}-twisted perturbed test function" ϕ^ϵ defined by:

$$\phi^\epsilon(t, X) := \begin{cases} \phi(t, X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon & \text{in} \quad (\frac{t_0}{2}, 2t_0) \times B_{\frac{1}{2}}(X_0) \\ U^\epsilon(t, X) & \text{outside,} \end{cases} \tag{4.3.24}$$

where $k_\epsilon \in \mathbb{Z}$ will be chosen later. We are going to prove that ϕ^ϵ is a supersolution of (4.3.13) in $Q_{r,r}(t_0, X_0)$ for some $r < \frac{1}{2}$ properly chosen and such that $Q_{r,r}(t_0, X_0) \subset (\frac{t_0}{2}, 2t_0) \times B_{\frac{1}{2}}(X_0)$. First, remark that since $U^+ - \phi$ attains a strict maximum at (t_0, X_0) and V is bounded, we can ensure that there exists $\epsilon_0 = \epsilon_0(r) > 0$ such that for $\epsilon \le \epsilon_0$

$$U^\epsilon(t, X) \le \phi(t, X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t, X)}{\epsilon}\right) - \gamma_r, \quad \text{in } \left(\frac{t_0}{3}, 3t_0\right) \times B_1(x_0) \setminus Q_{r,r}(t_0, x_0) \tag{4.3.25}$$

for some $\gamma_r = o_r(1) > 0$. Hence choosing $k_\epsilon = \lceil \frac{-\gamma_r}{\epsilon} \rceil$ we get $U^\epsilon \le \phi^\epsilon$ outside $Q_{r,r}(t_0, X_0)$.

Let us next study the equation. From (4.3.15), we deduce that $U^+(t, x, x_{N+1} + a) = U^+(t, x, x_{N+1}) + a$ for any $a \in \mathbb{R}$, from which we derive that $\partial_{x_{N+1}} F(t_0, X_0) = \partial_{x_{N+1}} \phi(t_0, X_0) = 1$. Then, there exists $r_0 > 0$ such that the map

$$\begin{array}{rccc} Id \times F: & Q_{r_0,r_0}(t_0, X_0) & \longrightarrow & \mathcal{U}_{r_0} \\ & (t, x, x_{N+1}) & \longmapsto & (t, x, F(t, x, x_{N+1})) \end{array}$$

is a C^1-diffeomorphism from $Q_{r_0,r_0}(t_0, X_0)$ onto its range $\mathcal{U}_{r_0}$. Let $G : \mathcal{U}_{r_0} \to \mathbb{R}$ be the map such that

$$\begin{array}{rccc} Id \times G: & \mathcal{U}_{r_0} & \longrightarrow & Q_{r_0,r_0}(t_0, X_0) \\ & (t, x, \xi_{N+1}) & \longmapsto & (t, x, G(t, x, \xi_{N+1})) \end{array}$$

is the inverse of $Id \times F$. Let us introduce the variables $\tau = t/\epsilon$, $Y = (y, y_{N+1})$ with $y = x/\epsilon$ and $y_{N+1} = F(t, X)/\epsilon$. Let us consider a test function ψ such that $\phi^\epsilon - \psi$ attains a global zero minimum at $(\bar{t}, \overline{X}) \in Q_{r_0,r_0}(t_0, X_0)$ and define

$$\Gamma^\epsilon(\tau, Y) = \frac{1}{\epsilon}[\psi(\epsilon\tau, \epsilon y, G(\epsilon\tau, \epsilon y, \epsilon y_{N+1})) - \phi(\epsilon\tau, \epsilon y, G(\epsilon\tau, \epsilon y, \epsilon y_{N+1}))] - k_\epsilon.$$

Then

$$\Gamma^\epsilon(\overline{\tau}, \overline{Y}) = V(\overline{\tau}, \overline{Y}) \quad \text{and} \quad \Gamma^\epsilon(\tau, Y) \le V(\tau, Y) \quad \text{for all } (\epsilon\tau, \epsilon Y) \in Q_{r_0,r_0}(t_0, X_0), \tag{4.3.26}$$

where $\overline{\tau} = \bar{t}/\epsilon$, $\overline{y} = \overline{x}/\epsilon$, $\overline{y}_{N+1} = F(\bar{t}, \overline{X})/\epsilon$, $\overline{Y} = (\overline{y}, \overline{y}_{N+1})$. From Proposition 4.3.4, we know that V is Lipschitz continuous w.r.t. y_{N+1} with Lipschitz constant L_η depending on η. This implies that

$$|\partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y})| \le L_\eta. \tag{4.3.27}$$

Simple computations yield with $P = (p, 1) \in \mathbb{R}^{N+1}$:

$$\begin{cases} \lambda_\eta^+ + \partial_\tau \Gamma^\epsilon(\overline{\tau}, \overline{Y}) = \partial_t \psi(\bar{t}, \overline{X}) + \left(1 - \partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right)(\partial_t \phi(t_0, X_0) - \partial_t \phi(\bar{t}, \overline{X})), \\ p + \nabla_y \Gamma^\epsilon(\overline{\tau}, \overline{Y}) = \nabla_x \psi(\bar{t}, \overline{X}) + \left(1 - \partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right)(\nabla_x \phi(t_0, X_0) - \nabla_x \phi(\bar{t}, \overline{X})), \\ \lambda_\eta^+ \overline{\tau} + P \cdot \overline{Y} + V(\overline{\tau}, \overline{Y}) = \frac{\phi^\epsilon(\bar{t}, \overline{X})}{\epsilon} - k_\epsilon. \end{cases} \tag{4.3.28}$$

Using (4.3.28) and (4.3.27), Equation (4.3.17) yields for any $\rho > 0$

$$\begin{aligned} \partial_t \psi(\bar{t}, \overline{X}) + o_r(1) \ge {} & L_0 + L_\eta + \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] \\ & - W'\left(\frac{\phi^\epsilon(\bar{t}, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) - o_\eta(1). \end{aligned} \tag{4.3.29}$$

We now use the following lemma whose proof is postponed:

Lemma 4.3.6. *For $\epsilon \le \epsilon_0(r) < r \le r_0$, we have*

$$\begin{aligned} \partial_t \psi(\bar{t}, \overline{X}) \ge {} & \mathcal{I}_1^{1,1}\left[\psi(\bar{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\bar{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] \\ & - W'\left(\frac{\phi^\epsilon(\bar{t}, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) - o_\eta(1) + o_r(1) + L_\eta. \end{aligned}$$

Let $r \le r_0$ be so small that $o_r(1) \ge -L_1/4$. Then, recalling that $L_\eta - o_\eta(1) \ge L_1/2$, for $\epsilon \le \epsilon_0(r)$ we have

$$\partial_t \psi(\bar t, \overline{X}) \ge \mathcal{I}_1^{1,1}\left[\psi(\bar t, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\bar t, \cdot, \overline{x}_{N+1}), \overline{x}\right] - W'\left(\frac{\phi^\epsilon(\bar t, \overline{X})}{\epsilon}\right)$$
$$+ \sigma\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) + \frac{L_1}{4},$$

and therefore ϕ^ϵ is a supersolution of (4.3.13) in $Q_{r,r}(t_0, X_0)$. Since $U^\epsilon \le \phi^\epsilon$ outside $Q_{r,r}(t_0, X_0)$, by the comparison principle, Proposition 4.2.3, we conclude that $U^\epsilon(t, X) \le \phi(t, X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon$ in $Q_{r,r}(t_0, X_0)$ and we obtain the desired contradiction by passing to the upper limit as $\epsilon \to 0$ at (t_0, X_0) using the fact that $U^+(t_0, X_0) = \phi(t_0, X_0)$: $0 \le -\gamma_r$.

Proof of Lemma 4.3.6. We call

$$L_0^1 = \int_{|x|\le 1} (\phi(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, X_0) - \nabla\phi(t_0, X_0)\cdot x)\mu(dx),$$
$$L_0^2 = \int_{|x|>1} (U^+(t_0, x_0 + x, x_{N+1}^0) - U^+(t_0, X_0))\mu(dx).$$

Then

$$L_0 = L_0^1 + L_0^2. \tag{4.3.30}$$

Keep in mind that $\overline{y}_N = \frac{F(\bar t, \overline{X})}{\epsilon}$. Since $\psi(t, X) = \phi(t, X) + \epsilon\Gamma^\epsilon\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon$, we have

$$\mathcal{I}_1^{1,1}\left[\psi^\epsilon(\bar t, \cdot, \overline{x}_{N+1}), \overline{x}\right] = I_1 + I_2, \tag{4.3.31}$$

where

$$I_1 = \int_{|x|\le 1} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y}) - \nabla_y\Gamma^\epsilon(\overline{\tau}, \overline{Y})\cdot\frac{x}{\epsilon}\right.$$
$$\left. - \partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau}, \overline{Y})\nabla_x F(\bar t, \overline{X})\cdot\frac{x}{\epsilon}\right)\mu(dx),$$
$$I_2 = \int_{|x|\le 1}\left(\phi(\bar t, \overline{x}+x, \overline{x}_{N+1}) - \phi(\bar t, \overline{X}) - \nabla\phi(\bar t, \overline{X})\cdot x\right)\mu(dx).$$

To show the result, we proceed in several steps. In what follows, we denote by C various positive constants independent of ϵ.

Step 1: We can choose ϵ_0 so small that for any $\epsilon \le \epsilon_0$ and any $\rho > 0$ small enough

$$I_1 \le \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + o_r(1) + C_\epsilon\rho.$$

Take $\rho > 0$, $\delta > \rho$ small and $R > 0$ large and such that $\epsilon R < 1$. Since g is even, we can write

$$I_1 = I_1^0 + I_1^1 + I_1^2 + I_1^3,$$

where

$$I_1^0 = \int_{|x|\le\epsilon\rho} \epsilon \left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y}) - \nabla_y \Gamma^\epsilon(\overline{\tau}, \overline{Y})\cdot\frac{x}{\epsilon}\right.$$
$$\left.-\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau}, \overline{Y})\nabla_x F(\bar t, \overline{X})\cdot\frac{x}{\epsilon}\right)\mu(dx),$$

$$I_1^1 = \int_{\epsilon\rho\le|x|\le\epsilon\delta} \epsilon \left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right)\mu(dx),$$

$$I_1^2 = \int_{\epsilon\delta\le|x|\le\epsilon R} \epsilon \left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right)\mu(dx),$$

$$I_1^3 = \int_{\epsilon R\le|x|\le 1} \epsilon \left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right)\mu(dx),$$

and

$$\mathcal{I}_1^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] = J_1 + J_2 + J_3,$$

where

$$J_1 = \int_{\rho<|z|\le\delta} (V(\overline{\tau}, \overline{y}+z, \overline{y}_{N+1}) - V(\overline{\tau}, \overline{Y}))\mu(dz),$$

$$J_1 = \int_{\delta<|z|\le R} (V(\overline{\tau}, \overline{y}+z, \overline{y}_{N+1}) - V(\overline{\tau}, \overline{Y}))\mu(dz),$$

$$J_3 = \int_{|z|>R} (V(\overline{\tau}, \overline{y}+z, \overline{y}_{N+1}) - V(\overline{\tau}, \overline{Y}))\mu(dz).$$

STEP 1.1: *Estimate of* I_1^0 *and* $\mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}]$.

Since Γ^ϵ is of class C^2, we have

$$|I_1^0|,\ |\mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}]| \le C_\epsilon \rho, \tag{4.3.32}$$

where C_ϵ depends on the second derivatives of Γ^ϵ.

STEP 1.2 *Estimate of* $I_1^1 - J_1$. Using (4.3.26) and the fact that g is even, we can estimate $I_1^1 - J_1$ as follows

$$I_1^1 - J_1 \le \int_{\rho<|z|\le\delta} \left[V\left(\overline{\tau}, \overline{y}+z, \frac{F(\bar t, \overline{x}+\epsilon z, \overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau}, \overline{y}+z, \frac{F(\bar t, \overline{x})}{\epsilon}\right)\right]\mu(dz)$$
$$= \int_{\rho<|z|\le\delta} \left\{\left[V\left(\overline{\tau}, \overline{y}+z, \frac{F(\bar t, \overline{x}+\epsilon z, \overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau}, \overline{y}+z, \frac{F(\bar t, \overline{x})}{\epsilon}\right)\right.\right.$$
$$\left. -\partial_{y_{N+1}} V\left(\overline{\tau}, \overline{y}+z, \frac{F(\bar t, \overline{X})}{\epsilon}\right)\nabla_x F(\bar t, \overline{X})\cdot z\right]$$
$$\left. + \left[\partial_{y_{N+1}} V(\overline{\tau}, \overline{y}+z, \overline{y}_{N+1}) - \partial_{y_{N+1}} V(\overline{\tau}, \overline{Y})\right]\nabla_x F(\bar t, \overline{X})\cdot z\right\}\mu(dz).$$

Next, using (4.3.20) and (4.3.21), we get

$$I_1^1 - J_1 \leq C \int_{|z|\leq\delta} (|z|^2 + |z|^{1+\alpha})\mu(dz) \leq C\delta^\alpha. \tag{4.3.33}$$

STEP 1.3 *Estimate of* $I_1^2 - J_2$. If L_η is the Lipschitz constant of V w.r.t. y_{N+1}, then

$$\begin{aligned}
I_1^2 - J_2 &\leq \int_{\delta<|z|\leq R} \left(V\left(\overline{\tau}, \overline{y}+z, \frac{F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau}, \overline{y}+z, \frac{F(\overline{t}, \overline{X})}{\epsilon}\right)\right)\mu(dz) \\
&\leq L_\eta \int_{\delta<|z|\leq R} \left| \frac{F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})}{\epsilon} - \frac{F(\overline{t}, \overline{X})}{\epsilon} \right| \mu(dz) \\
&\leq L_\eta \int_{\delta<|z|\leq R} \sup_{|z|\leq R} |\nabla_x F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})||z|\mu(dz).
\end{aligned}$$

Then

$$I_1^2 - J_2 \leq C \sup_{|z|\leq R} |\nabla_x F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})| \log(R/\delta) \tag{4.3.34}$$

STEP 1.4: *Estimate of* I_1^3 *and* J_3. Since V is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, we have

$$\begin{aligned}
I_1^3 &\leq \int_{R<|z|\leq\frac{1}{\epsilon}} \left(V\left(\overline{\tau}, \overline{y}+z, \frac{F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})}{\epsilon}\right) - V(\overline{\tau}, \overline{Y})\right)\mu(dz) \\
&\leq \int_{|z|>R} 2\|v\|_\infty \mu(dz) \leq \frac{C}{R}.
\end{aligned} \tag{4.3.35}$$

Similarly

$$J_3 \leq \frac{C}{R}. \tag{4.3.36}$$

Now, from (4.3.32), (4.3.33), (4.3.34), (4.3.35) and (4.3.36), we infer that

$$\begin{aligned}
I_1 &\leq \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + 2C_\epsilon\rho + C\delta^\alpha \\
&\quad + C \sup_{|z|\leq R} |\nabla_x F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})| \log\left(\frac{R}{\delta}\right) + \frac{C}{R}.
\end{aligned}$$

We choose $R = R(r)$ such $R \to +\infty$ as $r \to 0^+$, $\epsilon_0 = \epsilon_0(r)$ such that $R\epsilon_0(r) \leq r$ and $\delta = \delta(r) > 0$ such that $\delta \to 0$ as $r \to 0^+$ and $r\log(R/\delta) \to 0$ as $r \to 0^+$. With this choice, for any $\epsilon \leq \epsilon_0$ and any $\rho < \delta$

$$C\delta^\alpha + C \sup_{|z|\leq R} |\nabla_x F(\overline{t}, \overline{x}+\epsilon z, \overline{x}_{N+1})| \log\left(\frac{R}{\delta}\right) + \frac{C}{R} = o_r(1) \quad \text{as } r \to 0^+,$$

and Step 1 is proved.

Step 2: $I_2 \leq L_0^1 + o_r(1)$**.**

For $0 < \nu < 1$ we can split I_2 and L_0^1 as follows

$$\begin{aligned}
I_2 &= \int_{|x|\leq\nu} (\phi(\overline{t}, \overline{x}+x, \overline{x}_{N+1}) - \phi(\overline{t}, \overline{X}) - \nabla\phi(\overline{t}, \overline{X}) \cdot x)\mu(dx) \\
&\quad + \int_{\nu\leq|x|\leq 1} (\phi(\overline{t}, \overline{x}+x, \overline{x}_{N+1}) - \phi(\overline{t}, \overline{X}))\mu(dx) = I_2^1 + I_2^2,
\end{aligned}$$

$$\begin{aligned} L_0^1 = & \int_{|x|\le \nu} (\phi(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, X_0) - \nabla\phi(t_0, X_0)\cdot x)\mu(dx) \\ & + \int_{\nu\le|x|\le 1} (\phi(t_0, x_0+x, x_{N+1}^0) - \phi(t_0, X_0))\mu(dx) = T_1 + T_2. \end{aligned}$$

Since ϕ is of class C^2 we have

$$I_2^1, T_1 \le C\nu.$$

Using the Lipschitz continuity of ϕ we get

$$I_2^2 - T_2 = \int_{\nu<|x|\le 1} Cr\mu(dx) \le C\frac{r}{\nu}.$$

Hence, Step 2 follows choosing $\nu = \nu(r)$ such that $\nu \to 0$ and $r/\nu \to 0$ as $r \to 0^+$.
Step 3: $\mathcal{I}_1^{2,1}[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}] \le L_0^2 + o_r(1)$**.**

Remark that

$$U^\epsilon(\overline{t}, \overline{x}+x, \overline{x}_{N+1}) - \phi(\overline{t}, \overline{X}) - \epsilon V(\overline{\tau}, \overline{Y}) \le U^+(t_0, x_0+x, x_{N+1}^0) - \phi(t_0, X_0) + o_\epsilon(1) + o_r(1).$$

Then, recalling that $\phi(t_0, X_0) = U^+(t_0, X_0)$, for $\epsilon \le \epsilon_0$ we get

$$\mathcal{I}_1^{2,1}[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}] - L_0^2 \le o_r(1)$$

and Step 3 is proved.

Finally (4.3.30), (4.3.31), Steps 1, 2 and 3 give

$$\begin{aligned} \mathcal{I}_1^{1,1}[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}] + \mathcal{I}_1^{2,1}[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}] \le & \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] \\ & + L_0 + o_r(1) + C_\epsilon \rho. \end{aligned}$$

from which, using inequality (4.3.29) and letting $\rho \to 0^+$, we get for $\epsilon \le \epsilon_0$

$$\begin{aligned} \partial_t \psi(\overline{t}, \overline{X}) \ge & \mathcal{I}_1^{1,1}[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}] + \mathcal{I}_1^{2,1}[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}] - W'\left(\frac{\phi^\epsilon(\overline{t}, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) \\ & - o_\eta(1) + o_r(1) + L_\eta \end{aligned}$$

and this concludes the proof of the lemma. □

4.4 Building of Lipschitz sub and supercorrectors

In this section we construct sub and supersolutions of (4.3.16) that are Lipschitz w.r.t. y_{N+1}. As a byproduct, we will prove Theorem 4.1.1 and Proposition 4.3.3.

Proposition 4.4.1 (Lipschitz continuous sub and supercorrectors)**.** *Let λ be the quantity defined by Theorem 4.1.1. Then, for any fixed $p \in \mathbb{R}^N$, $P = (p, 1)$, $L \in \mathbb{R}$ and $\eta > 0$ small enough, there exist real numbers $\lambda_\eta^+(p, L)$, $\lambda_\eta^-(p, L)$, a constant $C > 0$ (independent of η, p and L) and bounded super and subcorrectors W_η^+, W_η^- i.e. respectively a super and a subsolution of* (4.3.16) *such that*

$$\lim_{\eta\to 0^+} \lambda_\eta^+(p, L) = \lim_{\eta\to 0^+} \lambda_\eta^-(p, L) = \lambda(p, L),$$

$\lambda_\eta^{\stackrel{+}{-}}$ *satisfy (i) and (ii) of Proposition 4.3.3 and for any* $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$

$$|W_\eta^{\stackrel{+}{-}}(\tau, Y)| \leq C. \tag{4.4.37}$$

Moreover $W_\eta^{\stackrel{+}{-}}$ *are Lipschitz continuous w.r.t.* y_{N+1} *and* α*-Hölder continuous w.r.t.* y *for any* $0 < \alpha < 1$*, with*

$$-1 \leq \partial_{y_{N+1}} W_\eta^{\stackrel{+}{-}} \leq \frac{\|W''\|_\infty}{\eta}, \tag{4.4.38}$$

$$< W_\eta^{\stackrel{+}{-}} >_y^\alpha \leq C_\eta. \tag{4.4.39}$$

In order to prove the proposition, for $\eta \geq 0$, $L \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $P = (p, 1)$, we introduce the problem

$$\begin{cases} \partial_\tau U = L + \mathcal{I}_1[U(\tau, \cdot, y_{N+1})] - W'(U + P \cdot Y) + \sigma(\tau, y) & \\ \qquad +\eta[a_0 + \inf_{Y'} U(\tau, Y') - U(\tau, Y)]|\partial_{y_{N+1}} U + 1| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ U(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{4.4.40}$$

4.4.1 Comparison principle

Proposition 4.4.2 (Comparison principle for (4.4.40))**.** *Let* $U_1 \in USC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ *and* $U_2 \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ *be respectively a viscosity subsolution and supersolution of* (4.4.40)*, then* $U_1 \leq U_2$ *on* $\mathbb{R}^+ \times \mathbb{R}^{N+1}$*.*

Proof. Let us define the functions $V_1(\tau, Y) := e^{-k\tau} U_1(\tau, Y)$ and $V_2(\tau, Y) := e^{-k\tau} U_2(\tau, Y)$, where $k := \|W''\|_\infty + 1$. It is easy to see that V_1 and V_2 are respectively sub and supersolution of

$$\begin{cases} \partial_\tau V = Le^{-k\tau} + \mathcal{I}_1[V(\tau, \cdot, y_{N+1})] + g(\tau, Y, V) & \\ \qquad +\eta[a_0 + e^{k\tau}(\inf_{Y'} V(\tau, Y') - V(\tau, Y))]|\partial_{y_{N+1}} V + e^{-k\tau}| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{4.4.41}$$

where $g(\tau, Y, V) = -e^{-k\tau} W'(e^{k\tau} V + P \cdot Y) - kV + e^{-k\tau} \sigma(\tau, y)$. Remark that, by the choice of k,

$$g(\tau, Y, V_1) - g(\tau, Z, V_2) \leq -(V_1 - V_2) + e^{-k\tau}(\|W''\|_\infty |P| + \|\sigma'\|_\infty)|Y - Z|. \tag{4.4.42}$$

To prove the comparison between U_1 and U_2, it suffices to show that $V_1(\tau, Y) \leq V_2(\tau, Y)$ for all $(\tau, Y) \in (0, T) \times \mathbb{R}^{N+1}$ and for any $T > 0$.

Suppose by contradiction that $M = \sup_{(\tau, Y) \in (0,T) \times \mathbb{R}^{N+1}} (V_1(\tau, Y) - V_2(\tau, Y)) > 0$ for some $T > 0$. Define for small $\nu_1, \nu_2, \beta, \delta > 0$ the function $\phi \in C^2((\mathbb{R}^+ \times \mathbb{R}^{N+1})^2)$ by

$$\phi(\tau, Y, s, Z) = \frac{1}{2\nu_1}|\tau - s|^2 + \frac{1}{2\nu_2}|Y - Z|^2 + \beta\psi(Y) + \frac{\delta}{T - \tau},$$

where ψ is defined as the function ψ_2 in the proof of Proposition 4.2.7. The supremum of $V_1(\tau, Y) - V_2(s, Z) - \phi(\tau, Y, s, Z)$ is attained at some point $(\overline{\tau}, \overline{Y}, \overline{s}, \overline{Z}) \in ((0, T) \times \mathbb{R}^{N+1})^2$. Standard arguments show that

$$(\overline{\tau}, \overline{Y}, \overline{s}, \overline{Z}) \to (\widehat{\tau}, \widehat{\tau}, \widehat{Y}, \widehat{Z}) \quad \text{as } \nu_1 \to 0,$$

$$V_1(\overline{\tau},\overline{Y})\to V_1(\widehat{\tau},\widehat{Y}),\ V_2(\overline{s},\overline{Z})\to V_2(\widehat{\tau},\widehat{Z})\quad \text{as } \nu_1\to 0,$$

where $(\widehat{\tau},\widehat{Y},\widehat{Z})$ is a maximum point of $V_1(\tau,Y)-V_2(\tau,Z)-\frac{1}{2\nu_2}|Y-Z|^2-\beta\psi(Y)-\frac{\eta}{T-\tau}$. Moreover, it is easy to see that

$$\limsup_{\nu_1\to 0}\inf_{Y'} V_1(\overline{\tau},Y')\le \inf_{Y'} V_1(\widehat{\tau},Y'),\ \liminf_{\nu_1\to 0}\inf_{Y'} V_2(\overline{s},Y')\ge \inf_{Y'} V_2(\widehat{\tau},Y').$$

Since V_1 and V_2 are respectively sub and supersolution of (4.4.41), for any $r>0$ we have

$$\begin{aligned}&\frac{\delta}{(T-\overline{\tau})^2}+\frac{\overline{\tau}-\overline{s}}{\nu_1}\\&\le Le^{-k\overline{\tau}}+\frac{C_Nr}{\nu_2}+\beta\mathcal{I}_1^{1,r}[\psi(\cdot,\overline{y}_{N+1}),\overline{y}]+\mathcal{I}_1^{2,r}[V_1(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+g(\overline{\tau},\overline{Y},V_1(\overline{\tau},\overline{Y}))\\&+\eta[a_0+e^{k\overline{\tau}}(\inf_{Y'}V_1(\overline{\tau},Y')-V_1(\overline{\tau},\overline{Y}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+\beta\partial_{y_{N+1}}\psi(\overline{Y})+e^{-k\overline{\tau}}\right|\end{aligned}\tag{4.4.43}$$

and

$$\begin{aligned}\frac{\overline{\tau}-\overline{s}}{\nu_1}&\ge Le^{-k\overline{s}}-\frac{C_Nr}{\nu_2}+\mathcal{I}_1^{2,r}[V_2(\overline{s},\cdot,\overline{z}_{N+1}),\overline{z}]+g(\overline{s},\overline{Z},V_2(\overline{s},\overline{Z}))\\&+\eta[a_0+e^{k\overline{s}}(\inf_{Y'}V_2(\overline{s},Y')-V_2(\overline{s},\overline{Z}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+e^{-k\overline{s}}\right|,\end{aligned}\tag{4.4.44}$$

where C_N is a constant depending on the dimension N. Since $(\overline{\tau},\overline{Y},\overline{s},\overline{Z})$ is a maximum point, we have

$$V_1(\overline{\tau},\overline{y}+x,\overline{y}_{N+1})-V_1(\overline{\tau},\overline{Y})\le V_2(\overline{s},\overline{z}+x,\overline{z}_{N+1})-V_2(\overline{s},\overline{Z})+\beta[\psi(\overline{y}+x,\overline{y}_{N+1})-\psi(\overline{Y})],$$

for any $x\in\mathbb{R}^N$, which implies that for any $r>0$

$$\mathcal{I}_1^{2,r}[V_1(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]\le \mathcal{I}_1^{2,r}[V_2(\overline{s},\cdot,\overline{z}_{N+1}),\overline{z}]+\beta\mathcal{I}_1^{2,r}[\psi(\cdot,\overline{y}_{N+1}),\overline{y}].$$

Then, subtracting (4.4.43) with (4.4.44) and letting $r\to 0^+$, we get

$$\begin{aligned}\frac{\delta}{(T-\overline{\tau})^2}&\le L(e^{-k\overline{\tau}}-e^{-k\overline{s}})+\beta\mathcal{I}_1[\psi(\cdot,\overline{y}_{N+1}),\overline{y}]+g(\overline{\tau},\overline{Y},V_1(\overline{\tau},\overline{Y}))-g(\overline{s},\overline{Z},V_2(\overline{s},\overline{Z}))\\&+\eta[a_0+e^{k\overline{\tau}}(\inf_{Y'}V_1(\overline{\tau},Y')-V_1(\overline{\tau},\overline{Y}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+\beta\partial_{y_{N+1}}\psi(\overline{Y})+e^{-k\overline{\tau}}\right|\\&-\eta[a_0+e^{k\overline{s}}(\inf_{Y'}V_2(\overline{s},Y')-V_2(\overline{s},\overline{Z}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+e^{-k\overline{s}}\right|.\end{aligned}$$

Next, letting $\nu_1\to 0$ and using (4.4.42), we obtain

$$\begin{aligned}&\frac{\delta}{(T-\widehat{\tau})^2}\\&\le\beta\mathcal{I}_1[\psi(\cdot,\widehat{y}_{N+1}),\widehat{y}]-(V_1(\widehat{\tau},\widehat{Y})-V_2(\widehat{\tau},\widehat{Z}))+e^{-k\widehat{\tau}}(\|W''\|_\infty|P|+\|\sigma'\|_\infty)|\widehat{Y}-\widehat{Z}|+C\beta\\&+\eta e^{k\widehat{\tau}}[\inf_{Y'}V_1(\widehat{\tau},Y')-\inf_{Y'}V_2(\widehat{\tau},Y')-(V_1(\widehat{\tau},\widehat{Y})-V_2(\widehat{\tau},\widehat{Z}))]\left|\frac{\widehat{y}_{N+1}-\widehat{z}_{N+1}}{\nu_2}+e^{-k\widehat{\tau}}\right|.\end{aligned}\tag{4.4.45}$$

It is easy to prove that

$$\limsup_{(\beta,\delta)\to(0,0)} (V_1(\hat{\tau},\hat{Y}) - V_2(\hat{\tau},\hat{Z})) \geq M \tag{4.4.46}$$

and

$$\frac{|\hat{Y}-\hat{Z}|^2}{\nu_2} \leq C,$$

where C is independent of β and δ. Up to subsequence, $\hat{\tau} \to \tau_0 \in [0,T]$ as $(\beta,\delta) \to (0,0)$ and by (4.4.46), we have

$$\begin{aligned}
&\limsup_{(\beta,\delta)\to(0,0)} \inf_{Y'} V_1(\hat{\tau},Y') - \inf_{Y'} V_2(\hat{\tau},Y') - (V_1(\hat{\tau},\hat{Y}) - V_2(\hat{\tau},\hat{Z}))\\
&\leq \inf_{Y'} V(\tau_0,Y') - \inf_{Y'} V_2(\tau_0,Y') - \sup_{Y'}(V_1(\tau_0,Y') - V_2(\tau_0,Y'))\\
&\leq 0.
\end{aligned}$$

Then, passing to the limit first as $(\beta,\delta) \to (0,0)$ and then as $\nu_2 \to 0$ in (4.4.45) we finally get the contradiction:

$$M \leq 0,$$

and this concludes the proof of the comparison theorem. □

4.4.2 Lipschitz regularity

Proposition 4.4.3 (Lipschitz continuity in y_{N+1})**.** *Suppose $\eta > 0$. Let $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ be the viscosity solution of* (4.4.40)*. Then U_η is Lipschitz continuous w.r.t. y_{N+1} and for almost every $(\tau,Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$*

$$-1 \leq \partial_{y_{N+1}} U_\eta(\tau,Y) \leq \frac{\|W''\|_\infty}{\eta}. \tag{4.4.47}$$

Proof. Let us define $\hat{U}(\tau,Y) = U(\tau,Y) + y_{N+1}$, then $\hat{U}$ satisfies

$$\begin{cases}
\partial_\tau \hat{U} = L + \mathcal{I}_1[\hat{U}(\tau,\cdot,y_{N+1})] - W'(\hat{U} + p\cdot y) + \sigma(\tau,y) & \\
\qquad +\eta[a_0 + \inf_{Y'}(\hat{U}(\tau,Y') - y'_{N+1}) - (\hat{U}(\tau,Y) - y_{N+1})]|\partial_{y_{N+1}}\hat{U}| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}\\
\hat{U}(0,Y) = y_{N+1} & \text{on } \mathbb{R}^{N+1}.
\end{cases} \tag{4.4.48}$$

We are going to prove that $\hat{U}$ is Lipschitz continuous w.r.t. y_{N+1} with

$$0 \leq \partial_{y_{N+1}} \hat{U}(\tau,Y) \leq 1 + \frac{\|W''\|_\infty}{\eta}.$$

By comparison, $\hat{U}(t,y,y_{N+1}) \leq \hat{U}(t,y,y_{N+1}+h)$ for $h \geq 0$, from which immediately follows that $\partial_{y_{N+1}}\hat{U} \geq 0$. In particular we can replace $|\partial_{y_{N+1}}\hat{U}|$ by $\partial_{y_{N+1}}\hat{U}$ in (4.4.48).

Let us now show that $\partial_{y_{N+1}}\hat{U} \leq 1 + \frac{\|W''\|_\infty}{\eta}$. We argue by contradiction by assuming that for some $T > 0$ the supremum of the function $\hat{U}(\tau,y,y_{N+1}) -$

$\widehat{U}(\tau, y, z_{N+1}) - K|y_{N+1} - z_{N+1}|$ on $[0,T] \times \mathbb{R}^{N+1}$ is strictly positive as soon as $K > 1 + \frac{\|W''\|_\infty}{\eta}$. Then for $\delta, \beta > 0$ small enough, M defined by

$$M = \max_{\substack{(\tau,y)\in[0,T]\times\mathbb{R}^N\\ y_{N+1},z_{N+1}\in\mathbb{R}}} \left(\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(\tau, y, z_{N+1}) - K|y_{N+1} - z_{N+1}| - \beta\psi(Y) - \frac{\delta}{T-\tau} \right),$$

where ψ is defined as the function ψ_2 in the proof of Proposition 4.2.7, is positive. For $j > 0$ let

$$M_j = \max_{\substack{\tau,s\in[0,T],y,z\in\mathbb{R}^N\\ y_{N+1},z_{N+1}\in\mathbb{R}}} \Big(\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(s, z, z_{N+1}) - K|y_{N+1} - z_{N+1}| - \beta\psi(Y) \\ - \frac{\delta}{T-\tau} - j|\tau - s|^2 - j|y-z|^2 \Big),$$

and let $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1}) \in ([0,T] \times \mathbb{R}^{N+1})^2$ be a point where M_j is attained. Classical arguments show that $M_j \to M$, $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1}) \to (\overline{\tau}, \overline{y}, \overline{y}_{N+1}, \overline{\tau}, \overline{y}, \overline{z}_{N+1})$ as $j \to +\infty$, where $(\overline{\tau}, \overline{y}, \overline{y}_{N+1}, \overline{z}_{N+1})$ is a point where M is attained.

Remark that $0 < \overline{\tau} < T$, moreover, since $\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) > \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})$ and $\widehat{U}$ is nondecreasing in y_{N+1}, it is

$$\overline{y}_{N+1} > \overline{z}_{N+1}. \tag{4.4.49}$$

In particular $y^j_{N+1} \neq z^j_{N+1}$ and $0 < s_j$, $\tau_j < T$ for j large enough. Hence, for $r > 0$, we obtain the following viscosity inequalities

$$\begin{aligned} &\frac{\delta}{(T-\tau_j)^2} + j(t_j - s_j) \\ &\le L_0 + C_N jr + \beta\mathcal{I}_1^{1,r}[\psi(\cdot, y^j_{N+1}), y^j] + \mathcal{I}_1^{2,r}[\widehat{U}(\tau^j, \cdot, y^j_{N+1}), y^j] \\ &- W'(\widehat{U}(\tau^j, y^j, y^j_{N+1}) + p\cdot y^j) + \sigma(\tau^j, y^j) + \eta[a_0 + \inf_{Y'}(\widehat{U}(\tau_j, Y') - y'_{N+1}) \\ &- (\widehat{U}(\tau^j, y^j, y^j_{N+1}) - y^j_{N+1})] \left(K\frac{y^j_{N+1} - z^j_{N+1}}{|y^j_{N+1} - z^j_{N+1}|} + \beta\partial_{y_{N+1}}\psi(y^j, y^j_{N+1}) \right), \end{aligned} \tag{4.4.50}$$

and

$$\begin{aligned} &j(t_j - s_j) \\ &\ge L_0 - C_N jr + \mathcal{I}_1^{2,r}[\widehat{U}(s^j, \cdot, z^j_{N+1}), z^j] - W'(\widehat{U}(s^j, z^j, z^j_{N+1}) + p\cdot z^j) + \sigma(s^j, z^j) \\ &+ \eta[a_0 + \inf_{Y'}(\widehat{U}(s_j, Y') - y'_{N+1}) - (\widehat{U}(s^j, z^j, z^j_{N+1}) - z^j_{N+1})]K\frac{y^j_{N+1} - z^j_{N+1}}{|y^j_{N+1} - z^j_{N+1}|}, \end{aligned} \tag{4.4.51}$$

where C_N is a constant depending on N. Since $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1})$ is a maximum point, we have

$$\begin{aligned} \widehat{U}(\tau^j, y^j + x, y^j_{N+1}) - \widehat{U}(\tau^j, y^j, y^j_{N+1}) &\le \widehat{U}(s^j, z^j + x, z^j_{N+1}) - \widehat{U}(s^j, z^j, z^j_{N+1}) \\ &+ \beta[\psi(y^j + x, y^j_{N+1}) - \psi(y^j, y^j_{N+1})] \end{aligned}$$

for any $x \in \mathbb{R}^N$, which implies that for any $r > 0$

$$\mathcal{I}_1^{2,r}[\widehat{U}(\tau^j,\cdot,y^j_{N+1}),y^j] \le \mathcal{I}_1^{2,r}[\widehat{U}(s^j,\cdot,z^j_{N+1}),z^j] + \beta\mathcal{I}_1^{2,r}[\psi(\cdot,y^j_{N+1}),y^j].$$

Hence, subtracting (4.4.50) with (4.4.51), sending $r \to 0^+$ and then $j \to +\infty$, we get

$$\begin{aligned}\frac{\delta}{(T-\overline{\tau})^2} &\le \beta\mathcal{I}_1[\psi(\cdot,\overline{y}_{N+1}),\overline{y}] + W'(\widehat{U}(\overline{\tau},\overline{y},\overline{z}_{N+1}) + p\cdot\overline{y}) - W'(\widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) + p\cdot\overline{y}) \\ &- \eta[\widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) - \widehat{U}(\overline{\tau},\overline{y},\overline{z}_{N+1}) - (\overline{y}_{N+1}-\overline{z}_{N+1})]K\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{|\overline{y}_{N+1}-\overline{z}_{N+1}|} \\ &+ \beta\partial_{y_{N+1}}\psi(\overline{y},\overline{y}_{N+1})\eta[a_0 + \inf_{Y'}(\widehat{U}(\overline{\tau},Y') - y'_{N+1}) - (\widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) - \overline{y}_{N+1})] \\ &\le \|W''\|_\infty|\widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) - \widehat{U}(\overline{\tau},\overline{y},\overline{z}_{N+1})| - \eta[\widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) - \widehat{U}(\overline{\tau},\overline{y},\overline{z}_{N+1}) \\ &- (\overline{y}_{N+1}-\overline{z}_{N+1})]K\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{|\overline{y}_{N+1}-\overline{z}_{N+1}|} + \beta C.\end{aligned}$$

Then, using (4.4.49) and that $K|\overline{y}_{N+1}-\overline{z}_{N+1}| < \widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) - \widehat{U}(\overline{\tau},\overline{y},\overline{z}_{N+1})$, for β small enough, we finally obtain

$$(\|W''\|_\infty + \eta - \eta K)(\widehat{U}(\overline{\tau},\overline{y},\overline{y}_{N+1}) - \widehat{U}(\overline{\tau},\overline{y},\overline{z}_{N+1})) \ge 0,$$

which is a contradiction for $K > 1 + \frac{\|W''\|_\infty}{\eta}$. □

4.4.3 Ergodicity

Proposition 4.4.4 (Ergodic properties)**.** *There exists a unique $\lambda_\eta = \lambda_\eta(p, L)$ such that the viscosity solution $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ of (4.4.40) satisfies:*

$$|U_\eta(\tau,Y) - \lambda_\eta\tau| \le C_3 \text{ for all } \tau > 0,\ Y \in \mathbb{R}^{N+1}, \tag{4.4.52}$$

where

$$C_3 = 5\lceil C_1\rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty,\ C_1 = \frac{2\|W'\|_\infty + 2\|\sigma\|_\infty + 2\|\mu_0\|_{L^1(\mathbb{R}^N\setminus B_{r_0})}}{c_0},$$

and c_0 is defined as in (H2). Moreover

$$L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0 \le \lambda_\eta \le L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0. \tag{4.4.53}$$

Proof. For simplicity of notations, in what follows we denote $U = U_\eta$ and $\lambda = \lambda_\eta$.

To prove the proposition we follow the proof of the analogue result in [68]. We proceed in three steps.

Step 1: existence The functions $W^+(\tau, Y) = C^+\tau$ and $W^-(\tau, Y) = C^-\tau$, where

$$C^{\pm} = L \pm \|W'\|_\infty \pm \|\sigma\|_\infty + \eta a_0,$$

are respectively sub and supersolution of (4.4.1). Then the existence of a unique solution of (4.4.1) follows from Perron's method.

Step 2: control of the oscillations w.r.t. space.
We want to prove that there exists $C_1 > 0$ such that

$$|U(\tau, Y) - U(\tau, Z)| \leq C_1 \quad \text{for all } \tau \geq 0,\ Y, Z \in \mathbb{R}^{N+1}. \tag{4.4.54}$$

STEP 2.1. For a given $k \in \mathbb{Z}^{N+1}$, we set $P \cdot k = l + \alpha$, with $l \in \mathbb{Z}$ and $\alpha \in [0, 1)$. The function $\widetilde{U}(\tau, Y) = U(\tau, Y + k) + \alpha$ is still a solution of (4.4.40), with $\widetilde{U}(0, Y) = \alpha$ Moreover

$$U(0, Y) = 0 \leq \widetilde{U}(0, Y) = \alpha \leq 1 = U(0, Y) + 1.$$

Then from the comparison principle for (4.4.40) and invariance by integer translations we deduce for all $\tau \geq 0$:

$$|U(\tau, Y + k) - U(\tau, Y)| \leq 1. \tag{4.4.55}$$

STEP 2.2. We proceed as in [68] by considering the functions

$$M(\tau) := \sup_{Y \in \mathbb{R}^{N+1}} U(\tau, Y), \quad m(\tau) := \inf_{Y \in \mathbb{R}^{N+1}} U(\tau, Y),$$

$$q(\tau) := M(\tau) - m(\tau) = \operatorname{osc} U(\tau, \cdot).$$

Let us assume that the extrema defining these functions are attained: $M(\tau) = U(\tau, Y^\tau)$, $m(\tau) = U(\tau, Z^\tau)$. If this is not the case, consider an ϵ-supremum and an ϵ-infimum and use a variational principle, such as Stegal's one for instance (see [38]).

It is easy ti see that $M(\tau)$ and $m(\tau)$ satisfy in the viscosity sense

$$\partial_\tau M \leq L + \mathcal{I}_1^{2,r}[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - W'(M + P \cdot Y^\tau) + \sigma(\tau, y^\tau) + \eta[a_0 + m(\tau) - M(\tau)],$$

$$\partial_\tau m \geq L + \mathcal{I}_1^{2,r}[U(\tau, \cdot, z^\tau_{N+1}), z^\tau] - W'(m + P \cdot Z^\tau) + \sigma(\tau, z^\tau) + \eta a_0,$$

for any $r > 0$.

Choose $r = r_0$, where r_0 is as in (H2). Then q satisfies in the viscosity sense

$$\begin{aligned}\partial_\tau q &\leq \mathcal{I}_1^{2,r_0}[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - \mathcal{I}_1^{2,r_0}[U(\tau, \cdot, z^\tau_{N+1}), z^\tau] - W'(M + P \cdot Y^\tau) \\ &\quad + W'(m + P \cdot Z^\tau) + \sigma(\tau, y^\tau) - \sigma(\tau, z^\tau) \\ &\leq \mathcal{I}_1^{2,r_0}[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - \mathcal{I}_1^{2,r_0}[U(\tau, \cdot, z^\tau_{N+1}), z^\tau] + 2\|W'\|_\infty + 2\|\sigma\|_\infty.\end{aligned}$$

Let us estimate the quantity $\mathcal{L}(\tau) := \mathcal{I}_1^{2,r_0}[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - \mathcal{I}_1^{2,r_0}[U(\tau, \cdot, z^\tau_{N+1}), z^\tau]$ from above by a function of q. Let us define $k^\tau \in \mathbb{Z}^{N+1}$ such that $Y^\tau - (Z^\tau + k^\tau) \in [0, 1)^{N+1}$ and let $\widetilde{Z}^\tau := Z^\tau + k^\tau$. Using successively (4.4.55) and the first inequality in (4.4.47), we obtain:

$$\begin{aligned}\mathcal{L}(\tau) &\leq \int_{|z|>r_0} (U(\tau, y^\tau + z, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu(dz) \\ &\quad - \int_{|z|>r_0} (U(\tau, \widetilde{z}^\tau + z, \widetilde{z}^\tau_{N+1}) - U(\tau, Z^\tau))\mu(dz) + \overline{\mu} \\ &\leq \int_{|z|>r_0} (U(\tau, y^\tau + z, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu(dz) \\ &\quad - \int_{|z|>r_0} (U(\tau, \widetilde{z}^\tau + z, y^\tau_{N+1}) - U(\tau, Z^\tau))\mu(dz) + 2\overline{\mu},\end{aligned}$$

where $\overline{\mu} = \|\mu_0\|_{L^1(\mathbb{R}^N \setminus B_{r_0}(0))}$. Now, let us introduce $c^\tau = \frac{y^\tau + \widetilde{z}^\tau}{2}$ and $\delta^\tau = \frac{y^\tau - \widetilde{z}^\tau}{2} \in [0, \frac{1}{2})^N$ so that $y^\tau = c^\tau + \delta^\tau$ and $\widetilde{z}^\tau = c^\tau - \delta^\tau$. Hence

$$\begin{aligned}
\mathcal{L}(\tau) &\leq 2\overline{\mu} + \int_{|z|>r_0} (U(\tau, c^\tau + z + \delta^\tau, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu(dz) \\
&\quad - \int_{|z|>r_0} (U(\tau, c^\tau + z - \delta^\tau, y^\tau_{N+1}) - U(\tau, Z^\tau))\mu(dz) \\
&\leq 2\overline{\mu} + \int_{|z-\delta^\tau|>r_0} (U(\tau, c^\tau + z, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu_0(z - \delta^\tau)dz \\
&\quad - \int_{|z+\delta^\tau|>r_0} (U(\tau, c^\tau + z, y^\tau_{N+1}) - U(\tau, Z^\tau))\mu_0(z + \delta^\tau)dz \\
&\leq 2\overline{\mu} - \int_{\{|z-\delta^\tau|>r_0\}\cap\{|z+\delta^\tau|>r_0\}} (U(\tau, Y^\tau) - U(\tau, Z^\tau)) \min\{\mu_0(z - \delta^\tau), \mu_0(z + \delta^\tau)\}dz \\
&\leq 2\overline{\mu} - c_0 q(\tau)
\end{aligned}$$

where c_0 is defined as in (H2). We conclude that q satisfies in the viscosity sense

$$\partial_\tau q(\tau) \leq 2\|W'\|_\infty + 2\|\sigma\|_\infty + 2\overline{\mu} - c_0 q(\tau),$$

with $q(0) = 0$, from which we obtain (4.4.54).

Step 3: control of the oscillations in time. We follow [68] by introducing the two quantities:

$$\lambda^+(T) := \sup_{\tau \geq 0} \frac{U(\tau + T, 0) - U(\tau, 0)}{T} \quad \text{and} \quad \lambda^-(T) := \inf_{\tau \geq 0} \frac{U(\tau + T, 0) - U(\tau, 0)}{T},$$

and proving that they have a common limit as $T \to +\infty$. First let us estimate $\lambda^+(T)$ from above. The function $U^+(t, Y) := U(\tau, 0) + C_1 + C^+ t$, is a supersolution of (4.4.40) if $C^+ = L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0$. Since $U^+(0, Y) \geq U(\tau, Y)$ if C_1 is as in (4.4.54), by the comparison principle for (4.4.40) in the time interval $[\tau, \tau + \tau_0]$, for any $\tau_0 > 0$ and $t \in [0, \tau_0]$ we get

$$U(\tau + t, Y) \leq U(\tau, 0) + C_1 + C^+ t. \tag{4.4.56}$$

Similarly

$$U(\tau + t, Y) \geq U(\tau, 0) - C_1 + C^- t, \tag{4.4.57}$$

where $C^- = L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0$. We then obtain for $\tau_0 = t = T$ and $y = 0$:

$$L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0 - \frac{C_1}{T} \leq \lambda^-(T) \leq \lambda^+(T) \leq L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0 + \frac{C_1}{T}. \tag{4.4.58}$$

By definition of $\lambda^{\pm}(T)$, for any $\delta > 0$, there exist $\tau^{\pm} \geq 0$ such that

$$\left| \lambda^{\pm}(T) - \frac{U(\tau^{\pm} + T, 0) - U(\tau^{\pm}, 0)}{T} \right| \leq \delta.$$

Let us consider $\alpha, \beta \in [0,1)$ such that $\tau^+ - \tau^- - \beta = k \in \mathbb{Z}$, and $U(\tau^+,0) - U(\tau^+ - k, 0) + \alpha \in \mathbb{Z}$. From (4.4.54) we have

$$\begin{aligned} U(\tau^+,Y) &\leq U(\tau^+,0) + C_1 \leq U(\tau^+ - k, Y) + 2C_1 + (U(\tau^+,0) - U(\tau^+ - k, 0)) \\ &\leq U(\tau^+ - k, Y) + 2\lceil C_1 \rceil + (U(\tau^+,0) - U(\tau^+ - k, 0) + \alpha). \end{aligned}$$

Since $\sigma(\cdot, y)$ and $W'(\cdot)$ are $\mathbb{Z}$-periodic, the comparison principle for (4.4.40) on the time interval $[\tau^+, \tau^+ + T]$ implies that:

$$U(\tau^+ + T, Y) \leq U(\tau^+ - k + T, Y) + 2\lceil C_1 \rceil + U(\tau^+,0) - U(\tau^+ - k, 0) + 1.$$

Choosing $Y = 0$ in the previous inequality we get

$$\begin{aligned} U(\tau^+ + T, 0) - U(\tau^+, 0) &\leq U(\tau^+ - k + T, 0) - U(\tau^+ - k, 0) + 2\lceil C_1 \rceil + 1 \\ &= U(\tau^- + \beta + T, 0) - U(\tau^- + \beta, 0) + 2\lceil C_1 \rceil + 1, \end{aligned}$$

and setting $t = \beta$ and $\tau = \tau^- + T$ in (4.4.56) and $\tau = \tau^-$ in (4.4.57) we finally obtain:

$$T\lambda^+(T) \leq T\lambda^-(T) + 4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty + 2\delta T.$$

Since this is true for any $\delta > 0$, we conclude that:

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty}{T}.$$

Now arguing as in [67] and [68], we conclude that there exist $\lim_{T\to+\infty} \lambda^{\pm}(T) =: \lambda$ and

$$|\lambda^{\pm}(T) - \lambda| \leq \frac{4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty}{T},$$

which implies that

$$|U(T,0) - \lambda T| \leq 4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty,$$

and then, using (4.4.54) we get (4.4.52). The uniqueness of λ follows from (4.4.52). Finally, (4.4.53) is obtained from (4.4.58) as $T \to +\infty$. □

4.4.4 Proof of Theorem 4.1.1

Let us consider the viscosity solution of (4.4.40) for $\eta = 0$. By Proposition 4.4.4 we know that there exists a unique λ such that $U(\tau, Y)/\tau$ converges to λ as τ goes to $+\infty$ for any $Y \in \mathbb{R}^{N+1}$. Moreover, by Proposition 4.2.6, $U(\tau, y, 0)$ is viscosity solution of (4.1.4). Hence, the theorem follows immediately from the uniqueness of the viscosity solution of (4.1.4).

4.4.5 Proof of Proposition 4.4.1

Let us denote by U_η^+ the solution of (4.4.40) with $a_0 = C_1$, where C_1 is defined as in (4.4.54), and by U_η^- the solution of (4.4.40) with $a_0 = 0$. Let $\lambda_\eta^+ = \lim_{\tau\to+\infty} \frac{U_\eta^+(\tau,Y)}{\tau}$ and $\lambda_\eta^- = \lim_{\tau\to+\infty} \frac{U_\eta^-(\tau,Y)}{\tau}$; the existence of λ_η^+ and λ_η^- is guaranteed by Proposition

4.4.4. By stability (see e.g. [22]), for $\eta \to 0^+$ the sequence $(U_\eta^+)_\eta$ converges to U solution of (4.4.40) with $\eta = 0$. Moreover by (4.4.53) the sequence $(\lambda_\eta^+)_\eta$ is bounded. Take a subsequence $\eta_n \to 0$ as $n \to +\infty$ such that $\lambda_{\eta_n}^+ \to \lambda_\infty$ as $n \to +\infty$. We want to show that $\lambda_\infty = \lambda$, where $\lambda = \lim_{\tau\to+\infty} \frac{U(\tau,Y)}{\tau}$. By the proof of Theorem 4.1.1, we know that λ is the same quantity defined in Theorem 4.1.1. Using (4.4.52), we get

$$\begin{aligned}|\lambda - \lambda_\infty| &\le \left|\lambda - \frac{U(\tau,0)}{\tau}\right| + \left|\frac{U(\tau,0)}{\tau} - \frac{U_{\eta_n}^+(\tau,0)}{\tau}\right| + \left|\frac{U_{\eta_n}^+(\tau,0)}{\tau} - \lambda_{\eta_n}^+\right| + |\lambda_{\eta_n}^+ - \lambda_\infty| \\ &\le \left|\lambda - \frac{U(\tau,0)}{\tau}\right| + \left|\frac{U(\tau,0)}{\tau} - \frac{U_{\eta_n}^+(\tau,0)}{\tau}\right| + \frac{C_3}{\tau} + |\lambda_{\eta_n}^+ - \lambda_\infty|\end{aligned}$$

where C_3 does not depend on n. Then, passing to the limit first as $n \to +\infty$ and then as $\tau \to +\infty$, we obtain that $\lambda = \lambda_\infty$. This implies that $\lambda_\eta^+ \to \lambda$ as $\eta \to 0$.

The same argument shows that $\lambda_\eta^- \to \lambda$ as $\eta \to 0$.

Now, we set

$$W_\eta^+(\tau,Y) := U_\eta^+(\tau,Y) - \lambda_\eta^+\tau$$

and

$$W_\eta^-(\tau,Y) := U_\eta^-(\tau,Y) - \lambda_\eta^-\tau.$$

Then, W_η^+ and W_η^- are respectively the desired super and subsolution.

Indeed, since by (4.4.54), $C_0 + \inf_{Y'} U_\eta^+(\tau,Y') - U_\eta^+(\tau,Y) \ge 0$, W_η^+ is supersolution of (4.3.16) with $\lambda = \lambda_\eta^+$. Moreover, by (4.4.52), W_η^+ is bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ uniformly w.r.t. η: $|W_\eta^+(\tau,Y)| \le C_3$ for all $(\tau,Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$.

By (4.4.47), W_η^+ is Lipschitz continuous w.r.t. y_{N+1} and $-1 \le \partial_{y_{N+1}} W_\eta^+ \le \frac{\|W''\|_\infty}{\eta}$. This implies that W_η^+ is also a viscosity subsolution of

$$\begin{cases} \lambda_\eta^+ + \partial_\tau V = L + \mathcal{I}_1[V(\tau,\cdot,y_{N+1})] - W'(V + \lambda_\eta^+\tau + P\cdot Y) + \sigma(\tau,y) & \\ \qquad\qquad +C_1(\|W''\|_\infty + \eta) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V(0,Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{4.4.59}$$

By Proposition 4.2.6, W_η^+ is supersolution of (4.3.16) and subsolution of (4.4.59) in $\mathbb{R}^+ \times \mathbb{R}^N$ for any $y_{N+1} \in \mathbb{R}$. Then by Proposition 4.2.7, W_η^+ is of class C^α w.r.t. y uniformly in y_{N+1} and η, for any $0 < \alpha < 1$.

Similar arguments show that W_η^- is subsolution of (4.3.16) with $\lambda = \lambda_\eta^-$, is bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, Lipschitz continuous w.r.t. y_{N+1} with $-1 \le \partial_{y_{N+1}} W_\eta^+ \le \frac{\|W''\|_\infty}{\eta}$ and Hölder continuous w.r.t. y. This concludes the proof of Proposition 4.4.1.

4.4.6 Proof of Proposition 4.3.3

The continuity of $\overline{H}(p,L)$ follows from stability of viscosity solutions of (4.1.4) (see e.g. [22]) and from (4.4.52). Indeed, let (p_n, L_n) be a sequence converging to (p_0, L_0) as $n \to +\infty$ and set $\lambda_n = \lambda(p_n, L_n)$, $n \ge 0$. By (4.4.52), we have for any $\tau > 0$

$$\left|\lambda_n - \frac{w_n(\tau,y)}{\tau}\right| \le \frac{C_3}{\tau}.$$

Stability of viscosity solutions of (4.1.4) implies that w_n converges locally uniformly in (τ, y) to a function w_0 which is a solution of (4.1.4) with $(p, L) = (p_0, L_0)$. This implies that $\limsup_{n\to+\infty} |\lambda_n - \lambda_0| \le \frac{2C_3}{\tau}$ for any $\tau > 0$. Hence, we conclude that $\lim_{n\to+\infty} \lambda_n = \lambda_0$.

Property (i) is an immediate consequence of (4.4.53).

The monotonicity in L of $\overline{H}(p, L)$ comes from the comparison principle.

Let us show (iii). Let v be the solution of (4.1.3) and $\lambda = \lambda(p, L)$. Set $\widetilde{v}(\tau, y) := v(\tau, -y)$. Remark that $\mathcal{I}_1[\widetilde{v}(\tau,\cdot), y] = \mathcal{I}_1[v(\tau,\cdot), -y]$. If $\sigma(\tau,\cdot)$ is even then $\widetilde{v}$ satisfies

$$\begin{cases} \lambda + \partial_\tau \widetilde{v} = \mathcal{I}_1[\widetilde{v}(\tau,\cdot), y] + L - W'(\widetilde{v} + \lambda t - p\cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ \widetilde{v}(0, y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

By the uniqueness of λ we deduce that $\lambda(L, p) = \lambda(L, -p)$, i.e. (iii).

Finally let us turn to (iv). Define $\widetilde{v}(\tau, y) := -v(\tau, -y)$. If $W'(\cdot)$ and $\sigma(\tau,\cdot)$ are odd functions, $\widetilde{v}$ satisfies

$$\begin{cases} -\lambda + \partial_\tau \widetilde{v} = \mathcal{I}_1[\widetilde{v}(\tau,\cdot), y] - L - W'(\widetilde{v} - \lambda t + p\cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ \widetilde{v}(0, y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

As before, we conclude that $\lambda(-L, p) = -\lambda(L, p)$, i.e. (iv).

4.5 Smooth approximate correctors

In this section, we prove the existence of approximate correctors that are smooth w.r.t. y_{N+1}, namely Proposition 4.3.4. We first need the following lemma:

Lemma 4.5.1. *Let $u_1, u_2 \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ be viscosity subsolutions (resp. supersolutions) of* (4.3.16) *in $\mathbb{R}^+ \times \mathbb{R}^N$, then $u_1 + u_2$ is viscosity subsolution (resp., supersolution) of*

$$\begin{cases} \lambda + \partial_\tau v = L + \mathcal{I}_1[v] - W'(u_1 + P\cdot Y + \lambda\tau) \\ \qquad\qquad -W'(u_2 + P\cdot Y + \lambda\tau) + \sigma(\tau, y) & \textit{in } \mathbb{R}^+ \times \mathbb{R}^N \\ v(0, y) = 0 & \textit{on } \mathbb{R}^N. \end{cases}$$

For the proof see Lemma 5.8 in [34].

Next, let us consider a positive smooth function $\rho : \mathbb{R} \to \mathbb{R}$, with support in $B_1(0)$ and mass 1. We define a sequence of mollifiers $(\rho_\delta)_\delta$ by $\rho_\delta(s) = \frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)$, $s \in \mathbb{R}$. Let W_η^+ (resp. W_η^-) be the Lipschitz supersolution (resp. subsolution) of (4.3.16) with $\lambda = \lambda_\eta^+$ (resp. $\lambda = \lambda_\eta^-$), whose existence is guaranteed by Proposition 4.4.1. We define

$$V_{\eta,\delta}^{\pm}(t, y, y_{N+1}) := W_\eta^{\pm}(t, y, \cdot) \star \rho_\delta(\cdot) = \int_{\mathbb{R}} W_\eta^{\pm}(t, y, z)\rho_\delta(y_{N+1} - z)dz. \quad (4.5.60)$$

Lemma 4.5.2. *The functions $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$ are respectively super and subsolution of*

$$\begin{cases} \lambda_\eta^{\pm} + \partial_\tau V_{\eta,\delta}^{\pm} = L + \mathcal{I}_1[V_{\eta,\delta}^{\pm}(\tau,\cdot, y_{N+1})] + \sigma(\tau, y) \\ \qquad\qquad - \int_{\mathbb{R}} W'(W_\eta^{\pm}(\tau, y, z) + p\cdot y + z + \lambda_\eta^{\pm}\tau)\rho_\delta(y_{N+1} - z)dz & \textit{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V_\eta^{\pm}(0, Y) = 0 & \textit{on } \mathbb{R}^{N+1}. \end{cases} \quad (4.5.61)$$

Proof. We prove the lemma for supersolutions. Let $Q_h^e = e+[-h/2,h/2)$, $\overline{\rho}_\delta(e,h) = \int_{Q_h^e}\rho_\delta(y)dy$ and

$$I_h(\tau,y,y_{N+1}) = \sum_{e\in h\mathbb{Z}} W_\eta^+(\tau,y,y_{N+1}-e)\overline{\rho}_\delta(e,h).$$

The function I_h is a discretization of the convolution integral and by classical results, converges uniformly to $V_{\eta,\delta}^+$ as $h\to 0$. By Proposition 4.2.6, W_η^+ is a viscosity supersolution of (4.3.16) also in $\mathbb{R}^+\times\mathbb{R}^N$. Then, by Lemma 4.5.1, for any $y_{N+1}\in\mathbb{R}$, $I_h(\tau,y,y_{N+1})$ is a supersolution of

$$\begin{cases} \lambda_\eta^+ + \partial_\tau V = L + \mathcal{I}_1[V(\tau,\cdot,y_{N+1})] + \sigma(\tau,y)\sum_{e\in h\mathbb{Z}}\overline{\rho}_\delta(e,h) \\ \qquad\qquad - \sum_{e\in h\mathbb{Z}} W'(W_\eta^+(\tau,y,y_{N+1}-e) \\ \qquad\qquad +p\cdot y + (y_{N+1}-e) + \lambda_\eta^+\tau)\overline{\rho}_\delta(e,h) & \text{in } \mathbb{R}^+\times\mathbb{R}^N \\ V(0,y)=0 & \text{on } \mathbb{R}^N. \end{cases}$$

Using the stability result for viscosity solution of non-local equations, see [22], we conclude that $V_{\eta,\delta}^+$ is supersolution of (4.5.61) in $\mathbb{R}^+\times\mathbb{R}^N$ and hence also in $\mathbb{R}^+\times\mathbb{R}^{N+1}$. □

4.5.1 Proof of Proposition 4.3.4

We first show that the functions $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$, defined in (4.5.60), are respectively super and subsolution of

$$\begin{cases} \lambda_\eta^{\overset{+}{-}} + \partial_\tau V_{\eta,\delta}^{\overset{+}{-}} = L + \mathcal{I}_1[V_{\eta,\delta}^{\overset{+}{-}}(\tau,\cdot,y_{N+1})] - W'(V_{\eta,\delta}^{\overset{+}{-}} + P\cdot Y + \lambda_\eta^{\overset{+}{-}}\tau) \\ \qquad\qquad +\sigma(\tau,y)\,\underset{+}{-}\,C_{\eta,\delta} & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1} \\ V_\eta^{\overset{+}{-}}(0,Y)=0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{4.5.62}$$

where $C_{\eta,\delta} = \|W''\|_\infty(2\delta\|W''\|_\infty/\eta+\delta)$. Using (4.4.38) and the properties of the mollifiers, we get

$$\begin{aligned}
&\left|W'(V_{\eta,\delta}^{\overset{+}{-}}(\tau,y,y_{N+1}) + p\cdot y + y_{N+1} + \lambda_\eta^{\overset{+}{-}}\tau) - \int_{\mathbb{R}} W'(W_\eta^{\overset{+}{-}}(\tau,y,z) + p\cdot y + z + \lambda_\eta^{\overset{+}{-}}\tau)\rho_\delta(y_{N+1}-z)dz\right| \\
&\le \int_{\mathbb{R}}\left|W'(V_{\eta,\delta}^{\overset{+}{-}}(\tau,y,y_{N+1}) + p\cdot y + y_{N+1} + \lambda_\eta^{\overset{+}{-}}\tau) - W'(W_\eta^{\overset{+}{-}}(\tau,y,z) + p\cdot y + z + \lambda_\eta^{\overset{+}{-}}\tau)\right|\rho_\delta(y_{N+1}-z)dz \\
&\le \|W''\|_\infty \int_{\mathbb{R}}\left[\left|V_{\eta,\delta}^{\overset{+}{-}}(\tau,y,y_{N+1}) - W_\eta^{\overset{+}{-}}(\tau,y,z)\right| + |y_{N+1}-z|\right]\rho_\delta(y_{N+1}-z)dz \\
&\le \|W''\|_\infty \int_{\mathbb{R}}\left[\int_{\mathbb{R}}\left|W_\eta^{\overset{+}{-}}(\tau,y,r) - W_\eta^{\overset{+}{-}}(\tau,y,z)\right|\rho_\delta(y_{N+1}-r)dr + |y_{N+1}-z|\right]\rho_\delta(y_{N+1}-z)dz \\
&\le \|W''\|_\infty \int_{\mathbb{R}}\left[\int_{|y_{N+1}-r|\le\delta}\frac{\|W''\|_\infty}{\eta}|r-z|\rho_\delta(y_{N+1}-r)dr + |y_{N+1}-z|\right]\rho_\delta(y_{N+1}-z)dz \\
&\le \|W''\|_\infty \int_{|y_{N+1}-z|\le\delta}\left[\frac{\|W''\|_\infty}{\eta}(|y_{N+1}-z|+\delta) + |y_{N+1}-z|\right]\rho_\delta(y_{N+1}-z)dz \\
&\le \|W''\|_\infty\left(2\delta\frac{\|W''\|_\infty}{\eta} + \delta\right)
\end{aligned}$$

From this estimate and Lemma 4.5.2, we deduce that $V^{+}_{\eta,\delta}$ and $V^{-}_{\eta,\delta}$ are respectively super and subsolution of (4.5.62). Now, we choose $\delta = \delta(\eta)$ such that $\|W''\|_\infty(2\delta\|W''\|_\infty/\eta + \delta) = o_\eta(1)$ as $\eta \to 0$ and define

$$V_\eta^{\overset{+}{-}}(\tau, Y) := V^{\overset{+}{-}}_{\eta,\delta(\eta)}(\tau, Y).$$

Then the functions $V_\eta^{\overset{+}{-}}$ are the desired super and subcorrectors. Indeed, we have already shown that they are super and subsolution of (4.3.17) with λ^+_η and λ^-_η satisfying (4.3.18). Properties (i) and (ii) of Proposition 4.3.3 can be shown as in the proof of the proposition. Finally, (4.3.19), (4.3.20) and (4.3.21) easily follow from (4.4.37), (4.4.38), (4.4.39) and the properties of the mollifiers. □

4.6 The Orowan's law

In this section we study the behavior of the effective Hamiltonian $\overline{H}(p, L)$ close to the origin, in dimension $N = 1$, when $\mathcal{I}_1$ is the half-Laplacian and $\sigma \equiv 0$. We want to prove that $\overline{H}(p, L) \sim c_0|p|L$ when p and L are very small, where c_0 is a positive constant to be made precise. This property is known in physics as the Orowan's law. In order to prove it, let us introduce a new corrector h, usually called *hull function*. For the precise definition of such a function we refer to [56] and references therein. For $p \neq 0$ and $L \in \mathbb{R}$, let w be the solution of (4.1.4) and let $u(\tau, y) = w(\tau, y) + py$. Let us define the function $h(z)$ such that $u(\tau, y) = h(\lambda\tau + py)$. We see that h is formally a solution of

$$\lambda h' = |p|\mathcal{I}_1[h] - W'(h) + L \quad \text{in } \mathbb{R}. \tag{4.6.63}$$

Moreover, by the ergodicity property of w, $|h(z) - z| \le C_3$ for any $z \in \mathbb{R}$. Let us fix $p_0 \in \mathbb{R} \setminus \{0\}$, $L_0 \in \mathbb{R}$ and let $p = \delta p_0$ and $L = \delta L_0$, where δ is a small parameter. The main idea to prove the result is to approximate h, for such p and L, by the following ansatz

$$\widetilde{h}(x) = \frac{L_0\delta}{\alpha} + \sum_{i=-\infty}^{+\infty}\left[\phi\left(\frac{x-i}{\delta|p_0|}\right) - \frac{1}{2}\right] + \delta\sum_{i=-\infty}^{+\infty}\psi\left(\frac{x-i}{\delta|p_0|}\right), \tag{4.6.64}$$

where $\alpha = W''(0) > 0$ (see later) and the functions ϕ and ψ are respectively the solutions of the following problems

$$\begin{cases} \mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \lim_{x\to-\infty}\phi(x) = 0, \quad \lim_{x\to+\infty}\phi(x) = 1, \quad \phi(0) = \frac{1}{2} & \\ \phi' > 0 & \text{in } \mathbb{R}, \end{cases} \tag{4.6.65}$$

and

$$\begin{cases} \mathcal{I}_1[\psi] = W''(\phi)\psi + \frac{L}{W''(0)}(W''(\phi) - W''(0)) + c\phi' \quad \text{in } \mathbb{R} \\ \lim_{x\to\overset{+}{-}\infty}\psi(x) = 0 \\ c = \frac{L}{\int_{\mathbb{R}}(\phi')^2}, \end{cases} \tag{4.6.66}$$

with $L = L_0$. Here and in what follows, $\mathcal{I}_1$ denote the half-Laplacian in dimension 1, i.e., $\mu(dy) = dy/|y|^2$. On the function W we assume the following hypothesis:

$$\begin{cases} W \in C^{4,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(v+1) = W(v) & \text{for any } v \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ \alpha = W''(0) > 0. \end{cases} \tag{4.6.67}$$

If (4.6.67) holds true, there exists a unique solution of (4.6.65) which is of class $C^{4,\beta}$, as shown by Cabré and Solà-Morales in [33]. Under (4.6.67), the existence of a solution of class $C^{3,\beta}$ of the problem (4.6.66) is proved by Gonzáles and Monneau in [62] .

We will show that the function (4.6.64) satisfies, up to small errors, the equation (4.6.63) with $\lambda = c_0|\delta p_0|\delta L_0$, where $c_0 = (\int_{\mathbb{R}}(\phi')^2)^{-1}$. This implies, by comparison, that $\overline{H}(\delta p_0, \delta L_0) \sim c_0|\delta p_0|\delta L_0$ as $\delta \to 0^+$. Precisely we have:

Proposition 4.6.1. *Assume* (4.6.67) *and let* p_0, $L_0 \in \mathbb{R}$. *Then*

$$\frac{\overline{H}(\delta p_0, \delta L_0)}{\delta^2} \to c_0|p_0|L_0 \quad \text{as } \delta \to 0^+. \tag{4.6.68}$$

Proof. Suppose $p_0 \neq 0$. For $L \in \mathbb{R}$, $\delta > 0$ and $n \in \mathbb{N}$ we define the sequence $\{s^L_{\delta,n}(x)\}_n$ by

$$s^L_{\delta,n}(x) = \frac{L\delta}{\alpha} + \sum_{i=-n}^{n} \phi\left(\frac{x-i}{\delta|p_0|}\right) + \delta \sum_{i=-n}^{n} \psi\left(\frac{x-i}{\delta|p_0|}\right) - n$$

where ϕ is a solution of (4.6.65) and ψ is a solution of (4.6.66). We consider the differential operator $NL_L^{\overline{\lambda}^L_\delta}$, defined on smooth functions as follows

$$NL_L^{\overline{\lambda}^L_\delta}[h] = \overline{\lambda}^L_\delta h' - \delta|p_0|\mathcal{I}_1[h] + W'(h) - \delta L,$$

where

$$\overline{\lambda}^L_\delta = \delta^2 c_0|p_0|L.$$

Then we have

Proposition 4.6.2. *Assume* (4.6.67). *For any* $x \in \mathbb{R}$ *there exists finite the limit*

$$h^L_\delta(x) = \lim_{n\to+\infty} s^L_{\delta,n}(x).$$

Moreover h^L_δ *has the following properties:*

(i) $h^L_\delta \in C^2(\mathbb{R})$ *and satisfies*

$$NL_L^{\overline{\lambda}^L_\delta}[h^L_\delta](x) = o(\delta),$$

where $\lim_{\delta\to 0} \frac{o(\delta)}{\delta} = 0$ *uniformly for* $x \in \mathbb{R}$*;*

(ii) There exists a constant $C > 0$ such that $|h(x) - x| \le C$ for any $x \in \mathbb{R}$.

The proof of Proposition 4.6.2 is postponed.

Remark 4.6.3. From (ii) of Proposition 4.6.2, we see that the function $h_\delta^L(x)$ goes to infinity like the power x. Then the integral $\mathcal{I}_1^2[h_\delta^L, x] = \int_{|y|>1}(h_\delta^L(x+y) - h_\delta^L(x))\frac{dy}{|y|^2}$, is not well defined in the sense of the Lesbegue integration. By $\mathcal{I}_1^2[h_\delta^L, x]$, we mean

$$\mathcal{I}_1^2[h_\delta^L, x] = \lim_{a\to+\infty}\int_{1<|y|<a}(h_\delta^L(x+y) - h_\delta^L(x))\frac{dy}{|y|^2}.$$

This definition coincides with the standard Lesbegue integral for integrable functions. In what follows we will consider the function $\overline{w}(\tau, y) = h_\delta^L(\delta p_0 y + \overline{\lambda}_\delta^L\tau) - \delta p_0 y$ which belongs to $C_b(\mathbb{R}^+ \times \mathbb{R})$ and then for which $\mathcal{I}_1^2$ is well defined and

$$\mathcal{I}_1^2[\overline{w}(\tau, \cdot), y] = \lim_{a\to+\infty}\int_{1<|z|<a}(\overline{w}(\tau, y+z) - \overline{w}(\tau, y))\frac{dz}{|z|^2} = \delta p_0\mathcal{I}_1^2[h_\delta^L, \delta p_0 y + \overline{\lambda}_\delta^L\tau].$$

Fix $\eta > 0$ and let $L = L_0 - \eta$. By (i) of Proposition 4.6.2, there exists $\delta_0 = \delta_0(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ we have

$$NL_{L_0}^{\overline{\lambda}_\delta^L}[h_\delta^L] = NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L] - \delta\eta < 0 \quad \text{in } \mathbb{R}. \tag{4.6.69}$$

Define the function $\overline{w}(\tau, y)$ by

$$\overline{w}(\tau, y) = h_\delta^L(\delta p_0 y + \overline{\lambda}_\delta^L\tau) - \delta p_0 y.$$

Then $w \in C_b(\mathbb{R}^+ \times \mathbb{R})$, since by (ii) of Proposition 4.6.2

$$|\overline{w}(\tau, y) - \overline{\lambda}_\delta^L\tau| \le \lceil C\rceil. \tag{4.6.70}$$

Moreover, by (4.6.69) and (4.6.70) $\overline{w}$ satisfies

$$\begin{cases} \overline{w}_\tau \le \mathcal{I}_1[\overline{w}] - W'(\overline{w} + \delta p_0 y) + \delta L_0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ \overline{w}(0, y) \le \lceil C\rceil & \text{on } \mathbb{R}. \end{cases}$$

Let $w(\tau, y)$ be the solution of (4.1.4) with $N = 1$, $p = \delta p_0$, $L = \delta L_0$ and $\sigma \equiv 0$, whose existence is ensured by Theorem 4.1.1, then from the comparison principle and the periodicity of W, we deduce that

$$\overline{w}(\tau, y) \le w(\tau, y) + \lceil C\rceil.$$

By the previous inequality and (4.6.70), we get

$$\overline{\lambda}_\delta^L\tau \le w(\tau, y) + 2\lceil C\rceil,$$

and dividing by τ and letting τ go to $+\infty$, we finally obtain

$$\delta^2 c_0|p_0|(L_0 - \eta) = \overline{\lambda}_\delta^L \le \overline{H}(\delta p_0, \delta L_0).$$

Similarly, it is possible to show that

$$\overline{H}(\delta p_0, \delta L_0) \le \delta^2 c_0|p_0|(L_0 + \eta).$$

We have proved that for any $\eta > 0$ there exists $\delta_0 = \delta_0(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ we have

$$\left|\frac{\overline{H}(\delta p_0, \delta L_0)}{\delta^2} - c_0|p_0|L_0\right| \le c_0|p_0|\eta,$$

i.e. (4.6.68), as desired. □

4.6.1 Proof of Proposition 4.6.2.

To prove the proposition we need the following two lemmas about the behavior of the functions ϕ and ψ at infinity. We denote by $H(x)$ the heaveside function defined by

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Then we have

Lemma 4.6.4. *Assume* (4.6.67)*. Let ϕ be the solution of* (4.6.65)*, then there exist constants $K_0, K_1 > 0$ such that*

$$\left|\phi(x) - H(x) + \frac{1}{\alpha\pi x}\right| \leq \frac{K_1}{x^2}, \quad \text{for } |x| \geq 1, \tag{4.6.71}$$

and for any $x \in \mathbb{R}$

$$0 < \frac{K_0}{1+x^2} \leq \phi'(x) \leq \frac{K_1}{1+x^2}, \tag{4.6.72}$$

$$-\frac{K_1}{1+x^2} \leq \phi''(x) \leq \frac{K_1}{1+x^2}, \tag{4.6.73}$$

$$-\frac{K_1}{1+x^2} \leq \phi'''(x) \leq \frac{K_1}{1+x^2}. \tag{4.6.74}$$

Lemma 4.6.5. *Assume* (4.6.67)*. Let ψ be the solution of* (4.6.66)*, then for any $L \in \mathbb{R}$ there exist constants K_2 and K_3, with $K_3 > 0$, depending on L such that*

$$\left|\psi(x) - \frac{K_2}{x}\right| \leq \frac{K_3}{x^2}, \quad \text{for } |x| \geq 1, \tag{4.6.75}$$

and for any $x \in \mathbb{R}$

$$-\frac{K_3}{1+x^2} \leq \psi'(x) \leq \frac{K_3}{1+x^2}, \tag{4.6.76}$$

$$-\frac{K_3}{1+x^2} \leq \psi''(x) \leq \frac{K_3}{1+x^2}. \tag{4.6.77}$$

We postpone the proof of the two lemmas to the end of the section.

For simplicity of notation we denote

$$x_i = \frac{x-i}{\delta|p_0|}, \quad \widetilde{\phi}(z) = \phi(z) - H(z).$$

We proceed by proving several claims.

Claim 1: *Let $x = i_0 + \gamma$, with $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, then*

$$\sum_{\substack{i=-n \\ i \neq i_0}}^{n} \frac{1}{x-i} \to -2\gamma \sum_{i=1}^{+\infty} \frac{1}{i^2 - \gamma^2} \quad \text{as } n \to +\infty,$$

$$\sum_{i=-n}^{i_0-1} \frac{1}{(x-i)^2} \to \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^2} \quad \text{as } n \to +\infty,$$

$$\sum_{i=i_0+1}^{n} \frac{1}{(x-i)^2} \to \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^2} \quad \textit{as } n \to +\infty.$$

We have

$$\sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{1}{x-i} = \sum_{i=-n}^{i_0-1} \frac{1}{i_0+\gamma-i} + \sum_{i=i_0+1}^{n} \frac{1}{i_0+\gamma-i} = \sum_{i=1}^{n+i_0} \frac{1}{i+\gamma} - \sum_{i=1}^{n-i_0} \frac{1}{i-\gamma}$$

$$= \begin{cases} \sum_{i=1}^{n} \frac{-2\gamma}{i^2-\gamma^2}, & \text{if } i_0 = 0 \\ \sum_{i=1}^{n-i_0} \frac{-2\gamma}{i^2-\gamma^2} + \sum_{i=n-i_0+1}^{n+i_0} \frac{1}{i+\gamma}, & \text{if } i_0 > 0 \\ \sum_{i=1}^{n+i_0} \frac{-2\gamma}{i^2-\gamma^2} - \sum_{i=n+i_0+1}^{n-i_0} \frac{1}{i-\gamma}, & \text{if } i_0 < 0 \end{cases} \to -2\gamma \sum_{i=1}^{+\infty} \frac{1}{i^2-\gamma^2} \quad \text{as } n \to +\infty.$$

Let us prove the second limit of the claim.

$$\sum_{i=-n}^{i_0-1} \frac{1}{(x-i)^2} = \sum_{i=1}^{n+i_0} \frac{1}{(i+\gamma)^2} \to \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^2} \quad \text{as } n \to +\infty.$$

Finally

$$\sum_{i=i_0+1}^{n} \frac{1}{(x-i)^2} = \sum_{i=1}^{n-i_0} \frac{1}{(i-\gamma)^2} \to \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^2} \quad \text{as } n \to +\infty,$$

and the claim is proved.

By Claim 1 $\sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{1}{x-i}$, $\sum_{i=-n}^{i_0-1} \frac{1}{(x-i)^2}$ and $\sum_{i=i_0+1}^{n} \frac{1}{(x-i)^2}$ are Cauchy sequences and then for $k > m > |i_0|$ we have

$$\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i} \to 0 \quad \text{as } m,k \to +\infty, \tag{4.6.78}$$

$$\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} \to 0 \quad \text{as } m,k \to +\infty, \tag{4.6.79}$$

and

$$\sum_{i=m+1}^{k} \frac{1}{(x-i)^2} \to 0 \quad \text{as } m,k \to +\infty. \tag{4.6.80}$$

Claim 2: *For any $x \in \mathbb{R}$ the sequence $\{s^L_{\delta,n}(x)\}_n$ converges as $n \to +\infty$.*
We show that $\{s^L_{\delta,n}(x)\}_n$ is a Cauchy sequence. Fix $x \in \mathbb{R}$ and let $i_0 \in \mathbb{Z}$ be the closest integer to x, then $x = i_0 + \gamma$ with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $|x-i| \geq \frac{1}{2}$ for $i \neq i_0$. Let δ be so small that $\frac{1}{\delta|p_0|} \geq 2$, then $\frac{|x-i|}{\delta|p_0|} \geq 1$ for $i \neq i_0$. Let $k > m > |i_0|$, using

(4.6.71) and (4.6.75) we get

$$\begin{aligned}
s^L_{\delta,k}(x)-s^L_{\delta,m}(x) &= \sum_{i=-k}^{-m-1}[\phi(x_i)+\delta\psi(x_i)]+\sum_{i=m+1}^{k}[\phi(x_i)+\delta\psi(x_i)]-(k-m)\\
&= \sum_{i=-k}^{-m-1}[(\phi(x_i)-1)+\delta\psi(x_i)]+\sum_{i=m+1}^{k}[\phi(x_i)+\delta\psi(x_i)]\\
&\le -\left(\frac{1}{\alpha\pi}-\delta K_2\right)\delta|p_0|\sum_{i=-k}^{-m-1}\frac{1}{x-i}+(K_1+\delta K_3)\delta^2|p_0|^2\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2}\\
&\quad -\left(\frac{1}{\alpha\pi}-\delta K_2\right)\delta|p_0|\sum_{i=m+1}^{k}\frac{1}{x-i}+(K_1+\delta K_3)\delta^2|p_0|^2\sum_{i=m+1}^{k}\frac{1}{(x-i)^2},
\end{aligned}$$

and

$$\begin{aligned}
s^L_{\delta,k}(x)-s^L_{\delta,m}(x) \ge &-\left(\frac{1}{\alpha\pi}-\delta K_2\right)\delta|p_0|\left(\sum_{i=-k}^{-m-1}\frac{1}{x-i}+\sum_{i=m+1}^{k}\frac{1}{x-i}\right)\\
&-(K_1+\delta K_3)\delta^2|p_0|^2\left(\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2}+\sum_{i=m+1}^{k}\frac{1}{(x-i)^2}\right).
\end{aligned}$$

Then from (4.6.78), (4.6.79), (4.6.80), we conclude that

$$|s^L_{\delta,k}(x)-s^L_{\delta,m}(x)|\to 0\quad\text{as } m,k\to+\infty,$$

as desired.

Claim 3: *The sequence $\{(s^L_{\delta,n})'\}_n$ converges on $\mathbb{R}$ as $n\to+\infty$, uniformly on compact sets.*

To prove the uniform convergence, it suffices to show that $\{(s^L_{\delta,n})'(x)\}_n$ is a Cauchy sequence uniformly on compact sets. Let us consider a bounded interval $[a,b]$ and let $x\in[a,b]$. For $\frac{1}{\delta|p_0|}\ge 2$ and $k>m>\max\{|a|,|b|\}$, by (4.6.72) and (4.6.76) we have

$$\begin{aligned}
(s^L_{\delta,k})'(x)-(s^L_{\delta,m})'(x) &= \frac{1}{\delta|p_0|}\sum_{i=-k}^{-m-1}[\phi'(x_i)+\delta\psi'(x_i)]+\frac{1}{\delta|p_0|}\sum_{i=m+1}^{k}[\phi'(x_i)+\delta\psi'(x_i)]\\
&\le (K_1+\delta K_3)\delta|p_0|\left[\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2}+\sum_{i=m+1}^{k}\frac{1}{(x-i)^2}\right]\\
&\le (K_1+\delta K_3)\delta|p_0|\left[\sum_{i=-k}^{-m-1}\frac{1}{(a-i)^2}+\sum_{i=m+1}^{k}\frac{1}{(b-i)^2}\right],
\end{aligned}$$

and

$$(s^L_{\delta,k})'(x)-(s^L_{\delta,m})'(x)\ge -K_3\delta^2|p_0|\left[\sum_{i=-k}^{-m-1}\frac{1}{(b-i)^2}+\sum_{i=m+1}^{k}\frac{1}{(a-i)^2}\right].$$

Then by (4.6.79) and (4.6.80)

$$\sup_{x\in[a,b]}|(s^L_{\delta,k})'(x)-(s^L_{\delta,m})'(x)|\to 0\quad\text{as } k,m\to+\infty,$$

and Claim 3 is proved.
Claim 4: *The sequence* $\{(s^L_{\delta,n})''\}_n$ *converges on* $\mathbb{R}$ *as* $n \to +\infty$*, uniformly on compact sets.*
Claim 4 can be proved like Claim 3. Indeed

$$(s^L_{\delta,n})''(x) = \frac{1}{\delta^2|p_0|^2}\sum_{i=-n}^{n}[\phi''(x_i) + \delta\psi''(x_i)]$$

and using (4.6.73) and (4.6.77), it is easy to show that $\{(s^L_{\delta,n})''\}_n$ is a Cauchy sequence uniformly on compact sets.
Claim 5: *For any* $x \in \mathbb{R}$ *the sequences* $\sum_{i=-n}^{n}\mathcal{I}_1[\phi, x_i]$ *and* $\sum_{i=-n}^{n}\mathcal{I}_1[\psi, x_i]$ *converge as* $n \to +\infty$*.*
We have

$$\mathcal{I}_1[\phi] = W'(\phi) = W'(\widetilde{\phi}) = W''(0)\widetilde{\phi} + O(\widetilde{\phi})^2.$$

Let $x = i_0 + \gamma$ with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, and $k > m > |i_0|$. From (4.6.71), (4.6.78), (4.6.79) and (4.6.80) we get

$$\sum_{i=-k}^{k}\mathcal{I}_1[\phi, x_i] - \sum_{i=-m}^{m}\mathcal{I}_1[\phi, x_i] = \sum_{i=-k}^{-m-1}[\alpha\widetilde{\phi}(x_i) + O(\widetilde{\phi}(x_i))^2] + \sum_{i=m+1}^{k}[\alpha\widetilde{\phi}(x_i) + O(\widetilde{\phi}(x_i))^2]$$
$$\leq -\frac{\delta|p_0|}{\pi}\left[\sum_{i=-k}^{-m-1}\frac{1}{x-i} + \sum_{i=m+1}^{k}\frac{1}{x-i}\right] + C\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2} + C\sum_{i=m+1}^{k}\frac{1}{(x-i)^2} \to 0,$$

as $m, k \to +\infty$, for some constant $C > 0$, and

$$\sum_{i=-k}^{k}\mathcal{I}_1[\phi, x_i] - \sum_{i=-m}^{m}\mathcal{I}_1[\phi, x_i]$$
$$\geq -\frac{\delta|p_0|}{\pi}\left[\sum_{i=-k}^{-m-1}\frac{1}{x-i} + \sum_{i=m+1}^{k}\frac{1}{x-i}\right] - C\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2} - C\sum_{i=m+1}^{k}\frac{1}{(x-i)^2} \to 0,$$

as $m, k \to +\infty$. Then $\sum_{i=-n}^{n}\mathcal{I}_1[\phi, x_i]$ is a Cauchy sequence, i.e. it converges.

Let us consider now $\sum_{i=-n}^{n}\mathcal{I}_1[\psi, x_i]$. We have

$$\mathcal{I}_1[\psi] = W''(\phi)\psi + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0)) + c\phi' = W''(\phi)\psi + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + O(\widetilde{\phi})^2 + c\phi'.$$

Let $R > 0$ be such that $0 < \frac{\alpha}{2} \leq W''(\phi(z)) \leq 2\alpha$ for $|z| \geq R$ and let δ be so small that $\frac{|x-i|}{\delta|p_0|} > R$ for $i \neq i_0$, then $0 < \frac{\alpha}{2} \leq W''(\phi(x_i)) \leq 2\alpha$ for $i \neq i_0$ and by (4.6.71), (4.6.72) and (4.6.75) we get

$$\sum_{i=-k}^{k}\mathcal{I}_1[\psi, x_i] - \sum_{i=-m}^{m}\mathcal{I}_1[\psi, x_i]$$
$$\leq \widetilde{C}\left[\sum_{i=-k}^{-m-1}\frac{1}{x-i} + \sum_{i=m+1}^{k}\frac{1}{x-i}\right] + C\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2} + C\sum_{i=m+1}^{k}\frac{1}{(x-i)^2},$$

and

$$
\begin{aligned}
&\sum_{i=-k}^{k} \mathcal{I}_1[\psi, x_i] - \sum_{i=-m}^{m} \mathcal{I}_1[\psi, x_i] \\
&\geq \widetilde{C}\left[\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i}\right] - C\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} - C\sum_{i=m+1}^{k} \frac{1}{(x-i)^2},
\end{aligned}
$$

for some $\widetilde{C} \in \mathbb{R}$ and $C > 0$, which ensures the convergence of $\sum_{i=-n}^{n} \mathcal{I}_1[\psi, x_i]$.

Claim 6: $-C\delta^2 \leq \lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L](x) \leq C\delta^2$, *where C is independent of x.*

Fix $x \in \mathbb{R}$, let $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $x = i_0 + \gamma$, let $\frac{1}{\delta|p_0|} \geq 2$ and $n > |i_0|$. Then we have

$$
\begin{aligned}
&NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L](x) \\
&= \frac{\overline{\lambda}_\delta^L}{\delta|p_0|} \sum_{i=-n}^{n} [\phi'(x_i) + \delta\psi'(x_i)] - \sum_{i=-n}^{n} [\mathcal{I}_1[\phi, x_i] + \delta\mathcal{I}_1[\psi, x_i]] \\
&+ W'\left(\frac{L\delta}{\alpha} + \sum_{i=-n}^{n} [\phi(x_i) + \delta\psi(x_i)]\right) - \delta L \\
&= \frac{\overline{\lambda}_\delta^L}{\delta|p_0|}\left\{\phi'(x_{i_0}) + \delta\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)]\right\} - \sum_{\substack{i=-n\\ i\neq i_0}}^{n} W'(\widetilde{\phi}(x_i)) - \delta\mathcal{I}_1[\psi, x_{i_0}] \\
&- \delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i] + W'\left(\frac{L\delta}{\alpha} + \sum_{i=-n}^{n}\left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right) - W'(\widetilde{\phi}(x_{i_0})) - \delta L \\
&= \delta c_o L\left\{\phi'(x_{i_0}) + \delta\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)]\right\} - W''(0)\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) - \delta\mathcal{I}_1[\psi, x_{i_0}] \\
&- \delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i] + W''(\phi(x_{i_0}))\left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right) \\
&+ \sum_{\substack{i=-n\\ i\neq i_0}}^{n} O(\widetilde{\phi}(x_i))^2 + O\left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right)^2 - \delta L \\
&= \delta c_o L\left\{\delta\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)]\right\} - W''(0)\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) - \delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i] \\
&+ W''(\phi(x_{i_0}))\left(\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right) + \delta\Big(-\mathcal{I}_1[\psi, x_{i_0}] + W''(\phi(x_{i_0}))\psi(x_{i_0}) \\
&+ \frac{L}{\alpha}W''(\phi(x_{i_0})) - L + c\phi'(x_{i_0})\Big) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} O(\widetilde{\phi}(x_i))^2
\end{aligned}
$$

$$+O\left(\frac{L\delta}{\alpha}+\delta\psi(x_{i_0})+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i)+\delta\psi(x_i)\right]\right)^2$$

$$=\delta c_0 L\left\{\delta\psi'(x_{i_0})+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\phi'(x_i)+\delta\psi'(x_i)\right]\right\}+(W''(\phi(x_{i_0}))-W''(0))\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\widetilde{\phi}(x_i)$$

$$-\delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\mathcal{I}_1[\psi,x_i]+W''(\phi(x_{i_0}))\delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\psi(x_i)+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}O(\widetilde{\phi}(x_i))^2$$

$$+O\left(\frac{L\delta}{\alpha}+\delta\psi(x_{i_0})+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i)+\delta\psi(x_i)\right]\right)^2$$

Let us bound the second member of the last equality, uniformly in x. From (4.6.72) and (4.6.76) it follows that

$$-\delta^3|p_0|^2K_3\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{1}{(x-i)^2}\le\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\phi'(x_i)+\delta\psi'(x_i)\right]\le\delta^2|p_0|^2(K_1+\delta K_3)\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{1}{(x-i)^2},$$

and then by Claim 1 we get

$$-C\delta^3\le\lim_{n\to+\infty}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\phi'(x_i)+\delta\psi'(x_i)\right]\le C\delta^2. \tag{4.6.81}$$

Here and henceforth, C denotes various positive constants independent of x.

Now, let us prove that

$$-C\delta^2\le\lim_{n\to+\infty}(W''(\phi(x_{i_0}))-W''(0))\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\widetilde{\phi}(x_i)\le C\delta^2. \tag{4.6.82}$$

By (4.6.71) we have

$$\left|\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\widetilde{\phi}(x_i)+\frac{\delta|p_0|}{\alpha\pi}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{1}{x-i}\right|\le K_1\delta^2|p_0|^2\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{1}{(x-i)^2}. \tag{4.6.83}$$

If $|\gamma|\ge\delta|p_0|$, then again from (4.6.71), $|\widetilde{\phi}(x_{i_0})+\frac{\delta|p_0|}{\alpha\pi\gamma}|\le K_1\frac{\delta^2|p_0|^2}{\gamma^2}$ which implies that

$$|W''(\widetilde{\phi}(x_{i_0}))-W''(0)|\le|W'''(0)\widetilde{\phi}(x_{i_0})|+O(\widetilde{\phi}(x_{i_0}))^2\le C\frac{\delta}{|\gamma|}+C\frac{\delta^2}{\gamma^2}.$$

By the previous inequality, (4.6.83) and Claim 1 we deduce that

$$\left|\lim_{n\to+\infty}(W''(\phi(x_{i_0}))-W''(0))\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\widetilde{\phi}(x_i)\right|\le C\left(\frac{\delta}{|\gamma|}+\frac{\delta^2}{\gamma^2}\right)(\delta|\gamma|+\delta^2)\le C\delta^2,$$

where C is independent of γ.

Finally, if $|\gamma| < \delta|p_0|$, from (4.6.83) and Claim 1 we conclude that

$$\left|\lim_{n\to+\infty}(W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i)\right| \leq C\delta|\gamma| + C\delta^2 \leq C\delta^2,$$

and (4.6.82) is proved.

Now, let us consider $\delta \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i]$. Let δ be so small that $\frac{\alpha}{2} \leq W''(\phi(x_i)) \leq \alpha$ for $i \neq i_0$. Then, since ψ satisfies (4.6.66), we have

$$\mathcal{I}_1[\psi, x_i] = W''(\phi(x_i))\psi(x_i) + \frac{L}{\alpha}W'''(0)\widetilde{\phi}(x_i) + O(\widetilde{\phi}(x_i))^2 + c\phi'^2(x_i) \leq \widetilde{C}\frac{\delta}{x-i} + C\frac{\delta^2}{(x-i)^2},$$

and

$$\mathcal{I}_1[\psi, x_i] \geq \widetilde{c}\frac{\delta}{x-i} - C\frac{\delta^2}{(x-i)^2},$$

for some constants $\widetilde{c}, \widetilde{C} \in \mathbb{R}$, $C > 0$. Then, from Claim 1 we deduce that

$$\left|\lim_{n\to+\infty} \delta \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i]\right| \leq C\delta^2. \tag{4.6.84}$$

Similar computations show that

$$\left|\lim_{n\to+\infty} W''(\phi(x_{i_0}))\delta \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \psi(x_i)\right| \leq C\delta^2. \tag{4.6.85}$$

Finally, still from (4.6.71), (4.6.75), and Claim 1 it follows that

$$\left|\lim_{n\to+\infty} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} (\widetilde{\phi}(x_i))^2 + O\left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right)^2\right| \leq C\delta^2. \tag{4.6.86}$$

Therefore, from (4.6.81), (4.6.82), (4.6.84), (4.6.85) and (4.6.86) we conclude that

$$-C\delta^2 \leq \lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L] \leq C\delta^2$$

with C independent of x and Claim 6 is proved.

Claim 7: $|h_\delta^L(x) - x| \leq C$ *for any* $x \in \mathbb{R}$.

Let $x = i_0 + \gamma$ with $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. Let $n > i_0$, then by (4.6.71) and

(4.6.75) we get

$$\begin{aligned} s^L_{\delta,n}(x)-x &= \frac{L\delta}{\alpha}+\phi(x_{i_0})+\delta\psi(x_{i_0})+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}[\phi(x_i)+\delta\psi(x_i)]-n-i_0-\gamma\\ &= \frac{L\delta}{\alpha}+\phi(x_{i_0})+\delta\psi(x_{i_0})+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}[\widetilde{\phi}(x_i)+\delta\psi(x_i)]-\gamma\\ &\le \frac{L\delta}{\alpha}+1+\delta\|\psi\|_\infty+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[-\left(\frac{1}{\alpha\pi}-\delta K_2\right)\frac{\delta|p_0|}{x-i}+(K_1+\delta K_3)\frac{\delta^2|p_0|^2}{(x-i)^2}\right]+\frac{1}{2}. \end{aligned}$$

Then, by Claim 1

$$h^L_\delta(x)-x=\lim_{n\to+\infty}s^L_{\delta,n}(x)-x\le C.$$

Similarly we can prove that

$$h^L_\delta(x)-x\ge -C,$$

which concludes the claim.

Now, let us show (i).

The function $h^L_\delta(x)=\lim_{n\to+\infty}s^L_{\delta,n}(x)$ is well defined for any $x\in\mathbb{R}$ by Claim 2. Moreover, by Claim 3 and 4 and classical analysis results, it is of class C^2 on $\mathbb{R}$ with

$$(h^L_\delta)'(x)=\lim_{n\to+\infty}(s^L_{\delta,n})'(x)=\lim_{n\to+\infty}\frac{1}{\delta|p_0|}\sum_{i=-n}^{n}\left[\phi'\left(\frac{x-i}{\delta|p_0|}\right)+\delta\psi'\left(\frac{x-i}{\delta|p_0|}\right)\right],$$

$$(h^L_\delta)''(x)=\lim_{n\to+\infty}(s^L_{\delta,n})''(x)=\lim_{n\to+\infty}\frac{1}{\delta^2|p_0|^2}\sum_{i=-n}^{n}\left[\phi''\left(\frac{x-i}{\delta|p_0|}\right)+\delta\psi''\left(\frac{x-i}{\delta|p_0|}\right)\right],$$

and the convergence of $\{s^L_{\delta,n}\}_n$, $\{(s^L_{\delta,n})'\}_n$ and $\{(s^L_{\delta,n})''\}_n$ is uniform on compact sets.

Let us show that for any $x\in\mathbb{R}$

$$\mathcal{I}_1[h^L_\delta,x]=\lim_{n\to+\infty}\mathcal{I}_1[s^L_{\delta,n},x]. \tag{4.6.87}$$

First, we prove that

$$\mathcal{I}^1_1[h^L_\delta,x]=\lim_{n\to+\infty}\mathcal{I}^1_1[s^L_{\delta,n},x]. \tag{4.6.88}$$

Fix $x\in\mathbb{R}$, we know that for any $y\in[-1,1]$, $y\neq 0$

$$\frac{s^L_{\delta,n}(x+y)-s^L_{\delta,n}(x)-(s^L_{\delta,n})'(x)y}{|y|^2}\to\frac{h^L_\delta(x+y)-h^L_\delta(x)-(h^L_\delta)'(x)y}{|y|^2}\quad\text{as } n\to+\infty.$$

By the uniform convergence of the sequence $\{(s^L_{\delta,n})''\}_n$ we have

$$\frac{|s^L_{\delta,n}(x+y)-s^L_{\delta,n}(x)-(s^L_{\delta,n})'(x)y|}{|y|^2}\le\sup_{z\in[x-1,x+1]}(s^L_{\delta,n})''(z)\le C,$$

where C is indipendent of n, and (4.6.88) follows from the dominate convergence Theorem.

Then, to prove (4.6.87) it suffices to show that

$$\mathcal{I}_1^2[h_\delta^L, x] = \lim_{n\to+\infty} \mathcal{I}_1^2[s_{\delta,n}^L, x].$$

From Claim 5 and (4.6.88), we know that for any $x \in \mathbb{R}$ there exists $\lim_{n\to+\infty} \mathcal{I}_1^2[s_{\delta,n}^L, x]$. For $a > 1$, we have

$$\mathcal{I}_1^2[s_{\delta,n}^L, x] = \int_{1\le|y|\le a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2} + \int_{|y|\ge a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2}.$$

By the uniform convergence of $\{s_{\delta,n}^L\}_n$ on compact sets

$$\lim_{n\to+\infty} \int_{1\le|y|\le a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2} = \int_{1\le|y|\le a} [h_\delta^L(x+y) - h_\delta^L(x)]\frac{dy}{|y|^2},$$

then there exists the limit

$$\lim_{n\to+\infty} \int_{|y|\ge a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2}.$$

Using (4.6.71) and (4.6.75) is not hard to prove that

$$\lim_{a\to+\infty} \lim_{n\to+\infty} \int_{|y|\ge a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2} = 0.$$

Then, we finally we get

$$\begin{aligned}
\lim_{n\to+\infty} \mathcal{I}_1^2[s_{\delta,n}^L, x] &= \lim_{a\to+\infty} \lim_{n\to+\infty} \mathcal{I}_1^2[s_{\delta,n}^L, x] \\
&= \lim_{a\to+\infty} \lim_{n\to+\infty} \int_{1\le|y|\le a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2} \\
&+ \lim_{a\to+\infty} \lim_{n\to+\infty} \int_{|y|>a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\frac{dy}{|y|^2} \\
&= \lim_{a\to+\infty} \int_{1\le|y|\le a} [h_\delta^L(x+y) - h_\delta^L(x)]\frac{dy}{|y|^2} \\
&= \mathcal{I}_1^2[h_\delta^L, x],
\end{aligned}$$

as desired.

Now we can conclude the proof of (i). Indeed, by Claim 2, Claim 3 and (4.6.87), for any $x \in \mathbb{R}$

$$NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L, x] = \lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L, x],$$

and Claim 6 implies that

$$NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L, x] = o(\delta), \quad \text{as } \delta \to 0,$$

where $\lim_{\delta\to 0} \frac{o(\delta)}{\delta} = 0$, uniformly for $x \in \mathbb{R}$. □

4.6.2 Proof of Lemma 4.6.4.

Properties (4.6.71) and (4.6.72) are proved in [62].

Let us show (4.6.73).

For $a > 0$, we denote by $\phi'_a(x) = \phi'\left(\frac{x}{a}\right)$. Remark that ϕ'_a is a solution of

$$\mathcal{I}_1[\phi'_a] = \frac{1}{a} W''(\phi_a)\phi'_a \quad \text{in } \mathbb{R}.$$

Since ϕ'' is bounded and of class $C^{2,\beta}$, $\mathcal{I}_1[\phi'']$ is well defined and by deriving twice the equation in (4.6.65) we see that ϕ'' is a solution of

$$\mathcal{I}_1[\phi''] = W''(\phi)\phi'' + W'''(\phi)(\phi')^2.$$

Let $\overline{\phi} = \phi'' - C\phi'_a$, with $C > 0$, then $\overline{\phi}$ satisfies

$$\begin{aligned}\mathcal{I}_1[\overline{\phi}] - W''(\phi)\overline{\phi} &= C\phi'_a\left(W''(\phi) - \frac{1}{a}W''(\phi_a)\right) + W'''(\phi)(\phi')^2\\ &= C\phi'_a\left(W''(\phi) - \frac{1}{a}W''(\phi_a)\right) + o\left(\frac{1}{1+x^2}\right),\end{aligned}$$

as $|x| \to +\infty$, by (4.6.72). Fix $a > 0$ and $R > 0$ such that

$$\begin{cases} W''(\phi) - \frac{1}{a}W''(\phi_a) > 0 & \text{on } \mathbb{R}\setminus[-R,R];\\ W''(\phi) > 0, & \text{on } \mathbb{R}\setminus[-R,R]. \end{cases} \tag{4.6.89}$$

Then from (4.6.72), for C large enough we get

$$\mathcal{I}_1[\overline{\phi}] - W''(\phi)\overline{\phi} \geq 0 \quad \text{on } \mathbb{R}\setminus[-R,R].$$

Choosing C such that moreover

$$\overline{\phi} < 0 \quad \text{on } [-R,R],$$

we can ensure that $\overline{\phi} \leq 0$ on $\mathbb{R}$. Indeed, assume by contradiction that there exists $x_0 \in \mathbb{R}\setminus[-R,R]$ such that

$$\overline{\phi}(x_0) = \sup_{\mathbb{R}} \overline{\phi} > 0.$$

Then

$$\begin{cases} \mathcal{I}_1[\overline{\phi}, x_0] \leq 0;\\ \mathcal{I}_1[\overline{\phi}, x_0] - W''(\phi(x_0))\overline{\phi}(x_0) \geq 0;\\ W''(\phi(x_0)) > 0, \end{cases}$$

from which

$$\overline{\phi}(x_0) \leq 0,$$

a contradiction. Therefore $\overline{\phi} \leq 0$ on $\mathbb{R}$ and then, by renaming the constants, from (4.6.72) we get $\phi'' \leq \frac{K_1}{1+x^2}$.

To prove that $\phi'' \geq -\frac{K_1}{1+x^2}$, we look at the infimum of the function $\phi'' + C\phi'_a$ to get similarly that $\phi'' + C\phi'_a \geq 0$ on $\mathbb{R}$.

To show (4.6.74) we proceed as in the proof of (4.6.73). Indeed, the function ϕ''' which is bounded and of class $C^{1,\beta}$, satisfies

$$\mathcal{I}_1[\phi'''] = W''(\phi)\phi''' + 3W'''(\phi)\phi'\phi'' + W^{IV}(\phi)(\phi')^3 = W''(\phi)\phi''' + o\left(\frac{1}{1+x^2}\right),$$

as $|x| \to +\infty$, by (4.6.72) and (4.6.73). Then, as before, for C and a large enough $\phi''' - C\phi'_a \leq 0$ and $\phi''' + C\phi'_a \geq 0$ on $\mathbb{R}$, which implies (4.6.74). □

4.6.3 Proof of Lemma 4.6.5.

Let us prove (4.6.75).

For $a > 0$ we denote by $\phi_a(x) = \phi\left(\frac{x}{a}\right)$, which is solution of

$$\mathcal{I}_1[\phi_a] = \frac{1}{a}W'(\phi_a) \quad \text{in } \mathbb{R}.$$

Let a and b be positive numbers, then making a Taylor expansion of the derivatives of W, we get

$$\begin{aligned}
\mathcal{I}_1[\psi - (\phi_a - \phi_b)] &= W''(\phi)\psi + \frac{L}{\alpha}(W''(\phi) - W''(0)) + c\phi' + \left(\frac{1}{b}W'(\phi_b) - \frac{1}{a}W'(\phi_a)\right) \\
&= W''(\phi)(\psi - (\phi_a - \phi_b)) + W''(\widetilde{\phi})(\phi_a - \phi_b) + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0)) \\
&\quad + c\phi' + \left(\frac{1}{b}W'(\widetilde{\phi}_b) - \frac{1}{a}W'(\widetilde{\phi}_a)\right) \\
&= W''(\phi)(\psi - (\phi_a - \phi_b)) + W''(0)(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + c\phi' \\
&\quad + W''(0)\left(\frac{1}{b}\widetilde{\phi}_b - \frac{1}{a}\widetilde{\phi}_a\right) + (\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2,
\end{aligned}$$

and then the function $\overline{\psi} = \psi - (\phi_a - \phi_b)$ satisfies

$$\begin{aligned}
\mathcal{I}_1[\overline{\psi}] - W''(\phi)\overline{\psi} = \alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + c\phi' + \alpha\left(\frac{1}{b}\widetilde{\phi}_b - \frac{1}{a}\widetilde{\phi}_a\right) \\
+ (\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2.
\end{aligned}$$

We want to estimate the right-hand side of the last equality. By Lemma 4.6.4, for $|x| \geq \max\{1, |a|, |b|\}$ we have

$$\alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} \geq -\frac{1}{\pi x}\left[(a - b) + \frac{L}{\alpha^2}W'''(0)\right] - \frac{K_1\alpha}{x^2}\left(a^2 + b^2 + \frac{|L|}{\alpha^2}|W'''(0)|\right).$$

Choose $a, b > 0$ such that $(a - b) + \frac{L}{\alpha^2}W'''(0) = 0$, then

$$\alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} \geq -\frac{C}{x^2},$$

for $|x| \geq \max\{1, |a|, |b|\}$. Here and in what follows, as usual C denotes various positive constants. From Lemma 4.6.4 we also derive that

$$\alpha\left(\frac{1}{b}\widetilde{\phi}_b - \frac{1}{a}\widetilde{\phi}_a\right) \geq -\frac{C}{x^2},$$

$$c\phi' \geq -\frac{C}{1 + x^2},$$

and

$$(\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2 \geq -\frac{C}{1 + x^2},$$

for $|x| \geq \max\{1, |a|, |b|\}$. Then we conclude that there exists $R > 0$ such that for $|x| \geq R$ we have

$$\mathcal{I}_1[\overline{\psi}] - W''(\phi)\overline{\psi} \geq -\frac{C}{1 + x^2}.$$

Now, let us consider the function $\phi'_d(x) = \phi'\left(\frac{x}{d}\right)$, $d > 0$, which is solution of

$$\mathcal{I}_1[\phi'_d] = \frac{1}{d} W''(\phi_d)\phi'_d \quad \text{in } \mathbb{R},$$

and denote

$$\overline{\overline{\psi}} = \overline{\psi} - \widetilde{C}\phi'_d,$$

with $\widetilde{C} > 0$. Then, for $|x| \geq R$ we have

$$\mathcal{I}_1[\overline{\overline{\psi}}] \geq W''(\phi)\overline{\psi} - \frac{\widetilde{C}}{d} W''(\phi_d)\phi'_d - \frac{C}{1+x^2} = W''(\phi)\overline{\overline{\psi}} + \widetilde{C}\phi'_d\left(W''(\phi) - \frac{1}{d}W''(\phi_d)\right) - \frac{C}{1+x^2}.$$

Let us choose $d > 0$ and $R_2 > R$ such that

$$\begin{cases} W''(\phi) - \frac{1}{d}W''(\phi_d) > 0 & \text{on } \mathbb{R}\setminus[-R_2, R_2]; \\ W''(\phi) > 0 & \text{on } \mathbb{R}\setminus[-R_2, R_2], \end{cases}$$

then from (4.6.72), for $\widetilde{C}$ large enough we get

$$\mathcal{I}_1[\overline{\overline{\psi}}] - W''(\phi)\overline{\overline{\psi}} \geq 0 \quad \text{on } \mathbb{R}\setminus[-R_2, R_2],$$

and

$$\overline{\overline{\psi}} < 0 \quad \text{on } [-R_2, R_2].$$

As in the proof of Lemma 4.6.4, we deduce that $\overline{\overline{\psi}} \leq 0$ on $\mathbb{R}$ and then

$$\psi \leq \frac{K_2}{x} + \frac{K_3}{x^2} \quad \text{for } |x| \geq 1,$$

for some $K_2 \in \mathbb{R}$ and $K_3 > 0$.

Looking at the function $\psi - (\phi_a - \phi_b) + \widetilde{C}\phi'_d$, we conclude similarly that

$$\psi \geq \frac{K_2}{x} - \frac{K_3}{x^2} \quad \text{for } |x| \geq 1,$$

and (4.6.75) is proved.

Now let us turn to (4.6.76). By deriving the first equation in (4.6.66), we see that the function ψ' which is bounded and of class $C^{2,\beta}$, is a solution of

$$\mathcal{I}_1[\psi'] = W''(\phi)\psi' + W'''(\phi)\phi'\psi + \frac{L}{\alpha}W'''(\phi)\phi' + c\phi'' \quad \text{in } \mathbb{R}.$$

Then the function $\overline{\psi}' = \psi' - C\phi'_a$, satisfies

$$\begin{aligned} \mathcal{I}_1[\overline{\psi}'] - W''(\phi)\overline{\psi}' &= C\phi'_a\left(W''(\phi) - \frac{1}{a}W''(\phi_a)\right) + W'''(\phi)\phi'\psi + \frac{L}{\alpha}W'''(\phi)\phi' + c\phi'' \\ &= C\phi'_a\left(W''(\phi) - \frac{1}{a}W''(\phi_a)\right) + O\left(\frac{1}{1+x^2}\right), \end{aligned}$$

by (4.6.72), (4.6.73) and (4.6.75), and as in the proof of Lemma 4.6.4, we deduce that for C and a large enough $\overline{\psi}' \leq 0$ on $\mathbb{R}$, which implies that $\psi' \leq \frac{K_3}{1+x^2}$. The inequality $\psi' \geq -\frac{K_3}{1+x^2}$ is obtained similarly by proving that $\overline{\psi}' + C\phi'_a \geq 0$ on $\mathbb{R}$.

Finally, with the same proof as before, using (4.6.72)-(4.6.76), we can prove the estimate (4.6.77) for the function ψ'' which is a bounded $C^{1,\beta}$ solution of

$$\begin{aligned}\mathcal{I}_1[\psi''] &= W''(\phi)\psi'' + 2W'''(\phi)\phi'\psi' + W^{IV}(\phi)(\phi')^2\psi + W'''(\phi)\phi''\psi + \frac{L}{\alpha}W'''(\phi)\phi'' \\ &\quad + \frac{L}{\alpha}W^{IV}(\phi)(\phi')^2 + c\phi''' \\ &= W''(\phi)\psi'' + O\left(\frac{1}{1+x^2}\right).\end{aligned}$$

□

Chapter 5

Rate of convergence in homogenization of local Hamilton-Jacobi equations

We consider homogenization problems for first order Hamilton-Jacobi equations with u^ϵ/ϵ periodic dependence, namely

$$\begin{cases} u_t^\epsilon + H\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{u^\epsilon}{\epsilon}, Du^\epsilon\right) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u^\epsilon(0,x) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{5.0.1}$$

with the following assumptions on the Hamiltonian H:

(H1) Periodicity: for any $(t,x,u,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$H(t+1, x+k, u+1, p) = H(t,x,u,p) \quad \text{for any } k \in \mathbb{Z}^N;$$

(H2) Regularity: $H : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and there exists a constant $C_1 > 0$ such that, for almost every $(t,x,u,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$|D_{(t,x)}H(t,x,u,p)| \le C_1(1+|p|), \quad |D_u H(t,x,u,p)| \le C_1, \quad |D_p H(t,x,u,p)| \le C_1;$$

(H3) $H(t,x,u,p) \to +\infty$ as $|p| \to +\infty$ uniformly for $(t,x,u) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$;

(H4) There exists a constant C such that for almost every $(t,x,u,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$|D_p H(t,x,u,p) \cdot p - H(t,x,u,p)| \le C.$$

Let us introduce the non-coercive Hamiltonian F defined by

$$F(t,x,y,p_x,p_y) = \begin{cases} |p_y| H(t,x,y,|p_y|^{-1} p_x), & \text{if } p_y \neq 0, \\ H_\infty(t,x,y,p_x), & \text{otherwise,} \end{cases} \tag{5.0.2}$$

where $H_\infty(t,x,u,p) = \lim_{s\to 0^+} sH(t,x,u,s^{-1}p)$. The function $U^\epsilon(t,x,y) := u^\epsilon(t,x) - y$ satisfies

$$\begin{cases} U_t^\epsilon + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon + y}{\epsilon}, D_x U^\epsilon, D_y U^\epsilon\right) = 0, & (t,x,y) \in (0,+\infty) \times \mathbb{R}^{N+1}, \\ U^\epsilon(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}. \end{cases} \tag{5.0.3}$$

We recall that in [18] Barles proves that under assumptions (H1)-(H4), the sequence U^ϵ converges to the solution U^0 of the following problem

$$\begin{cases} U^0_t + \overline{F}(D_x U^0, D_y U^0) = 0, & (t,x,y) \in (0,+\infty) \times \mathbb{R}^{N+1}, \\ U^0(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}, \end{cases} \tag{5.0.4}$$

where for $(p_x, p_y) \in \mathbb{R}^{N+1}$, $\overline{F}(p_x, p_y)$ is the unique number λ for which the cell problem

$$V_t + F(t,x,y,p_x + D_x V, p_y + D_y V) = \lambda \quad \text{in } \mathbb{R} \times \mathbb{R}^{N+1}. \tag{5.0.5}$$

admits bounded sub and supersolutions. This implies that the functions u^ϵ converge as $\epsilon \to 0$ to the solution u^0 of

$$\begin{cases} u^0_t + \overline{H}(Du^0) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u^0(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.0.6}$$

where $\overline{H}(p) = \overline{F}(p,-1)$.

The chapter is organized as follows: Section 5.1 is devoted to finding estimates on the rate of convergence as $\epsilon \to 0$. Section 5.2 is devoted to the numerical approximation of the effective Hamiltonian by Eulerian schemes. Finally, we present some numerical tests in Section 5.3.

5.1 An estimate on the rate of convergence when $\epsilon \to 0$

This section is devoted to the estimate of the rate of the uniform convergence of the solutions of (5.0.1) to the solution of the equation (5.0.6) in term of ϵ.

5.1.1 The main result

Theorem 5.1.1. *Assume (H1)-(H4) and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Let u^ϵ and u^0 be respectively the viscosity solutions of* (5.0.1) *and* (5.0.6)*. Then there exists a constant C, independent of $\epsilon \in (0,1)$, such that for any $T > 0$*

$$\sup_{[0,T]\times\mathbb{R}^N} |u^\epsilon(t,x) - u^0(t,x)| \le C e^T \epsilon^{\frac{1}{3}}. \tag{5.1.7}$$

If u_0 is affine then

$$\sup_{\mathbb{R}^+\times\mathbb{R}^N} |u^\epsilon(t,x) - u^0(t,x)| \le C\epsilon. \tag{5.1.8}$$

5.1.2 Preliminary results

In this section we recall some results that will be used later to obtain error estimates.

The assumptions (H1)-(H4) on H guarantee that F satisfies

(F1) Periodicity: for any $(t,x,y,p_x,p_y) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$

$$F(t+1, x+k, y+1, p_x, p_y) = F(t,x,y,p_x,p_y) \quad \text{for any } k \in \mathbb{Z}^N;$$

(F2) Regularity: $F : \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1} \to \mathbb{R}$ is Lipschitz continuous and there exists a constant $C_1 > 0$ such that, for almost every $(t,x,y,p_x,p_y) \in \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$

$$|D_{(t,x)}F(t,x,y,p_x,p_y)| \le C_1(|p_x|+|p_y|),\ |D_yF(t,x,y,p_x,p_y)| \le C_1|p_y|,$$

$$|D_{(p_x,p_y)}F(t,x,y,p_x,p_y)| \le C_1;$$

(F3) Coercivity: $F(t,x,y,p_x,p_y) \to +\infty$ as $|p_x| \to +\infty$ uniformly for $(t,x,y) \in \mathbb{R}\times\mathbb{R}^{N+1}$, $|p_y| \le R$, for any $R > 0$;

Remark that $F(t,x,y,0,0) = 0$. This and (F2) imply that for every $(t,x,y,p_x,p_y) \in \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$

$$|F(t,x,y,p_x,p_y)| \le C_1(|p_x|+|p_y|). \tag{5.1.9}$$

Moreover, by construction, F satisfies the "geometrical" assumption

(F4) For any $(t,x,y,p_x,p_y) \in \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$ and any $\lambda > 0$,

$$F(t,x,y,\lambda p_x,\lambda p_y) = \lambda F(t,x,y,p_x,p_y).$$

Assumption (F4) guarantees that (5.0.3) is invariant by any nondecreasing change $U \to \varphi(U)$, see [37] and [52], i.e., any function $V = \varphi(U^\epsilon)$, with φ nondecreasing is solution of

$$\begin{cases} V_t + F\left(\frac{t}{\epsilon},\frac{x}{\epsilon},\frac{U^\epsilon+y}{\epsilon},D_xV,D_yV\right) = 0, & (t,x,y) \in (0,+\infty)\times\mathbb{R}^{N+1}, \\ V(0,x,y) = \varphi(u_0(x)-y), & (x,y) \in \mathbb{R}^{N+1}. \end{cases}$$

Finally, note that (F3) and (F4) imply the existence of a positive constant C_2 such that

$$F(t,x,y,p_x,0) \ge C_2|p_x| \quad \text{for all } (t,x,y,p_x) \in \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^N. \tag{5.1.10}$$

In [18], in order to construct sub and supersolutions of (5.0.5), Barles introduces for $\alpha > 0$ the auxiliary equation

$$W_t^\alpha + F(t,x,y,p_x+D_xW^\alpha,p_y+D_yW^\alpha)+\alpha W^\alpha = 0, \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^{N+1}, \tag{5.1.11}$$

with F defined by (5.0.2), and shows that if (H1)-(H4) hold true, then (5.1.11) admits a unique continuous periodic viscosity solution. Moreover the limit of $\alpha W^\alpha(t,x,y)$ as $\alpha \to 0^+$ does not depend on (t,x,y) and the half-relaxed limits of $W^\alpha - \min W^\alpha$ provide a bounded subsolution and a bounded supersolution of (5.0.5), with $\lambda = -\lim_{\alpha\to 0^+}\alpha W^\alpha(t,x,y)$. We use the notation $P = (p_x,p_y) \in \mathbb{R}^{N+1}$ and $W^\alpha(x,y,P)$ for the unique solution of (5.1.11). We have the following proposition:

Proposition 5.1.2 (Barles, [18]). *For any $(t,x,y,P) \in \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$, $P = (p_x,p_y)$, the following estimates hold*

(i) $\min_{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{N+1}} -F(t,x,y,P) \le \alpha W^\alpha(t,x,y,P) \le \max_{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{N+1}} -F(t,x,y,P)$;

(ii) There exists a constant $K_1 > 0$ depending on $\|F(t,x,y,p_x,p_y)\|_\infty$ and C_2 such that

$$\max_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha - \min_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha \le K_1.$$

Further properties of $W^\alpha(x,y,P)$ are given in the following lemma:

Lemma 5.1.3. *For any $(t,x,y,P) \in \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$ the following estimates hold*

(i) $\alpha|D_P W^\alpha(t,x,y,P)| \le C_1$*, where C_1 is introduced in (F2);*

(ii) $|\alpha W^\alpha(t,x,y,P) + \overline{F}(P)| \le \alpha K_1$*, where K_1 is introduced in Proposition 5.1.2;*

(iii) $W^\alpha(t,x,y,0) \equiv 0$*;*

(iv) $\|D\overline{F}\|_\infty \le C_1$.

Proof. Let us fix $Q \in \mathbb{R}^{N+1}$. The Lipschitz continuity of F, i.e. (F2), implies that the function $W(t,x,y) = W^\alpha(t,x,y,P+Q)$ satisfies

$$W_t + F(t,x,y,P+DW) + \alpha W \le C_1|Q|$$

and then, by comparison

$$\alpha W(t,x,y) \le \alpha W^\alpha(t,x,y,P) + C_1|Q|.$$

A similar argument shows that $\alpha W(t,x,y) \ge \alpha W^\alpha(t,x,y,P) - C_1|Q|$. It then follows

$$\alpha|W^\alpha(t,x,y,P+Q) - W^\alpha(t,x,y,P)| \le C_1|Q|,$$

which proves (i).

Let us turn out to (ii). We claim that

$$\mu := \alpha \max_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha \ge -\overline{F}(P).$$

Indeed, $W^\alpha(t,x,y,P)$ is a supersolution of

$$W^\alpha_t + F(t,x,y,P+DW^\alpha) = -\mu.$$

Let V be a bounded subsolution of (5.0.5), then by comparison between $W^\alpha + \mu t$ and $V - \overline{F}(P)t$, we have

$$V(t,x,y) - W^\alpha(t,x,y) \le V(0,x,y) - W^\alpha(0,x,y) + t(\overline{F}(P) + \mu).$$

Since V and W^α are bounded, dividing by $t>0$ and letting t tend to $+\infty$, we obtain $\mu \ge -\overline{F}(P)$. Then from (ii) of Proposition 5.1.2, for $(t,x,y) \in \mathbb{R}\times\mathbb{R}^{N+1}$,

$$\alpha W^\alpha(t,x,y,P) \ge \alpha \min_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha \ge \alpha \max_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha - \alpha K_1 \ge -\overline{F}(P) - \alpha K_1.$$

A similar argument shows that

$$\alpha W^\alpha(t,x,y,P) + \overline{F}(P) \le \alpha K_1;$$

this concludes the proof of (ii).

Property (iii) follows from $F(t,x,y,0,0) = 0$ and the uniqueness of the periodic solution of (5.1.11).

Finally, (iv) is an immediate consequence of

$$\overline{F}(P) - \overline{F}(Q) \le 2\alpha K_1 + \alpha\|D_P W^\alpha\|_\infty |P-Q|$$

and of (i). □

We conclude this section by recalling some properties of the solutions u^0 and u^ϵ.

Proposition 5.1.4. *There exist constants $C_T, L > 0$ such that for any $(t,x),(s,y) \in [0,T] \times \mathbb{R}^N$*

$$|u^\epsilon(t,x)|,\ |u^0(t,x)| \le C_T, \tag{5.1.12}$$

$$|u^0(t,x) - u^0(s,y)| \le L(|t-s| + |x-y|). \tag{5.1.13}$$

Moreover, for any $t \in [0,T]$, the Lipschitz constant of $u^0(t,\cdot)$ is the Lipschitz constant of the initial datum u_0.

Proof. By comparison

$$|u^\epsilon(t,x) - u_0(x)| \le C_0 t$$

where $C_0 = \max_{x,y,|p|\le |u_0|_{1,\infty}} |H(x,y,p)|$. This implies (5.1.12) for u^ϵ. Similarly can be showed the same estimate for u^0.

The Lipschitz continuity of u^0 follows from the comparison principle for (5.0.6), see [13], Theorem III.3.7 and Remark III.3.8. □

5.1.3 Proof of the main result

This section is devoted to the proof of Theorem 5.1.1. We are going to show that for any $T > 0$

$$\sup_{[0,T]\times\mathbb{R}^{N+1}} |U^\epsilon(t,x,y) - U^0(t,x,y)| \le Ce^T \epsilon^{\frac{1}{3}},$$

where C does not depend on T. Since $U^\epsilon(t,x,y) = u^\epsilon(t,x) - y$ and $U^0(t,x,y) = u^0(t,x) - y$, this estimate automatically gives (5.1.7).

Let us consider a function $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties

$$\begin{cases} \phi'(s) > 0, & \text{for any } s \in \mathbb{R}, \\ \lim_{s\to+\infty} \phi(s) = 1, & \lim_{s\to-\infty} \phi(s) = 0, \\ |\phi(s) - \chi(s)|,\ |\phi'(s)| \le \frac{K_2}{1+s^2}, & \text{for any } s \in \mathbb{R}, \end{cases} \tag{5.1.14}$$

where we have denoted by $\chi(s)$ the heaviside function defined by

$$\chi(s) = \begin{cases} 1, & \text{for } s \ge 0, \\ 0, & \text{for } s < 0. \end{cases}$$

For $n \in \mathbb{N}$, ϵ, $\delta > 0$, let us define the function

$$\varphi_\epsilon^{n,\delta}(s) := \sum_{i=-n}^{n} \epsilon\phi\left(\frac{s - i\epsilon}{\delta}\right) - \epsilon(n+1).$$

Then we have:

Lemma 5.1.5. *Assume* (5.1.14). *Then for any $s \in \mathbb{R}$, the limit $\lim_{n\to+\infty} \varphi_\epsilon^{n,\delta}(s)$ exists and the function φ_ϵ^δ:*

$$\varphi_\epsilon^\delta(s) := \lim_{n\to+\infty} \varphi_\epsilon^{n,\delta}(s)$$

is of class C^1 with $(\varphi_\epsilon^\delta)'(s) > 0$ for any $s \in \mathbb{R}$. Moreover

$$\lim_{\delta\to 0^+} \varphi_\epsilon^\delta(s) = \begin{cases} (i-1)\epsilon + \phi(0)\epsilon, & \textit{if } s = i\epsilon, \\ i\epsilon, & \textit{if } i\epsilon < s < (i+1)\epsilon. \end{cases} \tag{5.1.15}$$

See the Appendix of this chapter for the proof of the lemma.

Let us define
$$\widetilde{U}^{\epsilon,\delta}(t,x,y) := \varphi_\epsilon^\delta(U^\epsilon(t,x,y)).$$
Since F satisfies the "geometrical" assumption (F4), the function $\widetilde{U}^{\epsilon,\delta}$ is solution of
$$\begin{cases} \widetilde{U}_t^{\epsilon,\delta} + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon+y}{\epsilon}, D_x\widetilde{U}^{\epsilon,\delta}, D_y\widetilde{U}^{\epsilon,\delta}\right) = 0, & (t,x,y)\in(0,T)\times\mathbb{R}^{N+1},\\ \widetilde{U}^{\epsilon,\delta}(0,x,y) = \varphi_\epsilon^\delta(u_0(x)-y), & (x,y)\in\mathbb{R}^{N+1}. \end{cases} \tag{5.1.16}$$
By stability of viscosity solutions, see e.g. [40], the limit $\widetilde{U}^\epsilon(t,x,y)$ of $\widetilde{U}^{\epsilon,\delta}(t,x,y)$ as $\delta\to 0^+$ is a discontinuous viscosity solution of (5.1.16) with initial datum $\varphi_\epsilon(u_0(x)-y)$, where $\varphi_\epsilon(s) = \lim_{\delta\to 0^+}\varphi_\epsilon^\delta(s)$. This means that $(\widetilde{U}^\epsilon)^* = \limsup^*_{\delta\to 0^+}\widetilde{U}^{\epsilon,\delta}$ (resp. $(\widetilde{U}^\epsilon)_* = \liminf_{*\,\delta\to 0^+}\widetilde{U}^{\epsilon,\delta}$) is a viscosity subsolution (resp. supersolution) of (5.1.16), and $(\widetilde{U}^\epsilon)^*(0,x,y) \le (\varphi_\epsilon)^*(u_0(x)-y)$ (resp. $(\widetilde{U}^\epsilon)_*(0,x,y) \ge (\varphi_\epsilon)_*(u_0(x)-y)$). Moreover, by (5.1.15)
$$\widetilde{U}^\epsilon(t,x,y) = \begin{cases} i\epsilon, & \text{if } i\epsilon < U^\epsilon(t,x,y) < (i+1)\epsilon,\\ (i-1)\epsilon + \phi(0)\epsilon, & \text{if } (t,x,y)\in \mathrm{Int}\{U^\epsilon = i\epsilon\}. \end{cases}$$
At the points $(t,x,y)\in\partial\{U^\epsilon = i\epsilon\}$, the value of $\widetilde{U}^\epsilon$ depends on the lower semi-continuous or the upper semi-continuous envelope that we consider in the definition of discontinuous viscosity solution. In particular, since U^ϵ is continuous, $\widetilde{U}^\epsilon$ has the following properties
$$|(\widetilde{U}^\epsilon)^*(t,x,y)-U^\epsilon(t,x,y)|,\ |(\widetilde{U}^\epsilon)_*(t,x,y)-U^\epsilon(t,x,y)| \le \epsilon \quad \text{for any } (t,x,y)\in[0,T]\times\mathbb{R}^{N+1} \tag{5.1.17}$$
and
$$D\widetilde{U}^\epsilon(t,x,y) = 0 \quad \text{if } U^\epsilon(t,x,y)\neq i\epsilon,\ i\in\mathbb{Z}. \tag{5.1.18}$$
Condition (5.1.18) implies that $\widetilde{U}^\epsilon$ is actually a solution of
$$\begin{cases} \widetilde{U}_t^{\epsilon} + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, D_x\widetilde{U}^{\epsilon}, D_y\widetilde{U}^{\epsilon}\right) = 0, & (t,x,y)\in(0,T)\times\mathbb{R}^{N+1},\\ \widetilde{U}^{\epsilon}(0,x,y) = \varphi_\epsilon(u_0(x)-y), & (x,y)\in\mathbb{R}^{N+1}. \end{cases}$$
Indeed, when $i\epsilon < U^\epsilon(t,x,y) < (i+1)\epsilon$, for some $i\in\mathbb{Z}$, the function $\widetilde{U}^\epsilon$ is constant in a neighborhood of (t,x,y). Then the result follows from the fact that $F(t,x,y,0)=0$. On the other hand, when $U^\epsilon(t,x,y) = i\epsilon$, by periodicity, $F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon+y}{\epsilon}, P\right) = F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, P\right)$.

In order to estimate $|U^\epsilon - U^0|$ it is convenient to estimate $|\widetilde{U}^\epsilon - U^0|$; indeed, $\frac{U^\epsilon}{\epsilon}$ does not any longer appear in the equation satisfied by $\widetilde{U}^\epsilon$.

Let us define $V^\epsilon(t,x,y) = e^{-t}\widetilde{U}^\epsilon(t,x,y)$ and $V^0(t,x,y) = e^{-t}U^0(t,x,y)$. The functions V^ϵ and V^0 are respectively solutions of
$$\begin{cases} V_t^\epsilon + V^\epsilon + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, D_xV^\epsilon, D_yV^\epsilon\right) = 0, & (t,x,y)\in(0,T)\times\mathbb{R}^{N+1},\\ V^\epsilon(0,x,y) = \varphi_\epsilon(u_0(x)-y), & (x,y)\in\mathbb{R}^{N+1}, \end{cases} \tag{5.1.19}$$
and

$$\begin{cases} V^0_t + V^0 + \overline{F}(D_x V^0, D_y V^0) = 0, & (t,x,y) \in (0,T) \times \mathbb{R}^{N+1}, \\ V^0(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}. \end{cases} \tag{5.1.20}$$

For alleviating the notations, let us denote a vector of $\mathbb{R}^{N+1}$ by $X = (x, x_{N+1})$, where $x \in \mathbb{R}^N$ and $x_{N+1} \in \mathbb{R}$. We first estimate from above the difference $(V^\epsilon)^* - V^0$: for this, let us introduce the auxiliary function

$$\begin{aligned} \Phi(t,X,s,Y) = (V^\epsilon)^*(t,X) - V^0(s,Y) - \epsilon W^\alpha\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, \frac{X-Y}{\epsilon^\beta}\right) \\ - \frac{|X-Y|^2}{2\epsilon^\beta} - \frac{|t-s|^2}{2\sigma} - \frac{r}{2}|X|^2 - \frac{\eta}{T-t}, \end{aligned} \tag{5.1.21}$$

where $\alpha = \epsilon^\theta$, θ, β, σ, r, $\eta \in (0,1)$ will be fix later on and β and θ satisfy

$$0 < \theta < 1 - \beta. \tag{5.1.22}$$

In view of (5.1.12), (5.1.17), (i) of Proposition 5.1.2 and (5.1.9),

$$\Phi(t,X,s,Y) \leq 2C_T + \epsilon + |x_{N+1} - y_{N+1}| + \frac{\epsilon}{\alpha} C_1 \frac{|X-Y|}{\epsilon^\beta} - \frac{|X-Y|^2}{2\epsilon^\beta} - \frac{r}{2}|X|^2$$

for all (t,X), $(s,Y) \in [0,T] \times \mathbb{R}^{N+1}$. Hence, Φ attains a global maximum at some point $(\bar{t}, \overline{X}, \sigma, \overline{Y}) \in ([0,T] \times \mathbb{R}^{N+1})^2$. Standard arguments show that $\bar{t}$, $\sigma < T$ for σ small enough.

Claim 1: There exists a constant $M_1 > 0$ independent of ϵ such that $\frac{|\bar{t}-\sigma|}{\sigma} \leq M_1(1 + |\overline{y}_{N+1}|)$.
The inequality $\Phi(\bar{t}, \overline{X}, \bar{t}, \overline{Y}) \leq \Phi(\bar{t}, \overline{X}, \sigma, \overline{Y})$ and Proposition (5.1.4) imply

$$\begin{aligned} \frac{|\bar{t}-\sigma|^2}{2\sigma} &\leq V^0(\bar{t}, \overline{Y}) - V^0(\sigma, \overline{Y}) \leq |e^{-\bar{t}} - e^{-\sigma}||U^0(\bar{t}, \overline{Y})| + e^{-\sigma}|U^0(\bar{t}, \overline{Y}) - U^0(\sigma, \overline{Y})| \\ &\leq |\bar{t} - \sigma|(C_T + |\overline{y}_{N+1}|) + L|\bar{t} - \sigma| \end{aligned}$$

from which Claim 1 follows.

Claim 2: There exists a constant $M_2 > 0$ independent of ϵ and T, such that $\frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta} \leq M_2$.
The inequality $\Phi(\bar{t}, \overline{X}, \sigma, \overline{X}) \leq \Phi(\bar{t}, \overline{X}, \sigma, \overline{Y})$ implies

$$\frac{|\overline{X}-\overline{Y}|^2}{\epsilon^\beta} \leq V^0(\sigma, \overline{X}) - V^0(\sigma, \overline{Y}) + \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, 0\right) - \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right).$$

Using (5.1.13),(i) of Lemma 5.1.3 and (5.1.22) we then infer

$$\begin{aligned} \frac{|\overline{X}-\overline{Y}|^2}{\epsilon^\beta} &\leq (L+1)|\overline{X}-\overline{Y}| + \frac{\epsilon}{\alpha} C_1 \frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta} = (L+1)|\overline{X}-\overline{Y}| + \epsilon^{1-\theta-\beta} C_1 |\overline{X}-\overline{Y}| \\ &\leq M_2 |\overline{X}-\overline{Y}|. \end{aligned}$$

This concludes the proof of Claim 2.

Claim 3: There exists a constant $M_3 > 0$ independent of ϵ such that $r|\overline{X}|^2 \le M_3$. The inequality $\Phi(\bar{t}, 0, \sigma, 0) \le \Phi(\bar{t}, \overline{X}, \sigma, \overline{Y})$ implies

$$\frac{r}{2}|\overline{X}|^2 \le (V^\epsilon)^*(\bar{t}, \overline{X}) - V^0(\sigma, \overline{Y}) + V^0(\sigma, 0) - (V^\epsilon)^*(\bar{t}, 0) + \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, 0, 0\right)$$
$$- \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - \overline{Y}}{\epsilon^\beta}\right).$$

Then, using (5.1.12), (5.1.17), Claims 1 and 2, (iii) of Lemma 5.1.3, (i) of Proposition 5.1.2 and (5.1.9), we deduce

$$\begin{aligned}
\frac{r}{2}|\overline{X}|^2 &\le e^{-\bar{t}}[U^\epsilon(\bar{t}, \overline{X}) - U^0(\sigma, \overline{Y})] + |e^{-\bar{t}} - e^{-\sigma}||U^0(\sigma, \overline{Y})| + \epsilon \\
&\quad + V^0(\sigma, 0) - (V^\epsilon)^*(\bar{t}, 0) - \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - \overline{Y}}{\epsilon^\beta}\right) \\
&\le 4C_T + M_2\epsilon^\beta + |\bar{t} - \sigma|(C_T + |\overline{y}_{N+1}|) + 2\epsilon + \frac{\epsilon}{\alpha}C_1\frac{|\overline{X} - \overline{Y}|}{\epsilon^\beta} \\
&\le C + 2\sigma M_1|\overline{y}_{N+1}|^2 \le C + 2\sigma M_1|\overline{X}|^2,
\end{aligned}$$

and Claim 3 follows by choosing $\sigma < \frac{r}{8M_1}$.

Now, suppose first that $\bar{t} = 0$, then

$$\begin{aligned}
&(V^\epsilon)^*(t, X) - V^0(t, X) - \epsilon W^\alpha\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, 0\right) - \frac{\eta}{T - t} - \frac{r}{2}|X|^2 \\
&\le (\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - V^0(\sigma, \overline{Y}) - \epsilon W^\alpha\left(0, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - \overline{Y}}{\epsilon^\beta}\right)
\end{aligned}$$

for any $(t, X) \in [0, T] \times \mathbb{R}^{N+1}$, from which, using (i) of Proposition 5.1.2, (iii) of Lemma 5.1.3, (5.1.9) and Claim 2, we deduce

$$(V^\epsilon)^*(t, X) - V^0(t, X) \le (\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - V^0(\sigma, \overline{Y}) + \frac{\eta}{T - t} + \frac{r}{2}|X|^2 + \epsilon^{1-\theta}C_1M_2.$$

Letting σ, η and r go to 0^+ and using (5.1.17) and Claim 2 we obtain

$$\begin{aligned}
(V^\epsilon)^*(t, X) - V^0(t, X) &\le (\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - (u_0(\overline{y}) - \overline{y}_{N+1}) + C\epsilon^{1-\theta} \\
&\le (\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - (u_0(\overline{x}) - \overline{x}_{N+1}) + (L + 1)|\overline{X} - \overline{Y}| + C\epsilon^{1-\theta} \\
&\le C(\epsilon^\beta + \epsilon^{1-\theta}) + \epsilon,
\end{aligned}$$

which implies

$$U^\epsilon(t, X) - U^0(t, X) \le Ce^t(\epsilon^\beta + \epsilon^{1-\theta}). \tag{5.1.23}$$

The same estimate can be showed if $\sigma = 0$.

Next, let us consider the case $\bar{t}, \sigma > 0$.

Claim 4: There exists a constant $C > 0$ independent of ϵ and T such that

$$\frac{\bar{t} - \sigma}{\sigma} + \frac{\eta}{(T - \bar{t})^2} + (V^\epsilon)^*(\bar{t}, \overline{X}) + \overline{F}\left(\frac{\overline{X} - \overline{Y}}{\epsilon^\beta}\right) \le C(\epsilon^{1-\theta-\beta} + \epsilon^\theta).$$

The function

$$(t,X) \to (V^\epsilon)^*(t,X) - \epsilon W^\alpha\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, \frac{X-\overline{Y}}{\epsilon^\beta}\right) - \frac{|X-\overline{Y}|^2}{2\epsilon^\beta} - \frac{r}{2}|X|^2 - \frac{|t-\sigma|^2}{2\sigma} - \frac{\eta}{T-t} \tag{5.1.24}$$

has a maximum at $(\overline{t}, \overline{X})$. By adding to Φ a smooth function vanishing with its first derivative at $(\overline{t}, \overline{X})$, we may assume the maximum is strict.

Next, for $j > 0$, let us introduce the function

$$\Psi_j(t,s,X,Y,Z) := (V^\epsilon)^*(t,X) - \epsilon W^\alpha\left(s, Y, \frac{Z-\overline{Y}}{\epsilon^\beta}\right) - \frac{|X-\overline{Y}|^2}{2\epsilon^\beta} - \frac{r}{2}|X|^2 - \frac{|t-\sigma|^2}{2\sigma} - \frac{\eta}{T-t} - \frac{j}{2}(|t-\epsilon s|^2 + |X-Z|^2 + |X-\epsilon Y|^2).$$

Let $P_j = (t_j, s_j, X_j, Y_j, Z_j)$ be a maximum point of Ψ_j on the set

$$A := \overline{B}(\overline{t}, 1) \times \overline{B}\left(\frac{\overline{t}}{\epsilon}, 1\right) \times \overline{B}(\overline{X}, 1) \times \overline{B}\left(\frac{\overline{X}}{\epsilon}, 1\right) \times \overline{B}(\overline{X}, 1).$$

Since $(\overline{t}, \overline{X})$ is a strict maximum point of (5.1.24), $t_j \to \overline{t}$, $s_j \to \frac{\overline{t}}{\epsilon}$, X_j, $Z_j \to \overline{X}$ and $Y_j \to \frac{\overline{X}}{\epsilon}$ as $j \to +\infty$. Then, for j large enough, P_j lies in the interior of A. Moreover, standard arguments show that

$$j|t_j - \epsilon s_j|^2, \quad j|X_j - Z_j|^2, \quad j|X_j - \epsilon Y_j|^2 \to 0 \quad \text{as } j \to +\infty. \tag{5.1.25}$$

Remark that this implies in addition that

$$2j|t_j - \epsilon s_j||X_j - \epsilon Y_j| \le j|t_j - \epsilon s_j|^2 + j|X_j - \epsilon Y_j|^2 \to 0 \quad \text{as } j \to +\infty. \tag{5.1.26}$$

Since $(V^\epsilon)^*$ and W^α are respectively viscosity subsolutions of (5.1.19) and supersolution of (5.1.11), we obtain

$$\begin{aligned} &\frac{t_j - \sigma}{\sigma} + \frac{\eta}{(T-t_j)^2} + j(t_j - \epsilon s_j) + (V^\epsilon)^*(t_j, X_j) \\ &+ F\left(\frac{t_j}{\epsilon}, \frac{X_j}{\epsilon}, \frac{X_j - \overline{Y}}{\epsilon^\beta} + rX_j + j(X_j - Z_j) + j(X_j - \epsilon Y_j)\right) \le 0 \end{aligned} \tag{5.1.27}$$

and

$$j(t_j - \epsilon s_j) + \alpha W^\alpha\left(s_j, Y_j, \frac{Z^j - \overline{Y}}{\epsilon^\beta}\right) + F\left(s_j, Y_j, \frac{Z^j - \overline{Y}}{\epsilon^\beta} + j(X_j - \epsilon Y_j)\right) \ge 0. \tag{5.1.28}$$

Subtracting (5.1.27) and (5.1.28) and using the Lipschitz continuity of F, assumption (F2), we get

$$\begin{aligned} &\frac{t_j - \sigma}{\sigma} + \frac{\eta}{(T-t_j)^2} + (V^\epsilon)^*(t_j, X_j) - \alpha W^\alpha\left(s_j, Y_j, \frac{Z^j - \overline{Y}}{\epsilon^\beta}\right) \\ &\le \frac{C_1}{\epsilon}(|t_j - \epsilon s_j| + |X_j - \epsilon Y_j|)\left(\frac{|Z^j - \overline{Y}|}{\epsilon^\beta} + j|X_j - \epsilon Y_j|\right) \\ &+ C_1\left(\frac{|X_j - Z_j|}{\epsilon^\beta} + r|X_j| + j|X_j - Z_j|\right). \end{aligned} \tag{5.1.29}$$

Let us estimate $j|X_j - Z_j|$. From the inequality $\Psi_j(t_j, s_j, X_j, Y_j, X_j) \leq \Psi_j(t_j, s_j, X_j, Y_j, Z_j)$ we deduce that

$$\frac{j}{2}|X_j - Z_j|^2 \leq \epsilon W^\alpha \left(s_j, Y_j, \frac{X^j - \overline{Y}}{\epsilon^\beta} \right) - \epsilon W^\alpha \left(s_j, Y_j, \frac{Z^j - \overline{Y}}{\epsilon^\beta} \right),$$

and using (i) of Lemma 5.1.3 we get

$$\frac{j}{2}|X_j - Z_j|^2 \leq C_1 \frac{\epsilon}{\alpha} \frac{|X_j - Z_j|}{\epsilon^\beta} = C_1 \epsilon^{1-\theta-\beta} |X_j - Z_j|.$$

Then

$$j|X_j - Z_j| \leq 2C_1 \epsilon^{1-\theta-\beta}. \tag{5.1.30}$$

Then, passing to the limsup as $j \to +\infty$ in (5.1.29) and taking into account Claim 2, (5.1.25) and (5.1.26), we obtain

$$\frac{\bar{t} - \sigma}{\sigma} + \frac{\eta}{(T - \bar{t})^2} + (V^\epsilon)^*(\bar{t}, \overline{X}) - \alpha W^\alpha \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - \overline{Y}}{\epsilon} \right) \leq C(\epsilon^{1-\theta-\beta} + r|\overline{X}|). \tag{5.1.31}$$

By Claim 3, $r|\overline{X}| \leq r^{\frac{1}{2}} M_3^{\frac{1}{2}}$, hence choosing $r > 0$ such that $r^{\frac{1}{2}} M_3^{\frac{1}{2}} \leq \epsilon^{1-\theta-\beta}$, we have $r|\overline{X}| \leq \epsilon^{1-\theta-\beta}$.

Finally, Claim 4 easily follows from (5.1.31), Claim 2 and the following inequality

$$-\alpha W^\alpha \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - \overline{Y}}{\epsilon^\beta} \right) \geq \overline{F} \left(\frac{\overline{X} - \overline{Y}}{\epsilon^\beta} \right) - \alpha K_1 \geq \overline{F} \left(\frac{\overline{X} - \overline{Y}}{\epsilon^\beta} \right) - K_1 \epsilon^\theta$$

which comes from (ii) of Lemma 5.1.3 .

Claim 5: There exists a constant $C > 0$ independent of ϵ and T such that

$$\frac{\bar{t} - \sigma}{\sigma} + V^0(\sigma, \overline{Y}) + \overline{F} \left(\frac{\overline{X} - \overline{Y}}{\epsilon^\beta} \right) \geq -C\epsilon^{1-\theta-\beta}.$$

The function

$$(s, Y) \to \phi(s, Y) := V^0(s, Y) + \epsilon W^\alpha \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - Y}{\epsilon^\beta} \right) + \frac{|\overline{X} - Y|^2}{2\epsilon^\beta} + \frac{|\bar{t} - s|^2}{2\sigma}$$

has a minimum at $(\sigma, \overline{Y})$, consequently $(0, 0) \in D^- \phi(\sigma, \overline{Y})$. If we set

$$\widetilde{V}(s, Y) := V^0(s, Y) + \frac{|\overline{X} - Y|^2}{2\epsilon^\beta} + \frac{|\bar{t} - s|^2}{2\sigma}, \quad \widetilde{W}(Y) := \epsilon W^\alpha \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X} - Y}{\epsilon^\beta} \right),$$

by properties of semijets of Lipschitz functions, see e.g. Lemma 2.4 in [36], there exists $Q \in \mathbb{R}^{N+1}$ such that

$$(0, Q) \in D^- \widetilde{V}(\sigma, \overline{Y}) = D^- V^0(\sigma, \overline{Y}) - \left(\frac{\bar{t} - \sigma}{\sigma}, \frac{\overline{X} - \overline{Y}}{\epsilon^\beta} \right) \quad - Q \in D^- \widetilde{W}(\overline{Y}).$$

Since V^0 is a supersolution of (5.1.20), we have

$$\frac{\bar{t}-\sigma}{\sigma} + V^0(\sigma, \overline{Y}) + \overline{F}\left(\frac{\overline{X}-\overline{Y}}{\epsilon^\beta} + Q\right) \geq 0. \tag{5.1.32}$$

By (i) of Lemma 5.1.3,

$$\left|\epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-Y}{\epsilon^\beta}\right) - \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-Z}{\epsilon^\beta}\right)\right| \leq \frac{\epsilon}{\alpha} C_1 \frac{|Y-Z|}{\epsilon^\beta} = C_1 \epsilon^{1-\theta-\beta}|Y-Z|,$$

from which we get the following estimate of Q:

$$|Q| \leq C_1 \epsilon^{1-\theta-\beta}. \tag{5.1.33}$$

Then, Claim 5 follows from (5.1.32) using estimate (5.1.33) and the Lipschitz continuity of $\overline{F}$ assured by (iv) of Lemma 5.1.3.

Claims 4 and 5 imply

$$(V^\epsilon)^*(\bar{t}, \overline{X}) - V^0(\sigma, \overline{Y}) \leq C(\epsilon^{1-\theta-\beta} + \epsilon^\theta),$$

for some constant C independent of ϵ and T. Since $(\bar{t}, \overline{X}, \sigma, \overline{Y})$ is a maximum point of Φ, we have

$$(V^\epsilon)^*(t, X) - V^0(t, X) \leq \Phi(\bar{t}, \overline{X}, \sigma, \overline{Y}) + \epsilon W^\alpha\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, 0\right) + \frac{r}{2}|X|^2 + \frac{\eta}{T-t},$$

for all $(t, X) \in [0, T] \times \mathbb{R}^{N+1}$. Then, by (iii) of Lemma 5.1.3

$$\begin{aligned}(V^\epsilon)^*(t, X) - V^0(t, X) &\leq (V^\epsilon)^*(\bar{t}, \overline{X}) - V^0(\sigma, \overline{Y}) - \epsilon W^\alpha\left(\frac{\bar{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right) + \frac{r}{2}|X|^2 + \frac{\eta}{T-t}\\
&\leq C(\epsilon^{1-\theta-\beta} + \epsilon^\theta) + \frac{\epsilon}{\alpha} C_1 \frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta} + \frac{r}{2}|X|^2 + \frac{\eta}{T-t}\\
&\leq C(\epsilon^{1-\theta-\beta} + \epsilon^\theta) + \frac{r}{2}|X|^2 + \frac{\eta}{T-t},\end{aligned}$$

for some positive constant C. Hence, sending r, $\eta, \to 0^+$ and taking into account (5.1.17), we get

$$U^\epsilon(t, X) - U^0(t, X) \leq Ce^t(\epsilon^{1-\theta-\beta} + \epsilon^\theta).$$

Then, from the previous estimate and (5.1.23), we can conclude that for all β, $\theta \in (0, 1)$ satisfying (5.1.22) we have

$$U^\epsilon(t, X) - U^0(t, X) \leq Ce^t(\epsilon^{1-\theta-\beta} + \epsilon^\theta + \epsilon^\beta),$$

for all $(t, X) \in [0, T] \times \mathbb{R}^{N+1}$. The optimal choice of the parameters is $\theta = \beta = \frac{1}{3}$, which gives

$$\sup_{[0,T]\times\mathbb{R}^{N+1}} (U^\epsilon(t, X) - U^0(t, X)) \leq C\epsilon^{\frac{1}{3}}.$$

The opposite inequality follows by similar arguments, replacing $(V^\epsilon)^*$ with V^0 and V^0 with $(V^\epsilon)_*$ in (5.1.21), and the proof of Theorem 5.1.1 in the general case is complete.

Now, let us consider the case when u_0 is affine. Let us suppose that $u_0(x) = p \cdot x + c_0$ for some $p \in \mathbb{R}^N$ and $c_0 \in \mathbb{R}$. In this case, the solution of (5.0.6) is $u^0(t,x) = p \cdot x + c_0 - \overline{H}(p)t$. Let $\overline{V}$ be a bounded viscosity supersolution of (5.0.5) with $p_x = p$ and $p_y = -1$. Let us define

$$V^\epsilon(t,X) = U^0(t,X) + \epsilon \overline{V}\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}\right).$$

Since $u_0(x) - y \geq \varphi_\epsilon(u_0(x) - y) - \epsilon$ then $V^\epsilon(0,X) \geq \varphi_\epsilon(u_0(x) - y) - (M+1)\epsilon$ where $M = \|\overline{V}\|_\infty$. Hence, it is easy to check that V^ϵ is a supersolution of

$$\begin{cases} V^\epsilon_t + F\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, D_X V^\epsilon\right) = 0, & (t,X) \in (0,T) \times \mathbb{R}^{N+1}, \\ V^\epsilon(0,X) = \varphi_\epsilon(u_0(x) - y) - (M+1)\epsilon, & (x,y) \in \mathbb{R}^{N+1}. \end{cases}$$

By comparison we get $V^\epsilon(t,X) \geq (\widetilde{U}^\epsilon)^*(t,X) - (M+1)\epsilon$ and this implies that $U^0(t,X) - U^\epsilon(t,X) \geq -C\epsilon$. A similar argument shows that $U^0(t,X) - U^\epsilon(t,X) \leq C\epsilon$ and this concludes the proof of the theorem. □

5.2 Approximation of the effective Hamiltonian by Eulerian schemes

In this section we give an approximation of the effective Hamiltonian $\overline{F}(P)$. To this end, we introduce an approximation scheme for the equation (5.1.11) and for simplicity we only discuss the case $N = 2$. Given N_X and N_t positive integers, we introduce $\Delta t = 1/N_t$, $h = 1/N_X$ and

$$\mathbb{R}^2_h := \{X_{i,j} = (x_i, y_j) \,|\, x_i = ih,\ y_j = jh,\ i,j \in \mathbb{Z}\},$$

$$\mathbb{R}_{\Delta t} := \{t_n = n\Delta t \,|\, n \in \mathbb{Z}\}.$$

An anisotropic mesh with steps h_1 and h_2 is possible too; we take $h_1 = h_2$ only for simplicity. We denote by $W^{n,P,\alpha}_{i,j}$ our numerical approximation of $W^{P,\alpha}$ at $(t_n, x_i, y_j) \in \mathbb{R}_{\Delta t} \times \mathbb{R}^2_h$. For (5.1.11) we consider the implicit Eulerian scheme of the form

$$\frac{W^{n+1,P,\alpha}_{i,j} - W^{n,P,\alpha}_{i,j}}{\Delta t} + \alpha W^{n+1,P,\alpha}_{i,j} + S(t_n, x_i, y_j, h, [W^{n+1,P,\alpha}]_{i,j}) = 0, \qquad (5.2.34)$$

where

$$\begin{aligned} &S(t_n, x_i, y_j, h, [W]_{i,j}) \\ &\quad = g(t_n, x_i, y_j, (\Delta^+_1 W)_{i,j} + p_x, (\Delta^+_1 W)_{i-1,j} + p_x, (\Delta^+_2 W)_{i,j} + p_y, (\Delta^+_2 W)_{i,j-1} + p_y) \end{aligned} \qquad (5.2.35)$$

and

$$(\Delta^+_1 W)_{i,j} = \frac{W_{i+1,j} - W_{i,j}}{h}, \quad (\Delta^+_2 W)_{i,j} = \frac{W_{i,j+1} - W_{i,j}}{h}.$$

We make the following assumptions on g:

(g1) Monotonicity: g is nonincreasing with respect to its fourth and sixth arguments, and nondecreasing with respect to its fifth and seventh arguments;

(g2) Consistency: for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $(q_x, q_y) \in \mathbb{R}^2$

$$g(t, x, y, q_x, q_x, q_y, q_y) = F(t, x, y, q_x, q_y).$$

(g3) Periodicity: for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $Q \in \mathbb{R}^4$

$$g(t + 1, x + 1, y + 1, Q) = g(t, x, y, Q);$$

(g4) Regularity: g is locally Lipschitz continuous and there exists $\widetilde{C}_1 > 0$ such that for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $Q \in \mathbb{R}^4$

$$|D_Q g(t, x, y, Q)| \leq \widetilde{C}_1;$$

(g5) Coercivity: there exist $\widetilde{C}_2$, $\widetilde{C}_3 > 0$ such that for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$, $(q_1, q_2) \in \mathbb{R}^2$

$$g(t, x, y, q_1, q_2, 0, 0) \geq \widetilde{C}_2(|q_1^-|^2 + |q_2^+|^2)^{\frac{1}{2}} - \widetilde{C}_3;$$

(g6) For any $t \in \mathbb{R}$, $(x, y_1), (x, y_2) \in \mathbb{R}^2$, $q_1, q_2 \in \mathbb{R}$

$$g(t, x, y_1, q_1, q_2, 0, 0) = g(t, x, y_2, q_1, q_2, 0, 0).$$

The points (g1)-(g4) are standard assumptions in the study of numerical schemes for Hamilton-Jacobi equations. The coercivity hypothesis (g5) can be substituted by the weaker condition

$$\lim_{q_1^+ + q_2^- \to +\infty} g(x, y, q_1, q_2, q_3, q_4) = +\infty$$

if g (and hence F) does not depend on time. If g is homogeneous of degree 1 w.r.t. Q, then the two coercivity conditions are equivalent.

As an example, we suppose that the Hamiltonian F is of the form $F(t, x, y, p_x, p_y) = a(t, x)|p_x| + b(t, x, y)|p_y|$, with a and b Lipschitz continuous functions and $a(t, x) \geq \widetilde{C}_2 > 0$; we consider a generalization of the Godunov scheme proposed in [94]:

$$\begin{aligned} &g(t, x, y, q_1, q_2, q_3, q_4) \\ &= a(t, x)[(q_1^-)^2 + (q_2^+)^2]^{\frac{1}{2}} + b^+(t, x, y)[(q_3^-)^2 + (q_4^+)^2]^{\frac{1}{2}} - b^-(t, x, y)[(q_3^+)^2 + (q_4^-)^2]^{\frac{1}{2}}. \end{aligned}$$

where $q^+ = \max(q, 0)$ and $q^- = (-q)^+$. Then hypothesis (g1)-(g6) are satisfied.

The following theorem is the discrete version of the analogous result in [18] for the exact solution $W^{P,\alpha}$ of (5.1.11).

Theorem 5.2.1. *Assume (g1)-(g6). Then we have*

(i) For any $P = (p_x, p_y) \in \mathbb{R}^2$, α, h, $\Delta t > 0$ *there exists a unique* $(W_{i,j}^{n,P,\alpha})$ *periodic solution of* (5.2.34)*;*

(ii) There exists a constant $\widetilde{K}_1$ depending on $\|F(\cdot,\cdot,\cdot,P)\|_\infty$, $\widetilde{C}_1$ in (g4), $\widetilde{C}_2$, $\widetilde{C}_3$ in (g5), p_x and p_y, but independent of α, h and Δt such that

$$\max_{i,j,n} W_{i,j}^{n,P,\alpha} - \min_{i,j,n} W_{i,j}^{n,P,\alpha} \le \widetilde{K}_1;$$

(iii) There exists a constant $\overline{F}_h^{\Delta t}(P)$ such that

$$\lim_{\alpha\to 0^+} \alpha W_{i,j}^{n,P,\alpha} = -\overline{F}_h^{\Delta t}(P) \quad \forall i,j,n; \tag{5.2.36}$$

(iv) $\overline{F}_h^{\Delta t}(P)$ is the unique number $\overline{\lambda}_h^{\Delta t}\in\mathbb{R}$ such that the equation

$$\frac{W_{i,j}^{n+1,P} - W_{i,j}^{n,P}}{\Delta t} + S(t_n, x_i, y_j, h, [W^{n+1,P}]_{i,j}) = \overline{\lambda}_h^{\Delta t} \tag{5.2.37}$$

admits a bounded solution.

Proof. A proof of the existence of a solution of (5.2.34) in the uniform grid on the torus with step h is given in [39].

Let us prove (ii). First, remark that by comparison with constants we have

$$\max_{i,j,n} |\alpha W_{i,j}^{n,P,\alpha}| \le C_0, \tag{5.2.38}$$

where $C_0 := \|F(\cdot,\cdot,\cdot,P)\|_\infty$. Next, let us define

$$\overline{W}_i^n := \max_j W_{i,j}^{n,P,\alpha}.$$

We claim that $\overline{W}_i^n$ satisfies

$$\frac{\overline{W}_i^{n+1} - \overline{W}_i^n}{\Delta t} + \alpha \overline{W}_i^{n+1} + \overline{S}(t_n, x_i, h, [\overline{W}^{n+1}]_i) \le 0,$$

where

$$\overline{S}(t_n, x_i, h, [W]_i) := \min_j g(t_n, x_i, y_j, (\Delta_1^+ W)_i + p_x, (\Delta_1^+ W)_{i-1} + p_x, p_y, p_y).$$

Indeed, for any i and n, denote by $\bar{j}_{(i,n)}$ the index j such that $\overline{W}_i^n = \max_j W_{i,j}^{n,P,\alpha} = W_{i,\bar{j}_{(i,n)}}^{n,P,\alpha}$, then

$$\frac{W_{i,\bar{j}_{(i,n+1)}}^{n+1,P,\alpha} - W_{i,\bar{j}_{(i,n+1)}}^{n,P,\alpha}}{\Delta t} \ge \frac{W_{i,\bar{j}_{(i,n+1)}}^{n+1,P,\alpha} - W_{i,\bar{j}_{(i,n)}}^{n,P,\alpha}}{\Delta t} = \frac{\overline{W}_i^{n+1} - \overline{W}_i^n}{\Delta t},$$

$$\begin{aligned}(\Delta_1^+ W^{n+1,P,\alpha})_{i,\bar{j}_{(i,n+1)}} &= \frac{W_{i+1,\bar{j}_{(i,n+1)}}^{n+1,P,\alpha} - W_{i,\bar{j}_{(i,n+1)}}^{n+1,P,\alpha}}{h} \le \frac{W_{i+1,\bar{j}_{i+1,n+1}}^{n+1,P,\alpha} - W_{i,\bar{j}_{(i,n+1)}}^{n+1,P,\alpha}}{h} \\ &= (\Delta_1^+ \overline{W}^{n+1})_i,\end{aligned}$$

$$
\begin{aligned}
(\Delta_1^+ W^{n+1,P,\alpha})_{i-1,\bar{j}_{(i,n+1)}} &= \frac{W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i-1,\bar{j}_{(i,n+1)}}}{h} \geq \frac{W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i-1,\bar{j}_{(i-1,n+1)}}}{h} \\
&= (\Delta_1^+ \overline{W}^{n+1})_{i-1},
\end{aligned}
$$

and

$$
(\Delta_2^+ W^{n+1,P,\alpha})_{i,\bar{j}_{(i,n+1)}} = \frac{W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}+1} - W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}}}{h} \leq 0,
$$

$$
(\Delta_2^+ W^{n+1,P,\alpha})_{i,\bar{j}_{(i,n+1)}-1} = \frac{W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}-1}}{h} \geq 0.
$$

Since $(W^{n,P,\alpha}_{i,j})$ satisfies (5.2.34), using the monotonicity assumption (g1), we get

$$
\begin{aligned}
&\frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \alpha \overline{W}^{n+1}_i + \overline{S}(t_n, x_i, h, [\overline{W}^{n+1}]_i) \\
&\leq \frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \alpha W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}} \\
&+ g(t_n, x_i, y_{\bar{j}_{(i,n+1)}}, (\Delta_1^+ \overline{W}^{n+1})_i + p_x, (\Delta_1^+ \overline{W}^{n+1})_{i-1} + p_x, p_y, p_y) \\
&\leq \frac{W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}} - W^{n,P,\alpha}_{i,\bar{j}_{(i,n+1)}}}{\Delta t} + \alpha W^{n+1,P,\alpha}_{i,\bar{j}_{(i,n+1)}} \\
&+ g(t_n, x_i, y_{\bar{j}_{(i,n+1)}}, (\Delta_1^+ W^{n+1,P,\alpha})_{i,\bar{j}_{(i,n+1)}} + p_x, (\Delta_1^+ W^{n+1,P,\alpha})_{i-1,\bar{j}_{(i,n+1)}} + p_x, \\
&(\Delta_2^+ W^{n+1,P,\alpha})_{i,\bar{j}_{(i,n+1)}} + p_y, (\Delta_2^+ W^{n+1,P,\alpha})_{i,\bar{j}_{(i,n+1)}-1} + p_y) \\
&\leq 0,
\end{aligned}
$$

as desired. Then, by (g4), (g5) and (5.2.38), we see that $\overline{W}^n_i$ satisfies

$$
\frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \widetilde{C}_2 \left(|[(\Delta_1^+ \overline{W}^{n+1})_i + p_x]^-|^2 + |[(\Delta_1^+ \overline{W}^{n+1})_{i-1} + p_x]^+|^2 \right)^{\frac{1}{2}} - \leq 0,
$$

where $K_1 = C_0 + \widetilde{C}_3 + 2\widetilde{C}_1 |p_y|$. In particular we infer that

$$
\overline{W}^{n+1}_i - \overline{W}^n_i \leq K_1 \Delta t,
$$

which implies that if $n \geq m$ then

$$
\overline{W}^n_i - \overline{W}^m_i \leq K_1 (n-m) \Delta t = K_1 (t_n - t_m). \tag{5.2.39}
$$

Next, let us consider

$$
\overline{\overline{W}}_i = \max_n \overline{W}^n_i.
$$

Similar arguments as before show that $\overline{\overline{W}}_i$ satisfies

$$
\widetilde{C}_2 \left(|[(\Delta_1^+ \overline{\overline{W}})_i + p_x]^-|^2 + |[(\Delta_1^+ \overline{\overline{W}})_{i-1} + p_x]^+|^2 \right)^{\frac{1}{2}} \leq K_1,
$$

which implies the existence of a constant $K_2 > 0$ depending on $C_0, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, p_x$ and p_y such that

$$\max_i |(\Delta_1^+ \overline{\overline{W}})_i| \leq K_2. \tag{5.2.40}$$

Now, let (i_1, n_1) and (i_2, n_2) be such that $\max_{i,n} \overline{W}_i^n = \overline{W}_{i_1}^{n_1}$ and $\min_{i,n} \overline{W}_i^n = \overline{W}_{i_2}^{n_2}$, and let n_{i_2} be such that $\overline{\overline{W}}_{i_2} = \max_n \overline{W}_{i_2}^n = \overline{W}_{i_2}^{n_{i_2}}$. By periodicity, we may take $|x_{i_1} - x_{i_2}| \leq 1$ and $0 \leq t_{n_{i_2}} - t_{n_2} \leq 1$. Then using (5.2.40) and (5.2.39), we get

$$\begin{aligned}
\overline{W}_{i_1}^{n_1} &= \overline{\overline{W}}_{i_1} \\
&\leq \overline{\overline{W}}_{i_2} + K_2 |x_{i_1} - x_{i_2}| \\
&\leq \overline{W}_{i_2}^{n_{i_2}} + K_2 \\
&\leq \overline{W}_{i_2}^{n_2} + K_1(t_{n_{i_2}} - t_{n_2}) + K_2 \\
&\leq \overline{W}_{i_2}^{n_2} + K_0.
\end{aligned}$$

Then we have proved that

$$\max_{i,n} \overline{W}_i^n - \min_{i,n} \overline{W}_i^n \leq K_0, \tag{5.2.41}$$

where K_0 depends only on $C_0, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, p_x$ and p_y.

Next, we consider the behavior of $W_{i,j}^{n,P,\alpha}$ in j. We claim that

$$\begin{aligned}
W_{i,j_1}^{n,P,\alpha} + p_y y_{j_1} \leq W_{i,j_2}^{n,P,\alpha} + p_y y_{j_2} \quad &\text{if } j_1 \geq j_2 \text{ and } p_y < 0, \\
W_{i,j_1}^{n,P,\alpha} = W_{i,j_2}^{n,P,\alpha} \quad &\text{for any } j_1, j_2 \text{ if } p_y = 0, \\
W_{i,j_1}^{n,P,\alpha} + p_y y_{j_1} \geq W_{i,j_2}^{n,P,\alpha} + p_y y_{j_2} \quad &\text{if } j_1 \geq j_2 \text{ and } p_y > 0.
\end{aligned} \tag{5.2.42}$$

Let us consider the case $p_y < 0$. Suppose by contradiction that

$$M := \max_{i,n,j_1 \geq j_2} (W_{i,j_1}^{n,P,\alpha} - W_{i,j_2}^{n,P,\alpha} + p_y(y_{j_1} - y_{j_2})) = W_{\bar{i},\bar{j}_1}^{\bar{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\bar{n},P,\alpha} + p_y(y_{\bar{j}_1} - y_{\bar{j}_2}) > 0.$$

Then $\bar{j}_1 \geq \bar{j}_2 + 1$. We have the following estimate

$$\begin{aligned}
(\Delta_1^+ W^{\bar{n},P,\alpha})_{\bar{i},\bar{j}_1} - (\Delta_1^+ W^{\bar{n},P,\alpha})_{\bar{i},\bar{j}_2} &= \frac{W_{\bar{i}+1,\bar{j}_1}^{\bar{n},P,\alpha} - W_{\bar{i},\bar{j}_1}^{\bar{n},P,\alpha}}{h} - \frac{W_{\bar{i}+1,\bar{j}_2}^{\bar{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\bar{n},P,\alpha}}{h} \\
&= \frac{W_{\bar{i}+1,\bar{j}_1}^{\bar{n},P,\alpha} - W_{\bar{i}+1,\bar{j}_2}^{\bar{n},P,\alpha}}{h} - \frac{W_{\bar{i},\bar{j}_1}^{\bar{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\bar{n},P,\alpha}}{h} \leq 0.
\end{aligned}$$

Similarly

$$(\Delta_1^+ W^{\bar{n},P,\alpha})_{\bar{i}-1,\bar{j}_1} \geq (\Delta_1^+ W^{\bar{n},P,\alpha})_{\bar{i}-1,\bar{j}_2},$$

and

$$\frac{W_{\bar{i},\bar{j}_1}^{\bar{n},P,\alpha} - W_{\bar{i},\bar{j}_1}^{\bar{n}-1,P,\alpha}}{\Delta t} \geq \frac{W_{\bar{i},\bar{j}_2}^{\bar{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\bar{n}-1,P,\alpha}}{\Delta t}.$$

Moreover, we have

$$
\begin{aligned}
(\Delta_2^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_1} + p_y &= \frac{W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_1+1} - W^{\overline{n}P,\alpha}_{\bar{i},\bar{j}_1}}{h} + p_y \\
&= \frac{W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_1+1} - W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_2}}{h} + p_y \frac{y_{\bar{j}_1+1} - y_{\bar{j}_2}}{h} - \frac{W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_1} - W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_2}}{h} - p_y \frac{y_{\bar{j}_1} - y_{\bar{j}_2}}{h} \le 0,
\end{aligned}
$$

similarly

$$
(\Delta_2^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_1-1} + p_y \ge 0, \quad (\Delta_2^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_2} + p_y \ge 0, \quad (\Delta_2^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_2-1} + p_y \le 0.
$$

Then, since $W^{\overline{n},P,\alpha}_{i,j}$ satisfies (5.2.34), using assumptions (g1) and (g6), we get

$$
\begin{aligned}
\alpha(W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_1} - W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_2}) &\le -g(t_{\overline{n}}, x_{\bar{i}}, y_{\bar{j}_1}, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_1} + p_x, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\bar{i}-1,\bar{j}_1} + p_x, 0, 0) \\
&\quad + g(t_{\overline{n}}, x_{\bar{i}}, y_{\bar{j}_2}, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_1} + p_x, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\bar{i}-1,\bar{j}_1} + p_x, 0, 0) = 0.
\end{aligned}
$$

This implies that

$$
0 < \alpha M = \alpha(W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_1} - W^{\overline{n},P,\alpha}_{\bar{i},\bar{j}_2} + p_y(y_{\bar{j}_1} - y_{\bar{j}_2})) \le \alpha p_y (y_{\bar{j}_1} - y_{\bar{j}_2}) < 0,
$$

which is a contradiction and this concludes the proof of (5.2.42) for $p_y < 0$. The case $p_y \ge 0$ can be treated in an analogous way.

Now, to prove (ii), we use the properties (5.2.41) and (5.2.42) of $W^{n,P,\alpha}_{i,j}$ and again we only consider the case $p_y < 0$. Let (i_1, j_1, n_1) and (i_2, j_2, n_2) be such that $W^{n_1,P,\alpha}_{i_1,j_1} = \max_{i,j,n} W^{n,P,\alpha}_{i,j}$ and $W^{n_2,P,\alpha}_{i_2,j_2} = \min_{i,j,n} W^{n,P,\alpha}_{i,j}$. Let $\bar{j}$ be such that $\overline{W}^{n_2}_{i_2} = W^{n_2,P,\alpha}_{i_2,\bar{j}}$. By periodicity, we can take $0 \le y_{\bar{j}} - y_{j_2} \le 1$ and $|x_{i_1} - x_{i_2}| \le 1$. Then

$$
\begin{aligned}
W^{n_1,P,\alpha}_{i_1,j_1} &= \overline{W}^{n_1}_{i_1} \\
&\le \overline{W}^{n_2}_{i_2} + K_0 \\
&= W^{n_2,P,\alpha}_{i_2,\bar{j}} + K_0 \\
&\le W^{n_2,P,\alpha}_{i_2,j_2} + p_y(y_{j_2} - y_{\bar{j}}) + K_0 \\
&\le W^{n_2,P,\alpha}_{i_2,j_2} - p_y + K_0,
\end{aligned}
$$

and this concludes the proof of (ii).

The property (iii) easily follows from (ii) and (5.2.38). Indeed, from (5.2.38), up to subsequence, $\alpha \min_{i,j,n} W^{n,P,\alpha}_{i,j}$ converges to a constant $-\overline{F}^{\Delta t}_h(P)$ as $\alpha \to 0^+$. Then from (ii), for any i, j, n, we get

$$
\begin{aligned}
|\alpha W^{n,P,\alpha}_{i,j} + \overline{F}^{\Delta t}_h(P)| &\le |\alpha \min_{i,j,n} W^{n,P,\alpha}_{i,j} + \overline{F}^{\Delta t}_h(P)| + \alpha |W^{n,P,\alpha}_{i,j} - \min_{i,j,n} W^{n,P,\alpha}_{i,j}| \\
&\le |\alpha \min_{i,j,n} W^{n,P,\alpha}_{i,j} + \overline{F}^{\Delta t}_h| + \alpha \widetilde{K}_1 \to 0 \quad \text{as } \alpha \to 0^+,
\end{aligned}
$$

and (iii) is proved.

Let us turn to (iv). Let us define $Z^{n,P,\alpha}_{i,j} = W^{n,P,\alpha}_{i,j} - \min_{i,j,n} W^{n,P,\alpha}_{i,j}$. By (ii), up to subsequence, $(Z^{n,P,\alpha}_{i,j})$ converges to a grid function $(Z^{n,P}_{i,j})$ as $\alpha \to 0^+$. The grid function $(Z^{n,P,\alpha}_{i,j})$ satisfies

$$\frac{Z^{n+1,P,\alpha}_{i,j} - Z^{n,P,\alpha}_{i,j}}{\Delta t} + \alpha Z^{n+1,P,\alpha}_{i,j} + S(t_n, x_i, y_j, h, [Z^{n+1,P,\alpha}]_{i,j}) = -\alpha \min_{i,j,n} W^{n,P,\alpha}_{i,j}.$$

Letting $\alpha \to 0^+$, since by (ii) $(Z^{n,P,\alpha}_{i,j})$ is bounded and $\alpha \min_{i,j,n} W^{n,P,\alpha}_{i,j} \to -\overline{F}^{\Delta t}_h$, we see that $(Z^{n,P}_{i,j})$ is a solution of (5.2.37) with $\overline{\lambda}^{\Delta t}_h = \overline{F}^{\Delta t}_h$.

To prove the uniqueness of a solution $(\overline{\lambda}^{\Delta t}_h, (W^{n,P}_{i,j}))$ of (5.2.37), we show that if there exists a subsolution $(U^{n,P}_{i,j})$ of (5.2.37) with $\overline{\lambda}^{\Delta t}_h = \lambda_1$ and a supersolution $(V^{n,P}_{i,j})$ of (5.2.37) with $\overline{\lambda}^{\Delta t}_h = \lambda_2$, then $\lambda_2 \le \lambda_1$.

Let $M = \max_{i,j,n}(U^{n,P}_{i,j} - V^{n,P}_{i,j}) = U^{n_0,P}_{i_0,j_0} - V^{n_0,P}_{i_0,j_0}$. Then

$$\frac{U^{n_0,P}_{i_0,j_0} - U^{n_0-1,P}_{i_0,j_0}}{\Delta t} \ge \frac{V^{n_0,P}_{i_0,j_0} - V^{n_0-1,P}_{i_0,j_0}}{\Delta t},$$

$$\begin{aligned}
&(\Delta^+_1 U^{n_0,P})_{i_0,j_0} \le (\Delta^+_1 V^{n_0,P})_{i_0,j_0}, \quad (\Delta^+_1 U^{n_0,P})_{i_0-1,j_0} \ge (\Delta^+_1 V^{n_0,P})_{i_0-1,j_0},\\
&(\Delta^+_2 U^{n_0,P})_{i_0,j_0} \le (\Delta^+_2 V^{n_0,P})_{i_0,j_0}, \quad (\Delta^+_2 U^{n_0,P})_{i_0,j_0-1} \ge (\Delta^+_2 V^{n_0,P})_{i_0,j_0-1}.
\end{aligned}$$

From the monotonicity of g,

$$\begin{aligned}
\lambda_1 \ge{}& \frac{U^{n_0,P}_{i_0,j_0} - U^{n_0-1,P}_{i_0,j_0}}{\Delta t} + g\Big(t_{n_0}, x_{i_0}, y_{j_0}, (\Delta^+_1 U^{n_0,P})_{i_0,j_0} + p_x, (\Delta^+_1 U^{n_0,P})_{i_0-1,j_0} + p_x,\\
&(\Delta^+_2 U^{n_0,P})_{i_0,j_0} + p_y, (\Delta^+_2 U^{n_0,P})_{i_0,j_0-1} + p_y\Big)\\
\ge{}& \frac{V^{n_0,P}_{i_0,j_0} - V^{n_0-1,P}_{i_0,j_0}}{\Delta t} + g\Big(t_{n_0}, x_{i_0}, y_{j_0}, (\Delta^+_1 V^{n_0,P})_{i_0,j_0} + p_x, (\Delta^+_1 V^{n_0,P})_{i_0-1,j_0} + p_x,\\
&(\Delta^+_2 V^{n_0,P})_{i_0,j_0} + p_y, (\Delta^+_2 V^{n_0,P})_{i_0,j_0-1} + p_y\Big)\\
\ge{}& \lambda_2.
\end{aligned}$$

This concludes the proof of (iv). □

We need a more precise estimate on the rate of convergence of $\alpha W^{n,\alpha,P}_{i,j}$ to $\overline{F}^{\Delta t}_h(P)$:

Proposition 5.2.2. *Assume (g1)-(g6). Then for any i, j, n*

$$|\alpha W^{n,\alpha,P}_{i,j} + \overline{F}^{\Delta t}_h(P)| \le \widetilde{K}_1 \alpha,$$

where $\widetilde{K}_1 = \widetilde{K}_1(P)$ is the constant in (ii) of Theorem 5.2.1.

Proof. As in the proof of (ii) of Lemma 5.1.3, the result follows from the comparison principle for (5.2.34) and (ii) of Theorem 5.2.1. □

Now, we are ready to show that the function $\overline{F}^{\Delta t}_h$ is actually an approximation of the effective Hamiltonian $\overline{F}$.

Proposition 5.2.3. *Assume (g1)-(g6). Let* $\overline{F}_h^{\Delta t}$ *be defined by* (5.2.36) *and let* $\overline{F}$ *be the effective Hamiltonian. Then, for any* $P \in \mathbb{R}^2$

$$\lim_{(\Delta t,h)\to(0,0)} \overline{F}_h^{\Delta t}(P) = \overline{F}(P)$$

uniformly on compact sets of $\mathbb{R}^2$.

Proof. To show the result we estimate $W^{P,\alpha}(t_n,x_i,y_j) - W_{i,j}^{n,P,\alpha}$. To this end, following the same proof as in [42] and [1], we assume that

$$\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n,x_i,y_j) - \alpha W_{i,j}^{n,P,\alpha}| = \sup_{i,j,n}(\alpha W^{P,\alpha}(t_n,x_i,y_j) - \alpha W_{i,j}^{n,P,\alpha}) = m \geq 0.$$

The case when $\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n,x_i,y_j) - \alpha W_{i,j}^{n,P,\alpha}| = \sup_{i,j,n}(\alpha W_{i,j}^{n,P,\alpha} - \alpha W^{P,\alpha}(t_n,x_i,y_j))$ is handled in a similar manner.

For simplicity of notations we omit the index P. Let us denote $W_{h,\Delta t}^{\alpha}(t_n, X_{i,j}) := W_{i,j}^{n,\alpha}$, $(t_n, X_{i,j}) \in \mathbb{R}_{\Delta t} \times \mathbb{R}_h^2$. For $(X,Y) \in \mathbb{R}^2 \times \mathbb{R}_h^2$ and $(t,s) \in \mathbb{R} \times \mathbb{R}_{\Delta t}$, consider the function

$$\Psi(t,X,s,Y) = \alpha W^{\alpha}(t,X) - \alpha W_{h,\Delta t}^{\alpha}(s,Y) + \left(5C_0 + \frac{m}{2}\right)\beta_\epsilon(t-s, X-Y),$$

where, as before, $C_0 = \|F(\cdot,\cdot,\cdot,P)\|_\infty$ and $\beta_\epsilon = \beta\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}\right)$ with β a non-negative smooth function such that

$$\begin{cases} \beta(t,X) = 1 - |X|^2 - |t|^2, & \text{if } |X|^2 + |t|^2 \leq \frac{1}{2}, \\ \beta \leq \frac{1}{2}, & \text{if } \frac{1}{2} \leq |X|^2 + |t|^2 \leq 1, \\ \beta = 0, & \text{if } |X|^2 + |t|^2 > 1. \end{cases}$$

We have the following lemma:

Lemma 5.2.4. *The function* Ψ *attains its maximum at a point* (t_0, X_0, s_0, Y_0) *such that*

(i) $\Psi(t_0,X_0,s_0,Y_0) \geq 5C_0 + \frac{3}{2}m$;

(ii) $\beta_\epsilon(t_0 - s_0, X_0 - Y_0) \geq \frac{3}{5}$.

For the proof, see Lemma 4.1 in [42].

Lemma 5.2.4 (ii) implies that

$$\beta_\epsilon(t_0 - s_0, X_0 - Y_0) = 1 - \left|\frac{X_0 - Y_0}{\epsilon}\right|^2 - \left|\frac{t_0 - s_0}{\epsilon}\right|^2.$$

Then, from the inequality $\Psi(s_0,Y_0,s_0,Y_0) \leq \Psi(t_0,X_0,s_0,Y_0)$ we deduce that

$$\left(5C_0 + \frac{m}{2}\right)\left(\left|\frac{X_0 - Y_0}{\epsilon}\right|^2 + \left|\frac{t_0 - s_0}{\epsilon}\right|^2\right) \leq \alpha W^{\alpha}(t_0,X_0) - \alpha W^{\alpha}(s_0,Y_0) \leq 2C_0. \tag{5.2.43}$$

This implies that $|t_0 - s_0| \to 0$ and $|X_0 - Y_0| \to 0$ as $\epsilon \to 0$. Moreover, since W^α and $W^\alpha_{h,\Delta t}$ are periodic, we can assume that (t_0, X_0, s_0, Y_0) lies in a compact set of $(\mathbb{R}\times\mathbb{R}^2)^2$. Hence, from (5.2.43) and the continuity of W^α we get that

$$\left|\frac{X_0 - Y_0}{\epsilon}\right|^2 + \left|\frac{t_0 - s_0}{\epsilon}\right|^2 \to 0 \quad \text{as } \epsilon \to 0. \tag{5.2.44}$$

Since (t_0, X_0) is a maximum point of $(t, X) \to \alpha W^\alpha(t, X) + \left(5C_0 + \frac{m}{2}\right)\beta_\epsilon(t - s_0, X - Y_0)$, we have

$$\begin{aligned} &-\frac{5C_0 + \frac{m}{2}}{\alpha}\partial_t\beta_\epsilon(t_0 - s_0, X_0 - Y_0) + \alpha W^\alpha(t_0, X_0) \\ &+ F\left(t_0, X_0, -\frac{5C_0 + m/2}{\alpha}D_X\beta_\epsilon(t_0 - s_0, X_0 - Y_0) + P\right) \le 0. \end{aligned} \tag{5.2.45}$$

Let i_0, j_0 and n_0 be such that $X_{i_0,j_0} = Y_0$ and $s_0 = t_{n_0}$. Since (s_0, Y_0) is a minimum point of $(s, Y) \to \alpha W^\alpha_{h,\Delta t}(s, Y) - (5C_0 + m/2)\,\beta_\epsilon(t_0 - s, X_0 - Y)$, we obtain

$$W^{n_0,\alpha}_{i_0+1,j_0} - W^{n_0,\alpha}_{i_0,j_0} \ge \frac{5C_0 + m/2}{\alpha}[\beta_\epsilon(t_0 - s_0, X_0 - Y_0 - he_1) - \beta_\epsilon(t_0 - s_0, X_0 - Y_0)],$$

where $e_1 = (1, 0)^T$. From the monotonicity of g,

$$\begin{aligned} &\frac{W^{n_0,\alpha}_{i_0,j_0} - W^{n_0-1,\alpha}_{i_0,j_0}}{\Delta t} + \alpha W^{n_0,\alpha}_{i_0,j_0} + g\left(s_0, Y_0, \frac{5C_0 + m/2}{\alpha}(\Delta_1^+\beta_\epsilon(t_0 - s_0, X_0 - \cdot))_{i_0,j_0} + p_x,\right. \\ &\left.(\Delta_1^+W^{n_0,\alpha})_{i_0-1,j_0} + p_x, (\Delta_2^+W^{n_0,\alpha})_{i_0,j_0} + p_y, (\Delta_2^+W^{n_0,\alpha})_{i_0,j_0-1} + p_y\right) \ge 0. \end{aligned} \tag{5.2.46}$$

But

$$|(\Delta_1^+\beta_\epsilon(t_0 - s_0, X_0 - \cdot))_{i_0,j_0} - e_1\cdot D_Y\beta_\epsilon(t_0 - s_0, X_0 - Y_0)| = \frac{h}{2}|e_1^T D^2_{YY}\beta_\epsilon(t_0 - s_0, X_0 - \overline{Y})e_1|,$$

for some $\overline{Y}$ belonging to the segment $(Y_0, Y_0 + he_1)$. Assuming h small enough, so that Lemma 5.2.4 (ii) implies that $|t_0 - s_0|^2 + |X_0 - Y_0|^2 + h^2 \le \frac{\epsilon^2}{2}$, we obtain that $D^2_{YY}\beta_\epsilon(t_0 - s_0, X_0 - \overline{Y}) = \frac{2}{\epsilon^2}I$, then

$$|(\Delta_1^+\beta_\epsilon(t_0 - s_0, X_0 - \cdot))_{i_0,j_0} - e_1 \cdot D_Y\beta_\epsilon(t_0 - s_0, X_0 - Y_0)| = \frac{h}{\epsilon^2}. \tag{5.2.47}$$

Now, (5.2.46), (5.2.47) and the monotonicity of g yield

$$\begin{aligned} &\frac{W^{n_0,\alpha}_{i_0,j_0} - W^{n_0-1,\alpha}_{i_0,j_0}}{\Delta t} + \alpha W^{n_0,\alpha}_{i_0,j_0} + g\left(s_0, Y_0, \frac{5C_0 + m/2}{\alpha}e_1 \cdot D_Y\beta_\epsilon(t_0 - s_0, X_0 - Y_0) + p_x,\right. \\ &\left.(\Delta_1^+W^{n_0,\alpha})_{i_0-1,j_0} + p_x, (\Delta_2^+W^{n_0,\alpha})_{i_0,j_0} + p_y, (\Delta_2^+W^{n_0,\alpha})_{i_0,j_0-1} + p_y\right) + \widetilde{C}_1 h\frac{5C_0 + m/2}{\epsilon^2\alpha} \ge 0. \end{aligned}$$

Repeating similar estimates for the other arguments in g and for the derivative with respect to time, we finally find that

$$\begin{aligned} &\frac{5C_0 + m/2}{\alpha}\partial_s\beta_\epsilon(t_0 - s_0, X_0 - Y_0) + \alpha W^{n_0,\alpha}_{i_0,j_0} + \\ &F\left(s_0, Y_0, \frac{5C_0 + m/2}{\alpha}D_Y\beta_\epsilon(t_0 - s_0, X_0 - Y_0) + P\right) + C\frac{h + \Delta t}{\epsilon^2\alpha} \ge 0, \end{aligned} \tag{5.2.48}$$

where C is independent of $h, \Delta t, \epsilon$ and α.

Subtracting (5.2.45) and (5.2.48) and using (F2) we get

$$\alpha W^\alpha(t_0, X_0) - \alpha W^\alpha_{h,\Delta t}(s_0, Y_0) \le C\frac{h+\Delta t}{\epsilon^2 \alpha} + \frac{C}{\alpha}\left|\frac{X_0 - Y_0}{\epsilon}\right|^2 + \frac{C}{\alpha}\left|\frac{t_0 - s_0}{\epsilon}\right|^2, \quad (5.2.49)$$

where C is independent of $h, \Delta t, \epsilon$ and α.

Choose $\epsilon = \epsilon(\Delta t, h)$ such that $\epsilon \to 0$ as $(\Delta t, h) \to (0,0)$ and $\frac{h+\Delta t}{\epsilon^2} \to 0$ as $(\Delta t, h) \to (0,0)$. From (i) of Lemma 5.2.4

$$\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W^{n,P,\alpha}_{i,j}| = m \le \sup \Psi - \left(5C_0 + \frac{m}{2}\right)\beta_\epsilon(t_0 - s_0, X_0 - Y_0)$$
$$= \alpha W^\alpha(t_0, X_0) - \alpha W^\alpha_{h,\Delta t}(s_0, Y_0).$$

Then from (5.2.49) and (5.2.44), we obtain

$$\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W^{n,P,\alpha}_{i,j}| \le \frac{C}{\alpha} o(1) \quad \text{as } (\Delta t, h) \to (0,0).$$

From the previous estimate, (ii) of Lemma 5.1.3 and Proposition 5.2.2 we finally obtain

$$|\overline{F}(P) - \overline{F}^{\Delta t}_h(P)| \le \widetilde{K}_1 \alpha + K_1 \alpha + \frac{C}{\alpha} o(1),$$

and letting $(h, \Delta t) \to (0,0)$, we find that

$$\limsup_{(\Delta t,h)\to(0,0)} |\overline{F}(P) - \overline{F}^{\Delta t}_h(P)| \le \widetilde{K}_1 \alpha + K_1 \alpha,$$

for any fixed $\alpha > 0$. This implies that $\lim_{(\Delta t,h)\to(0,0)} \overline{F}^{\Delta t}_h(P) = \overline{F}(P)$. Since $K_1 = K_1(P)$ and $\widetilde{K}_1 = \widetilde{K}_1(P)$ are bounded for P lying on compact subsets of $\mathbb{R}^2$, the convergence is uniform on compact sets. □

Remark 5.2.5. *If F is coercive, then we can get an estimate of the rate of convergence of $\overline{F}^{\Delta t}_h$ to $\overline{F}$. Indeed, we have:*

$$|\overline{F}^{\Delta t}_h - \overline{F}| \le (h + \Delta t)^{\frac{1}{2}},$$

see Proposition A.3 in [1].

We conclude this subsection by recalling the principal properties of $\overline{F}^{\Delta t}_h$.

Proposition 5.2.6. *Assume (g1)-(g6), (H1)-(H4). Then the approximate effective Hamiltonian $\overline{F}^{\Delta t}_h$ is Lipschitz continuous with a Lipschitz constant independent of h and Δt and for any $p_x \in \mathbb{R}$*

$$\overline{F}^{\Delta t}_h(p_x, 0) \ge C_2 |p_x|.$$

Proof. For the proof of the Lipschitz continuity of $\overline{F}$, see the proof of Proposition A.2 in [1].

Let us show the coercivity property. Let $(W_{i,j}^{n,P,\alpha})$ be a solution of (5.2.37) for $P = (p_x, 0)$. Let (i_0, j_0, n_0) be a maximum point of $(W_{i,j}^{n,P,\alpha})$, then

$$\frac{W_{i_0,j_0}^{n_0,P,\alpha} - W_{i_0,j_0}^{n_0-1,P,\alpha}}{\Delta t} \geq 0,\ (\Delta_1^+ W^{n_0,P,\alpha})_{i_0,j_0} \leq 0,\ (\Delta_1^+ W^{n_0,P,\alpha})_{i_0-1,j_0} \geq 0,$$

$$(\Delta_2^+ W^{n_0,P,\alpha})_{i_0,j_0} \leq 0,\ (\Delta_2^+ W^{n_0,P,\alpha})_{i_0,j_0-1} \geq 0.$$

By the monotonicity assumption (g1) and (5.1.10), we have

$$\overline{F}_h^{\Delta t}(p_x, 0) \geq g(t_{n_0}, x_{i_0}, y_{i_0}, p_x, p_x, 0, 0) = F(t_{n_0}, x_{i_0}, y_{i_0}, p_x, 0) \geq C_2|p_x|.$$

□

5.2.1 Long time approximation

A different way to approximate the effective Hamiltonian is given by the evolutive Hamilton-Jacobi equation

$$\begin{cases} V_t + F(t, x, y, p_x + D_x V, p_y + D_y V) = 0, & (t, x, y) \in (0, +\infty) \times \mathbb{R}^{N+1}, \\ V(0, x, y) = V_0(x, y), & (x, y) \in \mathbb{R}^{N+1}, \end{cases} \tag{5.2.50}$$

where V_0 is bounded and uniformly continuous on $\mathbb{R}^{N+1}$. Indeed, it is proved in [18] that (5.2.50) admits a unique solution V which is bounded and uniformly continuous on $[0, T] \times \mathbb{R}^{N+1}$ for any $T > 0$, and satisfies

$$\lim_{t \to +\infty} \frac{V(t, x, y)}{t} = -\overline{F}(P).$$

We approximate (5.2.50) by the implicit Eulerian scheme

$$\begin{aligned} &\tfrac{V_{i,j}^{n+1,P} - V_{i,j}^{n,P}}{\Delta t} + S(t_n, x_i, y_j, h, [V^{n+1,P}]_{i,j}) = 0 \\ &V_{i,j}^{0,P} = V_0(x_i, y_j), \end{aligned} \tag{5.2.51}$$

where S is defined as in (5.2.35). A proof of the existence of a solution $V = (V_{i,j}^{n,P})$ of (5.2.51) is given in [39] under assumptions (g1)-(g5).

Let $W = (W_{i,j}^{n,P,\alpha})$ be a solution of (5.2.37), then by comparison, there exist constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c} + W_{i,j}^{n,P,\alpha} - n\overline{F}_h^{\Delta t}(P)\Delta t \leq V_{i,j}^{n,P} \leq \overline{c} + W_{i,j}^{n,P,\alpha} - n\overline{F}_h^{\Delta t}(P)\Delta t.$$

Since W is bounded, this proves that

$$\lim_{n \to +\infty} \frac{V_{i,j}^{n,P}}{n\Delta t} = -\overline{F}_h^{\Delta t}(P).$$

5.2.2 Approximation of the homogenized problem

We now come back to the N-dimensional homogenized problem (5.0.6). From Theorem I.4.3 we know that if $\overline{H}$ is the effective Hamiltonian in (5.0.6), then $\overline{H}(p) = \overline{F}(p,-1)$ for any $p \in \mathbb{R}^N$. Hence, from Proposition 5.2.3, the discrete Hamiltonian

$$\overline{H}_h^{\Delta t}(p) := \overline{F}_h^{\Delta t}(p,-1),$$

is an approximation of $\overline{H}(p)$ for any $p \in \mathbb{R}^N$.

As in [1], we approximate (5.0.6) by the problem

$$\begin{cases} \partial_t u_{\Delta t,h} + \overline{H}_h^{\Delta t}(Du_{\Delta t,h}) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u_{\Delta t,h}(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.2.52}$$

where h and Δt are fixed, and u_0 is the same initial datum as in (5.0.6).

By Proposition 5.2.6 $\overline{H}_h^{\Delta t}$ is Lipschitz continuous and coercive, so (5.2.52) has a unique viscosity solution $u_{\Delta t,h}$ which is an approximation of the solution u^0 of (5.0.6):

Proposition 5.2.7. *Let u^0 and $u_{\Delta t,h}$ be respectively the viscosity solutions of* (5.0.6) *and* (5.2.52). *Then for any $T > 0$*

$$\sup_{[0,T]\times\mathbb{R}^N} |u_{\Delta t,h} - u^0| \to 0 \quad \textit{as } (\Delta t, h) \to (0,0). \tag{5.2.53}$$

Proof. If L is the Lipschitz constant of the initial datum u_0, then, by Proposition 5.1.4, the functions $u^0(t,x)$ and $u_{\Delta t,h}(t,x)$ are Lipschitz continuous with respect to x with same Lipschitz constant L, By Proposition 5.2.3 the approximate Hamiltonian $\overline{H}_h^{\Delta t}$ converges to $\overline{H}$ uniformly for $|p| \le L$. Hence (5.2.53) follows by the following proposition, which is a standard estimate in the regular perturbation theory of Hamilton-Jacobi equations (see Theorem VI.22.1 in [13])

Proposition 5.2.8. *If there exists $\eta > 0$ such that if H_i, $i = 1,2$, satisfy (H1)-(H3) with*

$$\|H_1 - H_2\|_\infty \le \eta,$$

and if u_i, $i = 1,2$, are viscosity solutions of

$$\begin{cases} u_t + H_i(Du) = 0, & (t,x) \in (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where u_0 is bounded and uniformly continuous on $\mathbb{R}^N$, then, for some constant C,

$$\|u_1 - u_2\|_\infty \le C\eta.$$

□

Remark 5.2.9. In order to compute numerically the approximation of u^0, we need further discretizations. Indeed, we have approximated $\overline{H}(p)$ by $\overline{H}_h^{\Delta t}(p)$ for any fixed $p \in \mathbb{R}^N$. Since it is not possible to compute $\overline{H}_h^{\Delta t}(p)$ for any p, one possibility is to introduce a triangulation of a bounded region of $\mathbb{R}^N$ and compute $\overline{H}_h^{\Delta t}(p_i)$, where

p_i are the vertices of the simplices and to approximate all the other values $\overline{H}_h^{\Delta t}(p)$ by $\overline{H}_{h,k}^{\Delta t}(p)$, where $\overline{H}_{h,k}^{\Delta t}$ is the linear interpolation of $\overline{H}_h^{\Delta t}$ and we denote by k the maximal diameter of the simplices. The solution $u_{\Delta t,h}^k$ of

$$\begin{cases} \partial_t u_{\Delta t,h}^k + \overline{H}_{h,k}^{\Delta t}(Du_{\Delta t,h}^k) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u_{\Delta t,h}^k(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.2.54}$$

is an approximation of $u_{\Delta t,h}$ as $k \to 0$ and hence, by Proposition 5.2.7, of u^0 as $(\Delta t, h, k) \to (0,0,0)$. Finally, discretizing (5.2.54) by means a monotone, consistent and stable approximation scheme, we can compute numerically an approximation of the solution u^0 of 5.0.6. See [1] for details.

5.3 Numerical Tests

The present paragraph is devoted to the description of numerical approximations of the effective Hamiltonian.

5.3.1 Results

First case

We discuss a one dimensional case where the Hamiltonian is

$$H(x,u,p) = 2\cos(2\pi x) + \sin(8\pi u) + (1 - \cos(6\pi x)/2)|p|.$$

We have used two approaches for computing the effective Hamiltonian.

(g1) Barles cell problem: the first approach consists of increasing the dimension and considering the long time behavior of the continuous viscosity solution w of

$$\begin{cases} w_t + F(x,y,p + D_x w, -1 + D_y w) = 0, & (t,x,y) \in (0,\infty) \times \mathbb{R} \times \mathbb{R}, \\ w(0,x,y) = 0, & (x,y) \in \mathbb{R} \times \mathbb{R} \end{cases} \tag{5.3.55}$$

where F is given by (5.0.2). In the present case, from the periodicity of H with respect to x and u, w is 1-periodic with respect to x and 1/4-periodic with respect to y. We know that when $t \to \infty$, $w(t,\cdot,\cdot)/t$ tends to a real number λ and that $\overline{H}(p) = -\lambda$.
For approximating (5.3.55) on a uniform grid, we have used an explicit Euler time marching method with a Godunov monotone scheme (see [48, 106]). A semi-implicit time marching scheme which allows for large time steps may be used as well, see [1], but very large time steps cannot be taken because of the periodic in time asymptotic behaviour of w.
Alternatively, we have also used the higher order method described in [78], see also [79]. It is a third order TVD explicit Runge-Kutta time marching method with a weighted ENO scheme in the spatial variables. This weighted ENO scheme is constructed upon and has the same stencil nodes as the third order ENO scheme but can be as as high as fifth order accurate in the smooth part of the solution.

(g2) Imbert-Monneau cell problem: when p is a rational number ($p = \frac{n}{q}$), instead of considering a problem posed in two space dimensions, one possible way of approximating the effective Hamiltonian $\overline{H}(p)$ is to consider the cell problem

$$\begin{cases} v_t + H(x, v + p \cdot x, p + Dv) = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ v(0, x) = 0, & x \in \mathbb{R}. \end{cases} \tag{5.3.56}$$

This problem has a unique continuous solution which is periodic of period q with respect to x (in fact, the smallest period of v may be a divisor of q). From [67] (Theorem 1), we know that there exists a unique real number λ such that $\frac{v(\tau,x)}{\tau}$ converges to λ as $\tau \to \infty$ uniformly in x, and that $\overline{H}(p) = -\lambda$. Moreover, when t is large, the function $v(t, x) - \lambda t$ becomes close to a periodic function of time. In what follows, (5.3.56) will be referred to as Imbert-Monneau cell problem. Note that the size of the period varies with p and may be arbitrary large. This is clearly a drawback of this approach which is yet the fastest one for one dimensional problems and moderate values of q.
For approximating (5.3.56) on a uniform grid, we have used either the abovementioned explicit Euler time marching method with a Godunov monotone scheme or the third order TVD explicit Runge-Kutta time marching method with a weighted ENO scheme in the spatial variable.

In Figure 5.3.56, we plot the graph of the effective Hamiltonian computed with the high order methods and both Imbert-Monneau and Barles cell problems. For Barles cell problems, the grid of the square $[0, 1] \times [0, 1/4]$ has 400×100 nodes and the time step is $1/1000$. For Imbert-Monneau cell problems, the grids in the x variable are uniform with a step of $1/400$ and the time step is $1/1000$. The two graphs are undistinguishable. It can be seen that the effective Hamiltonian is symmetric with respect to p and constant for small values of p, i.e. $|p| \lesssim 1.3$. The points where we have computed the effective Hamiltonian are concentrated near 1.3 where the slope of the graph changes. Our computations clearly indicate that the effective Hamiltonian is piecewise linear.
In order to show the convergence of $\frac{v(\tau,x)}{\tau}$ and $\frac{w(\tau,x)}{\tau}$, we take $p = 1.3$ so the space period of the Imbert-Monneau cell problem is 5. In Figure 5.3.56, we plot $\frac{\langle w(\tau)\rangle}{\tau}$ (left) and $\frac{\langle v(\tau)\rangle}{\tau}$ (right) as a function of τ, where $\langle v(\tau)\rangle$ is the median value of $v(\tau, \cdot)$ on a spatial period. Both functions converge to constants when $\tau \to \infty$ and the limit are close to each other (the error between the two scaled median values is smaller than 10^{-3} at $\tau \sim 60$ and we did not consider much longer times). In Figure 5.3.56, we plot the graphs of the functions $w(\tau, 0, 0) - \langle w(\tau)\rangle$ (left) and $v(\tau, 0) - \langle v(\tau)\rangle$ (right). We see that these functions become close to time-periodic. In Figure 5.3.56 (top), we plot the contour lines of the function $w(\tau, x, y)/\tau$ as a function of (x, y) for $\tau = 60$. In the bottom part of the figure we plot the graph of $y \to w(\tau, 0.13, y)/\tau$ for the same value of τ. We see that w has internal layers. In Figure 5.3.56, we plot the graph $x \to v(\tau, x)/\tau$ for $\tau = 60$. We first see that the function takes all its values in a small interval and has very rapid variations with respect to x (is nearly discontinuous). This does not contradict the theory, because there are no uniform estimates on the modulus of continuity of $v(\tau, \cdot)/\tau$.

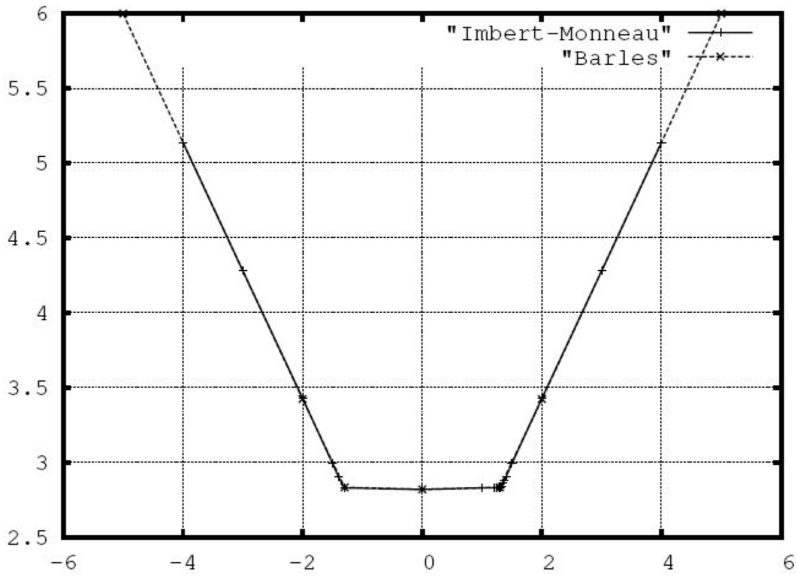

Figure 5.3.56. First case: the effective Hamiltonian as a function of p obtained with both Barles and Imbert-Monneau cell problems.

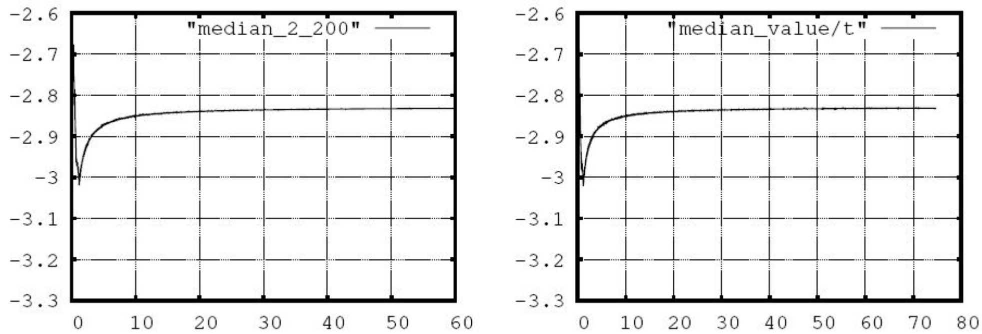

Figure 5.3.56. First case, $p = 1.3$. Left: the median value of $w(\tau, \cdot)/\tau$ on a period as a function of τ. Right: the median value of $v(\tau, \cdot)/\tau$ on a period as a function of τ

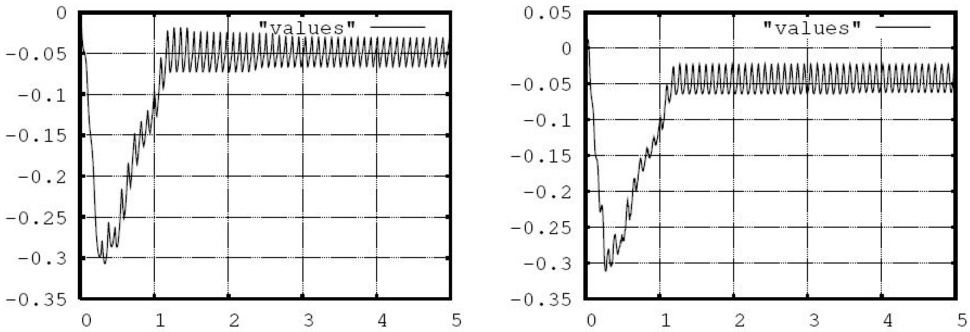

Figure 5.3.56. First case, $p = 1.3$: $w(\tau, 0, 0) - \langle w(\tau)\rangle$ (left) and $v(\tau, 0) - \langle v(\tau)\rangle$ (right) as a function of τ

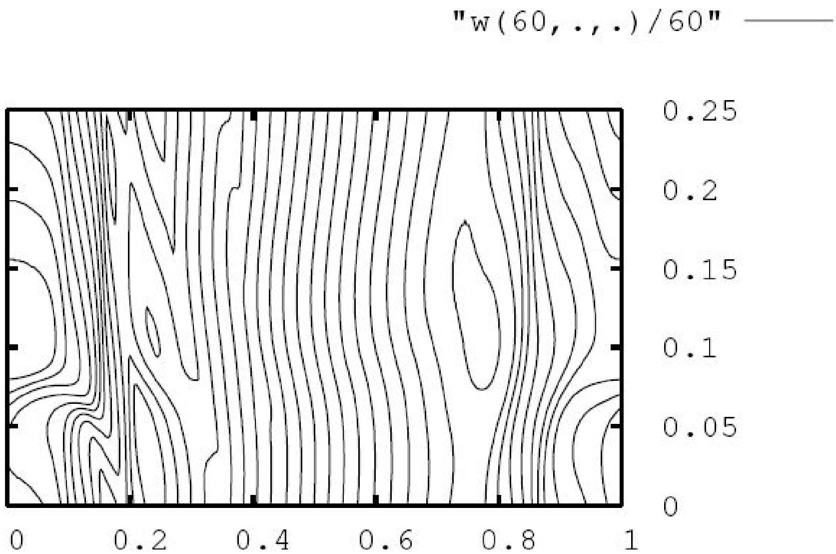

Figure 5.3.56. First case, Barles cell problem, $p = 1.3$. Top: contour lines of $w(\tau, \cdot)/\tau$ on a period as a function of (x, y). Bottom: the cross-section $x = 0.13$.

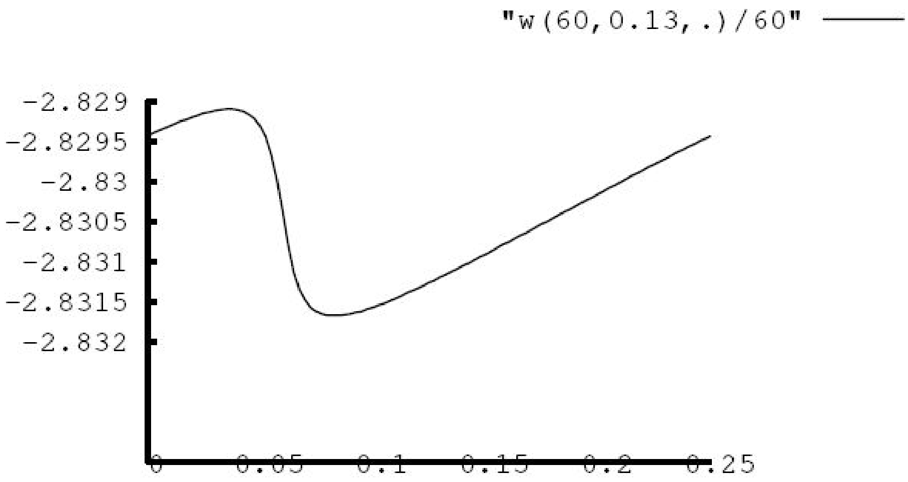

Figure 5.3.56. First case, Imbert-Monneau cell problem, $p = 1.3$: Top: Third order Runge Kutta/WENO scheme: $v(\tau, x)/\tau$ as a function of x for $\tau = 60$; the right part is a zoom. Bottom: same computation with Euler/Godunov scheme with the same grid parameters: some oscillations are smeared out, but the average value of the solution is well computed.

Second case

We consider a two dimensional problem, where the Hamiltonian is

$$\begin{aligned} H(x,u,p) &= \cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi(x_1 - x_2)) + \sin(2\pi u) \\ &\quad + \left(1 - \frac{\cos(2\pi x_1)}{2} - \frac{\sin(2\pi x_2)}{4}\right)|p|. \end{aligned}$$

For this case, only the Imbert-Monneau cell problems have been approximated on uniform grids with step 1/200. The time step is 0.005. In Figure 5.3.56, we plot the contours and the graph of the effective Hamiltonian computed with the high order method. We can see that the effective Hamiltonian is symmetric with respect to $p = (0,0)$, constant for small vectors p. In Figure 5.3.56, we plot $\frac{\langle v(\tau)\rangle}{\tau}$ as a function of τ. We see that this function converges when $\tau \to \infty$. In Figure 5.3.56, we plot the contours of $v(\tau, \cdot)/\tau$ for $\tau = 59.935$ and $p = (1,1)$. We see that for large values of τ, v is close to discontinuous.

5.4 Appendix

Proof of Lemma 5.1.5. To show that the sequence is convergent it suffices to show that for any $s \in \mathbb{R}$ $\varphi_\epsilon^{n,\delta}(s)$ is a Cauchy sequence. Fix $s \in \mathbb{R}$ and let $i_0 \in \mathbb{Z}$ be the closest integer to s, i.e., $s = i_0\epsilon + \gamma\epsilon$, with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. Let $k > m > |i_0|$, then,

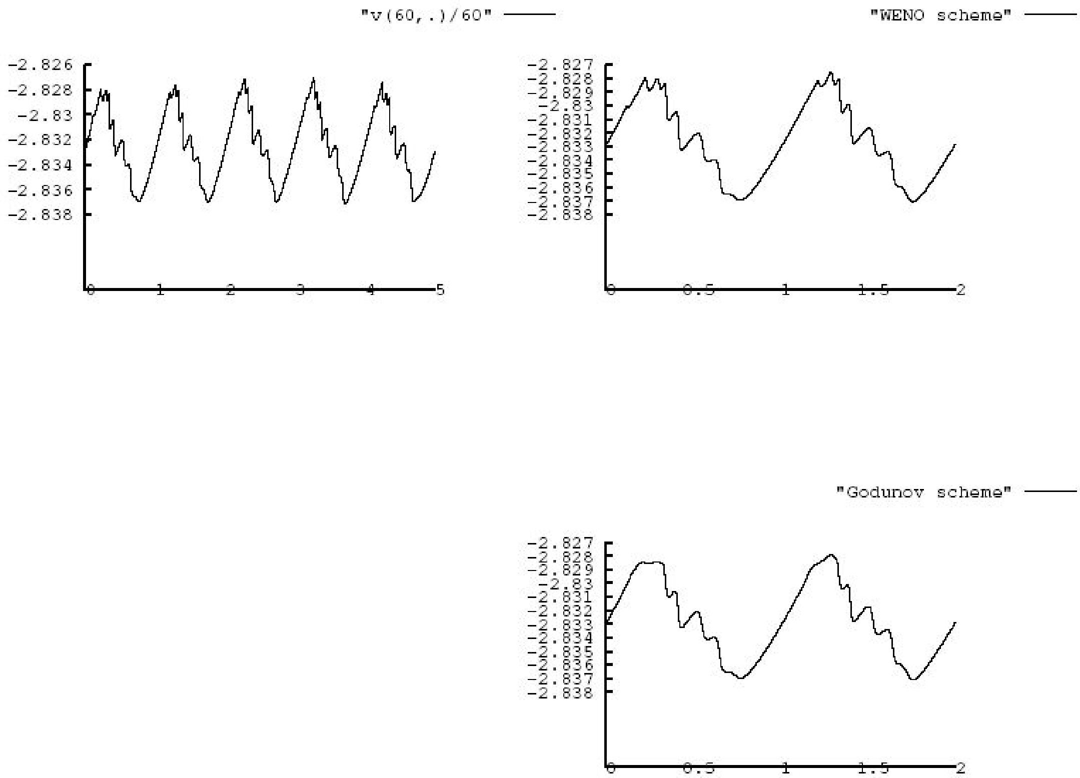

Figure 5.3.56. Second case, the effective Hamiltonian computed by solving Imbert-Monneau cell problems.

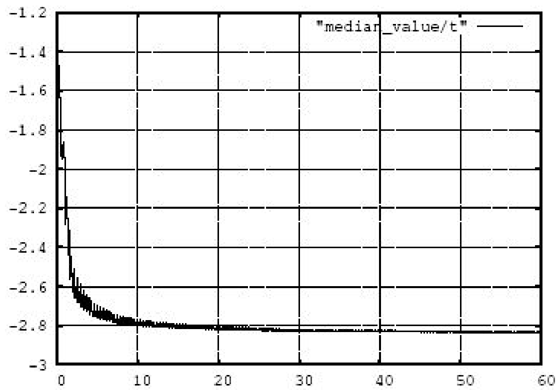

Figure 5.3.56. Second case, $p = (1, 1)$. The median value of $v(\tau, \cdot)/\tau$ on a period as a function of τ.

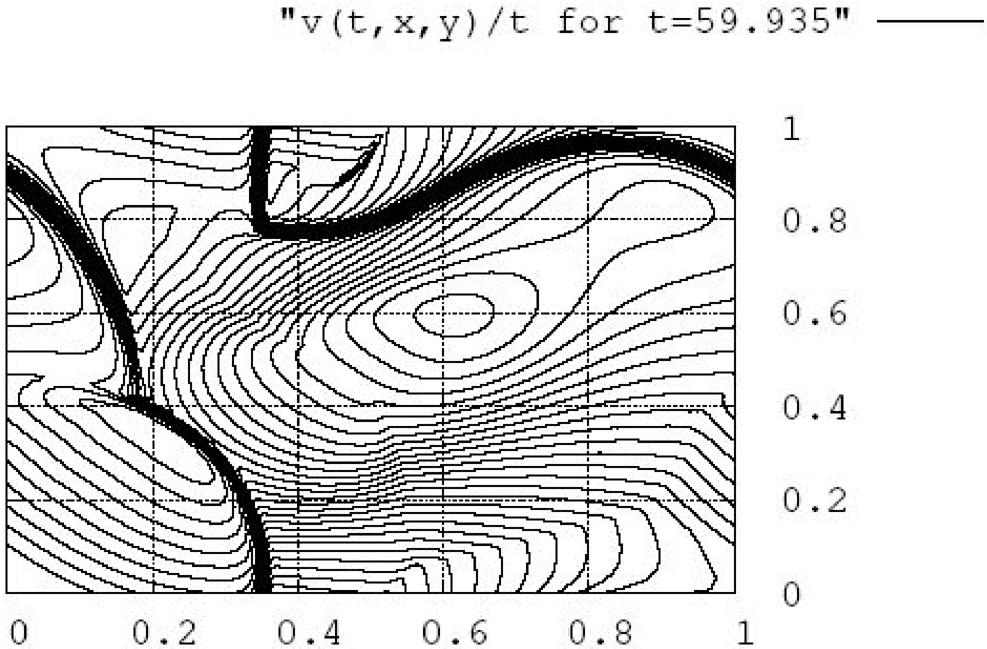

Figure 5.3.56. Second case, the contours of the solution of Imbert-Monneau cell problem for $p = (1,1)$ at time $\tau = 59.935$.

by assumptions (5.1.14) we have

$$\begin{aligned}
\varphi_\epsilon^{k,\delta}(s) - \varphi_\epsilon^{m,\delta}(s) &= \sum_{i=-k}^{-m-1} \epsilon\phi\left(\frac{s-\epsilon i}{\delta}\right) + \sum_{i=m+1}^{k} \epsilon\phi\left(\frac{s-\epsilon i}{\delta}\right) - \epsilon(k-m) \\
&= \sum_{i=-k}^{-m-1} \epsilon\left[\phi\left(\frac{s-\epsilon i}{\delta}\right) - 1\right] + \sum_{i=m+1}^{k} \epsilon\phi\left(\frac{s-\epsilon i}{\delta}\right) \\
&\le \epsilon K_2\delta^2 \sum_{i=-k}^{-m-1} \frac{1}{(s-\epsilon i)^2} + \epsilon K_2\delta^2 \sum_{i=m+1}^{k} \frac{1}{(s-\epsilon i)^2} \\
&= K_2\frac{\delta^2}{\epsilon} \sum_{i=-k}^{-m-1} \frac{1}{(i_0 - i+\gamma)^2} + K_2\frac{\delta^2}{\epsilon} \sum_{i=m+1}^{k} \frac{1}{(i_0 - i+\gamma)^2}.
\end{aligned}$$

Similarly, it can be showed that

$$\varphi_\epsilon^{k,\delta}(s) - \varphi_\epsilon^{m,\delta}(s) \ge -K_2\frac{\delta^2}{\epsilon} \sum_{i=-k}^{-m-1} \frac{1}{(i_0 - i+\gamma)^2} - K_2\frac{\delta^2}{\epsilon} \sum_{i=m+1}^{k} \frac{1}{(i_0 - i+\gamma)^2}.$$

Hence $|\varphi_\epsilon^{k,\delta}(s) - \varphi_\epsilon^{m,\delta}(s)| \to 0$ as $m,k \to +\infty$. Similar arguments show that the sequence $(\varphi_\epsilon^{\delta,n})'$ converge uniformly on compact sets of $\mathbb{R}$. This implies that $\varphi_\epsilon^{\delta}$ is of class C^1 with $(\varphi_\epsilon^{\delta})'(s) = \lim_{n\to+\infty}(\varphi_\epsilon^{\delta,n})'(s)$.

Now, let us show (5.1.15). Let $s = i_0\epsilon + \gamma\epsilon$ for some $i_0 \in \mathbb{Z}$ and $\gamma \in [0,1)$. Then

$$\begin{aligned}
\varphi_\epsilon^{n,\delta}(s) - i_0\epsilon &= \epsilon\left[\phi\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right] + \sum_{i=-n}^{i_0-1} \epsilon\left[\phi\left(\frac{i_0\epsilon + \gamma\epsilon - \epsilon i}{\delta}\right) - 1\right] + \sum_{i=i_0+1}^{n} \epsilon\phi\left(\frac{i_0\epsilon + \gamma\epsilon - \epsilon i}{\delta}\right) \\
&\le \epsilon\left[\phi\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right]\epsilon + \frac{\delta^2}{\epsilon} K_2 \sum_{i=-n}^{i_0-1} \frac{1}{(i_0 - i + \gamma)^2} + \frac{\delta^2}{\epsilon} K_2 \sum_{i=i_0+1}^{n} \frac{1}{(i - i_0 - \gamma)^2} \\
&= \epsilon\left[\phi\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right] + \frac{\delta^2}{\epsilon} K_2 \sum_{i=1}^{n+i_0} \frac{1}{(i+\gamma)^2} + \frac{\delta^2}{\epsilon} K_2 \sum_{i=1}^{n-i_0} \frac{1}{(i-\gamma)^2}.
\end{aligned}$$

Similarly

$$\varphi_\epsilon^{n,\delta}(s) - i_0\epsilon \ge \epsilon\left[\phi\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right] - \frac{\delta^2}{\epsilon} K_2 \sum_{i=1}^{n+i_0} \frac{1}{(i+\gamma)^2} - \frac{\delta^2}{\epsilon} K_2 \sum_{i=1}^{n-i_0} \frac{1}{(i-\gamma)^2}.$$

Letting $n \to +\infty$, we get

$$\left|\varphi_\epsilon^{\delta}(s) - i_0\epsilon - \epsilon\left[\phi\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right]\right| \le \frac{\delta^2}{\epsilon} K_2 \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^2} + \frac{\delta^2}{\epsilon} K_2 \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^2}.$$

If $\gamma > 0$ then $\phi\left(\frac{\gamma\epsilon}{\delta}\right) - 1 \to 0$ as $\delta \to 0^+$ and $\varphi_\epsilon^\delta(s) \to i_0\epsilon$ if $\delta \to 0^+$. If $\gamma = 0$, then $\varphi_\epsilon^\delta(s) \to (i_0 - 1)\epsilon + \phi(0)\epsilon$ if $\delta \to 0^+$ and (5.1.15) is proved. □

Appendix A

Notation list

$B_r(x)$ ball of radius r centered at x

$C(\overline{\Omega})$ set of continuous functions on $\overline{\Omega}$

$C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ set of continuous functions on $\mathbb{R}^+ \times \mathbb{R}^N$ which are bounded on $(0,T) \times \mathbb{R}^N$ for any $T > 0$

$C_x^\alpha((0,T) \times \mathbb{R}^N)$ space of continuous functions defined on $(0,T) \times \mathbb{R}^N$ that are bounded and with bounded seminorm $< u >_x^\alpha$

$d(x)$ distance function

$LSC(\overline{\Omega})$ set of lower semicontinuous functions on $\overline{\Omega}$

$LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ set of lower semicontinuous functions on $\mathbb{R}^+ \times \mathbb{R}^N$ which are bounded on $(0,T) \times \mathbb{R}^N$ for any $T > 0$

For $u : A \to \mathbb{R}$ and $x \in A$, $J^{2,+}u(x)$, $J^{2,-}u(x)$ are the second-order semi-jets defined by

$$J^{2,+}u(x) = \{(p,X) \in \mathbb{R}^N \times S(N) \,|\, u(y) \le u(x) + \langle p, y-x \rangle + \frac{1}{2}\langle X(y-x), y-x \rangle + o(|y-x|^2) \text{ as } y \to x,\ y \in A\}$$

$$J^{2,-}u(x) = \{(p,X) \in \mathbb{R}^N \times S(N) \,|\, u(y) \ge u(x) + \langle p, y-x \rangle + \frac{1}{2}\langle X(y-x), y-x \rangle + o(|y-x|^2) \text{ as } y \to x,\ y \in A\}$$

For $u : A \to \mathbb{R}$ and $x \in A$

$$\overline{J}^{2,+}u(x) = \{(p,X) \in \mathbb{R}^N \times S(N) \,|\, \exists (x_n, p_n, X_n) \in A \times \mathbb{R}^N \times S(N) \ (p_n, X_n) \in J^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X)\}$$

$$\overline{J}^{2,-}u(x) = \{(p,X) \in \mathbb{R}^N \times S(N) \,|\, \exists (x_n, p_n, X_n) \in A \times \mathbb{R}^N \times S(N) \ (p_n, X_n) \in J^{2,-}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X)\}$$

$\mathcal{M}^{\substack{+\\-}}_{a,A}(D^2u)$ Pucci's operators

$Q_{\tau,r}(t,x) = (t-\tau, t+\tau) \times B_r(x)$

S(N) space of real symmetric matrices $N \times N$

S^{N-1} unit sphere of $\mathbb{R}^N$

$USC(\overline{\Omega})$ set of upper semicontinuous functions on $\overline{\Omega}$

$USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ set of upper semicontinuous functions on $\mathbb{R}^+ \times \mathbb{R}^N$ which are bounded on $(0,T) \times \mathbb{R}^N$ for any $T > 0$

For $u : (0,T) \times \mathbb{R}^N \to \mathbb{R}$, $0 < T \leq +\infty$, for $0 < \alpha < 1$

$$< u >_x^\alpha := \sup_{\substack{(t,x),\,(t,x') \in (0,T) \times \mathbb{R}^N \\ x \neq x'}} \frac{|u(t,x) - u(t,x')|}{|x - x'|^\alpha}$$

For $X \in$ *S(N)*, $\|X\| = \sup\{|X\xi| \,|\, \xi \in \mathbb{R}^N,\ |\xi| \leq 1\} = \sup\{|\lambda| \,:\, \lambda \text{ is an eigenvalue of } X\}$

For $X \in$ *S(N)* $\mathrm{tr}(X)$ denotes the trace of X

For $\xi, \eta \in \mathbb{R}^N$, $\xi \otimes \eta$ denotes the matrix $(\xi_i \eta_j)_{ij}$

$\lfloor x \rfloor$ floor integer part of x

$\lceil x \rceil$ ceil integer part of x

$a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$

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