Geometric deformation functors for p-divisible groups

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Contents

The theory of deformations of Galois representations was first developed by Barry Mazur in the fundamental work [8]. The main purpose was the study of the absolute Galois group of $\mathbb{Q}$ through the study of its continuous representations into finite fields. The main idea was, starting from a representations $\bar{\rho}$ with values in $GL_n(\mathbb{F}_p)$, to give a collection of lifts of $\bar{\rho}$ to topological rings having $\mathbb{F}_p$ as a residue field such that all the lifts can be classified in a unique way from a universal one ρ_{univ} with values in an appropriate ring $R = R(\bar{\rho})$, depending only on $\bar{\rho}$ and on possible additional conditions imposed on the lifts; R is called the universal deformation ring of $\bar{\rho}$ and $Spec(R)$ is called the universal deformation space. Deformation theory started to be considered as a main tool after the proof of Fermat's last theorem given by Andrew Wiles in 1995; in his main work Wiles proved that the universal deformation ring R of a modular representation was isomorphic to an algebra T of Hecke operators on modular forms. The proof of this isomorphism, called the $R = T$ theorem, was the main tecnichal tool of Wiles's work. In the following years many other deformation problems with local conditions started to be studied, like the flat local deformation functor described by Ramakrishna. Also the concept of a framed deformation functor, which solved a problem of possible non-existence of the universal deformation ring, started to be used. In most recent times we mainly remind the work of Kisin, who used this framework in his works on the Fontaine-Mazur conjecture.

This work has the purpose of proving some results about a class of deformation problems called *geometric* (the definition is due to Kisin). In chapter 1 we give all the main definitions about the general deformation theory. In particular in section 1.4 we describe the functorial approach to deformation problems: this is a more theoretical, but useful approach to retrieve fundamental informations about possible computations of the universal deformation ring. Paragraph 1.55 introduce the concept of framed deformations and 1.6 describes possible examples of local conditions that can be imposed on the lifts of $\bar{\rho}$. The main references for this first part are [8] and [9]. In the second chapter we introduce the geometric objects where the representations we are considering come from: p-divisible groups. They are defined by a sequence of p-power-order group schemes having particular properties. After giving the main definitions and some examples we state the fundamental theorem by Tate about the structure of morphisms between p-divisible groups. Then we pass to describe a theorem of Schoof which will be the starting point of our discussion: it says that p-divisible groups composed by group schemes belonging to some subcategory $\underline{D}$ "do not deform" and their structure is rigidly defined. Our main objective will be to extablish a similar result for representations coming from such p-divisible groups. We can then define our deformation functor. We take an elliptic curve E with good supersingular reduction in p and semistable reduction in a prime $\ell \neq p$. We consider the representation $\bar{\rho}$ given by the natural $G_{\mathbb{Q}}$-action on the group scheme $E[p]$ of p-torsion points of E and look for deformations ρ of $\bar{\rho}$ which satisfy the following local conditions:

- ρ is odd, which means $det(\rho(c)) = -1$ for c the complex conjugation in $G_{\mathbb{Q}}$;
- ρ is flat at the prime p.
- ρ has semistable action at the prime ℓ, which means $(\rho(\sigma) - id)^2 = 0$ for every $\sigma \in I_\ell$, the inertia group of ℓ.

Moreover we work in the additional hypothesis (suggested by Schoof's theorem) that the extension group $Ext^1(E[p], E[p])$ has trivial p-torsion part. Our first purpose is showing that the universal deformation ring with these local conditions is isomorphic to $\mathbb{Z}_p$ and the universal representation is given by the Tate module of E. In paragraph 2.3 we describe an explicit example in which this situation occurs. We take E to be the Jacobian of the modular curve $X_0(11)$, $p = 2$ and $\ell = 11$. As shown in [12], this case satisfies all the hypotheses we have done. In paragraph 2.4 we prove the base case of our theorem and we also performe an explicit computation of the framed universal deformation ring, which will be also used later on in the paper.

Chapter 3 is dedicated to a generalisation of this result to representations not necessarily coming from elliptic curves and to higher dimension representations. The main problem is that in this case we do not have a canonical characteristic zero lift, like the Tate module, therefore our main concern will be to obtain a similar one using local-to-global arguments, mainly due to Kisin. We also have to consider only framed deformations, since we cannot ensure any more the existence of a universal ring, like in the elliptic curve case. After some recalls of Galois cohomology, the chapter describes the local deformation functors at the primes p, ℓ and the archimedean prime. In 3.2 we study the flat deformation functor at p using the approach of Ramakrishna [10], giving an explicit computation of the local deformation ring. In 3.3 we see that our condition of semistable action at ℓ belongs to the larger class of *Steinberg-type* conditions, with prescribed action on a 1-dimensional submodule of the representation module $V_{\bar{\rho}}$. Finally in 3.4 we make an explicit computation of the universal ring at archimedean primes. In 3.5 we give local-to-global arguments to pass from the local ring explored in the three prevous paragraphs to the global one and see that the local conditions we have imposed make our deformation functor belong to the class of *geometric* deformation functors; the arguments are mainly taken from [6]. The final paragraph contains our main original result: we consider a representation $\bar{\rho}$ which is direct sum of 2-dimensional representations $\bar{\rho}_j$ satisfying the local conditions we have defined and also impose the conditions that $Ext^1(V_{\bar{\rho}_i}, V_{\bar{\rho}_j})$ is trivial for all i, j and that all of the $\bar{\rho}_j$ admit a universal deformation ring; then we can compute the framed universal deformation ring of $\bar{\rho}$. The idea of the proof is first to compute the deformation ring of the single $\bar{\rho}_j$, using the local-to-global arguments and then perform an explicit computation of the framed deformation ring similar to the one used in the elliptic curve case in chapter 2. The framed ring turns out to be a power series ring over $\mathbb{Z}_p$ in a large number of variables.

Chapter 1

Deformation Theory

1.1 The absolute Galois group of a number field

Let K be a characteristic zero field and $\bar{K}$ an algebraic closure and denote as $G_K = Gal(\bar{K}/K)$ the absolute Galois group of K. G_K is a profinite topological group with the natural Krull topology and a base of open set is given by the subgroups $\{Fix(F)\}$ of G, given by the elements which fix a finite extension F of K contained in $\bar{K}$. However, very little of the structure of G_K is known in the general case.

Following the approach of [8],[9], we want to study the group G_K taking its continuous representations over smaller p-adic matrix groups and lifting them appropriately. Let us start from a simple example. Let $G = G_{\mathbb{Q}}$ and $\rho : G \to \mathbb{Z}/2\mathbb{Z}$ be a 1-dimensional continuous surjective representation. We want to give a lift of ρ to $\mathbb{Z}/4\mathbb{Z}$, that is a homomorphism $\rho' : G \to \mathbb{Z}/4\mathbb{Z}$ which reduces to ρ when composed with the natural projection. We have the following result

Proposition 1.1.1. *Let $\rho' : G \to \mathbb{Z}/4\mathbb{Z}$ be a set theoretic lift of ρ (which always exists as a map between sets). Then ρ' is a $G_{\mathbb{Q}}$-representation lifting ρ if and only if the 2-cocycle $C(s,t) = \rho'(st) - \rho'(t) - \rho'(s)$ is the zero map in $H^2(G, \mathbb{Z}/2\mathbb{Z})$.*

This proposition is very useful because the cohomology group $H^2(G, \mathbb{Z}/2\mathbb{Z})$ is easy to study. Moreover we can consider the natural restriction map

$$\theta : H^2(G, \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_p H^2(G_p, \mathbb{Z}/2\mathbb{Z}), \tag{1.1}$$

where G_p is the absolute Galois group of the local field $\mathbb{Q}_p$ and the sum is taken over all the rational primes (even the infinite). By Kummer's theory, θ is injective and $H^2(G_p, \mathbb{Z}/2\mathbb{Z})$ is nontrivial only for a finite number of primes, so we can solve the lifting problem quite easily. We can then use repeatedly the proposition and lift the representation to all the groups $\mathbb{Z}/2^m\mathbb{Z}$ and then take the inverse limit to have a characteristic zero representation.

The study of liftings of 1-dimensional representation is solved by Class Field Theory. What if we want to extend the same argument to representations of greater dimension? What comes out is that in this case the map $C(s,t)$ is an element of $H^2(G, Ker(\pi))$, where π is the natural projection from $GL_n(\mathbb{Z}/p^2\mathbb{Z})$ to $GL_n(\mathbb{F}_p)$; this kernel is the abelian group

$$\{I + pX|\ X \in M_n(\mathbb{F}_p)\} \tag{1.2}$$

and G acts on it by the adjoint action composed with ρ. It is therefore called $Ad(\rho)$. The problem is that this cohomology group is too big to be well known; in particular it has infinite dimension as $\mathbb{F}_p$-vector space in the general case. Moreover the restriction map θ is no longer injective, so we can not restrict to study the easier local cases. To avoid these problems, we restrict to study a finite dimensional subspace of $H^2(G, Ad(\rho))$.

Let K be a number field and S a finite set of primes of K. We denote by K_S the maximal algebraic extension of K in $\bar{K}$ which is unramified outside S and denote by $G_{K,S} = Gal(K_S/K)$. The fundamental property of $G_{K,S}$ is the following [8].

Lemma 1.1.2. *Let p be a prime number. If S is a finite set of primes of K containing all the primes over p, then $G_{K,S}$ satisfies the p-finiteness condition, which means that, for all open subgroups $H \subseteq G_{K,S}$, the set $Hom(H, \mathbb{Z}/p\mathbb{Z})$ of continuous homomorphisms is finite .*

The p-finiteness condition implies in particular that $H^2(G_{K,S}, Ad(\rho))$ is a finite dimensional $\mathbb{F}_p$-vector space. So the groups $G_{K,S}$ seem to be the right setting for our study.

1.2 The general representation setting

In this section we want to define our representation space. We fix a prime p and a finite field k of characteristic p. This will be the base field for our representation spaces. Let $W(k)$ be the ring of Witt vectors over k. Given a number field K, we take S to be a finite set of the primes of K containing the primes over p and the infinite primes. We simply denote as G the group $G_{K,S}$

Since $G_{K,S}$ is a profinite topological group and we want our representations to be continuous, we want the representation spaces to be profinite as well.

Definition 1.2.1. *A coefficient ring over k is a complete noetherian local $W(k)$-algebra A with residue field k. It naturally carries a complete profinite topology, such that*

$$A = \lim_{\leftarrow} A/m_A^n. \tag{1.3}$$

A base of the topology is given by the powers of m_A. A coefficient ring homomorphism is a continous homomorphism of $W(k)$-algebras which induces an isomorphism on the residue fields.

We can put a corresponding profinite topology on $GL_n(A)$ by

$$GL_n(A) = \lim_{\leftarrow} GL_n(A/m_A^{\nu}) \tag{1.4}$$

and a basis of open subgroups is given by the sets of matrices that, reduced modulo a power of m_A, become the identity.

In the following we will very often make use of a Schur-type result on our representations. Observe that the set of continous representations

$$\rho : G \to GL_n(A) \tag{1.5}$$

is in 1-to-1 correspondence with the set of continous homomorphisms of A-algebras

$$r : A[[G]] \to M_n(A) \tag{1.6}$$

where $A[[G]]$ is the completed group ring of G with coefficients in A, defined as

$$A[[G]] = \lim_{\leftarrow} A[G/G_0], \tag{1.7}$$

where G_0 runs over all the open normal subgroups of finite index in G and $A[G/G_0]$ is the usual group ring. The correspondance $r \mapsto \rho$ is obviously given by restriction.

Definition 1.2.2. *The underlying residual representations associated to ρ and r is the representation*

$$\bar{\rho} : G \to GL_n(k) \tag{1.8}$$

given by composing ρ with the natural projection of A to the residue field k.

Proposition 1.2.3. *The residual representation $\bar{\rho}$ associated to ρ is absolutely irreducible if ond only if the corresponding homomorphism r is surjective.*

Proof: The result follows from [1, Ch.8] and Nakayama's lemma.

Corollary 1.2.4. *If $\bar{\rho}$ is absolutely irreducible, then the centralizer of the image of ρ in $M_n(A)$ is the set of scalar matrices.*

Proof. Let M be a matrix of the centralizer. By profinite completion, M must lie in the center of the image of r; by the previous proposition, r is surjective, so M must lie in the center of $M_n(A)$, which is exactly the set of scalar matrices. □

1.3 Deformation theory

Let

$$h : A \to A' \tag{1.9}$$

be a coefficient-ring-homomorphism and

$$\tilde{h} : GL_n(A) \to GL_n(A') \tag{1.10}$$

be the induced homomorphism of matrices. If

$$\rho : G \to GL_n(A) \tag{1.11}$$

is a continous representation, then a *deformation* of ρ to the coefficient ring A' is an equivalence class of liftings

$$\rho' : G \to GL_n(A'), \tag{1.12}$$

where we say that two liftings $\rho_1^{'}, \rho_2^{'}$ are equivalent if there exists a matrix $M \in Ker(\tilde{h})$ such that

$$M^{-1}\rho_1^{'}(g)M = \rho_2^{'}(g) \tag{1.13}$$

for every $g \in G$. Any representation is, of course, a deformation of its residual.

We can reformulate the deformation problem in the language of categories. For a given coefficient ring A, Let $\underline{Ar}(A)$ be the category whose objects are artinian coefficient rings B together with a coefficient ring homomorphism $f : B \to A$ (f is sometimes called an A-augmentation) and whose morphisms are coefficient ring homomorphisms which commute with the augmentations. Let $\underline{\hat{A}r}(A)$ be the category of noetherian coefficient rings with A-augmentation. Clearly $\underline{Ar}(A)$ is a full subcategory of $\underline{\hat{A}r}(A)$. If $A = k$ we will omitt it in the notation.

Given a Galois representation

$$\rho : G \to GL_n(A), \tag{1.14}$$

we can define a functor $F_\rho : \underline{\hat{A}r}(A) \to \underline{Sets}$ which assigns to any object $B \in \underline{\hat{A}r}(A)$ the set of equivalence classes of deformations of ρ to A. Then our task will be the study of this functor.

The most interesting case will be when $A = k$ and $\rho = \bar{\rho}$ is a residual representation. In such a situation it takes only a finite amount of data to give the representation and, for fixed K, S, n, k, there is only a finite number of such representations, up to isomorphism classes. Attached to a residual representation $\bar{\rho}$ one can consider the whole panoply of $G_{K,S}$-representations which are deformations of $\bar{\rho}$. If we require some additional hypothesis on $\bar{\rho}$, then all this panoply comes from a single "universal deformation" with coefficients in a noetherian local complete ring with residue field k; an explicit description of this ring leads to a complete classification of all the Galois representations which are deformations of $\bar{\rho}$.

Theorem 1.3.1. *Let n be a positive integer and*

$$\bar{\rho} : G \to GL_n(k) \tag{1.15}$$

be an absolutely irreducible residual Galois representation, then there exists one and only one, up to canonical isomorphisms, coefficient-ring $R = R(\bar{\rho})$ with residue field k and a deformation

$$\rho_{univ} : G \to GL_n(R) \tag{1.16}$$

of $\bar{\rho}$ such that, for any coefficient ring A with residue field k and any deformation

$$\rho : G \to GL_n(A) \tag{1.17}$$

of $\bar{\rho}$ to A, there exists one and only one coefficient ring homomorphism $h : R \to A$ for which the composition the universal deformation ρ_{univ} with $\hat{h}$ is equal to ρ. In functorial terms, the functor

$$F_{\bar{\rho}} : \underline{\hat{Ar}} \to \underline{Sets} \tag{1.18}$$

is representable by R, that is,

$$F_{\bar{\rho}}(A) \simeq Hom_{W(k)-alg}(R, A), \tag{1.19}$$

where $W(k)$ is the ring of Witt vectors of k. R is called the universal deformation ring of $\bar{\rho}$ and ρ_{univ} is called the universal deformation.

We will give a proof of this theorem in the next sections, by using the functorial formulation.

We also want to give an alternative descriptions of the deformation problem, using the language of G-modules instead of group homomorphisms. Let V be a finite-dimensional k-vector space provided with a continuous G-action. If $A \in \hat{Ar}$, then we define a *deformation* of V to A to be a pair (V_A, ι_A), where V_A is a free A-module of finite rank with continuous G-action and $\iota_A : V_A \otimes_A k \simeq V$ is an isomorphism of G-modules. Then we can define a covariant functor

$$F_V : \underline{\hat{Ar}} \to \underline{Sets} \tag{1.20}$$

setting $F_V(A)$ to be the set of isomorphism classes of deformation of V to A.

By fixing a k-basis of V, we can identify the group $Aut_k(V)$ with $GL_n(k)$, where $n = dim_k(V)$ and the G-action on V with a corresponding residual representation $\bar{\rho} : G \to GL_n(k)$. This identification gives rise to a morphism of functors $F_V \to F_{\bar{\rho}}$ which is easily seen to be an isomorphism. In the following we will denote by V_ρ the G-module corresponing to a representation ρ via this identification.

We have a version of the universal deformation theorem also in this context.

Proposition 1.3.2. *Suppose the natural map*

$$k \to End_{k[G]}(V) \tag{1.21}$$

to be an isomorphism. Then the functor F_V is representable, that is, there exist a coefficient ring $R \in \underline{\hat{Ar}}$ and a finite free R-module V_R endowed with a continuous G-action which is a deformation of V to R and such that, for all $A \in \underline{\hat{Ar}}$ and $(V_A, \iota_A) \in F_V(A)$, there is a unique coefficient-ring homomorphism $R \to A$ which induces an isomorphism between V_A and $V_R \otimes_R A$.

Both the G-module and the representation approaches will be used in the rest of the paper, together with their categorical descriptions. They will both provide useful descriptions of the deformation setting, according to the different applications.

1.4 Functors and representability

We now want to recall some properties of the functor $F_{\bar{\rho}}$ and the main representability criteria. We only deal with covariant functors over the category $\underline{\hat{Ar}}(A)$ or a subcategory of it, if not explicitely stated otherwise. When A is the residue field k, we simply denote the category by $\underline{\hat{Ar}}$.

We have seen that any coefficient ring A may be written as

$$A = \lim_{\leftarrow} A/m_A^n, \tag{1.22}$$

where all the rings A/m_A^n are artinian, and we have that

$$Hom(R, A) = \lim_{\leftarrow} Hom(R, A/m_A^n). \tag{1.23}$$

These facts suggest the following definitions

Definition 1.4.1. *A functor $F : \underline{\hat{Ar}} \to \underline{Sets}$ is called continuous if it satisfies the property that*

$$F(A) = \lim_{\leftarrow} F(A/m_A^n) \tag{1.24}$$

for all $A \in \underline{\hat{Ar}}$.

A continuous functor is therefore uniquely determined by its restriction to $\underline{Ar}$.

Definition 1.4.2. *A functor $F : \underline{Ar} \to \underline{Sets}$ is called representable if there exists an object $R \in \hat{\underline{Ar}}$ such that $F(A) = Hom(R, A)$ for all $A \in \underline{Ar}$.*

To give some criteria for a functor to be representable we need to recall the definition of fiber product. Let $\underline{A}$ be a category. A *cartesian system* or *cartesian diagram* in $\underline{A}$ is a 5-uple (A, B, C, α, β), where A, B, C are objects in $\underline{A}$ and $\alpha : A \to C,\ \beta : B \to C$ morphisms in $\underline{A}$. Then the *fiber product* $A \times_C B$ of the cartesian system (A, B, C, α, β) is the set of couples $(a, b) \in A \times B$ such that $\alpha(a) = \beta(b)$. The fiber product comes with two natural projections π_A and π_B that make the diagram

$$\begin{CD} A \times_C B @>{\pi_B}>> B \\ @V{\pi_A}VV @VV{\beta}V \\ A @>{\alpha}>> C \end{CD}$$

commute.

Given a cartesian diagram

$$\begin{array}{ccc} A & & B \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & C & \end{array}$$

of objects and morphisms in $\underline{Ar}$, the fiber product $A \times_C B$ is an object of $\underline{Ar}$, too, because it is an artinian coefficient-subring of $A \times B$. This is not true for the bigger category $\hat{\underline{Ar}}$: for example, if we take $A = k[[x, y]],\ B = k,\ C = k[[x]]$, α be the map sending y to 0 and β be the inclusion, then the fiber product $A \times_C B$ is given by the subring $k \oplus yk[[x, y]]$, which is not noetherian.

Given a functor F and a cartesian diagram

$$\begin{array}{ccc} A & & B \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & C & \end{array}$$

in $\underline{Ar}$, we can naturally associate a map

$$h_F : F(A \times_C B) \to F(A) \times_{F(C)} F(B) \tag{1.25}$$

which is called the *Mayer-Vietoris map* of the functor F. Then we say that F satisfies the *Mayer-Vietoris property* if h_F is a bijection for all the cartesian diagrams

$$\begin{array}{ccc} A & & B \\ & \searrow^{\alpha} \quad \swarrow_{\beta} & \\ & C & \end{array}$$

in $\underline{Ar}$.

Lemma 1.4.3. *If the functor F is representable, then it satisfies the Mayer-Vietoris property.*

Proof. Obvious by the defintions. □

Now we need to define the tangent space of a functor. Let A be a coefficient ring and $\pi \in A$ a lift of the uniformizer of $W(k)$. We define the *Zariski cotangent space* of A as the k-vector space

$$t_A^* = m_A/(m_A^2, \pi) \tag{1.26}$$

and the *Zariski tangent space* of A is simply its k-dual

$$t_A = Hom_k(t_A^*, k) \tag{1.27}$$

Proposition 1.4.4. *There is a natural isomorphism of k-vector spaces*

$$t_A \simeq Hom_{W(k)}(A, k[\epsilon]) \tag{1.28}$$

where $k[\epsilon]$ with $\epsilon^2 = 0$ is the coefficient ring of dual numbers.

Proof. See [9] page 271. □

Definition 1.4.5. *Let F be a functor of artinian rings such that $F(k)$ is a singleton. We define the Zariski tangent space of F as the set*

$$t_F = F(k[\epsilon]). \tag{1.29}$$

Unfortunately we cannot guarantee that t_F has a natural structure of k-vector space. Anyway such a structure can be defined if the map h_F is a bijection for the diagram

$$\begin{array}{ccccc} k[\epsilon] & & & & k[\epsilon] \\ & \searrow^{\alpha} & & {}^{\alpha}\swarrow & \\ & & k & & \end{array}$$

where α is the natural projection. In this case we say that F satisfies the $(\mathbf{T}_k)$-hypothesis.

We can now give the main representability criteria

Theorem 1.4.6 (Grothendieck)**.** *Let $F : \underline{Ar} \to \underline{Sets}$ be a functor such that $F(k)$ is a singleton. Then F is representable if and only if it satisfies the Mayer-Vietoris property and the tangent space $F(k[\epsilon])$ is a finite dimensional k-vector space.*

This theorem is powerful, but checking the condition is too complicated in practical cases. Anyway we can cut drastically down the number of cartesian systems we have to check. We still need a definition.

Definition 1.4.7. *A morphism $\alpha : A \to C$ in $\underline{Ar}$ is said to be small if its kernel is a principal ideal of A annihilated by m_A.*

Theorem 1.4.8 (Schlessinger)**.** *Let $F : \underline{Ar} \to \underline{Sets}$ be a functor such that $F(k)$ is a singleton. Then F is representable if and only if the following conditions hold:*

- *(S1) if $\alpha : A \to C$ is small, then h_F is surjective;*
- *(S2) if $A = k[\epsilon]$, $C = k$ and α is the natural projection, then h_F is bijective (this in particular implies the $\mathbf{T}_k$-hypothesis for F);*
- *(S3) $F(k[\epsilon])$ is a finite dimensional k-vector space;*
- *(S4) if $A = B$ and α, β are equal and small, then h_F is bijective.*

Unfortunately, the deformation functor we want to study is not always representable (it does not usually satisfies the S4 condition). Anyway we can express a sort of "weak" version of representability. We need a preliminary definition.

Definition 1.4.9. *A morphism of functors $\xi : F_1 \to F_2$ is said to be smooth if, for any surjective map $A_1 \to A_2$ in $\underline{Ar}$, any element $\rho_1 \in F_1(A_2)$ and any lifting of $\rho_2 = \xi(\rho_1) \in F_2(A_2)$ to an element $\rho_2^{'} \in F_2(A_1)$, there exist an element $\rho_1^{'} \in F_1(A_1)$, which is a lifting of ρ_1 and such that $\xi(\rho_1^{'}) = \rho_2^{'}$. This condition is equivalent to ask that the natural map*

$$F_1(A_1) \to F_1(A_2) \times_{F_2(A_2)} F_2(A_1) \tag{1.30}$$

is surjective for all sujections $A_1 \to A_2$ in $\underline{Ar}$.

We can now define the weak notion of representability. Let F_R denote the representable functor represented by R.

Definition 1.4.10. *Let F be a covariant functor. A representable hull for F is a pair (R, ξ) where R is a coefficient ring and $\xi : F_R \to F$ is a smooth morphism of functors that induces an isomorphism on the Zariski tangent spaces.*

Corollary 1.4.11. *Let $F : \underline{Ar} \to \underline{Sets}$ be a functor such that $F(k)$ is a singleton. If F satisfies the conditions S1, S2, S3, then it has a representable hull (R, ξ), which is unique up to isomorphism (generally noncanonical).*

We also want to see when representability passes to subfunctors. Let $G \subseteq F$ be covariant functors from $\underline{Ar}$ to $\underline{Sets}$ such that $G(k) = F(k)$ is a singleton.

We say that G is *relatively representable* with respect to F if, for all cartesian systems (A, B, C, α, β) in $\underline{Ar}$, then the system

$$\begin{array}{ccccc} F(A \times_C B) & & & & G(A) \times_{G(C)} G(B) \\ & \overset{h_F}{\searrow} & & \overset{\subseteq}{\swarrow} & \\ & & F(A) \times_{F(C)} F(B) & & \end{array}$$

has fiber product isomorphic to $G(A \times_C B)$.

Lemma 1.4.12. *If G is a relatively representable functor with respect to F, then G satisfies Schlessinger's conditions and the $\mathbf{T_k}$-hypothesis if and only if F does. Moreover if F is representable by a ring R_F then G is representable by a quotient ring R_G of R_F.*

See [11] for a proof of Schlessinger's criterion, the corollary and the lemma.

We now go back to our case of deformation functors.

Proposition 1.4.13. *Let $\bar{\rho} : G \to GL_n(k)$ be a residual representation. Then the deformation functor $F_{\bar{\rho}} : \underline{\hat{Ar}} \to \underline{Sets}$ is continuous.*

Proof. Let $A_j = A/m_A^j$. We want to show that the natural map

$$i : F_{\bar{\rho}}(A) \to \lim_{\leftarrow} F_{\bar{\rho}}(A_j) \tag{1.31}$$

is bijective. We use the interpretation of representations as modules. If $V_{\bar{\rho}}$ is the free k-vector space of dimension n whith the G-action given by $\bar{\rho}$, then $F_{\bar{\rho}}(A)$ is the set of isomorphism classes of pairs (V, α) where V is a free A-module of rank n and $\alpha : V \otimes_A k \to V_{\bar{\rho}}$ is an isomorphism. By the same notations with the subscript j, we can describe the set $\{F_{\bar{\rho}}(A_j)\}$; we can choose a cofinal system $\{V_j, \alpha_j\}$ of these sets, such that there exist isomorphisms

$$\beta_j : V_{j+1} \otimes_{A_{j+1}} A_j \to V_j \tag{1.32}$$

such that $(1 \otimes \pi_j) \circ \alpha_{j+1} = \alpha_j \circ (\beta_j \otimes \pi_j)$, where π_j is the natural projection. If we consider the projective limit (V, α) of the cofinal system (V_j, α_j) with respect to the β_js, then V is a free A-module of rank n and $\alpha : V \otimes_A k \to V_{\bar{\rho}}$ an isomorphism, therefore this object lies in the image of i. Then the map is surjective.

To prove injectivity, let (V, α), (V', α') be two elements of $F_{\bar{\rho}}(A)$ having the same image, then we have isomorphisms $\gamma_j : V_j \to V_j^{'}$ such that $\alpha_j^{'} \circ (\gamma_j \otimes 1) = \alpha_j$, then the previous argument shows that the projective limit of the γ_js gives an isomorphism between (V, α) and $(V', \alpha^{'})$. □

Because of this result we can study the representability of this functor simply by restricting to the artinian subcategory $\underline{Ar}$.

Now we can prove representability for deformation functors

Theorem 1.4.14. *Let $\bar{\rho} : G \to GL_n(k)$ be a residual representation and $F_{\bar{\rho}}$ the associated functor. Then $F_{\bar{\rho}}$ satisfies the conditions S1, S2 and S3 and has therefore a representable hull (R, ξ) and the ring R is called the versal deformation ring of $F_{\bar{\rho}}$. Moreover, if the centralizer of the image of $\bar{\rho}$ consists only of scalar matrices, then $F_{\bar{\rho}}$ is representable and R is called the universal deformation ring of $F_{\bar{\rho}}$.*

Proof. Let

$$\begin{array}{ccc} A_1 & & A_2 \\ & \searrow^{\alpha_1} \quad {}^{\alpha_2}\swarrow & \\ & A_0 & \end{array}$$

be a diagram in $\underline{Ar}$ and let A_3 be its fiber product. Let E_i be the set of liftings of $\bar{\rho}$ to A_i and ρ_i a generic element of E_i for $i = 0, 1, 2$ respectively. We denote by $\Gamma_n(A_i)$ the kernel of the reduction map

$$\Gamma_n(A_i) = Ker(GL_n(A_i) \to GL_n(k)). \tag{1.33}$$

Then clearly $F_{\bar{\rho}}(A_i) = E_i/\Gamma_n(A_i)$. Finally we denote by $C(\bar{\rho})$ the centralizer of the image of $\bar{\rho}$.

We start from property $S1$. If α_1 is small, we consider two liftings ρ_1, ρ_2 of ρ_0 to A_1, A_2 respectively. Then there exists a matrix $M \in \Gamma_n(A_0)$ which conjugate the images of ρ_1 and ρ_2. Since α_1 is surjective, then also $\Gamma_n(A_1) \to \Gamma_n(A_0)$ is; therefore we can lift M to an element $\tilde{M} \in \Gamma_n(A_1)$. It follows that ρ_2 and $\tilde{M}^{-1}\rho_1\tilde{M}$ are group homomorphisms with the same image in $GL_n(A_0)$ and therefore they define an element $\rho_3 = (\tilde{M}^{-1}\rho_1\tilde{M}, \rho_2) \in E_3$. Then the deformation class of ρ_3 maps to the pair of deformation classes (ρ_1, ρ_2) via h_F, hence the map is sufjective.

We analyze the injectivity of $h_{F_{\bar{\rho}}}$. For every lifting $\bar{\rho}_i \in E_i$ let

$$G_i(\rho_i) = \{g \in \Gamma_n(A_i) |\ gh = hg\ \forall h \in Im(\rho_i)\}. \tag{1.34}$$

We want to show that, if the natural map $G_1(\rho_1) \to G_0(\rho_0)$ is surjective, then h_F is injective. Let $\rho_3^{'}, \rho_3^{''} \in E_3$ and suppose that they map via h_F to the elements $(\rho_1^{'}, \rho_2^{'}), (\rho_1^{''}, \rho_2^{''})$ which represent the same deformation class. Then there are elements $M_i \in \Gamma_n(A_i)$ for $i = 1, 2$ such that $\rho_i^{'} = M_i^{-1}\rho_i^{''}M_i$. Mapping down to E_0 we obtain that $\rho_0^{'} = \bar{M}_1^{-1}\rho_i^{''}\bar{M}_1 = \bar{M}_2^{-1}\rho_i^{''}\bar{M}_2$, so that $\bar{M}_1\bar{M}_2^{-1}$ commutes with the image of ρ_0 and therefore lies in $G_0(\rho_0)$.

By surjectivity of the natural map, we can find a matrix $N \in G_1(\rho_1)$ which reduces to $\bar{M}_1\bar{M}_2^{-1}$. Let $N_1 = N^{-1}M_1$. Then we have

$$N_1^{-1}\rho_1^{''}N_1 = M_1^{-1}N\rho_1^{''}N^{-1}M_1 = M_1^{-1}\rho_1^{''}M_1 = \rho_1^{'} \tag{1.35}$$

On the other hand passing to the image of N_1 in $\Gamma_n(A_0)$, we obtain

$$\bar{N}_1 = (\bar{M}_1\bar{M}_2^{-1})^{-1}M_1 = \bar{M}_2. \tag{1.36}$$

Since M_2 and N_1 have the same image in $\Gamma_n(A_0)$, the couple (N_1, M_2) defines an element $M \in \Gamma_n(A_3)$ and $M^{-1}\rho_3^{''}M = \rho_3^{'}$. Then the two elements are equivalent and we have injectivity fo h_F.

Properties $S2$ and $S3$ are immediate to prove. In fact if $A_0 = k,\ A_1 = k[\epsilon]$ and α_1 is the natural projection, then property $S1$ implies that h_F is surjective. Moreover, if $A_0 = k$ then $\Gamma_n(A_0)$ and therefore $G_0(\rho_0)$ contain only the identity. It follows that the natural map between the G_is is surjective and hence h_F is injective, proving $S2$. $S3$ follows trivially from property ϕ_p, because the property implies that there are only finitely many maps from $Ker(\bar{\rho})$ to $\Gamma_n(k[\epsilon])$.

Finally we deal with $S4$; for this property we need the condition that $C(\bar{\rho}) = k$. We want to show that, in this case, all the $G_i(\rho_i)$ consist of scalar matrices, therefore implying the injectivity of h_F (the surjectivity will follow from $S1$). We will use a sort of "induction" argument. The claim is clearly true if $A_0 = k$; we need to show that, if $A_1 \to A_0$ is small and $C(\rho_0) = A_0$ then $C(\rho_1) = A_1$. Let then $M \in C(\rho_1)$, then its projection $\bar{M} \in C(\rho_0)$ must be a scalar matrix. Then we have $M = \bar{M} + tN$ where t is a generator of $ker(\alpha)$ and $N \in M_n(A_1)$. Now, since M commutes with the image of ρ_1, we have that, for every $g \in G$

$$M\rho_1(g) = \rho_1(g)M \ \longrightarrow (\bar{M} + tN)\rho_1(g) = \rho_1(g)(\bar{M} + tN) \tag{1.37}$$

and since $\bar{M}$ and t are just scalars and commute with everything, we have that

$$N\rho_1(g) = \rho_1(g)N. \tag{1.38}$$

Reducing modulo the maximal idela m_{A_1} and using the fact that $C(\bar{\rho}) = k$, we have that $M = s + M_1$, where s is a scalar and M_1 has entries in m_{A_1}. But since α is a small map, we have that $tm_{A_1} = 0$. It follows that M must be itself a scalar and we are done.

Therefore we can apply Schlessinger's theorem and obtain the (versal or universal) deformation ring $R = R(\bar{\rho}) \in \underline{\hat{Ar}}$. As for the universal representation, we can consider a lifting

$$\rho_n : G \to GL_n(R/m_R^n) \tag{1.39}$$

and create a compatible family of these lifting. The universal deformation ρ_{univ} is simply the inverse limit of this family. □

The condition of being absolutely irreducible is too strong to be satisfied in the main interesting cases. Luckily it can be relaxed with the "trivial centralizer" condition.

Definition 1.4.15. *We say that $\bar{\rho}$ satisfies the trivial centralizer condition if*

$$End_{k[G]}(V(\bar{\rho})) = k, \tag{1.40}$$

that is, the centralizer of the image of $\bar{\rho}$ is given by the set of scalar matrices.

Proposition 1.4.16. *Let $\bar{\rho}$ a Galois representation satisfying the trivial centralizer condition. Then the functor $F_{\bar{\rho}}$ is representable.*

Proof. By the hypothesis and since $M_n(k)$ is finite dimensional, we can choose a finite set of element $g_1, ..., g_r$ in G such that the centralizer of this set is the set of scalar matrices. Let M_i be a lifting of $\bar{\rho}(g_i)$ to $M_n(W(k))$. If $A \in \underline{Ar}$ then let $M_n^0(A)$ be the ring $M_n(A)$ modulo the scalar matrices, which is a free A-module of rank n^2-1. By Nakayama's lemma, we have that the map $M_n^0(A) \to M_n(A)^r$ given by $M \mapsto MM_i - M_iM$ is injective and we take π_A to be a splitting of this injection. We consider the composite map

$$F_{\bar{\rho}}(A) \to M_n(A)^r \to M_n^0(A) \tag{1.41}$$

obtained sending a lift ρ to the matrices $\rho(g_i)$ $i = 1, \dots, r$ and composing with π_A. Since $M_n(^0(A) \simeq M_n^0(W(k)) \otimes_{W(k)} A$ and $\pi_A = \pi_{W(k)} \otimes id_A$, we say that ρ is *well placed* if its image via this map is given by $\pi_{W(k)}(M_1, \dots, M_r) \otimes 1$.

Now we use a lemma due to Faltings (the proof can be found in [5, Lemma 7.3]) which states that, for every $\rho \in F_{\bar{\rho}}(A)$, there exists a matrix $M \in GL_n(A)$ such that $M\rho M^{-1}$ is well placed and M is uniquely determined modulo $1+m_A$. We apply the lemma with $A = \tilde{R}$ the versal deformation ring of $\bar{\rho}$ and $\rho = \rho_{vers}$ the attached versal deformation. Then we obtain an attached well placed deformation ρ_0 and let R_0 be the smallest subalgebra of $\tilde{R}$ containing all the entries of $\rho(g)$ for all $g \in G$. Proving that R_0 is the universal ring of ρ is a straightforward computation. □

We want to give an example of representation satisfying the trivial centralizer condition but which is not absolutely irreducible. Consider a residual representation $\bar{\rho}$ of the form

$$\begin{pmatrix} \eta_1(g) & u(g) \\ 0 & \eta_2(g) \end{pmatrix} \tag{1.42}$$

which is not semisimple and such that at least one of the characters η_1, η_2 is nontrivial. The representation is clearly not absolutely irreducible, but its centralizer is trivial. See [9, pag.264] for details.

1.5 Framed deformations

Even if we limit to consider the trivial centralizer condition as our main condition of representability, there are a lot of fundamental representations which do not satisfy it. It happens very often to deal with Galois representations which can have only a versal deformation ring. To avoid this problem, we introduce a variant of the deformation functor which will be always representable.

Definition 1.5.1. *Let β be a fixed k-basis of V. A framed deformation of the couple (V, β) to a coefficient ring $A \in \underline{\hat{Ar}}$ is a triple (V_A, ι_A, β_A), where (V_A, ι_A) is a deformation of V to A and β_A is a basis of V_A lifting β (that is $\iota_A(\beta_A) = \beta$). In the language of homomorphisms, a framed deformation of $\bar{\rho}$ is a lifting ρ to $GL_2(A)$ (we do not require the equivalence under conjugation by the elements of the kernel). Therefore we define the framed deformation functor*

$$F_V^{\square} : \underline{\hat{Ar}} \to \underline{Sets} \tag{1.43}$$

to be the functor associating to a coefficient ring A the set of framed deformations of (V, β) to A.

The fundamental property of the framed deformation functor is that it is always representable

Theorem 1.5.2. *The framed deformation functor is representable by a complete noetherian $W(k)$-algebra $R^{\square}(\bar{\rho})$.*

Proof. Suppose first that G is a finite group and let

$$\langle g_1, ..., g_s | r_1(g_1, ..., g_s), ..., r_t(g_1, ..., g_s) \rangle \tag{1.44}$$

be a presentation. We define the ring

$$\tilde{R} = W(k)[X_{i,j}^k | i, j = 1, ..., n; k = 1, ..., s]/I \tag{1.45}$$

where I is the ideal generated by the elements of the matrices $r_l(X^1, ..., X^s) - id$ for $l = 1, ..., t$. Let J be the kernel of the map $\tilde{R} \to k$ which sends X^k to $\bar{\rho}(g_k)$ elementwise. We define $R^{\square}(\bar{\rho})$ to be the J-adic completion of $\tilde{R}$. It satisfies the condition of universal deformation ring practically by definition.

Suppose now G to be topologically finitely generated and write it as inverse limit $\lim G/H_n$ of finite groups. Let $g_1, ..., g_s$ be a set of topological generators of G and use their projections to obtain presentations for all the quotients G/H_n. Then, using the previous construction, we obtain rings $R_n^{\square}(\bar{\rho})$, which form an inverse system by the universal property. Then the inverse limit $R^{\square}(\bar{\rho})$ of the system is the desired ring. Note that noetherianity of this ring follows from the fact that the chosen generators are always the same.

If G is any profinite group, let $\tilde{G} = Ker(\bar{\rho})$. If ρ_A is a lifting of $\bar{\rho}$, then it factors through the kernel of the natural projection $GL_n(A) \to GL_n(k)$, which is a pro-p group. Let H be the normal closed subgroup of $\tilde{G}$ such that $\tilde{G}/H$ is the maximal pro-p quotient. By the defining property, it follows that H is also normal in G. The finiteness property Φ_p for G implies that $\tilde{G}/H$ is topologically finitely generated, therefore so is G/H. This enables to reduce to the previous case. $\square$

1.6 Tangent space and deformation conditions

We start this section by giving two other descriptions of the Zariski tangent space of a deformation functor $F_{\bar{\rho}}$, that will prove to be very useful.

Let $V_{\bar{\rho}}$ be the G-module associated to $\bar{\rho}$ and let $End(V_{\bar{\rho}})$ be the k-vector space of linear endomorphisms on $V_{\bar{\rho}}$. Then $End(V_{\bar{\rho}})$ becomes itself a G-module with the induced action

$$g.M(v) = \bar{\rho}(g)M\bar{\rho}(g)^{-1}(v) \tag{1.46}$$

for every $g \in G,\ M \in End(V_{\bar{\rho}}),\ v \in V_{\bar{\rho}}$. Since this action is no other than the classical adjoint action composed with $\bar{\rho}$, we will denote the resulting G-module as $Ad(\bar{\rho})$.

We also recall the concept of extension; given two finite dimensional k-vector spaces V, W provided with continuous k-linear G-action, an *extension of V by W* is a $k[G]$-modules E, such that the sequence

$$0 \longrightarrow W \xrightarrow{\alpha} E \xrightarrow{\beta} V \longrightarrow 0$$

is exact, where α, β are $k[G]$-module homomorphism. Two extensions E, E' are equivalent if there exists a $k[G]$-isomorphism $\gamma : E \to E'$ which makes the following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & V & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{\gamma} & & \downarrow{=} & & \\
0 & \longrightarrow & W & \xrightarrow{\alpha'} & E' & \xrightarrow{\beta'} & V & \longrightarrow & 0.
\end{array}$$

The set of equivalence classes of extensions is denoted by $Ext^1_{k[G]}(V, W)$. We say that an extension *splits* if it is equivalent to the trivial extension $V \oplus W$. Given two non-equivalent extensions E, E', we can define their *Baer sum* in the following way: let Γ be the fiber product of the cartesian diagram

$$\begin{array}{ccc}
E & & E' \\
 & {}_{\beta}\searrow \quad \swarrow_{\beta'} & \\
 & V &
\end{array}$$

and define an equivalence relation on Γ posing $(v + e, e') \simeq (e, v + e')$. The quotient $Y = \Gamma/\simeq$ is the Baer sum of E and E'; it is again an extension of V by W and the set $Ext^1_{k[G]}(V, W)$ is an abelian group with respect to this operation, with identity element given by the trivial extension $V \oplus W$. Since each extension is in particular a k-module, $Ext^1_{k[G]}(V, W)$ is also a k-vector space. See [16, Ch. 3.4], for a more detailed description of extensions and the Baer sum.

Theorem 1.6.1. *: The following k-vector spaces are isomorphic:*

- $Ext^1_{k[G]}(V_{\bar{\rho}}, V_{\bar{\rho}})$;
- $H^1(G, Ad(\bar{\rho}))$;
- $F_{\bar{\rho}}(k[\epsilon])$.

Proof. : We want to exibit a one-to-one correspondence among the three spaces. We start by considering an extension

$$0 \longrightarrow V_{\bar{\rho}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\bar{\rho}} \longrightarrow 0.$$

We consider a k-vector space homomorphism $\phi : V_{\bar{\rho}} \to E$ such that $\beta \circ \phi = id$; note that ϕ is generally not a $k[G]$-homomorphism itself because it does not preserve the G-action. Since β is a $k[G]$-homomorphism, the expression $\bar{\rho}(g)\phi\bar{\rho}(g)^{-1}(v) - \phi(v)$ lies in $Ker(\beta) = Im(\alpha)$ for every $g \in G$ and every $v \in V_{\bar{\rho}}$. Then, for every g, we can define the k-linear map

$$T_g \in End(V_{\bar{\rho}}), \quad T_g(v) = \alpha^{-1}(g\phi g^{-1}(v) - \phi(v)). \tag{1.47}$$

We want to show that the map $T : g \mapsto T_g$ is a cocycle. Writing explicitely, we have

$$\begin{aligned} T_{g_1 g_2} &= \alpha^{-1}(g_1 g_2 \phi {g_2}^{-1} {g_1}^{-1} - \phi) = \\ &= \alpha^{-1}(g_1 \phi {g_1}^{-1} - \phi) + \alpha^{-1}(g_1 g_2 \phi {g_2}^{-1} {g_1}^{-1} - g_1 \phi {g_1}^{-1}) = \\ &= T_{g_1} + g_1 T_{g_2}, \end{aligned} \tag{1.48}$$

where we have used that α is a $k[G]$-homomorphism and that the G-action on $End(V_{\bar{\rho}})$ is given by the $\bar{\rho}$-adjoint action. Therefore the correspondance $E \mapsto T$ gives a set map ξ between $Ext^1_{k[G]}(V_{\bar{\rho}, V_{\bar{\rho}}})$ and $H^1(G, Ad(\bar{\rho}))$.

First we need to show that ξ is well-defined. If E_1, E_2 are equivalent extension, γ a $k[G]$-isomorphism between them and ϕ_1, ϕ_2 the k-linear maps which are right-inverse of β_1, β_2 respectively. Then, using the fact that $\alpha_2 = \gamma\alpha_1$ we have

$$\begin{aligned} (T_2)_g - (T_1)_g &= \alpha_2^{-1}(g\phi_2 g^{-1} - \phi_2) - \alpha_1^{-1}(g\phi_1 g^{-1} - \phi_1) = \\ &= \alpha_1^{-1}\gamma^{-1}(g\phi_2 g^{-1} - \phi_2 - \gamma g \phi_1 g^{-1} + \gamma\phi_1) = \\ &= g\psi - \psi, \end{aligned} \tag{1.49}$$

where $\psi = \alpha_1^{-1}\gamma^{-1}(\phi_2 - \gamma\phi_1) \in Ad(\bar{\rho})$. Therefore the difference $T_2 - T_1$ is a coboundary and so ξ sends equivalent extensions in the same cohomology class.

Let us show the injectivity of ξ. If E_1, E_2 are extensions such that the respective images T_1, T_2 are the same class, then there exists a map $\psi \in Ad(\bar{\rho})$

such that $(T_2)_g - (T_1)_g = g\psi - \psi$. Let $e_1 \in E_1$; the element can be written uniquely as $e_1 = \alpha_1(v)+\phi_1(v')$ for some $v, v' \in V_{\bar{\rho}}$. We define a map $\gamma : E_1 \to E_2$ as $\gamma(e_1) = \alpha_2(v)+\phi_2(v')-\alpha_2(\psi(v'))$. Then it is immediate to see that $\alpha_2 = \gamma\alpha_1$ and that $\beta_1 = \beta_2\gamma$, therefore γ is an isomorphism between the extesions E_1 and E_2.

Finally we prove surjectivity. Let $g \mapsto C_g$ a cocycle and $E = V_{\bar{\rho}} \oplus \epsilon V_{\bar{\rho}} = V_{\bar{\rho}} \otimes_k k[\epsilon]$. We can look at $\bar{\rho}(g)$ as an element of $GL_n(k[\epsilon])$ via the natural inclusion. Therefore we define

$$\rho(g) = (Id + \epsilon C_g)\bar{\rho}(g); \tag{1.50}$$

ρ gives an action of G on E. Then we have

$$0 \longrightarrow V_{\bar{\rho}} \xrightarrow{\epsilon} E \xrightarrow{\beta} V_{\bar{\rho}} \longrightarrow 0$$

as an extension of $k[G]$-modules. Let ϕ be again the right-inverse of β. Then, building the map T associated to E according to the previous formula, we have

$$T_g(v) = \epsilon^{-1}((Id + \epsilon C_g)\bar{\rho}(g)\phi\bar{\rho}(g)^{-1}(v) - \phi(v)) = C_g(v) \tag{1.51}$$

and therefore the cocycle C_g lies in the image of ξ, which is surjective. It is immediate to see that ξ is also k-linear and therefore it is an isomorphism of vector spaces.

Let us consider now $F_{\bar{\rho}}(k[\epsilon])$. The representation ρ described above is clearly an element of the tangent space. Then we can consider the map which sends a cocycle C to the deformation class of $\rho(g) = (Id + \epsilon C_g)\bar{\rho}(g)$. Conversely, given a deformation ρ of $\bar{\rho}$ to $k[\epsilon]$, we can define a cocycle C_g by the formula $Id + \epsilon C_g = \rho(g)\bar{\rho}(g)$. The identity

$$(Id + \epsilon A)(Id + \epsilon C)(Id - \epsilon A) = (Id + \epsilon(A - \rho A\rho^{-1} + C))\rho \tag{1.52}$$

shows that the equivalence of deformations corresponds to equivalence of cohomology class. Therefore the theorem is proved.

□

Passing to the framed case, we can obtain the framed tangent space from the unframed one

Lemma 1.6.2. *The tangent space $F^{\square}_{\bar{\rho}}(k[\epsilon])$ fits the exact sequence of k-vector spaces*

$$0 \to Ad(\bar{\rho})/Ad(\bar{\rho})^G \to F^{\square}_{\bar{\rho}}(k[\epsilon]) \to F_{\bar{\rho}}(k[\epsilon]) \to 0. \tag{1.53}$$

In particular its dimension is finite and

$$dim_k F^{\square}_{\bar{\rho}}(k[\epsilon]) = dim_k F_{\bar{\rho}}(k[\epsilon]) + n^2 - dim_k Ad(\bar{\rho})^G. \tag{1.54}$$

Proof. Let $V_1 \in F_{\bar{\rho}}(k[\epsilon])$ and β a fixed basis of $V_{\bar{\rho}}$, then the set of $k[\epsilon]$-bases of V_1 lifting β is a k-vector space of dimension n^2. Let $\beta^{'}, \beta^{''}$ be two such bases. There is an isomorphism of framed deformations

$$(V_1, \beta^{'}) \simeq (V_1, \beta^{''}) \tag{1.55}$$

if and only if there is an automorphism of V_1 which is the identity mod ϵ and sends $\beta^{'}$ to $\beta^{''}$. This happens if and only if the fibers of the natural map

$$F^{\square}_{\bar{\rho}}(k[\epsilon]) \to F_{\bar{\rho}}(k[\epsilon]) \tag{1.56}$$

are $Ad(\bar{\rho})/Ad(\bar{\rho})^G$-torsors. This proves the lemma. □

Now that we have these new descriptions of the tangent space, we consider some particular types of deformations. One often studies deformation problems which are restricted by some conditions. We want to discuss the general form of these conditions and define them in a categorical way.

Let $\bar{\rho} : G \to GL_n(k)$ be a residual representation and let $\underline{F_n} = \underline{F_n(k, G)}$ be the category of pairs (A, V) where A is a coefficient ring and V is an A-module of rank n provided with a A-linear continuous G-action. A morphism in $\underline{F_n}$ is given by a pair of morphisms $A \to A'$ (of coefficient rings) and $V \to V'$ (of A-modules) inducing an isomorphism $V \otimes_A A' \simeq V'$ which is compatible with the G-action.

Definition 1.6.3. *A deformation condition $\underline{D}$ for $\bar{\rho}$ is a full subcategory of $\underline{F_n}$ which contains $(k, V_{\bar{\rho}})$ and satisfies the following properties:*

- *for any morphism $(A, V) \to (A', V')$ in $\underline{F_n}$, if (A, V) is in $\underline{D}$ then (A', V') is in $\underline{D}$;*
- *for any diagram*

$$A \xrightarrow{\alpha} C \xleftarrow{\beta} B$$

  in $\underline{Ar}$, then $(A \times_C B, V)$ is in $\underline{D}$ if and only if both (A, V_A) and (B, V_B) are, where V_A and V_B are the tensor products of V with respect to the natural projections of the fiber product;
- *for any morphism $(A, V) \to (A', V')$ in $\underline{F_n}$, if (A', V') is in $\underline{D}$ and $A \to A'$ is injective, then (A, V) is in $\underline{D}$.*

Given a deformation condition $\underline{D}$ for $\bar{\rho}$ and a lifting $\rho : G \to GL_n(A)$, we say that ρ is of type $\underline{D}$ and its deformation class is of type $\underline{D}$ if (A, V_ρ) is in $\underline{D}$. Therefore we can define a functor

$$F_{\underline{D}, \bar{\rho}} : \underline{Ar} \to \underline{Sets} \tag{1.57}$$

which is a subfunctor of $F_{\bar{\rho}}$ and sends an artinian coefficient ring A to the set of deformation classes of $\bar{\rho}$ to A which are of type $\underline{D}$. The functor can be naturally extended to $\underline{\hat{Ar}}$ by continuity.

Proposition 1.6.4. *If $\underline{D}$ is a deformation condition for $\bar{\rho}$, then the functor $F_{\underline{D},\bar{\rho}}$ is relatively representable with respect to $F_{\bar{\rho}}$. Therefore it satisfies the conditions S1, S2, S3 of Schlessinger's theorem and has a representable hull. If $\bar{\rho}$ is absolutely irreducible, then $F_{\underline{D},\bar{\rho}}$ is representable by a quotient ring of the universal ring representing $F_{\bar{\rho}}$.*

Proof. See [9, pag.290]. □

Since $F_{\underline{D},\bar{\rho}}$ is relatively representable, we may speak of the tangent space $F_{\underline{D},\bar{\rho}}(k[\epsilon])$, which is necessarily a vector subspace of $F_{\bar{\rho}}(k[\epsilon])$. We will also use the notations $H^1_{\underline{D}}(G, Ad(\bar{\rho}))$ and $Ext^1_{\underline{D}}(V, V)$ to denote the tangent space using the identifications given by k-linear isomorphisms in Theorem 1.6.1. A complete description of this tangent space attached to conditions is one of our main tasks.

Now we will give some examples of deformation conditions. Let $\bar{\rho}$ be a residual representation and χ its determinant. We consider the deformation classes of $\bar{\rho}$ having as determinant a lifting of χ to the appropriate coefficient ring (In the application χ will most often be the cyclotomic character). The subcategory $\underline{D}$ of $\underline{F_n}$ of pairs (A, V_ρ) where ρ is a deformation of $\bar{\rho}$ to A with determinant given by a lifting of χ to A is called the *fixed determinant* condition.

Proposition 1.6.5. *$\underline{D}$ is a deformation condition. Moreover, if $Ad^0(\bar{\rho}) \subseteq Ad(\bar{\rho})$ is the vector subspace of endomorphisms whose trace is zero, with the restriction of the $\bar{\rho}$-adjoint action on $Ad(\bar{\rho})$ and*

$$H^1(G, Ad^0(\bar{\rho}))' = Im(H^1(G, Ad^0(\bar{\rho})) \to H^1(G, Ad(\bar{\rho}))), \tag{1.58}$$

then we have

$$H^1_{\underline{D}}(G, Ad(\bar{\rho})) = H^1(G, Ad^0(\bar{\rho}))'. \tag{1.59}$$

The main deformation conditions are usually the ones arising from categorical restraints. We call $\underline{Rep_k(G)}$ the category of finite dimensional k-vector spaces provided with a continuous linear G-action. Let $\underline{P}$ be a full subcategory of $\underline{Rep_k(G)}$, which is closed by subobjects, quotients and direct sums. Then we can define a deformation condition starting from $\underline{P}$ by the following.

Definition 1.6.6. *A Ramakrishna's sucategory is a subcategory $\underline{P}$ of $\underline{Rep_k(G)}$ closed under formation of subobjects, quotients and direct sums.*

Proposition 1.6.7. *Let $\underline{P}$ be a Ramakrishna's subcategory. Let $\underline{D}$ be the subcategory of $\underline{F_n}$ given by the objects (A, V) of $\underline{F_n}$ such that V lies in $\underline{P}$. Then $\underline{D}$ is a deformation condition called a Ramakrishna's deformation condition.*

The main example of these Ramakrishna's deformation conditions is that of being "finite flat", that is, we ask the representation spaces of our deformations to be the generic fiber of a finite flat group scheme over $Spec(\mathbb{Z}_p)$. We will study this deformation condition in details in the next chapters.

Chapter 2

p-divisible groups and elliptic curves

2.1 Finite flat group schemes

In this section we list the main properties and definitions about finite flat group schemes. We drop the prooves, for which we refer to [13].

Definition 2.1.1. *Let $\underline{A}$ be a category. A group-object in $\underline{A}$, or a $\underline{A}$-group, is an object $G \in \underline{A}$ with a morphism $m : G \times G \to G$ such that the induced law $G(T) \times G(T) \to G(T)$ makes $G(T)$ a group for every element $T \in \underline{A}$. G is said to be commutative if $G(T)$ is an abelian group for all T. A homomorphism of group objects is a morphism $G \to G^{'}$ of objects in $\underline{A}$ such that $G(T) \to G^{'}(T)$ is a morphism of groups.*

Definition 2.1.2. *Let S be a base scheme and $\underline{Sch}_S$ be the category of schemes over S. A group scheme over S is a group object in the category $\underline{Sch}_S$. We denote the category of group schemes over S as $\underline{Gr}_S$.*

If S is affine, say $S = Spec(R)$, we may replace S with R in the notations. We will also omit S or R in the notation, when the contest makes it clear. Let $G = Spec(A)$ be an affine scheme over R, where A is an R-algebra. Then giving a structure of group schem over G is equivalent to give a structure of Hopf algebra over A, which is given by R-algebra homomorphisms

- $\tilde{m} : A \to A \otimes_R A$, called the comultiplication,
- $\tilde{\epsilon} : A \to R$ called the counit, or augmentation
- $\tilde{s} : A \to A$ called the coinverse, or the antipod.

In particular $Ker(\tilde{\epsilon})$ is an ideal I_G of A called the augmentation ideal of G

EXAMPLES:

1. Let $G_a = Spec(R[x])$ with x an indeterminate. We give G_a a structure of group scheme via the operations on $R[x]$
   - $\tilde{m}(x) = x \otimes 1 + 1 \otimes x$;
   - $\tilde{\epsilon}(x) = 0$;
   - $\tilde{s}(x) = -x$.

   G_a is called the additive group of R.

2. Let $G_m = Spec(R[x, x^{-1}])$. We give it a structure of group scheme via the operations
   - $\tilde{m}(x) = x \otimes x$;
   - $\tilde{\epsilon}(x) = 1$;
   - $\tilde{s}(x) = x^{-1}$.

   G_m is called the multiplicative group scheme over R

3. Let X be an abelian group and $R[X]$ be the group algebra of X over R. Then $G = Spec(R[X])$ is a group scheme with the same operations given in the previous case. Note that if $X = \mathbb{Z}$ we obtain G_m again and that if $X = \mathbb{Z}/n\mathbb{Z}$ we obtain the group scheme μ_n of the n-th roots of unity.

4. Let X be a group and X_S the disjoint union of copies of S indexed by X. Then $X_S(T)$ is identified with the set of locally constant functions $\phi : T \to X$. In particular, if T is nonempty, $X_S(T) = X$. It is called the constant group scheme associated to X and it is affine if and only if $S = Spec(R)$ and X is finite, in which case $X = Spec(Map(X, R))$.

Let S be alocally noetherian scheme. An S-scheme X is finite flat if and only if the sheaf O_X is locally free of finite rank, if and only if there exists a covering of S by affine open sets U such tath the morphisms $X|_U \to U$ are of the form $Spec(A) \to Spec(R)$ with A free of finte rank over R. This rank is a locally constant function on S called the order of X.

All the group scheme we will consider from now on will be finite flat.

Definition 2.1.3. *A group scheme Y over S is called etale if it is finite flat and, for each point $s \in S$, the fiber $Y_s = Y \times_S s$ is the spectrum of a separable algebra over the residue field of s.*

Proposition 2.1.4. *Let G_0 be the connected component of G containing the identity. Then G_0 is the spectrum of a henselian local R-algebra with the same residue field as R and it is a closed flat normal subgroup scheme of G such that the quotient $G_{et} = G/G_0$ is etale. Therefore we have an exact sequence*

$$0 \to G_0 \to G \to G_{et} \to 0, \tag{2.1}$$

which is called the connected-etale sequence for G. In particular G is connected if $G = G_0$ and $G_{et} = 0$, and G is etale if $G = G_{et}$ and $G_0 = 0$.

Given an R-module M, we denote by $M^* = Hom_{R-mod}(M, R)$. Consider then the dual Hopf algebra $A^* = Hom_{R-mod}(A, R)$; the operations obtained dualizing $\tilde{m}, \epsilon$ and s respectively turn A^* into a cocommutative Hopf algebra.

EXAMPLE: Let G be a constant group scheme over R associated with a finite group H. Then A is the ring of R-valued functions on H and A^* is the group algebra $R[H]$ of H over R. The pairing between them is given by

$$\langle \sum_{x \in H} r_x x, f \rangle = \sum_{x \in H} f_x f(x). \tag{2.2}$$

For a general G it is not true that A^* is the group algebra of $G(R)$, but we have the inclusion

$$G(R) = Hom_{R-alg}(A, R) \subseteq Hom_{R-mod}(A, R) = A^*, \tag{2.3}$$

which identifies $G(R)$ with the subgroup of the group-like elements of A^*, that is, the invertible elements $\lambda \in A^*$ such that $m(\lambda) = \lambda \otimes \lambda$.

Suppose now that the group scheme G is commutative. Then A^* is commutative and we can consider $G^* = Spec(A^*)$ as a finite flat commutative group scheme over R of the same order as G. G^* is called the *Cartier dual* of G and the functor sending G to its dual is an antiequivalence of the category of group schemes over R with itself.

EXAMPLE: Let G be a constant group scheme. Then its Cartier dual is given by the associated diagonalizable group scheme $D(G)$. In particular the dual of the constant group scheme associated to $\mathbb{Z}/n\mathbb{Z}$ is given by the group scheme μ_n of the n-th roots of unity.

2.2 p-divisible groups

Before following with the definition of our deformation functor, we want to recall the main definitions about p-divisible groups and some related statements. The main references for this part are [12],[14].

Definition 2.2.1. *Let p be a prime number, R a noetherian, integrally closed domain with fraction field K of characteristic 0. A finite flat group scheme over R is a group scheme $\mathfrak{G}$ which is finite flat over R. In particular it is affine. Let A be its coordinate ring. The rank of $\mathfrak{G}$ is the rank of A as R-algebra. The group scheme structure on $\mathfrak{G}$ translates in a cocommutative Hopf algebra structure on A, given by a comultiplication $\mu : A \to A \otimes_R A$, a counit $\epsilon : A \to R$ and a coinverse, or antipod, $i : A \to A$, which satisfy properties which are duals of properties of group operation on $\mathfrak{G}$.*

EXAMPLES:

- Let Γ be an abstract group and $A = Maps(\Gamma, R)$. The operations of comultiplication $\mu(f)(s,t) = f(st)$, counit $\epsilon(f)(s) = 1$ and coinverse $i(f)(s) = f(s^{-1})$ give a structure of Hopf Algebra over R for A, and therefore a group scheme structure, over Γ.

- Let $A = R[x]/(x^m - 1)$ for m a positive integer. Taking $\mu(x) = x \otimes x$, $\epsilon(x) = 1$ and $i(x) = x^{-1}$, we have a Hopf Algebra structure over A which gives the finite flat group scheme μ_m of m-th roots of unity.

Definition 2.2.2. *Let h be a positive integer. A p-divisible group over R of height h is an inductive system:*

$$G = (G_n, i_n), \tag{2.4}$$

where G_n is a finite flat group scheme over R of order p^{nh} and $i_n : G_n \to G_{n+1}$ is an injective morphism such that the sequence

$$0 \to G_n \stackrel{i_n}{\to} G_{n+1} \stackrel{p^n}{\to} G_{n+1} \tag{2.5}$$

is exact. A homomorphism between two p-divisible groups (G_n, i_n) and $(H_n, i_n^{'})$ is a set $f = \{f_n\}$ of morphisms of group schemes $f_n : G_n \to H_n$ such that $i_n^{'} \circ f_n = f_{n+1} \circ i_n$ for all n.

In the definition we are requiring G_n to be the kernel of the multiplication by p^n in G_{n+1}. In the following we frequently denote by G the inductive limit of the system and simply refer to G as a p-divisible group.

By iteration, we obtain the morphisms $i_{n,m} : G_n \to G_{n+m}$, that identify G_n with the kernel of the multiplication by p^n in all of the G_{n+m}, for $m \geq 1$. It follows that the homomorphism $p^m \in End(G_{m+n})$ factors through G_m and therefore, for all m, n, we can obtain a unique homomorphism $j_{m,n} : G_{m+n} \to G_n$ such that the sequence

$$0 \to G_m \stackrel{i_{m,n}}{\to} G_{m+n} \stackrel{j_{m,n}}{\to} G_n \to 0 \tag{2.6}$$

is exact and such that $i_{n,m} \circ j_{m,n} = p^m$.

We also introduce the concept of *dimension* of a p-divisible group. In order to do this, we need some definitions.

Definition 2.2.3. *A n-dimensional commutative formal Lie group Γ over R is a homomorphism from $A = R[[x_1, ..., x_n]]$ to $A \hat{\otimes} A$ described by a family of power series $f(y,z) = (f_i(y,z))_{i=1,...,n}$, where f_i is the image of x_i, that satisfies the following properties:*

- $x = f(0,x) = f(x,0)$ *(identity);*

- $f(x, f(y,z)) = f(f(x,y),z)$ *(associativity);*

- $f(x, y) = f(y, x)$ *(commutativity).*

We write $x * y = f(x, y)$ and define $\psi(x) = x * x * ... * x$ p times. Then ψ is an endomorphism of A which corresponds to the multiplication by p in Γ. We say that Γ is p-divisible if ψ is an isogeny. In this case, taking Γ_{p^n} to be the kernel of the n-th iteratation of ψ on Γ, it is immediate to verify that $\Gamma(p) = (\Gamma_{p^n}, i_n)$ is a p-divisible group of height h, where $p^h = deg(\psi)$, and it is connected (every Γ_{p^n} is a connected group scheme).

Proposition 2.2.4. *The association $\Gamma \mapsto \Gamma(p)$ is an equivalence of categories between commutative p-divisible formal Lie groups and connected p-divisible groups.*

The proof can be found in [14, prop. 1, pag. 162]. Now Let G be any p-divisible group. Then, taking the connected components of G_n and passing to the limit, we can find an exact sequence

$$0 \to G^0 \to G \to G^{et} \to 0 \tag{2.7}$$

where G^0 is connected and G^{et} is etale. Then we can define the *dimension* of G to be the dimension of the formal Lie group associated to G^0 via the equivalence of categories.

EXAMPLES: 1) Let E be an elliptic curve over R. Then the system $E[p^\infty] = (E[p^n], i_n)$ (where i_n is the obvious morphism) is a p-divisible group of dimension 1 and height 2.

2) Let $\mathbb{G}_m$ be the classical multiplicative group over R, then $\mathbf{G}_m(p) = (\mu_p^n = \mathbf{G}_m[p^n], i_n)$ is a p-divisible group of dimension and height equal to 1 called the group of roots of unity in R.

By using Cartier duality on group schemes, we can define the notion of dual p-divisible group G^*. If $G = (G_n, i_n)$ then we define $G^* = (G_n^*, i_n^*)$ where G_n^* is the Cartier dual of G_n and i_n^* is the map obtained dualizing the map $j_{1,n}$ defined in formula 2.6. Clearly G and G^* have the same height h and it can be proved that the sum of their dimensions must be equal to h. It follows immediatly that $dim(E[p^\infty]^*) = 1$ and $dim(\mathbf{G}_m(p)^*) = 0$.

We want to define a Galois action, and therefore a Galois representation, on some modules which are canonically attached to p-divisible groups. Inspired by the example of elliptic curves, we take the module $G_n(\bar{K})$ of the $\bar{K}$-valued points of G_n and $G(\bar{K})$ to be their direct limit. All of these sets have a canonical Galois action of $\Gamma_K = Gal(\bar{K}/K)$. Then, dualizing the maps i_n and j_n, we have new maps

$$i_n^* : G_n(\bar{K}) \to G_{n+1}(\bar{K}), \qquad j_n^* : G_{n+1}(\bar{K}) \to G_n(\bar{K}) \tag{2.8}$$

and so we can define the two Γ_K-modules

$$\Phi(G) = \lim_{\rightarrow} G_n(\bar{K}), \qquad T(G) = \lim_{\leftarrow} G_n(\bar{K}), \tag{2.9}$$

where the limits are taken with respect to the maps i_n's and j_n's respectively. We usually refer to $\Phi(G)$ as the *p-torsion module* of G and to $T(G)$ as the *Tate module* of G. They are both $\mathbb{Z}_p$ modules with a continous Γ_K-action and $T(G)$ is free over $\mathbb{Z}_p$ of rank h

The knowledge of one of these modules is equivalent to the knowledge of the other. In fact it can be proved that there are canonical isomorphisms

$$\Phi(G) \simeq T(G) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p), \qquad T(G) \simeq Hom_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Phi(G)). \tag{2.10}$$

Now we want to state Tate's main theorem on p-divisible groups:

Theorem 2.2.5 (Tate). *Let R be an integrally closed noetherian domain, whose field of fraction K has characteristic 0 and let G, H be p-divisible groups over R. Then any homomorphism $f : G \otimes_R K \to H \otimes_R K$ of the generic fiber uniquely extends to a homomorphism $f : G \to H$. In particular the canonical map*

$$Hom(G, H) \to Hom_{\mathbb{Z}_p[\Gamma_K]}(T(G), T(H)) \tag{2.11}$$

is bijective, that is the functor from the category of p-divisible groups over R to the category of $\mathbb{Z}_p$-modules with continous Γ_K-action is fully faithful.

Proof. See [14, Thm. 4, pag. 180-181] □

We end this section by stating a result that is the main inspirations for the constructions in the next chapters

Theorem 2.2.6 (Schoof). *Let R be as in the previous theorem and $\underline{D}$ be a full subcategory of p-group schemes which is closed by products, closed flat subgroup schemes and quotients by finite flat subgroup schemes. Let $G = \{G_n\}$ be a p-divisible group over R and suppose that $S = End(G)$ is a discrete valuation ring with uniformizer π and quotient field k. Moreover suppose that:*

- *every group scheme G_n is an element of $\underline{D}$;*
- *the map*

$$\eta : Hom_R(G[\pi], G[\pi]) \to Ext^1_{\underline{D}}(G[\pi], G[\pi]) \tag{2.12}$$

  associated to the exact sequence $0 \to G[\pi] \to G[\pi^2] \to G[\pi] \to 0$ is an isomorphism of 1-dimensional k-vector spaces.

Let $H = \{H_n\}$ be a p-divisible group over R such that:

- *every group scheme H_n is an object of $\underline{D}$;*
- *each H_n admits a filtration with finite flat subgroup schemes and successive quotients isomorphic to $G[\pi]$.*

Then H is isomorphic to G^g, for some $g \in \mathbb{N}$.

We want to give a proof of this theorem throughout a series of lemmas.

Lemma 2.2.7. *: Let $R, \underline{D}, G, S, k$ be as in the theorem. Suppose that the hypotheses of the theorem are satisfied. Then:*

1. *for all positive integers $j_1, j_2 \geq 1$, the natural map*
$$\xi : S/\pi^{j_2} S[\pi^{j_1}] \to Hom_R(G[\pi^{j_2}], G[\pi^{j_1}]) \tag{2.13}$$
is an isomorphism;

2. *for every positive integer j, the k-vector space $Ext^1_{\underline{D}}(G[\pi], G[\pi^j])$ is 1-dimensional and generated by the extension*
$$0 \longrightarrow G[\pi^j] \longrightarrow G[\pi^{j+1}] \xrightarrow{\pi^j} G[\pi] \longrightarrow 0.$$

Proof. : We start from a preliminar observation; if $f \in S$ and it is zero over $G[p^j]$ for some $j \geq 0$, then the induced morphism $T_p f$ over the Tate module $T_p G$ given by Tate's theorem is contained in $p^j T_p G$; but $p^j T_p G$ is isomorphic to $T_p G$ as a Galois module. It follows that there exists a Galois equivariant homomorphism $\gamma \in End(T_p G)$ such that $T_p f = p^j \gamma$. Then Tate's theorem implies that γ is induced by a morphism $g \in S$, that is, $f = p^j g$.

Let us now prove part 1. Let $f \in Ker(\xi)$. Then f is zero over $G[\pi^{j_2}]$. Let $a \geq 0$ such that $\pi^{j_2+a} = up^b$ for some $b \geq 0, u \in S^*$. Then $f\pi^a$ is zero over $G[p^b]$. By the previous observation we have $f\pi^a = p^b g$ for some $g \in S$; then f is a multiple of π^{j_2} and f is a trivial element of $S/\pi^{j_2} S[\pi^{j_1}]$.

To show that ξ is also surjective it is sufficient to show that left and right side are finite k-vector space of the same dimension. The left side has obviously dimension $min(j_1, j_2)$. From multiplicativity of orders in exact sequences, it follows that

$$dim_k Hom_R(G[\pi^{j_2}], G[\pi^{j_1}]) \leq \begin{cases} j_1 dim_k Hom_R(G[\pi^{j_2}], G[\pi]) \\ j_2 dim_k Hom_R(G[\pi], G[\pi^{j_1}]). \end{cases} \tag{2.14}$$

Therefore it suffices to prove that both the dimensions on the right side are equal to 1 for every j. We will prove by induction that the natural map $Hom_R(G[\pi], G[\pi]) \to Hom_R(G[\pi], G[\pi^j])$ is an isomorphism for every j and that the natural map $Ext^1_\Delta(G[\pi], G[\pi^{j+1}]) \to Ext^1_\Delta(G[\pi], G[\pi^j])$ is injective for every j, concluding the proof of part 1 and 2 of the lemma. For $j = 1$ the result follows from the hypothesis.

Consider now the commutative diagram with exact columns

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
G[\pi] & \xrightarrow{=} & G[\pi] & \xrightarrow{=} & G[\pi] & \xrightarrow{=} & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
G[\pi^2] & \longrightarrow & G[\pi^3] & \longrightarrow & G[\pi^4] & \longrightarrow & \dots \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\
G[\pi] & \longrightarrow & G[\pi^2] & \longrightarrow & G[\pi^3] & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

We apply the functor $Hom_R(G[\pi], -)$ and form the associated Ext long exact sequence, obtaining the following diagram with exact columns

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Hom_R(G[\pi], G[\pi]) & \xrightarrow{=} & Hom_R(G[\pi], G[\pi]) & \xrightarrow{=} & Hom_R(G[\pi], G[\pi]) & \xrightarrow{=} & \dots \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle f_1} & & \downarrow & & \\
Hom_R(G[\pi], G[\pi^2]) & \xrightarrow{g_1} & Hom_R(G[\pi], G[\pi^3]) & \longrightarrow & Hom_R(G[\pi], G[\pi^4]) & \longrightarrow & \dots \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_2} & & \downarrow & & \\
Hom_R(G[\pi], G[\pi]) & \xrightarrow{g_2} & Hom_R(G[\pi], G[\pi]) & \xrightarrow{g_3} & Hom_R(G[\pi], G[\pi]) & \longrightarrow & \dots \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle f_3} & & \downarrow & & \\
Ext^1_\Delta(G[\pi], G[\pi]) & \xrightarrow{=} & Ext^1_\Delta(G[\pi], G[\pi]) & \xrightarrow{=} & Ext^1_\Delta(G[\pi], G[\pi]) & \xrightarrow{=} & \dots \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_4} & & \downarrow & & \\
Ext^1_\Delta(G[\pi], G[\pi^2]) & \longrightarrow & Ext^1_\Delta(G[\pi], G[\pi^3]) & \longrightarrow & Ext^1_\Delta(G[\pi], G[\pi^4]) & \longrightarrow & \dots \\
\downarrow & & \downarrow{\scriptstyle f_5} & & \downarrow & & \\
Ext^1_\Delta(G[\pi], G[\pi]) & \longrightarrow & Ext^1_\Delta(G[\pi], G[\pi^2]) & \longrightarrow & Ext^1_\Delta(G[\pi], G[\pi^3]) & \longrightarrow & \dots
\end{array}
$$

The exacteness of the first column gives the result for $j = 2$. Let us observe the second column; g_2 must be an isomorphism because it is the same map as the first map in the first column, therefore even f_3 is an isomorphism. It follows that f_2 and f_4 must be the zero map and f_1 is an isomorphism and, finally, that g_1 is an isomorphism as well and that f_5 is injective. This proves the case $j = 3$. Proceeding inductively in every column we have the result for every j. Then the lemma is proved. □

Corollary 2.2.8. *: Suppose that the conditions of the lemma are satisfied. Then every group scheme in* $\underline{D}$ *which admits a filtration with closed flat subgroup schemes and successive subquotients isomorphic to* $G[\pi]$ *is isomorphic to a group scheme of the form*

$$\bigoplus_{i=1}^{r} G[\pi^{n_i}]. \tag{2.15}$$

Proof. : Let J be such a group scheme. We proceed by induction on the length of the filtration of J. If the length is 1, then $J = G[\pi]$ and we are done. Suppose now that the result is true for length r, then we have an exact sequence

$$0 \longrightarrow \oplus_{i=1}^{r} G[\pi^{n_i}] \longrightarrow J \longrightarrow G[\pi] \longrightarrow 0$$

therefore the extension class of J lies in

$$Ext^1_{\underline{D}}(G[\pi], \bigoplus_{i=1}^{r} G[\pi^{n_i}]) \simeq \bigoplus_{i=1}^{r} Ext^1_{\underline{D}}(G[\pi], G[\pi^{n_i}]) \tag{2.16}$$

By part 2 of the lemma, this extension space has dimension r over k and it is generated by extensions of the form

$$G[\pi^{n_j+1}] \times (\bigoplus_{i \neq j} G[\pi^{n_i}]), \tag{2.17}$$

which have all the required shape. Then we just need to show that the Baer sum of extensions of the required shape still has the same shape. By the definition of Baer sum (see chapter 6), it suffices to show that kernels and cokernels of morphisms between extensions of the required shape still have the same shape. By duality, we only need to deal with kernels.

Let then $g : \oplus_{i=1}^{r} G[\pi^{n_i}] \to \oplus_{j=1}^{s} G[\pi^{n_j}]$ be a morphism and K its kernel. By part 1 of the lemma, g is induced by a collection of elements $f_{i,j} \in S$. Let F be the matrix having $f_{i,j}$ as (i, j)-th element and let Π be the diagonal matrix with elements $\pi^{n_1}, \ldots, \pi^{n_r}$. Then K is isomorphic to the kernel of the restriction of F to $\oplus_{i=1}^{r} G[\pi^{n_i}]$ and we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & & K_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \oplus_{i=1}^{r} G[\pi^{n_i}] & \longrightarrow & G^r & \xrightarrow{\Pi} & G^r & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F\times\Pi} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & G^s & \longrightarrow & G^s \times G^r & \longrightarrow & G^r & \longrightarrow & 0.
\end{array}$$

Following the diagrams it follows that $K \simeq K_1$, so we can examine this last set. Let A be the matrix of the homomorphism $F \times \Pi$. Since S is a principal ideal domain, we can find invertible matrices $B \in GL_r(S)$ and $B' \in GL_{r+s}(S)$ such that $B'AB$ takes the form

$$\begin{pmatrix} g_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & g_r \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \tag{2.18}$$

where $g_i \in S$. Therefore K is isomorphic to the kernel of the $r \times r$ upper submatrices, which is of the required shape. This proves the corollary. □

Proof. of the theorem: By the corollary, we have that each group scheme $H[p^n]$ is isomorphic to a group scheme of the form $\oplus_{i=1}^{r} G[\pi^{n_i}]$. Therefore the set of $\bar{\mathbb{Q}}$-points of $H[p^n]$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^g$, where $g = dim(H)$. It follows that every summand of $H[p^n](\bar{\mathbb{Q}})$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. We have

$$H[p^n] \simeq \bigoplus_{i=1}^{r} G[\pi^{en}] \simeq G^r[p^n] \tag{2.19}$$

where e is the ramification index of S over $\mathbb{Z}_p$ and $redim(G[\pi]) = g$. Since $Hom_R(H[p^n], G[p^n])$ is finite, we can find a cofinal system of isomorphisms and then we have a global isomorphism $H \simeq G^g$ as required. □

2.3 Motivating example

Now we want to apply the previous results to a concrete case that is the main motivating example for our work. We want to put us in the setting of the theorem 8.5. The examples we are considering are inspired by [12].

Let E be an elliptic curve over $\mathbb{Q}$ and let p and ℓ two distinct prime numbers such that E has good supersingular reduction at p and semistable reduction at ℓ. Then we can look at the Galois module of p-torsion points $E[p]$ as a finite flat group scheme over the ring $\mathbb{Z}[1/\ell]$.

We denote by $\underline{Gr}$ the category of finite flat group schemes over $\mathbb{Z}[1/\ell]$ of p-power order and let $\underline{D}$ be the subcategory of $\underline{Gr}$ of the group schemes G for which we have $(\sigma - id)^2 = 0$ on $G(\bar{\mathbb{Q}})$ for all σ in the inertia group of any prime over ℓ. For the rest of the chapter, all the extensions are to be considered over the ring $\mathbb{Z}[1/\ell]$, if not specified otherwise.

Proposition 2.3.1. *Let $E, p, \ell, \underline{D}$ be as above. Then the following properties hold:*

1. *Every group scheme $E[p^n]$ is an object of $\underline{D}$;*
2. *Constant and diagonalisable group schemes of p-power order are objects of $\underline{D}$;*
3. *$\underline{D}$ is closed by Cartier duals, direct products, closed flat subgroup schemes and quotients by closed flat subgroup schemes. In particular, given two objects G_1 and G_2 of $\underline{D}$, the set of extension classes in $Ext^1(G_1, G_2)$ that are still objects of $\underline{D}$ form the subset of the $\underline{D}$-extensions*

$$Ext^1_{\underline{D}}(G_1, G_2), \tag{2.20}$$

which is a subgroup of the group $Ext^1_{\mathbb{Z}[1/\ell]}(G_1, G_2)$ of all the extensions in Gr. This subgroup is generally proper;

Proof. The proof can be found in [12, pag. 3-4]. □

Observe in particular that the subcategory of representations which come from a finite flat group scheme in $\underline{D}$ satisfies the same stability properties as $\underline{D}$ expressed in 2 and therefore it is a Ramakhrishna's subcategory (see Definition 1.6.7 and Proposition 1.6.8). It follows that the condition of coming from a finite flat group scheme which lies in $\underline{D}$ is a deformation condition for a Galois representation

Now we want to describe an explicit example which satisfies the hypothesis of theorem 8.5. The details can be found in [12, section 7]. Let $p = 2$, $\ell = 11$ and consider the modular curve $X_0(11)$ given by the Weierstrass equation

$$Y^2 + Y = X^3 - X^2 - 10X - 20. \tag{2.21}$$

We take $E = J_0(11)$ the Jacobian of this modular curve; then E has semistable reduction at 11 and good reduction at all the other primes and, in particular, it has supersingular reduction at 2. Then the 2-group scheme $E[2]$ is an object of $\underline{D}$ and the associated residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \to Aut(E[2](\bar{\mathbb{Q}})) \tag{2.22}$$

is surjective. The points of $E[2]$ generate the field $K = \mathbb{Q}(\sqrt{11}, \alpha)$, where α is a root of the polynomial $x^3 + x^2 + x - 1$ and $E[2]$ is simple and self-dual in $\underline{D}$.

Proposition 2.3.2. *The simple objects in the category $\underline{D}$ are the group schemes $\mathbb{Z}/2\mathbb{Z}$, μ_2 and $E[2]$.*

Proof. The proof can be found in [12, Prop. 7.1]. □

Now we have to consider the extensions of $E[2]$ by itself. We have trivially that the group $E[4]$ of 4-torsion points of E is such an extension and belongs to $\underline{D}$ because of the semistability at 11.

Proposition 2.3.3. *The set* $Ext^1_{\underline{D}}(E[2], E[2])$ *is a 1-dimensional* $\mathbb{F}_2$*-vector space, generated by the extension* $E[4]$*. Moreover we have that the subspace* $Ext^1_{\underline{D},2}(E[2], E[2])$ *of the extensions which are killed by 2 is trivial.*

Proof. The complete proof can be found in [12, prop. 7.2]. It consists of two steps: first it is shown that the $Ext^1_{\underline{D}}(E[2], E[2])$ is generated by $E[4]$ and $Ext^1_{\underline{D},2}(E[2], E[2])$ and then that this last subspace is trivial. □

Then we have found an explicit example which satisfies all the hypotheses of theorem 8.5. The 1-dimensionality of the extension module and the triviality of the annihilated-by-p submodule will be some of the main ingredients in the proof of the main theorem of the next chapter.

2.4 The Main results: elliptic curve case

Let E be an elliptic curve over $\mathbb{Q}$ with good supersingular reduction at the prime p and semistable reduction at the prime ℓ. Let $\underline{D}$ be the subcategory of p-power order $\mathbb{Z}[1/\ell]$-group schemes such that $(\sigma - id)^2 = 0$ on the set of $\bar{\mathbb{Q}}$-points for every σ in the inertia group of ℓ and let Δ be the Ramakrishna's categorical deformation condition attached to $\underline{D}$.

Let $G = G_{\mathbb{Q},S}$, where $S = \{p, \ell, \infty\}$ and let $\bar{\rho}$ be the representation of G given by the natural action on the p-torsion points of E. We consider the deformation functor $F_{\bar{\rho},S}$ which sends a coefficient ring A to the set of deformations ρ of $\bar{\rho}$ to A, which satisfy the following local conditions at S:

- ρ is odd: $det(\rho(c)) = -1$ for c any complex conjugation;
- ρ is flat at p;
- $(\rho(g) - id)^2 = 0$ for every $g \in I_\ell$.

Theorem 2.4.1 (Main Theorem: elliptic curve case)**.** *Suppose that the extension group* $Ext^1_{\underline{D}}(E[p], E[p])$ *is 1-dimensional over* $\mathbb{F}_p$ *and generated by the extension* $E[p^2]$*; in particular the subgroup* $Ext^1_{\underline{D},p}(E[p], E[p])$ *of the extensions which are killed by* p *is trivial. Then the funtor* $F_{\bar{\rho},S}$ *associated to the previous data is representable. Its universal deformation ring is isomorphic to* $\mathbb{Z}_p$*.*

Proof. : We start by examining the tangent space. We used the interpretation of the tangent space as extensions, that is,

$$F_{\bar{\rho},S}(\mathbb{F}_p[\epsilon]) \simeq Ext^1_{\mathbb{F}_p[G]}(V_{\bar{\rho}}, V_{\bar{\rho}}) \tag{2.23}$$

and we know by definition that $V_{\bar{\rho}} \simeq E[p](\bar{\mathbb{Q}})$. On the other hand we know by hypothesis that $Ext^1_{\underline{D}}(E[p], E[p])$ is 1-dimensional, generated by $E[p^2]$ and that the submodule of annihilated-by-p extension is trivial. Since we are asking our $\mathbb{F}_p[G]$-modules to be flat at p, the annihilated-by-p extensions remains trivial even when we pass to the generic fiber. It follows that $Ext^1_{\mathbb{F}_p[G]}(V_{\bar{\rho}}, V_{\bar{\rho}})$ is trivial too and so the tangent space is zero-dimensional. Therefore the universal deformation ring is a quotient of $\mathbb{Z}_p$ and its Krull dimension is ≤ 1. But we already know a deformation of $\bar{\rho}$ to $\mathbb{Z}_p$: it is given by the Tate module

$$T_pE = \lim_{\leftarrow} E[p^n], \tag{2.24}$$

which is a G-module with the action given by the inverse limit of the G-actions on $E[p^n]$. Therefore the universal deformation ring must be isomorphic to $\mathbb{Z}_p$. □

Corollary 2.4.2. : *The framed deformation functor $F^{\square}_{\bar{\rho},S}$ associated to $E[p]$ has universal deformation ring isomorphic to $\mathbb{Z}_p[[x_1, x_2, x_3]]$.*

Proof. : It follows immediatly from the Main Theorem and from the fact that the framed deformation functor is smooth over the unframed one of dimension $n^2 - 1$. □

If ρ_{univ} is the universal deformation of our $\bar{\rho}$, we can explicitely obtain the corresponding universal framed deformation by a "universal base change". We set

$$\tilde{\rho} = \left(Id + \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) \rho_{univ}(g) \left(Id + \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right)^{-1}. \tag{2.25}$$

Our goal is to modify $\tilde{\rho}$ by a scalar matrix, which does not change the framed deformation class, so that we can eliminate one of the x_i. We set

$$\left(Id + \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) \begin{pmatrix} 1+x & 0 \\ 0 & 1+x \end{pmatrix} = \begin{pmatrix} (1+x_1)(1+x) & x_2(1+x) \\ x_3(1+x) & (1+x_4)(1+x) \end{pmatrix} \tag{2.26}$$

for x an element of $W(k)[[x_1, x_2, x_3, x_4]]$. We choose $x = \frac{-x_4}{1+x_4}$ and change our variables setting

$$\tilde{x_1} = \frac{x_1 - x_4}{1 + x_4}, \quad \tilde{x_2} = \frac{x_2}{1 + x_4}, \quad \tilde{x_3} = \frac{x_3}{1 + x_4}. \tag{2.27}$$

In this new variables, $\tilde{\rho}$ takes the form

$$\tilde{\rho} = \left(Id + \begin{pmatrix} \tilde{x_1} & \tilde{x_2} \\ \tilde{x_3} & 0 \end{pmatrix}\right) \rho_{univ} \left(Id + \begin{pmatrix} \tilde{x_1} & \tilde{x_2} \\ \tilde{x_3} & 0 \end{pmatrix}\right)^{-1} \tag{2.28}$$

In the following we will omit the tilde simbol over the new variables and simply name them as the original x_i.

We need to show that $\tilde{\rho}$ is actually the universal framed deformation. To do this, we compute the tangent space $F^{\square}_{\bar{\rho},S}(k[\epsilon])$. Let $\alpha : W(k)[[x_1, x_2, x_3]] \to \mathbb{F}_p[\epsilon]$ be a coefficient-ring morphism; it gives rise to an element of the tangent space given by

$$\alpha \circ \tilde{\rho} = \left(Id + \begin{pmatrix} \alpha(x_1) & \alpha(x_2) \\ \alpha(x_3) & 0 \end{pmatrix} \right) \bar{\rho} \left(Id - \begin{pmatrix} \alpha(x_1) & \alpha(x_2) \\ \alpha(x_3) & 0 \end{pmatrix} \right) = \\ = \bar{\rho} + \left[\begin{pmatrix} \alpha(x_1) & \alpha(x_2) \\ \alpha(x_3) & 0 \end{pmatrix}, \bar{\rho} \right]. \quad (2.29)$$

Since we want the dimension of the tangent space to be 3 and therefore the $\alpha(x_i)$ to be independent and ordinary, we need that $\alpha(x_i) = \epsilon\alpha_i$ for some $\alpha_i \in k$.

Let $\alpha' : W(k)[[x_1, x_2, x_3]] \to \mathbb{F}_p[\epsilon]$ be another coefficient ring morphism and suppose that α and α' induce the same deformation class. Then we have $\alpha \circ \rho^{\square}_{univ} = \alpha' \circ \rho^{\square}_{univ}$, and therefore

$$\bar{\rho} + \epsilon \left[\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & 0 \end{pmatrix}, \bar{\rho} \right] = \bar{\rho} + \epsilon \left[\begin{pmatrix} \alpha_1^{'} & \alpha_2^{'} \\ \alpha_3^{'} & 0 \end{pmatrix}, \bar{\rho} \right] \\ \left[\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & 0 \end{pmatrix}, \bar{\rho} \right] = \left[\begin{pmatrix} \alpha_1^{'} & \alpha_2^{'} \\ \alpha_3^{'} & 0 \end{pmatrix}, \bar{\rho} \right] \\ \left[\begin{pmatrix} \alpha_1 - \alpha_1^{'} & \alpha_2 - \alpha_2^{'} \\ \alpha_3 - \alpha_3^{'} & 0 \end{pmatrix}, \bar{\rho} \right] = 0. \quad (2.30)$$

Since the centralizer of $\bar{\rho}$ is the set of scalar matrices, we can conclude that the matrix $\begin{pmatrix} \alpha_1 - \alpha_1^{'} & \alpha_2 - \alpha_2^{'} \\ \alpha_3 - \alpha_3^{'} & 0 \end{pmatrix}$ must be a scalar. This implies $\alpha_i = \alpha_i^{'}$ for $i = 1, 2, 3$, which means that the morphisms α and α' are identical. Therefore an element of the tangent space is determined uniquely by the parameters α_i and therefore its dimension over k is 3. Since $\rho^{\square}_{univ}$ is a deformation of $\bar{\rho}$ to $W(k)[[x_1, x_2, x_3]]$, it must be the universal one.

Our main task is to generalize these results to representations of higher dimensions and also to direct sum of low-degree representations. Anyway the main problem is that in this case we often do not have representability and we do not have a canonical way to build up a characteristic zero deformation like the Tate module. In the next chapter we will solve this problem and generalize the results using some local-to-global arguments, mainly due to Kisin.

Chapter 3

Local to global arguments

3.1 Galois cohomology

In this section we recall without proof some of the main result about cohomology of Galois groups. The main reference for this part is given by [15].

Let G be a (finite or profinite) group, X an abelian topologycal group provided with a continuous action of G (it will be called a *G-module* in the following). Let $H^i(G, X)$ denote the i-th cohomology group. If X is also a vector space over some field k, let h^i denote the dimension of $H^i(G, X)$ as a vector space over k.

Lemma 3.1.1 (Inflation-Restriction sequence)**.** *Let H be a closed normal subgroup of G. There exist a long exact sequence*

$$0 \to H^1(G/H, X^H) \to H^1(G, X) \to H^1(H, X)^{G/H} \to H^2(G/H, X^H) \to \dots \tag{3.1}$$

In particular, if p is a prime, $G = G_p = G_{\mathbb{Q}_p}$ and $H = I_p$, the inertia group of p, then $H^i(G_p/I_p, X^{I_p})$ is called the group of i-th unramified cohomolgy classes.

Corollary 3.1.2. *Suppose X is finite. Then $\#H^1(G_p/I_p, X^{I_p}) = \#H^0(G_p, X)$ and both of them are finite.*

Let X_1, X_2, X_3 be G-modules and let $\phi : X_1 \otimes X_2 \to X_3$ be a G-module homomorphism. Then the *cup product* is a map $\cup : H^1(G, X_1) \otimes H^1(G, X_2) \to H^2(G, X_3)$ such that, for every $f_k \in H^1(G, X_k)$, we have

$$f_1 \cup f_2(g_1, g_2) = \phi(f_1(g_1) \otimes g_1 f_2(g_2)) \tag{3.2}$$

Now we pass to describe the main result we will use.

Theorem 3.1.3 (Tate's local duality theorem)**.** *Let X be a finite G_p-module of cardinality n and $X^* = Hom(X, \mu_n)$. Then the following are true:*

1. *The group* $H^i(G_p, X)$ *is finite for each* $i \geq 0$ *and trivial for* $i \geq 3$.

2. *The cup product gives a non-degenerate pairing*

$$H^1(G_p, X) \times H^1(G_p, X^*) \to H^2(G_p, \mu_n) \simeq \mathbb{Q}/\mathbb{Z}. \tag{3.3}$$

3. *If* p *does not divide* n, *then the unramified classes* $H^1(G_p/I_p, X_p^{I_p})$ *and* $H^1(G_p/I_p, (X^*)^{I_p})$ *are the annihilators of each other under the pairing given by the cup product.*

Corollary 3.1.4. *Under the hypothesis of the theorem, we have that*

$$\frac{\#H^1(G_p, X)}{\#H^0(G_p, X)\#H^2(G_p, X)} = p^{v_p(\#X)}. \tag{3.4}$$

In particular, if X *is also a finite dimensional vecotr space over* $\mathbb{F}_p$, *we have that*

$$h^0 - h^1 + h^2 = -dim(X). \tag{3.5}$$

The left hand side is called the Euler-Poincarè characteristic of X *and denoted by* $c_{EP}(X)$.

We end this section by giving a description of the Poitou-Tate exact sequence. Let X be a finite $G_{\mathbb{Q}}$-module. Let Σ be a set of primes of $\mathbb{Q}$ containing the infinite prime, the primes dividing $\#X$ and the primes such that I_p does not act trivially on X. Since X is finite, there are only a finite number of primes for which this action is non trivial, therefore Σ can be taken to be a finite set, too. Let $\mathbb{Q}_\Sigma$ be the maximal extension of $\mathbb{Q}$ unramified outside Σ inside a fixed algebraic closure and $G_\Sigma = Gal(\mathbb{Q}_\Sigma/\mathbb{Q})$. Then we can look at X also as a G_Σ-module. Let

$$\alpha_r : H^r(G_\Sigma, X) \to \hat{H}^r(G_\mathbb{R}, X) \times \prod_{\ell \in \Sigma \setminus \{\infty\}} H^r(G_\ell, X) \tag{3.6}$$

be the map induced by restriction of cohomolgy, where $\hat{H}^0 = H^0/Norm(X)$, where $Norm(X)$ is the subgroup of norms given by the elements $\sum_{g \in G} gx$ for every $x \in X$, and $\hat{H}^i = H^i$ for $i > 0$. Tate's local duality theorem tells us that $\hat{H}^r(G_\mathbb{R}, X) \times \prod_H^r(G_\ell, X)$ is the dual of $\hat{H}^{2-r}(G_\mathbb{R}, X^*) \times \prod H^{2-r}(G_\ell, X^*)$, therefore dualizing the map α_r we obtain

$$\beta_r : \hat{H}^r(G_\mathbb{R}, X) \times \prod_{\ell \in \Sigma \setminus \{\infty\}} H^r(G_\ell, X) \to H^{2-r}(G_\Sigma, X^*)^\diamond \tag{3.7}$$

where $A^\diamond = Hom(A, \mathbb{Q}/\mathbb{Z})$.

Proposition 3.1.5. *1. There exists a non degenerate pairing*

$$Ker(\alpha_2) \times Ker(\alpha_1) \to \mathbb{Q}/\mathbb{Z} \tag{3.8}$$

2. α_0 is injective, β_2 is surjective and $Im(\alpha_r) = Ker(\beta_r)$.

The prevoius proposition gives rise to the following result

Proposition 3.1.6. *The following 9-term exact sequence is exact*

$$\begin{aligned} 0 \to H^0(G_\Sigma, X) \overset{\alpha_0}{\to} \hat{H}^0(G_\mathbb{R}, X) \times \prod_{\ell \in \Sigma \setminus \{\infty\}} H^0(G_\ell, X) \overset{\beta_0}{\to} H^2(G_\Sigma . X^*)^\diamond \\ \to H^1(G_\Sigma, X) \overset{\alpha_1}{\to} \prod_{\ell \in \Sigma} H^1(G_\ell, X) \overset{\beta_1}{\to} H^1(G_\Sigma, X^*)^\diamond \\ \to H^2(G_\Sigma, X) \overset{\alpha_2}{\to} \prod_{\ell \in \Sigma} H^2(G_\ell, X) \overset{\beta_2}{\to} H^0(G_\Sigma, X^*)^\diamond \to 0, \end{aligned} \tag{3.9}$$

where the unlabeled arrows are given by the non-degeneracy of the pairing in the previous proposition. This sequence is called the Poitou-Tate exact sequence for X.

3.2 The local flat deformation functor

In this section we want to deal with a local deformation condition which refers to the prime p, characteristic of the finite base field k. This condition was mainly studied by Ramakrishna in [10] and then generalised by Conrad in [3],[4] and Kisin in [6]. From now on, we will only deal with representations of degree 2.

Let F be a finite extension of $\mathbb{Q}_p$ and $\bar{\rho} : G_F \to GL_2(k)$ be a residual Galois representation. If ρ is a deformation of $\bar{\rho}$ to a coefficient ring A, we say that ρ is *flat* if there exists a finite flat group scheme X over the ring of integers O_F such that $V_\rho \simeq X(\bar{F})$, that is, V_ρ is the generic fiber of X.

Proposition 3.2.1. *The condition of being flat is a deformation condition.*

Proof. it suffices to show that the subcategory $\underline{\Delta}$ of flat deformations satisfies Ramakrishna's categorical conditions.

Let $0 \to T \to U \to V \to 0$ be a sequence of G-modules such that U is the generic fiber of a finite flat group scheme X over O_F. Then we can take the schematic closure X_1 of T in X (see [10, Lemma 2.1] for details) and $X_2 = X/X_1$ to see that also T and V are generic fibers of finite flat group schemes. This argument and the fact that a direct sum of finite flat group schemes is still a finite flat group scheme show that the subcategory of flat deformations is a deformation condition. □

If $\bar{\rho}$ satisfies the trivial centralizer condition $End_{k[G_F]}(V_{\bar{\rho}}) = k$, then the deformation functor which assigns to a coefficient ring the set of deformations of $\bar{\rho}$ which are flat, called the *flat deformation functor* and denoted as F^{fl}, is representable by a noetherian ring R_p^{fl}, which is called the *local flat universal deformation ring.* We want to give a proof of the main result of representability for this condition, which was proven by Ramakrishna for $p \neq 2$ and by Conrad for all cases. First we need some technical data

Definition 3.2.2. *Let ϕ denote the absolute Frobenius morphism $\phi(x) = x^p$. A Fontaine-Lafaille module is a $W(k)$-module M provided with a decreasing, exhaustive, separated filtration of $W(k)$-submodules $\{M_i\}$ such that, for every index i, there exists a ϕ-semilinear map $\phi_i : M_i \to M$ with the property that $\phi_i(x) = p\phi_{i+1}(x)$ for every $x \in M$.*

We denote by MF the category of Fontaine-Lafaille modules over $W(k)$. Moreover we denote by MF_{tor}^{f} the full subcategory of objects such that M has finite length and $\sum Im(\phi_i) = M$ and by $MF_{tor}^{f,j}$ the subcategory of objects such that $M_0 = M$ and $M_j = 0$. Finally we say that a Fontaine-Lafaille module is *connected* if the morphism ϕ_0 is nilpotent The main result about Fontaine-Lafaille modules (which we do not prove) is the following

Theorem 3.2.3 (Fontaine-Lafaille)**.** *For every $j \leq p$ there exists a faithful exact contravariant functor*

$$MF_{tor}^{f,j} \to Rep_{\mathbb{Z}_p}^{f}(G), \tag{3.10}$$

which is fully faithful if $j < p$ and becomes fully faithful when restricted to the subcategory of connected Fontaine-Lafaille modules if $j = p$. Morevoer $MF_{tor}^{f,2}$ is antiequivalent to the category of fintie flat group schemes over $W(k)$

Proof. See [6, Ch.8-9] for a proof and description of the functor. □

We say that a representation ρ has weight j if it comes from a Fontaine-Lafaille module lying in $MF_{tor}^{f,j}$ and we denote by $F_{\bar{\rho},j}$ the subfunctor of $F_{\bar{\rho}}$ given by deformations of $\bar{\rho}$ which are of weight j. It follows that if $\bar{\rho}$ is flat, then the functors F_2 and F_{fl} are the same, therefore we will identify them in the rest of the chapter.

We can now prove the main result for flat deformation functor. The proof is due to Ramakrishna for the case $p > 2$ (see [10, section 3]); then Conrad has shown (see [2]) that the proof works also in the case $p = 2$, since the Fontaine-Lafaille module used is connected.

Theorem 3.2.4. *Let $\bar{\rho} : G_{\mathbb{Q}_p} \to GL_2(k)$ be a flat residual Galois representation with trivial centralizer and such that $det(\bar{\rho}) = \chi$, where χ is the cyclotomic character. Then*

$$R_p^{fl}(\bar{\rho}) \simeq W(k)[[T_1, T_2]]. \tag{3.11}$$

Proof. We split the proof in two parts. Suppose first that $k = \mathbb{F}_p$ and $\bar{\rho}$ is the representation attached to the p-torsion points of an elliptic curve E over $\mathbb{Q}_p$ with good supersingular reduction. We proof the theorem in this particular case, where computations are relatively easy, and then pass to the general case.

In the particular case we have chosen, we know that $\bar{\rho}$ satisfies the trivial centralizer hypothesis and is of weight 2. We calculate the tangent space $F_2(\mathbb{F}_p[\epsilon])$. Viewing $\mathbb{F}_p[\epsilon]^2$ as a 4-dimensional $\mathbb{F}_p$-vector space, we can see and element $\rho \in F_2(\mathbb{F}_p[\epsilon])$ as a matrix

$$\rho(g) = \begin{pmatrix} \bar{\rho}(g) & 0 \\ R_g & \bar{\rho}(g) \end{pmatrix} \tag{3.12}$$

and such a representation gives clearly an element of $Ext^1_{2,p}(V_{\bar{\rho}}, V_{\bar{\rho}})$, the extensions in the category of weight 2 representations which are killed by p. It is immediate to check that equivalence of litings correspond to equivalent extensions.

Let M be the Fontaine-Lafaille module associated to $V_{\bar{\rho}}$ via 3.10. By full faithfulness of the functor, we have that $Ext^1_{2,p}(V_{\bar{\rho}}, V_{\bar{\rho}}) = Ext^1_{2,p}(M, M)$ and that $End_{MF}(M) = \mathbb{F}_p$.

We want to write the module in a compactified manner in terms of a 2×2-matrix X_M. For that we use the fact that M_1 is 1-dimensional (it will be proved shortly) and that $\phi_0(M_1) = 0$. Then we write

$$\phi_0 = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} * & \gamma \\ * & \delta \end{pmatrix}, \quad X_M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{3.13}$$

The matrix X_M encodes all the informations of the structure of M. We also want to write the elements of $Ext^1_{2,p}(M, M)$ via these matrices. If N is such an element, we have

$$X_N = \begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix}, \quad C \in M_2(\mathbb{F}_p). \tag{3.14}$$

The matrix C corresponds to an element of $Hom(M, M)$. If N' is another element of $Ext^1_{2,p}(M, M)$ and D is the 2×2 matrix in its upper triangular part, then it represents the same extension of N if and only if there exist a matrix $\begin{pmatrix} Id & R \\ 0 & Id \end{pmatrix} \in M_4(\mathbb{F}_p)$ such that

$$\begin{pmatrix} Id & R \\ 0 & Id \end{pmatrix} \begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix} = \begin{pmatrix} X_M & D \\ 0 & X_M \end{pmatrix} \begin{pmatrix} Id & R \\ 0 & Id \end{pmatrix} \tag{3.15}$$

and this happens if and only if $C - D = [R, X_M]$. Moreover R must preserve the filtration of M, because the isomorphism between N and N' does so. Let $\mathfrak{H}$ be the set of such matrices R. It follows that

$$Ext^1_{2,p}(M,M) \simeq Hom(M,M)/\{[R,X_M]: \ R \in \mathfrak{H}\} \tag{3.16}$$

Now we know that $dim_{\mathbb{F}_p} M_0 = 2$ and $dim_{\mathbb{F}_p} M_2 = 0$. If $dim_{\mathbb{F}_p} M_1 \neq 1$ than any endomorphism of M does not need to respect any filtration structure and therefore the centralizer of X_M in $M_2(\mathbb{F}_p)$, which has at least dimension 2, would belong to $End_{MF}(M)$; this is impossible because the endomorphism ring is 1-dimensional. Therefore $dim_{\mathbb{F}_p} M_1 = 1$.

Now we can compute the dimension of the tangent space: observe that $Hom(M,M)$ has dimension 4, the set of matrices R which preserves the filtration of M has dimension 3 and the kernel of the map $R \to [R, X_M]$ has dimension 1 (it is isomorphic to $End_{MF}(M)$). Therefore the tangent space has dimension $4-(3-1)=2$.

Now we have that $R_2(\bar{\rho}) = \mathbb{Z}_p[[T_1,T_2]]/I$. We count the number of $\mathbb{Z}_p/p^l$-valued points of the universal ring, which is the number of objects $N \in MF^{f,2}_{tor}$ which are free $\mathbb{Z}_p/p^l$-modules of rank 2. If N_p denotes the kernel of multiplication by p in N, then we need $N_p \simeq M$, in terms of matrices, since $X_N \equiv X_M \pmod{p}$. Since $X_N \in M_2(\mathbb{Z}_p/p^l)$ and we have to consider modulo p, there are $p^{4(l-1)}$ such matrices. We have to consider them modulo isomorphism. Now if $X_{N_1} \simeq X_{N_2}$, then there exists a matrix $R \in M_2(\mathbb{Z}_p/p^l)$ which respects the filtration of M such that $RX_{N_1} = X_{N_2}R$; there are $p^{3(l-1)}$ such matrices and p^{l-1} lie in the center of $M_2(\mathbb{Z}_p/p^l)$, therefore commute with all the X_N. So the number of $\mathbb{Z}_p/p^l$-valued points is $p^{4(l-1)}/(p^{3(l-1)}/p^{l-1}) = p^{2(l-1)}$. Observe that this is the same number of $\mathbb{Z}_p/p^l$-valued points of $\mathbb{Z}_p[[T_1,T_2]]$.

Let now $f \in I$ and $(x,y) \in (\mathbb{Z}_p/p^l)^2$, then $f(x,y) \equiv 0 \pmod{p^l}$ for every positive integer l. It follows that, taking liftings to characteristic zero, $f(x,y) = 0$ for all $(x,y) \in (p\mathbb{Z}_p)^2$ and therefore $f=0$. So $I=0$ and $R_2(\bar{\rho}) = \mathbb{Z}_p[[T_1,T_2]]$.

Now we can pass to the proof of the theorem in the general case and remove the hypothesis that $k = \mathbb{F}_p$ and that $\bar{\rho}$ is the representation coming from an elliptic curve. A lemma of Serre (whose proof can be found in [10]) tells us that $\bar{\rho}$ has restriction to inertia given by

$$\bar{\rho}|_I = \begin{pmatrix} \psi & 0 \\ 0 & \psi^p \end{pmatrix} \tag{3.17}$$

where ψ is a fundamental character of level 2. For such a representation we will compute both the "unrestricted" universal ring and the flat one. First of all we want to show that $H^2(G, Ad(\bar{\rho})) = 0$. By Tate local duality we have that $H^2(G, Ad(\bar{\rho})) = H^0(G, Ad(\bar{\rho})^*) = (Ad(\bar{\rho})^*)^G$. Let $\phi \in (Ad(\bar{\rho})^*)^G$, we want to show that its kernel is 4-dimensional and therefore $\phi = 0$. Let $R \in Ad(\bar{\rho})$, we have $\phi(gR) = g\phi(R)$, where the G-action is given by conjugacy composed with $\bar{\rho}$ on the left and by determinant on the right. It follows that, if $g \in I$, then $\bar{\rho}(g)R\bar{\rho}(g)^{-1} - det(\bar{\rho}(g))R \in Ker(\phi)$. Then, if we define the map

$$T_g : R \to \bar{\rho}(g)R\bar{\rho}(g)^{-1} - det(\bar{\rho}(g))R, \tag{3.18}$$

it suffices to show that there exists $g \in I$ such that $Ker(T_g) = 0$. We choose a g such that $\psi(g) = \alpha$ where α is an element of order $p^2 - 1$ in k^*. Then, taking explicit formulas

$$R = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \quad T_g(R) = \begin{pmatrix} x(1-\alpha^{p+1}) & y(\alpha^{1-p} - \alpha^{p+1}) \\ z(\alpha^{p-1} - \alpha^{p+1}) & w(1-\alpha^{p+1}) \end{pmatrix} \tag{3.19}$$

and the last matrix is zero if and only if $R = 0$. Then our claim is proved.

Now we use the formula for the Euler-Poincaré characteristic for $Ad(\bar{\rho})$. Let $h^i = dim(H^i(G, Ad(\bar{\rho})))$. We have

$$c_{EP}(Ad(\bar{\rho})) = h^0 - h^1 + h^2 = -dim_k Ad(\bar{\rho}). \tag{3.20}$$

We have that $h^2 = 0$, $h^0 = 1$ (it is the trivial centralizer condition) and $dim_k Ad(\bar{\rho}) = 4$, therefore $h^1 = 5$. It follows that the unrestricted universal ring for such a representation is isomorphic to $W(k)[[T_1, T_2, T_3, T_4, T_5]]$.

The flat deformation ring can be computed by means of calculations similar to the ones performed in the case $k = \mathbb{F}_p$, except that we have to consider Fontaine-Lafaille modules over $W(k)$ instead of $\mathbb{Z}_p$ and all the dimensions have to be computed over k. We obtain again that $R_2(\bar{\rho}) = W(k)[[T_1, T_2]]$. In particular $R_2(\bar{\rho})$ is a quotient of $R(\bar{\rho})$ and the surjective map between them has a 3-dimensional kernel. The theorem is therefore proved. □

Now we give a refinement of this result, which is due to Conrad [3].

Theorem 3.2.5. *Let $\bar{\rho}$ be as in the previous theorem and let $F^{fl,\chi}$ be the subfunctor of flat deformations of $\bar{\rho}$ which have fixed determinant χ. Then this functor is representable by the ring*

$$R_p^{fl,\chi}(\bar{\rho}) \simeq \mathbb{Z}_p[[T]]. \tag{3.21}$$

Proof. For the proof see [3, Ch. 4, Th.4.1.2]. □

EXAMPLE: Let E be an elliptic curve over $\mathbb{Q}_p$ that has supersingular reduction in p. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to GL_2(\mathbb{F}_p)$ be the representation coming from the Galois action on the p-torsion points of E. Then, by the results of [4], $\bar{\rho}$ is absolutely irreducible and therefore the functor $F_{\bar{\rho}}^{fl}$ is representable. Therefore, applying Ramakrishna's theorem, we have that the flat universal deformation ring is $\mathbb{Z}_p[[T_1, T_2]]$.

3.3 Steinberg representations at primes $\ell \neq p$

Now we want to analyse local conditions at finite primes which are different from p. We continue to assume that the representation space $V_{\bar{\rho}}$ has dimension 2.

Definition 3.3.1. *: A 2-dimensional representation $\bar{\rho} : G_\ell \to GL_2(k)$ is called of Steinberg type if it is a non-split extension of a character $\lambda : G_\ell \to k^*$ by the twist $\lambda(1) = \lambda \otimes \chi_p$ of λ by the p-adic cyclotomic character χ_p.*

A representation of Steinberg type has the matricial form

$$\bar{\rho}(g) = \begin{pmatrix} \lambda(1)(g) & * \\ 0 & \lambda(g) \end{pmatrix} \quad \forall g \in I_\ell \tag{3.22}$$

Observe that since $\ell \neq p$ the mod p cyclotomic character is unramified and, if p is not a square mod ℓ, it also happens that χ_p and its twists are trivial. We do not impose any ramification restriction on the character λ. Up to twisting by the inverse character of λ, we may assume that $det(\bar{\rho}) = \chi_p$ and that $V(\bar{\rho})(-1)^{G_\ell} \neq 0$, which means that there is a subrepresentation of dimension 1 on which G_ℓ acts via χ_p.

We define a subfunctor

$$L_{\bar{\rho}}^{\chi_p} : \underline{\hat{Ar}} \to \underline{Sets} \tag{3.23}$$

of the deformation functor $F_{\bar{\rho}}^{\chi_p}$ as

$$L_{\bar{\rho}}^{\chi_p}(A) = (V_A, L_A) \tag{3.24}$$

where

- V_A is a deformation of $\bar{\rho}$ to A.
- L_A is a submodule of rank 1 of V_A on which G_ℓ acts via χ_p.

We define in the same way the framed subfunctor $L_{\bar{\rho}}^{\chi_p,\square} : \underline{\hat{Ar}} \to \underline{Sets}$ as

$$L_{\bar{\rho}}^{\chi_p,\square}(A) = (V_A, \beta_A, L_A) \tag{3.25}$$

where

- (V_A, β_A) is a framed deformation of $\bar{\rho}$ to A.
- L_A is a submodule of rank 1 of V_A on which G_ℓ acts via χ_p.

This is the subfunctor corresponding to liftings of Steinberg type. In the following we work with the framed setting to avoid representability problems.

In order to deal with representability of deformations functors of Steingberg type, we need to recall the main definitions of formal schemes. Let R be a noetherian ring and I an ideal and assume that R is I-adically complete, so that we have

$$R = \lim_{\leftarrow} R/I^n. \tag{3.26}$$

We define a topologycal space $Spf(R)$ in the following way: given an element $f \in R$ and $\bar{f}$ its reduction modulo I, we define $D(\bar{f})$ to be the set of prime ideals of R/I not containing $\bar{f}$. Then the set $Spec(R/I)$ with the induced topology is called the *formal spectrum* of R, with respect to I, and denoted by $Spf(R)$. The sets $D(\bar{f})$ are a basis for the topology of $Spf(R)$.

For each $f \in R$ we define

$$R\langle f^{-1}\rangle = \lim_{\leftarrow} R[f^{-1}]/I^n \tag{3.27}$$

Then the assignment $D(\bar{f}) \mapsto R\langle f^{-1}\rangle$ defines a structure sheaf on $Spf(R)$.

Definition 3.3.2. *The affine formal scheme $Spf(R)$ over R with respect to I is the locally ringed space (X, O_X), where $X = Spec(R/I)$ and $O_X(D(\bar{f})) = R\langle f^{-1}\rangle$ for each $f \in R$.*

A noetherian formal scheme is a locally ringed space (X, O_X), where X is a topological space and O_X is a sheaf of rings over X such that each point $x \in X$ admits a neighborhood U such that $(U, O_X|_U)$ is isomorphic to an affine formal scheme $Spf(R)$.

A morphism of formal schemes is a pair $(f, f^*) : (X, O_X) \to (Y, O_Y)$, where $f : X \to Y$ is a continuous map of topologycal spaces and $f^* : O_Y \to f_* O_X$ is a morphism of sheaves.

If (X, O_X) is a scheme, we can obtain a formal scheme $\hat{X}$ by the following construction: let $I \subseteq O_X$ be an ideal sheaf and consider $\hat{X}$ the completion of X along I. Its underlying topological space is given by the subscheme Z of X defined by I and the structure sheaf is defined as before. A formal scheme obtained in this way is called *algebrizable*.

Finally, given a functor $\underline{Ar} \to \underline{Sets}$, we can pass to the opposite categories and obtain a functor $\underline{Ar}^\circ \to \underline{Sets}^\circ$; $\underline{Ar}^\circ$ is exactly the category of formal schemes on one point over $Spec(W(k))$ with residue field $Spec(k)$. Schlessinger's theorem then provides criteria for the functor to be representable by an object of $\underline{\hat{Ar}}^\circ$.

Let us now go back to deformation functors. We want to give a description of the representing object of $L_{\bar{\rho}}^{\chi_p,\square}$ using formal schemes. Let $R = R_{\bar{\rho}}^{\chi_p,\square}$ and V_R be the 2-dimensional module over R with the action given by the universal framed representation $\rho_{univ}^{\square}$. Let $\mathbb{P}(R)$ be the projectivization of V_R and $\hat{\mathbb{P}}(R)$ be its completion along the maximal ideal of R. We consider the closed subspace of $\hat{\mathbb{P}}(R)$ defined by the equations $gv - \chi(g)v = 0$ for every $g \in G_{ell}$ and each $v \in \hat{\mathbb{P}}(R)$. By formal GAGA, this subspace comes from a unique projective scheme $\mathfrak{L}$ over $Spf(R)$.

In the following we want to prove some properties of the scheme $\mathfrak{L}$. In particular we want to show that it is an affine scheme of the form $Spec(\tilde{R})$ for an appropriate ring $\tilde{R}$, which will be automathically the representing object of $L_{\bar{\rho}}^{\chi_p,\square}$, because of the defining property of $\mathfrak{L}$.

Lemma 3.3.3. *$\mathfrak{L}$ is formally smooth over $Spec(W(k))$ and its generic fiber $\mathfrak{L} \otimes_{W(k)} W(k)[1/p]$ is connected.*

Proof. Let $A_1 \to A_2$ be a sujective map in $\underline{Ar}$. An element $\eta_2 \in L_{\bar{\rho}}^{\chi_p,\square}(A_2)$ corresponds to an extension $c(\eta_2) \in Ext^1_{A_2[G]}(A_2, A_2(1))$. If η_1 is a lift of η_2 to $L_{\bar{\rho}}^{\chi_p,\square}(A_1)$, then the lift is uniquely determined by a lift of the class $c(\eta_2)$ to an element of $Ext^1_{A_1[G]}(A_1, A_1(1))$. Finding such a lift of extensions exists is equivalent to proving that the natural map

$$H^1(G, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} M \to H^1(G, M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)) \tag{3.28}$$

is an isomorphism for any $A \in \underline{Ar}$ and any A-module M. Since H^1 commutes with direct sums, it is sufficient to prove this result for $M = \mathbb{Z}/p^n\mathbb{Z}$. In this case the map is trivially injective and its cokernel is given by $H^2(G, \mathbb{Z}_p(1))[p^n]$; but, by Tate's local duality, $H^2(G, \mathbb{Z}_p(1))$ is the Pontryagin dual of $\mathbb{Q}_p/\mathbb{Z}_p$, which has no p^n-torsion. Therefore the map is an isomorphism.

Now we need to prove connectedness. We denote by $\mathfrak{L}[1/p]$ the generic fiber. By smoothness, the schemes $\mathfrak{L}[1/p]$, $\mathfrak{L}$ and $\mathfrak{L} \otimes_{W(k)} k$ have the same number of connected components and, as schemes over R, the same is true for $\mathfrak{L}$ and $Z = \mathfrak{L} \otimes_R k$; by [7, Prop.2.5.15] this scheme is either all of $\mathbb{P}(k)$, if the action of G_ℓ is trivial, or a single point. Therefore there is only one connected component. □

Before going on, we need a further notation. Given V a representation lifting $V_{\bar{\rho}_\ell}$ to some ring A, we denote by F_V^χ the subfunctor of $F_{\bar{\rho}_\ell}^\chi$ given by representations lifting V, too, and by $F_V^{\chi,\square}$ the corresponding framed deformation functor.

Lemma 3.3.4. *The natural morphism of functors $L_V^{\chi,\square} \to F_V^{\chi,\square}$ is fully faithful. In particular, if V is indecomposable, the morphism is an equivalence, F_V^χ is representable and its tangent space is 0-dimensional.*

Proof. Let $A \in \underline{Ar}$ and (V_A, β_A) be a framed deformation and L_A be a χ_p-invariant line in V_A. We need to show that L_A is unique. Indeed we have $Hom_{A[G]}(A(1), V_A/L_A)$ is trivial, since $det(V_A) = \chi$ and V_A/L_A is free of rank 1. Therefore we have $Hom_{A[G]}(A(1), V_A) = Hom_{A[G]}(A(1), L_A)$ and the uniqueness follows.

Suppose now that V is indecomposable, then in particular the unframed deformation functor F_V^χ is representable, too. We need to show that each deformation V_A contains an A-line L_A on which G_ℓ acts via χ. For this, it is enough to show that the tangent space is 0-dimensional, which implies that every deformation V_A is isomorphic to $V \otimes_k A$ and therefore inherits the trivial A-line from $V_{\bar{\rho}}$. By Tate's local duality (see Section 3.1) we have

$$h^1(G_\ell, Ad^0(V)) = h^0(G_\ell, Ad^0(V)) + h^2(G_\ell, Ad^0(V)(1)) \tag{3.29}$$

and that the two summands equal each other; therefore it is enough to show that $h^0(G_\ell, Ad^0(V)) = 0$. Since $\ell \neq 2$ we have the exact sequence

$$0 \to H^0(G_\ell, Ad^0(V)) \to H^0(G_\ell, Ad(V)) \to H^0(G_\ell, \mathbb{Q}_\ell) \to 0. \tag{3.30}$$

Trivially $H^0(G_\ell, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^0(G_\ell, Ad(V)) = \mathbb{Q}_\ell$ because V is indecomposable, therefore $H^0(G_\ell, Ad^0(V)) = 0$ and the lemma is proved.

□

Theorem 3.3.5. *Let $Spec(R_{\bar{\rho}}^{\chi,1,\square})$ be the image of the natural morphism $\mathfrak{L} \to Spec(R_{\bar{\rho}}^{\chi,\square})$. Then $R_{\bar{\rho}}^{\chi,1,\square}$ is a domain of dimension 4 and $R_{\bar{\rho}}^{\chi,1,\square}[1/p]$ is formally smooth over $W(k)[1/p]$. Moreover, for every $A \in \underline{Ar}$, a morphism $R_{\bar{\rho}}^{\chi,\square} \to A$ factors through $R_{\bar{\rho}}^{\chi,1,\square}$ if and only if the corresponding 2-dimensional representation is of Steinberg type.*

Proof. The scheme $\mathfrak{L}$ is smooth over $W(k)$ and connected. The ring $R_{\bar{\rho}}^{\chi,1,\square}$ is the ring of global section of $\mathfrak{L}$ over $R_{\bar{\rho}}^{\chi,\square}$, hence it must be a domain.

If we invert p, lemma 3.3 tells us that the generic fiber $\mathfrak{L}[1/p]$ is a closed subscheme of $Spec(R_{\bar{\rho}}^{\chi,\square}[1/p])$, then it must be isomorphic to $Spec(R_{\bar{\rho}}^{\chi,1,\square}[1/p])$; this proves that $R_{\bar{\rho}}^{\chi,1,\square}[1/p]$ is formally smooth over $W(k)[1/p]$.

We now calculate the dimension. Since $R_{\bar{\rho}}^{\chi,1,\square}$ has no nontrivial p-torsion, it is sufficient to calculate it on the generic fiber and add 1. Let V be an indecomposable point. By lemma 3.3, we have that F_V^{χ} is representable with tangent space of dimension 0, therefore the framed functor $F_V^{\chi,\square}$ has a tangent space of dimension 3. This proves the claim.

Finally, to prove the last statement, we used again the previous lemma. A morphism factors through $R_{\bar{\rho}}^{\chi,1,\square}$ if and only if it lifts to a unique point of $\mathfrak{L}$, that is, if and only if the corresponding representation space V has a 1-dimensional subrepresentation where G acts thorugh χ. The theorem is therefore proved. □

3.4 Computations of odd deformation rings

In this section we will deal with local conditions at the infinite places, computing explicitly the deformation ring. Let

$$\bar{\rho} : Gal(\mathbb{C}/\mathbb{R}) \to GL_2(k) \tag{3.31}$$

be a local representation at the infinite place with $det(\bar{\rho}(\gamma)) = -1 \in k$, where γ is a complex conjugation. Then, up to conjugation, $\bar{\rho}$ must be of one of these forms

1. $p > 2$, $\bar{\rho}(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
2. $p = 2$, $\bar{\rho}(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
3. $p = 2$, $\bar{\rho}(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The representation is determined by the image of γ, which is a matrix whose eigenvalues are 1 and -1 and therefore whose characteristic polynomial is x^2-1. Let $\mathfrak{M}(x^2-1)$ be the space of 2×2 matrices whose characteristic polynomial is x^2-1. Let $\mathfrak{N}$ its subset of matrices which lift to $\bar{\rho}(\gamma)$. Then we can consider the ring

$$R = W(k)[a,b,c,d]/I. \tag{3.32}$$

where I is an ideal encoding the condition on the characteristic polynomial. It is easy to see that $Spec(R) = \mathfrak{M}(x^2-1)$ and that its completion $\tilde{R}$ to the maximal ideal is the universal deformation ring of $\bar{\rho}$ and the universal deformation is given by

$$\rho_{univ}(\gamma) = \bar{\rho}(\gamma) + \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.33}$$

We do the calculations explicitly for the case 3 above (the other two cases being similar). Let $M = \rho_{univ}(\gamma)$, then, imposing the conditions $Tr(M) = 0$ and $det(M) = 1$, we have

$$\begin{aligned} R = W(k)[a,b,c,d]/((1+a)+(1+d),(1+a)(1+d)+1-bc) = \\ = W(k)[a,b,c]/(-(1+a)^2+1-bc) = \\ = W(k)[a,b,c]/(-2a-a^2-bc), \end{aligned} \tag{3.34}$$

and so $\tilde{R} \simeq W(k)[[a,b,c]]/(2a+a^2+bc)$. In particular, if we invert p, it is a regular ring of dimension 2 over $W(k)$. A similar computation gives the same result in the other two cases.

3.5 Local to global arguments

In this chapter we want to give a presentation of a global deformation ring in terms of local ones. The results we use are due to Kisin [6] and will contemplate both the framed and the unframed setting. In the application, the framed setting is mostly used, mainly to avoid representability problems in the local rings.

Let $\bar{\rho} : G_K \to GL_2(k)$ be a residual representation. Let S be a finite set of primes of K including the primes over p and the infinite prime and Σ a subset of S containing p and the infinite prime too; in many application we will have $\Sigma = S$. For each $v \in \Sigma$ we denote by K_v the completion of K to v and by $\bar{\rho}_v = \bar{\rho}|_{K_v}$. As before we denote by $\bar{\rho} = \bar{\rho}|_{G_{K,S}}$. Assume that $\bar{\rho}$ as well as all of the $\bar{\rho}_v$ for $v \in \Sigma$ satisfy the trivial centralizer hypothesis. Let $F_{\bar{\rho}}$ and $F_{\bar{\rho}_v}$ the deformation functors associated to $\bar{\rho}$ and $\bar{\rho}_v$ respectively; since all of the functors are representable, we denote as R_v^χ the local universal deformation functor of $\bar\rho_v$ with fixed determinant equal to χ and as R_S^χ the universal deformation functor of $\bar\rho$ with fixed determinant equal to χ. Finally we put

$$R_\Sigma^\chi = \hat\otimes_{v\in\Sigma} R_v^\chi. \tag{3.35}$$

Let

$$\theta_i : H^i(G_{K,S}, Ad^0(\bar\rho)) \to \prod_{v\in\Sigma} H^i(G_{K_v}, Ad^0(\bar\rho)) \tag{3.36}$$

be the usual restriction map. Following [6], we denote by r_i and t_i the dimensions of the kernel and cokernel of θ_i as k-vectorial spaces.

Let m_Σ^χ and m_S^χ be the maximal ideals of R_Σ^χ and R_S^χ respectively and let

$$\eta : m_\Sigma^\chi/((m_\Sigma^\chi)^2, p) \to m_S^\chi/((m_S^\chi)^2, p) \tag{3.37}$$

be the map between the dual tangent spaces. Then we have the following result

Theorem 3.5.1. *If the functors $F_{\bar\rho}$ and $F_{\bar\rho_v}$ are representable, then there exist elements $f_1, \dots, f_{t_1+r_2}$ lying in the maximal ideal of $R_\Sigma^\chi[[x_1, \dots, x_{r_1}]]$ such that*

$$R_S^\chi = R_\Sigma^\chi[[x_1, \dots, x_{r_1}]]/(f_1, \dots, f_{t_1+r_2}). \tag{3.38}$$

In particular $dim_{Krull} R_S^\chi \geq 1 + r_1 - r_2 - t_1$

Proof. Consider the quotient ring R_S^χ/m_Σ^χ; the tangent space of this ring is clearly the dual of $ker(\theta_1)$, therefore these two vector spaces have the same dimension. This proves the claim on the number of variables.

Let now $I = ker(\eta)$. There exists a surjection $R_{gl} = R_\Sigma^\chi[[x_1, ..., x_{r_1}]] \to R_S^\chi$ which induces a surjection on tangent spaces with kernel isomorphic to I. Denote by m_{gl} the maximal ideal of R_{gl} and by J the kernel of the surjection. Let ρ_S^χ be the universal deformation of $\bar\rho$ and consider a set theoretic lift ρ_{gl} of ρ_S^χ to the ring R_{gl}/Jm_{gl} with determinant χ. Define now a 2-cocycle

$$c : H^2(G_{K,S}, J/m_{gl}J \otimes_k Ad^0(\bar\rho)), \quad c(g_1, g_2) = \rho_{gl}(g_1 g_2)\rho_{gl}(g_2)^{-1}\rho_{gl}(g_1)^{-1}, \tag{3.39}$$

where we identify $J/m_{gl}J \otimes_k Ad^0(\bar\rho)$ with the kernel of the natural projection map $GL_2(R_{gl}/m_{gl}J) \to GL_2(R_{gl}/J)$. It is easy to see that the class of c in $H^2(G_{K,S}, Ad^0(\bar\rho)) \otimes_k J/m_{gl}J$ does not depend on ρ_{gl}, but only on the universal deformation ρ_S^χ and is trivial if and only if ρ_{gl} is a homomorphism.

Now, if we consider the restriction of c to $H^2(G_{K_p}, Ad^0(\bar\rho))$, this is the trivial cocycle, because $\rho_S^\chi|_{G_{K_p}}$ has a natural lifting to $GL_2(R_{gl})$. Then $c \in Ker(\theta_2) \otimes_k J/m_{gl}J$. Let $(J/m_{gl}J)^*$ denote the k-dual, then we obtain a map

$$\gamma : (J/m_{gl}J)^* \to Ker(\theta_2), \quad \gamma(u) = \langle c, u\rangle; \tag{3.40}$$

clearly $I^* \subseteq (J/\tilde m J)^*$, we claim that $Ker(\gamma) \subseteq I^*$.

Let $u \in Ker(\gamma)$ be a nonzero element; we denote by R^u_{gl} the push-out of $R_{gl}/m_{gl}J$ by u, so that $R^\chi_S \simeq R^u_{gl}/I^u$, with I^u an ideal of square zero and isomorphic to k as an R^u_{gl}-module. Since $u \in Ker(\gamma)$ we can find a representation $\rho_u : G_{\mathbb{Q},S} \to GL_2(\tilde{R}_u)$ with determinant χ which lifts ρ^χ_S. Then, by the universal property of R^χ_S the natural map $R^u_{gl} \to R^\chi_S$ has a section; it follows that $R^u_{gl} \simeq R^\chi_S \oplus I^u$ and $R^u_{gl}/pR^u_{gl} \simeq R^\chi_S/pR^\chi_S \oplus I_u$. Therefore the map $R^u_{gl} \to R^\chi_S$ does not reduce to an isomorphism on tangent spaces and it follows that the induced map

$$Ker(J/m_{gl}J \to I) \to J/m_{gl}J \to I^u \tag{3.41}$$

is not surjective and must be the zero map, that is, u factors through I and we have proved the claim.

Hence we have proved that $dim(J/m_{gl}J) = dim_k Ker(\gamma) + dim_k Im(\gamma) \leq dim(I) + r_2 = t_1 + r_2$ and we are done. $\square$

The hypotheses of the theorem are too strong for concrete applications, because they require all the functors to be representable. Therefore we want to establish a similar result in the framed setting. For the rings and ideals we have already defined, we simply add the $\square$ superscript to indicate that we are in the framed case

We need to define an auxiliary functor

$$F^{\chi,\square}_{\Sigma,S} : \underline{\hat{Ar}} \to \underline{Sets} \tag{3.42}$$

which associates to every coefficient ring A a deformation of $\bar{\rho}$ to A and a Σ-tuple of bases of V_A in the following way:

$$F^{\chi,\square}_{\Sigma,S}(A) = \{(V_A, \iota_A, (\beta_v)_{v\in\Sigma}) | (V_A, \iota_A) \in F^\chi_{\bar{\rho}}(A), \iota_A(\beta_v) = \beta \ \forall v \in \Sigma\}/\simeq \tag{3.43}$$

We have natural morphisms of functors

$$\begin{array}{ccc} F^{\chi,\square}_{\Sigma,S} & \longrightarrow & \prod_{v\in\Sigma} F_{\rho_v} \ , \\ \Big\downarrow & & \\ F^\chi_{\bar{\rho}} & & \end{array}$$

where the orizontal map is the restriction modulo each $v \in \Sigma$ and the vertical map is simply the forgetful functor which ignores bases. The following proposition describes the nature of these morphisms.

Proposition 3.5.2. *The natural morphism $F^{\chi,\square}_{\Sigma,S} \to F^\chi_{\bar{\rho}}$ is smooth and, passing to universal ring, we have an isomorphism*

$$R^{\chi,\square}_{\Sigma,S} \simeq R^\chi_S[[x_1, \ldots, x_{4|\Sigma|-1}]]. \tag{3.44}$$

Moreover the morphism $F^{\chi,\square}_{\Sigma,S} \to \prod_{v\in\Sigma} F_{\rho_v}$ *gives a homomorphism of universal rings*

$$R_{loc} = \hat{\otimes}_{v\in\Sigma} R^{\chi,\square}_v \to R^{\chi,\square}_{\Sigma,S}. \tag{3.45}$$

Proof. The smoothness and the dimension formula come from the smoothness of the framed deformation functor over the unframed one and the morphism of universal rings comes naturally from the morphism of functors. □

The passage to local rings is the key for computing $R^{\chi,\square}_{\Sigma,S}$. The use of framed deformations avoids the representability issues.

We can now state one of the main results of this approach. We need a generalization of the map θ_1, defined at the beginning of the chapter

Lemma 3.5.3 (Key lemma)**.** *Let*

$$\theta_1^{\square} : F^{\chi,\square}_{\Sigma,S}(k[\epsilon]) \to \bigoplus_{v\in\Sigma} F^{\chi,\square}_{\rho_v}(k[\epsilon]) \tag{3.46}$$

be the usual restriction map on tangent spaces and set $r = dim_k Ker(\theta_1^{\square})$ *and* $t = dim_k Ker(\theta_2) + dim_k coKer(\theta_1^{\square})$. *Then we have a presentation*

$$R^{\chi,\square}_{\Sigma,S} \simeq R_{loc}[[x_1, ..., x_r]]/(f_1, ..., f_t) \tag{3.47}$$

Proof. The proof is the same of theorem 3.5 in the unframed setting, simply substituting R^{χ}_S with $R^{\chi,\square}_{\Sigma,S}$ and the cohomological groups and the map θ with their framed counterparts.

□

Observe that r is an optimal value, while t is just an upper bound on the number of relations; for example some of the f_i may be trivial.

Now we need also to evaluate $\delta = dim_k coKer(\theta_2)$. Note that θ_2 is part of the Poitou-Tate sequence (see 3.1 for references) and that $H^2(G_v, Ad^0(\bar{\rho})) \simeq H^0(G_v, Ad^0(\bar{\rho})^*)^*$ by local Tate duality. Therefore

$$\begin{aligned} \delta = dim_k coKer(\theta_2) = \\ = dim_k Ker(H^0(G_{K,S}, Ad^0(\bar{\rho})^*) \to \bigoplus_{v\in S\setminus\Sigma} H^0(G_v, Ad^0(\bar{\rho})^*)). \end{aligned} \tag{3.48}$$

Note that $\delta = 0$ if $S\setminus\Sigma$ is non-empty, and therefore contains a finite prime, or if the image of $\bar{\rho}$ is non-solvable, and therefore $H^0(G_{K,S}, Ad^0(\bar{\rho})^*)$ is trivial.

The following result gives us a link between all the quantities we have defined

Theorem 3.5.4. *If* Σ *contains all the places above* p *and* ∞*, then* $r - t + \delta = |\Sigma| - 1$.

Proof. We will make use of the Tate's computation of the Euler-Poincaré characteristic (a proof of which can be found in [15])

$$c_{EP}(G, Ad^0(\bar{\rho})) = -[K:\mathbb{Q}]dim_k Ad^0(\bar{\rho}) + \sum_{v|\infty} h^0(G_v, Ad^0(\bar{\rho}_v)) \tag{3.49}$$

and the local version

$$c_{EP}(G_v, Ad^0(\bar{\rho}_v)) = \begin{cases} -dim_k Ad^0(\bar{\rho}_v)[K_v:\mathbb{Q}_p] & \text{if } v|p \\ h^0(G_v, Ad^0(\bar{\rho}_v)) & \text{if } v|\infty \\ 0 & \text{otherwise} \end{cases} \tag{3.50}$$

Then we have

$$\begin{aligned} r - t + \delta = dim_k Ker(\theta_1^{\square}) - dim_k coKer(\theta_1^{\square}) - dim_k Ker(\theta_2) + \\ + dim_k coKer(\theta_2) = dim_k F_{\Sigma,S}^{\chi,\square}(k[\epsilon]) - \sum_{v\in\Sigma} dim_k F_v(k[\epsilon]) - h^2(G, Ad^0(\bar{\rho})) + \\ + \sum_{v\in\Sigma} h^2(G_v, Ad^0(\bar{\rho}_v)). \end{aligned} \tag{3.51}$$

Now we evaluate the dimensions of tangent spaces and we have

$$\begin{aligned} h^1(G, Ad^0(\bar{\rho})) - h^0(G, Ad^0(\bar{\rho})) - 1 + |\Sigma|n^2 - h^2(G, Ad^0(\bar{\rho})) + \\ - \sum_{v\in\Sigma}(h^1(G_v, Ad^0(\bar{\rho}_v)) - h^0(G_v, Ad^0(\bar{\rho}_v)) - 1 + n^2 - h^2(G_v, Ad^0(\bar{\rho}_v))) = \\ = -c_{EP}(G, Ad^0(\bar{\rho})) + \sum_{v\in\Sigma} c_{EP}(G_v, Ad^0(\bar{\rho}_v)) + |\Sigma| - 1. \end{aligned} \tag{3.52}$$

Finally we use Tate's formulas for c_{EP} and we have

$$\begin{aligned} [K:\mathbb{Q}]dim_k Ad^0(\bar{\rho}) - \sum_{v|\infty} h^0(G_v, Ad^0(\bar{\rho}_v)) - \sum_{v|p}[K_v:\mathbb{Q}_p]dim_k Ad^0(\bar{\rho}_v) + \\ + \sum_{v|\infty} h^0(G_v, Ad^0(\bar{\rho}_v)) + |\Sigma| - 1 = |\Sigma| - 1. \end{aligned} \tag{3.53}$$

□

3.6 Geometric deformation rings

In this chapter we want to give some results about a particular class of deformation problems. We suppose that the field K is totally real and that our $\bar{\rho}$ is odd of dimension 2 and absolutely irreducible (so that the deformation functor is representable). The rest of the notation matches the one of the previous chapter.

For each $v \in \Sigma$ let $\bar{F}_v^{\chi,\square}$ be a representable subfunctor of $F_v^{\chi,\square}$ such that the corresponding representing ring $\bar{R}_v^{\chi,\square}$ (which is a quotient of $R_v^{\chi,\square}$) satisfies the following properties:

- $\bar{R}_v^{\chi,\square}$ is flat over $\mathbb{Z}_p$
- $\bar{R}_v^{\chi,\square}[1/p]$ is regular of dimension $\begin{cases} 3 & \text{if } v \neq p, \infty \\ 3 + [K_v : \mathbb{Q}_p] & \text{if } v|p \\ 2 & \text{if } v|\infty \end{cases}$

A deformation functor satisfying these properties will be called a *geometric deformation functor*. Such a functor satisfies the following properties:

- $\bar{R}_{loc} = \hat{\otimes}_{v\in\Sigma}\bar{R}_v^{\chi,\square}$ is flat over $\mathbb{Z}_p$ and its Krull dimension is $\geq 3|\Sigma| + 1$.
- The functors $\bar{F}_S^{\chi}$ and $F_{\Sigma,S}^{\chi,\square}$ are representable.
- The ring $\bar{R}_{\Sigma,S}^{\chi,\square}$ is isomorphic to $R_{\Sigma,S}^{\chi,\square}\hat{\otimes}_{v\in\Sigma}\bar{R}_{loc}$ and therefore

$$\bar{R}_{\Sigma,S}^{\chi,\square} \simeq \bar{R}_{loc}[[X_1, ..., X_r]]/(f_1, ..., f_t) \tag{3.54}$$

  with r, t defined as before. In particular the Krull dimension of $\bar{R}_{\Sigma,S}^{\chi,\square} \geq 4|\Sigma| - \delta$.

Since the map $\bar{F}_{\Sigma,S}^{\chi,\square} \to \bar{F}_S^{\chi}$ is smooth, we can obtain the following result

Theorem 3.6.1. *If $\delta = 0$, then $dim_{Krull}\bar{R}_S^{\chi} \geq 1$.*

3.7 The main result: dimension 2 case

Now we have all the necessary instruments to generalize the results of the previous chapter. Let $\bar{\rho}_1, \ldots, \bar{\rho}_n$ be representations of $G_{\mathbb{Q}}$ each with values in $GL_2(k)$, where k is a finite field of characteristic p, and such that each $V_{\bar{\rho}_i}$ is the generic fiber of a finite flat group scheme contained in a subcategory $\underline{D}$ closed by products, subobjects and quotients. We write

$$\bar{\rho} = \bar{\rho}_1 \oplus \ldots \oplus \bar{\rho}_n : G_{\mathbb{Q}} \to GL_{2n}(k). \tag{3.55}$$

It may happen that some of the $\bar{\rho}_i$ are isomorphic. Therefore we suppose that there are exactly r different representations among the $\bar{\rho}_i$ which are non-isomorphic and we assume them to be $\bar{\rho}_1, \ldots, \bar{\rho}_r$. Then we rewrite $\bar{\rho}$ as

$$\bar{\rho} = \bigoplus_{i=1}^{r} \bar{\rho}_i^{e_i}. \tag{3.56}$$

We want to extend the deformation functor of chapter 1 to this case. We start considering the single representation $\bar{\rho}_i$. Let $V_{\bar{\rho}_i}$ be the G-module associated to $\bar{\rho}_i$. We define the deformation functor $F_{\bar{\rho},\underline{D}} : \underline{Ar} \to \underline{Sets}$ which sends an artinian ring A to the set of deformation classes ρ_i of $\bar{\rho}_i$ to A such that

- ρ_i is p-flat over $\mathbb{Z}[1/\ell]$;
- ρ_i satisfies $(\rho_i(g) - Id)^2 = 0$ for every $g \in I_\ell$;
- ρ_i is odd;

and let $F_{\bar{\rho},\underline{D}} : \underline{Ar} \to \underline{Sets}$ be the deformation functor associated to $\bar{\rho}$ with the same local conditions.

Lemma 3.7.1. *$F_{\bar{\rho}_i,\underline{D}}$ is a geometric deformation functor.*

Proof. We need to show that our local conditions satisfy the geometric properties defined in section **??**. At the prime p we apply theorem 3.2.5 which tells us that the local ring is isomorphic to $\mathbb{Z}_p[[X]]$; in particoular its framed counterpart has dimension 4 over $\mathbb{Z}_p$, as in the geometric conditions. At the infinite prime, the result of section 3.4 tells us that the dimension over $\mathbb{Z}_p$ of the framed deformation ring is 2. Finally at the prime ℓ the condition that $(\rho_i(\sigma) - id)^2 = 0$ is equivalent to a Steinberg type condition with λ equal to the trivial character. Therefore theorem 3.3.5 gives us that the framed deformation ring has Krull dimension 4; in particular its dimension over $\mathbb{Z}_p$ is 3. It follows that all the conditions of being a geometric deformation functor are satisfied. Then we can apply theorem 3.6.1 and obtain that each $F_{\bar{\rho}_i}$ has a representing ring of Krull dimension at least 1; in particular each $\bar{\rho}_i$ has a lift ρ_i to characteristic zero. □

Theorem 3.7.2 (Main theorem: dimension 2 case)**.** *Suppose that:*

1. *$Ext^1_{\underline{D},p}(V_{\bar{\rho}_i}, V_{\bar{\rho}_j})$ of killed-by-p extensions is trivial for every $i, j = 1, \ldots, r$;*
2. *$Hom_G(V_{\bar{\rho}_i}, V_{\bar{\rho}_j}) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.*

Then the functor $F^{\square}_{\bar\rho,\underline{D}}$ is represented by a power series ring over $W(k)$ in N variables, where

$$N = 4n^2 - \sum_{i=1}^{r} e_i^2. \tag{3.57}$$

Proof. The representation $\bar\rho$ has the following matrix form

$$\begin{pmatrix} \begin{pmatrix} \bar\rho_1 & & \\ & \ddots & \\ & & \bar\rho_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \bar\rho_r & & \\ & \ddots & \\ & & \bar\rho_r \end{pmatrix} \end{pmatrix}. \tag{3.58}$$

We call $\bar T$ this matrix and $be\bar{t}a$ a k-basis of $V_{\bar\rho}$ in which $r\bar{h}o$ has this matrix form; $\bar T$ belongs to $M_h(k)$ where we denote by $h = 2n$. We also denote by $h_j = \sum_{i=1}^{j-1} 2e_i$.

By the lemma, we know that each $\bar\rho_i$ has a deformation ρ_i to $W(k)$; the hypothesis of triviality of extension set tells us that the tangent space of the deformation functor $F_{\bar\rho_i}$ is trivial, therefore the universal deformation ring is a quotient of $W(k)$. But we know that a deformation to $W(k)$ exists, given by ρ_i, therefore it must be the universal one.

Let then T be the matrix obtained by $\bar T$ replacing all the $\bar\rho_i$ with the respective ρ_i, V_ρ the associated representation module over $W(k)$ and β a basis of V_ρ lifting β in which T has the block-diagonal shape. We look for a framed deformation of $\bar T$ of the form

$$\tilde T = (1 + M(\underline{x}))T(1 + M(\underline{x}))^{-1} \tag{3.59}$$

where $M = M(\underline{x})$ is the matrix having a variable $x_{i,j}$ as (i,j)-th entry and $\underline{x}$ is the array of all such x. We write M as

$$\begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,r} \\ M_{2,1} & M_{2,2} & & \vdots \\ \vdots & & \ddots & \\ M_{r,1} & M_{r,2} & \dots & M_{r,r} \end{pmatrix}. \tag{3.60}$$

where $M_{i,j}$ is the $2e_i \times 2e_j$ submatrix given by

$$M_{i,j} = \begin{pmatrix} x_{h_i+1,h_j+1} & \dots & x_{h_i+1,h_{j+1}} \\ \vdots & \ddots & \vdots \\ x_{h_{i+1},h_j+1} & \dots & x_{h_{i+1},h_{j+1}} \end{pmatrix}. \tag{3.61}$$

Then we have that $(\tilde{T}, \beta(1+M))$ gives a framed deformation of $\bar{\rho}$ to the ring $R = W(k)[[x_{1,1}, \ldots, x_{2n,2n}]]$.

We want to modify our deformation by a linear transformations lying in the centralizer of ρ to kill some of the variables, as we did in the elliptic curve case. Consider the diagonal submatrices $M_{i,i}$; we can eventually subdivide it in 2×2 submatrices

$$\begin{pmatrix} M_{i,i}^{(1,1)} & \ldots & M_{i,i}^{(1,e_i)} \\ \vdots & \ddots & \vdots \\ M_{i,i}^{(e_i,1)} & \ldots & M_{i,i}^{(e_i,e_i)} \end{pmatrix}, \tag{3.62}$$

where

$$M_{i,i}^{(s,t)} = \begin{pmatrix} x_{h_i+2s-1,h_i+2t-1} & x_{h_i+2s-1,h_i+2t} \\ x_{h_i+2s,h_i+2t-1} & x_{h_i+2s,h_i+2t} \end{pmatrix}. \tag{3.63}$$

Since $M_{i,i}^{(s,t)}$ is a 2×2 matrix and we have the trivial Ext condition, we want apply a generalisation of the construction used Corollary 2.4.2. We look for a matrix $Y \in M_h(R)$ such that $1+Y$ commutes with T and the conjugation by $1+Y$ does not modify the framed deformation class. The hypothesis 2 on the mutual endomorphisms of the ρ_i implies that Y must be a block diagonal matrix of the form

$$Y = diag[Y_1, \ldots, Y_r] \tag{3.64}$$

where $Y_i = A_i \otimes id_2 \in M_{2e_i}(R)$ and $A_i \in M_{e_i}(R)$.

Now we need to choose properly the entries $\{a_{ist}\}_{s,t=1,\ldots,e_i}$ of the matrices Y_i. Let

$$(1+M)(1+Y)\tilde{T}(1+Y)^{-1}(1+M)^{-1} = (1+\tilde{M})\tilde{T}(1+\tilde{M})^{-1}. \tag{3.65}$$

Following Corollary 2.4.2, we set

$$a_{ist} = \frac{-x_{h_i+2s,h_i+2t}}{1+x_{h_i+2s,h_i+2t}}. \tag{3.66}$$

The resulting matrix $\tilde{M}$ has entries $\tilde{x}_{u,v}$ given by

$$\tilde{x}_{u,v} = \begin{cases} 0 & \text{if } u = h_i+2s,\ v = h_i+2t \\ \frac{x_{u,v}}{1+x_{h_i+2s,h_i+2t}} & \text{otherwise} \end{cases} \tag{3.67}$$

To make the notation easier, we rename $\tilde{M} = M$ and $\tilde{x}_{i,j} = x_{i,j}$. We call $(\tilde{\rho}, \tilde{\beta}(1+Y))$ the resulting framed deformation obtained at the end of this process.

The framed deformation $\tilde{\rho}$ has values in the ring

$$\tilde{R} = W(k)[[x_{1,1}, \ldots, x_{2n,2n}]]/(x_{h_i+2s,h_i+2t} : s,t = 1, \ldots, e_i,\ i = 1, \ldots, r) \tag{3.68}$$

We need to show that this is effectively the universal framed deformation. Observe that $\tilde{R}$ is a power series ring over $W(k)$ in exactly N variables. First we need to compute the dimension of the framed tangent space. We use the fact that the tangent space $F^{\square}_{\bar{\rho},S}(k[\epsilon])$ fits the exact sequence

$$0 \to F_{\bar{\rho},S}(k[\epsilon]) \to F^{\square}_{\bar{\rho},S}(k[\epsilon]) \to Ad(\bar{\rho})/Ad(\bar{\rho})^G \to 0 \tag{3.69}$$

and that the unframed tangent space is trivial, because of the triviality of the extension set. Note that

$$Ad(\bar{\rho})^G \;=\; End_G(\bar{\rho}_1^{e_1} \oplus \cdots \oplus \bar{\rho}_r^{e_r}) \;=\; \bigoplus_{i=1}^{r} End_G(\bar{\rho}_i^{e_i}) \;=\; \bigoplus_{i=1}^{r} M_{e_i}(k) \tag{3.70}$$

where we have used the hypothesis on the sets $Hom_G(V_{\bar{\rho}_i}, V_{\bar{\rho}_j})$.

Therefore we have

$$dim(F^{\square}_{\bar{\rho},S}(k[\epsilon])) = dim(Ad(\bar{\rho})) - dim(Ad(\bar{\rho})^G) =$$

$$= 4n^2 - \sum_{i=1}^{r} {e_i}^2 = N, \tag{3.71}$$

then the universal framed deformation ring $R^{\square}_{\bar{\rho},S}$ and $\tilde{R}$ have the same relative Krull dimension.

Now we use the universality of $R^{\square}_{\bar{\rho},S}$ that gives us a unique $W(k)$-algebra morphism $\pi : R^{\square}_{\bar{\rho},S} \to \tilde{R}$ such that $\hat{\pi} \circ \rho_{univ} = \tilde{\rho}$ where ρ_{univ} is the universal representation and $\hat{\pi}$ is the extension of π to GL_2. We have a diagram

$$\begin{array}{ccc} W(k)[[x_1, \ldots, x_N]] & \xrightarrow{\pi_1} & R^{\square}_{\bar{\rho},S} \\ & \searrow^{\pi_2} & \downarrow \pi \\ & & \tilde{R} \qquad \simeq W(k)[[y_1, \ldots, y_N]]. \end{array}$$

If the map π is surjective, since π_1 is surjective, too, it follows that π_2 is surjective, too. But π_2 is $W(k)$-algebra map between algebras of the same dimension and therefore it must be an isomorphism. But then π must be an isomorphism, too. The theorem is therefore proved, provided that π is surjective.

To prove that the map π is surjective, it is enough to show that the induced map on mod p tangent space

$$\tilde{\pi}_2 : Hom(\tilde{R}/p, k[\epsilon]) \to Hom(R^{\square}_{\bar{\rho},S}/p, k[\epsilon]) \tag{3.72}$$

is injective (because the functor $Hom(., k[\epsilon])$ is contravariant). Since $\tilde{R}$ is a power series ring over $W(k)$ in N variables, an element of $Hom(\tilde{R}/p, k[\epsilon])$ is given by a map which sends the variables $x_1, \ldots, x_N$ to elements $\epsilon\alpha_1, \ldots, \epsilon\alpha_N$ with $\alpha_1, \ldots, \alpha_N$ giving a basis for the $k[\epsilon]$-module V given by a representation lifting $V_{\bar{\rho}}$; different elements are given by different choices of the basis. Suppose then that two elements $(V, \{\alpha_i\}), (V, \{\alpha_j\})$ have the same image with respect to $\tilde{pi}_2$; it means that there exists a matrix $A \in GL_n(k[\epsilon])$ whose conjugation maps the basis $\{\alpha_i\}$ into $\{\alpha_j\}$ and A commutes with the representation, that is, lies in the centralizer of the image of $\bar{\rho}$. But, beacuse of the construction of the representation $\bar{\rho}$ such matrix must be the identity. Therefore the map is injective and the theorem is proved.

□

As an application of the theorem, consider an abelian variety A over $\mathbb{Q}$ which has good reduction in all but one prime ℓ, where it has semistable reduction. By [12, Th. 1.2], if $\ell = 11$ then A is isogenous to a product of copies of $E = J_0(11)$. Moreover A is supersingular at 2 and $A[2] \simeq E[2]^g$. Therefore we can look at the natural $G_{\mathbb{Q}}$-representation $\bar{\rho}_{A,2}$ on the 2-torsion points of A as product of g copies of the reperesentation $\bar{\rho}_{E,2}$. In formulas

$$\bar{\rho}_{A,2} = \bigoplus_{i=1}^{g} \bar{\rho}_{E,2}. \tag{3.73}$$

Then we can study the deformations of $\bar{\rho}_{A,2}$ from the ones of $\bar{\rho}_{E,2}$. Applyting the theorem we have that the functor $F_{\bar{\rho}_{A,2},\underline{D}}$ is represented by a power series ring over $\mathbb{Z}_p$ in $3g^2$ variables. Moreover, if we go trough the same construction as in the theorem, we have that the universal deformation is given taking the product of g copies of the $\mathbb{Z}_p$-representation given by the Tate module T_pE and then applying the transformation with the matrix M.

Bibliography

[1] N. Bourbaki, *Algebre*, Ch. VIII, Hermann, (1958).

[2] B. Conrad, F. Diamond and R. Taylor, Modularity of Potentially Barsotti-Tate Galois Representations, *J. Amer. Math. Soc.*, **12**, pp. 521-567, (1999).

[3] B. Conrad, Ramified Deformation Problems, *Duke Math. J.*, **97**, pp. 439-513, (1999).

[4] B. Conrad, The Flat Deformation Functor, in *Modular Forms and Fermat's Last Theorem*, pp. 373-420, Springer-Verlag, New York, 1998.

[5] B de Smit and H. W. Lenstra, Explicit Construction of Universal Deformation Rings, in *Modular Forms and Fermat's Last Theorem*, pp.313-326, Springer-Verlag, New York, 1998.

[6] J. M. Fontaine and G. Lafaille, Construction de Representations p-adiques, *Ann. Sci. Ecole Norm. Sup.*, **15**, pp. 547-608, (1983).

[6] M. Kisin, Modularity of 2-dimensional Galois Representations, *Current developments in mathematics 2005*, pp.191-230.

[7] M. Kisin, Moduli of Finite Flat Group Schemes and Modularity, *Annals of Mathematics*, **170**, pp.1084-1180, (2009)

[8] B. Mazur, Deforming Galois Representations, in *Galois Groups over* $\mathbb{Q}$, pp. 385-437, MSRI Publications, Vol. 16, Springer-Verlag, New York/Berlin/Heidelberg, 1987.

[9] B. Mazur, An Introduction to the Deformation Theory of Galois Representations, in *Modular Forms and Fermat's Last Theorem*, pp. 243-312, Springer-Verlag, New York, 1998.

[10] R. Ramakrishna, On a variation of Mazur's deformation functor, *Comp. Math.*, **87**, pp.269-286, (1993).

[11] M. Schlessinger, Functors of Artin rings, *Trans. Amer. Math. Soc.*, **130**, pp. 208-222, (1968).

[12] R. Schoof, Abelian varieties over **Q** with bad reduction in one prime only, *Comp. Math.*, **141**, 847-868, (2005).

[13] J. T. Tate, Finite Flat Group Schemes, in *Modular Forms and Fermat's Last Theorem*, pp. 121-154, Springer-Verlag, New York, 1998.

[14] J. T. Tate, p-divisible groups, in *Proceedings of a Conference on local fields*, pp. 158-183 Driebergen, 1966.

[15] L. Washington, Galois Cohomology, in *Modular Forms and Fermat's Last Theorem*, pp. 101-120, Springer-Verlag, New York, 1998.

[16] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.