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QUASI-LINEAR FORWARD-BACKWARD PARABOLIC EQUATIONS

Ph.D. Thesis in Mathematics

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Contents

Introduction

The initial-boundary value problem for the quasi-linear diffusion equation

$$u_t = \Delta\phi(u) \tag{1}$$

has a unique solution if the function ϕ is monotone increasing with $\phi' \geq c > 0$, such solution being, roughly speaking, as smooth as the function ϕ ([Be], [LSU]). On the other hand, if $\phi' \leq c < 0$, equation (1) is of *backward* parabolic type and, in view of the smoothing effect, the initial-boundary value problem for such an equation is in general *ill-posed*, since it may have a solution only for special initial data ([Pay]).

In this thesis we consider non-linearities ϕ whose main feature is their *non-monotone* character. In this case equation (1) is said to be a *forward-backward* parabolic equation, since it is well-posed forward in time at points such that $\phi'(s) > 0$, whereas it is ill-posed (forward in time) where $\phi'(s) < 0$. For, in the following the intervals where $\phi' > 0$ will be referred to as the *stable* phases, and the intervals where $\phi' < 0$ as the *unstable* phases of equation (1).

Most non-linearities ϕ considered in the literature belong to two different classes:

(i) a *cubic-like* ϕ satisfying the assumption

$$(H_1) \qquad \begin{cases} \phi(s) \to \pm\infty \quad \text{as } s \to \pm\infty, \\ \phi'(s) > 0 \quad \text{if } s < b \text{ and } s > c, \\ \phi'(s) < 0 \quad \text{if } b < s < c, \\ \phi''(b) \neq 0, \quad \phi''(c) \neq 0, \\ A := \phi(c) < \phi(b) =: B \end{cases}$$

(see Fig.1);

(ii) a function ϕ with *degeneration at infinity*, which satisfies the following assumption:

$$
(H_2) \qquad \begin{cases} \phi(s) > 0 \ \text{ if } s > 0, \ \ \phi(s) = -\phi(-s) \text{ if } s < 0, \\ \phi(0) = 0 \text{ and } \phi(s) \to 0 \text{ as } s \to +\infty, \\ \phi \in L^p(\mathbb{R}) \text{ for some } p \in [1, \infty), \\ \phi'(s) > 0 \ \text{ if } 0 \le s < 1, \ \ \phi'(s) < 0 \ \text{ if } s > 1, \\ \phi(1) = 1\,, \ \ \phi''(1) \neq 0 \end{cases}
$$

(see Fig.2). Both types are suggested by specific physical and biological models, as discussed in the following subsection.

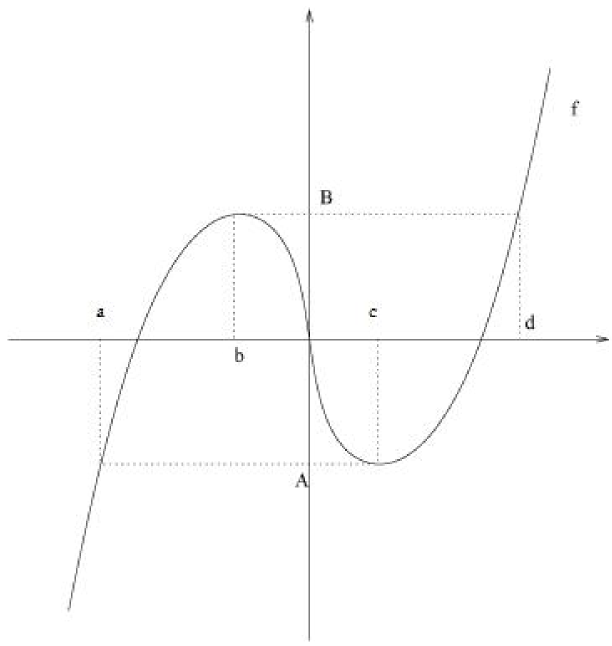

Figure 1: Assumption (H_1).

Motivations

Forward-backward parabolic equations with a cubic-like ϕ naturally arise in the theory of phase transitions. In this context the function u represents the *phase field*, whose values characterize the difference between the two phases (*e.g.*, see [BS]). Therefore the half-lines $(-\infty, b)$ and (c, ∞) correspond to stable phases, the interval (b, c) to an unstable phase (*e.g.*, see [MTT]), and equation (1) describes the dynamics of transition between stable phases.

Concerning assumption (H_2), various physical and biological phenomena modelled by means of equation (1) have been proposed in the literature, *e.g.* a continuum model for movements of biological organisms ([HPO]), and a

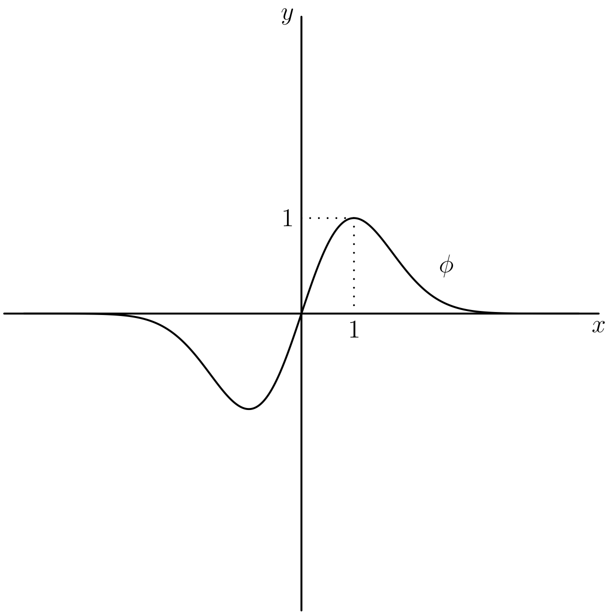

Figure 2: Assumption (H_2).

continuous approximation to a discrete model for *aggregating* populations ([Pa]). In the latter case a typical choice of the function ϕ is

$$\phi(u) = u \exp(-u),$$

where the unknown $u \geq 0$ represents the population density, while the transition probability (namely, the probability that an individual moves from its location) $p(u) = \exp(-u)$ models aggregation phenomena, for it is a decreasing function of u.

An independent, relevant motivation to study equation (1) subject to assumption (H_2) comes from the context of image processing. In 1990 P. Perona and J. Malik introduced an edge enhancement model, with the aim of denoising a given image u_0 while at the same time controlling blurring ([PM]). The non-linear diffusion equation they proposed, thereafter known as the *Perona-Malik equation*, reads

$$z_t = \operatorname{div}\left[\sigma(|\nabla z|)\nabla z\right]. \tag{2}$$

Typical choices of the function σ are $\sigma(s) = (1+s^2)^{-1}$, $\sigma(s) = \exp(-s)$. In the one-dimensional case, the equation reduces to

$$z_t = [\phi(z_x)]_x, \tag{3}$$

with $\phi(s) = s(1+s^2)^{-1}$ or $\phi(s) = s\exp(-s)$. Deriving equation (3) with respect to x and setting $u := z_x$ formally gives equation (1), with ϕ satisfying assumption (H_2).

In [BBDU] equation (3) arises as a mathematical model for heat transfer in a stably stratified turbolent shear flow. Here the temperature $w \geq 0$ satisfies the equation

$$w_t = [k w_x]_x \ , \tag{4}$$

and under fixed external conditions the function k only depends on the gradient of the temperature, namely

$$k = \sigma(w_x) \, . \tag{5}$$

Moreover, a typical choice of the function σ is $\sigma(s) = \frac{A}{B+s^2}$. Setting $\phi(s) := s\sigma(s)$ and combining (4)-(5) gives equation (3).

Finally, let us also mention that equation (3) with assumption (H_2) can be regarded as the formal L^2-gradient system associated with a *nonconvex* energy density ψ in one space dimension (in this case $\phi = \psi'$) of the form $\psi(s) = \log(1+s^2)$ ([BFG]). Analogously, the choice of the *double well potential* $\psi(s) = (1-s^2)^2$ leads naturally to equation (3) for a cubic-like ϕ satisfying assumption (H_1). Therefore the dynamics described by (3) (hence by equation (1)) in one space dimension is relevant to various settings, where nonconvex functionals arise (in this respect, see [Mü] for motivations in nonlinear elasticity).

How to regularize?

As already remarked, the lack of forward parabolicity in equation (1) under both assumptions $(H_1) - (H_2)$ gives rise to ill-posed problems. As a consequence, both development of singularities and lack of regularity can be expected, when considering initial data u_0 which take values in the unstable phase.

As a matter of fact, existence of solutions to the Neumann initial-boundary value problem for the Perona-Malik equation (3) has been proven if the derivative of the initial datum u_0 takes values in the stable phase ([KK]), while for large values of $|u_0'|$ no *global* C^1-solution exists ([G], [K]). This shows that even *local existence* of solutions (in some suitable functional space) to the initial-boundary value problem for equation (1) (or (3)) is a non-trivial problem (in this connection see also the numerical experiments in [BFG], [FGP], [NMS] and [SSW]).

On the other hand, the *uniqueness problem* is even more cumbersome. In the pioneering work [H] it was shown that, concerning the Neumann initial-boundary value problem for equation (3), infinitely many weak L^2-solutions can be constructed, if ϕ is a non-monotone piecewise linear function satisfying the coercivity condition $s\phi(s) \geq cs^2$ for some constant $c > 0$. This yields existence of infinitely many weak solutions to the forward-backward equation (1) under assumption (H_1). Although the assumptions in [H] are not satisfied if (H_2) holds, even in this case a general nonuniqueness result has been proven. In fact, the existence of infinitely many weak $W^{1,\infty}$-solutions for equation (3) (thus the existence of infinitely many L^∞-solutions for equation (1)) under assumption (H_2) was proven in [Z]. The techniques used in [Z] consist in rephrasing the Neumann problem for equation (3) into a partial differential inclusion problem, and are very different from ours (see the subsection below).

When dealing with phenomena as above, a widely accepted idea is that ill-posedness derives from neglecting some relevant information in the modelling of the physical phenomenon. Hence a general strategy is to restore this information by introducing additional relations, which define a *restricted class* of *admissible solutions* where the problem is expectedly well-posed. To this purpose, a natural approach to address equations (1), (3) is to modify the equation (and perhaps the boundary conditions) by introducing some physically sensible *regularization* which leads to a well-posed problem. Then the problem that arises is to describe the *limiting points* of the family of *approximate* solutions as the regularization parameter goes to zero. A natural question is whether such limiting points, obtained by means of the approximating process, define solutions (in some suitable sense depending on the regularization itself) to the initial-boundary value problem for the original ill-posed equation.

In this general framework, different regularizations have been proposed and investigated. Among them, let us first mention the *fourth-order* regularization, which leads to *the Cahn-Hilliard equation*

$$u_t = \Delta[\phi(u) - \varepsilon\Delta u]\,. \tag{6}$$

Equation (6) was introduced by Cahn in [C] for a non-linearity ϕ satisfying assumption (H_1), with the aim to describe isothermal phase separation of binary mixture quenched into an unstable homogeneous state.

Regularization (6) was used in [Sl] to address both the Dirichlet and Neumann initial-boundary value problems for equation (3), when ϕ satisfies assumption (H_1) (see [BFG] for the case (H_2)). Using the Young measure representation of composite weak limits (*e.g.*, see [GMS], [E2], [V]), it was proven that the family of approximate solutions to the regularized problems for (6) converges to a *measure-valued solution* of the initial-boundary value problem for the original unperturbed equation (3) (in this connection see also [Pl4]). Such a result is not surprising, for Young measures - and consequently measure-valued solutions - naturally arise when describing rapid oscillations that may appear in the limiting behaviour of solutions to nonlinear evolution equations ([D], [RH]).

A second, widely investigated regularization is the *pseudoparabolic* or *Sobolev* regularization, which leads to the equation

$$u_t = \Delta\phi(u) + \varepsilon\Delta u_t\,. \tag{7}$$

The term Δu_t can be interpreted by taking viscous relaxation effects into account (see [NP], [BFJ]).

The Neumann initial-boundary value problem for equation (7) was studied in [NP] under assumption (H_1), and in [Pa] under assumption (H_2). In both cases global existence and uniqueness of the solution u^ε is proven to hold in $L^\infty(Q_T)$ ($Q_T := \Omega \times (0,T)$) for any $\varepsilon > 0$. Moreover, solutions of equation (7) satisfy a class of *viscous entropy* inequalities, this parlance being suggested by a formal analogy with the entropy inequalities for *viscous conservation law* (see [E2], [MTT] and [Se]). As is well known, such entropy inequalities carry over to weak solutions of the Cauchy problem for the *first order hyperbolic conservation law* in the *vanishing viscosity* limit $\varepsilon \to 0$ (*e.g.*, see [Se]). These limiting entropy inequalities define the class of the *entropy solutions*, which is shown to be a well-posedness class for the original problem. Therefore, it is natural to wonder whether in the limit $\varepsilon \to 0$ it is possible to prove existence and uniqueness of suitably defined *weak entropy* solutions for the original equation (1).

In this direction, an exhaustive answer has been given in [Pl1] for the case of a cubic-like ϕ. In view of assumption (H_1), it turns out that the family $\{u^\varepsilon\}$ of solutions to the regularized Neumann initial-boundary value problem for equation (7) is uniformly bounded in the L^∞-norm, and the limiting points (u,v) of the families $\{u^\varepsilon\}\,,\ \{\phi(u^\varepsilon)\}$ satisfy in the weak sense the limiting equation

$$u_t = \Delta v \qquad \text{in } \ \mathcal{D}'(Q_T) \tag{8}$$

with initial datum u_0 and Neumann boundary conditions. Equation (8) would give a weak solution of the Neumann initial-boundary value problem for (1), if we had $v = \phi(u)$; however, no such conclusion can be drawn, due to the nonmonotone character of ϕ.

In this connection, in [Pl1] it is shown that the couple (u,v) is a *measure-valued* solution in the sense of Young measures to the unperturbed equation (1). With respect to the results in [Sl] for the Cahn-Hilliard regularization, the novel feature here is the study of the family $\{\tau^\varepsilon\}$ of Young measures associated to the approximate solutions u^ε, and the *characterization* of the *disintegration* $\nu_{(x,t)}$ of any Young measure τ obtained as the *narrow* limit of such measures (see [E1], [GMS], [V]). In particular, it is proven that the disintegration $\nu_{(x,t)}$ is an atomic measure given by the *superposition* of three Dirac masses concentrated on the branches $s_0,\ s_1,\ s_2$ of the equation $v = \phi(u)$. Hence the function u obtained as $\varepsilon \to 0$ has the following representation:

$$u = \sum_{i=0}^{2} \lambda_i s_i(v), \tag{9}$$

for some positive coefficients $\lambda_i \in L^\infty(Q_T)$ such that $\sum_{i=0}^{2} \lambda_i = 1$ (see [E2], [GMS], [V]). Equality (9) can be explained by saying that the function u takes the *fraction* λ_i of its value at (x, t) on the branch $s_i(v)$ of the graph of ϕ. Then the coefficients λ_i can be regarded as *phase fractions*, and u itself as a *superposition of different phases.*

Finally, the solutions (u, v) so obtained satisfy a class of suitable *limiting entropy* inequalities. This is why any couple (u, v) obtained from the Sobolev equation (7) via the above limiting procedure is said to be a *weak entropy measure-valued* solution of the initial-boundary value problem associated to equation (1).

In spite of the formal analogy with the case of hyperbolic conservation laws, no uniqueness result of weak entropy measure-valued solutions has been proven, although such solutions seem a natural candidate in this sense. In this respect it can be argued that the class of solutions considered in [Pl1] is still too wide, and that uniqueness results might be recovered when considering a more restricted class, defined by additional constraints. To this purpose, again for a function ϕ subject to assumption (H_1), in [EP] the choice of *two-phase entropy* solutions has been suggested. Roughly speaking, two-phase solutions of equation (1) occur when admitting transition *only* between stable phases. Such a transition is described by an interface which evolves in time, obeying suitable *admissibility conditions* (resulting from the entropy inequalities) which select *admissible jumps between the stable phases* (see [MTT]). Local existence and uniqueness of solutions of this kind have been proven in [MTT2] for the Cauchy problem associated to equation (1) in the case of a piecewise linear ϕ. However, it should be observed that such two-phase solutions *are not obtained* as limiting points of approximate solutions to some regularization of equation (1).

Finally, in [BBDU] the regularization

$$z_t = [\phi(z_x)]_x + \varepsilon[\psi(z_x)]_{xt} \tag{10}$$

has been proposed to address the Neumann initial-boundary value problem associated to equation (3) with ϕ satisfying (H_2). Here ψ is a nondecreasing smooth function with a *saturation* at infinity - namely, $\psi(s) \to \gamma \in \mathbb{R}$ as $s \to \infty$, so that equation (10) is regarded to as a *degenerate pseudoparabolic* approximation of equation (3). Observe that the usual transformation $u := z_x$ leads to a corresponding degenerate pseudoparabolic approximation for equation (1) under assumption (H_2).

Well-posedness of the Neumann initial-boundary value problem in any cylinder $Q_T = \Omega \times (0, T)$ for equation (10) has been studied in [BBDU] (here

$\Omega \subseteq \mathbb{R}$ is a bounded open interval). The main feature of the solutions $z^\varepsilon \in BV(Q_T)$, resulting from the degeneracy of ψ' at infinity, is the *formation of discontinuities in finite time, even for smooth initial data.* Moreover, at any fixed point x_0 the discontinuity jump $z^\varepsilon(x_0^+,t) - z^\varepsilon(x_0^-,t)$ is *nondecreasing* in time. This can be intrerpreted by saying that the *singular term* $z_x^{\varepsilon,(s)}$ (with respect to the Lebesgue measure) in the distributional derivative z_x prevails over the *regular (L^1-)term* $z_x^{\varepsilon,(r)}$ as time proceeds.

Outline of results

Within the above general framework, the present thesis addresses four main points, as outlined below. Each point, apart from the last one, corresponds to a paper either appeared or submitted.

(i) In Chapter 1 we consider the Sobolev regularization (7) of equation (1) in the case of a function ϕ subject to assumption (H_2). We wonder whether results analogous to those obtained in [Pl1] hold in the present case, and, if any difference occurs, what are the novel features deriving from assuming (H_2) instead of (H_1).

In this direction, let $\{u^\varepsilon\}$ be the family of approximate (positive) solutions to the Neumann initial-boundary value problem for the regularized equations (7) in any cylinder $Q_T := \Omega \times (0,T)$ and for any initial datum $u_0 \in L^\infty(\Omega),\ u_0 \geq 0$ (Ω being a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$). From the mathematical point of view, the main complication due to the specific shape of a non-linearity ϕ of "Perona-Malik type", in particular to its degeneragy at infinity, is the weakening of the a-priori estimates. Specifically, while for the functions $\phi(u^\varepsilon)$ and the *chemical potentials* $v^\varepsilon := \phi(u^\varepsilon) + \varepsilon u_t^\varepsilon$ uniform L^∞-estimates as in [Pl1] are proven to hold, the family $\{u^\varepsilon\}$ need not be uniformly bounded with respect to the L^∞-norm, thus only a uniform L^1-estimate is given. This implies that the limit of the family $\{u^\varepsilon\}$ as $\varepsilon \to 0$ can only be taken in a weaker sense with respect to [Pl1], namely in the space $\mathcal{M}^+(Q_T)$ of positive Radon measures over Q_T instead of $L^\infty(Q_T)$. In other words, any limiting point of the approximating family u^ε is a positive Radon measures $\tilde{u}$ on Q_T.

Nevertheless, using the idea of the *biting convergence* of "removing sets of small measure", and using the general properties of the narrow convergence for Young measures (*e.g.*, see [GMS], [E2] [V]), we can represent the Radon measure $\tilde{u}$ as the sum

$$\tilde{u} = u + \mu, \tag{11}$$

where $\mu \in \mathcal{M}^+(Q_T)$ is a positive Radon measure, in general not absolutely continuous with respect to the Lebesgue measure, and $u \in L^1(Q_T),\ u \geq 0$. On the other hand, the function u is proven to be a superposition of the stable branch s_1 and the unstable branch s_2 associated to the graph of ϕ (see Fig.2), namely

$$u = \begin{cases} \lambda s_1(v) + (1-\lambda)s_2(v) & \text{if } v > 0, \\ 0 & \text{if } v = 0 \end{cases} \tag{12}$$

for some $\lambda \in L^\infty(Q_T)$ such that $0 \leq \lambda \leq 1$. Here $v \geq 0$ is the limit of the family $\{\phi(u^\varepsilon)\}$ in the weak* topology of $L^\infty(Q_T)$. Clearly, this is the counterpart of the results proven in [Pl1] for a cubic-like ϕ. Hence the limiting equation obtained as $\varepsilon \to 0$ reads

$$(u+\mu)_t = \Delta v \qquad \text{in } \mathcal{D}'(Q_T), \tag{13}$$

the appearance of the measure μ depending on the degeneracy at infinity of the function ϕ of Perona-Malik type.

In analogy with the case of a cubic-like ϕ treated in [Pl1], we also can take the limit as $\varepsilon \to 0$ in the viscous entropy inequalities for the approximate solutions u^ε. Under additional restrictions due to the weaker a-priori estimates, we obtain entropy inequalities for the couple (u, v).

Concerning the measure μ, first we give qualitative properties of its support, then we prove the following "disintegration":

$$\iint_{\overline{Q}_T} f\, d\mu = \int_{(0,T)} dt \int_{\overline{\Omega}} f(x,t)\, d\tilde{\gamma}_t(x), \tag{14}$$

for any sufficiently regular f, where $\tilde{\gamma}_t \in \mathcal{M}^+(\overline{\Omega})$ is a positive Radon measure defined for a.e. $t \in (0,T)$. Finally, we show that the map $t \mapsto \tilde{\gamma}_t(E)$ is nondecreasing in $(0,T)$ for any Borel set $E \subseteq \overline{\Omega}$. This is the main qualitative feature of the singular term μ (or, equivalently, of its spatial disintegration $\tilde{\gamma}_t$). It suggests that in equation (13) the singular part μ prevails over the regular L^1-term u for large times (observe that the choice of $T > 0$ is arbitrary). In other words, it is reasonable to expect a general *coarsening effect*, since in the measure $u+\mu$ singularities can appear and spread as time goes on. This conjecture seems consistent with *concentration phenomena*, in agreement with the model interpretation of equation (1) under assumption (H_2), particularly concerning aggregation phenomena.

(*ii*) Chapter 2 deals with the degenerate pseudoparabolic regularization (10) of equation (3) in the case of a function ϕ subject to assumption (H_2). As already remarked, in [BBDU] existence and uniqueness of solutions to the Neumann initial-boundary value problem associated to (10) have been studied in any cylinder $Q_T = \Omega \times (0,T]$, $\Omega \subseteq \mathbb{R}$ being a bounded interval. In this framework, a solution is meant to be a couple $(z^\varepsilon, w^\varepsilon)$, where $z^\varepsilon \in L^\infty((0,T); BV(\Omega))$, $z^\varepsilon_x \in \mathcal{M}^+(Q_T)$, $z^\varepsilon_t \in L^2(Q_T)$ and $w^\varepsilon \in L^\infty((0,T); H^1_0(\Omega)) \cap C(\overline{Q}_T)$, $w^\varepsilon_t \in L^2((0,T); H^1_0(\Omega))$, such that

$$z_t^\varepsilon = \big[h(w^\varepsilon)\big]_x + \varepsilon\big[w^\varepsilon\big]_{tx} \qquad \text{in } \ L^2(Q_T) \tag{15}$$

with initial datum $z_0 \in BV(\Omega),\ z_0' \in \mathcal{M}^+(\Omega)$ (here $h := \phi \circ \psi^{-1}$).

Our first aim is to give a notion of solution to the Neumann initial-boundary value problem for (10) which is equivalent to that proposed in [BBDU] and, at the same time, more general. Precisely, denoting by $z_x^{\varepsilon,(r)}$ and $z_x^{\varepsilon,(s)}$ the regular and singular term of the spatial derivative z_x^ε with respect to the Lebesgue measure, we prove that

(a) $w^\varepsilon = \psi(z_x^{\varepsilon,(r)})$, $h(w^\varepsilon) = \phi(z_x^{\varepsilon,(r)})$,

(b) equation (15) reads

$$z_t^\varepsilon = \big[\phi(z_x^{\varepsilon,(r)})\big]_x + \varepsilon\big[\psi(z_x^{\varepsilon,(r)})\big]_{tx} \qquad \text{in } \ L^2(Q_T), \tag{16}$$

(c) $\operatorname{supp} z_x^{\varepsilon,(s)} = \Big\{(x,t) \in \overline{Q}_T \mid \psi(z_x^{\varepsilon,(r)})(x,t) = \gamma\Big\}$.

Observe also that deriving (16) with respect to x formally gives the following equation for the derivative z_x^ε

$$\big[z_x^\varepsilon\big]_t = \big[\phi(z_x^{\varepsilon,(r)})\big]_{xx} + \varepsilon\big[\psi(z_x^{\varepsilon,(r)})\big]_{txx} \qquad \text{in } \ \mathcal{D}'(Q_T) \tag{17}$$

which is a degenerate pseudoparabolic regularization for equation (1) under assumption (H_2).

Then, as in the case of the Sobolev regularization (7), we proceed to study the vanishing limit $\varepsilon \to 0$ in (16) (and consequently in (17)). In this direction, we only have general a-priori estimates in $BV(Q_T)$ for the family $\{z^\varepsilon\}$ - namely in $\mathcal{M}^+(Q_T)$ for the spatial derivatives z_x^ε. Hence, again the space of positive Radon measures seems a natural candidate to take the limit as $\varepsilon \to 0$, which leads to the limiting equations

$$z_t = v_x \qquad \text{in } \ L^2(Q_T), \tag{18}$$

$$\big[z_x\big]_t = v_{xx} \qquad \text{in } \ \mathcal{D}'(Q_T). \tag{19}$$

Here $z \in BV(Q_T)$ is the weak limit of the family $\{z^\varepsilon\}$ in $BV(Q_T)$, and $v \in L^\infty(Q_T) \cap L^2((0,T); H_0^1(\Omega)),\ v \geq 0$, is the limit of the family $\Big\{\phi(z_x^{\varepsilon,(r)})\Big\}$ in the weak* topology of $L^\infty(Q_T)$.

Arguing as in (i), we can use the general notion of Young measures, narrow and biting convergences, to prove the following decomposition of the Radon measure $z_x \in \mathcal{M}^+(Q_T)$:

$$z_x = Z + \mu, \tag{20}$$

where $\mu \in \mathcal{M}^+(Q_T)$ is a positive Radon measure, in general not absolutely continuous with respect to the Lebesgue measure, and $Z \in L^1(Q_T),\ Z \geq 0$, is a superposition of the two branches s_1, s_2 of the equation $v = \phi(Z)$, namely

$$Z = \begin{cases} \lambda s_1(v) + (1-\lambda)s_2(v) & \text{if } v > 0, \\ 0 & \text{if } v = 0 \end{cases}$$

for some $\lambda \in L^\infty(Q_T)$, $0 \leq \lambda \leq 1$. The above equality gives a clear analogy with a cubic-like function considered in [Pl1]. On the other hand, the measure μ can be "disintegrated" as the Lebesgue measure dt with respect to the time variable t, and as a positive Radon measure γ_t over Ω for *a.e.* $t > 0$ (see (14) in (*i*)), the map

$$t \mapsto \gamma_t(E)$$

being nondecreasing for any Borel set $E \subseteq \Omega$. This is the counterpart of the results decribed in (*i*) above for the (possibly) singular term $\tilde{\gamma}_t \in \mathcal{M}^+(\overline{\Omega})$. Finally, the novel feature here, due to the degenerating term $\varepsilon\big[\psi(z_x)\big]_{tx}$ in regularization (10), is the *characterization* of the support of the (possibly) singular measure $\gamma_t \in \mathcal{M}^+(\Omega)$ (hence of $\mu \in \mathcal{M}^+(Q_T)$). Precisely we prove that

$$\operatorname{supp} \gamma_t \subseteq \big\{x \in \overline{\Omega} \,|\; v(x,t) = 0\big\}$$

for *a.e.* $t > 0$.

(*iii*) In Chapter 3 we address the long-time behaviour of weak entropy measure-valued solutions (u, v) to the Neumann initial-boundary value problem for equation (1) under assumption (H_1) and in the one-dimensional case $\Omega = (0, 1)$. To this purpose, in view of the crucial estimate

$$\int_0^\infty \int_0^1 v_x^2 dxdt \leq C\,,$$

it is reasonable to expect that $v(\cdot, t)$ approaches a constant value $\overline{v}$ as time diverges. It is a natural question, whether this constant $\overline{v}$ is *uniquely* determined by the initial datum u_0 of the problem. In fact, since no uniqueness of measure-valued solutions to the Neumann initial-boundary value problem for (1) is known, the value $\overline{v}$ could depend on the solution itself (in this connection, see [MTT]). For any $u_0 \in L^\infty(0, 1)$ let

$$M_{u_0} := \int_0^1 u_0(x)dx. \tag{21}$$

Then, if $M_{u_0} < a$ (respectively, $M_{u_0} > d$; see Fig.1), we prove that $v(\cdot, t)$ and $u(\cdot, t)$ converge uniformly to $\phi(M_{u_0})$ and M_{u_0} respectively, as $t \to \infty$, $t \notin E_\delta$, where E_δ are sets of arbitrarily small - albeit not zero - Lebesgue measure. Observe that for $M_{u_0} < a$ and $M_{u_0} > d$ the constant $\overline{v}$ is uniquely determined by the initial datum u_0.

We cannot prove a similar result if $a \leq M_{u_0} \leq d$, since in this case the asymptotic behaviour of the coefficients λ_i in representation (9) plays a role. Prescisely, for any weak entropy solution (u, v) we can uniquely determine a constant $A \leq \overline{v} \leq B$, and three coefficients $\lambda_i^* \in L^\infty(0,1)$, such that $v(\cdot, t)$ converges to $\overline{v}$ in the strong topology of $C([0,1])$, and $u(\cdot, t)$ converges to $\overline{u}$,

$$\overline{u} = \sum_{i=0}^{2} \lambda_i^* s_i(\overline{v}),$$

a.e. in $(0,1)$, again as $t \to \infty$, $t \notin E_\delta$, E_δ being a set of arbitrarily small (Lebesgue) measure. In particular, for $a \leq M_{u_0} \leq d$ uniqueness of the constant $\overline{v}$ and of the coefficients λ_i^* only follows for any given weak entropy measure-valued solution (u, v) of the Neumann initial-boundary value problem for (1) - namely, different weak entropy solutions with the *same* initial datum u_0 might approach different values of $\overline{v}$ and $\overline{u}$.

(iv) Finally, in Chapter 4 we address the long-time behaviour of two-phase solutions to the Neumann initial-boundary value problem for equation (1), again in the one dimensional case $\Omega = (-1, 1)$ and for a cubic-like ϕ which satisfies assumption (H_1). The techniques are almost the same as those outlined in *(iii)* to study the asymptotic behaviour of general weak-entropy measure-valued solutions. However, some specific novel features arise, as explained below.

A two-phase solution to the Neumann initial-boundary value problem (in $Q = (-1,1) \times (0, \infty)$) for equation (1) is a triple (u, v, ξ) with the following properties (see Chapter 4, Definition 4.2.1, [MTT] and [MTT2]):

(α) (u, v) is a weak entropy measure valued solution of the Neumann initial-boundary value problem for (1) in Q and $\xi : [0, \infty) \to [-1, 1]$, $\xi(0) = 0$, is a Lipschitz-continuous function;
(β) $v \in C(\overline{Q}) \cap L^2((0,T); H^1(-1,1)$ for any $T > 0$ and $u \in L^\infty(Q)$,

$$u = s_i(v) \quad \text{in} \quad V_i \qquad (i = 1, 2)\,.$$

Here s_1, s_2 denote respectively the first and the second stable branch of the equation $v = \phi(u)$ (see Fig.1), and

$$V_1 := \{(x,t) \in Q \mid -1 < x < \xi(t)\}\ ,$$
$$V_2 := \{(x,t) \in Q \mid \xi(t) < x < 1\}\ .$$

Moreover $u \in C^{2,1}(V_i)$ $(i = 1, 2)$, where $C^{2,1}(V_i)$ denotes the space of continuous functions $f : V_i \to \mathbb{R}$ such that u_t, u_x, $u_{xx} \in C(V_i)$.

In view of (α)-(β), there holds:

(a) the couple (u, v) is a *classical* solution of the problem:

$$\begin{cases} u_t = \big[\phi(u)\big]_{xx} & \text{in } V_i, \\ u = u_0 & \text{in } \overline{V}_i \cap \{t = 0\} \end{cases}$$

$(i = 1, 2)$;

(b) for *a.e.* $t \geq 0$, $\xi'(t) \geq 0$ if $v(\xi(t), t) = A$, $\xi'(t) \leq 0$ if $v(\xi(t), t) = B$ and $\xi'(t) = 0$ if $A < v(\xi(t), t) < B$ (this is a consequence of the entropy inequalities).

In other words, in view of (α), for any fixed $t \in (0, \infty)$, the function $u(x, t)$ takes values in the first stable branch s_1 of the graph of ϕ for $x \in (-1, \xi(t))$, and in the second stable branch s_2 for $x \in (\xi(t), 1)$. Hence, the curve $\gamma = \{(\xi(t), t) \mid t \in [0, \infty)\}$ denotes the *interface* between stable phases, and by (b) the function u can *jump* between such phases only at the points (x, t) where $v(x, t)$ takes the values A, B.

As already remarked, uniqueness and *local* existence of two-phase solutions have been proven in [MTT2] for the Cauchy problem associated to (1) in $\mathbb{R} \times (0, T]$ (see also [MTT] for uniqueness of two-phase solutions to the Neumann initial-boundary value problem). *Global* existence for the same problem (or for the Neumann initial-boundary value problem) is being plenty addressed.

Assuming global existence, the long-time behaviour of such solutions has been investigated proving asymptotic results concerning both the couple (u, v) and the interface ξ. Let again M_{u_0} be defined by

$$M_{u_0} := \frac{1}{2} \int_{-1}^{1} u_0(x)\, dx$$

for any initial datum u_0, and let (u, v, ξ) be the two-phase solution of the Neumann initial boundary value problem for (1) with initial datum u_0. Then we prove that the function $v(\cdot, t)$ approaches a constant value $\overline{v}$ as $t \to \infty$ (in some sense made precise in Chapter 4). Moreover, there exists the limiting value of the interface

$$\xi^* := \lim_{t \to \infty} \xi(t),$$

and the following properties hold:

(1) if $M_{u_0} > d$ (respectively, $M_{u_0} < a$), then $\xi^* = -1$ (respectively, $\xi^* = 1$); in these cases $\overline{v} = \phi(M_{u_0})$ and $u(\cdot, t)$ approaches the value M_{u_0} as $t \to \infty$;

(2) if $a \leq M_{u_0} \leq d$, then $u(\cdot, t) \to \overline{u}$ as $t \to \infty$ (in some suitable sense), where

$$\overline{u} := \chi_{(-1, \xi^*)} s_1(\overline{v}) + \chi_{(\xi^*, 1)} s_2(\overline{v}).$$

Chapter 1

On a Class of Equations with Variable Parabolicity Direction

1.1 Introduction

In this chapter we study positive solutions to the Neumann initial-boundary value problem for the quasilinear *forward-backward* parabolic equation

$$u_t = \Delta\phi(u) \qquad \text{in } \Omega \times (0,T), \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Concerning the function $\phi \in C^2(\mathbb{R})$ we make the following assumption:

$$(H_1) \qquad \begin{cases} (i)\ \phi \text{ is bounded, } \phi^p \in L^1(\mathbb{R}) \text{ for some } p > 1; \\ (ii)\ \phi(0) = 0,\ \phi(u) > 0 \text{ for } u > 0,\ \phi(-u) = -\phi(u); \\ (iii)\ \phi \text{ is strictly increasing for } 0 < u < 1, \\ \qquad \text{strictly decreasing for } u > 1; \\ (iv)\ \phi'(0) \neq 0,\ \phi(u) \to 0 \text{ as } u \to +\infty \end{cases}$$

(see Fig.1.1). We always set $\phi(1) = 1$ in the following. Since the function ϕ is nonmonotone, equation (1.1.1) is *well-posed* whenever the solution u takes values in the interval $(0,1)$, yet it is *ill-posed* (forward in time) if $u \in (1, +\infty)$.

1.1.1 Motivations

Forward-backward parabolic equations naturally arise in the theory of *phase transitions*, where the function u represents the enthalpy, $\phi(u)$ the temperature of the medium and equation (1.1.1) follows from the Fourier law (*e.g*, see [BS]). In this case $\phi \in C^2(\mathbb{R})$ is a nonmonotone cubic-like function

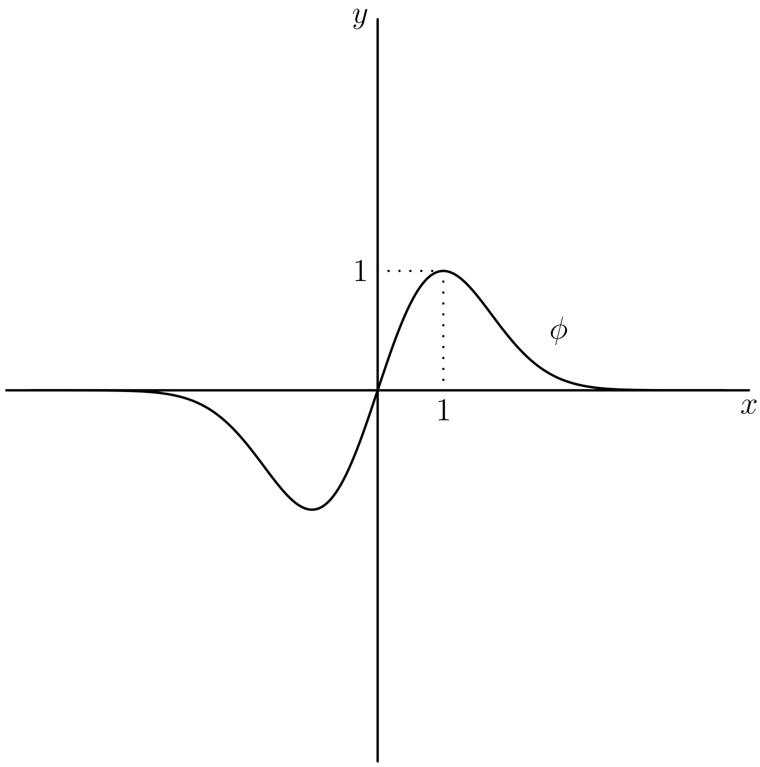

Figure 1.1: Assumption (H_1).

satisfying the following condition:

$$(H_2) \qquad \begin{cases} (i)\ \phi'(u) > 0 \quad \text{for } |u| > 1, \quad \phi'(u) < 0 \quad \text{for } |u| < 1; \\ (ii)\ \phi(\pm 1) = \mp 1, \quad \phi(u) \to \pm\infty \text{ as } u \to \pm\infty \end{cases}$$

(see Fig.1.2).

The two increasing branches $S_1 := \{(u, \phi(u)) \,|\, u \in (-\infty, -1)\}$, $S_2 := \{(u, \phi(u)) \,|\, u \in (1, +\infty)\}$ of the graph of ϕ correspond to *stable phases*, the decreasing branch $S_0 := \{(u, \phi(u)) \,|\, u \in (-1, 1)\}$ to the *unstable phase*. We shall use the same terminology if (H_1) holds (see below).

In one space dimension, equation (1.1.1) with $\phi(u) = u \exp(-u)$ (which satisfies assumption (H_1)) arises as a diffusion approximation to a discrete model for *aggregating* populations (see [Pa]). In this case the unknown $u \geq 0$ represents the population density, while the transition probability (*i.e.*, the probability that an individual moves from its location) $p(u) = \exp(-u)$ models aggregation phenomena, for it is a decreasing function of u.

An independent motivation to study equation (1.1.1) under assumption (H_1) is given by a mathematical model for heat transfer in a stably stratified turbulent shear flow in one space dimension (see [BBDU]). The temperature

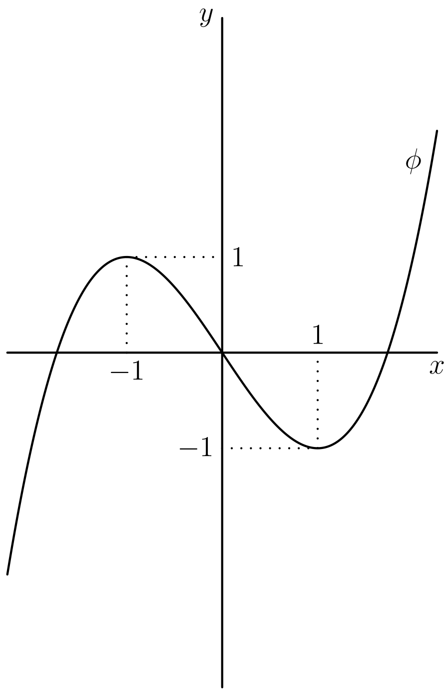

Figure 1.2: Assumption (H_2).

$w \geq 0$ satisfies the equation

$$w_t = (kw_x)_x \,;$$

under fixed external conditions the function k only depends on the gradient of the temperature, namely

$$k = \sigma(w_x)\,.$$

Moreover, a typical choice of the function σ is

$$\sigma(s) = \frac{A}{B+s^2} \qquad (A, B > 0)\,;$$

then the above equation reads

$$w_t = [\phi(w_x)]_x \tag{1.1.2}$$

with $\phi(s) := s\sigma(s)$. Deriving the above equation with respect to x and setting $u := w_x$ gives equation (1.1.1) (observe that $\phi(s) = s\sigma(s) = \frac{As}{B+s^2}$ satisfies assumption (H_1)).

It is worth observing that equation (1.1.2) with $\phi(s) = s\sigma(s)$ is the *one-dimensional Perona-Malik equation.* In the general n-dimensional case the Perona-Malik equation reads

$$w_t = \operatorname{div}\big[\sigma(|\nabla w|)\nabla w\big] \quad \text{in } \Omega \times (0, T) \qquad (\Omega \subseteq \mathbb{R}^n)\,; \tag{1.1.3}$$

typical choices of σ are either $\sigma(s) = (1+s^2)^{-1}$, or $\sigma(s) = \exp(-s)$ (see [PM]). If $n = 1$, the transformation $u = w_x$ gives a link between equations (1.1.1) and (1.1.2). Most results concerning equation (1.1.3) refer to the one-dimensional case. Existence of solutions to the Neumann initial-boundary value problem for equation (1.1.2) has been proved, if the derivative of the initial datum u_0 takes values in the stable phase (see [KK]), while for large values of $|u_0'|$ no *global* C^1-solution exists (see [G],[K]). Assuming homogeneous Neumann boundary conditions and smoothness of initial data, the existence of infinitely many weak $W^{1,\infty}$-solutions for the one-dimensional Perona-Malik equation has been proved in [Z] (this yields the existence of infinitely many weak L^1-solutions for equation (1.1.1)). The techniques used in [Z], where equation (1.1.2) is reformulated as a first order partial differential inclusion problem, are very different from those of the present approach.

Finally, observe that equation (1.1.2) can be regarded as the formal L^2-gradient system associated with a *nonconvex* energy density ψ in one space dimension (in this case $\phi = \psi'$); for instance, $\psi(s) = \log(1+s^2)$ holds for the Perona-Malik equation, or the *double well potential* $\psi(s) = (1-s^2)^2$ for a cubic nonlinearity. Therefore the dynamics described by equation (1.1.1) in one space dimension is relevant to various settings, where nonconvex functionals arise (*e.g.*, see [Mü] for motivations in nonlinear elasticity).

1.1.2 Outline of results

A natural approach to address equation (1.1.1) is to introduce some regularization. In this chapter, we associate with equation (1.1.1) the *pseudoparabolic* or *Sobolev* regularization

$$u_t = \Delta\phi(u) + \varepsilon\Delta u_t,$$

where ε is a positive parameter. Introducing the *chemical potential*

$$v := \phi(u) + \varepsilon u_t \qquad (\varepsilon > 0)\,, \tag{1.1.4}$$

we focus our attention on the initial-boundary value problem

$$\begin{cases} u_t = \Delta v & \text{in } \Omega \times (0,T] := Q_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T] \\ u = u_0 & \text{in } \Omega \times 0\,. \end{cases} \tag{1.1.5}$$

Let us mention that a different regularization, leading to the *Cahn-Hilliard equation*:

$$u_t = \Delta\phi(u) - \kappa\Delta^2 u \qquad (\kappa > 0)\,,$$

has been widely used (in particular, see [BFG], [Sl]). Both regularizations are physically meaningful (see [BFJ]), although the limiting dynamics of solutions expectedly depends on the regularization itself. Let us also recall that a *degenerate pseudoparabolic* regularization of equation (1.1.2), namely

$$w_t = [\phi(w_x)]_x + \varepsilon\chi(w_x)_{xt} \tag{1.1.6}$$

was used in [BBDU]; here χ is a smooth nonlinear function, $\chi'(s) > 0$ for $s > 0$, $\chi(s) \to \gamma \in \mathbb{R}$, $\chi'(s) \to 0$ as $s \to +\infty$. As before, in one space dimension deriving (1.1.6) with respect to x and setting $u = w_x$ gives a different regularization of equation (1.1.1).

Problem (1.1.4)-(1.1.5) was studied in [Pa], proving its well-posedness in the class of the bounded solutions for any $\varepsilon > 0$ (analogous results had been proved earlier in [NP], if (H_2) holds). Our main concern here is to investigate the vanishing viscosity limit of such solutions. In particular, a natural question is the following: describing the limiting points of the family $\{u^\varepsilon\}$ of solutions to (1.1.4)-(1.1.5) as $\varepsilon \to 0$ (in some suitable topology), can we define *weak*, or possibly *measure-valued* solutions to the Neumann initial-boundary value problem for the original ill-posed equation (1.1.1)? The latter reads:

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } Q_T \\ \dfrac{\partial}{\partial\nu}\phi(u) = 0 & \text{in } \partial\Omega \times (0, T] \\ u = u_0 & \text{in } \Omega \times \{0\}\ . \end{cases} \tag{1.1.7}$$

An exhaustive answer to the above question was given in [Pl1], if assumption (H_2) holds (see also [Pl2],[Pl3]). We outline below the main results of [Pl1] for convenience of the reader, aiming to point out the novel features deriving from assumption (H_1) - in particular, from the *degeneracy at infinity* of a nonlinearity ϕ "of Perona-Malik type".

Assumption (H_2)

Consider problem (1.1.4)-(1.1.5) under assumption (H_2). As proved in [NP], the following holds:

- for any $\varepsilon > 0$ and $u_0 \in L^\infty(\Omega)$ there exists a unique solution $(u^\varepsilon, v^\varepsilon)$ to problem (1.1.4)-(1.1.5), v^ε defined by (1.1.4);

- there exists a constant $C > 0$, which does not depend on ε, such that

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \le C\,; \tag{1.1.8}$$

$$\|v^\varepsilon\|_{L^2((0,T);H^1(\Omega))} + \|\sqrt{\varepsilon}u_t^\varepsilon\|_{L^2(Q_T)} \le C\,; \tag{1.1.9}$$

$$\|v^\varepsilon\|_{L^\infty(Q_T)} \le C\,. \tag{1.1.10}$$

In view of such uniform estimates of the family $\{(u^\varepsilon, v^\varepsilon)\}$, there exist sequences $\{u^{\varepsilon_k}\}$, $\{v^{\varepsilon_k}\}$ and a couple (u, v) with $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ such that:

$$u^{\varepsilon_k} \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T)\,, \tag{1.1.11}$$

$$v^{\varepsilon_k} \stackrel{*}{\rightharpoonup} v \quad \text{in } L^\infty(Q_T)\,, \tag{1.1.12}$$

$$v^{\varepsilon_k} \rightharpoonup v \quad \text{in } L^2((0,T); H^1(\Omega))\,. \tag{1.1.13}$$

Set $\varepsilon = \varepsilon_k$ in the weak formulation of problem (1.1.4)-(1.1.5), namely:

$$\iint_{Q_T} u^{\varepsilon_k} \psi_t \, dxdt = \iint_{Q_T} \nabla v^{\varepsilon_k} \cdot \nabla \psi \, dxdt - \int_\Omega u_0(x) \psi(x,0)\, dx \tag{1.1.14}$$

for any $\psi \in C^1(\overline{Q}_T)$, $\psi(\,.\,,T) = 0$ in Ω. Taking the limit as $k \to \infty$ in equality (1.1.14) and using (1.1.11)-(1.1.13) gives

$$\iint_{Q_T} (u\psi_t - \nabla v \cdot \nabla \psi)\, dxdt + \int_\Omega u_0 \psi(x,0)\, dx = 0 \tag{1.1.15}$$

for any ψ as above - namely, the couple (u, v) is a *weak solution* of problem (1.1.5).

Equation (1.1.15) would give a weak solution of problem (1.1.7), if we had $v = \phi(u)$; however, no such conclusion can be drawn from (1.1.11)-(1.1.13), due to the nonmonotone character of ϕ. Nevertheless, as proved in [Pl1], a weak solution of problem (1.1.7) *in the sense of Young measures* does exist. Consider the Young measure $\tau^k := \tau^{\varepsilon_k}$ associated to each u^{ε_k}; let τ denote the *narrow limit* of the sequence $\{\tau^k\}$ and $\nu_{(x,t)}$ its associated *disintegration*, defined for a.e. $(x,t) \in Q_T$ (see Definition 1.2.2 and Proposition 1.2.7 below). Since the sequence $\{u^{\epsilon_k}\}$ is uniformly bounded in $L^\infty(Q_T)$ (see (1.1.8)), for any $f \in C(\mathbb{R})$ there holds:

$$f(u^{\epsilon_k}) \stackrel{*}{\rightharpoonup} f^* \quad \text{in } L^\infty(Q_T)\,, \tag{1.1.16}$$

where

$$f^*(x,t) := \int_{\mathbb{R}} f(\xi)\, \nu_{(x,t)}(d\xi) \qquad \text{for } a.e.\ (x,t) \in Q_T \tag{1.1.17}$$

(*e.g.*, see [E1]).

The structure of the Young measure τ associated with the sequence $\{u^{\varepsilon_k}\}$ was investigated in [Pl1], proving that its disintegration $\nu_{(x,t)}$ is the *superposition of three Dirac masses* concentrated on the three branches of the equation $v = \phi(u)$. In fact, there holds:

$$\nu_{(x,t)}(\xi) = \sum_{i=0}^{2} \lambda_i(x,t) \delta\big(\xi - \beta_i(v(x,t))\big) \tag{1.1.18}$$

(for a.e. $(x,t) \in Q_T$ and any $\xi \in \mathbb{R}$) with some coefficients $\lambda_i \in L^\infty(Q_T)$, $\lambda_i \geq 0$ and $\sum_{i=0}^{2} \lambda_i = 1$; here we set $S_i := \{(\beta_i(v), v)\}$ $(i = 0, 1, 2)$.

By equality (1.1.18) there holds:

$$\int_{\mathbb{R}} \xi \nu_{(x,t)}(d\xi) = \sum_{i=0}^{2} \lambda_i(x,t)\beta_i(v(x,t)) = u(x,t) \tag{1.1.19}$$

(this follows from (1.1.11) and (1.1.17) choosing $f(\xi) = \xi$); moreover,

$$\int_{\mathbb{R}} \phi(\xi)\nu_{(x,t)}(d\xi) = \sum_{i=0}^{2} \lambda_i(x,t)\phi(\beta_i(v(x,t))) = v(x,t) \tag{1.1.20}$$

for a.e. $(x,t) \in Q_T$. Inserting equalities (1.1.19)-(1.1.20) in (1.1.15) we obtain:

$$\begin{aligned} \iint_{Q_T} \Big\{ \psi_t \int_{\mathbb{R}} \xi \nu_{(x,t)}(d\xi) - \nabla\psi \cdot \nabla \int_{\mathbb{R}} \phi(\xi)\nu_{(x,t)}(d\xi) \Big\} dxdt + \\ + \int_{\Omega} u_0(x)\psi(x,0)dx = 0\,. \end{aligned} \tag{1.1.21}$$

Equation (1.1.19) says that the limiting function u is the *barycenter* of the disintegration $\nu_{(x,t)}$ of the narrow limit τ; in view of (1.1.21), the measure τ can be regarded as a measure-valued solution of the limiting problem (1.1.7).

The crucial role of the uniform L^∞-estimate (1.1.8) is apparent from the above discussion. In turn, estimate (1.1.8) is an immediate consequence of the following result (see [NP]):

Let (H_2) *hold. Then any interval* $[u_1, u_2]$ *such that*

$$\phi(u_1) \leq \phi(u) \leq \phi(u_2) \qquad \text{if and only if } u \in [u_1, u_2] \tag{1.1.22}$$

is a positively invariant region *for problem* (1.1.4)-(1.1.5).

It is informative to sketch the proof of the above result. Set for any $g \in C^1(\mathbb{R})$, $g' \geq 0$:

$$G(u) := \int_0^u g(\phi(s))ds + k \qquad (k \in \mathbb{R})\,. \tag{1.1.23}$$

Let $\varepsilon > 0$ be fixed; let $(u^\varepsilon, v^\varepsilon)$ be the solution to problem (1.1.4)-(1.1.5). We have:

$$\begin{aligned} \frac{d}{dt}\int_{\Omega} G(u^\varepsilon(x,t))\,dx &= \int_{\Omega} g(\phi(u^\varepsilon))u_t^\varepsilon\,dx \\ = \int_{\Omega} g(v^\varepsilon)\Delta v^\varepsilon\,dx &+ \int_{\Omega} [g(\phi(u^\varepsilon)) - g(v^\varepsilon)]u_t^\varepsilon\,dx \\ = \int_{\Omega} \operatorname{div}(g(v^\varepsilon)\nabla v^\varepsilon)\,dx &- \int_{\Omega} g'(v^\varepsilon)|\nabla v^\varepsilon|^2\,dx \\ &+ \int_{\Omega} [g(\phi(u^\varepsilon)) - g(v^\varepsilon)]\frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon}\,dx. \end{aligned} \tag{1.1.24}$$

Since g is nondecreasing and

$$\frac{\partial v^\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T],$$

we obtain

$$\frac{d}{dt}\int_\Omega G(u^\varepsilon(x,t))\,dx \le -\int_\Omega g'(v^\varepsilon)|\nabla v^\varepsilon|^2\,dx \le 0 \quad \text{ in } (0,T)\,. \tag{1.1.25}$$

By a proper choice of the function g the result follows (see [NP] for details).

Clearly, the above proof of inequality (1.1.25) is independent from the specific shape of ϕ; yet, if (H_1) holds, a bounded interval $[u_1,u_2]$ is positively invariant only if $[u_1,u_2] \subseteq [0,1]$ (see Proposition 1.2.3). Therefore inequality (1.1.8) holds if $\|u_0\|_{L^\infty(\Omega)} \le 1$, but the family $\{u^\varepsilon\}$ need not be uniformly bounded in $L^\infty(Q_T)$ if $\|u_0\|_{L^\infty(\Omega)} > 1$.

However, it follows from (1.1.25) that the half-line $[0,\infty)$ is positively invariant (see Proposition 1.2.3). Then we get the following *conservation law* for *positive* solutions to (1.1.4)-(1.1.5):

$$\|u^\varepsilon(t)\|_{L^1(\Omega)} = \int_\Omega u^\varepsilon(x,t)\,dx = \int_\Omega u_0(x)\,dx = \|u_0\|_{L^1(\Omega)} \tag{1.1.26}$$

for any $t \in [0,T]$ - namely, a uniform L^1-estimate of the family $\{u^\varepsilon\}$, which will play a crucial role in the following.

Let us mention another important point. Arguing as in (1.1.24) we obtain the weak inequality:

$$\iint_{Q_T} \left\{G(u^\varepsilon)\psi_t - g(v^\varepsilon)\nabla\psi\cdot\nabla v^\varepsilon - \psi g'(v^\varepsilon)|\nabla v^\varepsilon|^2\right\} dxdt \ge 0 \tag{1.1.27}$$

for any $\psi \in C_c^\infty(Q_T)$, $\psi \ge 0$ (see Lemma 1.2.2). Inequality (1.1.27) is referred to as the *entropy inequality* for problem (1.1.4)-(1.1.5), in view of its analogy with the entropy inequality for the one-dimensional viscous conservation law (*e.g.*, see [Se]; see also [E2]). It was proved in [Pl1] that any weak solution (u,v) of problem (1.1.5) satisfies a limiting form of inequality (1.1.27) as $\varepsilon \to 0$; in fact, there holds:

$$\iint_{Q_T}\left\{G^*\psi_t - g(v)\nabla v\cdot\nabla\psi - g'(v)|\nabla v|^2\psi\right\} dxdt \ \ge\ 0 \tag{1.1.28}$$

for any ψ as above, where

$$G^*(x,t) := \sum_{i=0}^{2} \lambda_i G\big(\beta_i(v(x,t))\big) \qquad \text{for a.e. } (x,t) \in Q_T\,. \tag{1.1.29}$$

In view of the above discussion (in particular, see (1.1.18), (1.1.21) and (1.1.28)), we can think of the quintuple $u, v, \lambda_0, \lambda_1, \lambda_2$ as a *weak entropy solution in the sense of Young measures* of the limiting problem (1.1.7).

It was also proved in [Pl1] that the coefficients λ_i of such solutions (see (1.1.18)) have a remarkable monotonicity property with respect to time, which gives rise to a *hysteresis effect* in the mechanism of phase transitions; the latter is typical of phase changes described by a cubic-like nonlinearity ([EP]; see also [MTT]).

Assumption (H_1)

Let us now consider problem (1.1.4)-(1.1.5) under assumption (H_1). As before, for any $\varepsilon > 0$ and $u_0 \in L^\infty(\Omega)$ there exists a unique solution $(u^\varepsilon, v^\varepsilon)$ (see [Pa]). Assume $u_0 \geq 0$, as we always do in the following discussion; then the uniform L^1-estimate (1.1.26) and inequalities (1.1.9)-(1.1.10) hold (see Theorem 1.2.1 and Propositions 1.2.4-1.2.5).

As before, we can associate to each u^ε its Young measure τ^ε, introducing the narrow limit τ and its associated disintegration $\nu_{(x,t)}$. However, at variance from the previous case we cannot pass to the limit in the left-hand side of equality (1.1.14), since the family $\{u^\varepsilon\}$ need not be *equi-integrable* in the cylinder Q_T (thus relatively compact in the weak topology of L^1; see Proposition 1.2.7). This is the most relevant complication with respect to the case when (H_2) holds.

Nevertheless, using the idea of the *biting convergence* of "removing sets of small measure" (*e.g.*, see [GMS], [V]), we can associate to $\{u^{\varepsilon_k}\}$ an equi-integrable subsequence. More precisely, we can find a subsequence $\{u^{\varepsilon_j}\} \equiv \{u^{\varepsilon_{k_j}}\} \subseteq \{u^{\varepsilon_k}\}$, a decreasing sequence of measurable sets $A_j \subseteq Q_T$, $|A_j| \to 0$, and a measure $\mu \in \mathcal{M}(\overline{Q}_T)$ such that

$$\iint_{Q_T} u^{\varepsilon_j} \chi_{A_j} \psi \, dxdt \to \iint_{\overline{Q}_T} \psi \, d\mu \tag{1.1.30}$$

for any $\psi \in C(\overline{Q}_T)$, and

$$u^{\varepsilon_j} \chi_{Q_T \setminus A_j} \rightharpoonup u \qquad \text{in } L^1(Q_T)\,; \tag{1.1.31}$$

here $u \in L^1(Q_T)$ is the barycenter of the Young disintegration $\nu_{(x,t)}$, namely

$$u(x,t) := \int_{[0,\infty)} \xi \, \nu_{(x,t)}(d\xi) \qquad \text{for } a.e. \ (x,t) \in Q_T \tag{1.1.32}$$

(see Proposition 1.2.8; by χ_E we denote the characteristic function of any subset $E \subseteq Q_T$).

In view of (1.1.30)-(1.1.31), passing to the limit as $j \to \infty$ in equality (1.1.14) (written with $k = k_j$) gives:

$$\iint_{Q_T} u\psi_t \, dxdt + \iint_{\overline{Q}_T} \psi_t \, d\mu = \tag{1.1.33}$$

$$= \iint_{Q_T} \nabla v \cdot \nabla \psi \, dxdt - \int_\Omega u_0(x)\psi(x,0)\, dx$$

for any $\psi \in C^1(\overline{Q}_T)$ such that $\psi(\,.\,,T) = 0$ in Ω (see Theorem 1.2.9). Observe that the above equality reduces to (1.1.15) if $\mu = 0$; in fact, this is the case if the uniform L^∞-estimate holds, which implies equi-integrability of the family $\{u^\varepsilon\}$. Therefore the appearance of the measure μ is connected with assumption (H_1) - in particular, with the degeneracy of ϕ at infinity, which is a novel feature with respect to a cubic-like nonlinearity. It seems also related with possible *concentration phenomena*, in agreement with the model interpretation discussed above ([Pa]; in this connection, see the paragraph (β) below).

We can rephrase equation (1.1.33) by saying that the positive Radon measure $u + \mu \in \mathcal{M}(\overline{Q}_T)$ is a solution of the equation

$$(u + \mu)_t = \Delta v \qquad \text{in } \mathcal{D}'(Q_T)\,. \tag{1.1.34}$$

The properties of the *regular term* $u \in L^1(Q_T)$ are investigated in Subsection 1.2.3, those of the *singular term* $\mu \in \mathcal{M}(\overline{Q}_T)$ in Subsection 1.2.4; the main results are summarized below.

(α) The results concernig u are the counterpart of those in [Pl1] for a cubic-like ϕ. As in this case, we refer to the increasing branch

$$S_1 := \{(u, \phi(u)) \,|\, u \in [0,1]\} = \{(\beta_1(v), v) \,|\, v \in [0,1]\}$$

as the stable phase, to the decreasing branch

$$S_2 := \{(u, \phi(u) \,|\, u \in (1, +\infty)\} = \{(\beta_2(v), v) \,|\, v \in (0,1)\}$$

as the unstable one. As for the structure of the Young disintegration $\nu_{(x,t)}$ associated to the Young measure τ, we prove it to be (see Corollary 1.2.13):

- an *atomic* measure, whose support consists of the points $\beta_1(v(x,t))$ and $\beta_2(v(x,t))$, if $v(x,t) \neq 0$;
- the Dirac mass concentrated in $\beta_1(0) = 0$, if $v(x,t) = 0$

(recall that $v = v(x,t)$ is the weak*-limit in $L^\infty(Q_T)$ of both sequences $\{v^{\varepsilon_j}\}$, $\{\phi(u^{\varepsilon_j})\}$). Hence u is a *superposition* of the two phases S_1 and S_2, namely

$$u = \begin{cases} \lambda\beta_1(v) + (1-\lambda)\beta_2(v) & \text{for } v > 0, \\ 0 & \text{for } v = 0 \end{cases} \tag{1.1.35}$$

for some $\lambda \in L^\infty(Q_T)$, $0 \le \lambda \le 1$. In analogy with the cubic-like case, this can be expressed by saying that the function u takes the *fraction* λ of its value at (x,t) on the stable branch S_1, respectively the fraction $(1-\lambda)$ on the unstable branch S_2.

Using the above representation of u and the results of Subsection 1.2.2, we obtain the following inequality satisfied by the couple (u, v):

$$\iint_{Q_T} u\psi_t \, dxdt - \iint_{Q_T} \nabla v \cdot \nabla \psi \, dxdt + \int_{\Omega} u_0 \psi(x, 0) \, dx \geq 0 \qquad (1.1.36)$$

for any $\psi \in C^1(\overline{Q}_T)$, $\psi(\,.\,, T) = 0$ in Ω and $\psi \geq 0$ in Q_T (see Theorem 1.2.10). Observe that inequality (1.1.36) is not a consequence of the weak formulation (1.1.33) (in fact, no assumption on the sign of ψ_t is made).

By analogy with the cubic-like case, it is natural to ask whether the couple (u, v) in equality (1.1.33) satisfies a limiting entropy inequality. This is indeed the case, *if the family* $\{G(u^\varepsilon)\}$ *is equi-integrable in* Q_T (Theorem 1.2.16). Again, this restriction is due to the lack of equi-integrability of the family $\{u^\varepsilon\}$, thus to the weaker a priori estimates (L^1 instead of L^∞) available now. Nevertheless, monotonicity in time of the phase fraction λ can be proved also in the present case (see Theorem 1.2.15).

(β) In Subsection 1.2.4 we address the properties of the measure μ in equality (1.1.33). First we investigate the support of μ, making use of equality (1.1.33) itself (see Proposition 1.2.17). Secondly, we prove the following disintegration of μ:

$$\iint_{\overline{Q}_T} f \, d\mu = \int_{[0,T]} dt \int_{\overline{\Omega}} f(x,t) \, d\tilde{\gamma}_t(x) \quad \text{for any } f \in L^1(\overline{Q}_T, d\mu) \, ; \quad (1.1.37)$$

here $\tilde{\gamma}_t \in \mathcal{M}(\overline{\Omega})$ is a Radon measure defined for a.e. $t \in (0, T)$. We also show that there exists a unique $h \in L^\infty(0, T)$, $h \geq 0$ such that $\tilde{\gamma}_t = h(t)\gamma_t$ for a.e. $t \in (0, T)$; here γ_t is a probability measure over $\overline{\Omega}$ and a representative of h is

$$h(t) = \int_{\Omega} u_0(x) \, dx - \int_{\Omega} u(x,t) \, dx \qquad (1.1.38)$$

for a.e. $t \in (0, T)$ (see Propositions 1.2.18-1.2.19). Observe that the above equality also reads:

$$\int_{\Omega} u(x,t) \, dx + \int_{\overline{\Omega}} d\tilde{\gamma}_t(x) = \int_{\Omega} u_0(x) \, dx \, . \qquad (1.1.39)$$

A remarkable feature of the application $t \mapsto \tilde{\gamma}_t$ is its *nondecreasing* character. In fact, we prove (see Proposition 1.2.21):

$$\int_{\overline{\Omega}} \varphi(x) \, d\tilde{\gamma}_{t_1}(x) \leq \int_{\overline{\Omega}} \varphi(x) \, d\tilde{\gamma}_{t_2}(x) \qquad (1.1.40)$$

for any $\varphi \in C^1(\overline{\Omega})$ and a.e. $t_1, t_2 \in (0, T)$, $t_1 < t_2$; namely, the map $t \mapsto \tilde{\gamma}_t(E)$ is nondecreasing in $(0, T)$ for any Borel set $E \subseteq \overline{\Omega}$.

As a consequence of equalities (1.1.39)-(1.1.40), the function

$$t \mapsto \int_{\Omega} u(x,t) \, dx$$

is *nonincreasing in time*. Therefore, within the *constant* map from $(0,T)$ to $\mathbb{R}$, $t \mapsto \int_\Omega u(x,t)dx + \tilde{\gamma}_t(\overline{\Omega})$, there is a relative growth of the term $\tilde{\gamma}_t(\overline{\Omega})$ with respect to the term $\int_\Omega u(x,t)dx$ as time increases.

This suggests that in equation (1.1.34) the singular part μ prevails over the regular L^1-term u for large times[1]. In other words, it is reasonable to expect a general "coarsening" effect, since the absolutely continuous part of the measure $u + \mu$ decreases and possibly disappears, while singularities can appear and spread as time goes on. As already remarked, this conjecture seems consistent with the model interpretation of equation (1.1.1) (in particular, with its connection with the Perona-Malik equation).

1.2 Mathematical framework and results

1.2.1 Viscous regularization

Let us first give the following

Definition 1.2.1. *Let $u_0 \in L^\infty(\Omega)$. By a solution to problem* (1.1.4)-(1.1.5) *we mean any couple $u^\varepsilon \in C^1([0,T]; L^\infty(\Omega))$, $v^\varepsilon \in C([0,T]; C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega))$ with $p > n$, $\Delta v^\varepsilon \in C([0,T]; L^\infty(\Omega))$, which satisfies (1.1.4)-(1.1.5) in the classical sense. A solution is said to be global if it is a solution in Q_T for any $T > 0$.*

Concerning well-posedness of problem (1.1.4)-(1.1.5), the following result is well known (see [NP], [Pa] for the proof).

Theorem 1.2.1. *For any $u_0 \in L^\infty(\Omega)$ and $\varepsilon > 0$ there exists a unique global solution $(u^\varepsilon, v^\varepsilon)$ of problem* (1.1.4)-(1.1.5). *Moreover, there holds:*

$$\|\phi(u^\varepsilon)\|_{L^\infty(Q_T)} \leq 1\,, \qquad \|v^\varepsilon\|_{L^\infty(Q_T)} \leq 1\,. \tag{1.2.1}$$

Arguing as in the Introduction (see (1.1.24)) gives the following

Lemma 1.2.2. *Let $(u^\varepsilon, v^\varepsilon)$ be a solution of problem (1.1.4)-(1.1.5). Let $g \in C^1(\mathbb{R})$, $g' \geq 0$ and G be defined by (1.1.23). Then for any $t \in [0,T]$*

$$\int_\Omega G(u^\varepsilon(x,t))\,dx \leq \int_\Omega G(u_0(x))\,dx. \tag{1.2.2}$$

Moreover, for any $\psi \in C_c^\infty(Q_T)$, $\psi \geq 0$ the entropy inequality (1.1.27) *is satisfied.*

Concerning the existence of positively invariant regions for problem (1.1.4)-(1.1.5), the following result can be proven.

Proposition 1.2.3. *The half line $[0,+\infty)$ is positively invariant for problem* (1.1.4)-(1.1.5). *The same is true for any interval $[0,\bar{u}]$ with $\bar{u} \in (0,1]$.*

[1] Observe that the choice of $T > 0$ is arbitrary (see Theorem 1.2.1).

Remark 1.2.1. In view of the above result, the assumption $u_0 \geq 0$ implies $u^\varepsilon \geq 0$, thus $\phi(u^\varepsilon) \geq 0$ in Q_T (see (H_1)). Since for any $t \in [0,T]$ $v^\varepsilon \equiv v^\varepsilon(\,.\,,t)$ solves the problem:

$$\begin{cases} -\varepsilon\Delta v^\varepsilon + v^\varepsilon = \phi(u^\varepsilon)(\,.\,,t) & \text{in } \Omega \\ \dfrac{\partial v^\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

we also have $v^\varepsilon \geq 0$ in Q_T.

Concerning the initial data, in the sequel we always make the assumption:

$$(H_3) \qquad u_0 \in L^\infty(\Omega)\,, \ \ u_0 \geq 0\,.$$

Then from Proposition 1.2.3 we easily obtain the following a priori bound for the family $\{u^\varepsilon\}$.

Proposition 1.2.4. *Any positive solution to problem* (1.1.4)-(1.1.5) *satisfies equality* (1.1.26) *for each* $t \in [0,T]$.

We also have the following

Proposition 1.2.5. *Let $(u^\varepsilon, v^\varepsilon)$ solve problem (1.1.4)-(1.1.5). Then there exists a constant $C > 0$ such that inequality (1.1.9) holds for any $\varepsilon > 0$, $T > 0$.*

Set $C^1_b(\mathbb{R}) := \{f \in C^1(\mathbb{R}) \,|\, f,\ f' \text{ bounded}\}$. The following result plays an important role when studying the limiting behaviour of the family $\{u^\varepsilon\}$ as $\varepsilon \to 0$.

Proposition 1.2.6. *Let $f,\ g \in C^1_b(\mathbb{R})$; let $F := f(\phi)$ and G be defined by* (1.1.23). *Suppose*

$$F(u^\varepsilon) \overset{*}{\rightharpoonup} F^*, \qquad G(u^\varepsilon) \overset{*}{\rightharpoonup} G^*, \qquad F(u^\varepsilon)G(u^\varepsilon) \overset{*}{\rightharpoonup} H^*$$

in $L^\infty(Q_T)$, where $\{u^\varepsilon\}$ satisfies problem (1.1.4)-(1.1.5). Then $H^* = F^*G^*$.

The proof of Proposition 1.2.6 is almost the same as in [Pl1] (see also Chapter 2), thus we omit it.

Remark 1.2.2. The above assumption $G(u^\varepsilon) \overset{*}{\rightharpoonup} G^*$ would follow from the L^∞-estimate (1.1.8), if assumption (H_2) were satisfied. In the present case, since

$$|G(u)| = \left| \int_0^u g(\phi(s))\, ds \right| \leq \int_0^{+\infty} |g(\phi(s))|\, ds\,,$$

it is natural to assume $g \circ \phi \in L^1(\mathbb{R})$ to obtain boundedness of the family $\{G(u^\varepsilon)\}$ in $L^\infty(Q_T)$. Observe that any $g \in C^1_c(0,1)$ satisfies this condition;

in fact,

$$\begin{aligned} |G(u^\varepsilon)| &= \left|\int_0^{u^\varepsilon} g(\phi(s))\,ds\right| \le \int_0^{+\infty} |g(\phi(s))|\,ds \\ &\le \max_{\zeta\in[a,b]} |g'(\zeta)| \int_{\beta_1(a)}^{\beta_2(a)} |\phi(s)|\,ds < C\,. \end{aligned}$$

Here $0 < a < b < 1$ have been choosen so that $\operatorname{supp} g \subseteq [a,b]$, while $\beta_1(a)$, $\beta_2(a)$ denote the two solutions of the equation $\phi(u) = a$.

1.2.2 Vanishing viscosity limit

Let us recall the following

Definition 1.2.2. *Let τ^k, τ be Young measures on $Q_T \times \mathbb{R}$. We say that $\tau^k \to \tau$* narrowly, *if*

$$\int_{Q_T\times\mathbb{R}} \varphi\, d\tau^k \to \int_{Q_T\times\mathbb{R}} \varphi\, d\tau \tag{1.2.3}$$

for any $\varphi : Q_T \times \mathbb{R} \to \mathbb{R}$ bounded and measurable, $\varphi(x, t, \,.\,)$ continuous for a.e. $(x, t) \in Q_T$.

The following proposition is a consequence of the more general *Prohorov's theorem* (*e.g.*, see [V]).

Proposition 1.2.7. *Let u^ε denote the unique solution of problem* (1.1.4)-(1.1.5) *and τ^ε the associated Young measure ($\varepsilon > 0$). Then:*
(i) there exist a sequence $\{u^{\varepsilon_k}\} \subseteq \{u^\varepsilon\}$ and a Young measure τ on $Q_T \times \mathbb{R}$ such that $\tau^k \to \tau$ narrowly;
(ii) for any $f \in C(\mathbb{R})$ such that the sequence $\{f(u^{\varepsilon_k})\}$ is bounded in $L^1(Q_T)$ and equi-integrable there holds

$$f(u^{\varepsilon_k}) \rightharpoonup f^* \quad \text{in } L^1(Q_T); \tag{1.2.4}$$

here

$$f^*(x,t) := \int_{[0,+\infty)} f(\xi)\, \nu_{(x,t)}(d\xi) \qquad \text{for a.e. } (x,t) \in Q_T \tag{1.2.5}$$

and $\nu_{(x,t)}$ is the disintegration of the Young measure τ.

As pointed out in the Introduction, in general we cannot guarantee the equi-integrability of the sequence $\{u^{\varepsilon_k}\}$; hence Proposition 1.2.7-(ii) cannot be directly used with $f(u) = u$. However, we can associate to $\{u^{\varepsilon_k}\}$ an equi-integrable subsequence by removing sets of small measure; this is the content of the following proposition (*e.g.*, see [GMS], [V] for the proof).

Proposition 1.2.8. *Let the assumptions of Proposition 1.2.7 be satisfied. Then there exist a subsequence* $\{u^{\varepsilon_j}\} \equiv \left\{u^{\varepsilon_{k_j}}\right\} \subseteq \{u^{\varepsilon_k}\}$ *and a sequence of measurable sets* $\{A_j\}$,

$$A_j \subset Q_T, \qquad A_{j+1} \subset A_j \quad \text{for any } j \in \mathbb{N}, \quad |A_j| \to 0 \quad \text{as } j \to \infty,$$

such that the sequence $\left\{u^{\varepsilon_j} \chi_{Q_T \setminus A_j}\right\}$ *is equi-integrable. Moreover,* (1.1.31)-(1.1.32) *hold.*

From the above proposition we obtain the following

Theorem 1.2.9. *Let the assumptions of Proposition 1.2.7 be satisfied; let* $\{u^{\varepsilon_j}\}$, $\{A_j\}$ *be the sequences considered in Proposition 1.2.8.*
(i) Let $v \in L^1(Q_T)$ *be the* L^1*-weak limit of the sequence* $\{\phi(u^{\varepsilon_j})\}$*, whose existence is ensured by the first estimate in* (1.2.1) *and Proposition 1.2.7-(ii). Then* $v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ *and there holds:*

$$\begin{aligned} v^{\varepsilon_j} &\overset{*}{\rightharpoonup} v \qquad && \text{in} \quad L^\infty(Q_T), \\ v^{\varepsilon_j} &\rightharpoonup v && \text{in} \quad L^2((0,T); H^1(\Omega)), \end{aligned}$$

v^{ε_j} *being defined by* (1.1.4).
(ii) There exist a subsequence of $\{u^{\varepsilon_j}\}$*, denoted again* $\{u^{\varepsilon_j}\}$*, and a positive Radon measure* $\mu \in \mathcal{M}^+(\overline{Q}_T)$ *such that*

$$\iint_{Q_T} u^{\varepsilon_j} \chi_{A_j} \psi \, dxdt \to \iint_{\overline{Q}_T} \psi \, d\mu \tag{1.2.6}$$

for any $\psi \in C(\overline{Q}_T)$.
(iii) Let u *be the* L^1*-weak limiting function in* (1.1.31). *Then equality* (1.1.33) *holds for any* $\psi \in C^1(\overline{Q}_T)$ *such that* $\psi(\,.\,,T) = 0$ *in* Ω.

Since μ is a positive Radon measure on $\overline{Q}_T$, from (1.1.33) we get

$$\iint_{Q_T} (u\psi_t - \nabla v \cdot \nabla \psi) \, dxdt + \int_\Omega u_0(x)\psi(x,0) \, dx \le 0, \tag{1.2.7}$$

for any $\psi \in C^1(\overline{Q}_T)$ such that $\psi(\,.\,,T) = 0$ in Ω, $\psi_t \ge 0$ in Q_T. However, the sign assumption concerning ψ_t does not seem very natural; in this respect, the following theorem is expedient.

Theorem 1.2.10. *Let* (u, v) *be the couple given by Proposition 1.2.8 and Theorem 1.2.9. Then inequality (1.1.36) holds for any* $\psi \in C^1(\overline{Q}_T)$, $\psi \ge 0$ *in* Q_T *such that* $\psi(\,.\,,T) = 0$ *in* Ω.

1.2.3 Regular term

Let $\nu_{(x,t)}$ be the disintegration of the Young measure τ considered in Proposition 1.2.7, which holds for a.e. $(x,t) \in Q_T$. Following [Pl1], we assume the following condition to be satisfied.

Condition (S): *The functions β_1', β_2' are linearly independent on any open subset of the interval $(0,1)$.*

Let $I_1 \equiv [0,1]$, $I_2 \equiv (1,+\infty)$; set $\nu \equiv \nu_{(x,t)}$ for simplicity. For a.e. $(x,t) \in Q_T$ define two maps $\sigma_l \equiv \sigma_{(x,t);l} : C(\mathbb{R}) \to \mathbb{R}$ by setting

$$\int_{\mathbb{R}} f(\lambda)\,\sigma_l(d\lambda) \equiv\ \langle f, \sigma_l \rangle := \int_{I_l} (f \circ \phi)(\xi)\,\nu(d\xi) \qquad (l = 1,2)\,. \tag{1.2.8}$$

Set also

$$\sigma := \sigma_1 + \sigma_2\,. \tag{1.2.9}$$

It is immediately seen that σ_1, σ_2 are (positive) Radon measures on $\mathbb{R}$; in view of the above definitions, $\sigma \equiv \sigma_{(x,t)}$ is a probability measure on $\mathbb{R}$ for a.e. $(x,t) \in Q_T$. In analogy with [Pl1], the following lemma will be proven.

Lemma 1.2.11. *Let σ_1, σ_2 be the Radon measures defined by* (1.2.8)*. Then:*
(i) $\operatorname{supp}\sigma_l \subseteq [0,1]$ $(l = 1,2)$*;*
(ii) $\sigma_2(\{0\}) = 0$;
(iii) $f \circ \beta_l \in L^1([0,1], d\sigma_l)$ $(l = 1,2)$ *for any* $f \in C(\mathbb{R})$*, such that the sequence* $\{f(u^{\varepsilon_j})\}$ *is bounded in* $L^1(Q_T)$ *and equi-integrable.*

In view of Lemma 1.2.11-*(i)*, the support of the measure σ is contained in $[0,1]$. We also have:

$$\langle f, \sigma \rangle = \langle f, \sigma_1 \rangle + \langle f, \sigma_2 \rangle = \int_{[0,+\infty)} (f \circ \phi)(\xi)\,\nu(d\xi) \tag{1.2.10}$$

for any $f \in C(\mathbb{R})$; moreover,

$$\begin{aligned} \langle f, \nu \rangle &\equiv \int_{[0,+\infty)} f(\xi)\,\nu(d\xi) = \int_{I_1} f(\xi)\,\nu(d\xi) + \int_{I_2} f(\xi)\,\nu(d\xi) \\ &= \int_{I_1} [(f \circ \beta_1) \circ \phi](\xi)\,\nu(d\xi) + \int_{I_2} [(f \circ \beta_2) \circ \phi](\xi)\,\nu(d\xi) \\ &= \langle f \circ \beta_1, \sigma_1 \rangle + \langle f \circ \beta_2, \sigma_2 \rangle \end{aligned} \tag{1.2.11}$$

for any $f \in C(\mathbb{R})$ such that the sequence $\{f(u^{\varepsilon_j})\}$ is bounded in $L^1(Q_T)$ and equi-integrable (here use of (1.2.8) and Lemma 1.2.11-*(iii)* has been made).

The next theorem gives a useful representation of the measure σ.

Theorem 1.2.12. *The measure $\sigma \equiv \sigma_{(x,t)}$ is the Dirac mass concentrated at the point*

$$v(x,t) := \int_{[0,+\infty)} \phi(\xi)\, \nu_{(x,t)}(d\xi) = \langle \phi, \nu_{(x,t)} \rangle \tag{1.2.12}$$

for a.e. $(x,t) \in Q_T$.

Thanks to equations (1.2.10)-(1.2.11), Theorem 1.2.12 and Lemma 1.2.11-(*ii*), we obtain the following result, which describes the structure of the Young disintegration measure ν. The analogy with the cubic-like case investigated in [Pl1] (see (1.1.18)) should be observed.

Proposition 1.2.13. *Let $v \in L^\infty(Q_T) \cap L^2((0,T);H^1(\Omega))$ be the limiting function given by Theorem 1.2.9. Then for a.e. $(x,t) \in Q_T$ the measure $\nu_{(x,t)}$ is atomic. More precisely:*
(i) if $v(x,t) > 0$, then $\operatorname{supp} \nu_{(x,t)}$ consists of the points $\beta_1(v(x,t))$, $\beta_2(v(x,t))$;
(ii) if $v(x,t) = 0$, then $\operatorname{supp} \nu_{(x,t)} = \{0\}$.

From the above proposition we obtain the following

Theorem 1.2.14. *Let (u,v) be the couple mentioned in Theorem 1.2.9. Then:*
(i) there exists $\lambda \in L^\infty(Q_T)$, $0 \leq \lambda \leq 1$ such that equality (1.1.35) holds a.e. in Q_T;
(ii) there holds

$$\begin{aligned} &\phi(u^{\varepsilon_j}) \to v \qquad \text{in } L^p(Q_T) \text{ for any } p \in [1,\infty)\,, \\ &v^{\varepsilon_j} \to v \qquad \text{in } L^2(Q_T). \end{aligned}$$

The following monotonicity property of the coefficient λ in (1.1.35) can be proved; the proof is modeled after that in [Pl1], thus we omit it.

Theorem 1.2.15. *Assume $\phi''(1) \neq 0$. Let (u,v,λ) be the triple mentioned in Theorem 1.2.14; suppose*

$$0 < v \leq k < 1 \tag{1.2.13}$$

in some cylinder $Q_0 = \Omega_0 \times [\alpha,\beta]$, $\Omega_0 \subset \Omega$. Then the function $\lambda(x,\,.\,)$ is nondecreasing with respect to $t \in [\alpha,\beta]$, for a.e. $x \in \Omega_0$.

In view of Lemma 1.2.2, the solutions $(u^\varepsilon, v^\varepsilon)$ to problem (1.1.4)-(1.1.5) satisfy the entropy inequalities (1.1.27) for any $\varepsilon > 0$. The following theorem shows that, under suitable assumptions, this kind of inequalities is preserved in the viscous limit $\varepsilon \to 0$. The proof is similar to that given in [Pl1] (see also [MTT]) for the cubic-like case, thus it is omitted.

Theorem 1.2.16. *Let v, λ be the functions given by Theorem 1.2.9 and Theorem 1.2.14, respectively. Let G be defined by (1.1.23) with $g \in C^1(\mathbb{R})$, $g' \geq 0$ and let the family $\{G(u^\varepsilon)\}$ be equi-integrable in Q_T. Then there holds:*

$$\iint_{Q_T} \left(G^* \psi_t - g(v)\nabla v \cdot \nabla \psi - g'(v)|\nabla v|^2 \psi\right) dxdt \geq 0 \tag{1.2.14}$$

for any $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$, G^* being the L^1-weak limit of the sequence $\{G(u^{\varepsilon_j})\}$:

$$G^* = \begin{cases} \lambda G(\beta_1(v)) + (1-\lambda)G(\beta_2(v)) & \text{for } v > 0 \\ G(0) & \text{for } v = 0. \end{cases} \tag{1.2.15}$$

1.2.4 Singular term

Let us return to the measure μ encountered in Theorem 1.2.9. Some information concerning its support is given by the following proposition.

Proposition 1.2.17. *Let μ be the positive Radon measure mentioned in Theorem 1.2.9. Then:*
(i) μ is not a countable superposition of Dirac measures concentrated in points of $\overline{Q}_T$;
(ii) for any $t_0 \in [0,T]$ there holds $\mu(F_{t_0}) = 0$, where $F_{t_0} := \overline{\Omega} \times \{t_0\}$;
(iii) $\mu(E) = 0$ for any closed k-dimensional manifold $E \subset Q_T$ with $k < n-1$.

Remark 1.2.3. In view of Proposition 1.2.17-(*iii*) above, if $n \geq 3$ there holds $\mu(\{x_0\} \times [0,T]) = 0$ for any $x_0 \in \overline{\Omega}$.

Some qualitative properties of the measure μ are given below. To begin with, we observe that μ can be *disintegrated* in two measures, defined on $[0,T]$ and $\overline{\Omega}$ respectively; this is the content of the following proposition. The proof (which is a particular consequence of the more general Proposition 8 on p. 35 of [GMS], Vol. I) is omitted.

Proposition 1.2.18. *Let $\mu \in \mathcal{M}^+(\overline{Q}_T)$ be the measure mentioned in Theorem 1.2.9. Then there exists a measure $\lambda \in \mathcal{M}^+([0,T])$ and λ-a.e. in $[0,T]$ a measure $\gamma_t \in \mathcal{M}^+(\overline{\Omega})$ such that:*
(i) for any Borel set $E \subset \overline{Q}_T$ there holds

$$\mu(E) = \int_{[0,T]} \gamma_t(E_t)\, d\lambda(t),$$

where $E_t := \left\{x \in \overline{\Omega} \mid (x,t) \in E\right\}$;
(ii) for any $f \in L^1(\overline{Q}_T, d\mu)$ the function $f(t, .)$ belongs to $L^1(\overline{\Omega}, d\gamma_t)$ for $\lambda - a.e.$ $t \in [0,T]$ and there holds:

$$\iint_{\overline{Q}_T} f\, d\mu = \int_{[0,T]} d\lambda(t) \int_{\overline{\Omega}} f(x,t)\, d\gamma_t(x)\,.$$

Moreover, since $\mu(\overline{Q}_T) < \infty$, *we can choose* $\lambda(I) = \mu(\overline{\Omega} \times I)$ *for any* $I \subseteq [0,T]$, *and* $\gamma_t(\overline{\Omega}) = 1$ *for* $\lambda -$ *a.e.* $t \in [0,T]$.

The next proposition shows that $\lambda \in \mathcal{M}^+([0,T])$ is absolutely continuous with respect to the Lebesgue measure.

Proposition 1.2.19. (i) *There exists a unique* $h \in L^\infty(0,T)$, $h \geq 0$, *such that* $d\lambda = h\, dt$. *Moreover, equality* (1.1.38) *holds.*
(ii) *Set* $\tilde{\gamma}_t := h(t)\gamma_t \in \mathcal{M}(\overline{\Omega})$. *Then equality* (1.1.37) *holds.*

We can use the family of Radon measures $\{\tilde{\gamma}_t\}$ to improve the description of the limiting behaviour of the sequence $\{u^{\varepsilon_j}\}$ as $\varepsilon_j \to 0$. Precisely, the following theorem holds.

Theorem 1.2.20. *Let assumption of Theorem 1.2.9 be satisfied. Let* $u \in L^1(Q_T)$ *be the limiting function given by Theorems 1.2.9-1.2.14. Let* $\tilde{\gamma}_t \in \mathcal{M}(\overline{\Omega})$ *be the Radon measure given by Proposition 1.2.19-(ii) for a.e.* $t \in (0,T)$. *Then:*
(i) *for any* $\varphi \in C(\overline{\Omega})$

$$\int_\Omega (u^{\varepsilon_j}\chi_{Q_T\setminus A_j})(x,\,.\,)\varphi(x)\,dx \overset{*}{\rightharpoonup} \int_\Omega u(x,\,.\,)\varphi(x)\,dx \qquad \text{in } L^\infty(0,T),$$

$$\int_\Omega (u^{\varepsilon_j}\chi_{A_j})(x,\,.\,)\varphi(x)\,dx \overset{*}{\rightharpoonup} \int_{\overline{\Omega}} \varphi(x)\,d\tilde{\gamma}_t(x) \qquad \text{in } L^\infty(0,T)\,;$$

(ii) *set*

$$W_j^\varphi(t) := \int_\Omega u^{\varepsilon_j}(x,t)\varphi(x)\,dx \tag{1.2.16}$$

for any $\varphi \in C^1(\overline{\Omega})$. *Then the sequence* $\left\{W_j^\varphi\right\}$ *strongly converges in* $C([0,T])$ *to the function*

$$W^\varphi(t) := \int_\Omega u(x,t)\varphi(x)\,dx + \int_{\overline{\Omega}} \varphi(x)\,d\tilde{\gamma}_t(x), \qquad t \in [0,T]. \tag{1.2.17}$$

Moreover, for any $t \in [0,T]$

$$\begin{aligned} &\int_\Omega u(x,t)\,\varphi(x)\,dx + \int_{\overline{\Omega}} \varphi(x)\,d\tilde{\gamma}_t(x) \qquad\qquad (1.2.18)\\ = \;& -\int_0^t ds \int_\Omega \nabla v(x,s)\cdot\nabla\varphi(x)\,dx + \int_\Omega \varphi(x)\,u_0(x)\,dx. \end{aligned}$$

Remark 1.2.4. *Equation* (1.2.18) *in the above theorem implies that for any* $\varphi \in C^1(\overline{\Omega})$ *the function* $t \mapsto W^\varphi(t)$ *belongs to the space* $W^{1,2}(0,T)$ *(since* $v \in L^2(0,T;H^1(\Omega))$*), with weak derivative given by*

$$W_t^\varphi(t) := -\int_\Omega \nabla v(x,t)\cdot\nabla\varphi(x)\,dx.$$

We also observe that there is a formal analogy between equation (1.1.33) *and equation* (1.2.18)*, hence a natural question is whether equation* (1.2.18) *can be deduced directly by equation* (1.1.33)*. Actually, it is not so. In fact, from equation* (1.1.34) *we obtain for any* $\psi \in C^1(\overline{Q}_T)$*:*

$$\begin{aligned} & \int_\Omega u(x,t)\psi(x,t)\,dx + \int_{\overline{\Omega}} \psi(x,t)\,d\tilde{\gamma}_t(x) - \int_\Omega \psi(x,0)u_0(x)\,dx + \\ & - \int_0^t \int_\Omega u(x,s)\psi_t(x,s)\,dxds - \int_0^t \int_\Omega \psi_t(x,s)d\mu \qquad (1.2.19) \\ = & \int_0^t \int_{\overline{\Omega}} (u+\mu)_t \psi(x,s) = -\int_0^t ds \int_\Omega \nabla v(x,s)\cdot\nabla\psi(x,s)\,dx. \end{aligned}$$

Thus, this shows that equation (1.1.33) *follows from* (1.2.19) *by choosing* $t = T$ *and* $\psi(\,.\,,T) = 0$ *in* Ω*, while* (1.2.19) *implies equation* (1.2.18) *choosing* $\psi(x,t) = \psi(x)$*.*

In view of the above results, from Theorem 1.2.10 we can deduce the following monotonicity property of the family $\{\tilde{\gamma}_t\}$, whose interpretation has been pointed out in the Introduction.

Proposition 1.2.21. *For any* $\varphi \in C^1(\overline{\Omega})$*,* $\varphi \geq 0$ *and for a.e.* $0 \leq t_1 < t_2 \leq T$ *there holds:*

$$\int_{\overline{\Omega}} \varphi(x)\,d\tilde{\gamma}_{t_1}(x) \leq \int_{\overline{\Omega}} \varphi(x)\,d\tilde{\gamma}_{t_2}(x)\,. \qquad (1.2.20)$$

1.3 Viscous regularization: Proofs

Proof of Lemma 1.2.2. The proof of inequality (1.1.25), which plainly implies (1.2.2), has been given in the Introduction. Concerning inequality (1.1.27), for any $\psi \in C_c^\infty(Q_T)$, $\psi \geq 0$, there holds

$$\begin{aligned} \frac{d}{dt}\int_\Omega G(u^\varepsilon)\psi\,dx &= \int_\Omega [G(u^\varepsilon)]_t\psi\,dx + \int_\Omega G(u^\varepsilon)\psi_t\,dx \qquad (1.3.1) \\ &= \int_\Omega g(\phi(u^\varepsilon))u_t^\varepsilon\psi\,dx + \int_\Omega G(u^\varepsilon)\psi_t\,dx \\ &\leq \int_\Omega \psi g(v^\varepsilon)\Delta v^\varepsilon\,dx + \int_\Omega G(u^\varepsilon)\psi_t\,dx. \end{aligned}$$

Using the Neumann boundary condition, we have

$$\begin{aligned} \int_\Omega \psi g(v^\varepsilon)\Delta v^\varepsilon\,dx &= \int_\Omega \mathrm{div}\,(\psi g(v^\varepsilon)\nabla v^\varepsilon)\,dx - \int_\Omega \nabla(\psi g(v^\varepsilon))\cdot\nabla v^\varepsilon\,dx \\ &= -\int_\Omega \left\{g(v^\varepsilon)\nabla\psi\cdot\nabla v^\varepsilon + \psi g'(v^\varepsilon)|\nabla v^\varepsilon|^2\right\}dx. \quad (1.3.2) \end{aligned}$$

Integrating (1.3.1) with respect to time and using (1.3.2) gives inequality (1.1.27). □

Proof of Proposition 1.2.3. (i) Choose $g \in C^1(\mathbb{R})$ such that $g(s) < 0$, $g'(s) > 0$ if $s < 0$, $g(s) \equiv 0$ if $s \geq 0$. By assumption (H_1) we have $G(u) > 0$ if $u \in (-\infty, 0)$, $G(u) \equiv 0$ if $u \geq 0$ (here we choose $k = 0$ in the definition (1.1.23)). By inequality (1.2.2) we obtain

$$0 \leq \int_\Omega G(u^\varepsilon(x,t))\,dx \leq \int_\Omega G(u_0(x))\,dx = 0$$

for any $t \in [0,T]$. This implies $G(u(\,.\,,t)) = 0$, thus $u(\,.\,,t) \geq 0$ a.e. in Ω for any $t \in [0,T]$ and the first claim follows.

(ii) If $\bar{u} < 1$, set $M := \phi(\bar{u})$ and choose $g \in C^1(\mathbb{R})$, $g' \geq 0$ such that $g(s) < 0$ if $s < 0$, $g(s) = 0$ if $s \in [0, M]$, $g(s) > 0$ if $s > M$. It is easily seen that $G(u) \geq 0$ for any $u \in \mathbb{R}$, $G(u) = 0$ if $u \in [0, \bar{u}]$ and $G(u) > 0$ for $u \in \mathbb{R} \setminus [0, \bar{u}]$. By inequality (1.2.2) we obtain now $G(u(\,.\,,t)) = 0$, thus $u(\,.\,,t) \in [0, \bar{u}]$ a.e. in Ω for any $t \in [0,T]$.

The case $\bar{u} = 1$ can be treated in a similar way. Define $\tilde{\phi} \in \mathrm{Lip}(\mathbb{R})$ as follows:

$$\tilde{\phi}(s) := \begin{cases} \phi(s) & \text{if } 0 \leq s \leq 1 \\ s & \text{if } s > 1, \end{cases}$$

then consider the solution $\tilde{u}^\varepsilon$ of the correspondent problem (1.1.4)-(1.1.5). Arguing as above shows that $\tilde{u}^\varepsilon \leq 1$ uniformly in Q_T, thus $\tilde{u}^\varepsilon = u^\varepsilon$ in Q_T for any $T > 0$; hence the conclusion follows. □

Proof of Proposition 1.2.4. Integrating with respect to x the first equation in (1.1.5) and using the Neumann boundary conditions we obtain:

$$\frac{d}{dt}\int_\Omega u^\varepsilon\,dx = \int_\Omega u^\varepsilon_t\,dx = \int_{\partial\Omega} \frac{\partial v^\varepsilon}{\partial \nu}\,d\sigma = 0$$

for any $t \in [0,T]$. This implies

$$\int_\Omega u^\varepsilon(x,t)\,dx = \int_\Omega u_0(x)\,dx$$

for any $t \in [0,T]$ and $\varepsilon > 0$. Finally, assumption $u_0 \geq 0$ in Ω implies $u^\varepsilon \geq 0$ in Q_T (see Proposition 1.2.3); hence the conclusion. □

The following proof is almost the same as in [Pl1], [MTT]; we give it for convenience of the reader.

Proof of Proposition 1.2.5. Choosing $g(s) = s$ in equation (1.1.24) gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega dx \int_0^{u^\varepsilon(x,t)} \phi(s)\,ds \quad &= \quad \int_\Omega [\phi(u^\varepsilon) - v^\varepsilon]\frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon}\,dx \\ &\quad + \int_\Omega \mathrm{div}(v^\varepsilon \nabla v^\varepsilon)\,dx - \int_\Omega |\nabla v^\varepsilon|^2\,dx. \end{aligned}$$

In view of equation (1.1.4) and of the Neumann boundary conditions, we get

$$-\frac{d}{dt}\int_\Omega dx \left(\int_0^{u^\varepsilon(x,t)} \phi(s)\,ds\right) = \int_\Omega \varepsilon (u^\varepsilon_t)^2\,dx + \int_\Omega |\nabla v^\varepsilon|^2\,dx.$$

Integrating the above equality on $(0,T)$ (for any $T>0$) gives

$$
\begin{aligned}
&\int_0^T\int_\Omega \left[\varepsilon (u_t^\varepsilon)^2 + |\nabla v^\varepsilon|^2\right] dxdt \\
= \ &\int_\Omega dx \left(\int_0^{u_0(x)} \phi(s)\, ds\right) - \int_\Omega dx \left(\int_0^{u^\varepsilon(x,T)} \phi(s)\, ds\right) \\
\le \ &\int_\Omega dx \left(\int_0^{u_0(x)} \phi(s)\, ds\right) ;
\end{aligned}
$$

here use of assumption (H_3) and Proposition 1.2.3 has been made. Hence the result follows. □

1.4 Vanishing viscosity limit: Proofs

Proof of Theorem 1.2.9. (i) By the first estimate in (1.2.1) the sequence $\{\phi(u^{\varepsilon_j})\}$ is bounded in $L^\infty(Q_T)$; hence $v \in L^\infty(Q_T)$ and

$$\phi(u^{\varepsilon_j}) \overset{*}{\rightharpoonup} v \qquad \text{in} \quad L^\infty(Q_T)$$

as $j \to \infty$. By the second estimate in (1.2.1), also the sequence $\{v^{\varepsilon_j}\}$ is bounded in $L^\infty(Q_T)$, hence weakly* relatively compact in this space. On the other hand, for any $\varphi \in L^2(Q_T)$ there holds:

$$
\begin{aligned}
&\left|\iint_{Q_T} (v^{\varepsilon_j}\varphi - v\varphi)\, dxdt\right| \qquad (1.4.1)\\
\le \ &\iint_{Q_T} |v^{\varepsilon_j} - \phi(u^{\varepsilon_j})||\varphi|\, dxdt + \left|\iint_{Q_T} (\phi(u^{\varepsilon_j}) - v)\varphi\, dxdt\right| \\
\le \ &\varepsilon_j^{1/2} \|\varepsilon_j^{1/2} u_t^{\varepsilon_j}\|_{L^2(Q_T)} \|\varphi\|_{L^2(Q_T)} + \left|\iint_{Q_T} (\phi(u^{\varepsilon_j}) - v)\varphi\, dxdt\right| .
\end{aligned}
$$

In view of (1.1.9), passing to the limit with respect to $j \to \infty$ in (1.4.1) gives

$$v^{\varepsilon_j} \rightharpoonup v \qquad \text{in} \quad L^2(Q_T),$$

hence weakly* in $L^\infty(Q_T)$.

Moreover, in view of estimate (1.1.9), the sequence $\{v^{\varepsilon_j}\}$ is uniformly bounded in $L^2((0,T);H^1(\Omega))$, thus $v \in L^2((0,T);H^1(\Omega))$ and there holds:

$$v^{\varepsilon_j} \rightharpoonup v \qquad \text{in} \quad L^2((0,T);H^1(\Omega)). \qquad (1.4.2)$$

(ii) Since the sequence $\{u^{\varepsilon_j}\}$ is bounded in $L^1(Q_T)$ (see (1.1.26)), the same holds for the sequence $\{u^{\varepsilon_j}\chi_{A_j}\}$, too. For simplicity, set

$$\mu_j := u^{\varepsilon_j}\chi_{A_j}\,, \qquad \tilde\mu_j := \begin{cases} \mu_j & \text{in } \ Q_T \\ 0 & \text{in } \ \mathbb{R}^{n+1}\setminus Q_T \end{cases} .$$

It follows that

$$\|\tilde{\mu}_j\|_{L^1(\mathbb{R}^{n+1})} = \|\mu_j\|_{L^1(Q_T)} < C\,,$$

hence there exist a subsequence of $\{\tilde{\mu}_j\}$, denoted again $\{\tilde{\mu}_j\}$, and a Radon measure $\mu \in \mathcal{M}(\mathbb{R}^{n+1})$ such that

$$\iint_{\mathbb{R}^{n+1}} \tilde{\mu}_j \psi \, dxdt \to \iint_{\mathbb{R}^{n+1}} \psi \, d\mu, \tag{1.4.3}$$

for any $\psi \in C_c(\mathbb{R}^{n+1})$ (*e.g.* , see [GMS]). Clearly,

$$\operatorname{supp} \mu \subseteq \overline{Q}_T\,; \tag{1.4.4}$$

moreover, since $\overline{Q}_T \subset \mathbb{R}^{n+1}$ is compact, for any $\psi \in C(\overline{Q}_T)$ we can find $\tilde{\psi} \in C_c(\mathbb{R}^{n+1})$ such that $\tilde{\psi} = \psi$ in $\overline{Q}_T$. Then by (1.4.3)-(1.4.4) the claim follows.

(iii) Set

$$u^{\varepsilon_j} = u^{\varepsilon_j} \chi_{Q_T \setminus A_j} + u^{\varepsilon_j} \chi_{A_j} \quad (j \in \mathbb{N})$$

in the weak formulation (1.1.14) of problem (1.1.4)-(1.1.5) (here recall that $\{u^{\varepsilon_j}\} \equiv \left\{u^{\varepsilon_{k_j}}\right\} \subseteq \{u^{\varepsilon_k}\}$). Fix any $\psi \in C^1(\overline{Q}_T)$, $\psi(\,.\,,T) = 0$ in Ω; in view of (1.2.6), (1.4.2) and in view of Proposition 1.2.8, passing to the limit as $j \to \infty$ in (1.1.14) gives equality (1.1.33). Hence the conclusion follows. □

The proof of Theorem 1.2.10 will be given at the end of Section 1.5.

1.5 Regular term: Proofs

Proof of Lemma 1.2.11. (i) Choose $f \in C(\mathbb{R})$ such that

- (a) $f(\lambda) > 0$ if $\lambda \in (1, +\infty)$,
- (b) $f(\lambda) = 0$ if $\lambda \in [0, 1]$.

By (1.2.1) and since $u^\varepsilon \geq 0$ there holds $0 \leq \phi(u^\varepsilon) \leq 1$, thus $f(\phi(u^\varepsilon)) = 0$ a.e. in Q_T. Then from equalities (1.2.5), (1.2.8) we obtain:

$$0 = \int_{[0,+\infty)} (f \circ \phi)(\xi) \, \nu(d\xi) = \sum_{l=1,2} \langle f, \sigma_l \rangle \ .$$

Since $f \geq 0$ on $\mathbb{R}$, this implies $\langle f, \sigma_l \rangle = 0$, thus $f = 0$ for $\sigma_l - a.e.$ $\lambda \in \mathbb{R}$ $(l = 1, 2)$; hence the claim follows.

(ii) For any $h \in \mathbb{N}$, we consider the function $f_h \in C([0, 1])$ defined by setting

$$f(\lambda) := \begin{cases} -h\lambda + 1 & \text{for } \lambda \in [0, 1/h) \\ 0 & \text{for } \lambda \in [1/h, 1]\,; \end{cases} \tag{1.5.1}$$

observe that $f_h \geq 0$, $f_h(\lambda) \to \chi_{\{0\}}(\lambda)$ as $h \to \infty$ for any $\lambda \in [0,1]$. Moreover,

$$\begin{aligned} 0 \leq \langle f_h, \sigma_2 \rangle &= \int_{[0,1]} f_h \, \sigma_2(d\lambda) = \int_{(1,+\infty)} (f_h \circ \phi)(\xi) \, \nu(d\xi) \qquad (1.5.2) \\ &= \int_{(\beta_2(1/h),+\infty)} (-h\phi(\xi) + 1) \, \nu(d\xi) \leq \int_{(\beta_2(1/h),+\infty)} \nu(d\xi). \end{aligned}$$

Since $\chi_{(\beta_2(1/h),+\infty)}(\xi) \to 0$ for any $\xi \in [0,+\infty)$, passing to the limit with respect to $h \to \infty$ in (1.5.2) proves the claim.
(*iii*) Consider any $f \in C(\mathbb{R})$ such that the sequence $\{f(u^{\varepsilon_j})\}$ is bounded in $L^1(Q_T)$ and equi-integrable; then $f \in L^1(\mathbb{R}^+, d\nu)$ by Proposition 1.2.7-(*ii*). Clearly, $|f| \circ \beta_1 \in C([0,1]) \subseteq L^1([0,1], d\sigma_1)$; then by (1.2.8) and claim (*i*) above we get:

$$\int_{I_1} (|f| \circ \beta_1 \circ \phi)(\xi) \, \nu(d\xi) = \int_{[0,1]} (|f| \circ \beta_1)(\lambda) \, \sigma_1(d\lambda). \qquad (1.5.3)$$

Moreover, $|f| \circ \beta_2 \in C((0,1])$, thus, in view of claim (*ii*), it is σ_2-measurable. Then by (1.2.8) we obtain (see also (1.2.11)):

$$\begin{aligned} &\int_{[0,1]} (|f| \circ \beta_2)(\lambda) \, \sigma_2(d\lambda) = \int_{I_2} (|f| \circ \beta_2 \circ \phi)(\xi) \, \nu(d\xi) \qquad (1.5.4) \\ = &\int_{[0,+\infty)} |f|(\xi) \, \nu(d\xi) - \int_{[0,1]} (|f| \circ \beta_1)(\lambda) \, \sigma_1(d\lambda) < +\infty. \end{aligned}$$

This concludes the proof. □

The proof of Theorem 1.2.12 needs two preliminary results. The first one is an easy consequence of Proposition 1.2.6 and Proposition 1.2.7-(*ii*).

Lemma 1.5.1. *Let $\nu_{(x,t)}$ be the disintegration of the Young measure τ given by Proposition 1.2.7. Let F, G be as in Proposition 1.2.6; suppose the family $\{G(u^\varepsilon)\}$ to be bounded in $L^\infty(Q_T)$. Then for a.e. $(x,t) \in Q_T$*

$$\begin{aligned} &\left(\int_{[0,+\infty)} F(\xi) \, \nu_{(x,t)}(d\xi) \right) \left(\int_{[0,+\infty)} G(\xi) \, \nu_{(x,t)}(d\xi) \right) = \\ = &\int_{[0,+\infty)} F(\xi) G(\xi) \, \nu_{(x,t)}(d\xi). \end{aligned}$$

The proof of the second result is almost the same as in [Pl1]; we give it for convenience of the reader. In this connection, consider the *nonincreasing* functions

$$\rho_l(\lambda) := \sigma_l([\lambda, 1]) \,, \quad \rho_{l,A}(\lambda) := \sigma_l([\lambda, 1] \cap A) \,,$$

where $l = 1, 2$, $\lambda \in [0,1]$ and $A \subseteq [0,1]$. Then the following holds.

Lemma 1.5.2. *Let $A \subset [0,1]$ be compact and $\sigma(A) > 0$. Then*

$$M(\lambda) - M_A(\lambda) = N_A \quad \text{for a.e. } \lambda \in (0,1), \tag{1.5.5}$$

where

$$\begin{aligned} M &:= \left(\beta_1' - \beta_2'\right)^{-1} \sum_{l=1}^{2} \beta_l' \rho_l \,, \\ M_A &:= [\sigma(A)]^{-1} \left(\beta_1' - \beta_2'\right)^{-1} \sum_{l=1}^{2} \beta_l' \rho_{l,A} \,, \\ N_A &:= [\sigma(A)]^{-1} \sigma_2(A) - \sigma_2\left([0,1]\right). \end{aligned}$$

Proof. Since A is compact, there exists a sequence $\{f_h\} \subset C^1([0,1])$, $f_h \geq 0$, $f_h = 1$ on A, such that

$$f_h(\lambda) \to \chi_A(\lambda) \quad \text{for any } \lambda \in [0,1]$$

as $h \to \infty$. Fix $g \in C_c^1(0,1)$; consider the function G defined by (1.1.23). In view of Remark 1.2.2, the family $\{G(u^\varepsilon)\}$ is uniformly bounded in Q_T. Set $F_h := f_h(\phi)$; by Proposition 1.2.6 and Lemma 1.5.1 we obtain:

$$\begin{aligned} &\left(\int_{[0,+\infty)} (f_h \circ \phi)(\xi)\, \nu(d\xi)\right) \left(\int_{[0,+\infty)} G(\xi)\, \nu(d\xi)\right) \\ = &\int_{[0,+\infty)} G(\xi)(f_h \circ \phi)(\xi)\, \nu(d\xi). \end{aligned}$$

Using (1.2.11), the above equation reads:

$$\left(\int_{[0,1]} f_h(\lambda)\, \sigma(d\lambda)\right) \sum_{l=1}^{2} \langle G \circ \beta_l, \sigma_l \rangle = \sum_{l=1}^{2} \langle f_h(G \circ \beta_l), \sigma_l \rangle \,.$$

Letting $h \to \infty$ gives

$$\sigma(A) \sum_{l=1}^{2} \int_{[0,1]} G(\beta_l(\lambda))\, \sigma_l(d\lambda) \;=\; \sum_{l=1}^{2} \int_A G(\beta_l(\lambda)) \sigma_l(d\lambda) \,. \tag{1.5.6}$$

Observe that for $\lambda > 0$

$$(G \circ \beta_1)(\lambda) = \int_0^{\beta_1(\lambda)} g(\phi(s))\, ds = \int_0^{\lambda} g(\zeta) \beta_1'(\zeta)\, d\zeta \,,$$

$$\begin{aligned} (G \circ \beta_2)(\lambda) &= \int_0^{\beta_2(\lambda)} g(\phi(s))\, ds \;= \int_0^1 g(\phi(s))\, ds + \int_1^{\beta_2(\lambda)} g(\phi(s))\, ds \\ &= \int_0^1 g(\zeta) \beta_1'(\zeta)\, d\zeta - \int_\lambda^1 g(\zeta) \beta_2'(\zeta)\, d\zeta \\ &= \int_0^\lambda g(\zeta) \beta_2'(\zeta)\, d\zeta + \int_0^1 \left(\beta_1'(\zeta) - \beta_2'(\zeta)\right) g(\zeta)\, d\zeta \,. \end{aligned}$$

Since $g \in C_c^1(0,1)$, the function $G \circ \beta_2$ can be continuously extended to $\lambda = 0$, so that $G \circ \beta_2 \in C([0,1])$ and for any $\lambda \in [0,1]$ there holds:

$$(G \circ \beta_l)(\lambda) = c_l + \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta\,, \tag{1.5.7}$$

where

$$c_1 := 0, \quad c_2 := \int_0^1 g(\zeta)(\beta_1'(\zeta) - \beta_2'(\zeta))\, d\zeta\,. \tag{1.5.8}$$

Using (1.5.7)-(1.5.8), equality (1.5.6) reads:

$$\begin{aligned} &\sigma(A) \sum_{l=1}^2 \int_{[0,1]} \left(c_l + \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta \right) \sigma_l(d\lambda) \\ &= \sum_{l=1}^2 \int_A \left(c_l + \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta \right) \sigma_l(d\lambda). \end{aligned} \tag{1.5.9}$$

Concerning the left-hand side of (1.5.9), we have:

$$\begin{aligned} &\sigma(A) \sum_{l=1}^2 \int_{[0,1]} \left(c_l + \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta \right) \sigma_l(d\lambda) \\ &= \sigma(A) \sum_{l=1}^2 \left(c_l \langle 1, \sigma_l \rangle + \int_{[0,1]} \sigma_l(d\lambda) \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta \right) \\ &= \sigma(A) \sum_{l=1}^2 \left(c_l \langle 1, \sigma_l \rangle + \int_0^1 d\zeta\, g(\zeta)\beta_l'(\zeta) \int_{[\zeta,1]} \sigma_l(d\lambda) \right) \\ &= \sigma(A) \sum_{l=1}^2 \left(c_l \langle 1, \sigma_l \rangle + \int_0^1 g(\zeta)\beta_l'(\zeta)\rho_l(\zeta)\, d\zeta \right). \end{aligned} \tag{1.5.10}$$

As for the right-hand side, there holds:

$$\begin{aligned} &\sum_{l=1}^2 \int_A \left(c_l + \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta \right) \sigma_l(d\lambda) \\ = &\sum_{l=1}^2 \left(c_l\sigma_l(A) + \int_{[0,1]} \chi_A(\lambda)\, \sigma_l(d\lambda) \int_0^\lambda g(\zeta)\beta_l'(\zeta)\, d\zeta \right) \\ = &\sum_{l=1}^2 \left(c_l\sigma_l(A) + \int_0^1 d\zeta\, g(\zeta)\beta_l'(\zeta) \int_{[\zeta,1]} \chi_A(\lambda)\, \sigma_l(d\lambda) \right) \\ = &\sum_{l=1}^2 \left(c_l\sigma_l(A) + \int_0^1 \rho_{l,A}(\zeta) g(\zeta)\beta_l'(\zeta)\, d\zeta \right). \end{aligned} \tag{1.5.11}$$

By (1.5.10)-(1.5.11) equality (1.5.9) reads:

$$\sigma(A)\sum_{l=1}^{2}\left(c_l\,\langle 1,\sigma_l\rangle+\int_0^1 (g\beta_l'\rho_l)(\zeta)\,d\zeta\right)= \tag{1.5.12}$$
$$=\sum_{l=1}^{2}\left(c_l\sigma_l(A)+\int_0^1 (g\beta_l'\rho_{l,A})(\zeta)\,d\zeta\right),$$

namely (see (1.5.8))

$$\int_0^1 g(\zeta)\Big\{\big[\,\langle 1,\sigma_2\rangle-\sigma_2(A)[\sigma(A)]^{-1}\big](\beta_1'-\beta_2')+$$
$$+\sum_{l=1}^{2}\beta_l'\big[\rho_l-\rho_{l,A}[\sigma(A)]^{-1}\big]\Big\}(\zeta)\,d\zeta=0\,.$$

Since $g\in C_c^1(0,1)$ is arbitrary, we also have:

$$\sum_{l=1}^{2}\beta_l'\big[\rho_l-[\sigma(A)]^{-1}\rho_{l,A}\big]=\big[[\sigma(A)]^{-1}\sigma_2(A)-\langle 1,\sigma_2\rangle\big]\,(\beta_1'-\beta_2')$$

for *a.e.* $\zeta\in(0,1)$. Dividing by $\beta_1'-\beta_2'$ (which is positive in $(0,1)$) both members of the above equality we obtain (1.5.5). This completes the proof. □

Proof of Theorem 1.2.12. Set

$$\lambda_0:=\min\{\lambda\in[0,1]\,|\,\lambda\in\operatorname{supp}\sigma\}\,.$$

If $\lambda_0=1$, the claim is obvious. Let $\lambda_0<1$; choose $A=[\lambda_0,\lambda_0+\delta]$ with $\delta>0$ suitably small. Then $\sigma(A)\neq 0$, $M_A(\lambda)=0$ if $\lambda\in(\lambda_0+\delta,1)$; moreover, by equation (1.5.5) we have

$$M(\lambda)=N_A$$

for *a.e* $\lambda\in(\lambda_0+\delta,1)$. Since N_A does not depend on λ and δ is arbitrary, we obtain

$$M(\lambda)=N_{\{\lambda_0\}} \tag{1.5.13}$$

for *a.e.* $\lambda\in(\lambda_0,1)$.

Consider any compact $A\subset[\lambda_0,1)$; there exists an interval $(\lambda^*,1)$ such that

$$A\cap(\lambda^*,1)=\emptyset.$$

In the interval $(\lambda^*,1)$ we have $M_A(\lambda)\equiv 0$, hence by (1.5.5) and (1.5.13)

$$N_A=N_{\{\lambda_0\}}. \tag{1.5.14}$$

Again in view of (1.5.5), equalities (1.5.13)-(1.5.14) imply $M_A(\lambda) = 0$ for *a.e.* $\lambda \in (\lambda_0, 1)$ and for any compact $A \subset [\lambda_0, 1)$, namely

$$\sum_{l=1}^{2} \beta_l'(\lambda)\sigma_l([\lambda, 1] \cap A) = 0 \quad \text{for } a.e.\ \lambda \in (\lambda_0, 1). \tag{1.5.15}$$

Consider any closed interval $A = [\alpha, \beta] \subset (\lambda_0, 1)$. If $\lambda \in (\lambda_0, \alpha)$ we have $\sigma_l([\lambda, 1] \cap A) = \sigma_l(A)$. Hence, by equation (1.5.15), it follows that

$$\sum_{l=1}^{2} \beta_l'(\lambda)\sigma_l(A) = 0 \quad \text{for } a.e.\ \lambda \in (\lambda_0, \alpha). \tag{1.5.16}$$

Since the functions β_1' and β_2' are continuous in (λ_0, α), equality (1.5.16) holds for any $\lambda \in (\lambda_0, \alpha)$; by Condition *(S)*, this implies $\sigma_1(A) = \sigma_2(A) = 0$. Since α and β are arbitrary, it follows that the support of σ consists at most of two points, namely $\lambda_0, \{1\}$. Let us prove that $\operatorname{supp} \sigma = \{\lambda_0\}$, ruling out the latter possibility.

By contradiction, let $\{1\} \in \operatorname{supp} \sigma$; choose $A = \{1\}$ in (1.5.5). There holds

$$0 < \sigma(A) < \sigma([0, 1]) \tag{1.5.17}$$

and

$$\rho_l(\lambda) = \sigma_l([\lambda, 1]) = \sigma_l([\lambda, 1] \cap A) = \rho_{l,A}(\lambda) = \sigma_l(A),$$

for $\lambda \in (\lambda_0, 1)$. Hence, by (1.5.5), we obtain

$$\begin{aligned} 0 \;=\; & \beta_1'(\lambda)\left[\sigma_1(A) - \sigma_1(A)[\sigma(A)]^{-1} - \sigma_2(A)[\sigma(A)]^{-1} + \sigma_2([0, 1])\right] + \\ & + \beta_2'(\lambda)\left[\sigma_2(A) - \sigma_2(A)[\sigma(A)]^{-1} + \sigma_2(A)[\sigma(A)]^{-1} - \sigma_2([0, 1])\right], \end{aligned}$$

for any $\lambda \in (\lambda_0, 1)$. By Condition (*S*) it follows that:

$$\begin{cases} \sigma_1(A) - [\sigma(A)]^{-1}\sigma_1(A) - [\sigma(A)]^{-1}\sigma_2(A) + \sigma_2([0, 1]) = 0 \\ \sigma_2(A) - [\sigma(A)]^{-1}\sigma_2(A) + [\sigma(A)]^{-1}\sigma_2(A) - \sigma_2([0, 1]) = 0. \end{cases}$$

The above equalities imply $\sigma(A) = 1$, a contradiction with (1.5.17) (recall that by (1.2.10) σ is a probability measure). This proves that $\operatorname{supp} \sigma = \{\lambda_0\}$, thus σ is the Dirac mass concentrated at the point λ_0.

The above conclusion holds for the measure $\sigma \equiv \sigma_{(x,t)}$, for a.e. $(x, t) \in Q_T$. Taking the dependence on (x, t) into account and using (1.2.10) with $f(\lambda) = \lambda$, we have:

$$\lambda_0(x, t) = \langle \lambda, \sigma_{(x,t)} \rangle = \langle \phi, \nu_{(x,t)} \rangle = v(x, t)\,,$$

$v(x, t)$ being defined by (1.2.12). This completes the proof. □

Proof of Proposition 1.2.13. By Theorem 1.2.12 the measure $\sigma_{(x,t)}$ is the Dirac mass concentrated at the point $v(x, t)$. Let us distinguish two cases, namely $v(x, t) > 0$ and $v(x, t) = 0$.

(i) If $v(x,t) > 0$, equation (1.2.10) implies that $\sigma_{1(x,t)}$ and $\sigma_{2(x,t)}$ have the following form:

$$\sigma_{1(x,t)} = \lambda(x,t)\, \delta_{v(x,t)}, \qquad \sigma_{2(x,t)} = \big(1 - \lambda(x,t)\big)\, \delta_{v(x,t)}$$

for some $\lambda \in L^\infty(Q_T)$, $\lambda \geq 0$ in Q_T. Then by equation (1.2.11) there holds:

$$\begin{aligned} \int_{[0,+\infty)} f(\xi)\nu_{(x,t)}(d\xi) &= \sum_{l=1}^{2} \big\langle f \circ \beta_l, \sigma_{l,(x,t)} \big\rangle \qquad (1.5.18) \\ &= \lambda(x,t) f(\beta_1(v((x,t))) + \big(1 - \lambda(x,t)\big) f(\beta_2(v(x,t))), \end{aligned}$$

for any $f \in C(\mathbb{R})$ such that $f(u^{\varepsilon_j})$ is bounded in $L^1(Q_T)$ and equi-integrable (see Lemma 1.2.11-(iii)).
(i) If $v(x,t) = 0$, by Lemma 1.2.11 we get $\sigma_{1,(x,t)} = \sigma_{(x,t)}$ and

$$\int_{[0,+\infty)} f(\xi)\nu_{(x,t)}(d\xi) = \big\langle f \circ \beta_1, \sigma_{(x,t)} \big\rangle = f(\beta_1(0)) = f(0)\,. \qquad (1.5.19)$$

Then the conclusion follows. □

Proof of Theorem 1.2.14. (i) Equality (1.1.35) is a direct consequence of Propositions 1.2.8, 1.2.13 (see (1.5.18)-(1.5.19)).
(ii) In view of Propositions 1.2.7 and 1.2.13, for any $p > 1$

$$[\phi(u^{\varepsilon_j})]^p \overset{*}{\rightharpoonup} v^p \quad \text{in } L^\infty(Q_T)\,.$$

Hence

$$\phi(u^{\varepsilon_j}) \to v \quad \text{in } L^p(Q_T)$$

for any $p \in [1, \infty)$ (*e.g.*, see [GMS]), thus the claim follows from estimate (1.1.9). This completes the proof. □

Now we can prove Theorem 1.2.10.

Proof of Theorem 1.2.10. Denote by G_θ the function defined by (1.1.23) with $g(v) = v^\theta$ ($\theta \in (0,1)$) and $k = 0$, namely

$$G_\theta(u) := \int_0^u [\phi(s)]^\theta ds.$$

Let $\psi \in C^1(\overline{Q}_T)$, $\psi \geq 0$, $\psi(\,.\,,T) = 0$ in Ω. As in the proof of Lemma 1.2.2, we get

$$\begin{aligned} \frac{d}{dt}\int_\Omega G_\theta(u^\varepsilon)\psi\, dx &= \int_\Omega G_\theta(u^\varepsilon)\psi_t\, dx + \int_\Omega [G_\theta(u^\varepsilon)]_t \psi\, dx \qquad (1.5.20) \\ &\leq \int_\Omega G_\theta(u^\varepsilon)\psi_t\, dx + \int_\Omega (v^\varepsilon)^\theta \Delta v^\varepsilon \psi\, dx \end{aligned}$$

(recall that $v^\varepsilon \geq 0$ by Remark 1.2.1). We proceed in three steps.

Step 1. For any $\varepsilon > 0$ and $\theta \in (0,1)$ there holds:

$$\iint_{Q_T} \left\{ G_\theta(u^\varepsilon)\psi_t - \frac{1}{\theta+1}\nabla(v^\varepsilon)^{\theta+1}\cdot\nabla\psi \right\} dxdt + \qquad (1.5.21)$$
$$+ \quad \int_\Omega G_\theta(u_0)(x)\psi(x,0)\,dx \geq 0.$$

In fact, inequality (1.5.21) plainly follows from (1.5.20), if we show that

$$\int_\Omega (v^\varepsilon)^\theta \Delta v^\varepsilon \psi\,dx \leq -\frac{1}{\theta+1}\int_\Omega \nabla(v^\varepsilon)^{\theta+1}\cdot\nabla\psi\,dx. \qquad (1.5.22)$$

For any $k \in \mathbb{N}$ the function $\left(v^\varepsilon + \frac{1}{k}\right)^\theta$ is in $H^1(\Omega)$, hence we have:

$$\begin{aligned} & \int_\Omega \left(v^\varepsilon + \frac{1}{k}\right)^\theta \Delta v^\varepsilon \psi\,dx \\ = & -\int_\Omega \left(v^\varepsilon + \frac{1}{k}\right)^\theta \nabla v^\varepsilon\cdot\nabla\psi\,dx - \theta\int_\Omega \left(v^\varepsilon + \frac{1}{k}\right)^{\theta-1}|\nabla v^\varepsilon|^2\psi\,dx \\ \leq & -\int_\Omega \left(v^\varepsilon + \frac{1}{k}\right)^\theta \nabla v^\varepsilon\cdot\nabla\psi\,dx. \end{aligned}$$

Passing to the limit with respect to $k \to \infty$ in the above inequality gives

$$\int_\Omega (v^\varepsilon)^\theta \Delta v^\varepsilon \psi\,dx \leq -\int_\Omega (v^\varepsilon)^\theta \nabla v^\varepsilon\cdot\nabla\psi\,dx.$$

Observe that $(v^\varepsilon)^{\theta+1} \in H^1(\Omega)$ and $\nabla[(v^\varepsilon)^{\theta+1}] = (\theta+1)(v^\varepsilon)^\theta\nabla v^\varepsilon$: hence inequality (1.5.22), thus (1.5.21) follows.
Step 2. Let us prove that for any $\theta \in (0,1)$

$$\iint_{Q_T} \left\{ G^*_\theta\psi_t - \frac{1}{\theta+1}\nabla v^{\theta+1}\cdot\nabla\psi \right\} dxdt +$$
$$+ \int_\Omega G_\theta(u_0)(x)\psi(x,0)\,dx \geq 0, \qquad (1.5.23)$$

where G^*_θ is the L^1-weak limit of the sequence $\{G_\theta(u^{\varepsilon_j})\}$ (see (1.2.15)). To this purpose, we study separately the different terms of (1.5.21) (written with $\varepsilon = \varepsilon_j$) as $\varepsilon_j \to 0$.
(*i*) By assumption (H_1) $\phi^p \in L^1(\mathbb{R})$ for some $p > 1$, hence $\phi^\theta \in L^{\frac{p}{\theta}}(\mathbb{R})$ $(\theta \in (0,1))$. Then for any $u \geq 0$

$$|G_\theta(u)| \leq \int_0^u \phi^\theta(s)\,ds \leq \left(\int_0^u \phi^{\theta\frac{p}{\theta}}(s)\,ds\right)^{\frac{\theta}{p}} (u)^{\frac{p-\theta}{p}} \leq \|\phi^p\|^{\frac{\theta}{p}}_{L^1(\mathbb{R})}(u)^{\frac{p-\theta}{p}}.$$

Since the sequence $\{u^{\varepsilon_j}\}$ is bounded in $L^1(Q_T)$, by the above inequality the sequence $\{G_\theta(u^{\varepsilon_j})\}$ is bounded in $L^{\frac{p}{p-\theta}}(Q_T)$, hence weakly compact in this space. In particular, this implies (possibly passing to a subsequence):

$$G_\theta(u^{\varepsilon_j}) \rightharpoonup G^*_\theta \qquad \text{in } L^1(Q_T)\,,$$

$$\iint_{Q_T} G_\theta(u^{\varepsilon_j})\psi_t \, dxdt \to \iint_{Q_T} G^*_\theta \psi_t \, dxdt.$$

(ii) Observe that

$$\begin{aligned}
\|(v^{\varepsilon_j})^{\theta+1}\|^2_{L^2(0,T;H^1(\Omega))} &= \iint_{Q_T} [(v^{\varepsilon_j})^{2\theta+2} + |\nabla (v^{\varepsilon_j})^{\theta+1}|^2] dxdt \\
&\leq |Q_T| + (\theta+1)^2 \iint_{Q_T} (v^{\varepsilon_j})^{2\theta} |\nabla v^{\varepsilon_j}|^2 dxdt \\
&\leq |Q_T| + 4\|v^{\varepsilon_j}\|_{L^2(0,T;H^1(\Omega))} \leq C\,;
\end{aligned}$$

here use of estimates (1.1.9), (1.2.1) has been made. Hence, possibly passing to a subsequence, there exists $w \in L^2(0,T;H^1(\Omega))$ such that

$$(v^{\varepsilon_j})^{\theta+1} \rightharpoonup w \quad \text{in} \quad L^2(0,T;H^1(\Omega))$$

as $j \to \infty$. Since by Theorem 1.2.14-(ii) $v^{\varepsilon_j} \to v$ in $L^2(Q_T)$, it follows that $w = v^{\theta+1}$. Therefore

$$(v^{\varepsilon_j})^{\theta+1} \rightharpoonup v^{\theta+1} \qquad \text{in } L^2(0,T;H^1(\Omega))$$

as $j \to \infty$, whence

$$\frac{1}{\theta+1} \iint_{Q_T} \nabla (v^{\varepsilon_j})^{\theta+1} \cdot \nabla \psi \, dxdt \to \iint_{Q_T} \frac{1}{\theta+1} \iint_{Q_T} \nabla v^{\theta+1} \cdot \nabla \psi \, dxdt.$$

Step 3. Finally, we pass to the limit with respect to $\theta \to 0$ in inequality (1.5.23). Again, we consider separately its different terms.

(i) By (1.2.15) there holds:

$$G^*_\theta = \begin{cases} \lambda \int_0^{\beta_1(v)} [\phi(s)]^\theta ds + (1-\lambda) \int_0^{\beta_2(v)} [\phi(s)]^\theta ds & \text{if } v > 0 \\ 0 & \text{if } v = 0, \end{cases}$$

Plainly, this implies $G^*_\theta(x,t) \to u(x,t)$ as $\theta \to 0$, for a.e. $(x,t) \in Q_T$. Moreover, a.e. in Q_T there holds

$$|G^*_\theta| \leq \begin{cases} \lambda\beta_1(v) + (1-\lambda)\beta_2(v) & \text{if } v > 0 \\ = 0 & \text{if } v = 0\,; \end{cases}$$

hence by (1.1.35) we have $|G^*_\theta| \leq u \in L^1(Q_T)$. It follows that

$$G^*_\theta \to u \qquad \text{in } L^1(Q_T) \tag{1.5.24}$$

as $\theta \to 0$. It is similarly seen that

$$G_\theta(u_0) \to u_0 \quad \text{in } L^1(\Omega). \tag{1.5.25}$$

(*ii*) From *Step 2* above we get

$$\|v^{\theta+1}\|^2_{L^2(0,T;H^1(\Omega))} \leq C$$

for any $\theta \in (0,1)$, with a constant C independent of θ; hence the family $\{v^{\theta+1}\}$ is weakly compact in $L^2(0,T;H^1(\Omega))$. Observe that $v^{\theta+1} \to v$ in $L^2(Q_T)$ as $\theta \to 0$. This implies

$$v^{\theta+1} \rightharpoonup v \qquad \text{in } \ L^2(0,T;H^1(\Omega))\,,$$

$$\frac{1}{\theta+1}\iint_{Q_T} \nabla v^{\theta+1}\cdot\nabla\psi\,dxdt \to \iint_{Q_T} \nabla v\cdot\nabla\psi\,dxdt \tag{1.5.26}$$

as $\theta \to 0$. In view of (1.5.24)-(1.5.26), passing to the limit as $\theta \to 0$ in (1.5.23) gives the claim. □

1.6 Singular term: Proofs

Proof of Proposition 1.2.17. (*i*) Consider any $(x_0,t_0) \in \Omega\times(0,T)$. Let $I_r \equiv [t_0-r, t_0+r]$ and $B(x_0,r) \subset \mathbb{R}^n$ be the n-dimensional ball with center in x_0 and radius r. Choose r such that

$$I_{2r} \subset (0,T) \quad \text{and} \quad B(x_0,3r) \subset\subset \Omega.$$

By standard arguments there exist $\eta \in C^1_c(0,T)$, $\rho \in C^\infty_c(\Omega)$ with the following properties:

(*a*) $\eta(t)=1$ for $t\in I_r$, $\rho(x)=1$ for $x \in B(x_0,r)$;
(*b*) $0\leq\eta(t)\leq 1$ for any $t\in(0,T)$, $0\leq\rho(x)\leq 1$ for any $x\in\Omega$;
(*c*) $\operatorname{supp}\eta \subseteq I_{2r}$, $\operatorname{supp}\rho\subseteq \overline{B(x_0,3r)}$;
(*d*) $\left|\frac{\partial\rho}{\partial x_i}(x)\right| \leq \frac{C}{r}$ for any $x\in\Omega$ $(i=1,...,n)$.

Set

$$\psi(x,t) := \rho(x)\tilde{\eta}(t),$$

where

$$\tilde{\eta}(t) := -\int_t^T \eta(s)\,ds. \tag{1.6.1}$$

Clearly, $\psi\in C^1(\overline{Q}_T)$ and $\psi_t \geq 0$. Then from (1.1.33) we obtain:

$$\begin{aligned}
&\iint_{B(x_0,r)\times I_r} u\,dxdt + \iint_{B(x_0,r)\times I_r} d\mu \\
\leq\ & \iint_{Q_T} \tilde{\eta}(t)\nabla v\cdot\nabla\rho\,dxdt - \tilde{\eta}(0)\int_\Omega u_0(x)\rho(x)\,dx \\
\leq\ & |\tilde{\eta}(0)|\int_0^T\int_{B(x_0,3r)} |\nabla v||\nabla\rho|\,dxdt + 4r\int_{B(x_0,3r)} u_0(x)\rho(x)\,dx \\
\leq\ & C_1 r r^{-1}\|\nabla v\|_{L^2(Q_T)} r^{n/2} + C_2\|u_0\|_{L^\infty(\Omega)} r^{n+1}.
\end{aligned}$$

Then there exists $C > 0$ such that for small values of r

$$\iint_{B(x_0,r)\times I_r} d\mu \leq C r^{n/2}\,;$$

letting $r \to 0$ claim (i) follows in this case. The case $(x_0, t_0) \in \partial Q_T$ can be dealt with in a similar way.
(ii) Given any $t_0 \in (0, T)$, choose $\psi(x,t) = \tilde{\eta}(t)$ as test function in (1.1.33), with $\tilde{\eta}$ defined by (1.6.1). We get

$$\begin{aligned}\iint_{\Omega\times I_r} u\,dxdt + \iint_{\overline{\Omega}\times I_r} d\mu &\leq \iint_{Q_T} \eta u\,dxdt + \iint_{\overline{Q}_T} \eta\,d\mu \\ &\leq -\tilde{\eta}(0)\int_\Omega u_0(x)\,dx \leq 4r\|u_0\|_{L^1(\Omega)}.\end{aligned}$$

The cases $t_0 = 0$ and $t_0 = T$ are dealt with similarly, thus the claim follows.
(iii) Let $E \subset Q_T$ be a k-dimensional closed manifold. Then for any $(x_0, t_0) \in E$ there exist an open neighbourhood U_0 of (x_0, t_0) and a map

$$F : U_0 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+1-k}, \qquad F = (F^1, F^2, \ldots F^{n+1-k}),$$

such that:
 (a) $E \cap U_0 = \{(x,t) \in U_0 \mid F(x,t) = 0\}$;
 (b) the derivative $DF(x_0, t_0)$ has maximal rank, *i.e.* equal to $n + 1 - k$.
Set $x_{n+1} \equiv t$. By (b) above there holds

$$\frac{\partial(F^1, F^2, \ldots F^{n+1-k})}{\partial(x_{i_1}, x_{i_2}, \ldots x_{i_{n+1-k}})}(x_0, t_0) = a \neq 0$$

for some $\{i_1, i_2, \ldots i_{n+1-k}\} \subset \{1, 2, \ldots n+1\}$. For sake of simplicity, assume

$$\{i_1, i_2, \ldots i_{n+1-k}\} = \{k+1, k+2, \ldots n+1\}\,.$$

Consider the function

$$G : U_0 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \qquad G(x_1, x_2, \ldots x_{n+1}) = (y_1, y_2, \ldots y_{n+1}),$$

defined as follows:

$$\begin{cases} y_1 := x_1 - x_{01} \\ \ldots \\ y_k := x_k - x_{0k} \\ y_{k+1} := F^1(x_1, \ldots x_{n+1}) \\ \ldots \\ y_{n+1} := F^{n+1-k}(x_1, \ldots x_{n+1}). \end{cases}$$

Hence, $G(x_0, t_0) = 0$ and

$$G(E \cap U_0) = \left\{(y_1, \ldots y_{n+1}) \in \mathbb{R}^{n+1} \mid y_{k+1} = \ldots = y_{n+1} = 0\right\}.$$

Moreover, by (b) the function G is local diffeomorphism near (x_0, t_0). For any $R,\ r > 0$ consider the sets

$$B_k(0,R) := \left\{ (y_1,\,..\,y_k) \in \mathbb{R}^k \mid \sqrt{y_1^2 +\,..\,+ y_k^2} < R \right\},$$
$$Q_{n+1-k}(0,r) := \{ (y_{k+1},\,..\,y_{n+1}) \mid |y_i| < r \}\ ;$$

define also

$$\mathcal{N}_r^R := B_k(0,R) \times Q_{n+1-k}(0,r)\,,$$
$$E_0^R := G^{-1}(B_k(0,R) \times \{\underbrace{0,\ldots,0}_{n+1-k}\}) \subseteq G^{-1}(\mathcal{N}_r^R)\,;$$

observe that $E_0^R \subseteq E$ is a neighbourhood of (x_0, t_0) in E.

Consider the map $\tilde{\varphi} : \mathcal{N}_r^R \to \mathbb{R}$,

$$\tilde{\varphi}(y_1,\,..\,y_{n+1}) := \tilde{\varphi}_k(y_1,\,..\,y_k)\,\tilde{\varphi}_{k+1}(y_{k+1}) \cdots \tilde{\varphi}_{n+1}(y_{n+1}),$$

where the functions $\tilde{\varphi}_i$ satisfy the following properties:

(a) $\tilde{\varphi}_k \in C_c^\infty(B_k(0,R)),\ \tilde{\varphi}_i \in C_c^\infty(-r,r)\ (i = k+1,\,..\,n+1)$;

(b) $0 \le \tilde{\varphi}_i \le 1\ (i = k,\,..\,n+1)$;

(c) $\tilde{\varphi}_k \equiv 1$ in $B_k(0,R/2),\ \tilde{\varphi}_i \equiv 1$ in $[-r/2, r/2],\ i = k+1,\,..\,n+1$;

(d) $|\nabla\tilde{\varphi}_k| \le \frac{C}{R},\ \left|\frac{d\tilde{\varphi}_i}{dy_i}\right| \le \frac{C}{r}\ (i = k+1,\,..\,n+1)$.

Set $\varphi := \tilde{\varphi} \circ G$; recall that for $r,\ R$ suitably small the map

$$G : G^{-1}(\mathcal{N}_r^R) \to \mathcal{N}_r^R$$

is a diffeomorphism. Hence $\varphi \in C_c^\infty(G^{-1}(\mathcal{N}_r^R)),\ \varphi \equiv 1$ in $G^{-1}(\mathcal{N}_{r/2}^{R/2})$ and

$$\left|\frac{\partial\varphi}{\partial x_j}\right| = \left|\sum_{h=1}^{n+1} \frac{\partial\tilde{\varphi}}{\partial y_h}\frac{\partial y_h}{\partial x_j}\right| \le C_1\frac{1}{R} + C_2\frac{1}{r} \qquad (l = 1 \ldots n+1) \tag{1.6.2}$$

for some $C_1,\ C_2 > 0$ (which depend on the map G). Choose

$$\psi(x,t) = -\int_t^T \varphi(x,s)\,ds \tag{1.6.3}$$

as test function in equation (1.1.33). Then we get:

$$\begin{aligned}
&\iint_{G^{-1}(\mathcal{N}_{r/2}^{R/2})} u\,dxdt + \iint_{G^{-1}(\mathcal{N}_{r/2}^{R/2})} d\mu \le \qquad (1.6.4)\\
&\le \iint_{Q_T} u\varphi\,dxdt + \iint_{Q_T} \varphi\,d\mu\\
&= \iint_{Q_T} \nabla v \cdot \nabla\psi\,dxdt - \int_\Omega u_0\psi(x,0)\,dx.
\end{aligned}$$

Since

$$\begin{aligned}
\iint_{Q_T} \nabla v \cdot \nabla \psi \, dxdt &= \iint_{Q_T} \nabla v \cdot \nabla \left(\int_0^t \varphi(x,s)\, ds \right) dxdt \\
&\le \|v\|_{L^2(0,T;H^1(\Omega))} \left(\int_0^T \int_\Omega \left(\int_0^t |\nabla \varphi| \, ds \right)^2 dxdt \right)^{1/2} \\
&\le \|v\|_{L^2(0,T;H^1(\Omega))} T^{1/2} \left(\iint_{G^{-1}(\mathcal{N}_r^R)} |\nabla \varphi|^2 \, dxdt \right)^{1/2} \\
&\le \frac{C}{r} |G^{-1}(\mathcal{N}_r^R)|^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
-\int_\Omega u_0 \psi(x,0)\, dx &= \int_\Omega u_0 \left(\int_0^T \varphi(x,s)\, ds \right) dx \qquad (1.6.5) \\
&\le \|u_0\|_{L^\infty(\Omega)} \int_0^T \int_\Omega \varphi(x,s)\, ds \le C|G^{-1}(\mathcal{N}_r^R)|,
\end{aligned}$$

from (1.6.4) we obtain:

$$\begin{aligned}
&\iint_{G^{-1}(\mathcal{N}_{r/2}^{R/2})} u \, dxdt + \mu\big(G^{-1}(\mathcal{N}_{r/2}^{R/2})\big) \\
\le\ & \frac{C}{r} |G^{-1}(\mathcal{N}_r^R)|^{1/2} + C|G^{-1}(\mathcal{N}_r^R)| \\
\le\ & CR^k r^{\frac{n+1-k}{2}-1} + CR^k r^{n+1-k} \le C_R r^{\frac{n-1-k}{2}}.
\end{aligned}$$

Passing to the limit in the above inequality as $r \to 0$ gives

$$\mu(E_0^{R/2}) \le \lim_{r\to 0} \mu\big(G^{-1}(\mathcal{N}_{r/2}^{R/2})\big) \le \lim_{r \to 0} C_R r^{\frac{n-1-k}{2}} = 0\,;$$

in view of the compactness of E, the conclusion follows. □

Proof of Proposition 1.2.19. For any $I \subset [0,T]$ Proposition 1.2.18 gives:

$$\int_I d\lambda(t) = \int_I \gamma_t(\overline{\Omega})\, d\lambda(t) = \mu(\overline{\Omega} \times I) \le 2\|u_0\|_{L^1(\Omega)} |I|\,,$$

the last estimate following by Proposition 1.2.17-(*ii*). This shows that the measure λ is absolutely continuous with respect to the Lebesgue measure on $[0,T]$, thus there exists $h \in L^1(0,T)$, $h \ge 0$, such that $d\lambda = h\, dt$.

Fix $t_0 \in (0,T)$ and choose $\eta_\sigma \in C_c^\infty(0,T)$ with the following properties:
(a) $0 \le \eta_\sigma \le 1$, $\eta_\sigma \equiv 1$ in $[t_0 - r, t_0 + r]$,
(b) $\operatorname{supp} \eta_\sigma \subseteq [t_0 - r - \sigma, t_0 + r + \sigma]$ with r, $\sigma > 0$ suitably small. Choosing

$$\tilde{\eta}_\sigma(t) := -\int_t^T \eta_\sigma(s)\, ds$$

as test function in equation (1.1.33) and taking the limit as $\sigma \to 0$ gives

$$\iint_{\overline{\Omega}\times[t_0-r,t_0+r]} d\mu = -\iint_{\Omega\times[t_0-r,t_0+r]} u\, dxdt + 2r\int_\Omega u_0\, dx,$$

In view of Proposition 1.2.18, the above equality reads:

$$\int_{t_0-r}^{t_0+r} h(t)\,\gamma_t(\overline{\Omega})\, dt \;=\; \int_{t_0-r}^{t_0+r} h(t)\, dt = -\int_{t_0-r}^{t_0+r} dt\int_\Omega u\, dx + 2r\int_\Omega u_0\, dx.$$

Dividing by $2r$ and letting $r \to 0$ we obtain equality (1.1.38) for a.e. $t \in (0,T)$. Since $u \geq 0$ in Q_T, from (1.1.38) we get

$$h(t) \leq \int_\Omega u_0(x)\, dx$$

for a.e. $t \in (0,T)$, thus $h \in L^\infty(0,T)$. This completes the proof. □

Proof of Theorem 1.2.20. (i) Fix any $\varphi \in C(\overline{\Omega})$; set

$$\begin{aligned} W_j^{1,\varphi}(t) &:= \int_\Omega (u^{\varepsilon_j}\chi_{Q_T\setminus A_j})(x,t)\varphi(x)\, dx\,, \\ W_j^{2,\varphi}(t) &:= \int_{\overline{\Omega}} (u^{\varepsilon_j}\chi_{A_j})(x,t)\varphi(x)\, dx \qquad (j\in\mathbb{N})\,. \end{aligned}$$

In view of estimate (1.1.26) the sequences $\{W_j^{1,\varphi}\}$, $\{W_j^{2,\varphi}\}$ are bounded in $L^\infty(0,T)$; hence (possibly extracting a subsequence)

$$W_j^{1,\varphi} \overset{*}{\rightharpoonup} W^{1,\varphi}, \qquad W^{2,\varphi} \overset{*}{\rightharpoonup} W^{2,\varphi} \qquad \text{in } \; L^\infty(0,T)$$

for some $W^{1,\varphi},\; W^{2,\varphi} \in L^\infty(0,T)$. By (1.1.31) there holds

$$W^{1,\varphi}(t) = \int_\Omega u(x,t)\,\varphi(x)\, dx \qquad \text{for } a.e.\ t\in(0,T). \tag{1.6.6}$$

On the other hand, the weak convergence of $\left\{u^{\varepsilon_j}\chi_{A_j}\right\}$ to μ in $\mathcal{M}(\overline{Q}_T)$ (see (1.2.6)) and equation (1.1.37) imply

$$W^{2,\varphi}(t) = \int_{\overline{\Omega}} \varphi(x)\, d\tilde{\gamma}_t(x) \qquad \text{for } a.e.\ t\in(0,T). \tag{1.6.7}$$

(ii) Let us show that the sequence $\left\{W_j^\varphi\right\}$ (W_j^φ defined by (1.2.16)) belongs to $C([0,T])$ and is relatively compact in this space. We have

$$|W_j^\varphi(t)| \leq \int_\Omega |\varphi(x)| u^{\varepsilon_j}(x,t)\, dx \leq \|\varphi\|_{C(\overline{\Omega})}\|u_0\|_{L^1(\Omega)}$$

for any $t \in [0,T]$, $j \in \mathbb{N}$ (here use of estimate (1.1.26) has been made). Moreover, using equation (1.1.5), we get

$$\begin{aligned} |W_j^\varphi(t_1) - W_j^\varphi(t_2)| &= \int_{t_1}^{t_2} dt \left| \int_\Omega \nabla v^{\varepsilon_j} \cdot \nabla\varphi \, dx \right| \\ &\leq \|\nabla v^{\varepsilon_j}\|_{L^2(Q_T)} \|\varphi\|_{C^1(\overline{\Omega})} |\Omega|^{1/2} |t_1 - t_2|^{1/2} \,; \end{aligned}$$

hence the claim follows.

By the above inequality and Ascoli-Arzelà Theorem, we conclude that $W_j^\varphi \to W^\varphi \in C([0,T])$, where

$$W^\varphi(t) := \int_\Omega \varphi(x)\, u(x,t)\, dx + \int_{\overline{\Omega}} \varphi(x)\, d\tilde{\gamma}_t(x)$$

by step (i) above.

Finally, from the weak formulation of problem (1.1.5) we get

$$\begin{aligned} &\int_\Omega \varphi(x)\, u^{\varepsilon_j}(x,t)\, dx = \\ &= -\int_0^t ds \int_\Omega \nabla v^{\varepsilon_j}(x,s) \cdot \nabla\varphi(x)\, dx + \int_\Omega \varphi(x)\, u_0(x)\, dx \end{aligned}$$

for any $t \in [0,T]$, hence equation (1.2.18) follows as $j \to \infty$. This completes the proof. □

Proof of Proposition 1.2.21. Fix any $\varphi \in C^1(\overline{\Omega})$, $\varphi \geq 0$; let $\eta \in \mathrm{Lip}([0,T])$, $\eta \geq 0$, $\eta(T) = 0$. We can choose

$$\psi(x,t) = \varphi(x)\eta(t)$$

both in equation (1.1.33) and in inequality (1.1.36). This obtains:

$$\iint_{\overline{Q}_T} \eta_t\varphi\, d\mu + \iint_{Q_T} [\eta_t\varphi u - \eta\nabla v \cdot \nabla\varphi]\, dxdt + \eta(0)\int_\Omega \varphi\, u_0\, dx = 0\,,$$

$$\iint_{Q_T} [\eta_t\varphi u - \eta\nabla v \cdot \nabla\varphi]\, dxdt + \eta(0)\int_\Omega \varphi\, u_0\, dx \geq 0\,.$$

This implies

$$\iint_{\overline{Q}_T} \eta_t\varphi\, d\mu \leq 0,$$

namely (using Proposition 1.2.18)

$$\int_0^T \eta_t(t) W^{2,\varphi}(t)\, dt \leq 0 \qquad (1.6.8)$$

for any η as above, the function $W^{2,\varphi} \in L^\infty(0,T)$ being defined by (1.6.7).

Fix $0 < t_1 < t_2 < T$; consider $\eta \in \text{Lip}([0,T])$ defined as follows:

$$\eta(t) := \begin{cases} (t - t_1 + r/2)/r & \text{if } t \in (t_1 - r/2, t_1 + r/2) \\ 1 & \text{if } t \in [t_1 + r/2, t_2 - r/2] \\ -(t - t_2 - r/2)/r & \text{if } t \in (t_2 - r/2, t_2 + r/2), \end{cases}$$

with $r > 0$ suitably small. Using η as test function in inequality (1.6.8) gives

$$\frac{1}{r}\int_{t_1-r/2}^{t_1+r/2} W^{2,\varphi}(t)\,dt \leq \frac{1}{r}\int_{t_2-r/2}^{t_2+r/2} W^{2,\varphi}(t)\,dt\,.$$

Thus as $r \to 0$ we get

$$W^{2,\varphi}(t_1) \leq W^{2,\varphi}(t_2)\,,$$

and the conclusion follows. □

Chapter 2

Degenerate pseudoparabolic regularization of a forward-backward parabolic equation

2.1 Introduction

In this chapter we consider the initial-boundary value problem

$$\begin{cases} u_t = [\varphi(u_x)]_x & \text{in } \Omega \times (0,T] =: Q \\ \varphi(u_x) = 0 & \text{in } \partial\,\Omega \times (0,T] \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{2.1.1}$$

Here $T > 0$, $\Omega \subset \mathbb{R}$ is a bounded interval and φ is a nonmonotone function, which satisfies for some constant $\alpha > 0$ the following assumption:

$$(H_1) \qquad \begin{cases} (i) \quad \varphi \in C^2(\mathbb{R}) \cap L^1(\mathbb{R}),\ \varphi(0) = 0,\ \varphi(s) \to 0 \quad \text{as } s \to \infty\,; \\ (ii)\ \ 0 < \varphi(s) \le \varphi(\alpha) \ \text{ for } s > 0,\ \varphi(s) < 0 \ \text{ for } s < 0\,; \\ (iii)\ \varphi'(s) > 0 \ \text{ for } 0 < s < \alpha,\ \varphi'(s) < 0 \ \text{ for } s > \alpha\,. \end{cases}$$

In view of assumption $(H_1) - (iii)$, the first equation in (2.1.1) is of *forward-backward* parabolic type. Its main feature is to be ill-posed whenever the solution u_x takes values in the interval (α, ∞) where $\varphi' < 0$.

Problem (2.1.1) independently arises in mathematical models of oceanography [BBDU] and image processing [PM]. By the change of unknown $v := u_x$, it reduces to a model for aggregating populations in population dynamics [Pa]. Under different assumptions on φ, it also arises in the theory of phase transitions (in this connection, see [E2], [MTT] and references therein).

Several regularizations of forward-backward parabolic equations have been proposed on physical grounds and mathematically investigated (in particular, see [BFG], [Pa], [NP], [Sl])). In this chapter we make use of the regularization proposed in [BBDU] to take memory effects into account, namely

$$u_t = [\varphi(u_x)]_x + \epsilon[\psi(u_x)]_{xt}\,. \tag{2.1.2}$$

The function ψ is related to φ and satisfies the following assumption:

$$(H_2) \qquad \begin{cases} (i) \quad \psi \in C^3(\mathbb{R}),\ \psi' > 0 \text{ in } \mathbb{R},\ \psi(-s) = -\psi(s), \\ \qquad \psi(s) \to \gamma \text{ as } s \to \infty \ \text{ for some } \gamma \in (0,\infty)\,; \\ (ii) \ |\varphi'| \le k_1\psi' \text{ in } \mathbb{R} \ \text{ for some } k_1 > 0\,; \\ (iii) \quad \left|\left(\frac{\varphi'}{\psi'}\right)'\right| \le k_2\psi' \text{ in } \mathbb{R} \ \text{ for some } k_2 > 0\,. \end{cases}$$

Observe that $(H_2)-(i)$ implies $\psi'(s) \to 0$ as $s \to \infty$. Hence ψ' is not bounded away from zero, and equation (2.1.2) is *degenerate pseudoparabolic*.
Concerning the initial data u_0, in [BBDU] the following assumption was made:

$$(H_3) \qquad \begin{cases} (i) \quad u_0 \in BV(\Omega)\,; \\ (ii) \ \ u_0 \ \text{ nondecreasing in } \Omega\,. \end{cases}$$

Assumptions $(H_1) - (H_3)$ are always made below. Following [BBDU], under the above we address for any $\epsilon > 0$ the initial-boundary value problem

$$\begin{cases} u_t = [\varphi(u_x)]_x + \epsilon[\psi(u_x)]_{xt} & \text{in } Q \\ \varphi(u_x) = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{2.1.3}$$

The purpose of the present chapter is twofold:

Step (i). First we study problem (2.1.3) for fixed $\epsilon > 0$. Existence and uniqueness of solutions to problem (2.1.3) have been proved in [BBDU]; in this framework, a solution of (2.1.3) is meant to be a couple (u^ϵ, w^ϵ), where $u^\epsilon \in L^\infty((0,T); BV(\Omega))$, $u^\epsilon(\cdot,t)$ is non-decreasing for *a.e.* $t \in (0,T)$, $u^\epsilon_t \in L^2(Q)$ and $w^\epsilon \in L^\infty((0,T); H^1_0(\Omega)) \cap C(\overline{Q})$, $w^\epsilon_t \in L^2((0,T); H^1_0(\Omega))$, such that

$$u^\epsilon_t = h(w^\epsilon)_x + \epsilon w^\epsilon_{tx} \qquad \text{in } \ L^2(Q) \tag{2.1.4}$$

with initial datum u_0 (here $h := \varphi \circ \psi^{-1}$). We show that the definition of solution made in [BBDU] (see Definition 2.2.2 below) can be actually interpreted in an alternative - and equivalent - way. Precisely, denoting by $u^{\epsilon,(r)}_x$ and $u^{\epsilon,(s)}_x$ the regular and singular term of the spatial derivative u^ϵ_x with respect to the Lebesgue measure, we prove that:
(a) $w^\epsilon = \psi\big(u^{\epsilon,(r)}_x\big)$, $h(w^\epsilon) = \varphi\big(u^{\epsilon,(r)}_x\big)$ *a.e.* in Q;
(b) equation (2.1.4) reads

$$u^\epsilon_t = \varphi\big(u^{\epsilon,(r)}_x\big)_x + \epsilon\psi\big(u^{\epsilon,(r)}_x\big)_{tx} = v^\epsilon_x \qquad \text{in } \ L^2(Q), \tag{2.1.5}$$

where

$$v^\epsilon := \varphi\big(u_x^{\epsilon,(r)}\big) + \epsilon\big[\psi\big(u_x^{\epsilon,(r)}\big)\big]_t\,;$$

(c) the support of the singular part $u_x^{\epsilon,(s)}$ is characterized as follows:

$$\operatorname{supp} u_x^{\epsilon,(s)} = \big\{(x,t) \in \overline{Q} \mid \psi\big(u_x^{\epsilon,(r)}\big)(x,t) = \gamma\big\}\,.$$

Observe also that deriving (2.1.5) with respect to x gives the following equation for the derivative u_x^ϵ

$$\big[u_x^\epsilon\big]_t = \big[\varphi\big(u_x^{\epsilon,(r)}\big)\big]_{xx} + \epsilon\big[\psi\big(u_x^{\epsilon,(r)}\big)\big]_{txx} = v_{xx}^\epsilon \qquad \text{in } \mathcal{D}'(Q). \tag{2.1.6}$$

Step (ii). Then we investigate the limit of solutions of (2.1.3) as $\epsilon \to 0$. In this direction, concerning the family $\{v^\epsilon\}$ we show that there exists a constant $C > 0$ such that

$$\|v^\epsilon\|_{L^\infty(Q)},\ \|v^\epsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C.$$

On the other hand, for the family $\{u^\epsilon\}$ in general we only have a-priori estimates in $BV(Q)$ - namely in $\mathcal{M}^+(Q_T)$ for the spatial derivatives u_x^ϵ. Thus, the space of positive Radon measures seems a natural candidate to take the limit as $\epsilon \to 0$ in problems of (2.1.3). In particular we obtain the limiting equations

$$u_t = v_x \qquad \text{in } L^2(Q), \tag{2.1.7}$$

and also

$$\big[u_x\big]_t = v_{xx} \qquad \text{in } \mathcal{D}'(Q). \tag{2.1.8}$$

Here, for some sequence $\epsilon_j \to 0$, $u \in BV(Q)$ is the weak limit of the sequence $\{u^{\epsilon_j}\}$ in $BV(Q)$, and $v \in L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$, $v \geq 0$, is the limit of both the sequences $\{v^{\epsilon_j}\}$ and $\Big\{\varphi(u_x^{\epsilon_j,(r)})\Big\}$ in the weak* topology of $L^\infty(Q)$.

Moreover, we can use the general notion of Young measures, narrow and biting convergences, to prove the following decomposition of the Radon measure $u_x \in \mathcal{M}^+(Q_T)$:

$$u_x = Z + \mu, \tag{2.1.9}$$

where $\mu \in \mathcal{M}^+(Q_T)$ is a positive Radon measure, in general not absolutely continuous with respect to the Lebesgue measure, and $Z \in L^1(Q)$, $Z \geq 0$, is a superposition of the two branches s_1, s_2 of the equation $v = \varphi(z)$ $(v \geq 0)$, namely

$$Z = \begin{cases} \lambda s_1(v) + (1-\lambda) s_2(v) & \text{if } v > 0 \\ 0 & \text{if } v = 0 \end{cases}$$

(see Theorem 2.2.7). Moreover, denoting by $< \cdot, \cdot >$ the duality map between $\mathcal{M}^+(Q)$ and the space $C_c(Q)$, in Theorem 2.2.9 we show that the following *disintegration* of the measure μ holds:

$$< \mu, f > = \int_0^T < \tilde{\gamma}_t, f(\cdot, t) > dt\,;$$

here $\tilde{\gamma}_t$ is a positive Radon measure over Ω for *a.e.* $t > 0$ and the map

$$t \mapsto \tilde{\gamma}_t(E)$$

is non-decreasing for any Borel set $E \subseteq \Omega$.

Finally, concerning the support of the (possibly) singular measure $\tilde{\gamma}_t \in \mathcal{M}^+(\Omega)$ (hence of $\mu \in \mathcal{M}^+(Q)$), in Theorem 2.2.10 we prove that

$$\operatorname{supp} \tilde{\gamma}_t \subseteq \big\{ x \in \overline{\Omega} \,|\; v(x,t) = 0 \big\}$$

for *a.e.* $t > 0$.

2.2 Mathematical framework and results

2.2.1 The case $\epsilon > 0$

In the sequel we denote by $\mathcal{M}^+(Q)$ the space of positive Radon measures on Q, and by $< \cdot, \cdot >$ the duality map between $\mathcal{M}^+(Q)$ and the space $C_c(Q)$ of continuous functions $f : Q \to \mathbb{R}$ with compact support. Let $C^1_c(Q)$ be the space of C^1 functions $f : Q \to \mathbb{R}$ with compact support.

Let us make the following definition.

Definition 2.2.1. *A function $u^\epsilon : \overline{Q} \to \mathbb{R}$ is a solution of problem* (2.1.3), *if there holds:*

(i) $u^\epsilon \in L^\infty((0,T); BV(\Omega))$, $u^\epsilon(\cdot, t)$ is nondecreasing for a.e. $t \in (0,T)$, and $u^\epsilon_t \in L^2(Q)$;

(ii) $\varphi\big(u^{\epsilon,(r)}_x\big)$, $\psi\big(u^{\epsilon,(r)}_x\big) \in C(\overline{Q}) \cap L^\infty((0,T); H^1_0(\Omega))$, and $[\psi\big(u^{\epsilon,(r)}_x\big)]_t \in L^2((0,T); H^1_0(\Omega))$, where $u^{\epsilon,(r)}_x$ denote the density of the absolutely continuous part (with respect to the Lebesgue measure) of the Radon measure $u^\epsilon_x \in \mathcal{M}^+(Q)$.

(iii) the equation

$$u^\epsilon_t = [\varphi\big(u^{\epsilon,(r)}_x\big)]_x + \epsilon[\psi\big(u^{\epsilon,(r)}_x\big)]_{xt} \tag{2.2.1}$$

is satisfied in $L^2(Q)$*, and there holds*

$$\iint_Q \{u^\epsilon \zeta_t + [\varphi(u_x^{\epsilon,(r)})]_x \zeta + \epsilon[\psi(u_x^{\epsilon,(r)})]_{xt} \zeta\} dxdt = \tag{2.2.2}$$
$$= \; - \int_\Omega u_0(x)\zeta(x,0)\, dx$$

for any $\zeta \in C^1(\overline{Q})$, $\zeta(\cdot, T) = 0$ *in* Ω.

The following result will be proven.

Theorem 2.2.1. *Let assumptions* $(H_1) - (H_3)$ *be satisfied. Then for any* $\epsilon > 0$ *there exists a unique solution* u^ϵ *of problem* (2.1.3)*. Moreover, there holds*

$$\big[u_x^{\epsilon,(r)} + u_x^{\epsilon,(s)}\big]_t = \big[\varphi\big(u_x^{\epsilon,(r)}\big)\big]_{xx} + \epsilon\big[\psi\big(u_x^{\epsilon,(r)}\big)\big]_{xxt} \quad \text{in } \mathcal{D}'(Q)\,; \tag{2.2.3}$$

here $u_x^{\epsilon,(r)} \in L^1(Q)$, $u_x^{\epsilon,(s)} \in \mathcal{M}^+(Q)$ *denote the density of the absolutely continuous part, respectively the singular part (with respect to the Lebesgue measure) of the Radon measure* u_x^ϵ.

It is informative to compare Definition 2.2.1 with an alternative definition of solution to problem (2.1.3), which was used in [BBDU]. Define a function $h : [-\gamma, \gamma] \to \mathbb{R}$ by setting

$$h(z) := \begin{cases} \varphi \circ \psi^{-1}(z) & \text{if } |z| < \gamma \\ 0 & \text{if } |z| = \gamma\,. \end{cases} \tag{2.2.4}$$

Definition 2.2.2. *A couple of functions* $u^\epsilon, w^\epsilon : \overline{Q} \to \mathbb{R}$ *is a solution of problem* (2.1.3)*, if there holds:*

(i) $u^\epsilon \in L^\infty((0,T); BV(\Omega))$, $u^\epsilon(\cdot, t)$ *is nondecreasing for a.e.* $t \in (0,T)$*, and* $u_t^\epsilon \in L^2(Q)$*;*

(ii) $w^\epsilon \in C(\overline{Q}) \cap L^\infty((0,T); H_0^1(\Omega))$ *such that* $|w^\epsilon| \le \gamma$ *in* Q*, and*

$$w^\epsilon(x,t) = \lim_{h \to 0^+} \psi\left(\frac{u^\epsilon(x+h,t) - u^\epsilon(x^+,t)}{h}\right) = \tag{2.2.5}$$
$$= \lim_{h \to 0^+} \psi\left(\frac{u^\epsilon(x-h,t) - u^\epsilon(x^-,t)}{h}\right)$$

for any $x \in \Omega$ *and* $t > 0$ *(here* $u^\epsilon(x^\pm, t) := \lim_{\eta \to 0^+} u(x \pm \eta, t)$*). Moreover,* $w_t^\epsilon \in L^2((0,T); H_0^1(\Omega))$*;*

(iii) the equation

$$u_t^\epsilon = [h(w^\epsilon)]_x + \epsilon w_{xt}^\epsilon \tag{2.2.6}$$

is satisfied in $L^2(Q)$*, and there holds*

$$\iint_Q \{u^\epsilon \zeta_t + [h(w^\epsilon)]_x \zeta + \epsilon w^\epsilon_{xt} \zeta\} dxdt = \tag{2.2.7}$$
$$= \quad -\int_\Omega u_0(x)\zeta(x,0)dx$$

for any $\zeta \in C^1(\overline{Q}),\ \zeta(\cdot,T) = 0$ *in* Ω.

The following well-posedness result was proven in [BBDU].

Theorem 2.2.2. *Let assumptions* $(H_1) - (H_3)$ *be satisfied. Then for any* $\epsilon > 0$ *there exists a unique solution* (u^ϵ, w^ϵ) *of problem* (2.1.3) *in the sense of Definition 2.2.2.*

The equivalence between Definitions 2.2.1 and 2.2.2 is an immediate consequence of the following statement.

Theorem 2.2.3. *Let assumptions* $(H_1) - (H_3)$ *be satisfied. Let* (u^ϵ, w^ϵ) *be the solution of problem* (2.1.3) *in the sense of Definition 2.2.2, whose existence is asserted in Theorem 2.2.2. Then*

$$u_x^{\epsilon,(r)} = \psi^{-1}(w^\epsilon) \quad a.e.\ in\ Q\,, \tag{2.2.8}$$

$$\operatorname{supp} u_x^{\epsilon,(s)} = \big\{(x,t) \in \overline{Q} \mid w^\epsilon(x,t) = \gamma\big\}\,. \tag{2.2.9}$$

Moreover, the set $\operatorname{supp} u_x^{\epsilon,(s)}$ *has Lebesgue measure* $|\operatorname{supp} u_x^{\epsilon,(s)}| = 0$.

For any $\epsilon > 0$ set

$$v^\epsilon := \varphi\big(u_x^{\epsilon,(r)}\big) + \epsilon\big[\psi(u_x^{\epsilon,(r)})\big]_t\,. \tag{2.2.10}$$

Observe that equation (2.2.1) simply reads

$$u_t^\epsilon = v_x^\epsilon\,. \tag{2.2.11}$$

Inspired by [Pl1], we will show that for any $\epsilon > 0$ there exists a set $F^\epsilon \subseteq (0,T)$ of Lebesgue measure $|F^\epsilon| = 0$ such that the couple $(u_x^{\epsilon,(r)}, v^\epsilon)$, satisfies the *entropy inequality*:

$$\int_0^1 G(u_x^{\epsilon,(r)})(x,t_2)\zeta(x,t_2)dx - \int_\Omega G(u_x^{\epsilon,(r)})(x,t_1)\zeta(x,t_1)dx \leq$$
$$\leq \quad \int_{t_1}^{t_2}\int_\Omega [G(u_x^{\epsilon,(r)})\zeta_t - g(v^\epsilon)v_x^\epsilon\zeta_x]dxdt \tag{2.2.12}$$

for any $t_1, t_2 \in (0,T)\backslash F^\epsilon$ with $t_1 < t_2$ and any $\zeta \in C^1([0,T]; H_0^1(\Omega)\cap H^2(\Omega))$, $\zeta \geq 0,\ \zeta_{xx} \leq 0$ (see Proposition 2.3.17). Here

$$G(\lambda) := \int_0^\lambda (g\circ\varphi)(s)ds \tag{2.2.13}$$

and g is an *arbitrary* function in $C^1(\mathbb{R})$ such that $g' \geq 0$, $g \equiv 0$ in $[0, S_g]$, for some $S_g > 0$.

2.2.2 Letting $\epsilon \to 0$

Set

$$S_1 := \{(\zeta, \varphi(\zeta)) | \zeta \in [0, \alpha]\} \equiv \{(s_1(v), v) | v \in [0, \varphi(\alpha)]\}, \tag{2.2.14}$$

$$S_2 := \{(\zeta, \varphi(\zeta)) | \zeta \in (\alpha, \infty)\} \equiv \{(s_2(v), v) | v \in (0, \varphi(\alpha))\}; \tag{2.2.15}$$

the above sets will be referred to as the *stable* branch, respectively the *unstable* of the graph of φ. Following [Pl1], we always assume in the sequel:

Condition (S): *The functions s_1', s_2' are linearly independent on any open subset of the interval $(0, \varphi(\alpha))$.*

Let u^ϵ be the unique solution (in the sense of Definition 2.2.1) of problem (2.1.3), whose existence is asserted by Theorem 2.2.1. Our purpose is to study the behaviour and the limiting points of the families $\{u^\epsilon\}$, $\{v^\epsilon\}$ and $\{\psi(u_x^{\epsilon,(r)})\}$ as $\epsilon \to 0$. To this aim, in Lemmata 2.3.10-2.3.12 it is shown that there exists a constant $C > 0$, which does not depend on ϵ, such that

$$\|u_x^\epsilon\|_{\mathcal{M}^+(Q)} \leq C. \tag{2.2.16}$$

$$\|u_t^\epsilon\|_{L^2(Q)} \leq C; \tag{2.2.17}$$

$$\|v^\epsilon\|_{L^\infty(Q)} \leq C, \tag{2.2.18}$$

$$\|v^\epsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C. \tag{2.2.19}$$

Observe that inequality (2.2.16) implies

$$\|u_x^{\epsilon,(r)}\|_{L^1(Q)} \leq C, \qquad \|u_x^{\epsilon,(s)}\|_{\mathcal{M}^+(Q)} \leq C \tag{2.2.20}$$

for some constant $C > 0$ independent of ϵ. Also, inequalities (2.2.16)-(2.2.17) imply that the family $\{u^\epsilon\}$ is bounded in $BV(Q)$. Hence there exist a subsequence $\{\epsilon_k\}$, $\epsilon_k \to 0$, and a couple of functions $u \in BV(Q)$ with $u_t \in L^2(Q)$, $v \in L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$ such that

$$u^{\epsilon_k} \rightharpoonup u \quad \text{in } BV(Q), \tag{2.2.21}$$

$$u_x^{\epsilon_k} \overset{*}{\rightharpoonup} u_x \quad \text{in } \mathcal{M}^+(Q), \tag{2.2.22}$$

$$u_t^{\epsilon_k} \rightharpoonup u_t \quad \text{in } L^2(Q), \tag{2.2.23}$$

$$v^{\epsilon_k} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(Q), \tag{2.2.24}$$

$$v^{\epsilon_k} \rightharpoonup v \quad \text{in } L^2((0,T); H_0^1(\Omega)). \tag{2.2.25}$$

It will also be proven (see Lemma 2.4.3) that

$$\varphi\big(u_x^{\epsilon_k,(r)}\big) \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(Q). \tag{2.2.26}$$

Observe that (2.2.21) implies

$$u^{\epsilon_k} \to u \quad \text{in } L^1_{loc}(Q)\,. \tag{2.2.27}$$

The above remarks allow to take the limit as $\epsilon_k \to 0$ in equality (2.2.2) written with $\epsilon = \epsilon_k$, namely

$$\iint_Q \left\{u^{\epsilon_k}\,\zeta_t + v_x^{\epsilon_k}\,\zeta\right\} dxdt = -\int_\Omega u_0(x)\zeta(x,0)\,dx\,,$$

thus obtaining

$$\iint_Q \left\{u\,\zeta_t + v_x\,\zeta\right\} dxdt = -\int_\Omega u_0(x)\zeta(x,0)\,dx \tag{2.2.28}$$

for any $\zeta \in C^1(\overline{Q})$, $\zeta(\cdot,T) = 0$ in Ω. This can be expressed by saying that the couple (u,v) is a weak solution of the problem

$$\begin{cases} u_t = v_x & \text{in } Q \\ v = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{2.2.29}$$

2.2.3 Structure of u_x

Were $v = \varphi(u_x)$, equation (2.2.28) would give a weak solution of problem (2.1.1). However, no such conclusion can be drawn from (2.2.21)-(2.2.25), in view of the nonmonotone character of φ. Nevertheless, the structure of the limiting measure $u_x \in \mathcal{M}^+(Q)$ (see (2.2.22)) can be studied in considerable detail by *Young measures* techniques. To this purpose, let us first recall the following definition ([GMS], [V]).

Definition 2.2.3. *Let τ_k, τ be Young measures on $Q \times \mathbb{R}$ ($k \in \mathbb{N}$). The sequence $\{\tau_k\}$ converges to τ* narrowly*, if*

$$\int_{Q\times\mathbb{R}} \psi\,d\tau_k \to \int_{Q\times\mathbb{R}} \psi\,d\tau \tag{2.2.30}$$

for any $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and measurable, $\psi(x,t,\cdot)$ continuous for a.e. $(x,t) \in Q$.

Consider the family $\{\tau_\epsilon\}$ of Young measures associated to $\{u_x^{\epsilon,(r)}\}$. In view of (2.2.20) and the Prohorov Theorem (*e.g.* see [V]), we have the following result.

Proposition 2.2.4. *Let u^ϵ be the unique solution of problem* (2.1.3)*, and τ_ϵ the Young measure associated to the density $u_x^{\epsilon,(r)}$ of the absolutely continuous part of the Radon measure $u_x^\epsilon \in \mathcal{M}^+(Q)$ ($\epsilon > 0$). Then:*
(i) there exist a sequence $\{u_x^{\epsilon_k,(r)}\} \subseteq \{u_x^{\epsilon,(r)}\}$ and a Young measure τ on $Q_T \times \mathbb{R}$ such that $\tau_k \to \tau$ narrowly (here $\tau_k \equiv \tau_{\epsilon_k}$);
(ii) for any $f \in C(\mathbb{R})$ such that the sequence $\{f(u_x^{\epsilon_k,(r)})\}$ is bounded in $L^1(Q)$ and equi-integrable there holds

$$f\left(u_x^{\epsilon_k,(r)}\right) \rightharpoonup f^* \quad \text{in } L^1(Q); \tag{2.2.31}$$

here

$$f^*(x,t) := \int_{[0,+\infty)} f(\xi)\, d\nu_{(x,t)}(\xi) \qquad \text{for a.e. } (x,t) \in Q \tag{2.2.32}$$

and $\nu_{(x,t)}$ is the disintegration of the Young measure τ.

In general, the sequence $\{u_x^{\epsilon_k,(r)}\}$ need not be equi-integrable in the cylinder Q; hence Proposition 2.2.4-(*ii*) cannot be applied with $f(z) = z$. However, we can associate to $\{u_x^{\epsilon_k,(r)}\}$ an equi-integrable subsequence *by removing sets of small measure*. This is the content of the following theorem, which easily follows from the Biting Lemma (*e.g.*, see [GMS], [V] for the proof; here and in the sequel we denote by $|E|$ the Lebesgue measure of any measurable set $E \subseteq \mathbb{R}$).

Theorem 2.2.5. *Let the assumptions of Proposition 2.2.4 be satisfied. Then there exist a subsequence $\{u_x^{\epsilon_j,(r)}\} \equiv \{u_x^{\epsilon_{k_j},(r)}\} \subseteq \{u_x^{\epsilon_k,(r)}\}$ and a sequence of measurable sets $\{A_j\}$,*

$$A_j \subset Q, \qquad A_{j+1} \subset A_j \quad \text{for any } j \in \mathbb{N}, \quad |A_j| \to 0 \quad \text{as } j \to \infty,$$

such that the sequence $\{u_x^{\epsilon_j,(r)} \chi_{Q\setminus A_j}\}$ is equi-integrable. Moreover,
(i) there holds

$$u_x^{\epsilon_j,(r)} \chi_{Q\setminus A_j} \rightharpoonup Z \quad \text{in } L^1(Q)\,, \tag{2.2.33}$$

where $Z \in L^1(Q)$ is the barycenter of the Young disintegration $\nu_{(x,t)}$, namely

$$Z(x,t) := \int_{[0,\infty)} \xi \, d\nu_{(x,t)}(\xi) \qquad \text{for a.e. } (x,t) \in Q\,; \tag{2.2.34}$$

(ii) there exists a measure $\mu_1 \in \mathcal{M}^+(Q)$ such that

$$u_x^{\epsilon_j,(r)} \chi_{A_j} \overset{*}{\rightharpoonup} \mu_1 \quad \text{in } \mathcal{M}^+(Q)\,. \tag{2.2.35}$$

Concerning the family $\{u_x^{\epsilon,(s)}\}$, the second estimate in (2.2.20) immediately gives the following result.

Theorem 2.2.6. *Let u^ϵ be the unique solution of problem* (2.1.3)*, and $u_x^{\epsilon,(s)}$ the singular part of the Radon measure $u_x^\epsilon \in \mathcal{M}^+(Q)$ ($\epsilon > 0$). Then there exist a subsequence $\{u_x^{\epsilon_j,(s)}\}$ and a measure $\mu_2 \in \mathcal{M}^+(Q)$ such that*

$$u_x^{\epsilon_j,(s)} \overset{*}{\rightharpoonup} \mu_2 \quad in\ \mathcal{M}^+(Q)\,. \tag{2.2.36}$$

Observe that (2.2.35) and (2.2.36) read

$$\iint_Q u_x^{\epsilon_j,(r)} \chi_{A_j} \zeta \, dxdt \to < \mu_1, \zeta > , \qquad \iint_Q u_x^{\epsilon_j,(s)} \zeta \, dxdt \to < \mu_2, \zeta > \tag{2.2.37}$$

for any $\zeta \in C_c(Q)$. Let $u_x \in \mathcal{M}^+(Q)$ be the limit of the sequence $u_x^{\epsilon_k}$ in the weak* topology of $\mathcal{M}^+(Q)$ (see (2.2.22)). In view of (2.2.33), (2.2.35) and (2.2.36), it follows that

$$u_x = Z + \mu\,, \tag{2.2.38}$$

where Z is the barycenter of the Young disintegration of the limiting measure τ (see (2.2.34)) and

$$\mu := \mu_1 + \mu_2\,. \tag{2.2.39}$$

Let us observe that the triple (Z, μ, v), where v denotes the limiting function in (2.2.24)-(2.2.26), satisfies the equality

$$(Z + \mu)_t = v_{xx} \quad \text{in } \mathcal{D}'(Q)\,. \tag{2.2.40}$$

In fact, equality (2.2.3) reads

$$\iint_Q \left\{ u_x^{\epsilon,(r)} \zeta_t - v_x^\epsilon \zeta_x \right\} dxdt + \ < u_x^{\epsilon,(s)}, \zeta_t > = 0$$

for any $\zeta \in C_c^\infty(Q)$ (see (2.2.10)). By (2.2.25), Theorems 2.2.5-2.2.6 and (2.2.39), letting $\epsilon \to 0$ gives

$$\iint_Q [Z\zeta_t - v_x\, \zeta_x] dxdt + < \mu, \zeta_t > = 0$$

Remark 2.2.1. *Although Z can be regarded as the density of an absolutely continuous measure (with respect to the Lebesgue measure), we do not know whether this measure and the measure μ are mutually singular. Therefore the representation* (2.2.38) *need not coincide with the Lebesgue decomposition of u_x.*

Concerning the function Z, we shall prove the following result.

Theorem 2.2.7. *There exists* $\lambda \in L^\infty(Q)$, $0 \leq \lambda \leq 1$, *such that*

$$Z = \begin{cases} \lambda s_1(v) + (1-\lambda) s_2(v) & \text{if } v > 0 \\ 0 & \text{if } v = 0 \end{cases} \tag{2.2.41}$$

a.e. in Q. *Here* s_1, s_2 *are defined by* (2.2.14)-(2.2.15) *and* v *is the limiting function in* (2.2.24)-(2.2.25).

The proof of Theorem 2.2.7 relies on the following characterization of the disintegration $\nu_{(x,t)}$ of the measure τ (see Proposition 2.2.4-(i))

$$\nu_{(x,t)} = \begin{cases} \lambda(x,t)\delta_{s_1(v(x,t))} + (1-\lambda(x,t))\delta_{s_2(v(x,t))} & \text{if } v(x,t) > 0 \\ \delta_0 & \text{if } v(x,t) = 0\,, \end{cases} \tag{2.2.42}$$

which holds for almost every $(x,t) \in Q$. The proof is adapted from [Pl1], [Sm].

Further we investigate the properties of the measure μ defined in (2.2.39). A remarkable feature of μ is its *nondecreasing character* with respect to time; this is the content of the following theorem.

Theorem 2.2.8. *There holds:*

$$\iint_Q \left\{ Z\zeta_t - v_x\,\zeta_x \right\} dxdt \geq 0 \tag{2.2.43}$$

For any $\zeta \in C^1([0,T]; H^1_0(\Omega) \cap H^2(\Omega))$, $\zeta(\cdot,0) = \zeta(\cdot,T) = 0$ *in* Ω, $\zeta \geq 0$ *and* $\zeta_{xx} \leq 0$.

In view of Theorem 2.2.8, in equality (2.2.40) the singular part μ of the measure in the left-hand side prevails over the regular L^1-term Z as time progresses. This produces a general *"coarsening" effect*, since the absolutely continuous part decreases and possibly disappears, while singularities can appear and spread as time goes on. Such effect seems consistent with the model interpretation of equation (2.1.1), and with the results proven in [BBDU] for the case $\epsilon > 0$.

Let us next prove a disintegration result concerning the measure μ. For any subset $E \subseteq Q$ denote by $E_t := \left\{ x \in \Omega \,|\, (x,t) \in E \right\}$ its section at the time $t \in (0,T)$. Then we can prove the following result.

Theorem 2.2.9. *Let* μ *be the measure defined in* (2.2.39). *Then for a.e.* $t \in (0,T)$ *there exists a measure* $\tilde{\gamma}_t \in \mathcal{M}^+(\Omega)$ *such that:*
(i) *for any Borel set* $E \subseteq Q$ *there holds*

$$\mu(E) = \int_0^T \tilde{\gamma}_t(E_t)\, dt\,;$$

moreover, for any $f \in C_c(Q)$ there holds:

$$< \mu, f > = \int_0^T < \tilde{\gamma}_t, f(\cdot, t) > dt\,; \tag{2.2.44}$$

(ii) for any $\rho \in H_0^1(\Omega) \cap H^2(\Omega)$, $\rho \geq 0$, $\rho_{xx} \leq 0$ in Ω, there holds

$$\langle \tilde{\gamma}_{t_1}, \rho \rangle \leq \langle \tilde{\gamma}_{t_2}, \rho \rangle \tag{2.2.45}$$

for almost every $t_1, t_2 \in (0, T)$, $t_1 < t_2$.

Finally, the following theorem holds.

Theorem 2.2.10. *For a.e. $t \in (0,T)$ let $\tilde{\gamma}_t \in \mathcal{M}^+(\Omega)$ be the Radon measure given by Theorem 2.2.9 and v the limiting function in* (2.2.24)-(2.2.26)*. Let the following assumption be satisfied:*

$$(H_4) \qquad s^2 \psi'(s) \leq k_3 \quad \textit{for some } k_3 > 0\,.$$

Then there exists a subset $E \subseteq (0,T)$ of zero Lebesgue measure such that

$$\operatorname{supp} \tilde{\gamma}_t \subseteq \big\{ x \in \overline{\Omega} \mid v(x,t) = 0 \big\}$$

for any $t \in (0,T) \setminus E$.

2.3 The case $\epsilon > 0$: Proofs.

Let us recall for further purposes the proof of the existence part of Theorem 2.2.2. This was obtained approximating problem (2.1.3) by the nondegenerate problem

$$(P_\kappa^\epsilon) \qquad \begin{cases} u_t = [\varphi_\kappa(u_x)]_x + \epsilon[\psi_\kappa(u_x)]_{xt} & \text{in } Q \\ \varphi_\kappa(u_x) = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_{0\kappa} & \text{in } \Omega \times \{0\} \end{cases}$$

for any $\kappa > 0$, then letting $\kappa \to 0$. Concerning φ_κ, ψ_κ and $u_{0\kappa}$ the following was assumed:

$$(A) \qquad \begin{cases} (i) \;\; \varphi_\kappa(0) = 0\,,\; \varphi_\kappa \to \varphi, \;\; \psi_\kappa \to \psi \text{ in } C^3_{loc}(\mathbb{R}) \text{ as } \kappa \to 0\,; \\ (ii) \;\; 0 < \varphi_\kappa(s) \leq \varphi_\kappa(\alpha) \text{ for } s > 0, \; \varphi_\kappa(s) < 0 \text{ for } s < 0\,; \\ (iii) \; \psi_\kappa \text{ odd}, \;\; \psi' + \kappa \leq \psi'_\kappa \leq \psi' + 2\kappa \text{ in } \mathbb{R}, \;\; \psi''_\kappa \in L^\infty(\mathbb{R})\,; \\ (iv) \; |\varphi'_\kappa| \leq k_1\, \psi'_\kappa \quad \left| \left(\frac{\varphi'_\kappa}{\psi'_\kappa} \right)' \right| \leq k_2\, \psi'_\kappa \text{ on } \mathbb{R}\,, \;\; \varphi_k \in L^1(\mathbb{R}); \\ (v) \;\; u_{0\kappa} \in C^\infty(\overline{\Omega}), \; u'_{0\kappa} \geq 0 \text{ in } \Omega, \; u'_{0\kappa}(0) = u'_{0\kappa}(1) = 0, \\ \qquad u_{0\kappa} \to u_0 \text{ in } L^1(\Omega) \text{ as } \kappa \to 0\,, \; \|u'_{0\kappa}\|_{L^1(\Omega)} \leq \|u'_0\|_{\mathcal{M}^+(\Omega)}\,. \end{cases}$$

It is easily seen that under the above hypotheses problem (P^ϵ_κ) has a unique solution $u^\epsilon_\kappa \in C([0,T];C^{2+l}(\overline{\Omega})) \cap C^1((0,T];C^{2+l}(\overline{\Omega}))$ for any $\kappa > 0$ and $l \in \mathbb{N}$ [BBDU]. Moreover, the following holds.

Lemma 2.3.1. *Let assumption* (A) *be satisfied. Then:*
(i) *there holds*

$$\int_\Omega u^\epsilon_\kappa(x,t)dx = \int_\Omega u_{0\,\kappa}(x)dx \quad \textit{for any } t > 0\,; \tag{2.3.1}$$

(ii) $u^\epsilon_{\kappa x}(\cdot,t) \geq 0$ *in* Ω .

The next step is to obtain uniform a priori estimates of the sequences $\{u^\epsilon_\kappa\}$ and $\{\psi_\kappa(u^\epsilon_{\kappa x})\}$ ($\epsilon > 0$ fixed); this is the content of the following three lemmata. We prove the first two for future reference, while referring the reader to [BBDU] for the proof of the third.

Lemma 2.3.2. *Let* (A) *be satisfied. Then there exists a constant* $C > 0$ *such that for any* $\kappa > 0$

$$\|u^\epsilon_\kappa\|_{L^\infty(Q)} \leq C\,, \tag{2.3.2}$$

$$\|u^\epsilon_{\kappa x}\|_{L^\infty((0,T);L^1(\Omega))} \leq C\,. \tag{2.3.3}$$

Moreover, the constant C *is independent of* ϵ.

Proof. Inequality (2.3.2) follows from (2.3.3). To prove the latter, set

$$v^\epsilon_\kappa := \varphi_\kappa(u^\epsilon_{\kappa x}) + \epsilon\,[\psi_\kappa(u^\epsilon_{\kappa x})]_t\,, \tag{2.3.4}$$

and observe that deriving with respect to x the equation in (P^ϵ_κ) gives

$$u^\epsilon_{\kappa xt} = v^\epsilon_{\kappa xx} \quad \text{in } Q\,. \tag{2.3.5}$$

From (2.3.4)-(2.3.5) we obtain the equality

$$v^\epsilon_\kappa = \varphi_\kappa(u^\epsilon_{\kappa x}) + \epsilon\psi'_\kappa(u^\epsilon_{\kappa x})v^\epsilon_{\kappa xx}\,.$$

Then for any $t \in (0,T)$, $v^\epsilon_\kappa(\cdot,t)$ solves the problem

$$\begin{cases} z - \epsilon\,[\psi'_\kappa(u^\epsilon_{\kappa x}(\cdot,t))]\,z_{xx} = \varphi(u^\epsilon_{\kappa x}(\cdot,t)) & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.3.6}$$

Since by assumption $\psi'_\kappa \geq \psi' + \kappa \geq \kappa$, and $u^\epsilon_{\kappa x} \geq 0$ by Lemma 2.3.1-(ii), by the maximum principle we obtain

$$0 \leq v^\epsilon_\kappa(\cdot,t) \leq \varphi_\kappa(\alpha) \quad \text{in } \Omega \tag{2.3.7}$$

(here use of assumption (A)-(ii) has been made). In view of the boundary condition $v^\epsilon_\kappa(\cdot,t)=0$ on $\partial\Omega$ $(t\geq 0)$ we also have

$$\frac{\partial v^\epsilon_\kappa}{\partial\nu}(\cdot,t)\leq 0 \quad \text{on } \partial\Omega$$

for any $t\in(0,T)$, where $\frac{\partial}{\partial\nu}$ denotes the outer derivative at $\partial\Omega$. Then integrating with respect to x and t equation (2.3.5) we obtain

$$\int_\Omega u^\epsilon_{\kappa x}(x,t)dx \leq \int_\Omega u'_{0\kappa}(x)dx\,.$$

Since $u^\epsilon_{\kappa x}\geq 0$ by Lemma 2.3.1-(ii), the result follows. □

Lemma 2.3.3. *Let (A) be satisfied. Then there exists a constant $C>0$ such that for any $\kappa>0$*

$$\iint_Q \frac{\left[\psi_\kappa(u^\epsilon_{\kappa x})\right]^2_t}{\psi'_\kappa(u^\epsilon_{\kappa x})}\,dxdt \leq \frac{C}{\epsilon}\,, \tag{2.3.8}$$

and

$$\|u^\epsilon_{\kappa t}\|_{L^2(Q)} = \|v^\epsilon_{\kappa x}\|_{L^2(Q)}\leq C\,, \tag{2.3.9}$$

where the function v^ϵ_k is defined by (2.3.4). *Moreover, the constant C is independent of ϵ.*

Proof. From (2.3.4)-(2.3.5) we obtain plainly

$$\frac{d}{dt}\int_\Omega dx\int_0^{u^\epsilon_{\kappa x}}\varphi_\kappa(s)ds = -\int_\Omega \epsilon\psi'_\kappa(u^\epsilon_{\kappa x})(u^\epsilon_{\kappa xt})^2dx - \int_\Omega (v^\epsilon_{\kappa x})^2dx\,.$$

Integrating the above equality with respect to t gives

$$\begin{aligned}
&\iint_Q (v^\epsilon_\kappa)^2_x + \epsilon\psi'_\kappa(u^\epsilon_{\kappa x})(u^\epsilon_{\kappa xt})^2dxdt = \qquad (2.3.10)\\
&= \int_\Omega dx\int_0^{u'_{0\kappa}(x)}\varphi_\kappa(s)ds - \int_\Omega dx\int_0^{u^\epsilon_{\kappa x}(x,T)}\varphi_\kappa(s)ds \leq\\
&\leq \int_\Omega dx\int_0^{u'_{0\kappa}(x)}\varphi_\kappa(s)ds
\end{aligned}$$

(here use of Lemma 2.3.1-(ii) has been made). Since $u^\epsilon_{\kappa t}=v^\epsilon_{\kappa x}$ (see the equation in (P^ϵ_k)) and $\varphi_k\in L^1(\mathbb{R})$ by assumption (A)-(iv), the result follows. □

Remark 2.3.1. *Observe that by assumptions $(H_2)-(i)$ and $(A)-(iii)$ inequality* (2.3.8) *implies*

$$\|\left[\psi_\kappa(u^\epsilon_{\kappa x})\right]_t\|_{L^2(Q)}\leq \frac{C}{\sqrt{\epsilon}}\,. \tag{2.3.11}$$

Lemma 2.3.4. *Let (A) be satisfied. Then there exists a constant $C > 0$ such that for any $\kappa > 0$*

$$\|\psi_\kappa(u^\epsilon_{\kappa x})\|_{L^\infty((0,T);H^1_0(\Omega))} \le C\,, \tag{2.3.12}$$

$$\|\,[\psi_\kappa(u^\epsilon_{\kappa x})]_{xt}\,\|_{L^2(Q)} \le C\,, \tag{2.3.13}$$

$$\|\varphi_\kappa(u^\epsilon_{\kappa x})\|_{L^\infty((0,T);H^1_0(\Omega))} \le C\,. \tag{2.3.14}$$

Remark 2.3.2. *Let us mention that the constant $C > 0$ in inequalities* (2.3.2)-(2.3.3), (2.3.8)-(2.3.9) *does not depend on ϵ, whereas it does in inequalities* (2.3.12)-(2.3.14).

Corollary 2.3.5. *Let (A) be satisfied. Then there exists a constant $C > 0$ such that for any $\kappa > 0$*

$$\|\psi_\kappa(u^\epsilon_{\kappa x})\|_{H^1(Q)} \le C\,, \tag{2.3.15}$$

$$\|\psi_\kappa(u^\epsilon_{\kappa x})\|_{C^{1/2}(Q)} \le C\,, \tag{2.3.16}$$

where $C^{1/2}(Q)$ denotes the Banach space of Hölder continuous functions with exponent $1/2$ in Q endowed with the usual norm.

Proof. Inequality (2.3.15) follows from (2.3.11) and (2.3.12). Inequality (2.3.16) is an easy consequence of the same inequalities and (2.3.13). □

Following [BBDU], let us now draw some consequences of the above estimates. In view of (2.3.3) and (2.3.9), the family $\{u^\epsilon_\kappa\}$ is uniformly bounded in $W^{1,1}(Q) \cap L^\infty((0,T);W^{1,1}(\Omega))$. Hence by compact embedding and a diagonal argument there exist a sequence $\kappa_j \to 0$ and a function $u^\epsilon \in BV(Q) \cap L^\infty((0,T);BV(\Omega))$ with $u^\epsilon_t \in L^2(Q)$, such that

$$u^\epsilon_{\kappa_j} \to u^\epsilon \quad \text{in } L^1(Q)\,, \tag{2.3.17}$$

$$u^\epsilon_{k_j x} \overset{*}{\rightharpoonup} u^\epsilon_x \quad \text{in } \mathcal{M}^+(Q)\,, \tag{2.3.18}$$

$$u^\epsilon_{\kappa_j}(\cdot,t) \to u^\epsilon(\cdot,t) \quad \text{in } L^1(\Omega) \text{ for a.e. } t \in (0,T)\,, \tag{2.3.19}$$

$$u^\epsilon_{\kappa_j t} \rightharpoonup u^\epsilon_t \quad \text{in } L^2(Q)\,. \tag{2.3.20}$$

Observe that (2.3.1) and (2.3.19) imply

$$\int_\Omega u^\epsilon(x,t)dx = \int_\Omega u_0(x)dx \quad \text{for a.e. } t \in (0,T)\,. \tag{2.3.21}$$

Moreover, by estimates (2.3.12), (2.3.13) and (2.3.15) there exists $w^\epsilon \in L^\infty((0,T);H^1_0(\Omega)) \cap H^1(Q)$, with $w^\epsilon_t \in L^2((0,T);H^1_0(\Omega))$, such that

$$\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right) \rightharpoonup w^\epsilon \quad \text{in } H^1(Q)\,, \tag{2.3.22}$$

$$\left[\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right)\right]_t \rightharpoonup w^\epsilon_t \quad \text{in } L^2((0,T);H^1_0(\Omega))\,. \tag{2.3.23}$$

In view of (2.3.16), we can assume $w^\epsilon \in C(\overline{Q})$ and

$$\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right) \to w^\epsilon \quad \text{in } C(\overline{Q})\,. \tag{2.3.24}$$

Concerning the sequence $\{\varphi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right)\}$ we can now prove the following lemma, where the function h is defined by (2.2.4).

Lemma 2.3.6. *Let (A) be satisfied. Then*

$$\varphi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right) \rightharpoonup h(w^\epsilon) \quad \textit{in} \;\; L^2((0,T);H^1_0(\Omega))\,, \tag{2.3.25}$$

$$\varphi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right) \to h(w^\epsilon) \qquad \textit{in } C(\overline{Q})\,. \tag{2.3.26}$$

Proof. By inequality (2.3.14), possibly extracting a subsequence, also denoted by $\{\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})\}$, there exists $z \in L^2((0,T);H^1_0(\Omega))$ such that

$$\varphi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right) \rightharpoonup z \quad \text{in } \;L^2((0,T);H^1_0(\Omega))\,.$$

Let us define:

$$h_{\kappa_j} := \varphi_{\kappa_j} \circ \psi^{-1}_{\kappa_j}\,.$$

We have:

$$\begin{aligned} &\left[\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_t = \left[h_{\kappa_j}\left(\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right)\right]_t = \\ =\;& h'_{\kappa_j}\left(\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right)\left[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_t \end{aligned}$$

and

$$\begin{aligned} &\left[\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_{tx} = \left[h_{\kappa_j}\left(\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right)\right]_{tx} = \\ =\;& \left[h'_{\kappa_j}\left(\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right)\left[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_t\right]_x = \\ =\;& h''_{\kappa_j}\left(\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right)\left[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_x\left[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_t + \\ &+ h'_{\kappa_j}\left(\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right)\left[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_{tx}\,. \end{aligned}$$

Moreover, observe that (2.3.13) implies that there exists a constant $C > 0$ such that

$$\left\|\left[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_t\right\|_{L^2((0,T);L^\infty(\Omega))} \le C\,. \tag{2.3.27}$$

In view of assumption (A)-(iv), (2.3.12)-(2.3.13) and (2.3.27), there exists a constant $\tilde{C} > 0$, independent of κ_j, such that

$$\left\|\left[\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_{tx}\right\|_{L^2(Q)} \le \tilde{C}\,, \quad \left\|\left[\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})\right]_t\right\|_{L^2((0,T);L^\infty(\Omega))} \le \tilde{C}\,. \tag{2.3.28}$$

By (2.3.14) and (2.3.28) we obtain

$$\left\| \varphi_{\kappa_j}\left(u^{\epsilon}_{\kappa_j x}\right)\right\|_{C^{1/2}(\overline{Q})} \le C\,,$$

hence (possibly extracting another subsequence) we have

$$\varphi_{\kappa_j}\left(u^{\epsilon}_{\kappa_j x}\right) \to z \quad \text{in} \quad C(\overline{Q})\,.$$

On the other hand, from the inequality on $\mathbb{R}$:

$$\begin{aligned}&\left|\varphi_{\kappa_j}\left(u^{\epsilon}_{\kappa_j x}\right) - h(w^{\epsilon})\right| \le \left|h_{\kappa_j}\left(\psi_{\kappa_j}\left(u^{\epsilon}_{\kappa_j x}\right)\right) - h_{\kappa_j}(w^{\epsilon})\right| + \\ &+ \left|h_{\kappa_j}(w^{\epsilon}) - h(w^{\epsilon})\right| \le k_1 \left|\psi_{\kappa_j}\left(u^{\epsilon}_{\kappa_j x}\right) - w^{\epsilon}\right| + \\ &+ \left|h_{k_j}(w^{\epsilon}) - h(w^{\epsilon})\right|\end{aligned}$$

we obtain

$$\varphi_{\kappa_j}\left(u^{\epsilon}_{\kappa_j x}\right) \to h(w^{\epsilon}) \quad a.e. \text{ in } Q$$

(here use of assumption (A)-(i) and (A)-(iv) has been made). Hence $z = h(w^{\epsilon})$ *a.e.* in Q and (2.3.26) follows. □

In view of the above remarks, taking the limit as $j \to \infty$ in the weak formulation of problem $(P^{\epsilon}_{\kappa_j})$ we see that the couple $(u^{\epsilon}, w^{\epsilon})$ (with u^{ϵ} as in (2.3.17) and w^{ϵ} as in (2.3.22)) solves problem (2.1.3) in the sense of Definition 2.2.2. Uniqueness was proven in [BBDU], while monotonicity in space follows from Lemma 2.3.1-(ii) and the above convergence results (see (2.3.18)). Hence Theorem 2.2.2 follows.

It is also easily seen that:

Lemma 2.3.7. *Let (A) be satisfied. Then*

$$\psi(u^{\epsilon}_{\kappa_j x}) \to w^{\epsilon} \quad \text{in } L^{\infty}((0,T);L^1(\Omega))\,, \tag{2.3.29}$$

$$\psi(u^{\epsilon}_{\kappa_j x}) \to w^{\epsilon} \qquad a.e. \text{ in } Q. \tag{2.3.30}$$

Proof. Assumption $(A)-(iii)$ implies that

$$\psi(u^{\epsilon}_{\kappa_j x}) + \kappa_j u^{\epsilon}_{\kappa_j x} \le \psi_{\kappa_j}(u^{\epsilon}_{\kappa_j x}) \le \psi(u^{\epsilon}_{\kappa_j x}) + 2\kappa_j u^{\epsilon}_{\kappa_j x} \tag{2.3.31}$$

(recall that $u^{\epsilon}_{\kappa_j x} \ge 0$ by Lemma 2.3.1-(ii)). Then we have

$$\|\psi(u^{\epsilon}_{\kappa_j x}) - \psi_{\kappa_j}(u^{\epsilon}_{\kappa_j x})\|_{L^{\infty}((0,T);L^1(\Omega))} = \tag{2.3.32}$$

$$= \sup_{t\in(0,T)} \int_{\Omega} \left[\psi_{\kappa_j}(u^{\epsilon}_{\kappa_j x}) - \psi(u^{\epsilon}_{\kappa_j x})\right](x,t)dx \le 2\kappa_j \|u^{\epsilon}_{\kappa_j x}\|_{L^{\infty}((0,T);L^1(\Omega))}\,.$$

From (2.3.3) and (2.3.32) convergence (2.3.29) follows. As $j \to \infty$ (possibly extracting a subsequence, still denoted $\{u^{\epsilon}_{\kappa_j x}\}$), this also gives (2.3.30). □

Remark 2.3.3. *Observe that* (2.3.24) *and the left inequality in* (2.3.31) *imply* $w^\epsilon \geq 0$ *in* $\overline{Q}$ *(since* $u^\epsilon_{\kappa_j x} \geq 0$*), whereas* (2.3.30) *and the fact that* $w^\epsilon \in C(\overline{Q})$ *give* $w^\epsilon \leq \gamma$ *in* $\overline{Q}$*, for* $0 \leq \psi < \gamma$ *in* $[0,\infty)$*.*

Proposition 2.3.9 below deals with the behaviour of the family $\{u^\epsilon_{\kappa_j x}\}$ of solutions to (P^ϵ_κ) in the limit $\kappa_j \to 0$. Let us first prove the following lemma.

Lemma 2.3.8. *Let* (A) *be satisfied. Let* $\{\eta_{\kappa_j}\}$ *be the sequence of Young measures associated to the sequence* $\{u^\epsilon_{\kappa_j x}\}$ *above. Then:*
(i) there exists a Young measure η *such that as* $\kappa_j \to 0$

$$\eta_{\kappa_j} \to \eta \quad \text{narrowly in } Q\,; \tag{2.3.33}$$

(ii) the disintegration $\nu_{(x,t)}$ *of the Young measure* η *is the Dirac mass concentrated at the point* $\psi^{-1}(w^\epsilon(x,t))$*, namely*

$$\nu_{(x,t)} = \delta_{\psi^{-1}(w^\epsilon(x,t))} \quad \text{for a.e. } (x,t) \in Q\,. \tag{2.3.34}$$

Proof. *(i)* Follows from inequality (2.3.3) and the Prohorov's theorem (see [V]).
(ii) In view of (2.3.29), the sequence $\{\psi(u^\epsilon_{\kappa_j x})\}$ is bounded in $L^1(Q)$, hence by Prohorov's theorem the associated sequence of Young measures $\{\chi_{\kappa_j}\}$ converges narrowly to a Young measure χ. Let $\sigma_{(x,t)}$ denote the disintegration of the Young measure χ for *a.e.* $(x,t) \in Q$. By the very definition of the sequences $\{\eta_{\kappa_j}\}$, $\{\chi_{\kappa_j}\}$ and of disintegration, for any $f \in C_c(\mathbb{R})$ we have

$$\iint_Q \phi(x,t) \left\{ \int_{[0,\infty)} f(\xi)\, d\nu_{(x,t)}(\xi) \right\} dxdt = \tag{2.3.35}$$

$$= \lim_{j\to\infty} \iint_Q \phi(x,t) f\left(u^\epsilon_{\kappa_j x}(x,t)\right) dxdt =$$

$$= \lim_{j\to\infty} \iint_Q \phi(x,t) \left(f \circ \psi^{-1}\right) \left(\psi\left(u^\epsilon_{\kappa_j x}\right)(x,t)\right) dxdt =$$

$$= \iint_Q \phi(x,t) \left\{ \int_{[0,\infty)} \left(f \circ \psi^{-1}\right)(\xi)\, d\sigma_{(x,t)}(\xi) \right\} dxdt$$

for any $\phi \in C^1_c(Q)$. On the other hand, since $\psi(u^\epsilon_{\kappa_j x}) \to w^\epsilon$ *a.e.* in Q (see (2.3.30)), the disintegration $\sigma_{(x,t)}$ of χ is the Dirac mass concentrated at the point $w^\epsilon(x,t)$, namely

$$\sigma_{(x,t)} = \delta_{w^\epsilon(x,t)} \tag{2.3.36}$$

(see [V, Proposition 1]). Then from equalities (2.3.35)-(2.3.36) we obtain

$$\int_{[0,\infty)} f(\xi)\, d\nu_{(x,t)}(\xi) = f\left(\psi^{-1}(w^\epsilon(x,t))\right)$$

for *a.e.* $(x,t) \in Q$, whence the result follows. □

Proposition 2.3.9. *Let* (A) *be satisfied. Then:*
(i) $\psi^{-1}(w^\epsilon) \in L^1(Q)$ *and there exists a subsequence of* $\left\{u^\epsilon_{\kappa_j x}\right\}$*, denoted again* $\left\{u^\epsilon_{\kappa_j x}\right\}$*, such that:*

$$u^\epsilon_{\kappa_j x} \to \psi^{-1}(w^\epsilon) \qquad \text{a.e. in } Q\,; \tag{2.3.37}$$

(ii) *the set*

$$\mathcal{S}^\epsilon := \left\{(x,t) \in \overline{Q} \mid w^\epsilon(x,t) = \gamma\right\} \tag{2.3.38}$$

has zero Lebesgue measure.

Proof. (i) The limit (2.3.37) follows from equality (2.3.34) by Proposition 1 in [V]. Since $u^\epsilon_{\kappa_j x} \geq 0$ (see Lemma 2.3.1-(ii)), by (2.3.37), inequality (2.3.3) and the Fatou Lemma we obtain

$$\iint_Q \psi^{-1}(w^\epsilon)dxdt \leq \liminf_{\kappa_j \to \infty} \iint_Q u^\epsilon_{\kappa_j x} dxdt \leq C\,.$$

Therefore $\psi^{-1}(w^\epsilon) \in L^1(Q)$.

(ii) Set

$$B^\epsilon_n := \left\{(x,t) \in \overline{Q} \mid w^\epsilon(x,t) \geq \gamma - \frac{1}{n}\right\} \qquad (n \in \mathbb{N})\,. \tag{2.3.39}$$

Then

$$B^\epsilon_{n+1} \subseteq B^\epsilon_n\,, \quad \mathcal{S}^\epsilon = \bigcap_{n=1}^{\infty} B^\epsilon_n\,, \quad |\mathcal{S}^\epsilon| = \lim_{n\to\infty} |B^\epsilon_n|\,, \tag{2.3.40}$$

where $|\cdot|$ denotes the Lebesgue measure. Since $\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right) \to w^\epsilon$ uniformly in $\overline{Q}$, thus in B^ϵ_n (see (2.3.24)), there holds

$$\sup_{(x,t)\in B^\epsilon_n} \left|\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right)(x,t) - w^\epsilon(x,t)\right| < \frac{1}{n}$$

for any $\kappa_j > 0$ sufficiently small. From the above inequality and (2.3.31) we obtain

$$u^\epsilon_{\kappa_j x} > \psi^{-1}\left(\gamma - \frac{1}{2n} - 2\kappa_j u^\epsilon_{\kappa_j x}\right) \quad \text{in } B^\epsilon_n\,. \tag{2.3.41}$$

On the other hand, by Lemma 2.3.1-(ii) and (2.3.3) there exists a subsequence, denoted again $\{\kappa_j\}$, such that $\kappa_j u^\epsilon_{\kappa_j x} \to 0$ *a.e.* in Q, thus

$$\psi^{-1}\left(\gamma - \frac{1}{2n} - 2\kappa_j u^\epsilon_{\kappa_j x}\right) \to \psi^{-1}\left(\gamma - \frac{1}{2n}\right) \quad \text{a.e. in } B^\epsilon_n\,.$$

Then by the Lebesgue Theorem we have

$$\iint_{B^\epsilon_n} \psi^{-1}\left(\gamma - \frac{1}{2n} - 2\kappa_j u^\epsilon_{\kappa_j x}\right) dxdt \to \psi^{-1}\left(\gamma - \frac{1}{2n}\right) |B^\epsilon_n| \tag{2.3.42}$$

for any $n \in \mathbb{N}$. In view of (2.3.41)-(2.3.42), we obtain

$$\psi^{-1}\left(\gamma - \frac{1}{2n}\right)|B_n^\epsilon| = \lim_{\kappa_j \to 0} \iint_{B_n^\epsilon} \psi^{-1}\left(\gamma - \frac{1}{2n} - 2\kappa_j u_{\kappa_j x}^\epsilon\right) dxdt \leq$$

$$\leq \liminf_{\kappa_j \to 0} \iint_{B_n^\epsilon} u_{\kappa_j x}^\epsilon \leq C \tag{2.3.43}$$

thus

$$|B_n^\epsilon| < \frac{C}{\psi^{-1}\left(\gamma - \frac{1}{2n}\right)}$$

for some constant $C > 0$ and any $n \in \mathbb{N}$. Letting $n \to \infty$ in the previous inequality the conclusion follows. □

We can now prove Theorem 2.2.3.

Proof of Theorem 2.2.3. Fix any $\epsilon > 0$ and set

$$\mathcal{R}^\epsilon := \left\{(x,t) \in Q \mid w^\epsilon(x,t) < \gamma\right\}.$$

Since $w^\epsilon \in C(\overline{Q})$, $\mathcal{R}^\epsilon$ is open in Q. Let $\zeta \in C_c(\mathcal{R}^\epsilon)$; denote by K the support of ζ. Since w^ϵ is continuous in $\overline{Q}$, thus in K, there exists

$$M_K := \max_{(x,t) \in K} w^\epsilon(x,t) < \gamma\,.$$

Set $\delta_K := \gamma - M_K$. Since $\psi_{\kappa_j}\left(u_{\kappa_j x}^\epsilon\right) \to w$ uniformly in $C(\overline{Q})$ (see (2.3.24)), there holds

$$\max_K \psi_{\kappa_j}\left(u_{\kappa_j x}^\epsilon\right) \leq M_K + \frac{\delta_K}{2} = \gamma - \frac{\delta_K}{2}$$

for any κ_j sufficiently small. In view of the left inequality in (2.3.31), this plainly implies

$$u_{\kappa_j x}^\epsilon \leq \psi^{-1}\left(\gamma - \frac{\delta_K}{2}\right) \qquad \text{in } K\,,$$

if κ_j is sufficiently small.
From the latter inequality and the limit (2.3.37), by the Lebesgue Theorem we obtain

$$\iint_Q u_{\kappa_j x}^\epsilon \zeta\, dxdt \to \iint_Q \psi^{-1}(w^\epsilon)\zeta\, dxdt \qquad \text{for any } \zeta \in C_c(\mathcal{R}^\epsilon)\,. \tag{2.3.44}$$

On the other hand, in view of (2.3.3) and (2.3.18), there holds

$$\iint_Q u_{\kappa_j x}^\epsilon \zeta\, dxdt \to < u_x^\epsilon, \zeta > = \iint_Q u_x^{\epsilon,(r)} \zeta\, dxdt + < u_x^{\epsilon,(s)}, \zeta > \tag{2.3.45}$$

for any $\zeta \in C_c(Q)$. From (2.3.44)-(2.3.45) we obtain the equality

$$< u_x^{\epsilon,(s)}, \zeta > = \iint_Q \left\{\psi^{-1}(w^\epsilon) - u_x^{\epsilon,(r)}\right\} \zeta\, dxdt \qquad \text{for any } \zeta \in C_c(\mathcal{R}^\epsilon)\,,$$

which entails:
(*i*) $u_x^{\epsilon,(s)}(K) = 0$ for any compact subset $K \subseteq \mathcal{R}^\epsilon$, hence for any $\epsilon > 0$ we have $u_x^{\epsilon,(s)}(\mathcal{R}^\epsilon) = 0$ and $\operatorname{supp} u_x^{\epsilon,(s)} = \overline{Q} \setminus \mathcal{R}^\epsilon$ (because $\overline{Q} \setminus \mathcal{R}^\epsilon$ is closed). Let $\mathcal{S}^\epsilon$ be the closed set defined by (2.3.38) and observe that $\overline{Q} \setminus \mathcal{R}^\epsilon = \mathcal{S}^\epsilon \cup \{\partial Q \setminus \{\mathcal{S}^\epsilon \cap \partial Q\}\}$. Let us show that any $(x_0, t_0) \in \partial Q \setminus \{\mathcal{S}^\epsilon \cap \partial Q\}$ does not belong to $\operatorname{supp} u_x^{\epsilon,(s)}$. In fact for any (x_0, t_0) as above there holds $w^\epsilon(x_0, t_0) < \gamma$, hence by the continuity of w^ϵ, for any $\delta > 0$, sufficiently small, there exists $U_{0,\delta} \subseteq \overline{Q}$, $(x_0, t_0) \in U_0$, such that

$$w^\epsilon(x, t) \leq w^\epsilon(x_0, t_0) + \delta \leq \gamma - \delta \,.$$

We can suppose that

$$U_{0,\delta} = \overline{B}_{\delta^2}(x_0, t_0) \cap \overline{Q} \,,$$

where $B_{\delta^2}(x_0, t_0)$ denotes the ball centered at (x_0, t_0) and radius δ^2 (see (2.3.16)). Arguing as above, we can use the uniform convergence (2.3.24) to prove that:

$$u_x^{\epsilon,(s)}(B_{\delta^2}(x_0, t_0) \cap Q) = 0$$

for any $\delta > 0$, sufficiently small. This implies that $(x_0, t_0) \notin \operatorname{supp} u_x^{\epsilon,(s)}$, namely:

$$\operatorname{supp} u_x^{\epsilon,(s)} = \mathcal{S}^\epsilon$$

for any $\epsilon > 0$; then (2.2.9) follows. Finally, by Proposition 2.3.9-(*ii*) $\mathcal{S}^\epsilon$ has zero Lebesgue measure.
(*ii*) $u_x^{\epsilon,(r)} = \psi^{-1}(w^\epsilon)$ *a.e.* in $\mathcal{R}^\epsilon$, thus in Q. Then the conclusion follows. □

Let us now prove Theorem 2.2.1.

Proof of Theorem 2.2.1. The existence of a unique solution to problem (2.1.3) is an obvious consequence of Theorems 2.2.2-2.2.3. To prove (2.2.3), observe that for any $\kappa > 0$ $u_{\kappa x}^\epsilon$ satisfies the problem

$$\begin{cases} U_t = [\varphi_\kappa(U)]_{xx} + \epsilon[\psi_\kappa(U)]_{xxt} & \text{in } Q \\ U = 0 & \text{in } \partial\Omega \times (0, T] \\ U = u'_{0\kappa} & \text{in } \Omega \times \{0\} \,. \end{cases}$$

Then for any $\zeta \in C_c^\infty(Q)$ there holds

$$\iint_Q \{u_{\kappa x}^\epsilon \zeta_t - \varphi_\kappa\big(u_{\kappa x}^\epsilon\big)_x \zeta_x - \epsilon\psi_\kappa\big(u_{\kappa x}^\epsilon\big)_{tx} \zeta_x\} dxdt = 0 \,.$$

In view of (2.3.18), (2.3.23) and (2.3.25) letting $\kappa \to 0$ obtains

$$\iint_Q \Big\{u_x^{\epsilon,(r)} \zeta_t - \varphi\big(u_x^{\epsilon,(r)}\big)_x \zeta_x - \epsilon\psi\big(u_x^{\epsilon,(r)}\big)_{tx} \zeta_x\Big\} dxdt + \, < u_x^{\epsilon,(s)}, \zeta_t > = 0 \,.$$

This completes the proof. □

Remark 2.3.4. *Consider for any $n \in \mathbb{N}$ the complement in Q of the set* (2.3.39)*, namely*

$$A_n^\epsilon := \Big\{(x,t) \in Q \mid \psi(u_x^{\epsilon,(r)}) < \gamma - \frac{1}{n}\Big\} \qquad (n \in \mathbb{N})\,; \tag{2.3.46}$$

(recall that by (2.2.8) $w^\epsilon = \psi\big(u_x^{\epsilon,(r)}\big)$ *a.e. in Q). Then for any $j \in \mathbb{N}$ sufficiently large there holds:*

$$u_{k_j x}^\epsilon \le \psi^{-1}\left(\gamma - \frac{1}{2n}\right) \quad \text{in } A_n^\epsilon\,. \tag{2.3.47}$$

In fact, fix any $\epsilon > 0$. Since

$$\psi_{k_j}(u_{k_j x}^\epsilon) \to \psi(u_x^{\epsilon,(r)}) \quad \text{in } C(\overline{Q})$$

as $k_j \to 0$ (see (2.2.8) *and* (2.3.24)*), we have*

$$\psi_{k_j}(u_{k_j x}^\epsilon) \le \gamma - \frac{1}{2n} \quad \text{in } \; A_n^\epsilon\,.$$

Then assumption (A) and Lemma 2.3.1-(ii) give

$$\psi(u_{k_j x}^\epsilon) \le k_j u_{k_j x}^\epsilon + \psi(u_{k_j x}^\epsilon) \le \psi_{k_j}(u_{k_j x}^\epsilon) \le \gamma - \frac{1}{2n} \quad \text{in } A_n^\epsilon\,.$$

This proves the claim.

Lemma 2.3.10. *For any $\epsilon > 0$ the function v^ϵ defined by* (2.2.10) *belongs to $L^\infty(Q) \cap L^2((0,T);H_0^1(\Omega))$, and the following estimates hold:*

$$0 \le v^\epsilon \le \varphi(\alpha)\,, \tag{2.3.48}$$

$$\|v_x^\epsilon\|_{L^2(Q)} \le C \tag{2.3.49}$$

for some constant $C > 0$, which does not depend on ϵ.

Proof. By (2.3.23) and (2.3.25) there holds

$$v_{\kappa_j}^\epsilon \rightharpoonup h(w^\epsilon) + \epsilon w_t^\epsilon \quad \text{in } L^2((0,T);H_0^1(\Omega)) \tag{2.3.50}$$

as $\kappa_j \to 0$, $v_{\kappa_j}^\epsilon$ being defined by (2.3.4). By Proposition 2.3.9-(ii) there holds $w^\epsilon < \gamma$, thus $h(w^\epsilon) = \varphi \circ \psi^{-1}(w^\epsilon) = \varphi\big(u_x^{\epsilon,(r)}\big)$ *a.e.* in Q (see (2.2.4) and (2.2.8)). Hence, by (2.3.50) we obtain

$$v_{\kappa_j}^\epsilon \rightharpoonup v^\epsilon \quad \text{in } L^2((0,T);H_0^1(\Omega))\,.$$

Then inequality (2.3.48) is a consequence of assumption (A)-(i) and (2.3.7), since

$$0 \le \lim_{\kappa_j \to 0} \iint_Q \{\varphi_{\kappa_j}(\alpha) - v_{\kappa_j}^\epsilon\}\zeta dxdt = \iint_Q \{\varphi(\alpha) - v^\epsilon\}\zeta dxdt$$

for any $\zeta \in L^2(Q)$, $\zeta \ge 0$. On the other hand, inequality (2.3.49) follows from (2.3.9) by the lower semicontinuity of the norm (see also Remark 2.3.2); hence the result follows. □

Lemma 2.3.11. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ there holds:*

$$\iint_Q \frac{\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t^2}{\psi'(u_x^{\epsilon,(r)})}\, dxdt \leq \frac{C}{\epsilon} \tag{2.3.51}$$

Proof. By (2.3.8) there exists $g \in L^2(Q)$ such that (possibly extracting a subsequence) there holds:

$$\frac{\left[\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right)\right]_t}{\sqrt{\psi'_{\kappa_j}(u^\epsilon_{\kappa_j x})}} \rightharpoonup g \quad \text{in} \quad L^2(Q). \tag{2.3.52}$$

Let $\mathcal{S}^\epsilon$, A^ϵ_n denote the sets defined by (2.3.38), respectively (2.3.46). Since A^ϵ_n is open, $\mathcal{S}^\epsilon$ is closed and

$$A^\epsilon_n \subseteq A^\epsilon_{n+1}\,, \quad \mathcal{S}^\epsilon = \bigcap_{n=1}^{\infty} B^\epsilon_n$$

(see (2.3.40)), for any $\zeta \in C^1_c(Q \setminus \mathcal{S}^\epsilon)$ there exists $n \in \mathbb{N}$ such that $\operatorname{supp}\zeta \subseteq A^\epsilon_n$. Then by inequality (2.3.47) we obtain

$$0 \leq \frac{1}{\psi'(u^\epsilon_{\kappa_j x})} \leq \frac{1}{\psi'\left(\psi^{-1}\left(\gamma - \frac{1}{2n}\right)\right)}$$

in A^ϵ_n for any $n \in \mathbb{N}$. This implies

$$\frac{\left[\psi_{\kappa_j}\left(u^\epsilon_{\kappa_j x}\right)\right]_t}{\sqrt{\psi'_{\kappa_j}(u^\epsilon_{\kappa_j x})}}\,\zeta \rightharpoonup \frac{\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t}{\sqrt{\psi'\left(u_x^{\epsilon,(r)}\right)}}\,\zeta \quad \text{in} \quad L^2(Q)$$

for any $\zeta \in C^1_c(Q \setminus \mathcal{S}^\epsilon)$ (here use of (2.3.23) and (2.3.37) has been made). Hence

$$g = \frac{\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t}{\sqrt{\psi'\left(u_x^{\epsilon,(r)}\right)}} \quad a.e. \text{ in } \; Q$$

since $|\mathcal{S}^\epsilon| = 0$ by Proposition 2.3.9-(ii). Then inequality (2.3.51) follows from (2.3.8) by the lower semicontinuity of the norm. □

Lemma 2.3.12. *Inequalities* (2.2.16) *and* (2.2.17) *hold.*

Proof. Observe that, in view of estimate (2.3.3), the family $\{u^\epsilon_x\}$ is bounded in $\mathcal{M}^+(Q)$), hence (2.2.16) holds. Moreover,

$$u^\epsilon_t = v^\epsilon_x \qquad a.e. \text{ in } Q$$

(see equation (2.2.1) and (2.2.10)), hence estimates (2.3.49) gives (2.2.17). □

The next proposition deals with the regularity of v^ϵ and $\left(u_x^{\epsilon,(r)}\right)_t$.

Proposition 2.3.13. *Let v^ϵ be the function defined by* (2.2.10) *and for any $n \in \mathbb{N}$ let A_n^ϵ be the set defined by* (2.3.46). *Then for any $n \in \mathbb{N}$ there holds v_{xx}^ϵ, $\left(u_x^{\epsilon,(r)}\right)_t \in L^2(A_n^\epsilon)$ and*

$$v_{xx}^\epsilon = \left(u_x^{\epsilon,(r)}\right)_t = \frac{\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t}{\psi'\left(u_x^{\epsilon,(r)}\right)} \qquad \text{a.e. in } A_n^\epsilon\,. \tag{2.3.53}$$

Moreover,

$$v_{k_j xx}^\epsilon \rightharpoonup v_{xx}^\epsilon, \quad u_{k_j xt}^\epsilon \rightharpoonup \left(u_x^{\epsilon,(r)}\right)_t \quad \text{in} \quad L^2(A_n^\epsilon).$$

Proof. Observe that

$$u_{\kappa_j xt}^\epsilon = v_{\kappa_j xx}^\epsilon = \frac{[\psi_{\kappa_j}(u_{\kappa_j x}^\epsilon)]_t}{\psi_{\kappa_j}'(u_{\kappa_j x}^\epsilon)} \tag{2.3.54}$$

(here use of (2.3.5) and (2.3.6) has been made). By (2.3.47) in Remark 2.3.4 we have:

$$\begin{aligned} \iint_{A_n^\epsilon} \left(u_{\kappa_j xt}^\epsilon\right)^2 dxdt &= \iint_{A_n^\epsilon} (v_{\kappa_j xx}^\epsilon)^2 dxdt = \\ &= \iint_{A_n^\epsilon} \left(\frac{[\psi_{\kappa_j}(u_{\kappa_j x}^\epsilon)]_t}{\psi_{\kappa_j}'(u_{\kappa_j x}^\epsilon)}\right)^2 dxdt \leq \\ &\leq \|[\psi_{\kappa_j}(u_{\kappa_j x}^\epsilon)]_t\|_{L^2(Q)}^2 \left(\frac{1}{\psi'\left(\psi^{-1}\left(\gamma - \frac{1}{2n}\right)\right)}\right)^2 \leq \\ &\leq \frac{C}{\epsilon}\left(\frac{1}{\psi'\left(\psi^{-1}\left(\gamma - \frac{1}{2n}\right)\right)}\right)^2, \end{aligned} \tag{2.3.55}$$

the last estimate in the previous equality following by (2.3.11). Inequality (2.3.55) implies that the families $\left\{v_{\kappa_j xx}^\epsilon\right\}$ and $\left\{u_{\kappa_j xt}^\epsilon\right\}$ are uniformly bounded in $L^2(A_n^\epsilon)$, hence v_{xx}^ϵ, $\left(u_x^{\epsilon,(r)}\right)_t \in L^2(A_n^\epsilon)$, and

$$v_{\kappa_j xx}^\epsilon \rightharpoonup v_{xx}^\epsilon\,, \quad u_{\kappa_j xt}^\epsilon \rightharpoonup \left(u_x^{\epsilon,(r)}\right)_t \quad \text{in } L^2(A_n^\epsilon)$$

as $\kappa_j \to 0$, for any $n \in \mathbb{N}$. Finally, in view of (2.3.23), (2.3.37) and (2.3.47), we obtain equality (2.3.53). □

For any $g \in C^1(\mathbb{R})$, set

$$G_\kappa(\lambda) := \int_0^\lambda g \circ \varphi_\kappa(s) ds. \tag{2.3.56}$$

Proposition 2.3.14. *For any $g \in C^1(0,\varphi(\alpha))$, $g \equiv 0$ in $[0,S_g]$ for some $S_g > 0$, let G be the function defined by (2.2.13). Then for any $\epsilon > 0$ there exists a set $E^\epsilon \subseteq Q$ of Lebesgue measure $|E^\epsilon| = 0$ such that there holds:*
(i) $G(u_x^{\epsilon,(r)}) \in L^\infty(Q)$ and there holds:

$$G_{\kappa_j}(u^\epsilon_{\kappa_j x})(x,t) \to G(u_x^{\epsilon,(r)})(x,t)$$

for any $(x,t) \in Q \setminus E^\epsilon$;
(ii) there exists

$$\begin{aligned} \big[G(u_x^{\epsilon,(r)})\big]_t &= g(\varphi\big(u_x^{\epsilon,(r)}\big))\frac{\big[\psi\big(u_x^{\epsilon,(r)}\big)\big]_t}{\psi'\big(u_x^{\epsilon,(r)}\big)} \equiv \\ &\equiv g(\varphi\big(u_x^{\epsilon,(r)}\big))(u_x^{\epsilon,(r)})_t \qquad in\ L^2(Q)\,. \end{aligned} \tag{2.3.57}$$

Moreover,

$$[G_{\kappa_j}(u^\epsilon_{\kappa_j x})]_t \rightharpoonup \big[G(u_x^{\epsilon,(r)})\big]_t \quad in\ L^2(Q). \tag{2.3.58}$$

Proof. (i) Fix any $\epsilon > 0$. Let $E^\epsilon \subseteq Q$ be the set of Lebesgue measure $|E^\epsilon| = 0$ such that (2.3.37) holds for any $(x,t) \in Q \setminus E^\epsilon$. In view of (2.2.8), (2.3.37) and Assumption (A), we have

$$G_{\kappa_j}(u^\epsilon_{\kappa_j x})(x,t) \to G(u_x^{\epsilon,(r)})(x,t) \quad \text{for any } (x,t) \in Q \setminus E^\epsilon,$$

where G_{κ_j} and G are defined by (2.3.56), (2.2.13), respectively. Moreover, since $g \equiv 0$ in $[0,S_g]$, we have:

$$|G_{\kappa_j}(u^\epsilon_{\kappa_j x})| \le \left| \int_0^{u^\epsilon_{\kappa_j x}} g(\varphi_\kappa(s))ds \right| \le \int_{s_{\kappa_j 1}(S_g)}^{s_{\kappa_j 2}(S_g)} g(\varphi_{\kappa_j}(s))ds \le C_g \tag{2.3.59}$$

(here $s_{\kappa_j 1}$ and $s_{\kappa_j 2}$ denote the stable and unstable branch of the equation $v = \varphi_k(z)$, respectively). Hence, $G(u_x^{\epsilon,(r)}) \in L^\infty(Q)$.

(ii) Fix any $g \in C^1(0,\varphi(\alpha))$, $g \equiv 0$ in $[0,S_g]$ for some $S_g > 0$. Consider the family $\{u^\epsilon_{\kappa_j}\}$ of the solutions to $(P^\epsilon_{\kappa_j})$. We have

$$\begin{aligned} [G(u^\epsilon_{\kappa_j x})]_t &= g(\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x}))(u^\epsilon_{\kappa_j x})_t = \\ &= g(\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x}))\frac{[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})]_t}{\psi'_{\kappa_j}(u^\epsilon_{\kappa_j x})}\chi_{\left\{\psi_{\kappa_j}(u^\epsilon_{\kappa_j x}) \le \psi_{\kappa_j}(s_{\kappa_j 2}(S_g))\right\}}. \end{aligned} \tag{2.3.60}$$

Moreover, in view of Assumption (A)-(i), we can suppose that for any κ_j small enough there holds $\psi_{\kappa_j}(s_{\kappa_j 2}(S_g)) \le \psi(s_2(S_g)) + \rho$ for some $0 < \rho < \frac{\gamma - \psi(s_2(S_g))}{4}$ (here s_2 denote the unstable branche of the equation $v = \varphi(z)$). Hence,

$$\left\{\psi_{\kappa_j}(u^\epsilon_{\kappa_j x}) \le \psi_{\kappa_j}(s_{\kappa_j 2}(S_g))\right\} \subseteq \left\{\psi_{\kappa_j}(u^\epsilon_{\kappa_j x}) \le \psi(s_2(S_g)) + \rho\right\}.$$

On the other hand, in view of (2.3.24) we have:

$$\{\psi_{\kappa_j}(u^\epsilon_{\kappa_j x}) \leq \psi(s_2(S_g)) + \rho\} \subseteq \{\psi(u^{\epsilon,(r)}_x) \leq \psi(s_2(S_g)) + 2\rho\} \subseteq A^\epsilon_{\delta^g}$$

where δ^g is chosen so that

$$\psi(s_2(S_g)) + \frac{\gamma - \psi(s_2(S_g))}{2} \leq \gamma - \delta^g.$$

Thus,

$$[G(u^\epsilon_{\kappa_j x})]_t = g(\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x}))\frac{[\psi_{\kappa_j}(u^\epsilon_{\kappa_j x})]_t}{\psi'_{\kappa_j}(u^\epsilon_{\kappa_j x})}\chi_{A^\epsilon_{\delta^g}}$$

and the claim follows by (2.2.8), (2.3.26) and Lemma 2.3.13. □

Lemma 2.3.15. *For any $g \in C^1_c(0, \varphi(\alpha))$ and $\kappa > 0$ let G_k be the function defined by (2.3.56). Then there exists a constant $C_g > 0$ (independent of κ and ϵ) such that*

$$\int_0^T \left|\int_\Omega [G_\kappa(u^\epsilon_{\kappa x})]_t h dx\right| dt \leq C_g\big(\|h\|_{L^\infty(\Omega)} + \|h_x\|_{L^2(\Omega)}\big) \tag{2.3.61}$$

for any $h \in C^1_c(\Omega)$.

Proof. Fix any $g \in C^1_c(0, \varphi(\alpha))$ and let $a_g,\ b_g \in (0, \varphi(\alpha))$ be such that

$$\operatorname{supp} g = [a_g, b_g] \subset (0, \varphi(\alpha)).$$

Let v^ϵ_κ be the function defined by (2.3.4). In view of (2.3.5) and (2.3.6), we have:

$$\begin{aligned}
&\int_0^T \left|\int_\Omega [G_\kappa(u^\epsilon_{\kappa x})]_t h dx\right| dt = \int_0^T \left|\int_\Omega g(\varphi_\kappa(u^\epsilon_{\kappa x}))u^\epsilon_{\kappa xt} h dx\right| dt \leq \\
\leq\ & \int_0^T \left|\int_\Omega [g(\varphi_\kappa(u^\epsilon_{\kappa x})) - g(v^\epsilon_\kappa)]u^\epsilon_{\kappa xt} h dx\right| dt + \int_0^T \left|\int_\Omega g(v^\epsilon_\kappa)v^\epsilon_{\kappa xx} h dx\right| dt \leq \\
\leq\ & \int\int_Q \|g'\|_{C([a_g,b_g])}\frac{\psi_\kappa(u^\epsilon_{\kappa x})^2_t}{\psi'_\kappa(u^\epsilon_{\kappa x})}|h| dx dt + \\
&+ \int\int_Q [|g(v^{\epsilon_\kappa})||v^{\epsilon_\kappa}_x||h_x| + |h||g'(v^{\epsilon_\kappa})|(v^\epsilon_{\kappa x})^2] dx dt \leq \\
\leq\ & C_g(\|h\|_{L^\infty(Q)} + \|h_x\|_{L^2(Q)}),
\end{aligned} \tag{2.3.62}$$

the last estimate following by (2.3.49) and (2.3.51). This concludes the proof. □

Proposition 2.3.16. *For any $g \in C^1_c(0, \varphi(\alpha))$ let G be the function defined by (2.2.13). Then for any $h \in C^1_c(\Omega)$ there holds:*

$$\int_0^T \left|\int_\Omega G(u^{\epsilon,(r)}_x)_t h dx\right| dt \leq C_g(\|h\|_{L^\infty(Q)} + \|h_x\|_{L^2(Q)}), \tag{2.3.63}$$

for some $C_g > 0$ independent of ϵ.

Proof. For any $\epsilon > 0$, $\kappa_j > 0$ and $h \in C^1_c(\Omega)$ set

$$\Gamma^\epsilon_{\kappa_j}(t) := \int_\Omega [G_{\kappa_j}(u^\epsilon_{\kappa_j x})]_t(x,t)\, h(x)dx.$$

By (2.3.58) there holds

$$\Gamma^\epsilon_{\kappa_j} \rightharpoonup \Gamma^\epsilon \quad \text{in} \quad L^1(0,T)$$

as $\kappa_j \to 0$, where

$$\Gamma^\epsilon(t) := \int_\Omega \left[G\big(u^{\epsilon,(r)}_x\big)\right]_t (x,t)\, h(x)dx.$$

Thus, inequality (2.3.63) is an easy consequence of (2.3.61). □

Proposition 2.3.17. *Let $g \in C^1([0,\varphi(\alpha)])$, $g' \geq 0$, $g \equiv 0$ in $[0,S_g]$ for some $S_g > 0$, and consider the function G defined by* (2.2.13) *in terms of g. Then, for any $\epsilon > 0$ there exists a set $F^\epsilon \subseteq (0,T)$ of Lebesgue measure $|F^\epsilon| = 0$ such that inequalities* (2.2.12) *hold.*

Proof. Fix any $\epsilon > 0$ and any $\zeta \in C^1([0,T]; H^1_0(\Omega)\cap H^2(\Omega))$, $\zeta \geq 0$, $\zeta_{xx} \leq 0$ in Q. Consider the family $\{u^\epsilon_\kappa\}$ of solutions to problem (P^ϵ_κ) and let G_κ be the funcions defined by (2.3.56) for any $g \in C^1([0,\varphi(\alpha)])$. Assume that $g \equiv 0$ in $[0,S_g]$, for some $S_g > 0$, and assume that $g' \geq 0$. We have

$$\begin{aligned} \frac{d}{dt}\int_\Omega G_{\kappa_j}(u^\epsilon_{\kappa_j x})\zeta dx &= \int_\Omega G_{\kappa_j}(u^\epsilon_{\kappa_j x})\zeta_t dx + \\ &+ \int_\Omega g(\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x}))u^\epsilon_{\kappa_j xt}\zeta dx\,. \end{aligned} \tag{2.3.64}$$

Let $v^\epsilon_{\kappa_j}$ be the function defined by (2.3.4). Since $u^\epsilon_{\kappa_j xt} = v^\epsilon_{\kappa_j xx}$ (see (2.3.5)), we obtain

$$\begin{aligned} &\int_\Omega g(\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x}))u^\epsilon_{\kappa_j xt}\zeta dx = \int_\Omega g(v^\epsilon_{\kappa_j})v^\epsilon_{\kappa_j xx}\zeta dx + \\ &+ \int_\Omega \left[g(\varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})) - g(v^\epsilon_{\kappa_j})\right]\frac{v^\epsilon_{\kappa_j} - \varphi_{\kappa_j}(u^\epsilon_{\kappa_j x})}{\epsilon\psi'_{\kappa_j}(u^\epsilon_{\kappa_j x})}\zeta dx \leq \\ \leq\ & -\int_\Omega g(v^\epsilon_{\kappa_j})v^\epsilon_{\kappa_j x}\zeta_x dx - \int_\Omega g'(v^\epsilon_{\kappa_j})(v^\epsilon_{\kappa_j x})^2\zeta dx \leq \\ \leq\ & -\int_\Omega g(v^\epsilon_{\kappa_j})\zeta_x v^\epsilon_{\kappa_j x}dx = \int_\Omega G(v^\epsilon_{\kappa_j})\zeta_{xx}dx. \end{aligned} \tag{2.3.65}$$

Here

$$G(v^\epsilon_{\kappa_j}) := \int_0^{v^\epsilon_{\kappa_j}} g(s)ds. \tag{2.3.66}$$

Integrating equality (2.3.64) with respect to t and using (2.3.65) gives

$$\int_\Omega G_{\kappa_j}(u^\epsilon_{\kappa_j x}(x,t_2))\zeta(x,t_2)dx - \int_\Omega G_{\kappa_j}(u^\epsilon_{\kappa_j x}(x,t_1))\zeta(x,t_1)dx \leq$$
$$\leq \int_{t_1}^{t_2}\int_\Omega G_{\kappa_j}(u^\epsilon_{\kappa_j x})\zeta_t dxdt + \int_{t_1}^{t_2}\int_\Omega G(v^\epsilon_{\kappa_j})\zeta_{xx} dxdt \tag{2.3.67}$$

for any $t_1 < t_2 \leq T$. Since $G(\lambda)$ is a convex function on $\mathbb{R}$ (by the assumption $g' \geq 0$), there holds

$$G(v^\epsilon_{\kappa_j}) \geq g(v^\epsilon)(v^\epsilon_{\kappa_j} - v^\epsilon) + G(v^\epsilon),$$

hence, in view of (2.3.50) we obtain

$$\int_{t_1}^{t_2}\int_\Omega G(v^\epsilon)\zeta_{xx}dxdt \geq \liminf_{\kappa_j\to 0}\int_{t_1}^{t_2}\int_\Omega G(v^\epsilon_{\kappa_j})\zeta_{xx}dxdt \tag{2.3.68}$$

(here use of assumption $\zeta_{xx} \leq 0$ has been made). Let $E^\epsilon \subseteq Q$ be the set of zero Lebesgue-measure given by Proposition 2.3.14. Then there exists $F^\epsilon \subseteq (0,T)$, $|F^\epsilon| = 0$ such that for any $t \in (0,T) \setminus F^\epsilon$ the set

$$E^{\epsilon,t} = \{x \in \Omega \mid (x,t) \in E^\epsilon\} \subseteq \Omega$$

has Lebesgue measure $|E^{\epsilon,t}| = 0$. Moreover, for any $t \in (0,T) \setminus F^\epsilon$ there holds

$$G(u^\epsilon_{\kappa_j x}(\cdot,t)) \to G(u^{\epsilon,(r)}_x(\cdot,t)) \quad a.e. \text{ in } \Omega \tag{2.3.69}$$

(see Proposition 2.3.14-(i)). By (2.3.68), (2.3.69) and Proposition 2.3.14-(i), passing to the limit with respect to $\kappa_j \to 0$ in (2.3.67) gives

$$\int_\Omega G(u^{\epsilon,(r)}_x(x,t_2))\zeta(x,t_2)dx - \int_\Omega G(u^{\epsilon,(r)}_x(x,t_1))\zeta(x,t_1)dx \leq$$
$$\leq \int_{t_1}^{t_2}\int_\Omega G(u^{\epsilon,(r)}_x)\zeta_t dxdt + \int_{t_1}^{t_2}\int_\Omega G(v^\epsilon)\zeta_{xx}dxdt =$$
$$= \int_{t_1}^{t_2}\int_\Omega G(u^{\epsilon,(r)}_x)\zeta_t dxdt - \int_{t_1}^{t_2}\int_\Omega g(v^\epsilon)v^\epsilon_x\zeta_x dxdt \tag{2.3.70}$$

and this concludes the proof. □

2.4 Vanishing Viscosity Limit: proofs

To prove Theorem 2.2.7 we need some technical preliminaries. As a first step, consider the orthonormal basis of $L^2(\Omega)$ which is formed by the eigenfunctions $\eta_h \in H^1_0(\Omega)$ of the operator $-\Delta$ with homogeneous Dirichlet conditions. Let $\{\mu_h\}$ be the corresponding sequence of eigenvalues. For any $\epsilon > 0$, let Π_ϵ be the operator defined as follows

$$\Pi_\epsilon f := \sum_{h:\,\epsilon\mu_h\leq 1} f_h\eta_h\,, \qquad f_h = (f,\eta_h)_{L^2(\Omega)}, \tag{2.4.1}$$

for any $f \in L^2(\Omega)$. In this way we have introduced a family of orthogonal projection operators which is used in the following result.

Lemma 2.4.1. *There exists $C > 0$ such that, for any $\kappa > 0$, $\epsilon > 0$ there holds*

$$\|\Pi_\epsilon \, \varphi_\kappa(u^\epsilon_{\kappa x})\|_{L^2((0,T);H^1_0(\Omega))} + \epsilon^{-1/2} \|(I - \Pi_\epsilon)\varphi_\kappa(u^\epsilon_{\kappa x})\|_{L^2(Q)} \leq C. \quad (2.4.2)$$

Proof. Fix any ϵ, $\kappa > 0$, fix any $t \in (0,T)$ and for simplicity set

$$\varphi(x) := \varphi_\kappa(u^\epsilon_{\kappa x})(x,t), \qquad v(x) := v^\epsilon_\kappa(x,t), \qquad \psi_t(x) := \big[\psi_\kappa(u^\epsilon_{\kappa x})\big]_t(x,t),$$

where v^ϵ_κ is defined by (2.3.4). We have:

$$\begin{aligned} \varphi_h &= \int_\Omega \varphi(x)\, \eta_h(x) dx = && (2.4.3)\\ &= -\epsilon \int_\Omega \psi_t(x)\, \eta_h(x) dx + \int_\Omega v(x)\, \eta_h(x) dx = \\ &= -\epsilon [\psi_t]_h + v_h. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Pi_\epsilon \, \varphi\|^2_{H^1_0(\Omega)} = \sum_{\epsilon\mu_h \leq 1} \mu_h \varphi_h^2 \leq & \qquad (2.4.4)\\ \leq \sum_{\epsilon\mu_h \leq 1} \left[2\mu_h v_h^2 + 2\mu_h \epsilon^2 [\psi_t]_h^2\right] \leq & \\ \leq \sum_{h=1}^{\infty} 2\mu_h v_h^2 + \sum_{h=1}^{\infty} 2\epsilon [\psi_t]_h^2 = & \\ = 2\int_\Omega \left[\big(v^\epsilon_{\kappa x}\big)^2 + \epsilon \big[\psi(u^\epsilon_{\kappa x})\big]^2_t \right] (x,t) dx \,, & \end{aligned}$$

and,

$$\begin{aligned} \epsilon^{-1} \|(I - \Pi_\epsilon)\varphi\|^2_{L^2(\Omega)} = \sum_{\epsilon\mu_h > 1} \epsilon^{-1} \varphi_h^2 \leq & \qquad (2.4.5)\\ \leq 2\epsilon^{-1} \sum_{\epsilon\mu_h > 1} [v_h^2 + \epsilon^2 [\psi_t]_h^2] \leq & \\ \leq \sum_{\epsilon\mu_h > 1} 2\mu_h v_h^2 + \sum_{\epsilon\mu_h > 1} 2\epsilon [\psi_t]_h^2 \leq & \\ \leq 2\int_\Omega \left[\big(v^\epsilon_{\kappa x}\big)^2 + \epsilon \big[\psi(u^\epsilon_{\kappa x})\big]^2_t \right] (x,t) dx. & \end{aligned}$$

In view of estimate (2.3.49) and (2.3.51), integrating (2.4.4) and (2.4.5) with respect to t gives (2.4.2). □

For any $f \in C(\mathbb{R})$ set

$$F(\lambda) := f(\varphi(\lambda)). \tag{2.4.6}$$

The following proposition will be crucial in the investigation of the viscosity limit $\epsilon \to 0$.

Proposition 2.4.2. *Fix any $g \in C^1_c(0, \varphi(\alpha))$, $f \in C^1(\mathbb{R})$ and let G, F be the functions defined by* (2.2.13) *and* (2.4.6)*, respectively. Suppose that there exists $C > 0$ such that $\|f\|_{L^\infty(\mathbb{R})} \le C$, $\|f'\|_{L^\infty(\mathbb{R})} \le C$. Finally, assume that $G(u_x^{\epsilon,(r)}) \overset{*}{\rightharpoonup} G^*$, $F(u_x^{\epsilon,(r)}) \overset{*}{\rightharpoonup} F^*$ and $G(u_x^{\epsilon,(r)})F(u_x^{\epsilon,(r)}) \overset{*}{\rightharpoonup} H^*$ in $L^\infty(Q)$. Then*

$$H^* = G^* F^*. \tag{2.4.7}$$

Remark 2.4.1. *Observe that for any $g \in C^1([0, \varphi(\alpha)])$, $g(0) = 0$, the family $\left\{G\big(u_x^{\epsilon,(r)}\big)\right\}$ is uniformly bounded in $L^\infty(Q)$. In fact for a.e. $(x,t) \in Q$ there holds:*

$$\begin{aligned} |G(u_x^{\epsilon,(r)})(x,t)| &= \left| \int_0^{u_x^{\epsilon,(r)}(x,t)} g(\varphi(\lambda)) d\lambda \right| \le \\ &\le \left| \int_0^\infty |g(\varphi(\lambda))| d\lambda \right| \le \\ &\le \max_{\xi \in [0,\varphi(\alpha)]} |g'(\xi)| \int_{\mathbb{R}} |\varphi(\lambda)| d\lambda \le C \end{aligned}$$

since $\varphi \in L^1(\mathbb{R})$ by assumption (H_1)-(i).

Proof of Proposition 2.4.2. Following [Pl1], we set

$$F^\epsilon := f(\Pi_\epsilon\, \varphi(u_x^{\epsilon,(r)})) \tag{2.4.8}$$

and observe that, passing to the limit with respec to $\kappa_j \to 0$ in inequality (2.4.2) gives

$$\|\Pi_\epsilon\, \varphi\big(u_x^{\epsilon,(r)}\big)\|_{L^2((0,T);H^1_0(\Omega))} + \epsilon^{-1/2} \|(I - \Pi_\epsilon)\varphi\big(u_x^{\epsilon,(r)}\big)\|_{L^2(Q)} \le C, \tag{2.4.9}$$

(here use of Lemma 2.3.6 has been made). Since $\|f'\|_{L^\infty(\mathbb{R})}$ is bounded, we have

$$\begin{aligned} \|F^\epsilon - F(u_x^{\epsilon,(r)})\|_{L^2(Q)} &= \|f(\Pi_\epsilon\, \varphi\big(u_x^{\epsilon,(r)}\big)) - f(\varphi\big(u_x^{\epsilon,(r)}\big))\|_{L^2(Q)} \le \\ &\le \|f'\|_{L^\infty(\mathbb{R})} \|(I - \Pi_\epsilon)\varphi\big(u_x^{\epsilon,(r)}\big)\|_{L^2(Q)} \to 0 \end{aligned}$$

as $\epsilon \to 0$ by (2.4.9). Moreover, the family $\big\{G(u_x^{\epsilon,(r)})\big\}$ is uniformly bounded in Q (see Remark 2.4.1), hence the previous inequality implies

$$\|G(u_x^{\epsilon,(r)})F^\epsilon - G(u_x^{\epsilon,(r)})F(u_x^{\epsilon,(r)})\|_{L^2(Q)} \to 0$$

as $\epsilon \to 0$. Thus, in order to prove (2.4.7), it suffices to show that

$$\int\int_Q F^\epsilon G(u_x^{\epsilon,(r)})h\,dxdt \to \int\int_Q F^* G^* h\,dxdt\,, \tag{2.4.10}$$

as $\epsilon \to 0$, for any $h \in C_c^1(Q)$. To this purpose, assume for simplicity $\Omega = (0,1)$, set

$$\Gamma^\epsilon(x,t) := \int_0^x G(u_x^{\epsilon,(r)})(\xi,t)d\xi \qquad \text{for } \ a.e.\ t \in (0,T) \tag{2.4.11}$$

and observe that:

$$\int\int_Q F^\epsilon G(u_x^{\epsilon,(r)})h\,dxdt = -\int\int_Q \Gamma^\epsilon (F^\epsilon h)_x dxdt\,. \tag{2.4.12}$$

In view of (2.4.9), there holds

$$\|F_x^\epsilon\|_{L^2(Q)} \le \|f'\|_{L^\infty(\mathbb{R})}\|[\Pi_\epsilon\varphi(u_x^{\epsilon,(r)})]_x\|_{L^2(Q)} \le C_f\,,$$

hence $F^* \in L^2((0,T);H^1(\Omega))$ and

$$F_x^\epsilon \rightharpoonup F_x^* \quad \text{in} \ \ L^2(Q) \tag{2.4.13}$$

as $\epsilon \to 0$. Then, for any $\phi \in C_c^1(\Omega)$ and for $a.e.$ $t \in (0,T)$, set

$$\Lambda_\phi^\epsilon(t) := \int_\Omega G(u_x^{\epsilon,(r)})(\xi,t)\phi(\xi)d\xi. \tag{2.4.14}$$

In view of (2.3.63) we have:

$$\|\Lambda_\phi^\epsilon\|_{W^{1,1}(0,T)} \le C_{g,\phi}.$$

Thus, for any $\phi \in C_c^1(\Omega)$ there exist a sequence $\epsilon_k \to 0$ and $\Lambda_\phi \in L^1(0,T)$ such that

$$\int_0^T |\Lambda_\phi^{\epsilon_k} - \Lambda_\phi|dt \to 0. \tag{2.4.15}$$

On the other hand, since we have assumed $G(u_x^{\epsilon,(r)}) \overset{*}{\rightharpoonup} G^*$ in $L^\infty(Q)$ as $\epsilon \to 0$, there holds

$$\Lambda_\phi(t) \equiv \int_\Omega G^*(\xi,t)\phi(\xi)d\xi \tag{2.4.16}$$

for $a.e.$ $t \in (0,T)$, and the whole family $\left\{\Lambda_\phi^\epsilon\right\}$ satisfies (2.4.15). In other words, we have:

$$\int_0^T \left|\int_\Omega G(u_x^{\epsilon,(r)})\phi\,d\xi - \int_\Omega G^*\phi\,d\xi\right| dt \to 0 \tag{2.4.17}$$

for any $\phi \in C_c^1(\Omega)$. Since $C_c^1(\Omega)$ is dense in $L^1(\Omega)$, by means of (2.4.17) there holds

$$\int_0^T \left| \int_\Omega G(u_x^{\epsilon,(r)}) \chi_{(0,x)} \, d\xi - \int_\Omega G^* \chi_{(0,x)} \, d\xi \right| dt \to 0$$

for any $x \in (0,1)$ (recall that we have assumed $\Omega = (0,1)$), namely

$$\int_0^T |\Gamma^\epsilon(x,t) - \Gamma^*(x,t)| dt \to 0 \tag{2.4.18}$$

(see (2.4.11)) for any $x \in \Omega$. Here

$$\Gamma^*(x,t) := \int_0^x G^*(\xi,t) d\xi \tag{2.4.19}$$

for *a.e.* $t \in (0,T)$. In view of (2.4.18) and since the family $\{\Gamma^\epsilon\}$ is uniformly bounded in $L^\infty(Q)$, we have

$$\Gamma^\epsilon \to \Gamma^* \quad \text{in} \quad L^1(Q).$$

Thus, eventually up to a sequence $\epsilon_k \to 0$, there holds

$$\Gamma^{\epsilon_k}(x,t) \to \Gamma^*(x,t) \quad \text{for } a.e. \;\; (x,t) \in Q. \tag{2.4.20}$$

Let us conclude the proof. In view of (2.4.20) and since the family $\{\Gamma^\epsilon\}$ is uniformly bounded in $L^\infty(Q)$, there holds

$$\Gamma^{\epsilon_k} \to \Gamma^* \quad \text{in} \quad L^2(Q).$$

Therefore, by (2.4.13) we obtain:

$$F_x^{\epsilon_k} \Gamma^{\epsilon_k} \rightharpoonup F_x^* \Gamma^* \quad \text{in} \quad L^2(Q)$$

Γ^* being defined by (2.4.19). Hence, for any $h \in C_c^1(Q)$ the right-hand side in (2.4.12) (written for $\epsilon = \epsilon_k$) converges to

$$-\int\int_Q (F^* h)_x \Gamma^* dxdt = \int\int_Q F^* G^* h \, dxdt$$

(see (2.4.19)) and the claim follows. □

Lemma 2.4.3. *Let $v \in L^\infty(Q)$ be the limit of the sequence $\{v^{\epsilon_k}\}$ in the weak* topology of $L^\infty(Q)$ (see* (2.2.24)*). Then* (2.2.26) *holds.*

Proof. Observe that

$$\|\varphi(u_x^{\epsilon,(r)}) - v^\epsilon\|_{L^2(Q)} = \epsilon^{1/2} \left\| \epsilon^{1/2} \left[\psi(u_x^{\epsilon,(r)})\right]_t \right\|_{L^2(Q)} \to 0 \tag{2.4.21}$$

as $\epsilon \to 0$ (here use of (2.3.51) has been made). By (2.4.21) we obtain

$$\varphi\big(u_x^{\epsilon_k,(r)}\big) \rightharpoonup v \quad \text{in} \quad L^2(Q) \tag{2.4.22}$$

where v is the limit of the sequence $\{v^{\epsilon_k}\}$ in the weak topology of $L^\infty(Q)$. On the other hand, the family $\Big\{\varphi(u_x^{\epsilon_k,(r)})\Big\}$ is uniformly bounded in $L^\infty(Q)$. Hence, eventually up to a subsequence ϵ_{k_j}, there holds

$$\varphi\big(u_x^{\epsilon_{k_j},(r)}\big) \overset{*}{\rightharpoonup} \tilde{v} \quad \text{in} \quad L^\infty(Q) \tag{2.4.23}$$

for some $\tilde{v} \in L^\infty(Q)$. Finally, by (2.4.22) $\tilde{v} = v$ *a.e.* in Q and the whole sequence $\Big\{\varphi\big(u_x^{\epsilon_k,(r)}\big)\Big\}$ satisfies (2.4.23). □

Now we can prove Theorem 2.2.7.

Proof of Theorem 2.2.7. Let τ be the Young measure obtained as narrow limit of the sequence τ_{ϵ_k} of Young measures associated to the functions $u_x^{\epsilon_k,(r)}$ (see Proposition 2.2.4). Let $\nu_{(x,t)}$ be the disintegration of τ, which holds for a.e. $(x,t) \in Q$. Our purpose is to give a characterization of the probability measure $\nu_{(x,t)}$ for *a.e.* $(x,t) \in Q$. In this direction, fix any $(x,t) \in Q$, set $I_1 := [0,\alpha]$, $I_2 := (\alpha,+\infty)$ and $\nu := \nu_{(x,t)}$ for simplicity. Then define two maps $\sigma_l \equiv \sigma_{(x,t);\,l} : C(\mathbb{R}) \to \mathbb{R}$ by setting

$$\int_{\mathbb{R}} f(\lambda)\,d\sigma_l(\lambda) \equiv \langle \sigma_l, f\rangle := \int_{I_l} (f\circ\varphi)(\xi)\,d\nu(\xi) \qquad (l=1,2)\,. \tag{2.4.24}$$

Then σ_1, σ_2 are (positive) Radon measures on $\mathbb{R}$.

Step 1. Concerning σ_l, $l=1,2$, it is easily seen that:
(*i*) $\operatorname{supp}\sigma_l \subseteq [0,\varphi(\alpha)]$ $(l=1,2)$;
(*ii*) $\sigma_2(\{0\}) = 0$;
(*iii*) let s_1, s_2 be the stable and unstable branch of the equation $v=\varphi(u)$ (see (2.2.14)-(2.2.15)); then for any $f \in C(\mathbb{R})$, such that the sequence $\{f(u_x^{\epsilon_k,(r)})\}$ is bounded in $L^1(Q)$ and equi-integrable, the function $f\circ s_l \in L^1([0,\varphi(\alpha)],d\sigma_l)$ $(l=1,2)$ (*e.g.*, see [Sm]).

Then set

$$\sigma := \sigma_1 + \sigma_2\,. \tag{2.4.25}$$

In view of the above definitions, we have

$$\langle \sigma, f\rangle = \langle \sigma_1, f\rangle + \langle \sigma_2, f\rangle = \int_{[0,+\infty)} (f\circ\varphi)(\xi)\,d\nu(\xi) \tag{2.4.26}$$

for any $f \in C(\mathbb{R})$, hence $\sigma \equiv \sigma_{(x,t)}$ is a probability measure on $\mathbb{R}$ for a.e. $(x,t) \in Q$. In view of *(ii)* the support of the measure σ is contained in $[0, \varphi(\alpha)]$; moreover ν and σ satisfy the following relation:

$$\begin{aligned} \langle \nu, f \rangle &\equiv \int_{[0,+\infty)} f(\xi)\, d\nu(\xi) = \int_{I_1} f(\xi)\, d\nu(\xi) + \int_{I_2} f(\xi)\, d\nu(\xi) = \\ &= \int_{I_1} [(f \circ s_1) \circ \varphi](\xi)\, d\nu(\xi) + \int_{I_2} [(f \circ s_2) \circ \varphi](\xi)\, d\nu(\xi) = \\ &= \langle \sigma_1, f \circ s_1 \rangle + \langle \sigma_2, f \circ s_2 \rangle \qquad (2.4.27) \end{aligned}$$

for any $f \in C(\mathbb{R})$ such that the sequence $\{f(u_x^{\epsilon_j,(r)})\}$ is bounded in $L^1(Q)$ and equi-integrable (here use of (2.4.24) and *Step 1-(iii)* has been made).

Step 2. For *a.e.* $(x,t) \in Q$ the measure $\sigma_{(x,t)}$ is the Dirac mass concentrated at the point

$$v(x,t) := \int_{[0,+\infty)} \varphi(\xi)\, d\nu_{(x,t)}(\xi) = \langle \nu_{(x,t)}, \varphi \rangle . \qquad (2.4.28)$$

Observe that v is the weak* limit of the sqeunce $\{\varphi(u_x^{\epsilon_k,(r)}\}$ in $L^\infty(Q)$ (see (2.2.31)-(2.2.32)).

Let us give a sketch of the proof (see [Pl1] and [Sm] for further details). In view of Proposition 2.4.2 and (2.2.31)-(2.2.32), for *a.e.* $(x,t) \in Q$ we obtain

$$\begin{aligned} &\left(\int_{[0,+\infty)} F(\xi)\, d\nu_{(x,t)}(\xi) \right) \left(\int_{[0,+\infty)} G(\xi)\, d\nu_{(x,t)}(\xi) \right) = \\ &= \int_{[0,+\infty)} F(\xi) G(\xi)\, d\nu_{(x,t)}(\xi) \qquad (2.4.29) \end{aligned}$$

for any G, F defined by (2.2.13) and (2.4.6), in correspondence of $f \in C^1(\mathbb{R})$ with $\|f\|_{L^\infty(\mathbb{R})}$, $\|f'\|_{L^\infty(\mathbb{R})}$ bounded and $g \in C_c^1(0, \varphi(\alpha))$.

Fix any $(x,t) \in Q$ such that (2.4.29) holds and set $\sigma \equiv \sigma_{(x,t)}$, $\nu \equiv \nu_{(x,t)}$. Let $A \subseteq [0, \varphi(\alpha)]$ be any compact such that $\sigma(A) > 0$. Since A is compact, there exists a sequence $\{f_h\} \subset C([0, \varphi(\alpha)])$, $f_h \geq 0$, $f_h = 1$ on A, such that

$$f_h(\lambda) \to \chi_A(\lambda) \quad \text{for any } \lambda \in [0, \varphi(\alpha)]$$

as $h \to \infty$. Set $F_h := f_h(\varphi)$. In view of (2.4.29) we have

$$\begin{aligned} &\left(\int_{[0,+\infty)} (f_h \circ \varphi)(\xi)\, d\nu(\xi) \right) \left(\int_{[0,+\infty)} G(\xi)\, d\nu(\xi) \right) = \\ &= \int_{[0,+\infty)} G(\xi) (f_h \circ \varphi)(\xi)\, d\nu(\xi). \end{aligned}$$

Using (2.4.27), the above equation reads:

$$\langle \sigma, f_h \rangle \sum_{l=1}^{2} \langle \sigma_l, G \circ s_l \rangle = \sum_{l=1}^{2} \langle \sigma_l, f_h (G \circ s_l) \rangle \,,$$

and letting $h \to \infty$ gives

$$\sigma(A) \sum_{l=1}^{2} \int_{[0,\varphi(\alpha)]} G(s_l(\lambda))\, d\sigma_l(\lambda) \quad = \quad \sum_{l=1}^{2} \int_A G(s_l(\lambda)) d\sigma_l(\lambda) \,.$$

Writing the above equality in a suitable way gives the following equation

$$M(\lambda) - M_A(\lambda) = N_A \;\; \text{for } a.e.\ \lambda \in (0, \varphi(\alpha)), \tag{2.4.30}$$

where

$$M(\lambda) := \left(s_1'(\lambda) - s_2'(\lambda)\right)^{-1} \sum_{l=1}^{2} s_l' \sigma_l \left([\lambda, \varphi(\alpha)]\right) \,,$$

$$M_A(\lambda) := [\sigma(A)]^{-1} \left(s_1'(\lambda) - s_2'(\lambda)\right)^{-1} \sum_{l=1}^{2} s_l' \sigma_l \left([\lambda, \varphi(\alpha)] \cap A\right) \,,$$

$$N_A := [\sigma(A)]^{-1} \sigma_2(A) - \sigma_2 \left([0, \varphi(\alpha)]\right)$$

(see [Pl1] and [Sm] for details).

Then set

$$\lambda_0 := \min\big\{\lambda \in [0, \varphi(\alpha)] \,|\, \lambda \in \operatorname{supp} \sigma\big\}.$$

If $\lambda_0 = \varphi(\alpha)$, the claim is obvious. Assume $\lambda_0 < \varphi(\alpha)$ and choose $A_\delta = [\lambda_0, \lambda_0 + \delta]$ with $\delta > 0$ small enough. Then $\sigma(A_\delta) \neq 0$ and $M_{A_\delta}(\lambda) = 0$ if $\lambda \in (\lambda_0 + \delta, \varphi(\alpha))$. Therefore by equation (2.4.30) we have

$$M(\lambda) = N_{A_\delta} \quad \text{for } a.e.\ \lambda \in (\lambda_0 + \delta, \varphi(\alpha)).$$

Since N_{A_δ} does not depend on λ and δ is arbitrary, we obtain

$$M(\lambda) = N_{\{\lambda_0\}} \quad \text{for } a.e.\ \lambda \in (\lambda_0, \varphi(\alpha)) \,. \tag{2.4.31}$$

Then observe that for any compact $A \subset [\lambda_0, \varphi(\alpha))$ there exists an interval $(\lambda^*, \varphi(\alpha))$ such that

$$A \cap (\lambda^*, \varphi(\alpha)) = \emptyset \,.$$

Therefore in the interval $(\lambda^*, \varphi(\alpha))$ we have $M_A(\lambda) \equiv 0$, hence in view of (2.4.30) and (2.4.31) we have:

$$N_A = N_{\{\lambda_0\}} \,. \tag{2.4.32}$$

Using (2.4.30) again, observe that equalities (2.4.31)-(2.4.32) imply $M_A(\lambda) = 0$ for *a.e.* $\lambda \in (\lambda_0, \varphi(\alpha))$ and for any compact $A \subset [\lambda_0, \varphi(\alpha))$, namely

$$\sum_{l=1}^{2} s_l'(\lambda)\sigma_l([\lambda, \varphi(\alpha)] \cap A) = 0 \quad \text{for } a.e.\ \lambda \in (\lambda_0, \varphi(\alpha)). \tag{2.4.33}$$

Consider any closed interval $A = [\beta_1, \beta_2] \subset (\lambda_0, \varphi(\alpha))$. If $\lambda \in (\lambda_0, \beta_1)$ we have $\sigma_l([\lambda, \varphi(\alpha)] \cap A) = \sigma_l(A)$. Hence, by equation (2.4.33), it follows that

$$\sum_{l=1}^{2} s_l'(\lambda)\sigma_l(A) = 0 \quad \text{for } a.e.\ \lambda \in (\lambda_0, \beta_1). \tag{2.4.34}$$

Since the functions s_1' and s_2' are continuous in (λ_0, β_1), equality (2.4.34) holds for any $\lambda \in (\lambda_0, \beta_1)$; by Condition *(S)* there holds $\sigma_1(A) = \sigma_2(A) = 0$. Since β_1 and β_2 are arbitrary, it follows that the support of σ consists at most of two points, namely $\{\lambda_0\}$ and $\{\varphi(\alpha)\}$. Finally, by means of Condition (*S*) again, the latter possibility is ruled out (see [Sm]).

Step 3. Let us conclude the proof: in view of *Steps 1-2* and (2.4.26), for *a.e.* $(x,t) \in Q$ the measures $\sigma_{1(x,t)}$ and $\sigma_{2(x,t)}$ have the following form:

$$\sigma_{1(x,t)} = \begin{cases} \lambda(x,t)\,\delta_{v(x,t)} & \text{if } v(x,t) > 0 \\ \delta_0 & \text{if } v(x,t) = 0 \end{cases}$$

$$\sigma_{2(x,t)} = \begin{cases} \big(1 - \lambda(x,t)\big)\,\delta_{v(x,t)} & \text{if } v(x,t) > 0 \\ 0 & \text{if } v(x,t) = 0 \end{cases}$$

for some $\lambda \in L^\infty(Q)$, $\lambda \geq 0$ in Q. By (2.2.31)-(2.2.32) and equality (2.4.27) we obtain representation (2.2.42). Finally equality (2.2.41) is a consequence of (2.2.34) and (2.2.42). □

Proposition 2.4.4. *Let $v \in L^\infty(Q)$ be the weak* limit of the sequence $\{\varphi(u_x^{\epsilon,(r)})\}$ in $L^\infty(Q)$. Then, there exists a subsequence $\{\epsilon_j\} \subseteq \{\epsilon_k\}$, $\epsilon_j \equiv \epsilon_{k_j}$, such that there holds*

$$\varphi(u_x^{\epsilon_j,(r)}) \to v \qquad a.e.\ in\ Q. \tag{2.4.35}$$

Proof. Observe that (2.2.31), (2.2.32) and (2.2.42) imply that

$$|\varphi(u_x^{\epsilon_j,(r)})|^p \rightharpoonup |v|^p \quad \text{in } L^1(Q),$$

for any $1 < p < \infty$, namely also

$$\|\varphi(u_x^{\epsilon_j,(r)})\|_{L^p(Q)} \to \|v\|_{L^p(Q)}.$$

Hence

$$\varphi(u_x^{\epsilon_j,(r)}) \to v \quad \text{in } L^p(Q)$$

for any $1 \leq p < \infty$ (*e.g.*, see [B]) and this concludes the proof. $\square$

In the following theorem we prove a refinement "at fixed time" of the disintegration formula (2.2.42).

Theorem 2.4.5. *Let $\{\epsilon_j\} \subseteq \{\epsilon_k\}$ be the subsequence given by Proposition 2.4.4. For a.e. $t > 0$, let $\{\tau^t_{\epsilon_j}\}$ be the family of Young measures associated to the sequence $\{u_x^{\epsilon_j,(r)}(\cdot,t)\}$. Then there exists a set $F \subseteq (0,T)$ of Lebesgue measure $|F| = 0$ such that for any $t \in (0,T)\setminus F$ there exists a Young measure τ^t such that*

$$\tau^t_{\epsilon_j} \to \tau^t \quad \textit{narrowly in } \Omega \times \mathbb{R}\,. \tag{2.4.36}$$

Moreover, for a.e. $x \in \Omega$ the disintegration ν^t_x of τ^t is given by

$$\nu^t_x = \begin{cases} \lambda(x,t)\delta_{s_1(v(x,t))} + (1-\lambda(x,t))\delta_{s_2(v(x,t))} & \textit{if } v(x,t) > 0 \\ \delta_0 & \textit{if } v(x,t) = 0\,. \end{cases} \tag{2.4.37}$$

Here $v(\cdot,t)$ and $\lambda(\cdot,t)$ are the values at fixed t of the functions considered in (2.2.42).

Proof. In view of Proposition 2.4.4, there exists a set $F^1 \subseteq (0,T)$ of Lebesgue measure $|F^1| = 0$ such that:

$$\varphi(u_x^{\epsilon_j,(r)}(\cdot,t)) \to v(\cdot,t) \qquad a.e. \text{ in } \Omega, \tag{2.4.38}$$

for any $t \in (0,T) \setminus F^1$. For any $\epsilon_j > 0$ let $F^{\epsilon_j} \subseteq (0,T)$ be the set of zero Lebesgue-measure given by Proposition 2.3.17, such that the entropy inequalities (2.2.12) hold for any t_1, $t_2 \in (0,T) \setminus F^{\epsilon_j}$. Set

$$F^2 := \bigcup_{h\in\mathbb{N}} F^{\epsilon_j}\,, \qquad F := F^1 \cup F^2\,.$$

Thus, $F \subseteq (0,T)$ has Lebesgue measure $|F| = 0$.

For any $t \in (0,T) \setminus F$ there exists a subsequence $\{\epsilon_{j,t}\} \subseteq \{\epsilon_j\}$, such that

$$\chi_{\left\{0\leq u_x^{\epsilon_{j,t},(r)}(\cdot,t)\leq\alpha\right\}} \overset{*}{\rightharpoonup} \lambda^t \quad \text{in } L^\infty(\Omega) \tag{2.4.39}$$

for some $\lambda^t \in L^\infty(\Omega)$, $0 \leq \lambda^t \leq 1$.

Fix any $t \in (0,T) \setminus F$ and observe that for any $f \in C_c(\mathbb{R})$ we can write:

$$\begin{aligned} f(u_x^{\epsilon_{j,t},(r)}(\cdot,t)) &= (f\circ s_1 \circ\varphi)(u_x^{\epsilon_{j,t},(r)}(\cdot,t))\chi_{\left\{0\leq u_x^{\epsilon_{j,t},(r)}(\cdot,t)\leq\alpha\right\}} + \\ &\quad +(f\circ s_2\circ\varphi)(u_x^{\epsilon_{j,t},(r)}(\cdot,t))\chi_{\left\{u_x^{\epsilon_{j,t},(r)}(\cdot,t)>\alpha\right\}} \end{aligned} \tag{2.4.40}$$

$a.e.$ in Ω. In view of (2.4.38) and (2.4.39) we obtain:

$$f(u_x^{\epsilon_{j,t},(r)}(\cdot,t)) \overset{*}{\rightharpoonup} \lambda^t(\cdot)(f\circ s_1)(v(\cdot,t)) + \\ +(1-\lambda^t(\cdot))(f\circ s_2)(v(\cdot,t)) \quad \text{in } L^\infty(\Omega)\,. \tag{2.4.41}$$

This implies that for any $t\in(0,T)\setminus F$ the sequence $\left\{\tau^t_{\epsilon_{j,t}}\right\}$ of Young measures associated to the sequence $\{u^{\epsilon_{j,t}}(\cdot,t)\}$ converges narrowly to a Young measure τ^t over $\Omega\times\mathbb{R}$ whose disintegration $\nu^t_{(\cdot)}$ is of the form:

$$\nu^t_x = \begin{cases} \lambda^t(x)\delta_{s_1(v(x,t))} + (1-\lambda^t(x))\delta_{s_2(v(x,t))} & \text{if } v(x,t)>0 \\ \delta_0 & \text{if } v(x,t)=0 \end{cases} \tag{2.4.42}$$

for $a.e.$ $x\in\Omega$. Let us show that for $a.e.$ $x\in\Omega$ the coefficient $\lambda^t(x)$ is the value at fixed t of the function $\lambda(x,t)$, given by Theorem 2.2.7 - which implies that the *whole* sequence $\{\tau^{\epsilon_j}\}$ satisfies (2.4.36) and (2.4.37). To this purpose, fix any $\overline{g}\in C^1([0,\varphi(\alpha)])$, $\overline{g}'\geq 0$, $\overline{g}\equiv 0$ in $[0,S_{\overline{g}}]$ for some $S_{\overline{g}}>0$, and consider inequalities (2.2.12) with $g=\overline{g}$, namely:

$$\int_\Omega \overline{G}(u^\epsilon_x)(x,t_2)\zeta(x,t_2)dx - \int_\Omega \overline{G}(u_x^{\epsilon,(r)})(x,t_1)\zeta(x,t_1)dxdx \leq \\ \leq \int_{t_1}^{t_2}\int_\Omega \overline{G}(u_x^{\epsilon,(r)})\zeta_t dxdt - \int_{t_1}^{t_2}\int_\Omega \overline{g}(v^\epsilon)v^\epsilon_x\zeta_x\,dxdt \tag{2.4.43}$$

for any $t_1,\,t_2\in(0,T)\setminus F^\epsilon$, $t_1<t_2$, and for any $\zeta\in C^1([0,T];H^1_0(\Omega)\cap H^2(\Omega))$, $\zeta\geq 0$, $\zeta_{xx}\leq 0$. Here $\overline{G}$ is defined by (2.2.13) in correspondence of $\overline{g}$. Fix any $\overline{t}\in(0,T)\setminus F$ (so that (2.4.41)-(2.4.42) hold).

Then for any $f\in H^1_0(\Omega)\cap H^2(\Omega)$, $f\geq 0$, $f_{xx}\leq 0$ and for any $r>0$ set:

$$\zeta^r(x,t) = h^r(t)f(x)\,,$$

where

$$h^r(t) := \begin{cases} 0 & \text{if } |\,t-\overline{t}|\,>r, \\ \frac{1}{r}(t-\overline{t})+1 & \text{if } t\in[\overline{t}-r,\overline{t}], \\ -\frac{1}{r}(t-\overline{t})+1 & \text{if } t\in(\overline{t},\overline{t}+r]\,. \end{cases} \tag{2.4.44}$$

By standard arguments of approximation by smooth functions we can choose ζ^r as test function in inequalities (2.4.43) written for $t_1=\overline{t}-r$, $t_2=\overline{t}$ and $\epsilon=\epsilon_{j,\overline{t}}$. We obtain

$$\int_\Omega \overline{G}\big(u_x^{\epsilon_{j,\overline{t}},(r)}\big)(x,\overline{t})f(x)dx \leq \\ \leq \frac{1}{r}\int_{\overline{t}-r}^{\overline{t}}\int_\Omega \overline{G}\big(u_x^{\epsilon_{j,\overline{t}},(r)}\big)fdxdt - \int_{\overline{t}-r}^{\overline{t}}\int_\Omega h^r\overline{g}(v^{\epsilon_{j,\overline{t}}})v_x^{\epsilon_{j,\overline{t}}}f_xdxdt\,. \tag{2.4.45}$$

Let us take the limit as $\epsilon_{j,\overline{t}}\to 0$ in the above inequalities. In this direction, observe that by estimate (2.3.51), there holds

$$\|v^{\epsilon_{j,\overline{t}}} - \varphi(u_x^{\epsilon_{j,\overline{t}},(r)})\|_{L^2(Q)} \to 0,$$

hence

$$v^{\epsilon_{j,\bar{t}}} \to v \quad \text{in} \quad L^2(Q)$$

as $\epsilon_{j,\bar{t}} \to 0$ (here use of (2.4.35) has been made). Therefore, the limit as $\epsilon_{j,\bar{t}} \to 0$ in inequalities (2.4.45) gives

$$\int_\Omega \left[\lambda^{\bar{t}}(x)(\overline{G} \circ s_1)(v(x,\bar{t})) + (1-\lambda^{\bar{t}})(\overline{G} \circ s_2)(v(x,\bar{t}))\right] f(x)dx \leq$$

$$\leq \frac{1}{r}\int_{\bar{t}-r}^{\bar{t}} \int_\Omega \overline{G}^* f dx dt - \int_{\bar{t}-r}^{\bar{t}} \int_\Omega h^r \overline{g}(v) v_x f_x \tag{2.4.46}$$

for any $f \in H_0^1(\Omega) \cap H^2(\Omega)$, $f \geq 0$, $f_{xx} \leq 0$, where

$$\overline{G}^* = \begin{cases} \lambda \overline{G}(s_1(v)) + (1-\lambda)\overline{G}(s_2(v)) & \text{if } v > 0 \\ \overline{G}(0) \equiv 0 & \text{if } v = 0 \end{cases} \tag{2.4.47}$$

(here use of (2.2.42), (2.4.41), Remark 2.4.1 and Proposition 2.2.4 has been made). We can assume that for any $f \in H_0^1(\Omega) \cap H^2(\Omega)$, $f \geq 0$, $f_{xx} < 0$, we have:

$$\lim_{r \to 0} \frac{1}{r} \int_{\bar{t}-r}^{\bar{t}} \int_\Omega \overline{G}^*(x,t) f(x)\, dx dt = \int_\Omega \overline{G}^*(x,\bar{t}) f(x)\, dx\,,$$

$$\lim_{r \to 0} \frac{1}{r} \int_{\bar{t}}^{\bar{t}+r} \int_\Omega \overline{G}^*(x,t) f(x)\, dx dt = \int_\Omega \overline{G}^*(x,\bar{t}) f(x)\, dx\,,$$

for *a.e.* $\bar{t} \in (0,T) \setminus F$. Then we take the limit as $r \to 0$ in (2.4.46) and obtain:

$$\int_\Omega [\lambda^{\bar{t}}(x)(\overline{G} \circ s_1)(v(x,\bar{t})) + (1-\lambda^{\bar{t}})\overline{G} \circ s_2)(v(x,\bar{t}))] f(x)dx \leq$$

$$\leq \int_\Omega \overline{G}^*(x,\bar{t}) f(x) dx$$

for any f as above. By analogous arguments also the reverse inequality can be proven, therefore we have:

$$\lambda^{\bar{t}}(x)\overline{G}(s_1(v(x,\bar{t}))) + (1-\lambda^{\bar{t}})\overline{G}(s_2(v(x,\bar{t}))) = \overline{G}^*(x,\bar{t})$$

for *a.e.* $x \in \Omega$. In view of (2.4.47) the above equality gives

$$\lambda^{\bar{t}}(x) = \lambda(x,\bar{t})$$

for *a.e.* $x \in \Omega$ and for any $\bar{t} \in (0,T) \setminus F$, thus the conclusion follows. □

As a consequence of the above theorem, for any $t \in (0,T) \setminus F$, where $F \subseteq (0,T)$, $|F| = 0$ is the set given by Theorem 2.4.5, there holds:

$$\nu_x^t = \nu_{(x,t)} \quad \text{for } a.e.\ x \in \Omega\,, \tag{2.4.48}$$

where $\nu^t_{(\cdot)}$ is defined by (2.4.37) in Theorem 2.4.5 and $\nu_{(\cdot,\cdot)}$ is the disintegration associated to the limiting Young measure τ over $Q \times \mathbb{R}$ given by Proposition 2.2.4 and (2.2.42). In view of Theorem 2.4.5 and by the general properties of Young measures, for any $t \in (0,T) \setminus F$ and for any $f \in C(\mathbb{R})$, such that the sequence $\left\{f(u_x^{\epsilon_j,(r)}(\cdot,t))\right\}$ is bounded in $L^1(\Omega)$ and equi-integrable, there holds:

$$f(u_x^{\epsilon_j,(r)}(\cdot,t)) \rightharpoonup f^{*,t}(\cdot) \quad \text{in } L^1(\Omega), \tag{2.4.49}$$

where

$$f^{*,t}(x) = \begin{cases} \left[\lambda f(s_1(v)) + (1-\lambda) f(s_2(v))\right](x,t) & \text{if } v(x,t) > 0, \\ f(0) & \text{if } v(x,t) = 0 \end{cases} \tag{2.4.50}$$

for *a.e.* $x \in \Omega$ (see [GMS], [V]). Finally, letting $\epsilon_j \to 0$ in the entropy inequalities (2.2.12) gives the following result.

Theorem 2.4.6. *For any $g \in C^1(\mathbb{R})$ let G be the function defined by* (2.2.13)*. Let $F \subseteq (0,T)$ be the set of zero Lebesgue-measure given by Theorem 2.4.5. Then for any $g \in C^1([0,\varphi(\alpha)])$, $g \equiv 0$ in $[0,S_g]$ for some $S_g > 0$ and $g' \geq 0$ there holds*

$$\begin{aligned} &\int_0^1 G^*(x,t_2)\zeta(x,t_2)dx - \int_\Omega G^*(x,t_1)\zeta(x,t_1)dx \leq \\ \leq\; &\int_{t_1}^{t_2}\int_\Omega \left[G^*\zeta_t - g(v)v_x\zeta_x\right](x,t)dxdt, \end{aligned} \tag{2.4.51}$$

for any $t_1 < t_2$, $t_1, t_2 \in (0,T) \setminus F$, and for any $\zeta \in C^1([0,T]; H^1_0(\Omega) \cap H^2(\Omega))$, $\zeta \geq 0$, $\zeta_{xx} \leq 0$. Here

$$G^* = \begin{cases} \lambda G(s_1(v)) + (1-\lambda) G(s_2(v)) & \text{if } v > 0 \\ 0 & \text{if } v = 0 \end{cases} \tag{2.4.52}$$

a.e. in Q.

Proof. Consider any $g \in C^1([0,\varphi(\alpha)])$, $g' \geq 0$, $g \equiv 0$ in $[0,S_g]$, for some $S_g > 0$. Let $\{\epsilon_j\}$ be the sequence given by Proposition 2.4.4. Observe that the family $\left\{G(u_x^{\epsilon_j,(r)})\right\}$ is bounded in $L^\infty(Q)$ (see Remark 2.4.1). Hence, in view of (2.2.31), (2.2.32) and (2.2.42) we have

$$G(u_x^{\epsilon_j,(r)}) \overset{*}{\rightharpoonup} G^* \quad \text{in } L^\infty(Q), \tag{2.4.53}$$

where G^* is defined by (2.4.52). Moreover, in view of Theorem 2.4.5 for any $t \in (0,T) \setminus F$ there holds

$$G(u_x^{\epsilon_j,(r)}(\cdot,t)) \overset{*}{\rightharpoonup} G^*(\cdot,t) \quad \text{in } L^\infty(\Omega) \tag{2.4.54}$$

(see (2.4.49) and (2.4.50)). Finally, by means of (2.3.51) we obtain

$$\|v^{\epsilon_j} - \varphi(u_x^{\epsilon_j,(r)})\|_{L^2(Q)} = \|\epsilon_j \psi(u_x^{\epsilon_j,(r)})_t\|_{L^2(Q)} \to 0 \tag{2.4.55}$$

as $j \to \infty$. Hence, in view of Proposition 2.4.4 there holds:

$$v^{\epsilon_j} \to v \quad \text{in } L^2(Q)\,. \tag{2.4.56}$$

By (2.2.25) and (2.4.53)-(2.4.56), passing to the limit with respect to $\epsilon_j \to 0$ in the entropy inequalities (2.2.12) gives (2.4.51) (see [MTT], [Pl1] for further details). □

2.5 Structure of u_x: Proofs

Proof of Theorem 2.2.8. Consider the sequence $\{g_n\} \subseteq C^1([0,\varphi(\alpha)])$, defined as follows

$$g_n(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2n] \\ 2ns - 1 & \text{if } s \in (1/2n, 1/n) \\ 1 & \text{if } s \in [1/n, \varphi(\alpha)]\,. \end{cases}$$

By standard arguments of regularization and approximation with smooth functions, we can write the entropy inequalities (2.4.51) for $g = g_n$. We obtain

$$\iint_Q [G_n^* \zeta_t - g_n(v) v_x \zeta_x] dxdt \geq 0, \tag{2.5.1}$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega) \cap H^2(\Omega)),\ \zeta \geq 0,\ \zeta_{xx} \leq 0,\ \zeta(\cdot,0) = \zeta(\cdot,T) = 0$ in Ω. Recall that

$$G_n^* = \begin{cases} \lambda \int_0^{s_1(v)} g_n(\varphi((s))ds + (1-\lambda) \int_0^{s_2(v)} g_n(\varphi((s))ds & \text{if } v > 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Thus, $G_n^* \leq Z \in L^1(Q)$ and $G_n^* \to Z$ *a.e.* in Q as $n \to \infty$ (because $g_n(s) \to 1$ for any $s \in (0, \varphi(\alpha))$). This implies that

$$\iint_Q G_n^* \zeta_t dxdt \to \iint_Q Z \zeta_t dxdt \tag{2.5.2}$$

as $n \to \infty$ for any ζ as above. Moreover, observe that

$$g_n(v) v_x = \left[\int_0^v g_n(s) ds \right]_x \tag{2.5.3}$$

and

$$\|g_n(v) v_x\|_{L^2(Q)} \leq \|v_x\|_{L^2(Q)}\,.$$

The above estimate implies that the sequence $\{g_n(v)v_x\}$ is weakly relatively compact in $L^2(Q)$. In view of (2.5.3) and since for *a.e.* $(x,t) \in Q$

$$\int_0^{v(x,t)} g_n(s)ds \to v$$

as $n \to \infty$, there holds

$$g_n(v)v_x \rightharpoonup v_x \quad \text{in } L^2(Q). \tag{2.5.4}$$

Using (2.5.2) and (2.5.4), passing to the limit as $n \to \infty$ in (2.5.1) gives (2.2.43). □

Proof of Theorem 2.2.9. There exists a measure $\lambda \in \mathcal{M}^+(0,T)$, and for λ-*a.e.* $t \in (0,T)$ a measure $\gamma_t \in \mathcal{M}^+(\Omega)$ such that:

(a) for any Borel set $E \subset Q$ there holds

$$\mu(E) = \int_0^T \gamma_t(E_t)\, d\lambda(t),$$

where $E_t := \{x \in \Omega \,|\, (x,t) \in E\}$;

(b) for any $f \in C_c(Q)$ there holds:

$$\iint_Q f\, d\mu = \int_0^T d\lambda(t) \int_\Omega f(x,t)\, d\gamma_t(x) \tag{2.5.5}$$

(this is a consequence of the more general Proposition 8 on p. 35 of [GMS], Vol. I). Moreover, since $\mu(Q) < \infty$, we can choose $\lambda(I) = \mu(\Omega \times I)$ for any $I \subset (0,T)$, and $\gamma_t(\Omega) = 1$ for $\lambda - a.e.\ t \in (0,T)$.

(i) Let us prove that the measure $\lambda \in \mathcal{M}^+(0,T)$ is absolutely continuous with respect to the Lebesgue measure. To this purpose, fix any $0 < t_0 < T$ and consider the interval $I_r := [t_0 - r, t_0 + r]$. Choose $r > 0$ such that $I_{2r} := [t_0 - 2r, t_0 + 2r] \subset (0,T)$. Then there exists $\eta_r \in C_c^1(I_{2r})$ such that $\eta \equiv 1$ in I_r, $0 \le \eta_r \le 1$, and $\operatorname{supp} \eta_r \subseteq I_{2r}$. Set

$$\tilde{\eta}_r(t) = \int_0^t \eta_r(s)ds - \int_0^{t_0+2r} \eta_r(s)ds. \tag{2.5.6}$$

Consider the family $\{u_\kappa^\epsilon\}$ of solutions to problem (P_κ^ϵ) and let v_κ^ϵ be the function defined by (2.3.4) for any ϵ, $\kappa > 0$. Recall that in the proof of Lemma 2.3.2 we have shown that $v_\kappa^\epsilon(\cdot,t) \in H_0^1(\Omega)$, $v^{\epsilon_\kappa}(\cdot,t) > 0$ in Ω for any $t \in (0,T)$. Hence, there holds:

$$v_{\kappa x}^\epsilon(1,t) < 0, \quad v_{\kappa x}^\epsilon(0,t) > 0 \tag{2.5.7}$$

for any $t \in (0,T)$. In view of assumption (A)-(v), (2.3.5) and (2.5.7), there holds

$$\int_{t_0-2r}^{t_0+2r}\int_\Omega u^\epsilon_{\kappa x}(x,t)\eta_r(t)dxdt = -\int_{t_0-2r}^{t_0+2r}\tilde{\eta}_r(t)\int_\Omega v^\epsilon_{\kappa xx}dxdt +$$
$$-\tilde{\eta}_r(0)\int_\Omega u'_{0,\kappa}dx \le 4r\int_\Omega u'_{0,\kappa}dx \tag{2.5.8}$$

(observe that $\tilde{\eta}_r(t) \le 0$ for any t and $|\tilde{\eta}(0)| \le 4r$). Passing to the limit in (2.5.8) first as $\kappa \to 0$, then as $\epsilon \to 0$ gives

$$\int_{t_0-r}^{t_0+r}\int_\Omega Z(x,t)dxdt + \int_{t_0-r}^{t_0+r}\int_\Omega d\mu \le 4r\|u'_0\|_{\mathcal{M}^+(\Omega)}\,. \tag{2.5.9}$$

In view of (2.5.5), the above inequality reads

$$\int_{t_0-r}^{t_0+r}d\lambda(t) = \int_{t_0-r}^{t_0+r}d\lambda(t)\int_\Omega d\gamma_t(x) \le \tag{2.5.10}$$
$$\le \quad 4r\|u'_0\|_{\mathcal{M}^+(\Omega)} - \int_{t_0-r}^{t_0+r}\int_\Omega Z(x,t)dxdt$$

(recall that $d\gamma_t$ is a probability measure for λ-*a.e.* $t \in (0,T)$). Thus, $d\lambda = h(t)dt$ for some $h \in L^1(0,T)$. On the other hand, $h \in L^\infty(0,T)$, since by (2.5.10) we have

$$h(t) \le 2\|u'_0\|_{\mathcal{M}^+(\Omega)} - \|Z(\cdot,t)\|_{L^1(\Omega)}$$

for *a.e.* $t > 0$ (recall that by assumption $u'_0 \in \mathcal{M}^+(\Omega)$ it follows $Z \ge 0$ *a.e.* in Q). Setting

$$\tilde{\gamma}_t := h(t)\gamma_t$$

for *a.e.* $t \in (0,T)$ gives claim (i).

(ii) By (2.2.40) and inequality (2.2.43) there holds

$$\langle \mu, \zeta_t\rangle \le 0, \tag{2.5.11}$$

for any $\zeta \in C^1_c([0,T];H^1_0(\Omega)\cap H^2(\Omega))$, $\zeta \ge 0$, $\zeta_{xx} \le 0$, $\zeta(\cdot,0) = \zeta(\cdot,T) = 0$ in Ω. Fix any $0 < t_1 < t_2$ and consider $\eta_r \in \mathrm{Lip}([0,\infty))$ defined as follows:

$$\eta_r(t) := \begin{cases} \frac{1}{r}(t - t_1 + \frac{r}{2}) & \text{if } t \in (t_1 - \frac{r}{2}, t_1 + \frac{r}{2}) \\ 1 & \text{if } t \in [t_1 + \frac{r}{2}, t_2 - \frac{r}{2}] \\ -\frac{1}{r}(t - t_2 - \frac{r}{2}) & \text{if } t \in (t_2 - \frac{r}{2}, t_2 + \frac{r}{2}), \end{cases}$$

with $r > 0$ such that $[t_1 - \frac{r}{2}, t_2 + \frac{r}{2}] \subset (0,T)$. For any $\rho \in H^1_0(\Omega)\cap H^2(\Omega)$, $\rho \ge 0$, $\rho_{xx} \le 0$, choose $\psi^r(x,t) := \eta_r(t)\rho(x)$ as test function in inequality (2.5.11). In wiew of (2.2.44) we obtain:

$$\frac{1}{r}\int_{t_1-\frac{r}{2}}^{t_1+\frac{r}{2}}\langle\tilde{\gamma}_t,\rho\rangle\,dt \le \frac{1}{r}\int_{t_2-\frac{r}{2}}^{t_2+\frac{r}{2}}\langle\tilde{\gamma}_t,\rho\rangle\,dt,$$

whence as $r \to 0$ we get

$$\langle \tilde{\gamma}_{t_1}, \rho \rangle \le \langle \tilde{\gamma}_{t_2}, \rho \rangle \ .$$

□

To prove Theorem 2.2.10 we need some preliminary results. The first one is the following technical Lemma.

Lemma 2.5.1. *Let $f \in L^2((0,T); H^1(\Omega))$, where $\Omega \subseteq \mathbb{R}$ is a bounded interval. Then there exists a set $H \subseteq (0,T)$ of Lebesgue measure $|H| = 0$ such that for any $t_0 \in (0,T) \setminus H$ there holds:*

$$\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} f(x_0, t)\, dt = f(x_0, t_0) \tag{2.5.12}$$

for any $x_0 \in \Omega$.

Proof. Set $Q := \Omega \times (0,T)$. Since $f_x \in L^2(Q)$, there exists a set $H^1 \subseteq (0,T)$ of Lebesgue measure $|H^1| = 0$ such that for any $t_0 \in (0,T) \setminus H^1$ there holds $f(\cdot, t_0) \in H^1(\Omega)$ and

$$\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} dt \int_\Omega f_x^2(x,t) dx = \int_\Omega f_x^2(x, t_0)\, dx := C(t_0)\,. \tag{2.5.13}$$

On the other hand, we can find a dense and countable set $D \subseteq \Omega,\ D = \{x_k\}$ such that for any $x_k \in D$ the map

$$t \longmapsto f(x_k, t)$$

belongs to the space $L^1(0,T)$. Therefore for any $x_k \in D$ there exists a set $H^k \subseteq (0,T)$, $|H^k| = 0$, such that

$$\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} f(x_k, t) dt = f(x_k, t_0) \tag{2.5.14}$$

for any $t_0 \in (0,T) \setminus H^k$. Set:

$$H := H^1 \cup H^2\,, \quad H^2 := \left(\bigcup_{k \in \mathbb{N}} H^k \right) .$$

Fix any $t_0 \in (0,T) \setminus H$ and then fix any $x_0 \in \Omega$. Since D is dense and countable in Ω, for any $\varepsilon > 0$ there exists $x_0^\varepsilon \in D$ such that

$$|x_0 - x_0^\varepsilon|^{\frac{1}{2}} < \frac{\varepsilon}{6\sqrt{C(t_0)}} \tag{2.5.15}$$

(here $C(t_0) > 0$ is defined by (2.5.13)). Observe that

$$\begin{aligned}\frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0,t)-f(x_0,t_0)]\,dt &= \frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0,t)-f(x_0^\varepsilon,t)]\,dt+\\ &+\frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0^\varepsilon,t)-f(x_0^\varepsilon,t_0)]\,dt+\\ &+\frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0^\varepsilon,t_0)-f(x_0,t_0)]\,dt\,.\end{aligned}$$

Let us study the three term in the right-hand side of the above equality. In view of (2.5.15), we have:

$$\begin{aligned}&\left|\frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0,t)-f(x_0^\varepsilon,t)]\,dt\right| = \frac{1}{h}\left|\int_{t_0}^{t_0+h}dt\int_{x_0^\varepsilon}^{x_0}f_x(x,t)\,dx\right| \leq\\ &\leq \left(\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega f_x^2\,dxdt\right)^{\frac{1}{2}}|x_0-x_0^\varepsilon|^{\frac{1}{2}} \leq \frac{\varepsilon}{3}\,. \qquad (2.5.16)\end{aligned}$$

hor any $h \leq \overline{h}^1(\varepsilon, t_0)$ (here use of (2.5.13) has been made). Moreover,

$$\left|\frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0^\varepsilon,t)-f(x_0^\varepsilon,t_0)]\,dt\right| \leq \frac{\varepsilon}{3} \qquad (2.5.17)$$

for any $h \leq \overline{h}^2(\varepsilon, x_0, t_0)$ by (2.5.14) (recall that $x_0^\varepsilon \in D$). Finally, there holds:

$$\begin{aligned}&\frac{1}{h}\left|\int_{t_0}^{t_0+h}[f(x_0^\varepsilon,t)-f(x_0^\varepsilon,t_0)]\,dt\right| = |f(x_0^\varepsilon,t)-f(x_0^\varepsilon,t_0)| \leq\\ &\leq \left(\int_\Omega f_x^2(x,t_0)\,dx\right)^{\frac{1}{2}}|x_0-x_0^\varepsilon|^{\frac{1}{2}} \leq \frac{\varepsilon}{3} \qquad (2.5.18)\end{aligned}$$

the last inequality being a consequence of (2.5.13) and (2.5.15). Set

$$\overline{h}(\varepsilon,x_0,t_0) = \min\left\{\overline{h}^1(\varepsilon,t_0),\overline{h}^2(\varepsilon,x_0,t_0)\right\}\,.$$

In view of (2.5.16)-(2.5.18), there holds

$$\frac{1}{h}\int_{t_0}^{t_0+h}[f(x_0,t)-f(x_0,t_0)]\,dt < \varepsilon$$

hor any $h \leq \overline{h}(\varepsilon, x_0, t_0)$. This concludes the proof. □

Next, arguing as in the proof of Theorem 2.2.9, we can decompose the positive Radon measure $u_x^{\epsilon,(s)}$ in the following way

$$< u_x^{\epsilon,(s)}, \phi > = \int_0^\infty \langle \tilde{\gamma}_t^\epsilon, \phi(\cdot,t)\rangle\;dt \qquad (2.5.19)$$

for any $\phi \in C_c(Q)$, for some $\tilde{\gamma}_t^\epsilon \in \mathcal{M}^+(\Omega)$ defined for *a.e.* $t \in (0, T)$.

Remark 2.5.1. *Observe that by* (2.2.3) *there holds:*

$$\int_0^T h_t(t)\left\{\left(\int_\Omega u_x^{\epsilon,(r)}(x,t)\phi(x)dx\right)+\langle\tilde{\gamma}_t^\epsilon,\phi\rangle\right\}dt =$$
$$= \int_0^T h(t)dt\int_\Omega v_x^\epsilon(x,t)\phi_x(x)dx \tag{2.5.20}$$

for any $\phi \in C_c^1(\Omega)$ *and* $h \in C_c^1(0,T)$*. Since by* (2.2.19) *the map*

$$t \longmapsto \int_\Omega v_x^\epsilon(x,t)\phi_x(x)dx$$

belongs to the space $L^2(0,T)$ *for any* $\epsilon > 0$*, it follows that the function*

$$t \longmapsto \left(\int_\Omega u_x^{\epsilon,(r)}(x,t)\phi(x)dx\right)+\langle\tilde{\gamma}_t^\epsilon,\phi\rangle$$

belongs to $H^1(0,T) \subseteq C([0,T])$ *for any* $\phi \in C_c^1(\Omega)$*.*

The following lemma holds.

Lemma 2.5.2. *For any* $\epsilon > 0$ *there exists a set* $H^\epsilon \subset (0,T)$*, of zero Lebesgue measure such that:*
(i) for any $t \in (0,T) \setminus H^\epsilon$*,* $v^\epsilon(\cdot,t) \in H_0^1(\Omega)$ *(here* v^ϵ *is defined by* (2.2.10)*);*
(ii) for any $\epsilon > 0$*,* $t \in (0,T) \setminus H^\epsilon$ *and for any* $\delta > 0$*, set*

$$B_\delta^\epsilon(t) := \left\{x \in \overline{\Omega} \mid v^\epsilon(x,t) \geq \delta\right\}. \tag{2.5.21}$$

Then for any $\epsilon > 0$ *there holds*

$$\operatorname{supp}\tilde{\gamma}_t^\epsilon \cap B_\delta^\epsilon(t) = \emptyset\,. \tag{2.5.22}$$

Proof. (*i*) Since $v^\epsilon \in L^2((0,T);H_0^1(\Omega))$ (see (2.2.19)), it follows that, for any $\epsilon > 0$, there exists a set $H^{(1,\epsilon)} \subset (0,T)$ of Lebesgue measure $|H^{(1,\epsilon)}| = 0$, such that $v^\epsilon(\cdot,t) \in H_0^1(\Omega)$ for any $t \in (0,T) \setminus H^{(1,\epsilon)}$. This gives claim (*i*). Moreover, since $\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t \in L^2((0,T);H_0^1(\Omega))$, we can find a set $H^{(2,\epsilon)} \subseteq (0,T)$ of Lebesgue-measure $|H^{(2,\epsilon)}| = 0$ such that for any $t \in (0,T) \setminus H^{(2,\epsilon)}$ there holds $\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t(\cdot,t) \in H_0^1(\Omega) \subseteq C(\overline{\Omega})$ and:

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h}\left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t(x_0,t)dt = \left[\psi\left(u_x^{\epsilon,(r)}\right)\right]_t(x_0,t_0) \tag{2.5.23}$$

for any $x_0 \in \Omega$ (see Proposition 2.5.1). Set

$$H^\epsilon := H^{(1,\epsilon)} \cup H^{(2,\epsilon)}\,.$$

(ii) Fix any $t_0 \in (0,T) \setminus H^\epsilon$ and for any $\delta > 0$ let $B^\epsilon_\delta(t_0) \subset \Omega$ be the set defined by (2.5.21) in correspondence of t_0. In view of Theorem 2.2.3 and decomposition (2.5.19), there holds

$$\operatorname{supp} \tilde{\gamma}^\epsilon_t \equiv \left\{ x \in \Omega \mid \psi(u_x^{\epsilon,(r)})(x,t) = \gamma \right\}.$$

Fix any $\epsilon > 0$ and suppose that there exist $t_0 \in (0,T) \setminus H^\epsilon$ and $x_0 \in \Omega$ such that

$$B^\epsilon_\delta(t_0) \cap \left\{ x \in \Omega \mid \psi(u_x^{\epsilon,(r)})(x,t_0) = \gamma \right\} \supseteq \{x_0\}. \tag{2.5.24}$$

Let $I_r(x_0)$ denote the interval centered at x_0 and length r. We have:

$$\begin{aligned} &\frac{1}{r}\int_{I_r(x_0)} \psi(u_x^{\epsilon,(r)})(x,t_0+h)dx - \frac{1}{r}\int_{I_r(x_0)} \psi(u_x^{\epsilon,(r)})(x,t_0)dx = \\ = \ &\frac{1}{r}\int_{t_0}^{t_0+h}\int_{I_r(x_0)} [\psi(u_x^{\epsilon,(r)})]_t(x,t)dxdt\,, \end{aligned}$$

hence in the limit as $r \to 0$,

$$\psi(u_x^{\epsilon,(r)})(x_0,t_0+h)dx - \psi(u_x^{\epsilon,(r)})(x_0,t_0)dx = \int_{t_0}^{t_0+h} [\psi(u_x^{\epsilon,(r)})]_t(x_0,t)dxdt$$

(recall that $\psi\big(u_x^{\epsilon,(r)}\big) \in C(\overline{Q})$ and $\big[\psi\big(u_x^{\epsilon,(r)}(\cdot,t)\big)\big]_t \in C(\overline{\Omega})$ for *a.e.* $t \in (0,T)$). Observe that by (2.5.24) there holds $\psi\big(u_x^{\epsilon,(r)}\big)(x_0,t_0) = \gamma$. Therefore we have:

$$\varphi(u_x^{\epsilon,(r)})(x_0,t_0) = 0\,. \tag{2.5.25}$$

Moreover, in [BBDU] it is proved that if $\psi(u_x^{\epsilon,(r)})(x_0,t_0) = \gamma$, then there holds $\psi(u_x^{\epsilon,(r)})(x_0,t_0+h) = \gamma$ for any $h > 0$. Therefore we obtain

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h} [\psi(u_x^{\epsilon,(r)})]_t(x_0,t)dt = 0\,,$$

namely:

$$[\psi(u_x^{\epsilon,(r)})]_t(x_0,t_0) = 0$$

(here use of (2.5.23) has been made). On the other hand, by our assumption $x_0 \in B^\epsilon_\delta(t_0)$ hence we have:

$$\begin{aligned} &\delta \le v^\epsilon(x_0,t_0) = \varphi(u_x^{\epsilon,(r)})(x_0,t_0) + \epsilon[\psi(u_x^{\epsilon,(r)})]_t(x_0,t_0) = \\ = \ &\epsilon[\psi(u_x^{\epsilon,(r)})]_t(x_0,t_0) = 0, \end{aligned}$$

(see also (2.5.25)) which gives a contradiction. □

Remark 2.5.2. *In view of Lemma 2.5.2, for any* $\epsilon > 0$ *and for any* $g \in C^1([0, \varphi(\alpha)])$, $g' \geq 0$, $g \equiv 0$ *in* $[0, S_g]$ *for some* $S_g > 0$, *there holds:*

$$g(v^\epsilon(\cdot,t))v^\epsilon_{xx}(\cdot,t) \equiv g(v^\epsilon(\cdot,t))\frac{\left[\psi(u_x^{\epsilon,(r)}(\cdot,t))\right]_t}{\psi'(u_x^{\epsilon,(r)}(\cdot,t))} \in L^1(\Omega)\,, \text{ and}$$

$$\int_\Omega g(\varphi(u_x^{\epsilon,(r)}(x,t)))\frac{\left[\psi(u_x^{\epsilon,(r)}(x,t))\right]_t}{\psi'(u_x^{\epsilon,(r)}(x,t))}\zeta(x,t)\,dx \leq$$

$$\leq \; -\int_\Omega g(v^\epsilon(x,t))v^\epsilon_x(x,t)\zeta_x(x,t)\,dx$$

for a.e. $t \in (0,T)$ *and for any* $\zeta \in C^1([0,T]; H^1_0(\Omega))$, $\zeta \geq 0$ *(by the same arguments used to prove (2.3.65) in Proposition 2.3.17). This implies:*

(i) the entropy inequalities (2.4.51) and inequalities (2.2.43) hold for any $\zeta \in C^1([0,T]; H^1_0(\Omega))$, $\zeta \geq 0$ *(see the proof of Theorems 2.2.8-2.4.6);*
(ii) for a.e. $t \in (0,T)$ *let* $\tilde{\gamma}_t \in \mathcal{M}^+(\Omega)$ *be the Radon measure given by Theorem 2.2.9. Then* $\langle \tilde{\gamma}_{t_1}, f\rangle \leq \langle \tilde{\gamma}_{t_2}, f\rangle$ *for a.e.* $t_1 \leq t_2$ *and for any* $f \in C^1_c(\Omega)$, $f \geq 0$.

Proposition 2.5.3. *Let* Z *be the function defined by* (2.2.34), (2.2.41) *and* $\tilde{\gamma}_t \in \mathcal{M}^+(\Omega)$ *be the Radon measure given by Theorem 2.2.9-(i) for a.e.* $t \in (0,T)$. *Let* $\{\epsilon_j\}$, $\epsilon_j \to 0$, *be the sequence given by Proposition 2.4.4. Then for any* $t \in (0,T)$ *there holds*

$$\int_\Omega u_x^{\epsilon_j,(r)}(x,t)\phi(x,t)dx + \left\langle \tilde{\gamma}_t^{\epsilon_j}, \phi\right\rangle \to \int_\Omega Z(x,t)\phi(x)dx + \langle \tilde{\gamma}_t, \phi\rangle \qquad (2.5.26)$$

for any $\phi \in C^1_c(\Omega)$, *as* $\epsilon_j \to 0$.

Proof. Fix any $\phi \in C^1_c(\Omega)$ and observe that the function

$$U^{\epsilon_j}_\phi(t) := \left(\int_\Omega u_x^{\epsilon_j,(r)}(x,t)\phi(x)dx\right) + \left\langle \tilde{\gamma}_t^{\epsilon_j}, \phi\right\rangle$$

belongs to the space $H^1(0,T)$ (see Remark 2.5.1). By (2.5.20) it follows that

$$U^{\epsilon_j}_\phi(t) = \frac{1}{t}\int_0^t \left(\int_\Omega u_x^{\epsilon,(r)}(x,s)\phi(x)\,dxds + \langle \tilde{\gamma}^\epsilon_s, \phi\rangle\right)ds +$$

$$- \quad \frac{1}{t}\int_0^t \int_\Omega s\, v^\epsilon_x(x,s)\phi_x(x)dxds\,,$$

hence estimates (2.2.16) and (2.2.19) give

$$\|U^{\epsilon_j}_\phi\|_{C([0,T])} \leq C$$

for some C independent of ϵ_j. Moreover, by (2.2.19) (see also Remark 2.5.1), we obtain

$$|U^{\epsilon_j}(t_2) - U^{\epsilon_j}(t_1)| \leq C_\phi \|v_x^{\epsilon_j}\|_{L^2(Q)} |t_2 - t_1|^{1/2} \leq C_\phi |t_1 - t_2|^{1/2},$$

where the constant C_ϕ does not depend on ϵ_j. Then the sequence $\left\{U_\phi^{\epsilon_j}\right\}$ is relatively compact in $C([0,T])$, and the conclusion follows. □

Proposition 2.5.4. *Let* $\{\epsilon_j\}$, $\epsilon_j \to 0$, *be the sequence given by Proposition 2.4.4. Then there exists a subset* $E^1 \subseteq (0,T)$ *of Lebesgue measure* $|E^1| = 0$, *with the following property: for any* $t \in (0,\infty)\setminus E^1$ *there exists a subsequence* $\{\epsilon_{j,t}\} \subseteq \{\epsilon_j\}$ *(depending on* t*) such that*

$$\int_\Omega \left\{ (v_x^{\epsilon_{j,t}})^2 + \epsilon_{j,t} \frac{[\psi(u_x^{\epsilon_{j,t},(r)})]_t^2}{\psi'(u_x^{\epsilon_{j,t},(r)})} \right\} (x,t)\, dx \leq C(t) < \infty\,, \tag{2.5.27}$$

$$v^{\epsilon_{j,t}}(\cdot,t) \to v(\cdot,t) \qquad \text{in } C(\overline{\Omega})\,. \tag{2.5.28}$$

Proof. In view of estimates (2.3.49), (2.3.51) and using the Fatou Lemma, we have

$$\int_0^T \liminf_{j\to\infty} \left(\int_\Omega \left[(v_x^{\epsilon_j})^2 + \epsilon_j \frac{\left[\psi(u_x^{\epsilon_j,(r)})\right]_t^2}{\psi'(u_x^{\epsilon_j,(r)})} \right] (x,t)\, dx \right) dt \leq C.$$

The above estimate implies that

$$\liminf_{j\to\infty} \left(\int_\Omega \left[(v_x^{\epsilon_j})^2 + \epsilon_j \frac{\left[\psi(u_x^{\epsilon_j,(r)})\right]_t^2}{\psi'(u_x^{\epsilon_j,(r)})} \right] (x,t)\, dx \right)$$

belongs to the space $L^1(0,T)$, hence there exists a set $\tilde{E}^1 \subset (0,T)$, $|\tilde{E}^1| = 0$, such that, for any $t \in (0,T)\setminus \tilde{E}^1$, claim (2.5.27) holds for some subsequence $\{\epsilon_{j,t}\} \subseteq \{\epsilon_j\}$, which depends on t.

Let $F \subseteq (0,T)$ be the set of zero Lebesgue measure given by Theorem 2.4.5 and set

$$E^1 := \tilde{E}^1 \cup F\,.$$

Now, fix any $t \in (0,T]\setminus E^1$ and observe that estimate (2.5.27) implies that the sequence $\{v^{\epsilon_{j,t}}(\cdot,t)\}$ is uniformly bounded in $C(\overline{\Omega})$ and equi-continuous. On the other hand, for any $t \in (0,T)\setminus E^1$ there holds $v^{\epsilon_j,t}(\cdot,t) \to v$ *a.e.* in Ω. In fact

$$v^{\epsilon_{j,t}}(\cdot,t) = \varphi(u_x^{\epsilon_{j,t},(r)})(\cdot,t) + \epsilon_{j,t}\big[\psi(u_x^{\epsilon_{j,t},(r)})\big]_t(\cdot,t)\,,$$

and by Proposition 2.4.4 and (2.5.27) we obtain

$$\varphi(u_x^{\epsilon_{j,t},(r)})(\cdot,t) \to v \quad \textit{a.e.} \text{ in } \Omega\,,$$

and

$$\epsilon_{j,t}\big[\psi(u_x^{\epsilon_{j,t},(r)})\big]_t(\cdot,t)\to 0 \quad a.e. \text{ in } \Omega$$

Therefore the whole sequence $\{v^{\epsilon_{j,t}}(\cdot,)\}$ converges uniformly in $\overline{\Omega}$, namely:

$$v^{\epsilon_{j,t}}(\cdot,t)\to v(\cdot,t) \qquad \text{in } C(\overline{\Omega})$$

and this concludes the proof. □

Now we can prove Theorem 2.2.10.

Proof of Theorem 2.2.10. For any $\epsilon_j > 0$ let $H^{\epsilon_j} \subseteq (0,T)$ be the set of zero Lebesgue measure given by Proposition 2.5.2. Finally, let $E^1 \subseteq (0,T)$, $|E^1| = 0$ be the set given by Proposition 2.5.4. Set:

$$E := E^1 \cup E^2, \quad E^2 := \left(\bigcup_j H^{\epsilon_j}\right).$$

Fix any $t \in (0,\infty)\setminus E$ and for any $\delta > 0$, set

$$B_\delta(t) := \big\{x \in \overline{\Omega} \,|\, v(x,t) \geq \delta\big\}; \tag{2.5.29}$$

consider the sequence $\{\epsilon_{j,t}\}$ given by Proposition 2.5.4, so that (2.5.27) and (2.5.28) hold. In view of the uniform convergence

$$v^{\epsilon_{j,t}}(\cdot,t)\to v(\cdot,t) \qquad \text{in } C(\overline{\Omega}),$$

it follows that

$$v^{\epsilon_{j,t}}(\cdot,t) \geq v(\cdot,t) - \frac{\delta}{2} \geq \frac{\delta}{2} \quad \text{in } B_\delta(t)$$

for any $\epsilon_{j,t}$ small enough. Therefore in view of Lemma 2.5.2 there holds:

$$B_\delta(t) \cap \operatorname{supp} \tilde{\gamma}_t^{\epsilon_{j,t}} = \emptyset$$

for any $\epsilon_{j,t}$ small enough. Moreover, by (2.5.27) and (2.5.28) we obtain:

$$\begin{aligned}
&\frac{\delta^2}{4}\int_{B_\delta(t)} (u_x^{\epsilon_{j,t},(r)})^2(x,t)dx \leq \int_{B_\delta(t)} \big[(u_x^{\epsilon_{j,t},(r)})^2 (v^{\epsilon_{j,t}})^2\big](x,t)dx \leq \\
\leq\ & 2\int_{B_\delta(t)} \Big[(u^{\epsilon_{j,t},(r)})^2 \varphi(u_x^{\epsilon_{j,t},(r)})^2\Big](x,t)dx \\
&+\epsilon_{j,t}\int_{B_\delta(t)} \left[(u_x^{\epsilon_{j,t},(r)})^2 \psi'(u_x^{\epsilon_{j,t},(r)}) \frac{\big[\psi(u_x^{\epsilon_{j,t},(r)})\big]_t^2}{\psi'(u_x^{\epsilon_{j,t},(r)})}\right](x,t)dx\,.
\end{aligned} \tag{2.5.30}$$

Observe that the assumption (H_1)-(i) implies that there exists a constant $C > 0$ such that:

$$\|u_x^{\epsilon_{j,t},(r)}(\cdot,t)\varphi(u_x^{\epsilon_{j,t},(r)})(\cdot,t)\|_{L^\infty(\Omega)} \leq C\,. \tag{2.5.31}$$

In view of assumption (H_4) and estimate (2.5.27), it follows that

$$\begin{aligned}
&\epsilon_{j,t}\int_\Omega (u_x^{\epsilon_{j,t},(r)})^2\psi'(u_x^{\epsilon_{j,t},(r)})\frac{\left[\psi(u_x^{\epsilon_{j,t},(r)})\right]_t^2}{\psi'(u_x^{\epsilon_{j,t},(r)})}dx \leq \qquad (2.5.32)\\
\leq\;& k_3\int_\Omega \epsilon_{j,t}\frac{\left[\psi(u_x^{\epsilon_{j,t},(r)})\right]_t^2}{\psi'(u_x^{\epsilon_{j,t},(r)})}dx \leq C(t)<\infty.
\end{aligned}$$

Estimates (2.5.30)-(2.5.32) imply that the sequence $\left\{u_x^{\epsilon_{j,t},(r)}(\cdot,t)\right\}$ is weakly relatively compact in $L^1(B_\delta(t))$, hence convergent to $Z(\cdot,t)$ in the weak topology of this space (here use of Theorem 2.4.5 has been made). In other words

$$\begin{aligned}
&\int_{B_\delta(t)} u_x^{\epsilon_{j,t},(r)}(x,t)\phi(x)dx+\int_{B_\delta(t)}\phi(x)d\tilde{\gamma}_t^{\epsilon_{j,t}} = \qquad (2.5.33)\\
=\;&\int_{B_\delta(t)} u_x^{\epsilon_{j,t},(r)}(x,t)\phi(x)dx \to \int_{B_\delta(t)} Z(x,t)\phi(x)dx,
\end{aligned}$$

for any $\phi\in C_c(\Omega)$. On the other hand, setting

$$B_\delta^a(t):=\{x\in\Omega\,|\;v(x,t)>\delta\}\subseteq B_\delta(t)$$

Proposition 2.5.3 gives

$$\begin{aligned}
&\lim_{\epsilon_{j,t}\to 0}\int_{B_\delta(t)} u_x^{\epsilon_{j,t},(r)}(x,t)\phi(x)dx+\int_{B_\delta(t)}\phi(x)d\tilde{\gamma}_t^{\epsilon_{j,t}} =\\
=\;&\int_{B_\delta(t)} Z(x,t)\phi(x)dx+\int_{B_\delta(t)}\phi(x)d\tilde{\gamma}_t,
\end{aligned}$$

for any $\phi\in C_c^1(B_\delta^a(t))$. Hence, in view of (2.5.33) we obtain:

$$\int_{B_\delta^a(t)}\phi(x)d\tilde{\gamma}_t=0,$$

for any $\phi\in C_c^1(B_\delta^a(t))$, for any $\delta>0$. This implies that $\tilde{\gamma}_t(B_\delta^a(t))=0$ for any $\delta>0$, namely:

$$\tilde{\gamma}_t(B(t))=0\,,\quad B(t)=\{x\in\Omega\,|\;v(x,t)>0\} \qquad (2.5.34)$$

(because $B(t)$ is an open set and the family $\{B_\delta^a(t)\}_\delta$ for $\delta=\frac{1}{n}$, $n\in\mathbb{N}$, is an increasing sequence of open sets such that $\cup_n B_{1/n}^a(t)=B(t)$). By (2.5.34) the claim follows. □

Chapter 3

Long-time behaviour of solutions to a forward-backward parabolic equation

3.1 Introduction

In this chapter we study the long-time behaviour of solutions to the quasilinear *forward-backward* parabolic problem

$$\begin{cases} u_t = [\phi(u)]_{xx} & \text{in } (0,1)\times(0,\infty) := Q_\infty \\ [\phi(u)]_x = 0 & \text{in } \{0,1\}\times(0,\infty) \\ u = u_0 & \text{in } (0,1)\times\{0\}\ . \end{cases} \tag{3.1.1}$$

Here $u_0 \in L^\infty(0,1)$ and $\phi \in C^2(\mathbb{R})$ is a nonmonotone, cubic-like function satisfying the following conditions:

$$(H) \quad \begin{cases} (i)\ \phi'(u) > 0 \quad \text{for } u < b \text{ and } u > c, \quad \phi'(u) < 0 \quad \text{for } b < u < c; \\ (ii)\ A := \phi(c) < \phi(b) =: B, \quad \phi(u) \to \pm\infty \text{ as } u \to \pm\infty; \\ (iii)\ \phi''(b) \neq 0,\ \phi''(c) \neq 0. \end{cases}$$

We also denote by $a \in (-\infty, b)$ and $d \in (c, \infty)$ the roots of the equation $\phi(u) = A$, respectively $\phi(u) = B$ (see Fig.3.1).

Problem (3.1.1) with a cubic-like ϕ arises in the theory of phase transitions (see below for the physical motivation of different choices of ϕ). In this context the function u represents the phase field, whose values characterize the difference between the two phases (*e.g.*, see [BS]). The half-lines $(-\infty, b)$ and (c, ∞) correspond to stable phases and the interval (b, c) to an unstable phase (*e.g.*, see [MTT]). Therefore

$$S_1 := \{(u, \phi(u)) |\ u \in (-\infty, b)\} \equiv \{(s_1(v), v) |\ v \in (-\infty, B)\}\,,$$

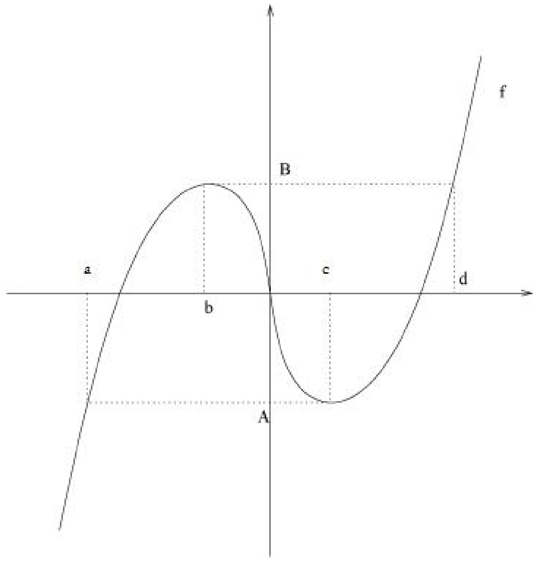

Figure 3.1: Assumption (H).

and

$$S_2 := \{(u, \phi(u)) |\, u \in (c, \infty)\} \equiv \{(s_2(v), v) |\, v \in (A, \infty)\}$$

are referred to as the stable branches, and

$$S_0 := \{(u, \phi(u)) |\, u \in (b, c)\} \equiv \{(s_0(v), v) |\, v \in (A, B)\}$$

as the unstable branch of the graph of ϕ. Beside (H), we always make the following assumption:

Condition (S): *The functions s_1', s_2' and s_0' are linearly independent on any open subset of the interval (A, B).*

In what follows, we always consider *weak entropy measure-valued solutions* to problem (3.1.1), whose existence and relevant properties were investigated in [Pl1] (see Definition 3.2.1). They are obtained as limiting points as $\varepsilon \to 0$ of the family $\{u^\varepsilon\}$ of solutions to the regularized equation

$$u_t = [\phi(u)]_{xx} + \varepsilon u_{xxt} \qquad (\varepsilon > 0)\,, \tag{3.1.2}$$

considered in the half-strip $(0,1)\times(0,\infty)$ with the same initial and boundary conditions as in (3.1.1). As proved in [NP], such solutions satisfy a family of *viscous entropy inequalities*, whose limit as $\varepsilon \to 0$ exists in a suitable sense ([Pl1]; see Section 3.2 below). In [NP] the long-time behaviour of the solution u^ε was studied for fixed $\varepsilon > 0$.

Let us mention that other regularizations of forward-backward equations, beside that considered in (3.1.2), have been used. The equation

$$w_t = [\phi(w_x)]_x \tag{3.1.3}$$

arises both in image reconstruction problems (as the one-dimensional version of the Perona-Malik equation; see [PM]), and as a mathematical model for heat transfer in a stably stratified turbulent shear flow in one space dimension (see [BBDU]). In these cases a typical choice of the function ϕ is $\phi(s) = \frac{As}{B+s^2}$ $(A, B > 0)$, or $\phi(s) = s\exp(-s)$. Observe that the transformation $u = w_x$ reduces equation (3.1.3) to the equation $u_t = [\phi(u)]_{xx}$. If $\phi(s) = s\exp(-s)$, the latter has been proposed as a mathematical model for aggregating populations (see [Pa]). Using the regularization (3.1.2), results analogous to those for problem (3.1.1) with a cubic-like ϕ have been proved in [Pa] for the case $\varepsilon > 0$, and in [Sm] for the limiting case $\varepsilon \to 0$. A different regularization of (3.1.3), namely

$$w_t = [\phi(w_x)]_x + \varepsilon[\chi(w_x)]_{xt} \tag{3.1.4}$$

was used in [BBDU]; here χ is a smooth nonlinear function, such that $\chi'(s) > 0$ for $s > 0$, $\chi(s) \to \gamma \in \mathbb{R}$ and $\chi'(s) \to 0$ as $s \to \infty$. In addition, the regularization of (3.1.3) leading to the fourth-order equation

$$w_t = [\phi(w_x)]_x - \kappa w_{xxxx} \qquad (\kappa > 0) \tag{3.1.5}$$

has been also investigated (see [BFG], [Sl]; observe that the change of unknown $u = w_x$ reduces equation (3.1.5) to the one-dimensional Cahn-Hilliard equation). While the regularizations (3.1.2), (3.1.4) take *time-delay* effects into account, (3.1.5) arises when considering *non-local spatial* effects. It is conceivable that both regularizations are physically meaningful (see [BFJ]), although the limiting dynamics of solutions expectedly depends on the regularization itself.

It was proved in [Sl] that measure-valued solutions of the Neumann initial-boundary value problem for equation (3.1.3) can be defined by taking a suitable limit as $\kappa \to 0$ of solutions to the corresponding problem for (3.1.5), in the same way as for $u_t = [\phi(u)]_{xx}$ letting $\varepsilon \to 0$ in (3.1.2) (however, such solutions do not seem to satisfy any entropy inequality). The long-time behaviour of such solutions was also studied, yet under assumptions on ϕ which are not satisfied if assumption (H) holds.

The chapter is organized as follows. In Section 3.2 we describe our results and the methods of proofs. Precise statements are given in Section 3.3 (see also Subsection 3.4.2). Sections 3.4 and 3.5 are essentially devoted to proofs.

3.2 Outline of results

Following [Pl1] (see also [EP], [MTT]) we give the following definition.

Definition 3.2.1. *By a weak entropy measure-valued solution of* (3.1.1) *in* Q_∞ *we mean any quintuple* u, λ_0, λ_1, $\lambda_2 \in L^\infty(Q_\infty)$, $v \in L^\infty(Q_\infty) \cap L^2((0,T); H^1(0,1))$ *for any* $T > 0$ *such that:*

(i) $\sum_{i=0}^{2} \lambda_i = 1,\ \lambda_i \geq 0$ *and there holds:*

$$u = \sum_{i=0}^{2} \lambda_i s_i(v) \tag{3.2.1}$$

with $\lambda_1 = 1$ *if* $v < A$, $\lambda_2 = 1$ *if* $v > B$*;*

(ii) for any $T > 0$*, set* $Q_T := (0,1) \times (0,T)$*; then for any* $T > 0$ *the couple* (u, v) *satisfies the equality*

$$\int\int_{Q_T} \{u\psi_t - v_x\psi_x\}\, dxdt + \int_0^1 u_0(x)\psi(x,0)dx = 0 \tag{3.2.2}$$

for any $\psi \in C^1(\overline{Q}_T),\ \psi(\cdot, T) = 0$ *in* $(0,1)$*;*

(iii) for any $g \in C^1(\mathbb{R})$*, set*

$$G(\lambda) := \int^{\lambda} g(\phi(s))ds. \tag{3.2.3}$$

Then, for any $T > 0$ *the entropy inequality*

$$\int\int_{Q_T} \{G^*\psi_t - g(v)v_x\psi_x - g'(v)v_x^2\psi\}\, dxdt + \\ + \int_0^1 G(u_0)\psi(x,0)dx \geq 0 \tag{3.2.4}$$

is satisfied for any $\psi \in C^1(\overline{Q}_T),\ \psi \geq 0,\ \psi(\cdot,T) = 0$ *in* $(0,1)$*, and* $g \in C^1(\mathbb{R}),\ g' \geq 0$.
Here, $G^* \in L^\infty(Q_\infty)$ *is defined by*

$$G^*(x,t) := \sum_{i=0}^{2} \lambda_i(x,t) G(s_i(v(x,t))) \tag{3.2.5}$$

for a.e. $(x,t) \in Q_\infty$*.*

Let us also make the following:

Definition 3.2.2. *By a steady state solution of* (3.1.1) *we mean any quintuple* $\overline{u},\ \lambda_0^*,\ \lambda_1^*,\ \lambda_2^* \in L^\infty(0,1),\ \overline{v} \in \mathbb{R}$ *such that* $0 \leq \lambda_i^* \leq 1,\ \sum_{i=0}^{2} \lambda_i^* = 1$ *and*

$$\overline{u} = \sum_{i=0}^{2} \lambda_i^* s_i(\overline{v}) \tag{3.2.6}$$

with $\lambda_1^* = 1$ *if* $\overline{v} < A$, $\lambda_2^* = 1$ *if* $\overline{v} > B$*. Observe that* $\overline{u}$ *is constant if* $\overline{v} < A,\ \overline{v} > B$*.*

In [Pl1] the existence of weak entropy measure-valued solutions of problem (3.1.1) was proved; let us briefly outline the proof for further reference.

Consider for any $\varepsilon > 0$ the *regularized* problem:

$$\begin{cases} u_t^\varepsilon = v_{xx}^\varepsilon & \text{in } Q_\infty \\ v_x^\varepsilon = 0 & \text{in } \{0,1\} \times (0,\infty) \\ u = u_0 & \text{in } (0,1) \times \{0\}, \end{cases} \tag{3.2.7}$$

where

$$v^\varepsilon := \phi(u^\varepsilon) + \varepsilon u_t^\varepsilon . \tag{3.2.8}$$

Global existence and uniqueness of the solution u^ε to problem (3.2.7) were proved in [NP].

Moreover, concerning the families $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ the following a priori estimates were proved to hold:

$$\| u^\varepsilon \|_{L^\infty(Q_\infty)} \leq C, \tag{3.2.9}$$

$$\| v^\varepsilon \|_{L^\infty(Q_\infty)} \leq C, \tag{3.2.10}$$

$$\| v_x^\varepsilon \|_{L^2(Q_\infty)} + \| \sqrt{\varepsilon} u_t^\varepsilon \|_{L^2(Q_\infty)} \leq C, \tag{3.2.11}$$

for some $C > 0$ independent of ε. The proof of the above estimates makes use of the equality

$$\begin{aligned} &\int_0^1 G(u^\varepsilon)(x,t_2)\psi(x,t_2)dx - \int_0^1 G(u^\varepsilon)(x,t_1)\psi(x,t_1)dx = \qquad (3.2.12)\\ = &\int_{t_1}^{t_2}\int_0^1 \psi_t G(u^\varepsilon)dxdt + \int_{t_1}^{t_2}\int_0^1 \psi[g(\phi(u^\varepsilon)) - g(v^\varepsilon)]\frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon}dxdt + \\ &- \int_{t_1}^{t_2}\int_0^1 g(v^\varepsilon)\psi_x v_x^\varepsilon dxdt - \int_{t_1}^{t_2}\int_0^1 \psi g'(v^\varepsilon)(v_x^\varepsilon)^2 dxdt, \end{aligned}$$

which holds for any $t_1 < t_2$, $\psi \in C^1(\overline{Q}_\infty)$, $g \in C^1(\mathbb{R})$ and G defined by (3.2.3). For any $T > 0$, choosing in (3.2.12) $\psi \in C^1(\overline{Q}_T)$, $\psi \geq 0$, $\psi(\cdot,T) = 0$ and $g' \geq 0$ also gives the so-called viscous entropy inequality

$$\begin{aligned} &\iint_{Q_T} \left\{G(u^\varepsilon)\psi_t - g(v^\varepsilon)v_x^\varepsilon\psi_x - g'(v^\varepsilon)(v_x^\varepsilon)^2\psi\right\} dxdt + \\ &+ \int_0^1 G(u_0)\psi(x,0)dx \geq 0, \qquad (3.2.13) \end{aligned}$$

which thus holds for any nondecreasing sufficiently regular g.

Relying on estimates (3.2.9), (3.2.10) and (3.2.11), it was shown in [Pl1] that, eventually up to a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, in any cylinder Q_T the sequence $\{\tau^{\varepsilon_k}\}$ of Young measures associated to the functions u^{ε_k} converges in the *narrow topology* over $Q_T \times \mathbb{R}$ to a Young measure τ (*e.g.*, see [V]), whose disintegration $\nu_{(x,t)}$ is a superposition of the three Dirac masses concentrated on the branches $S_1,\ S_2,\ S_0$ of the graph of ϕ. In other words there exist $\lambda_i \in L^\infty(Q_\infty)\ \ (i=0,1,2),\ 0 \le \lambda_i \le 1,\ \sum_{i=0}^2 \lambda_i = 1$, such that there holds:

$$\nu_{(x,t)} = \sum_{i=0}^{2} \lambda_i(x,t)\delta_{s_i(v(x,t))}, \tag{3.2.14}$$

where $\lambda_1 = 1$ if $v < A$, $\lambda_2 = 1$ if $v > B$ and $v \in L^\infty(Q_\infty)$ is the limit of both the sequences $\{\phi(u^{\varepsilon_k})\}$ and $\{v^{\varepsilon_k}\}$ in the weak* topology of $L^\infty(Q_\infty)$ (see (3.2.8) and (3.2.11)). By the properties of the narrow convergence of Young measures, for any $f \in C(\mathbb{R})$ there holds:

$$f(u^{\varepsilon_k}) \overset{*}{\rightharpoonup} f^* \qquad \text{in } L^\infty(Q_\infty), \tag{3.2.15}$$

where

$$f^*(x,t) = \sum_{i=0}^{2} \lambda_i(x,t) f(s_i(v(x,t))) \tag{3.2.16}$$

(*e.g.*, see [GMS] and [V]). In particular, there holds $u^{\varepsilon_k} \overset{*}{\rightharpoonup} u$ in $L^\infty(Q_\infty)$, $u = \sum_{i=0}^2 \lambda_i s_i(v)$. Moreover, in view of estimate (3.2.11), we have $v \in L^2((0,T); H^1(0,1))$ and $v^{\varepsilon_k} \rightharpoonup v$ in $L^2((0,T); H^1(0,1))$ for any $T > 0$. Finally, passing to the limit as $\varepsilon_k \to 0$ in the weak formulation of problems (3.2.7) and in inequalities (3.2.13) gives equation (3.2.2) and the entropy inequalities (3.2.4), respectively.

This shows that *global* weak entropy measure-valued solutions of problem (3.1.1) do exist, hence it is meaningful to investigate their long-time behaviour.

The chapter is organized as follows:

(α) in Subsection 3.3.1 we claim that, for any weak entropy measure-valued solution (u, v) of problem (3.1.1), not necessarily obtained by means of the Sobolev regularization (3.1.2), there exists a set $F \subseteq (0,\infty)$ of Lebesgue measure $|F| = 0$ such that the following inequalities:

$$\int_0^1 G^*(x,t_1)\varphi(x)dx - \int_0^1 G^*(x,t_2)\varphi(x)dx \ge \tag{3.2.17}$$
$$\ge \int_{t_1}^{t_2}\int_0^1 [g(v)v_x\varphi_x + g'(v)v_x^2\varphi]dxdt$$

hold for any $t_1, t_2 \in (0,\infty) \setminus F,\ t_1 < t_2,\ \varphi \in C^1([0,1]),\ \varphi \ge 0$, and $g \in C^1(\mathbb{R}),\ g' \ge 0$ (see Theorem 3.3.1). Here the function G^* is defined by (3.2.3) and (3.2.5). In particular, choosing $\varphi \equiv 1$ in the above equalities gives

$$\int_0^\infty dt \int_0^1 v_x^2(x,t)dx \le C. \tag{3.2.18}$$

for some constant $C > 0$ (by setting $g(\lambda) = \lambda$); moreover,

$$\int_0^1 G^*(x, t_2)dx \leq \int_0^1 G^*(x, t_1)dx \qquad (3.2.19)$$

for any $t_1 \leq t_2$, $t_1, t_2 \in (0, \infty) \setminus F$ and for any *non-decreasing* g.

Inequalities (3.2.17)-(3.2.19) will play a crucial role in the study of the asymptotic behaviour in time of the solutions to problem (3.1.1).

In Subsection 3.4.2 we address the case of weak entropy measure-valued solutions (u, v) of problem (3.1.1) *obtained* as limiting points of the families $\{u^\varepsilon\}$, $\{v^\varepsilon\}$ of solutions to the regularized problems (3.2.7) (here for any $\varepsilon > 0$ the function v^ε is defined by (3.2.8)). As already remarked, the estimates and convergence results proved in [Pl1] in the limit $\varepsilon \to 0$ hold in the cylinder Q_∞, and do not give any information about the behaviour of the family $\{u^\varepsilon(\cdot, t)\}$ for fixed $t > 0$. In this connection, we claim (see Proposition 3.4.3 and Theorem 3.4.4) that there exists a subset $\tilde{F} \subseteq (0, \infty)$, of Lebesgue measure $|\tilde{F}| = 0$, such that for any $t \in (0, \infty) \setminus \tilde{F}$ the Young measures associated to the functions $u^\varepsilon(\cdot, t)$ (which are uniformly bounded in $L^\infty(0, 1)$) converge narrowly to a Young measure τ^t with disintegration

$$\nu_x^t = \sum_{i=0}^{2} \lambda_i(x, t)\delta_{s_i(v(x,t))} \qquad (3.2.20)$$

for *a.e.* $x \in (0, 1)$. Here $\lambda_i(\cdot, t)$, $v(\cdot, t)$ are the values at fixed t of the functions considered in (3.2.14).

(β) Then we proceed to investigate the long-time behaviour of any weak entropy measure-valued solution (u, v) to problem (3.1.1). In this direction, first we observe that in view of inequalities (3.2.18) the map

$$t \longmapsto \int_0^1 v_x^2(x, t)dx$$

belongs to the space $L^1(0, \infty)$, hence

$$\int_T^\infty dt \int_0^1 v_x^2(x, t)dx \to 0 \qquad \text{as } T \to \infty. \qquad (3.2.21)$$

In view of (3.2.21), in Theorem 3.3.5 we show that there exists a unique constant $\overline{v} \in \mathbb{R}$ such that for any diverging sequence $\{t_n\}$ there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and a set $E \subseteq (0, \infty)$ of Lebesgue measure $|E| = 0$, so that

$$v(\cdot, t + t_{n_k}) \to \overline{v} \quad \text{in } \; C([0, 1]) \qquad (3.2.22)$$

for any $t \in (0, \infty) \setminus E$. The value of $\overline{v}$ depends on the average M_{u_0} of the initial datum u_0 to problem (3.1.1),

$$M_{u_0} := \int_0^1 u_0(x)dx. \qquad (3.2.23)$$

In fact, as a consequence of the homogeneous Neumann boundary condition in (3.1.1), the following conservation law holds:

$$\int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx \qquad \text{for any } t > 0 \tag{3.2.24}$$

and using (3.2.24) we prove that

(i) $a \leq M_{u_0} \leq d$ if and only if $A \leq \overline{v} \leq B$;

(ii) if $M_{u_0} < a$ (respectively, $M_{u_0} > d$), then $\overline{v} = \phi(M_{u_0})$;

(see Fig.3.1). Observe that for $M_{u_0} < a$ and $M_{u_0} > d$ the constant $\overline{v}$ is uniquely determined by the initial datum u_0, precisely by its average over $(0,1)$ - namely, $\overline{v}$ does not change for any weak entropy solution of problem (3.1.1) with the same initial datum u_0. This is a remarkable feature, for no uniqueness of measure valued solutions to problem (3.1.1) is known. Unfortunately, we do not prove the same result for $a \leq M_{u_0} \leq d$: in this case we only deduce the uniqueness of the constant $\overline{v}$ for any given weak entropy measure-valued solution (u,v) of problem (3.1.1) - namely, the value of $\overline{v}$ might depend on the particular choice of the couple (u,v).

Concerning the long-time behaviour of $u(\cdot,t)$, we have to distinguish the cases $a \leq M_{u_0} \leq d$ and $M_{u_0} < a,\ M_{u_0} > d$.
In fact when $a \leq M_{u_0} \leq d$, we have to take into account the long-time behaviour of the coefficients λ_i. Precisely, for any $i = 0,1,2$ there exists a unique $\lambda_i^* \in L^\infty(0,1)$, $\lambda_i^* \geq 0$, $\sum_{i=0}^2 \lambda_i^* = 1$, such that for any diverging and non-decreasing sequence $\{t_n\}$ there holds

$$\lambda_i(x, t + t_{n_k}) \to \lambda_i^*(x) \qquad \text{for } a.e.\ x \in (0,1) \tag{3.2.25}$$

for any $t \in (0,\infty) \setminus E$, where $\{t_{n_k}\} \subseteq \{t_n\}$ and $E \subseteq (0,\infty)$ are respectively any subsequence and any set of zero Lebesgue-measure (whose existence is assured by Theorem 3.3.5) such that (3.2.22) holds (see Proposition 3.3.4 and Proposition 3.5.4). The coefficients λ_i^* are uniquely determined by any fixed weak entropy measure-valued solution (u,v) of problem (3.1.1), that is, they do not depend on the sequence $\{t_n\}$. Thus, in view of (3.2.22) and (3.2.25) we obtain:

$$u(\cdot, t + t_{n_k}) \to \overline{u} \qquad a.e. \text{ in } (0,1), \tag{3.2.26}$$

for any $t \in (0,\infty) \setminus E$, where

$$\overline{u} := \sum_{i=0}^{2} \lambda_i^* s_i(\overline{v}) \tag{3.2.27}$$

(see Theorem 3.3.6-(i)).

On the other hand, when $M_{u_0} < a$ (respectively $M_{u_0} > d$), by the uniform convergence $v(\cdot, t + t_n) \to \phi(M_{u_0})$, using standard arguments of positively invariant regions we show that there exists $T > 0$ so large that $v(\cdot, t) < A$ (respectively, $v(\cdot, t) > B$) in $(0, 1)$ for *a.e.* $t \geq T$ (see Theorem 3.3.5-(*ii*)). Thus, by (3.2.1) we have

$$u(\cdot, t) = s_1(v(\cdot, t)) \qquad \text{in } (0, 1) \tag{3.2.28}$$

(respectively, $u(\cdot, t) = s_2(v(\cdot, t))$ in $(0, 1)$) for *a.e.* $t \geq T$. Arguing as in the case $a \leq M_{u_0} \leq d$, for any diverging and non-decreasing sequence $\{t_n\}$ we denote by $\{t_{n_k}\} \subseteq \{t_n\}$ and $E \subseteq (0, \infty)$ respectively any subsequence and any set of zero Lebesgue-measure such that (3.2.22) holds. In view of the above remarks, we have:

$$u(\cdot, t + t_{n_k}) \to M_{u_0} \quad \text{in } \; C([0, 1]) \tag{3.2.29}$$

for any $t \in (0, \infty) \setminus E$ (see Theorem 3.3.6-(*ii*)).

Then, given any weak entropy measure-valued solution (u, v) of problem (3.1.1) we wonder whether there exists the limit as $t \to \infty$, in some suitable topology, of the families $u(\cdot, t)$ and $v(\cdot, t)$. In fact, in view of the above remarks, for any non-decreasing sequence $\{t_n\}$, $t_n \to \infty$, there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and a set $E \subseteq (0, \infty)$, $|E| = 0$ such that $v(\cdot, t + t_{n_k}) \to \overline{v}$ in $C([0, 1])$, and $u(\cdot, t + t_{n_k}) \to \overline{u}$ *a.e.* in $(0, 1)$ or $u(\cdot, t + t_{n_k}) \to M_{u_0}$ uniformly in $[0, 1]$, only for $t \in \mathbb{R}^+ \setminus E$; observe that the set E, in general, depends on the sequence $\{t_n\}$. A natural question is the following: is it possible to prove that E is independent of the choice of $\{t_n\}$? In other words, we are interested in proving the existence of the limits

$$v(\cdot, t) \to \overline{v} \qquad \text{in } C([0, 1]), \tag{3.2.30}$$

$$u(\cdot, t) \to \overline{u} \qquad \textit{a.e.} \text{ in } (0, 1), \tag{3.2.31}$$

(in the case $a \leq M_{u_0} \leq d$) and

$$u(\cdot, t) \to M_{u_0} \qquad \text{in } \; C([0, 1]) \tag{3.2.32}$$

(in the cases $M_{u_0} < a$, $M_{u_0} > d$) as $t \to \infty$, $t \in \mathbb{R}^+ \setminus E^*$, for some $E^* \subseteq (0, \infty)$ of Lebesgue measure, $|\,E^*| = 0$. To address this point, for any $k \in \mathbb{N}$ consider the sets:

$$B_k := \left\{ t \in (0, \infty) | \; \int_0^1 v_x^2(x, t) dx < k \right\}, \tag{3.2.33}$$

and

$$A_k := \left\{ t \in (0, \infty) | \; \int_0^1 v_x^2(x, t) dx \geq k \right\} \equiv (0, \infty) \setminus B_k. \tag{3.2.34}$$

Then, $A_{k+1} \subseteq A_k$, and, in view of estimate (3.2.18),

$$| A_k| \leq \frac{C}{k} \to 0 \qquad \text{as } k \to \infty. \tag{3.2.35}$$

This implies that

$$A_\infty := \bigcap_{k=1}^{\infty} A_k \tag{3.2.36}$$

has Lebesgue measure $|A_\infty| = 0$, thus $E^* = A_\infty$ would be a natural choice. However, we can only prove a slightly weaker result, showing that for any $k > 0$, the limits (3.2.30)-(3.2.32) hold as $t \to \infty,\ t \in B_k$ (see Theorem 3.3.7). In other words, for any $\delta > 0$, we can find a set $A_{1/\delta}$ such that $|A_{1/\delta}| \leq \delta$, and convergences (3.2.30)-(3.2.32) hold for $t \to \infty,\ t \in (0, \infty) \setminus A_{1/\delta}$.

Finally, in view of Definition 3.2.2, the couple $(\overline{u}, \overline{v})$ (in the case $a \leq M_{u_0} \leq d$) and the couple $(M_{u_0}, \phi(M_{u_0}))$ (in the cases $M_{u_0} < a,\ M_{u_0} > d$) are steady state solutions of problem (3.1.1).

3.3 Mathematical frameworks and results

3.3.1 A priori estimates

The following theorem is a consequence of the entropy inequalities (3.2.4).

Theorem 3.3.1. *Let (u, v) be a weak entropy measure-valued solution of problem (3.1.1). Then there exists a set $F \subseteq (0, \infty)$ of Lebesgue measure $|F| = 0$ such that inequalities (3.2.17) hold for any $t_1,\ t_2 \in (0, \infty) \setminus F$.*

By Theorem 3.3.1 we obtain the following results.

Corollary 3.3.2. *Let (u, v) be a weak entropy measure-valued solution of problem (3.1.1). Then there exists a constant $C > 0$ such that estimate (3.2.18) holds.*

Corollary 3.3.3. *Let (u, v) be a weak entropy measure-valued solution of problem (3.1.1) and let F be the set given by Theorem 3.3.1. For any $g \in C^1(\mathbb{R})$, let G^* be the function defined by (3.2.5). Then there exists*

$$L_g := \lim_{\substack{t \to \infty \\ t \in (0,\infty) \setminus F}} \int_0^1 G^*(x, t)\, dx, \tag{3.3.1}$$

for any non-decreasing g.

Finally, we give a property of monotonicity in time of the coefficients $\lambda_i(x,t)$ for *a.e.* $x \in (0,1)$. Analogous results in this direction have been proved in [Pl1], showing that $\lambda_1(x,t)$ (repectively, $\lambda_2(x,t)$) is non-decreasing with respect to t in any cylinder of the form $I \times (t_1, t_2)$, $I \subseteq (0,1)$ whenever v is strictly less that B (respectively, strictly larger than A). However if the latter assumption is dropped, a weaker result of monotonicity is still valid. This is the content of the following proposition.

Proposition 3.3.4. *Let (u,v) be a weak entropy measure-valued solution of problem* (3.1.1)*. Let $t_1 < t_2 \in (0,\infty) \setminus F$ where F is the set of zero Lebesgue-measure given by Theorem 3.3.1. Then:*

(i) if $v(\cdot, t_j) \leq B^* < B$ in $(0,1)$, $j = 1,2$, we have

$$\lambda_1(x,t_2) \geq \lambda_1(x,t_1) \tag{3.3.2}$$

for a.e. $x \in (0,1)$;

(ii) if $v(\cdot, t_j) \geq A^* > A$ in $(0,1)$, $j = 1,2$, we have

$$\lambda_2(x,t_2) \geq \lambda_2(x,t_1) \tag{3.3.3}$$

for a.e. $x \in (0,1)$.

3.3.2 Large-time behaviour of weak entropy solutions

In what follows we denote by (u,v) a weak entropy measure-valued solution of problem (3.1.1). We begin by the following result, which is a consequence of estimate (3.2.18) and the conservation law (3.2.24).

Theorem 3.3.5. *Let (u,v) be a weak entropy measure-valued solution of problem* (3.1.1) *with initial datum u_0 and let M_{u_0} be defined by* (3.2.23)*. Then there exists a unique constant $\overline{v} \in \mathbb{R}$ such that for any diverging sequence $\{t_n\}$ there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and a set $E \subseteq (0,\infty)$ of Lebesgue measure $|E| = 0$, so that there holds*

$$v(\cdot, t + t_{n_k}) \to \overline{v} \quad \text{in} \quad C([0,1]) \tag{3.3.4}$$

for any $t \in (0,\infty) \setminus E$. Moreover,

(i) $a \leq M_{u_0} \leq d$ if and only if $A \leq \overline{v} \leq B$;

(ii) if $M_{u_0} < a$ or $M_{u_0} > d$, then

$$\overline{v} = \phi(M_{u_0}). \tag{3.3.5}$$

Finally, if $M_{u_0} < a$ (respectively, $M_{u_0} > d$), for any $\epsilon > 0$ there exists $T > 0$ such that $v(\cdot,t) \leq A - \epsilon$ (respectively, $v(\cdot,t) \geq B + \epsilon$) in $[0,1]$ for any $t \in (T,\infty) \setminus F$. Here F is the set of zero Lebesgue-measure given by Theorem 3.3.1.

Remark 3.3.1. *The set $E \subseteq (0,\infty)$ of zero Lebesgue-measure given by Theorem 3.3.5 in correspondence of any diverging sequence $\{t_n\}$ depends on the sequence itself.*

Next, for any diverging sequence $\{t_n\}$ and for *a.e.* $t > 0$, consider the sequence $\{u(\cdot, t+t_n)\}$, where

$$u(x,t+t_n) = \sum_{i=0}^{2} \lambda_i(x,t+t_n) s_i(v(x,t+t_n)) \quad \text{for } \; a.e. \;\; x \in (0,1) \qquad (3.3.6)$$

(see (3.2.1)). In the following theorem we show that $u(\cdot, t+t_n)$ approaches for *a.e.* $t > 0$ a time-independent function $\overline{u}$, uniquely determined by the couple (u, v) itself.

Theorem 3.3.6. *Let (u, v) be a weak entropy measure-valued solution of problem* (3.1.1) *with initial datum u_0, let M_{u_0} be defined by* (3.2.23) *and let $\overline{v}$ be the constant given by Theorem 3.3.5. Then:*

(i) if $a \leq M_{u_0} \leq d$, for any $i = 0, 1, 2$ there exists a unique $\lambda_i^* \in L^\infty(0,1)$, $\lambda_i^* \geq 0$ and $\sum_{i=0}^{2} \lambda_i^* = 1$ such that for any diverging and non-decreasing sequence $\{t_n\}$ there holds:

$$u(\cdot, t+t_{n_k}) \to \overline{u} := \sum_{i=0}^{2} \lambda_i^* s_i(\overline{v}) \qquad a.e. \; in \; (0,1) \qquad (3.3.7)$$

for any $t \in (0,\infty) \setminus E$, where $\{t_{n_k}\} \subseteq \{t_n\}$ and $E \subseteq (0,\infty)$ are respectively any subsequence and any set of zero Lebesgue-measure (whose existence is assured by Theorem 3.3.5) such that (3.3.4) *holds;*

(ii) if $M_{u_0} < a$ and $M_{u_0} > d$, for any diverging and non-decreasing sequence $\{t_n\}$ there holds:

$$u(\cdot, t+t_{n_k}) \to M_{u_0} \qquad in \;\; C([0,1]) \qquad (3.3.8)$$

for any $t \in (0,\infty) \setminus E$, where $\{t_{n_k}\} \subseteq \{t_n\}$ and $E \subseteq (0,\infty)$ are respectively any subsequence and any set of zero Lebesgue-measure (whose existence is assured by Theorem 3.3.5) such that (3.3.4) *holds. Moreover, if $M_{u_0} < a$ (respectively, $M_{u_0} > d$) there exists $T > 0$ such that $u(\cdot, t) = s_1(v(\cdot, t))$ in* $(0,1)$ *(respectively, $u = s_2(v(\cdot,t))$ if $M_{u_0} > d$) for any $t \in (T, \infty) \setminus F$. Here F is the set of zero Lebesgue-measure given by Theorem 3.3.1.*

Observe that the coefficients λ_i^* given by Theorem 3.3.6 do not depend on the sequence $\{t_n\}$.

Theorem 3.3.5 and Theorem 3.3.6 address the asymptotic behaviour in time of $v(\cdot, t+t_n)$ and $u(\cdot, t+t_n)$ along any diverging sequence $\{t_n\}$ and for any

$t \in (0,\infty) \setminus E$, where E is a set of Lebesgue measure $|E| = 0$, possibly *depending* on the choice of $\{t_n\}$ itself. As stated in Section 3.2, we wonder whether we can refine the results of Theorem 3.3.5 and Theorem 3.3.6 finding a *fixed* set E^* of Lebesgue measure $|E^*| = 0$, such that

$$v(\cdot,t_n) \to \overline{v}\,, \quad u(\cdot,t_n) \to \overline{u} \;\; (\text{or } M_{u_0}) \tag{3.3.9}$$

in the respective topologies, for any sequence $\{t_n\} \subseteq (0,\infty) \setminus E^*$. A slightly weaker result in this direction is the content of the following theorem. Precisely, we show that convergences (3.3.9) hold only except for sets of arbitrarily small - albeit non-zero - Lebesgue measure.

Theorem 3.3.7. *Let* (u,v) *be a weak entropy measure-valued solution of problem* (3.1.1) *with initial datum* u_0*. For any* $k > 0$*, let* $B_k \subseteq \mathbb{R}^+$ *be the set defined by* (3.2.33)*. Let* M_{u_0} *be defined by* (3.2.23)*, let* $\overline{v}$ *be the constant given by Theorem 3.3.5 and let* λ_i^* *be the functions given by Theorem 3.3.6. Let* F *be the set given by Theorem 3.3.1. Then for any diverging and non-decreasing sequence* $\{t_n\} \subseteq B_k \setminus F$ *there holds:*

$$v(\cdot,t_n) \to \overline{v} \qquad \text{in } C([0,1]). \tag{3.3.10}$$

Moreover,

(i) if $a \leq M_{u_0} \leq d$*, then*

$$u(\cdot,t_n) \to \overline{u} \qquad \text{a.e. in } (0,1) \tag{3.3.11}$$

where $\overline{u} \in L^\infty(0,1)$ *is the function defined in* (3.3.7)*;*

(ii) if $M_{u_0} < a$ *or* $M_{u_0} > d$*, then*

$$u(\cdot,t_n) \to M_{u_0} \qquad \text{in } C([0,1]). \tag{3.3.12}$$

The couple $(\overline{u},\overline{v})$ in (3.3.10)-(3.3.11) (in the case $a \leq M_{u_0} \leq d$) and the couple $(M_{u_0}, \phi(M_{u_0}))$ (in the cases $M_{u_0} < a$, $M_{u_0} > d$) are steady state solutions of problem (3.1.1) (see Definition 3.2.2). The following theorem is an immediate consequence of Theorem 3.3.7.

Theorem 3.3.8. *Let* (u,v) *be a weak entropy measure-valued solution of problem* (3.1.1)*. For any* $k > 0$*, let* $B_k \subseteq \mathbb{R}^+$ *be the set defined by* (3.2.33)*. Then for any diverging and non-decreasing sequence* $\{t_n\} \subseteq B_k$ *the couple*

$$(u(\cdot,t_n), v(\cdot,t_n))$$

converges to a steady state solution of (3.1.1) *(in a sense made precise by Theorem 3.3.7).*

3.4 Proof of results of Subsection 3.3.1 and improved results on the Sobolev regularization

3.4.1 Proof of results of Subsection 3.3.1

The proof of Theorem 3.3.1 needs the following lemma.

Lemma 3.4.1. *There exists a set $E \subseteq Q_\infty$, of Lebesgue measure $|\,E| = 0$, such that for any $(x,t) \in Q_\infty \setminus E$ there holds:*

$$\frac{1}{r^2}\int_{I_r(t)}\int_{I_r(x)} |\, G^*(\xi,s) - G^*(x,t)|\, d\xi ds \to 0 \quad \textit{as } r \to 0, \tag{3.4.1}$$

where $G^* \in L^\infty(Q_\infty)$ is any function defined by (3.2.5) *for any $g \in C^1(\mathbb{R})$. Here $I_r(t)$, $I_r(x)$ denote the intervals of length r centered at $t > 0$ and $x \in (0,1)$, respectively.*

Remark 3.4.1. *The importance of Lemma 3.4.1 can be explained as follows. Since the function $G^* \in L^\infty(Q_\infty)$ for any $g \in C^1(\mathbb{R})$, there exists a set $E_{G^*} \subseteq Q_\infty$, $|\,E_{G^*}| = 0$, in general depending on G^*, such that* (3.4.1) *holds for any $t \in Q_\infty \setminus E_{G^*}$ (e.g., see [GMS]). The main result in our context is that we can find a set $E \subseteq Q_\infty$, $|\,E| = 0$, so that* (3.4.1) *is satisfied for any $t \in Q_\infty \setminus E$ and for any choice of the function G^* - namely, the set E is independent of G^*.*

Proof of Lemma 3.4.1. Since $v_x \in L^2_{loc}(Q_\infty)$ and $v,\ \lambda_i,\ s_i(v) \in L^\infty(Q_\infty)$ $(i = 0,1,2)$, there exists a set $E \subseteq Q_\infty$ of Lebesgue measure $|E| = 0$, such that there hold:

$$\frac{1}{r^2}\int_{I_r(t)}\int_{I_r(x)} |\, v_x(\xi,s) - v_x(x,t)|^2 d\xi ds \to 0, \tag{3.4.2}$$

$$\frac{1}{r^2}\int_{I_r(t)}\int_{I_r(x)} |\, v(\xi,s) - v(x,t)|^p d\xi ds \to 0, \tag{3.4.3}$$

$$\frac{1}{r^2}\int_{I_r(t)}\int_{I_r(x)} |\, s_i(v(\xi,s)) - s_i(v(x,t))|^p d\xi ds \to 0, \tag{3.4.4}$$

$$\frac{1}{r^2}\int_{I_r(t)}\int_{I_r(x)} |\, \lambda_i(\xi,s) - \lambda_i(x,t)|^p d\xi ds \to 0 \tag{3.4.5}$$

as $r \to 0$, for any $(x,t) \in Q_\infty \setminus E$ (*e.g.*, see [GMS]) and for any $p \in [1,\infty)$.

Thus, fix any $(x,t) \in Q_\infty \setminus E$, let G^* be the function defined by (3.2.3) and (3.2.5) for any $g \in C^1(\mathbb{R})$ and let $I_r^{G^*}$ denote the integral in (3.4.1). To begin with, observe that

$$I_r^{G^*} \le \frac{1}{r^2}\sum_{i=0}^{2}\int_{I_r(t)}\int_{I_r(x)} \Bigg| \lambda_i(\xi,s)\int_{s_i(v(x,t))}^{s_i(v(\xi,s))} g(\phi(\lambda))d\lambda +$$

$$+[\lambda_i(\xi,s) - \lambda_i(x,t)]\int^{s_i(v(x,t))} g(\phi(\lambda))d\lambda \Bigg| d\xi ds,$$

hence

$$I_r^{G^*} \leq \frac{1}{r^2}\sum_{i=0}^{2}\int_{I_r(t)}\int_{I_r(x)}\left|\lambda_i(\xi,s)\int_{s_i(v(x,t))}^{s_i(v(\xi,s))} g(\phi(\lambda))d\lambda\right| d\xi ds + \quad (3.4.6)$$

$$+\frac{1}{r^2}\sum_{i=0}^{2}\int_{I_r(t)}\int_{I_r(x)}|\lambda_i(\xi,s)-\lambda_i(x,t)|\left|\int^{s_i(v(x,t))} g(\phi(\lambda))d\lambda\right| d\xi ds.$$

In view of (3.4.5), the last integral in the right-hand side of (3.4.6) converges to zero as $r \to 0$. Finally, observe that

$$\left|\int_{s_i(v(x,t))}^{s_i(v(\xi,s))} |g(\phi(\lambda))|d\lambda\right| \leq \|g\|_{L^\infty(-C,C)}|s_i(v(\xi,s)) - s_i(v(x,t))|\,,$$

where C is chosen so that $\|v\|_{L^\infty(Q_\infty)} \leq C$. Therefore, by (3.4.4) passing to the limit as $r \to 0$ in the first term of the right-hand side of (3.4.6) gives

$$\frac{1}{r^2}\sum_{i=0}^{2}\int_{I_r(t)}\int_{I_r(x)}\lambda_i(\xi,s)\left|\int_{s_1(v(x,t))}^{s_i(v(\xi,s))}|g(\phi(\lambda))|d\lambda\right| d\xi ds \leq$$

$$\leq \quad \|g\|_{L^\infty(-C,C)}\sum_{i=0}^{2}\frac{1}{r^2}\int_{I_r(t)}\int_{I_r(x)}|\,s_i(v(\xi,s)) - s_i(v(x,t))|\,d\xi ds \to 0.$$

This concludes the proof. □

Lemma 3.4.2. *Let* (u,v) *be a weak entropy measure-valued solution of problem* (3.1.1) *and let* G^* *be the function defined by* (3.2.5) *for any* $g \in C^1(\mathbb{R})$*). Then there exists* $F \subseteq (0,\infty)$ *of Lebesgue measure* $|F| = 0$*, such that for any* $g \in C^1(\mathbb{R})$*,* $g' \geq 0$*, there holds*

$$n\int_{t-\frac{1}{n}}^{t}\int_0^1 G^*(\xi,s)\varphi(\xi)d\xi ds \to \int_0^1 G^*(\xi,t)\varphi(\xi)d\xi \qquad (3.4.7)$$

and

$$n\int_{t}^{t+\frac{1}{n}}\int_0^1 G^*(\xi,s)\varphi(\xi)d\xi ds \to \int_0^1 G^*(\xi,t)\varphi(\xi)d\xi \qquad (3.4.8)$$

as $n \to \infty$*, for any* $\varphi \in C^1([0,1])$*,* $\varphi \geq 0$*, and for any* $t \in (0,\infty)\setminus F$*.*

Proof. Let $E \subseteq Q_\infty$ be the set of zero Lebesgue-measure given by Lemma 3.4.1. There exists $F \subseteq (0,\infty)$, $|\,F| = 0$, such that for any $t \in (0,\infty)\setminus F$

$$E^t := \{x \in (0,1)|\ (x,t) \in E\} \subseteq (0,1) \qquad (3.4.9)$$

has Lebesgue measure $|\,E^t| = 0$.

Let us address (3.4.7) ((3.4.8) can be proved in an analogous way). Fix any $t \in (0,\infty) \setminus F$ and for any $n \in \mathbb{N}$ consider the function $\Gamma_n(\xi)$, $\xi \in (0,1)$, defined as follows:

$$\Gamma_n(\xi) := n \int_{t-\frac{1}{n}}^{t} G^*(\xi, s) ds. \tag{3.4.10}$$

Since $G^* \in L^\infty(Q_\infty)$, we have

$$\|\Gamma_n\|_{L^\infty(0,1)} \leq \|G^*\|_{L^\infty(Q_\infty)}$$

for any $n \in \mathbb{N}$. Thus, there exists $G^t \in L^\infty(0,1)$ such that, eventually up to a subsequence, there holds

$$\Gamma_n \overset{*}{\rightharpoonup} G^t \quad \text{in} \;\; L^\infty(0,1) \tag{3.4.11}$$

as $n \to \infty$.

For any $n > 0$ and $k > 0$, consider the functions:

$$h^{n,k}(s) := \begin{cases} h^n(s) = n(s-t)+1 & \text{if } s \in [t-\frac{1}{n}, t], \\ h^k(s) = -k(s-t)+1 & \text{if } s \in (t, t+\frac{1}{k}], \end{cases} \tag{3.4.12}$$

and, for *a.e.* $x \in (0,1)$,

$$\varphi^{x,k}(\xi) := \begin{cases} 0 & \text{if } |\,\xi - x| \; > \frac{1}{k}, \\ k^2(\xi - x) + k & \text{if } \xi \in [x - \frac{1}{k}, x], \\ -k^2(\xi - x) + k & \text{if } \xi \in (x, x+\frac{1}{k}]. \end{cases} \tag{3.4.13}$$

Denote by S_k the square

$$S_k := (x - \frac{1}{k}, x + \frac{1}{k}) \times (t, t + \frac{1}{k}).$$

Choosing

$$\psi^{n,k}(\xi, s) := h^{n,k}(s)\varphi^{x,k}(\xi) \tag{3.4.14}$$

as test function in the entropy inequalities (3.2.4) gives

$$\begin{aligned} & n \int_{t-\frac{1}{n}}^{t} \int_{x-\frac{1}{k}}^{x+\frac{1}{k}} G^*(\xi,s)\varphi^{x,k}(\xi) d\xi ds - k \int\int_{S_k} G^* \varphi^{x,k} d\xi ds \geq \\ \geq \; & \int_{t-\frac{1}{n}}^{t+\frac{1}{k}} \int_{x-\frac{1}{k}}^{x+\frac{1}{k}} g(v) \varphi^{x,k}_\xi v_\xi h^{n,k} d\xi ds \end{aligned} \tag{3.4.15}$$

for any $g \in C^1(\mathbb{R})$, $g' \geq 0$. In view of (3.4.11), taking the limit $n \to \infty$ in (3.4.15) gives

$$\begin{aligned} & \int_{x-\frac{1}{k}}^{x+\frac{1}{k}} G^t(\xi)\varphi^{x,k}(\xi) d\xi - k \int\int_{S_k} G^* \varphi^{x,k} d\xi ds \geq \\ \geq \; & \int\int_{S_k} g(v) \varphi^{x,k}_\xi v_\xi h^k d\xi ds. \end{aligned} \tag{3.4.16}$$

We study the limit $k \to \infty$ in the above inequality for any fixed $x \in (0,1) \setminus E^t$ (here, for any $t \in F$, $E^t \subseteq (0,1)$ is the set of zero Lebesgue-measure defined by (3.4.9) in correspondence of t). By Lemma 3.4.1 we have:

$$k \int\int_{S_k} G^* \varphi^{x,k} d\xi ds \to G^*(x,t) \tag{3.4.17}$$

as $k \to \infty$. Concerning the second term in the right-hand side of (3.4.16), there holds

$$\begin{aligned}\int\int_{S_k} g(v) h^k v_\xi \varphi_\xi^{x,k} d\xi ds &= k^2 \int_t^{t+\frac{1}{k}} \int_{x-\frac{1}{k}}^{x} h^k g(v) v_\xi d\xi ds + \\ &- k^2 \int_t^{t+\frac{1}{k}} \int_x^{x+\frac{1}{k}} h^k g(v) v_\xi d\xi ds,\end{aligned} \tag{3.4.18}$$

and the right-hand side of (3.4.18) converges to

$$\frac{g(v(x,t))v_x(x,t)}{2} - \frac{g(v(x,t))v_x(x,t)}{2} = 0$$

as $k \to \infty$ (here use of (3.4.2) and (3.4.3) has been made). Hence, (3.4.17)-(3.4.18) imply that for any $x \in (0,1) \setminus E^t$ (hence for *a.e.* $x \in (0,1)$) there holds:

$$\lim_{k\to\infty} \int_0^1 G^t(\xi) \varphi^{x,k}(\xi)\, d\xi \geq G^*(x,t)\,.$$

Since

$$G^t(x) = \lim_{k\to\infty} \int_0^1 G^t(\xi) \varphi^{x,k}(\xi) d\xi\,,$$

for *a.e.* $x \in (0,1)$, there holds:

$$G^t(x) \geq G^*(x,t) \tag{3.4.19}$$

for *a.e.* $x \in (0,1)$ and for any $g \in C^1(\mathbb{R})$, $g' \geq 0$ and G^* defined by (3.2.5). Let us prove the reverse inequality. To this purpose, for any $n > 0$ and $k > 0$, consider the functions:

$$z^{n,k}(s) := \begin{cases} k(s - t + \frac{1}{n}) + 1 & \text{if } s \in [t - \frac{1}{n} - \frac{1}{k}, t - \frac{1}{n}], \\ -n(s - t + \frac{1}{n}) + 1 & \text{if } s \in (t - \frac{1}{n}, t], \end{cases} \tag{3.4.20}$$

and, for *a.e.* $x \in (0,1)$,

$$\zeta^{x,k}(\xi) := \begin{cases} 0 & \text{if } |\,\xi - x| > \frac{1}{k}, \\ k^2(\xi - x) + k & \text{if } \xi \in [x - \frac{1}{k}, x], \\ -k^2(\xi - x) + k & \text{if } \xi \in (x, x + \frac{1}{k}]. \end{cases} \tag{3.4.21}$$

Choose

$$\Psi^{n,k}(\xi, s) := z^{n,k}(s) \zeta^{x,k}(\xi) \tag{3.4.22}$$

as test function in the entropy inequalities (3.2.4). We obtain

$$
\begin{aligned}
& k\int_{t-\frac{1}{n}-\frac{1}{k}}^{t-\frac{1}{n}}\int_{x-\frac{1}{k}}^{x+\frac{1}{k}} G^*(\xi,s)\zeta^{x,k}(\xi)d\xi ds - n\int_{t-\frac{1}{n}}^{t}\int_{x-\frac{1}{k}}^{x+\frac{1}{k}} G^*\zeta^{x,k}d\xi ds \geq \\
\geq\; & \int_{t-\frac{1}{n}-\frac{1}{k}}^{t}\int_{x-\frac{1}{k}}^{x+\frac{1}{k}} g(v)\zeta_\xi^{x,k}v_\xi z^{n,k}d\xi ds \qquad (3.4.23)
\end{aligned}
$$

for any $g \in C^1(\mathbb{R}),\ g' \geq 0$. Denote by S_k the square

$$
S_k := (x-\frac{1}{k}, x+\frac{1}{k}) \times (t-\frac{1}{k}, t).
$$

In view of (3.4.11), taking the limit $n \to \infty$ in (3.4.23) gives

$$
\begin{aligned}
& k\int\int_{S_k} G^*\zeta^{x,k}d\xi ds - \int_{x-\frac{1}{k}}^{x+\frac{1}{k}} G^t(\xi)\zeta^{x,k}(\xi)d\xi \geq \\
\geq\; & \int\int_{S_k} g(v)\zeta_\xi^{x,k}v_\xi \tilde{z}^k d\xi ds, \qquad (3.4.24)
\end{aligned}
$$

where $\tilde{z}^k(s) = k(s-t)+1$ for $s \in (t-\frac{1}{k}, t)$. Arguing as above, taking the limit $k \to \infty$ in (3.4.24) gives

$$
G^t(x) = \lim_{k\to\infty}\int_0^1 G^t(\xi)\zeta^{x,k}(\xi)\,d\xi \leq G^*(x,t) \qquad (3.4.25)
$$

for *a.e.* $x \in (0,1)$, for any $g \in C^1(\mathbb{R}),\ g' \geq 0$ and G^* defined by (3.2.5). Thus (3.4.7) follows. □

Proof of Theorem 3.3.1. Fix any $t_1,\ t_2 \in (0,\infty)\setminus F$, where F is the set given in Lemma 3.4.2. Suppose $t_1 < t_2$ and consider the function

$$
h^n(t) := \begin{cases} 0 & \text{if } t < t_1 - \frac{1}{n}, \\ n(t-t_1)+1 & \text{if } t \in [t_1-\frac{1}{n}, t_1], \\ 1 & \text{if } t \in (t_1, t_2), \\ n(t_2-t)+1 & \text{if } t \in [t_2, t_2+\frac{1}{n}], \\ 0 & \text{if } t > t_2+\frac{1}{n}. \end{cases} \qquad (3.4.26)
$$

For any choice of $\varphi \in C^1([0,1]),\ \varphi \geq 0$, choosing $\psi^n(x,t) := \varphi(x)h^n(t)$ as test function in the entropy inequalities (3.2.4) gives

$$
\begin{aligned}
& n\int_{t_1-\frac{1}{n}}^{t_1}\int_0^1 G^*(x,t)\varphi(x)dx - n\int_{t_2}^{t_2+\frac{1}{n}}\int_0^1 G^*(x,t)\varphi(x)dx \geq \\
\geq\; & \int_{t_1-\frac{1}{n}}^{t_2+\frac{1}{n}}\int_0^1 h^n[g(v)v_x\varphi_x + g'(v)v_x^2\varphi]dxdt \qquad (3.4.27)
\end{aligned}
$$

for any $g \in C^1(\mathbb{R}),\ g' \geq 0$ and G^* defined by (3.2.5). In view of (3.4.7)-(3.4.8) in Lemma 3.4.2, passing to the limit as $n \to \infty$ in (3.4.27) gives (3.2.17) and the claim follows. □

Proof of Corollary 3.3.2. Write inequalities (3.2.17) with $t_1 = 0$, $\varphi(\cdot) \equiv 1$ in $(0,1)$ and $g(s) = s$. We obtain:

$$\int_0^T \int_0^1 v_x^2 dxdt \leq \int_0^1 \left(\int_0^{u_0(x)} \phi(s)ds \right) dx +$$
$$- \sum_{i=0}^{2} \int_0^1 \lambda_i(x,T) \left(\int_0^{s_i(v(x,T))} \phi(s)ds \right) dx \leq C$$

for any $T \in (0,\infty) \setminus F$, since $v \in L^\infty(Q_\infty)$ (here F is the set given by Theorem 3.3.1). Taking the limit as $T \to \infty$ in the above inequality gives estimate (3.2.18). □

Proof of Corollary 3.3.3. Write inequalities (3.2.17) with $\varphi(\cdot) \equiv 1$ in $(0,1)$ and $g \in C^1(\mathbb{R}),\ g' \geq 0$. We obtain:

$$\int_0^1 G^*(x,t_1)dx - \int_0^1 G^*(x,t_2)dx \geq \int_{t_1}^{t_2} \int_0^1 g'(v)v_x^2 dxdt \geq 0$$

for any $t_1 < t_2 \in (0,\infty) \setminus F$, where F is the set given by Theorem 3.3.1 and G^* is the function defined by (3.2.5) in terms of g. The above inequality implies that the map

$$t \longmapsto \int_0^1 G^*(x,t)dx$$

is non-increasing in $(0,\infty) \setminus F$ for any $g \in C^1(\mathbb{R}),\ g' \geq 0$. By standard arguments of approximation with smooth functions, the assumption $g \in C^1(\mathbb{R})$ can be dropped. □

Proof of Proposition 3.3.4. Let $t_1 < t_2 \in (0,\infty) \setminus F$ and assume $v(\cdot,t_j) \leq B^* < B$ in $(0,1)$ for $j = 1,2$ (the case $v(\cdot,t_j) \geq A^* > A$ can be treated in an analogous way). Following [Pl1], for any $\rho > 0$ set

$$g_\rho(\lambda) := \begin{cases} 0 & \text{if } \lambda \leq B - \rho, \\ \rho^{-1/2} & \text{if } \lambda > B - \rho \end{cases} \tag{3.4.28}$$

and let G_ρ^* be the function defined by (3.2.5) in terms of G_ρ, where

$$G_\rho(\lambda) := \int_0^\lambda g_\rho(\phi(s))ds\,.$$

Since g_ρ is non-decreasing, using standard arguments of approximation with smooth functions, we can use it in inequality (3.2.17) and obtain

$$\int_0^1 G_\rho^*(x,t_1)\varphi(x)dx - \int_0^1 G_\rho^*(x,t_2)\varphi(x)dx \geq \qquad (3.4.29)$$
$$\geq \int_{t_1}^{t_2}\int_0^1 g_\rho(v)v_x\varphi_x dxdt$$

for any $\varphi \in C_c^\infty(0,1)$, $\varphi \geq 0$. For any ρ such that $B-\rho > B^*$, we have

$$\begin{aligned} G_\rho^*(x,t_j) &= \sum_{i=0}^{2}\lambda_i(x,t_j)\int_0^{s_i(v(x,t_j))} g_\rho(\phi(s))ds = \\ &= \lambda_1(x,t_j)\int_{s_0(B-\rho)}^{s_1(B-\rho)} \rho^{-1/2}ds \qquad (3.4.30)\end{aligned}$$

for $j = 1,2$ (here use of assumption $v(\cdot,t_j) \leq B^* < B$ has been made). On the other hand, since $\phi''(b) \neq 0$, we have

$$\int_{s_0(B-\rho)}^{s_1(B-\rho)} \rho^{-1/2}ds \to -C \qquad (3.4.31)$$

as $\rho \to 0$. Here $C > 0$ is a constant depending on the value $\phi''(b)$ (see also [Pl1]).

Moreover, using a standard argument of positively invariant regions (*e.g.*, see [NP] and [MTT]), it is easily seen that

$$v(\cdot,t) \leq B \qquad (3.4.32)$$

for *a.e.* $t \geq t_1$. Hence,

$$\begin{aligned} &\left|\int_{t_1}^{t_2}\int_0^1 g_\rho(v)v_x\varphi_x dxdt\right| = \qquad (3.4.33) \\ = &\left|\int_{t_1}^{t_2}\int_0^1 \varphi_{xx}\left(\int_0^v g_\rho(s)ds\right)dxdt\right| \leq \\ \leq &\ \rho^{-1/2}\int_{t_1}^{t_2}\int_{\{v(\cdot,t)>B-\rho\}} (v-B+\rho)|\varphi_{xx}|\,dxdt \leq \\ \leq &\ \rho^{1/2}\int_{t_1}^{t_2}\int_0^1 |\varphi_{xx}|\,dxdt \to 0 \end{aligned}$$

as $\rho \to 0$, the last inequality being a consequence of (3.4.32). Observe that (3.4.33) shows that the right-hand side in (3.4.29) converges to zero as $\rho \to 0$. Concerning the first member of (3.4.29), by (3.4.31) it easily seen that

$$\lim_{\rho\to 0}\int_0^1 [G_\rho^*(x,t_1) - G_\rho^*(x,t_2)]\varphi(x)dx = \qquad (3.4.34)$$
$$= -C\int_0^1 [\lambda_1(x,t_1) - \lambda_1(x,t_2)]\varphi(x)dx$$

for any $\varphi \in C_c^\infty(0,1),\ \varphi \geq 0$. Thus, by (3.4.33) and (3.4.34) passing to the limit as $\rho \to 0$ in (3.4.29) gives

$$\int_0^1 [\lambda_1(x,t_2) - \lambda_1(x,t_1)]\varphi(x)dx \geq 0 \tag{3.4.35}$$

for any $\varphi \in C_c^\infty(0,1),\ \varphi \geq 0$. This implies (3.3.2). □

3.4.2 More about the Sobolev regularization and the vanishing viscosity limit

Let (u, v) be a weak entropy measure-valued solution of problem (3.1.1) obtained as limiting point of the solutions $u^\varepsilon,\ v^\varepsilon$ to the regularized problems (3.2.7) (here for any $\varepsilon > 0$ the function v^ε is defined by (3.2.8)). Precisely, there exists a sequence $\{\varepsilon_k\},\ \varepsilon_k \to 0$ such that

$$\begin{aligned} &u^{\varepsilon_k} \overset{*}{\rightharpoonup} u = \sum_{i=0}^{2} \lambda_i s_i(v) \ \text{ in } \ L^\infty(Q_\infty), \\ &v^{\varepsilon_k},\, \phi(u^{\varepsilon_k}) \overset{*}{\rightharpoonup} v, \ \text{ in } \ L^\infty(Q_\infty), \\ &v_x^{\varepsilon_k} \rightharpoonup v_x \ \text{ in } \ L^2(Q_\infty). \end{aligned}$$

Moreover, we can assume:

$$\phi(u^{\varepsilon_k}) \to v \quad a.e. \text{ in } \ Q_\infty \tag{3.4.36}$$

(*e.g.*, see [Pl1]). The following proposition is a direct consequence of (3.4.36).

Proposition 3.4.3. *Let $v \in L^\infty(Q_\infty)$ be the limit of the sequence $\{\phi(u^{\varepsilon_k})\}$ in the weak* topology of $L^\infty(Q_\infty)$. For any $t > 0$, denote by $\{\tau^t_{\varepsilon_k}\}$ the sequence of the Young measures associated to the family $\{u^{\varepsilon_k}(\cdot,t)\}$. Then there exists $F_1 \subseteq (0,\infty),\ |F_1| = 0$ such that for any $t \in (0,\infty) \setminus F_1$ there exist a subsequence $\{\varepsilon_{k,t}\} \subseteq \{\varepsilon_k\}$ and a Young measure τ^t over $(0,1)\times\mathbb{R}$ so that:*

$$\tau^t_{\varepsilon_{k,t}} \to \tau^t \qquad \textit{narrowly.} \tag{3.4.37}$$

Moreover, for any $t \in (0,\infty)\setminus F_1$ there exist $\lambda_i^t \in L^\infty(0,1)$ $(i = 0,1,2)$, $0 \leq \lambda_i^t \leq 1$, $\sum_{i=0}^2 \lambda_i^t = 1$, such that the disintegration ν_x^t of τ^t is of the form

$$\nu_x^t = \sum_{i=0}^{2} \lambda_i^t(x)\delta_{s_i(v(x,t))}, \tag{3.4.38}$$

for a.e. $x \in (0,1)$, where $\lambda_1^t(x) = 1$ if $v(x,t) < A$ and $\lambda_2^t(x) = 1$ if $v(x,t) > B$.

Proof. In view of (3.4.36), there exists a set $F_1 \subseteq (0,\infty)$, $|F_1| = 0$, such that

$$\phi(u^{\varepsilon_k})(x,t) \to v(x,t) \qquad \text{for } a.e.\ x \in (0,1) \tag{3.4.39}$$

and for any $t \in (0,\infty) \setminus F_1$. Thus, for any $t \in (0,\infty) \setminus F_1$ the Young measures associated to the sequence $\{\phi(u^{\varepsilon_k})(\cdot,t)\}$ converge in the narrow topology over $(0,1) \times \mathbb{R}$ to a Young measure whose disintegration σ_x^t is the Dirac mass concentrated at the point $v(x,t)$ - namely

$$\sigma_x^t = \delta_{v(x,t)} \quad \text{for } a.e.\ x \in (0,1) \tag{3.4.40}$$

(see [GMS] and [V]). On the other hand, since $\| u^{\varepsilon_k}(\cdot,t)\|_{L^\infty(0,1)} \le C$, for any $t \in (0,\infty) \setminus F^1$ there exists a subsequence $\{\varepsilon_{k,t}\} \subseteq \{\varepsilon_k\}$ such that the Young measures associated to the sequence $\{u^{\varepsilon_{k,t}}(\cdot,t)\}$ converge to a Young measure τ^t in the narrow topology of $(0,1) \times \mathbb{R}$. For $a.e.$ $x \in (0,1)$ let ν_x^t denote the disintegration of the Young measure τ^t, at any fixed $t \in (0,\infty) \setminus F_1$.

Fix any $t \in (0,\infty) \setminus F_1$, consequently fix any $x \in (0,1)$, and write for simplicity

$$\sigma \equiv \sigma_x^t, \quad v(x,t) \equiv v \quad \text{and} \quad \nu \equiv \nu_x^t.$$

Arguing as in [Pl1] and using the general properties of the narrow convergence of Young measures (e.g., see [V]), for any $f \in C(\mathbb{R})$ there holds

$$f(v) = \int_{\mathbb{R}} f(\zeta) d\sigma(\zeta) = \int_{\mathbb{R}} (f \circ \phi)(\lambda) d\nu(\lambda), \tag{3.4.41}$$

the first equality in the above equation following by (3.4.40). Then decompose the measure σ in three measures σ_i $(i = 0,1,2)$, namely

$$\sigma = \sum_{i=0}^{2} \sigma_i,$$

where

$$\begin{aligned}
\int_{\mathbb{R}} f(\zeta) d\sigma_1(\zeta) &:= \int_{(-\infty,b)} (f \circ \phi)(\lambda) d\nu(\lambda)\,, \\
\int_{\mathbb{R}} f(\zeta) d\sigma_0(\zeta) &:= \int_{[b,c]} (f \circ \phi)(\lambda) d\nu(\lambda)\,, \\
\int_{\mathbb{R}} f(\zeta) d\sigma_2(\zeta) &:= \int_{(c,\infty)} (f \circ \phi)(\lambda) d\nu(\lambda)
\end{aligned}$$

for any $f \in C(\mathbb{R})$. Here b, c are defined as in Fig.3.1. Clearly, in view of (3.4.40) we easily obtain

$$\sigma_i = \lambda_i \delta_v \;\; (i = 0,1,2) \tag{3.4.42}$$

for some coefficients $0 \le \lambda_i \le 1$, such that $\sum_{i=0}^{2} \lambda_i = 1$. Here in general $\lambda_i = \lambda_i^t(x)$, hence for any fixed $t \in (0,\infty) \setminus F_1$, $\lambda_i^t \in L^\infty(0,1)$.

We can now conclude the proof, giving the characterization (3.4.38) of the measure ν. In fact, in view of (3.4.42) we easily obtain the following relation between the measures σ_i and ν,

$$\begin{aligned}
&\int_{\mathbb{R}} f(\lambda) d\nu(\lambda) = \int_{(-\infty,b)} (f \circ s_1 \circ \phi)(\lambda) d\nu(\lambda) + \\
&+ \int_{[b,c]} (f \circ s_0 \circ \phi)(\lambda) d\nu(\lambda) + \int_{(c,\infty)} (f \circ s_2 \circ \phi)(\lambda) d\nu(\lambda) = \\
= \ &\int_{\mathbb{R}} (f \circ s_1)(\zeta) d\sigma_1(\zeta) + \int_{\mathbb{R}} (f \circ s_0)(\zeta) d\sigma_0(\zeta) + \int_{\mathbb{R}} (f \circ s_2)(\zeta) d\sigma_2(\zeta) = \\
= \ &\lambda_1 f(s_1(v)) + \lambda_0 f(s_0(v)) + \lambda_2 f(s_2(v))
\end{aligned}$$

for any $f \in C(\mathbb{R})$. In other words, ν is an atomic measure concentrated on the three branches of the equation $v = \phi(u)$ and (3.4.38) follows. □

In [Pl1] it is proved that the sequence of the Young measures associated to the family $\{u^{\varepsilon_k}\}$ converges in the narrow topology of the Young measures on $Q_T \times \mathbb{R}$ to a measure τ whose disintegration $\nu_{(x,t)}$ is given by (3.2.14) (for any $T > 0$). Hence, a natural question is the following: is it possible to show that for *a.e.* $t > 0$ there holds $\nu^t_{(\cdot)} = \nu_{(\cdot\,,t)}$ *a.e.* in $(0,1)$? In this connection, in view of (3.2.14) and (3.4.38), it suffices to prove that for *a.e.* $t > 0$

$$\lambda_i^t(x) = \lambda_i(x,t) \qquad \text{if } A < v(x,t) < B, \tag{3.4.43}$$

$$\lambda_1^t(x) = \lambda_1(x,t) \qquad \text{if } v(x,t) = A \tag{3.4.44}$$

and

$$\lambda_2^t(x) = \lambda_2(x,t) \qquad \text{if } v(x,t) = B \tag{3.4.45}$$

for *a.e.* $x \in (0,1)$. In fact, observe that for $v(x,t) = A$ there holds

$$\begin{aligned}
\nu_x^t &= \lambda_1^t(x)\delta_{s_1(A)} + (1 - \lambda_1^t(x))\delta_{s_0(A)}, \\
\nu_{(x,t)} &= \lambda_1(x,t)\delta_{s_1(A)} + (1 - \lambda_1(x,t))\delta_{s_0(A)}
\end{aligned}$$

and for $v(x,t) = B$

$$\begin{aligned}
\nu_x^t &= \lambda_2^t(x)\delta_{s_2(B)} + (1 - \lambda_2^t(x))\delta_{s_0(B)}, \\
\nu_{(x,t)} &= \lambda_2(x,t)\delta_{s_2(B)} + (1 - \lambda_2(x,t))\delta_{s_0(B)}.
\end{aligned}$$

The proof of equalities (3.4.43)-(3.4.45) is the content of the following theorem.

Theorem 3.4.4. *There exists* $\tilde{F} \subseteq (0,\infty)$, $|\tilde{F}| = 0$, *such that for any* $t \in (0,\infty) \setminus \tilde{F}$ *equalities* (3.4.43)-(3.4.45) *hold.*

Proof. Let $F_1 \subseteq (0,\infty)$ be the set of zero Lebesgue-measure given by Proposition 3.4.3. Observe that, in view of Proposition 3.4.3 and using the general properties of the narrow convergence of Young measures (*e.g.*, see [GMS], [V]), for any $f \in C(\mathbb{R})$ and for any $t \in (0,\infty) \setminus F_1$ we have:

$$f(u^{\varepsilon_{k,t}}(\cdot,t)) \overset{*}{\rightharpoonup} f^t \quad \text{in} \quad L^\infty(0,1), \tag{3.4.46}$$

where $\{\varepsilon_{k,t}\} \subseteq \{\varepsilon_k\}$ is the subsequence given by Proposition 3.4.3 in correspondence of any $t \in (0,\infty) \setminus F^1$ and

$$f^t(x) = \sum_{i=0}^{2} \lambda_i^t(x) f(s_i(v(x,t))) \tag{3.4.47}$$

for *a.e.* $x \in (0,1)$. Here λ_i^t is the function given by Proposition 3.4.3 ($i = 0,1,2$).
Let $F \subseteq (0,\infty)$ be the set of zero Lebesgue measure given by Theorem 3.3.1 and set

$$\tilde{F} := F \cup F_1.$$

Clearly, $\tilde{F}$ has Lebesgue measure $|\tilde{F}| = 0$. Fix any $t \in (0,\infty) \setminus \tilde{F}$ and define

$$h^n(s) = n(s-t) + 1 \qquad \text{if } t - \frac{1}{n} \leq s \leq t.$$

Write the viscous equalities (3.2.12) for $t_1 = t - \frac{1}{n}$, $t_2 = t$, $\varepsilon = \varepsilon_{k,t}$ and test function

$$\psi^n(x,s) := h^n(s)\varphi(x),$$

for any $\varphi \in C^1([0,1])$, $\varphi \geq 0$. Moreover, assuming in (3.2.12) $g \in C^1(\mathbb{R})$ and $g' \geq 0$, we obtain:

$$\begin{aligned} \int_0^1 G(u^{\varepsilon_{k,t}}(x,t))\varphi(x)dx \leq n \int_{t-\frac{1}{n}}^{t} \int_0^1 G(u^{\varepsilon_{k,t}})\varphi \, dxds \, + \\ - \int_{t-\frac{1}{n}}^{t} \int_0^1 h^n(s) g(v^{\varepsilon_{k,t}}) v_x^{\varepsilon_{k,t}} \varphi_x dxds, \end{aligned} \tag{3.4.48}$$

where G is the function defined by (3.2.3) in terms of g. In view of (3.4.46)-(3.4.47) and (3.2.15)-(3.2.16), passing to the limit as $\varepsilon_{k,t} \to 0$ in the above inequalities gives

$$\int_0^1 G^t(x)\varphi(x)dx \leq n \int_{t-\frac{1}{n}}^{t} \int_0^1 G^* \varphi dxds - \int_{t-\frac{1}{n}}^{t} \int_0^1 h^n g(v) v_x \varphi_x dxds, \tag{3.4.49}$$

for any $\varphi \in C^1([0,1])$, $\varphi \geq 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$. Here G^* is the function defined by (3.2.5) and

$$G^t = \sum_{i=0}^{2} \lambda_i^t G(s_i(v)) \quad a.e. \text{ in } (0,1) \tag{3.4.50}$$

(see (3.4.46)-(3.4.47)). On the other hand, by (3.4.7) in Lemma 3.4.2, taking the limit as $n \to \infty$ in (3.4.49) gives

$$\int_0^1 G^t(x)\varphi(x)dx \leq \int_0^1 G^*(x,t)\varphi(x)dx$$

for any φ and g as above. This implies

$$G^t(x) \leq G^*(x,t)$$

for *a.e.* $x \in (0,1)$. In an analogous way we can prove the reverse inequality, hence for any $g \in C^1(\mathbb{R}),\ g' \geq 0$ we have:

$$G^t(x) = G^*(x,t) \tag{3.4.51}$$

for *a.e.* $x \in (0,1)$, where G^t is defined by (3.4.50) and G^* is defined by (3.2.5). By approximation arguments, equality (3.4.51) holds for any *non-decreasing* g, hence for any $g \in BV(\mathbb{R})$. Precisely we obtain:

$$\sum_{i=0}^{2} \lambda_i^t(x) \int^{s_i(v(x,t))} g(\phi(\lambda))d\lambda = \sum_{i=0}^{2} \lambda_i(x,t) \int^{s_i(v(x,t))} g(\phi(\lambda))d\lambda$$

for *a.e.* $x \in (0,1)$ and for any $g \in BV(\mathbb{R})$. The above equalities implies (3.4.43)-(3.4.45) (see Lemma 3.5.2 and Lemma 3.5.3 in the following section).
□

As a consequence of the above result, for any $t \in (0,\infty) \setminus \tilde{F}$ the *whole* sequence $\{\tau^t_{\varepsilon_k}\}$ of Young measures associated to the functions $u^{\varepsilon_k(\cdot,t)}$ converges in the narrow topology over $(0,1) \times \mathbb{R}$. Using the general properties of the narrow convergence of Young measures, the following result holds.

Proposition 3.4.5. *Let $\tilde{F} \subseteq (0,\infty)$ be the set of zero Lebesgue-measure given by Theorem 3.4.4. Then for any $t \in (0,\infty) \setminus \tilde{F}$ and for any $f \in C(\mathbb{R})$, we have*

$$f(u^{\varepsilon_k}(\cdot,t)) \overset{*}{\rightharpoonup} f^*(\cdot,t) \quad \text{in} \quad L^\infty(0,1),$$

where $f^*(x,t)$ is defined by (3.2.16)*.*

3.5 Proof of results of Section 3.3.2

Most proofs of the results in Section 3.3.2 make use of the following technical lemmas.

Let $BV(\mathbb{R})$ denote the space of the functions with bounded total variation on $\mathbb{R}$.

Lemma 3.5.1. *Let $v_1,\ v_2 \in [A, B],\ 0 \le a_i \le 1,\ 0 \le b_i \le 1\ (i = 1, 2)$, such that*

$$\begin{aligned} & a_1 \int_0^{s_1(v_1)} g(\phi(s))ds + b_1 \int_0^{s_2(v_1)} g(\phi(s))ds + \\ & +(1 - a_1 - b_1) \int_0^{s_0(v_1)} g(\phi(s))ds = \\ = \ & a_2 \int_0^{s_1(v_2)} g(\phi(s))ds + b_2 \int_0^{s_2(v_2)} g(\phi(s))ds + \\ & +(1 - a_2 - b_2) \int_0^{s_0(v_2)} g(\phi(s))ds, \end{aligned} \tag{3.5.1}$$

for any $g \in BV(\mathbb{R})$. Then $v_1 = v_2$.

Proof. For simplicity, assume that $v_2 > v_1$ and let us distinguish the cases $v_2 > 0,\ v_2 \le 0$.

(*i*) If $v_2 > 0$, set

$$\overline{v} := \max\{0, v_1\}$$

and then fix any $v \in (\overline{v}, v_2)$. For any $n \in \mathbb{N}$, set

$$g_n(\lambda) := n\chi_{[v,v+1/n]}(\lambda). \tag{3.5.2}$$

Equality (3.5.1) with $g = g_n$ gives

$$\begin{aligned} & a_1 n[s_0(v + 1/n) - s_0(v)] + a_1 n[s_1(v) - s_1(v + 1/n)] = \\ = \ & a_2 n[s_0(v + 1/n) - s_0(v)] + b_2 n[s_2(v + 1/n) - s_2(v)] + \\ & +(1 - a_2 - b_2)n[s_0(v + 1/n) - s_0(v)]. \end{aligned} \tag{3.5.3}$$

Let us take the limit as $n \to \infty$ in (3.5.3). We obtain

$$\begin{aligned} & a_1[s_0'(v) - s_1'(v)] = \\ = \ & a_2 s_0'(v) + b_2 s_2'(v) + (1 - a_2 - b_2)s_0'(v) \end{aligned} \tag{3.5.4}$$

for any $v \in (\overline{v}, v_2)$. Hence, in view of Condition (**S**), there holds

$$\begin{cases} a_1 = 0 \\ b_2 = 0 \\ a_1 + b_2 = 1, \end{cases} \tag{3.5.5}$$

which gives an absurd. This concludes the proof in the case $v_2 > 0$.

(*ii*) If $v_2 \le 0$, again fix any $v \in (v_1, v_2)$ and for any $n \in \mathbb{N}$ let g_n be the function defined by (3.5.2). Equality (3.5.1) with $g = g_n$ gives

$$\begin{aligned} & a_1 n[s_1(v) - s_1(v + 1/n)] + b_1 n[s_0(v) - s_0(v + 1/n)] + \\ & +(1 - a_1 - b_1)n[s_0(v) - s_0(v + 1/n)] = \\ = \ & b_2 n[s_0(v) - s_0(v + 1/n)] + b_2 n[s_2(v + 1/n) - s_2(v)]. \end{aligned} \tag{3.5.6}$$

Thus, we pass to the limit with respect to $n \to \infty$ in (3.5.6) and obtain

$$\begin{aligned} & -a_1 s_1'(v) - b_1 s_0'(v) - (1 - a_1 - b_1) s_0'(v) = \\ = \quad & -b_2 s_0'(v) + b_2 s_2'(v) \end{aligned} \tag{3.5.7}$$

for any $v \in (v_1, v_2)$. Again, (3.5.7) and Condition (**S**) imply

$$\begin{cases} a_1 = 0 \\ b_2 = 0 \\ a_1 + b_2 = 1, \end{cases} \tag{3.5.8}$$

and the claim follows. □

Lemma 3.5.2. *Let* $v \in (A, B),\ 0 \le a_i \le 1,\ 0 \le b_i \le 1\ (i = 1, 2)$, *be such that equality*

$$\begin{aligned} & a_1 \int_0^{s_1(v)} g(\phi(s))ds + b_1 \int_0^{s_2(v)} g(\phi(s))ds + \\ & +(1 - a_1 - b_1) \int_0^{s_0(v)} g(\phi(s))ds = \\ = \quad & a_2 \int_0^{s_1(v)} g(\phi(s))ds + b_2 \int_0^{s_2(v)} g(\phi(s))ds + \\ & +(1 - a_2 - b_2) \int_0^{s_0(v)} g(\phi(s))ds \end{aligned} \tag{3.5.9}$$

holds for any $g \in BV(\mathbb{R})$. *Then* $a_1 = a_2$ *and* $b_1 = b_2$.

Proof. In (3.5.9), choose

$$g(\lambda) := \chi_{[v,B]}(\lambda) \qquad \text{if } v \ge 0, \tag{3.5.10}$$

$$g(\lambda) := \chi_{[A,v]}(\lambda) \qquad \text{if } v < 0. \tag{3.5.11}$$

We have:

$$(a_1 - a_2) \int_{s_0(v)}^{s_1(v)} ds = 0 \qquad \text{if } v \ge 0, \tag{3.5.12}$$

and

$$(b_1 - b_2) \int_{s_0(v)}^{s_2(v)} ds = 0 \qquad \text{if } v < 0. \tag{3.5.13}$$

Thus, (3.5.12)-(3.5.13) imply

$$\begin{cases} a_1 = a_2 & \text{if } v \ge 0, \\ b_1 = b_2 & \text{if } v < 0. \end{cases} \tag{3.5.14}$$

Moreover, choosing $g(\lambda) \equiv 1$ in (3.5.9) gives

$$(b_1 - b_2)(s_2(v) - s_0(v)) = 0 \qquad \text{if } v \geq 0, \tag{3.5.15}$$
$$(a_1 - a_2)(s_1(v) - s_0(v)) = 0 \qquad \text{if } v < 0. \tag{3.5.16}$$

Hence,

$$\begin{cases} b_1 = b_2 & \text{if } v \geq 0, \\ a_1 = a_2 & \text{if } v < 0. \end{cases} \tag{3.5.17}$$

This concludes the proof. □

Lemma 3.5.3. *Let us consider equality* (3.5.9) *for $v = A$ and $v = B$. Then,*

$$\begin{cases} a_1 = a_2 & \textit{if } v = A, \\ b_1 = b_2 & \textit{if } v = B. \end{cases} \tag{3.5.18}$$

Proof. Observe that $s_0(A) = c = s_2(A)$, $s_0(B) = b = s_1(B)$ (see Fig.3.1). Hence equality (3.5.9) reads

$$\begin{aligned} a_1 \int_0^a g(\phi(s))ds + (1 - a_1) \int_0^c g(\phi(s))ds = \\ = \quad a_2 \int_0^a g(\phi(s))ds + (1 - a_2) \int_0^c g(\phi(s))ds, \end{aligned} \tag{3.5.19}$$

if $v = A$ (recall that $a = s_1(A)$), and

$$\begin{aligned} b_1 \int_0^d g(\phi(s))ds + (1 - b_1) \int_0^b g(\phi(s))ds = \\ = \quad b_2 \int_0^d g(\phi(s))ds + (1 - b_2) \int_0^b g(\phi(s))ds, \end{aligned} \tag{3.5.20}$$

if $v = B$ (recall that $d = s_2(B)$). Equalities (3.5.19)-(3.5.20) imply (3.5.18) and the claim follows. □

Proof of Theorem 3.3.5. Let $\{t_n\} \subseteq (0, \infty)$ be any diverging sequence. Observe that

$$\int_0^\infty \int_0^1 v_x^2(x, t + t_n)dxdt = \int_{t_n}^\infty \int_0^1 v_x^2(x, s)dxds,$$

thus, in view of (3.2.18) we have

$$\int_0^\infty \int_0^1 v_x^2(x, t + t_n)dxdt \to 0 \qquad \text{as } n \to \infty. \tag{3.5.21}$$

This implies that there exist a subsequence $\{t_{n_k}\}$ and a set $E \subseteq (0, \infty)$ of Lebesgue measure $|E| = 0$ such that

$$\int_0^1 v_x^2(x, t + t_{n_k})dx \to 0 \qquad \text{as } k \to \infty \tag{3.5.22}$$

for any $t \in (0, \infty) \setminus E$. We proceed as follows.

Step (α). For any diverging sequence $\{t_n\}$, let $\{t_{n_k}\} \subseteq \{t_n\}$ and $E \subseteq (0, \infty)$ be respectively any subsequence and any set of zero Lebesgue-measure such that (3.5.22) holds for any $t \in (0, \infty) \setminus E$. Then we show that the sequence $\{v(\cdot, t + t_{n_k})\}$ converges uniformly in $(0, 1)$ to a constant $\overline{v}_{t_n}$ (possibly dependending on the choice of the sequence $\{t_n\}$) for any $t \in (0, \infty) \setminus E$.

Step (β). We prove that the constant $\overline{v}_{t_n}$ given in *Step* (α) does not depend on the choice of the diverging sequence $\{t_n\}$. In other words $\overline{v}_{t_n} = \overline{v}$ for any sequence $\{t_n\}$.

Proof of Step (α). Fix any diverging sequence $\{t_n\}$ and let $\{t_{n_k}\} \subseteq \{t_n\}$ and $E \subseteq (0, \infty)$ be respectively any subsequence and any set of zero Lebesgue-measure such that (3.5.22) holds for any $t \in (0, \infty) \setminus E$. Arguing by contradiction, suppose that we can find two subsequences $\{t_{n,1}\}, \ \{t_{n,2}\} \subseteq \{t_{n_k}\}$ and $t_1, \ t_2 \in (0, \infty) \setminus E$, such that

$$\liminf_{n \to \infty} \|v(\cdot, t_1 + t_{n,1}) - v(\cdot, t_2 + t_{n,2})\|_{C([0,1])} > 0. \tag{3.5.23}$$

Observe that by (3.5.22) we have

$$\int_0^1 v_x^2(x, t_j + t_{n,j})dx \to 0 \qquad (j = 1, 2). \tag{3.5.24}$$

Moreover, since $v(\cdot, t_j + t_{n,j}) \in H^1(0, 1) \subseteq C([0, 1])$ and

$$|\, v(x_2, t_j + t_{n,j}) - v(x_1, t_j + t_{n,j})| \leq \left(\int_0^1 v_x^2(x, t_j + t_{n,j})dx \right)^{1/2} |\, x_2 - x_1|^{1/2} \tag{3.5.25}$$

for any $x_1 \neq x_2 \in (0, 1)$, we deduce by the Ascoli-Arzelà Theorem that the sequence $\{v(\cdot, t_j + t_{n,j})\}$ is relatively compact in $C([0, 1])$ for $j = 1, 2$ (here use of (3.5.24) has been made). Hence, possibly passing to a subsequence, we have

$$v(\cdot, t_j + t_{n,j}) \to v^j \qquad \text{in } C([0, 1]) \qquad (j = 1, 2). \tag{3.5.26}$$

Observe that by (3.5.24) and (3.5.25) v^1 and v^2 are constant. Let us show that:

$$v^1 = v^2 := \overline{v}_{t_n}, \tag{3.5.27}$$

which is in clear contradiction with (3.5.23) and concludes the proof of *Step* (α).

In this direction, first observe that the sequences $\{\lambda_i(\cdot, t_j + t_{n,j})\}$ are uniformly bounded in $L^\infty(0,1)$ for $j = 1,2$ and $i = 0,1,2$. Hence, eventually passing to a subsequence, we can suppose that

$$\lambda_i(\cdot, t_j + t_{n,j}) \overset{*}{\rightharpoonup} \lambda_i^{*,j} \qquad \text{in } L^\infty(0,1) \qquad (j = 1,2) \tag{3.5.28}$$

for some $0 \le \lambda_i^{*,j} \le 1$, $\lambda_1^{*,j} = 1$ if $v^j < A$, $\lambda_2^{*,j} = 1$ if $v^j > B$ and $\sum_{i=0}^{2} \lambda_i^{*,j} = 1$ *a.e.* in $(0,1)$. Since by representation (3.2.1) we have

$$u(\cdot, t_j + t_{n,j}) = \sum_{i=0}^{2} \lambda_i(\cdot, t_j + t_{n,j}) s_i(v(\cdot, t_j + t_{n,j})) \quad \text{in} \quad (0,1)$$

(for $j = 1,2$), by means of (3.5.26) and (3.5.28) we obtain

$$u(\cdot, t_j + t_{n,j}) \overset{*}{\rightharpoonup} \sum_{i=0}^{2} \lambda_i^{*,j}(\cdot) s_i(v^j) \qquad \text{in} \quad L^\infty(0,1) \tag{3.5.29}$$

for $j = 1,2$. Thus, using the above convergence and the conservation law (3.2.24) gives

$$\sum_{i=0}^{2} s_i(v^j) \int_0^1 \lambda_i^{*,j}(x) dx = M_{u_0} \qquad (j = 1,2) \tag{3.5.30}$$

where M_{u_0} is defined by (3.2.23). Let us distinguish the cases $a \le M_{u_0} \le d$ and $M_{u_0} < a$, $M_{u_0} > d$.

If $a \le M_{u_0} \le d$, observe that $v^j < A$ (and $v^j > B$) in (3.5.30) gives a contradiction. In fact, $v^j < A$ would imply $\lambda_1^{*,j} = 1$ in $(0,1)$. Therefore (3.5.30) would reduce to

$$s_1(v^j) = M_{u_0}\,.$$

On the other hand, $v^j < A$ implies $s_1(v^j) < a$, which gives an absurd since we have assumed $a \le M_{u_0} \le d$. Clearly, with the same arguments, it is easily seen that $v^j \le B$ in the case $a \le M_{u_0} \le d$. Hence, $v^j \in [A, B]$ for $j = 1,2$. Moreover, by (3.5.26) and (3.5.28), for any non-decreasing g and G^* defined by (3.2.5) in terms of g there holds

$$G^*(\cdot, t_j + t_{n,j}) \overset{*}{\rightharpoonup} \sum_{i=0}^{2} \lambda_i^{*,j}(\cdot) \int^{s_i(v^j)} g(\phi(s)) ds \quad \text{in} \quad L^\infty(0,1)\,. \tag{3.5.31}$$

On the other hand, since there exists

$$\lim_{\substack{t \to \infty \\ t \in (0,\infty) \setminus F}} \int_0^1 G^*(x,t) dx =: L_g$$

for any non-decreasing g (see (3.3.1) in Corollary 3.3.3), there holds

$$\lim_{n\to\infty}\int_0^1 G^*(x,t_1+t_{n,1})dx = \lim_{n\to\infty}\int_0^1 G^*(x,t_2+t_{n,2})dx$$

for any non-decreasing g, hence for any $g \in BV(\mathbb{R})$. Using (3.5.31) the above equality reads

$$\sum_{i=0}^{2}\left(\int_0^1 \lambda_i^{*,1}(x)dx\right)\int^{s_i(v^1)} g(\phi(s))ds = \tag{3.5.32}$$

$$= \sum_{i=0}^{2}\left(\int_0^1 \lambda_i^{*,2}(x)dx\right)\int^{s_i(v^2)} g(\phi(s))ds$$

for any $g \in BV(\mathbb{R})$, thus $v^1 = v^2$ by Lemma 3.5.1. This proves equality (3.5.27) and concludes the proof of *Step (α)* in the case $a \le M_{u_0} \le d$.

Now suppose $M_{u_0} < a$ (the case $M_{u_0} > d$ can be treated in an analogous way). Arguing as in the case $a \le M_{u_0} \le d$, it is easily seen that equation (3.5.30) with $M_{u_0} < a$ implies $v^j < A$ for $j = 1, 2$. Thus, since for $v^j < A$ we have $\lambda_1^{*,j} = 1$ ($j = 1, 2$), equation (3.5.30) reduces to

$$s_1(v^j) = M_{u_0}. \tag{3.5.33}$$

This implies $v^1 = v^2$ - namely (3.5.27) - and concludes the proof of *Step* (α) also in the case $M_{u_0} < a$.

Proof of Step (β). Now suppose that there exist $\overline{v}_{t_n^1} \ne \overline{v}_{t_n^2}$ and two diverging sequences $\{t_n^1\}$, $\{t_n^2\}$ such that

$$v(\cdot,t_j+t_n^j) \to \overline{v}_{t_n^j} \qquad \text{in } C([0,1]) \qquad (j=1,2), \tag{3.5.34}$$

for some $t_1,\ t_2 \in (0,\infty)$. Here $\overline{v}_{t_n^j}$ is the constant given by *Step* (α) in correspondence of the diverging sequence $\left\{t_n^j\right\}$, $j = 1, 2$ (see equality (3.5.27)). Arguing as in the previous step, we can assume that, eventually passing to a subsequence, the sequences $\left\{\lambda_i(\cdot,t_j+t_n^j)\right\}$ converge to some $\lambda_i^{*,j} \in L^\infty(0,1)$ in the weak* topology of $L^\infty(0,1)$, $i = 0, 1, 2$ and $j = 1, 2$. Again, $0 \le \lambda_i^{*,j} \le 1$, $\sum_{i=0}^{2}\lambda_i^{*,j} = 1$, $\lambda_1^{*,j} = 1$ if $\overline{v}_{t_n^j} < A$ and $\lambda_2^{*,j} = 1$ if $\overline{v}_{t_n^j} > B$. Therefore, concerning the sequence $u(\cdot,t_j+t_n^j)$ ($j = 1, 2$) convergenge (3.5.29) holds in correspondence of each sequence $\left\{t_j+t_n^j\right\}$. Consequently the conservation law (3.2.24) gives equation (3.5.30). Again, we distinguish the cases $a \le M_{u_0} \le d$ and $M_{u_0} < a,\ M_{u_0} > d$.

If $a \le M_{u_0} \le d$ we can argue as in *Step* (α), proving that $\overline{v}_{t_n^1}$ and $\overline{v}_{t_n^2}$ satisfy equation (3.5.32) - namely $\overline{v}_{t_n^1} = \overline{v}_{t_n^2}$ by Lemma 3.5.1.

On the other hand, if $M_{u_0} < a$ (the case $M_{u_0} > B$ is analogous) we can proceed as in the proof of *Step* (α) showing that equation (3.5.30) implies $\overline{v}_{t_n^j} < A$ for $j = 1, 2$, hence $s_1(\overline{v}_{t_n^1}) = s_1(\overline{v}_{t_n^2}) = M_{u_0}$ (recall that if $\overline{v}_{t_n^j} < A$ then $\lambda_1^{*,j} \equiv 1$ in (3.5.29)).

Finally, let us prove the last claim in Theorem 3.3.5-(ii). In this direction, assume $M_{u_0} < a$ (the case $M_{u_0} > d$ can be treated in a similar way). In view of the above remarks, there exists a nondecreasing sequence $\{t_n\}$, $t_n \to \infty$, such that $v(\cdot, t_n) \to \phi(M_{u_0})$ in $C([0,1])$, and, by our assumption, $\phi(M_{u_0}) < A$. This means that, for any fixed $\varepsilon > 0$ small enough, there exists $N > 0$ such that

$$v(x, t_n) \leq \phi(M_{u_0}) - 2\epsilon < A - \epsilon, \qquad \text{for any } t_n \geq t_N. \tag{3.5.35}$$

Let $g_A \in C^1(\mathbb{R})$ be the non-decreasing function on $\mathbb{R}$, defined as follows:

$$g_A(\lambda) = \begin{cases} (\lambda - A + \epsilon)^2 & \text{if } \lambda \geq A - \epsilon, \\ 0 & \text{if } \lambda < A - \epsilon, \end{cases} \tag{3.5.36}$$

and set

$$G_A(\lambda) := \int_{s_1(A-\epsilon)}^{\lambda} g_A(\phi(s))ds.$$

Using g_A in inequality (3.2.17) with test function $\varphi \equiv 1$ in $(0,1)$, gives

$$\int_0^1 G_A^*(x,t)dx \leq \int_0^1 G_A^*(x, t_N)dx \equiv 0, \tag{3.5.37}$$

for any $t \in (0, \infty) \setminus F$, $t \geq t_N$, where F is the set given by Theorem 3.3.1 (the last equality in (3.5.37) being a consequence of (3.5.35) and (3.5.36)). Since $G_A^*(x,t) > 0$ if $v(x,t) > A - \epsilon$, by inequality (3.5.37) the claim follows. □

The proof of Theorem 3.3.6 needs some preliminary results. The techniques used and the results concerning the characterization of the behaviour of the sequence $\{u(\cdot, t + t_n)\}$ defined by (3.3.6) for large values of t_n are quite different in the cases $a \leq M_{u_0} \leq d$ and $M_{u_0} < a$, $M_{u_0} > d$, respectively.

In fact, observe that if $a \leq M_{u_0} \leq d$ we have to take into account the behaviour of the sequences $\{\lambda_i(\cdot, t + t_n)\}$ in (3.3.6), hence in this case the first step is to study the long-time behaviour of these sequences for any diverging $\{t_n\}$ and for *a.e.* $t > 0$ (see Proposition 3.5.4 below).

On the other hand, when $M_{u_0} < a$ (or $M_{u_0} > d$), in view of Theorem 3.3.5-(ii) and in view of (3.2.1), we have that (3.3.6) reduces to $u(\cdot, t + t_n) = s_1(v(\cdot, t + t_n))$ (respectively, $u(\cdot, t + t_n) = s_2(v(\cdot, t + t_n))$) for large values of t_n.

To begin with, using Proposition 3.3.4 we proceed to study the long-time behaviour of the coefficients λ_i. Precisely, the following proposition holds.

Proposition 3.5.4. *Let* (u, v) *be a weak entropy measure-valued solution of problem* (3.1.1) *with initial datum* u_0*. Assume* $a \le M_{u_0} \le d$*, where* M_{u_0} *is defined by* (3.2.23)*, and let* $\overline{v} \in [A, B]$ *be the constant given by Theorem 3.3.5. Then:*

(i) if $A < \overline{v} < B$*, for any* $i = 0, 1, 2$ *there exists a unique* $\lambda_i^* \in L^\infty(0, 1)$ $(i = 0, 1, 2)$*,* $0 \le \lambda_i^* \le 1$*,* $\sum_{i=0}^{2} \lambda_i^* = 1$ *a.e. in* $(0, 1)$*, such that for any diverging and non-decreasing sequence* $\{t_n\}$ *there holds:*

$$\lambda_i(\cdot, t + t_{n_k}) \to \lambda_i^*(\cdot) \qquad a.e.\ in\ (0, 1) \tag{3.5.38}$$

for any $t \in (0, \infty) \setminus E$*, where* $\{t_{n_k}\} \subseteq \{t_n\}$ *and* $E \subseteq (0, \infty)$ *are respectively any subsequence and any set of zero Lebesgue-measure (whose existence is assured by Theorem 3.3.5) such that* (3.3.4) *holds.*

(ii) if $\overline{v} = B$*, there exists a unique* $\lambda_2^* \in L^\infty(0, 1)$*,* $0 \le \lambda_2^* \le 1$*, such that for any diverging and non-decreasing sequence* $\{t_n\}$ *there holds:*

$$\lambda_2(\cdot, t + t_{n_k}) \to \lambda_2^*(\cdot) \qquad a.e.\ in\ (0, 1) \tag{3.5.39}$$

for any $t \in (0, \infty) \setminus E$*, where* $\{t_{n_k}\} \subseteq \{t_n\}$ *and* $E \subseteq (0, \infty)$ *are respectively any subsequence and any set of zero-Lebesgue-measure as in (i);*

(iii) if $\overline{v} = A$*, there exists a unique* $\lambda_1^* \in L^\infty(0, 1)$*,* $0 \le \lambda_1^* \le 1$*, such that for any diverging and non-decreasing sequence* $\{t_n\}$ *there holds:*

$$\lambda_1(\cdot, t + t_{n_k}) \to \lambda_1^*(\cdot) \qquad a.e.\ in\ (0, 1) \tag{3.5.40}$$

for any $t \in (0, \infty) \setminus E$*, where* $\{t_{n_k}\} \subseteq \{t_n\}$ *and* $E \subseteq (0, \infty)$ *are respectively any subsequence and any set of zero Lebesgue-measure as in (i).*

Proof. Let $a \le M_{u_0} \le d$, hence $\overline{v} \in [A, B]$ by Theorem 3.3.5. Fix any non-decreasing diverging sequence $\{t_n\}$ and then fix any subsequence of $\{t_n\}$ (which we will continue to denote by $\{t_n\}$) and any set $E \subseteq (0, \infty)$, $|E| = 0$ (whose existence is assured by Theorem 3.3.5) such that $v(\cdot, t + t_n) \to \overline{v}$ in $C([0, 1])$ for any $t \in (0, \infty) \setminus E$. This implies that for any fixed $\epsilon > 0$ small enough, and for any $t \in (0, \infty) \setminus E$ there exists $N \in \mathbb{N}$, in general dependending on t and $\{t_n\}$, such that:

$$\overline{v} - \epsilon \le v(x, t + t_n) \le \overline{v} + \epsilon \tag{3.5.41}$$

for any $x \in (0, 1)$ and for any $n \ge N$. Let us consider separately the cases $A < \overline{v} < B$, $\overline{v} = A$ and $\overline{v} = B$.

(i) Assume $A < \overline{v} < B$. Then in view of (3.5.41) and by Proposition 3.3.4, for any $t \in (0, \infty) \setminus E$ there holds

$$\lambda_1(\cdot, t + t_n) \le \lambda_1(\cdot, t + t_{n+1}),$$
$$\lambda_2(\cdot, t + t_n) \le \lambda_2(\cdot, t + t_{n+1})$$

for any $n \geq N$ (because we can suppose in (3.5.41) $A + \epsilon \leq \overline{v} - \epsilon$ and $\overline{v} + \varepsilon \leq B - \varepsilon$ for some $\varepsilon > 0$ small enough). This implies that for any $t \in (0, \infty) \setminus E$ there exists $\lambda_i^{*,t} \in L^\infty(0,1)$ such that

$$\lambda_i(x, t + t_n) \to \lambda_i^{*,t}(x), \quad \text{for } a.e.\ x \in (0,1) \qquad (i = 0, 1, 2). \tag{3.5.42}$$

Let us show that the coefficients $\lambda_i^{*,t}$ do not depend on t. To this purpose, fix $t_1 < t_2$. Suppose that

$$\lambda_i(\cdot, t_j + t_n) \to \lambda_i^{*,t_j}(\cdot) \quad a.e.\text{ in } (0,1) \tag{3.5.43}$$

as $n \to \infty$ $(j = 1, 2)$. Observe that the uniform convergence of $v(t_j + t_n)$ to $\overline{v}$ as $n \to \infty$ proved in Theorem 3.3.5 (here $j = 1, 2$), and (3.5.43) imply that

$$G^*(\cdot, t_j + t_n) \overset{*}{\rightharpoonup} \sum_{i=0}^{2} \lambda_i^{*,t_j} \int^{s_i(\overline{v})} g(\phi(s))ds \tag{3.5.44}$$

as $n \to \infty$, $j = 1, 2$. Here G^* is any function defined by (3.2.5) in terms of any non-decreasing g.

By (3.5.41) and in view of Proposition 3.3.4, we have

$$\lambda_1(x, t_2 + t_n) \geq \lambda_1(x, t_1 + t_n), \tag{3.5.45}$$

$$\lambda_2(x, t_2 + t_n) \geq \lambda_2(x, t_1 + t_n) \tag{3.5.46}$$

for $a.e.$ $x \in (0,1)$ for n large enough (because we have assumed $t_1 < t_2$). Observe that properties (3.5.45) and (3.5.46) hold in correspondence of both the coefficients λ_1 and λ_2 since we have assumed $A < \overline{v} < B$ (see Proposition 3.3.4). This implies

$$\lambda_1^{*,t_2} \geq \lambda_1^{*,t_1}, \quad \lambda_2^{*,t_2} \geq \lambda_2^{*,t_1} \qquad a.e.\text{ in } (0,1). \tag{3.5.47}$$

On the other hand, for any $g \in BV(\mathbb{R})$, there holds

$$\lim_{n\to\infty} \int_0^1 G^*(x, t_1 + t_n)dx = \lim_{n\to\infty} \int_0^1 G^*(x, t_2 + t_n)dx$$

(see Corollary 3.3.3), namely

$$\begin{aligned} \sum_{i=0}^{2} \left(\int_0^1 \lambda_i^{*,t_2}(x)dx \right) \int^{s_i(\overline{v})} g(\phi(s))ds &= \\ = \sum_{i=0}^{2} \left(\int_0^1 \lambda_i^{*,t_1}(x)dx \right) \int^{s_i(\overline{v})} g(\phi(s))ds & \end{aligned} \tag{3.5.48}$$

(here use of (3.5.44) has been made). Equality (3.5.48) implies that

$$\int_0^1 \lambda_i^{*,t_1}(x)dx = \int_0^1 \lambda_i^{*,t_2}(x)dx \qquad (i = 1, 2) \tag{3.5.49}$$

(see Lemma 3.5.2), hence in view of (3.5.47) we have $\lambda_i^{*,t_1} = \lambda_i^{*,t_2}$ ($i = 0, 1, 2$) and we can set:

$$\lambda_i^{*,t} \equiv \lambda_i^{*,t_n}$$

in (3.5.42), the coefficients λ_i^{*,t_n} possibly depending on the sequence $\{t_n\}$.

Then we show that the coefficients λ_i^{*,t_n} are independent of the sequence $\{t_n\}$. To this purpose, suppose that there exist $\{t_n^1\}$, $\{t_n^2\}$, non-decreasing, such that

$$\lambda_i(x, t + t_n^j) \to \lambda_i^{*,j}(x) \quad \text{for } a.e.\ x \in (0,1),\ t \geq 0 \qquad (j = 1, 2). \quad (3.5.50)$$

Assume that

$$\liminf_{n\to\infty}(t_n^2 - t_n^1) \geq 0,$$

and fix any $t_1,\ t_2 \in \mathbb{R}^+$, such that

$$\liminf_{n\to\infty}(t_2 + t_n^2 - t_1 - t_n^1) > 0. \quad (3.5.51)$$

Thus, for n large enough, $t_2 + t_n^2 \geq t_1 + t_n^1$. It follows that, arguing as above we obtain $\lambda_i^{*,1} = \lambda_i^{*,2}$ in $(0,1)$ ($i = 0, 1, 2$) and the claim follows.

(*ii*) Assume that $\overline{v} = A$ (the case $\overline{v} = B$ is analogous). Again, by (3.5.41) and in view of Proposition 3.3.4, for any $t \in (0, \infty) \setminus E$, there exists $\lambda_1^{*,t} \in L^\infty(0,1)$ such that

$$\lambda_1(x, t + t_n) \to \lambda_1^{*,t}, \quad \text{for } a.e.\ x \in (0,1). \quad (3.5.52)$$

Then we fix $t_1 < t_2$ and show that there holds $\lambda_1^{*,t_1} \equiv \lambda_1^{*,t_2}$. To begin with, observe that (3.5.41) and Proposition 3.3.4 give

$$\lambda_1(x, t_2 + t_n) \geq \lambda_1(x, t_1 + t_n) \quad (3.5.53)$$

for n large enough, hence

$$\lambda_1^{*,t_2} \geq \lambda_1^{*,t_1} \qquad a.e. \text{ in } (0,1) \quad (3.5.54)$$

(because $\overline{v} = A < B$ and $t_1 < t_2$). On the other hand, the same arguments used in (*i*), Corollary 3.3.3 and Lemma 3.5.3 give:

$$\int_0^1 \lambda_1^{*,t_1} dx = \int_0^1 \lambda_1^{*,t_2} dx, \quad (3.5.55)$$

hence $\lambda_1^{*,t_1} = \lambda_1^{*,t_2}$.
Finally, arguing as in the case $A < \overline{v} < B$, it is easily seen that the coefficient λ_1^* does not depend on the sequence $\{t_n\}$. □

Proof of Theorem 3.3.6. Fix any non-decreasing diverging sequence $\{t_n\}$ and then fix any subsequence of $\{t_n\}$ (which we will continue to denote by $\{t_n\}$) and any set $E \subseteq (0,\infty)$, $|E| = 0$ (whose existence is assured by Theorem 3.3.5) such that $v(\cdot, t+t_n) \to \overline{v}$ in $C([0,1])$ for any $t \in (0,\infty) \setminus E$.

(i) Assume $a \le M_{u_0} \le d$. Then for any $t \in (0,\infty) \setminus E$ we have $\lambda_i(\cdot, t + t_n) \to \lambda_i^*(\cdot)$ *a.e.* in $(0,1)$, where $\lambda_i^* \in L^\infty(0,1)$ are the functions uniquely determined by Proposition 3.5.4. Thus, in view of representation (3.3.6) $u(\cdot, t+t_n) \to \overline{u}(\cdot)$ *a.e.* in $(0,1)$ and for any $t \in (0,\infty) \setminus E$, where $\overline{u} \in L^\infty(0,1)$ is the function defined by (3.3.7).

(ii) Now consider the case $M_{u_0} < a$ (if $M_{u_0} > d$ we proceed in an analogous way). By definition (3.2.1) and Theorem 3.3.5-(ii), we have

$$u(\cdot, t+t_n) = s_1(v(\cdot, t+t_n)) \quad \text{in} \quad (0,1) \tag{3.5.56}$$

for large values of t_n. Thus $u(\cdot, t+t_n) \to s_1(\phi(M_{u_0})) = M_{u_0}$ uniformly in $[0,1]$ for any $t \in (0,\infty) \setminus E$ by the uniform convergence $v(\cdot, t+t_n) \to \phi(M_{u_0})$ (see equality (3.3.5)). □

Proof of Theorem 3.3.7. Fix any $k > 0$ and consider any non-decreasing sequence $\{t_n\} \subseteq B_k \setminus F$, $t_n \to \infty$ as $n \to \infty$. In view of definition (3.2.33),

$$\sup_{n\in\mathbb{N}} \int_0^1 v_x^2(x, t_n)dx < k. \tag{3.5.57}$$

Arguing as in the proof of Theorem 3.3.5 it is easily seen that (3.5.57) implies that, eventually up to a subsequence, there holds

$$v(\cdot, t_n) \to w \quad \text{in} \quad C([0,1]), \tag{3.5.58}$$

for some $w \in C([0,1])$. On the other hand, we can find two non-decreasing and diverging sequences $\left\{s_n^1\right\}$, $\left\{s_n^2\right\}$ such that $s_n^1 \le t_n \le s_n^2$, $|t_n - s_n^j| \le 1$ and $v(\cdot, s_n^j) \to \overline{v}$ uniformly in $[0,1]$ $(j = 1,2)$. Writing inequalities (3.2.17) first between s_n^1 and t_n, then between t_n and s_n^2 gives

$$\int_0^1 G^*(x, s_n^1)\varphi(x)dx - \int_0^1 G^*(x, t_n)\varphi(x)dx \ge \int_{s_n^1}^{t_n} \int_0^1 g(v)v_x\varphi_x dxdt,$$
$$\int_0^1 G^*(x, t_n)\varphi(x)dx - \int_0^1 G^*(x, s_n^2)\varphi(x)dx \ge \int_{t_n}^{s_n^2} \int_0^1 g(v)v_x\varphi_x dxdt$$

for any $g \in C^1(\mathbb{R})$, $g' \ge 0$, $\varphi \in C^1([0,1])$, $\varphi \ge 0$ and G^* defined by (3.2.5). We take the limit as $n \to \infty$ in the above inequalities and obtain (for *a.e.* $x \in (0,1)$)

$$\sum_{i=0}^{2} \overline{\lambda}_i(x) \int^{s_i(w(x))} g(\phi(s))ds = \sum_{i=0}^{2} \lambda_i^*(x) \int^{s_i(\overline{v})} g(\phi(s))ds$$

for any $g \in C^1(\mathbb{R})$, $g' \geq 0$ (hence for any $g \in BV(\mathbb{R})$). Here, for any $i = 0,1,2$, λ_i^* is the function given by Proposition 3.5.4 and $\overline{\lambda}_i$ is some function such that

$$\lambda_i(\cdot, t_n) \overset{*}{\rightharpoonup} \overline{\lambda}_i(\cdot) \ \text{ in } \ L^\infty(0,1)$$

(eventually up to a subsequence). By Lemma 3.5.1 we obtain

$$w(x) = \overline{v} \quad \text{for any} \ \ x \in [0,1].$$

Thus, the the whole sequence $\{v(\cdot, t_n)\}$ converges to $\overline{v}$ in the strong topology of $C([0,1])$ - namely (3.3.10) follows. Concerning the sequence $\{u(\cdot, t_n)\}$ we have to distinguish the cases $a \leq M_{u_0} \leq d$ and $M_{u_0} < a$, $M_{u_0} > d$.

(i) Assume $a \leq M_{u_0} \leq d$. Observe that, in view of the uniform convergence (3.3.10) we can use Proposition 3.3.4 and obtain

$$\begin{aligned}
&\lambda_i(x, t_{n+1}) \geq \lambda_i(x, t_n), && \text{if } A < \overline{v} < B \quad (i = 1,2),\\
&\lambda_1(x, t_{n+1}) \geq \lambda_1(x, t_n), && \text{if } \overline{v} = A,\\
&\lambda_2(x, t_{n+1}) \geq \lambda_2(x, t_n), && \text{if } \overline{v} = B
\end{aligned}$$

for *a.e.* $x \in (0,1)$ and for n large enough. Hence, arguing as in the proof of Theorem 3.3.6 gives:

$$\begin{aligned}
&\lambda_i(x, t_{n+1}) \to \lambda_i^*(x), && \text{if } A < \overline{v} < B \quad (i = 1,2),\\
&\lambda_1(x, t_{n+1}) \to \lambda_1^*(x), && \text{if } \overline{v} = A,\\
&\lambda_2(x, t_{n+1}) \to \lambda_2^*(x), && \text{if } \overline{v} = B
\end{aligned}$$

for *a.e.* $x \in (0,1)$, where the coefficients λ_i^* are uniquely determined by Proposition 3.5.4. Observe that the above convergences and (3.3.10) imply (3.3.11) and this concludes the proof in the case $a \leq M_{u_0} \leq d$.

(ii) Now assume $M_{u_0} < a$ (if $M_{u_0} > d$ the claim follows by similar arguments). Recall that in this case $\overline{v} = \phi(M_{u_0}) < A$ (see (3.3.5) in Theorem 3.3.5). Moreover, in view of Theorem 3.3.5-(ii) again, there holds $v(\cdot, t_n) \leq A_{M_{u_0}} < A$ in $(0,1)$ for n large enough. Hence

$$u(x, t_n) = s_1(v(x, t_n)) \quad \text{for any } x \in (0,1). \tag{3.5.59}$$

Observe that for any $x \in (0,1)$ there holds

$$\begin{aligned}
|\, s_1(v(x,t_n)) - M_{u_0}| \ &\equiv \ |\, s_1(v(x,t_n)) - s_1(\phi(M_{u_0}))| \ \leq \\
&\leq \ C_{M_{u_0}} \|\, v(\cdot, t_n) - \phi(M_{u_0}) \|_{C([0,1])}
\end{aligned} \tag{3.5.60}$$

where

$$C_{M_{u_0}} := \|\, s_1' \|_{L^\infty(\phi(M_{u_0}) - \epsilon, \phi(M_{u_0}) + \epsilon)} < \infty$$

for some fixed $\epsilon > 0$, small enough. In fact, by assumption $M_{u_0} < a$ we can choose ϵ such that $\phi(M_{u_0}) + \epsilon < A$ (recall that $s_1'(A) = +\infty$), hence

$$\|\, s_1' \|_{L^\infty(\phi(M_{u_0}) - \epsilon, \phi(M_{u_0}) + \epsilon)} < \infty.$$

Since the right-hand side in (3.5.60) approaches zero as $n \to \infty$, the uniform convergence (3.3.12) holds. □

Chapter 4

Long-time behaviour of two-phase solutions

4.1 Introduction

In this chapter we consider the Neumann initial-boundary value problem for the equation

$$u_t = \big[\phi(u)\big]_{xx} \qquad \text{in } \ Q := (-1,1)\times(0,\infty) \tag{4.1.1}$$

where the function ϕ satisfies the following assumption

$$(H_1) \qquad \begin{cases} \phi'(u) > 0 & \text{if } u \in (-\infty, b)\cup(c,\infty), \\ \phi'(u) < 0 & \text{if } u \in (b,c), \\ B := \phi(b) > \phi(c) =: A, \ \phi(u) \to \pm\infty \text{ as } u \to \pm\infty, \\ \phi''(b) \neq 0, \ \phi''(c) \neq 0. \end{cases} \tag{4.1.2}$$

We also denote by $a \in (-\infty, b)$ and $d \in (c,\infty)$ the roots of the equation $\phi(u) = A$, respectively $\phi(u) = B$ (see Fig.4.1).

In view of the non-monotone character of the non-linearity ϕ, equation (4.1.1) is of *forward-backward* parabolic type, since it is well-posed forward in time at the points where $\phi' > 0$ and it is ill-posed where $\phi' < 0$. In this connection, we denote by

$$S_1 := \{(u,\phi(u)) \mid u \in (-\infty,b)\} \equiv \{(s_1(v),v) \mid v \in (-\infty,B)\}$$

and

$$S_2 := \{(u,\phi(u)) \mid u \in (c,\infty)\} \equiv \{(s_2(v),v) \mid v \in (A,\infty)\}$$

the stable branches of the equation $v = \phi(u)$, whereas

$$S_0 := \{(u,\phi(u)) \mid u \in (b,c)\} \equiv \{(s_0(v),v)) \mid v \in (A,B)\}$$

is referred to as the unstable branch.

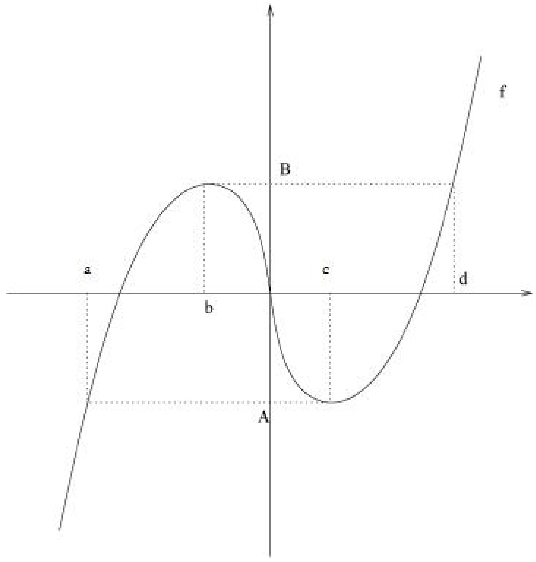

Figure 4.1: Assumption (H_1).

4.1.1 Motivations and related problems

Equation (4.1.1) with a function ϕ satisfying assumption (H_1) naturally arises in the theory of phase transitions. In this context, u represents the phase field and equation (4.1.1) describes the evolution between stable phases. With a non-linearity ϕ of a different shape, in particular for a ϕ which *vanishes* at infinity, equation (4.1.1) describes models in population dynamics ([Pa]), image processing ([PM]) and gradient systems associated with non-convex functionals ([BFG]).

The initial-boundary value problem for equation (4.1.1) (either under Dirichlet or Neumann boundary conditions) has been widely addressed in the literature. Most techniques consist in *modifying* the (possibly) ill-posed equation (hence the boundary conditions) with some regularization which leads to a well-posed problem. A natural question is whether the approximating solutions define a solution (in some suitable sense, depending on the regularization itself) of (4.1.1) as the regularization parameter goes to zero. Many regularizations of equation (4.1.1) have been proposed and investigated (see [BBDU], [NP], [Sl]). Among them, let us mention the *pseudoparabolic* or *Sobolev* regularization

$$u_t = \Delta\phi(u) + \varepsilon\Delta u_t, \tag{4.1.3}$$

which has been studied in [NP] for the corresponding Neumann initial-boundary value problem in $Q_T := \Omega \times (0, T)$, for any $T > 0$. In [Pl1] it is shown that the limiting points of the family of the approximating solutions $(u^\varepsilon,\ \phi(u^\varepsilon))$ are *weak entropy measure-valued solutions* (u, v) of the Neumann initial-boundary value problem in Q_T for the original equation (4.1.1). Precisely, it is shown that the couple (u, v) obtained in the limit $\varepsilon \to 0$ satisfies the following properties:

(i) $u \in L^\infty(Q_T),\ v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ and

$$u = \sum_{i=0}^{2} \lambda_i s_i(v)$$

for some $\lambda_i \in L^\infty(Q_T)$, $0 \le \lambda_i \le 1$ and $\sum_{i=0}^{2} \lambda_i = 1$;

(ii) the couple (u, v) solves in the weak sense the equation

$$u_t = \Delta v \qquad \text{in} \ \ \mathcal{D}'(Q_T); \tag{4.1.4}$$

(iii) the couple (u, v) satisfies the following class of *entropy inequalities*:

$$\int\int_{Q_T} [G^* \psi_t - g(v)\nabla v \nabla \psi + g'(v)|\nabla v|^2 \psi] dxdt + \\ + \int_\Omega G(u_0)\psi(x,0)dx \ge 0$$

for any $\psi \in C^1(\overline{Q}_T),\ \psi \ge 0,\ \psi(\cdot, T) \equiv 0$. Here, for any $g \in C^1(\mathbb{R}),\ g' \ge 0$,

$$G(\lambda) := \int^\lambda g(\phi(s))ds$$

and

$$G^* = \sum_{i=0}^{2} \lambda_i G(s_i(v)).$$

Actually, uniqueness in the class of weak entropy measure-valued solutions to the Neumann initial-boundary value problem for equation (4.1.1) is unknown, albeit this class seems a natural candidate in this sense, in view of the entropy inequalities (see also [H] and [Z] for general results of nonuniqueness). A natural question is whether uniqueness can be recovered by introducing some additional constraints. To this purpose, *two-phase* solutions have been introduced in [EP] and investigated in [MTT2] (see also [MTT]). Roughly speaking, a two-phase solution of the Neumann initial-boundary value problem associated to equation (4.1.1) in $Q_T = (-1,1) \times (0,T)$ is a weak entropy measure-valued solution (u, v) (in the sense of [Pl1]) which describes transitions only between stable phases. Such solutions exhibit a smooth interface $\xi : [0,T] \to [-1,1]$ such that

$$
\begin{array}{ll}
u = s_1(v) & \text{in } \; \{(x,t) \in Q_T \mid \; -1 \le x < \xi(t)\} \\
u = s_2(v) & \text{in } \; \{(x,t) \in Q_T \mid \xi(t) < x \le 1\} \;,
\end{array}
$$

where s_1 and s_2 denote the first and the second stable branch of the equation $v = \phi(u)$. It is worth observing that the interface ξ evolves obeying admissibility conditions which follows from the entropy inequalities (see Definition 4.2.1 in Subsection 4.2.1).

Uniqueness and local existence of two-phase solutions of the Cauchy problem for equation (4.1.1) under assumption (H_1) has been proved in [MTT2] (the proof of similar results for the Neumann initial-buondary value problem was outlined in [MTT]). Actually, *global* existence of such solutions is not known, albeit it is plenty addressed.

Assuming global exixtence, we investigate the long-time behaviour of two-phase solutions to the Neumann initial-boundary value problem for equation (4.1.1), proving asymptotic results concerning both $v(\cdot,t)$ and the interface $\xi(t)$.

4.2 Mathematical framework and results

4.2.1 Properties and Basic Estimates

Consider the initial-boundary value problem

$$
\begin{cases}
u_t = \big[\phi(u)\big]_{xx} & \text{in } \; (-1,1)\times(0,\infty) := Q, \\
\big[\phi(u)\big]_x = 0 & \text{in } \; \{-1,1\}\times(0,\infty), \\
u = u_0 & \text{in } \; (-1,1)\times\{0\}\,,
\end{cases}
\tag{4.2.1}
$$

where $u_0 \in L^\infty(-1,1)$ satisfies the following assumption

$$
(A) \qquad \begin{cases}
u_0 \le b \; \text{ in } \; (-1,0)\,, \;\; u_0 \ge c \; \text{ in } \; (0,1)\,, \\
\phi(u_0) \in C([-1,1]).
\end{cases}
$$

Following [MTT], we give the definition of *two-phase* solutions to problem (4.2.1).

Denote by $C^{2,1}(Q)$ the set of functions $f \in C(Q)$ such that f_x, f_{xx}, $f_t \in C(Q)$.

Definition 4.2.1. *By a two-phase solution of problem* (4.2.1) *we mean any triple* (u, v, ξ) *such that:*

(i) $u \in L^\infty(Q)$, $v \in L^\infty(Q) \cap L^2((0,T); H^1(-1,1))$ *for any* $T > 0$ *and* $\xi : [0,\infty) \to [-1,1]$, $\xi \in C^1([0,\infty))$, $\xi(0) = 0$;

(ii) set

$$V_1 := \{(x,t) \in Q \mid -1 \le x < \xi(t)\,,\ t \in [0,\infty)\}\,, \tag{4.2.2}$$

$$V_2 := \{(x,t) \in Q \mid \xi(t) < x \le 1\,,\ t \in [0,\infty)\} \tag{4.2.3}$$

and

$$\gamma := \partial V_1 \cap \partial V_2 = \{(\xi(t), t) \mid t \in (0,\infty)\}\,. \tag{4.2.4}$$

Then, $u \in C^{2,1}(V_1) \cap C^{2,1}(V_2)$, $v(\cdot,t) \in C([-1,1])$ *for any* $t \ge 0$, *and there holds*

$$u = s_i(v) \qquad \text{a.e. in } V_i \qquad (i = 1,2); \tag{4.2.5}$$

(iii) for any $t \ge 0$ *there exist finite the limits*

$$\lim_{\eta \to 0} v_x(\xi(t) \pm \eta, t) := v_x(\xi(t)^{\pm}, t); \tag{4.2.6}$$

(iv) for any $T > 0$ *set* $Q_T := (-1,1) \times (0,T)$. *Then for any* $T > 0$ *there holds:*

$$\int\!\!\int_{Q_T} [u\psi_t - v_x\psi_x]dxdt + \int_{-1}^{1} u_0(x)\psi(x,0)dx = 0\,; \tag{4.2.7}$$

for any $\psi \in C^1(\overline{Q}_T)$, $\psi(\cdot,T) \equiv 0$ *in* $[-1,1]$;

(v) for any $g \in C^1(\mathbb{R})$, *set*

$$G(\lambda) := \int^{\lambda} g(\phi(s))ds; \tag{4.2.8}$$

then, for any $T > 0$ *and under the assumption* $g' \ge 0$, *the entropy inequalities*

$$\int\!\!\int_{Q_T} [G(u)\psi_t - g(v)v_x\psi_x - g'(v)v_x^2\psi]dxdt + \tag{4.2.9}$$

$$+ \quad \int_{-1}^{1} G(u_0(x))\psi(x,0)dx \ge 0$$

hold for any $\psi \in C^1(\overline{Q}_T)$, $\psi \ge 0$ *and* $\psi(\cdot,T) \equiv 0$ *in* $(-1,1)$.

Remark 4.2.1. *Observe that, in view of Definition 4.2.1, the following properties hold.*
(i) The function $v(\,.\,,t) \in H^1(-1,1)$ *for any* $t \geq 0$*. Moreover, the couple* (u,v) *is a classical solution of*

$$\begin{cases} u_t = \left[\phi(u)\right]_{xx} & \text{in } V_i, \\ u = u_0 & \text{in } \overline{V}_i \cap \{t=0\} \end{cases}$$

($i = 1, 2$*);*
(ii) the Rankine-Hugoniot condition

$$\xi' = -\frac{[v_x]}{[u]} \tag{4.2.10}$$

holds a.e. on γ*. Here* $[h] := h(\xi(t)^+, t) - h(\xi(t)^-, t)$ *denotes the jump across* γ *of any piecewise continuous function* h*;*
(iii) by the entropy inequalities (4.2.9)*, it follows that*

$$\xi'[G(u)] \geq -g(v)[v_x] \qquad \text{a.e. on } \gamma,$$

for any G *defined by* (4.2.8) *in terms of* $g \in C^1(\mathbb{R})$*,* $g' \geq 0$*. Observe that the above condition implies that*

$$\begin{cases} \xi' \geq 0 & \text{if } v = A, \\ \xi' \leq 0 & \text{if } v = B, \\ \xi' = 0 & \text{if } v \neq A, \ v \neq B. \end{cases} \tag{4.2.11}$$

Namely, jumps between the stable phases s_1 *and* s_2 *occur only at the points* (x,t) *where the function* $v(x,t)$ *takes the values* A *(jumps from* s_2 *to* s_1*) or* B *(jumps from* s_1 *to* s_2*).*

Uniqueness and local existence of two-phase solutions have been studied in [MTT2] for the Cauchy problem, under suitable assumptions on the initial datum u_0 and for a piecewise function ϕ. In [MTT] uniqueness of two-phase solutions to the Neumann initial-boundary value problem for equation (4.1.1) is proven. As already stated in the introduction, actually no resut concerning *global* existence of two-phase solutions (either for the Cauchy problem or for the Neumann initial-boundary value problem) is known, albeit it is plenty object of investigation. However, assuming global existence, the long-time behaviour of two-phase solutions to problem (4.2.1) presents very nice features and novelties with respect to the general case of weak entropy measure-valued solutions (see Chapter 3). Let us give more details.

To begin with, some a-priori estimates are in order. For any initial datum u_0 set

$$M_{u_0} := \frac{1}{2}\int_{-1}^{1} u_0(x)dx. \tag{4.2.12}$$

By the homogeneous Neumann boundary conditions in (4.2.1), we deduce the following result.

Proposition 4.2.1. *Let $u_0 \in L^\infty(-1,1)$ and let (u,v,ξ) be the two-phase solution of problem* (4.2.1) *with initial datum u_0. Then the following conservation law holds*

$$\frac{1}{2}\int_{-1}^{1} u(x,t)dx \equiv M_{u_0} \tag{4.2.13}$$

for any $t \geq 0$.

On the other hand, in view of the entropy inequalities (4.2.9), we obtain the two following results, whose role will be crucial in the latter.

Proposition 4.2.2. *Let (u,v,ξ) be a two-phase solution of problem* (4.2.1) *and for any $g \in C^1(\mathbb{R})$, let G be the function defined by* (4.2.8)*. Then:*

(i) for any $t_1 < t_2$ and for any $\varphi \in C^1([-1,1])$, $\varphi \geq 0$, there holds

$$\int_{-1}^{1} G(u(x,t_1))\varphi(x)\,dx - \int_{-1}^{1} G(u(x,t_2))\varphi(x)\,dx \geq \tag{4.2.14}$$

$$\geq \int_{t_1}^{t_2}\int_{-1}^{1} [g(v)v_x\varphi_x + g'(v)v_x^2\varphi]dxdt$$

for any $g \in C^1(\mathbb{R})$, $g' \geq 0$;

(ii) there exists

$$L_g := \lim_{t\to\infty}\int_{-1}^{1} G(u)(x,t)dx \tag{4.2.15}$$

for any non-decreasing g.

Proposition 4.2.3. *Let (u,v,ξ) be a two-phase solution of problem* (4.2.1)*. Then there exists $C > 0$ such that*

$$\int_{0}^{\infty}\int_{-1}^{1} v_x^2(x,t)dxdt \leq C. \tag{4.2.16}$$

4.2.2 Long-time behaviour

In the latter we denote by (u,v,ξ) any two-phase solution of problem (4.2.1). We begin by the following proposition.

Proposition 4.2.4. *Let (u, v, ξ) be the two-phase solution of problem* (4.2.1) *with initial datum u_0 and let M_{u_0} be defined by* (4.2.12)*. Then there exists a unique constant v^* such that for any diverging sequence $\{t_n\}$ there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and a set $E \subseteq (0, \infty)$ of Lebesgue measure $|E| = 0$, so that:*

$$v(\cdot, t + t_{n_k}) \to v^* \qquad in \;\; C([-1, 1]) \tag{4.2.17}$$

for any $t \in (0, \infty) \setminus E$. Moreover,

(i) $A \le v^* \le B$ if and only if $a \le M_{u_0} \le d$;

(ii) if $M_{u_0} < a$ (respectively $M_{u_0} > d$) then $v^* = \phi(M_{u_0})$ and for any $\varepsilon > 0$ there exists $T > 0$ such that $v(\cdot, t) < A - \varepsilon$ (respectively $v(\cdot, t) > B + \varepsilon$) in $[-1, 1]$ for any $t \ge T$.

The first step in the investigation of the long-time behaviour of two-phase solutions of problem (4.2.1) is the study of the interface $\xi(t)$ as t diverges. This is the content of the following theorem.

Theorem 4.2.5. *Let (u, v, ξ) be the two-phase solution of problem* (4.2.1) *with initial datum u_0, let M_{u_0} be defined by* (4.2.12) *and let v^* be the constant given by Proposition 4.2.4. Then, there exists*

$$\lim_{t \to \infty} \xi(t) =: \xi^*. \tag{4.2.18}$$

Moreover,

(i) if $A < v^* < B$ there exists $T > 0$ such that $\xi(t) = \xi^*$ for any $t \ge T$;

(ii) if $v^* < A$ (respectively, $v^* > B$) then $\xi^* = 1$ (respectively, $\xi^* = -1$) and there exists $T > 0$ such that $\xi(t) = 1$ (respectively, $\xi(t) = -1$) for any $t \ge T$.

Remark 4.2.2. *As a consequence of Proposition 4.2.4 and Theorem 4.2.5, when considering initial data u_0 of problem* (4.2.1) *with mass*

$$M_{u_0} < a \quad (or \;\; M_{u_0} > d),$$

there exists $T > 0$ such that for any $t \ge T$ there holds:

$$u(\cdot, t) = s_1(v(\cdot, t)) \quad (u(\cdot, t) = s_2(v(\cdot, t))$$

in $[-1, 1]$ (here (u, v, ξ) is the two-phase solution of (4.1.1) *with initial datum u_0).*

Now our aim is to estabilish whether, for any two-phase solution (u, v, ξ) of (4.2.1), there exists the limit as $t \to \infty$, in some suitable topology, of the families $v(\cdot, t)$ and $u(\cdot, t)$. In this direction, for any $k \in \mathbb{N}$ consider the sets

$$B_k := \left\{ t \in (0, \infty) \,|\, \int_{-1}^{1} v_x^2(x, t) dx < k \right\}, \tag{4.2.19}$$

and

$$A_k := (0,\infty) \setminus B_k = \left\{ t \in (0,\infty) \mid \int_{-1}^{1} v_x^2(x,t)dx \geq k \right\}. \qquad (4.2.20)$$

Observe that, $A_{k+1} \subseteq A_k$, $|A_k| \leq C/k$ by estimate (4.2.16), hence

$$\left| \bigcap_{k=1}^{\infty} A_k \right| = \lim_{k\to\infty} |A_k| = 0.$$

The following theorem describes the long-time behaviour of the function $v(\cdot,t)$ along any diverging sequence $\{t_n\}$.

Theorem 4.2.6. *Let (u,v,ξ) be the two-phase solution of problem* (4.2.1) *with initial datum u_0, let M_{u_0} be defined by* (4.2.12) *and let v^* be the constant given by Proposition 4.2.4. For any $k \in \mathbb{N}$, let B_k, $A_k \subseteq (0,\infty)$ be the sets defined by* (4.2.19) *and* (4.2.20)*, respectively. Then,*

(i) for any diverging sequence $\{t_n\} \subseteq B_k$ there holds

$$v(\cdot,t_n) \to v^* \qquad \text{in } C([-1,1]); \qquad (4.2.21)$$

(ii) for any diverging sequence $\{t_n\} \subseteq A_k$ there holds

$$v(\cdot,t_n) \to v^* \qquad \text{in } L^p(-1,1) \qquad (4.2.22)$$

for any $1 \leq p < \infty$.

The next step is the investigation of the long-time behaviour of the function $u(\cdot,t)$. Since by (4.2.2)-(4.2.5) in Definition 4.2.1

$$u(\cdot,t) = \chi_{(-1,\xi(t))}s_1(v(\cdot,t)) + \chi_{(\xi(t),1)}s_2(v(\cdot,t)) \qquad \text{in } (-1,1),$$

we have to take into account the asymptotic behaviour of the interface $\xi(t)$ (here χ_E denotes the characteristic function of any set $E \subseteq (-1,1)$). Therefore, combining Theorem 4.2.5 and Theorem 4.2.6 we show that the $u(\cdot,t)$ approaches the function u^*, where

$$u^* = \begin{cases} \chi_{(-1,\xi^*)}s_1(v^*) + \chi_{(\xi^*,1)}s_2(v^*) & \text{if } a \leq M_{u_0} \leq d \\ M_{u_0} & \text{if } M_{u_0} < a,\ M_{u_0} > d, \end{cases} \qquad (4.2.23)$$

as $t \to \infty$. This is the content of the following theorem.

Theorem 4.2.7. *Let* (u, v, ξ) *be the two-phase solution of problem* (4.2.1) *with initial datum* u_0. *Let* M_{u_0} *be defined by* (4.2.12), *let* ξ^* *be the constant given by Theorem 4.2.5 and let* u^* *be the function defined by* (4.2.23). *For any* $k \in \mathbb{N}$, *let* B_k, $A_k \subseteq (0, \infty)$ *be the sets defined by* (4.2.19) *and* (4.2.20), *repectively. Then,*

(i) for any diverging sequence $\{t_n\} \subseteq B_k$ *there holds*

$$u(x, t_n) \to u^* \qquad \textit{for any} \;\; x \in [-1, 1] \setminus \{\xi^*\} \tag{4.2.24}$$

if $a \leq M_{u_0} \leq d$; *otherwise*

$$u(\cdot, t_n) \to u^* \equiv M_{u_0} \qquad \textit{in} \;\; C([-1, 1]) \tag{4.2.25}$$

if $M_{u_0} < a, \; M_{u_0} > d$;

(ii) for any diverging sequence $\{t_n\} \subseteq A_k$ *there holds*

$$u(\cdot, t_n) \to u^* \qquad \textit{in} \; L^p(-1, 1) \tag{4.2.26}$$

for any $1 \leq p < \infty$.

Remark 4.2.3. *Convergences in Theorem 4.2.6-(ii) and Theorem 4.2.7-(ii) hold also in the weak* topology of the space* $L^\infty(-1, 1)$.

4.3 Proofs of Section 4.2.1

Proof of Proposition 4.2.1. Fix any $t > 0$ and for any $n \in \mathbb{N}$ set

$$h_n^t(s) = \begin{cases} 1 & \text{if } t \in [0, t), \\ -n(s - t - \frac{1}{n}) & \text{if } s \in [t, t + \frac{1}{n}]. \end{cases} \tag{4.3.1}$$

Choosing

$$\psi_n(x, s) := h_n^t(s)$$

as test function in the weak formulation (4.2.7) gives

$$n \int_t^{t+\frac{1}{n}} \int_{-1}^1 u(x, t) dx = \int_{-1}^1 u_0(x) dx,$$

hence (4.2.13) in the limit $n \to \infty$. This concludes the proof. □

Proof of Proposition 4.2.2 (i) Consider any $t_1 < t_2$ and for any $n \in \mathbb{N}$ set

$$h_n(t) = \begin{cases} n(t - t_1 + \frac{1}{n}) & \text{if } t \in [t_1 - \frac{1}{n}, t_1], \\ 1 & \text{if } t \in (t_1, t_2), \\ -n(t - t_2 - \frac{1}{n}) & \text{if } t \in [t_2, t_2 + \frac{1}{n}]. \end{cases}$$

Fix any $\varphi \in C^1([-1,1]),\ \varphi \geq 0$ and choose

$$\psi_n(x,t) := h_n(t)\varphi(x)$$

as test function in the entropy inequalities (4.2.9). We obtain

$$n\int_{t_1-1/n}^{t_1} dt \int_{-1}^{1} G(u)\varphi\, dx - n\int_{t_2}^{t_2+1/n} dt \int_{-1}^{1} G(u)\varphi\, dx \geq$$
$$\geq \int_{t_1-1/n}^{t_2+1/n} \int_{-1}^{1} h_n[g(v))v_x\varphi_x + \varphi g'(v)v_x^2]dxdt,$$

for any $g \in C^1(\mathbb{R}),\ g' \geq 0$. Hence, taking the limit as $n \to \infty$ in the previous inequality gives (4.2.14).

(ii) Observe that choosing $\varphi(x) \equiv 1$ in inequalities (4.2.14) gives

$$\int_{-1}^{1} G(u(x,t_1))dx \geq \int_{-1}^{1} G(u(x,t_2))dx \tag{4.3.2}$$

for any $t_1 \leq t_2$ and for any $g \in C^1(\mathbb{R}),\ g' \geq 0$ (recall that G is defined in terms of g by (4.2.8)). By standard arguments of approximation with smooth functions, the assumption $g \in C^1(\mathbb{R})$ can be dropped. Inequalities (4.3.2) imply that the map

$$t \mapsto \int_{-1}^{1} G(u(x,t))dx$$

is nonincreasing in $(0,\infty)$ for any non-decreasing g, hence the claim follows. □

Proof of Proposition 4.2.3. Let us choose in inequalities (4.2.14) $g(\lambda) = \lambda$ and $\varphi(\cdot) \equiv 1$ in $[-1,1]$. We obtain

$$\int_0^T \int_{-1}^{1} v_x^2(x,t)dxdt \leq \int_{-1}^{1} I(u_0)dx - \int_{-1}^{1} I(u(x,T))dx, \tag{4.3.3}$$

where

$$I(\lambda) := \int^{\lambda} \phi(s)ds.$$

Since $u \in L^\infty(Q)$ (see Definition 4.2.1-(i)) and $T > 0$ is arbitrary, inequalities (4.3.3) imply estimate (4.2.16). □

4.4 Proofs of Section 4.2.2

Most proofs of the results in Section 4.2.2 need the following technical results.

Let $BV(\mathbb{R})$ denote the space of real functions which have bounded total variation on $\mathbb{R}$.

Proposition 4.4.1. *Let* $v^1,\ v^2 \in [A,B]$ *and* $\xi^1,\ \xi^2 \in [-1,1]$ *be such that*

$$\begin{aligned} & (\xi^1+1)\int_0^{s_1(v^1)} g(\phi(s))ds + (1-\xi^1)\int_0^{s_2(v^1)} g(\phi(s))ds = \\ = \ & (\xi^2+1)\int_0^{s_1(v^2)} g(\phi(s))ds + (1-\xi^2)\int_0^{s_2(v^2)} g(\phi(s))ds, \end{aligned}$$

for any $g \in BV(\mathbb{R})$. *Then,* $v^1 = v^2$ *and* $\xi^1 = \xi^2$.

The proof of Proposition 4.4.1 is almost the same as in [ST] (see also Chapter 3), thus we omit it.

In order to prove Proposition 4.2.4, we begin by the following proposition.

Proposition 4.4.2. *Let* (u, v, ξ) *be the two-phase solution of problem* (4.2.1) *with initial datum* u_0 *and let* M_{u_0} *be defined by* (4.2.12). *Then, there exists a unique constant* v^* *such that*

$$v(\cdot, t_n) \to v^* \qquad in\ C([-1,1]) \tag{4.4.1}$$

for any diverging sequence $\{t_n\}$ *such that*

$$\int_{-1}^{1} v_x^2(x,t_n)dx \to 0 \qquad as\ n \to \infty. \tag{4.4.2}$$

Proof. Observe that for any diverging sequence $\{t_n\}$ satisfying (4.4.2) there exists a constant $k > 0$ such that:

$$\begin{aligned} |v(x_2,t_n) - v(x_1,t_n)| \ & \leq \ \left(\int_{-1}^{1} v_x^2(x,t_n)dx\right)^{1/2} |x_2 - x_1|^{1/2} \leq \\ & \leq \ k^{1/2}|x_2-x_1|^{1/2}, \end{aligned} \tag{4.4.3}$$

for any $x_1,\ x_2 \in [-1,1]$ and for any $n \in \mathbb{N}$ large enough. Moreover,

$$\|v(\cdot,t_n)\|_{C([-1,1])} \leq C. \tag{4.4.4}$$

(see Definition 4.2.1-(i)). Estimates (4.4.3) and (4.4.4) imply that the sequence $\{v(\cdot,t_n)\}$ is equi-continuous and uniformly bounded in $C([-1,1])$. We proceed in two steps.

(α) First we show that the sequence $\{v(\cdot, t_n)\}$ converges uniformly $[-1,1]$ to a constant v^{t_n}, possibly depending on $\{t_n\}$.

(β) Then we prove that v^{t_n} is independent of the choice of the sequence $\{t_n\}$. In other words there exists a unique $v^* \in \mathbb{R}$ such that (4.4.1) holds.

(α) Suppose that there exist two subesequences $\{t_n^1\}$, $\{t_n^2\} \subseteq \{t_n\}$ such that

$$\liminf_{n\to\infty} \|v(\cdot, t_n^1) - v(\cdot, t_n^2)\|_{C([-1,1])} \geq \delta \tag{4.4.5}$$

for some $\delta > 0$. On the other hand, we can assume that (eventually passing to subsequences)

$$v(\,.\,, t_n^j) \to v^j \qquad \text{in } C([-1,1]), \qquad (j = 1,2) \tag{4.4.6}$$

for some constants v^1, $v^2 \in [-1,1]$ (here use of (4.4.2) and (4.4.3) has been made). Moreover, we can suppose that

$$\xi(t_n^j) \to \xi^j \qquad \text{as } n \to \infty \qquad (j = 1,2). \tag{4.4.7}$$

Let us show that

$$v^1 = v^2 := v^{t_n}, \quad \xi^1 = \xi^2 = \xi^{t_n}. \tag{4.4.8}$$

In view of Definition 4.2.1-(ii) we have:

$$G(u(\cdot, t_n^j)) = \chi_{(-1,\xi(t_n^j))} G(s_1(v(\cdot, t_n^j))) + \chi_{(\xi(t_n^j),1)} G(s_2(v(\cdot, t_n^j)))\,, \tag{4.4.9}$$

hence by (4.4.6)-(4.4.7) there holds

$$G(u(\cdot, t_n^j)) \overset{*}{\rightharpoonup} \chi_{(-1,\xi^j)} G(s_1(v^j)) + \chi_{(\xi^j,1)} G(s_2(v^j)) \qquad (j = 1,2) \tag{4.4.10}$$

for any G defined by (4.2.8) in terms of any $g \in BV(\mathbb{R})$. Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \int_{-1}^{1} G(u(x, t_n^j))dx = \\ = \quad &(\xi^j + 1) G(s_1(v^j)) + (1 - \xi^j) G(s_2(v^j)) \qquad (j = 1,2). \end{aligned}$$

On the other hand, for any G defined by (4.2.8) in terms of any $g \in BV(\mathbb{R})$, by (4.2.15) there holds

$$\lim_{n\to\infty} \int_{-1}^{1} G(u(x, t_n^1))dx = \lim_{n\to\infty} \int_{-1}^{1} G(u(x, t_n^2))dx\,,$$

namely:

$$\begin{aligned} &(\xi^1 + 1) G(s_1(v^1)) + (1 - \xi^1) G(s_2(v^1)) = \\ = \quad &(\xi^2 + 1) G(s_1(v^2)) + (1 - \xi^2) G(s_2(v^2))\,. \end{aligned}$$

The above equality implies (4.4.8) (see Proposition 4.4.1) which is in clear contradiction with (4.4.5).

(β) Now assume that there exist two diverging sequences $\{t_n^1\}$ and $\{t_n^2\}$ satisfying (4.4.2) such that

$$v(\,.\,,t_n^j) \to v^j \qquad \text{in } C([-1,1]), \qquad (j=1,2) \tag{4.4.11}$$

for some constants $v_1,\ v_2$. Moreover, we can suppose that

$$\xi(t_n^j) \to \xi^j \qquad \text{as } n \to \infty, \tag{4.4.12}$$

Arguing as in *Step (α)* gives equality

$$\begin{aligned} &(\xi^1+1)G(s_1(v^1)) + (1-\xi^1)G(s_2(v^1)) = \\ =\ &(\xi^2+1)G(s_1(v^2)) + (1-\xi^2)G(s_2(v^2)) \end{aligned}$$

for any G defined by (4.2.8) in terms of any $g \in BV(\mathbb{R})$. This implies $v^1 = v^2$ (see Proposition 4.4.1) and the claim follows. □

Proof of Proposition 4.2.4. For any diverging sequence $\{t_n\}$, set

$$v_{t_n}(x,t) := v(x,t+t_n) \qquad \text{for } x \in [-1,1],\ \ t \geq 0.$$

Since

$$\int_0^\infty \int_{-1}^1 (v_{t_n})_x^2(x,t)dxdt = \int_{t_n}^\infty \int_{-1}^1 v_x^2(x,s)dxds \to 0$$

as $n \to \infty$ (see (4.2.16)), there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and a set $E \subseteq (0,\infty)$ of Lebesgue measure $|E| = 0$ such that:

$$\int_{-1}^1 v_x^2(x,t+t_{n_k})dx \to 0$$

for any $t \in (0,\infty) \setminus E$. Hence, by Proposition 4.4.2 convergence (4.2.17) follows.

Fix any $\{t_n\},\ t_n \to \infty$ such that $v(\cdot,t_n)$ converge uniformly to v^* in $[-1,1]$. The conservation law (4.2.13) implies

$$(1+\xi^*)s_1(v^*) + (1-\xi^*)s_2(v^*) = 2M_{u_0} \tag{4.4.13}$$

where M_{u_0} is defined by (4.2.12) and ξ^* is some value in $[-1,1]$ such that, eventually up to a subsequence, $\xi(t_n) \to \xi^*$. Thus:

(*i*) if $a \leq M_{u_0} \leq d$, suppose $v^* < A$ (hence $\xi^* = 1$), so that (4.4.13) reduces to

$$a > s_1(v^*) = M_{u_0},$$

which gives an absurd. Analogously we can show that $v^* \leq B$. Hence $v^* \in [A, B]$ in this case.
If $M_{u_0} < a$ (the case $M_{u_0} > d$ is analogous), suppose that $v^* \geq A$. Again, in view of (4.4.13), we obtain

$$\begin{aligned} 2a &\leq (\xi^* + 1)s_1(A) + (1 - \xi^*)s_1(A) \leq \\ &\leq (\xi^* + 1)s_1(v^*) + (1 - \xi^*)s_2(v^*) = 2M_{u_0}, \end{aligned}$$

which gives a contradiction.
(ii) Finally, let us prove the last claim of Proposition 4.2.4 (again in the case $M_{u_0} < a$). In this direction, fix any $\{t_n\}$, $t_n \to \infty$ such that

$$v(\cdot, t_n) \to v^* \qquad \text{in } C([-1, 1]).$$

It follows that, for any $\varepsilon > 0$ small enough, there exists $\overline{n} \in \mathbb{N}$, such that

$$v(\cdot, t_n) \leq v^* - 2\varepsilon \leq A - \varepsilon \tag{4.4.14}$$

for any $n \geq \overline{n}$. Set

$$T := t_{\overline{n}},$$

and

$$g_{A-\varepsilon}(s) := \begin{cases} 0 & \text{if } s \leq A - \varepsilon, \\ > 0 & \text{if } s > A - \varepsilon\,. \end{cases}$$

Assume that $g_{A-\varepsilon}$ is non-decreasing on $\mathbb{R}$. Observe that

$$G_{A-\varepsilon}(\lambda) := \int_{s_1(A-\varepsilon)}^{\lambda} g_{A-\varepsilon}(\phi(s))ds = \begin{cases} 0 & \text{if } \lambda \leq s_1(A - \varepsilon), \\ > 0 & \text{if } \lambda > s_1(A - \varepsilon)\,. \end{cases} \tag{4.4.15}$$

In view of (4.2.14), for any $t \geq T$ we obtain

$$\begin{aligned} 0 &\leq \int_{-1}^{\xi(t)} G_{A-\varepsilon}(s_1(v(x,t)))dx + \int_{\xi(t)}^{1} G_{A-\varepsilon}(s_2(v(x,t)))dx \leq \\ &\leq \int_{-1}^{1} G_{A-\varepsilon}(s_1(v(x,T)))dx = 0 \end{aligned} \tag{4.4.16}$$

(here use of Definition 4.2.1-(ii), (4.4.14) and (4.4.15) has been made), which implies $v(\cdot, t) \leq A - \varepsilon$ for any $t \geq T$. This concludes the proof. □

The following Lemma gives properties of monotonicity in time of the inteface $\xi(t)$.

Lemma 4.4.3. *Let* (u, v, ξ) *be the two-phase solution of problem* (4.2.1) *with initial datum* u_0 *and let* v^* *be the constant given by Proposition 4.4.2. Then there exists* $T > 0$ *such that the map*

$$t \mapsto \xi(t)$$

for $t \geq T$ is:

(i) non-decreasing if $v^* < B$;

(ii) non-increasing if $v^* > A$.

Proof. (i) Assume $v^* < B$. Consider any sequence $\{t_n\}$, $t_n \to \infty$, such that

$$v(\cdot, t_n) \to v^* \qquad \text{in } C([-1,1])$$

(here use of Proposition 4.2.4 has been made). Since $v^* < B$, there exists $\overline{n} \in \mathbb{N}$ such that $v(\cdot, t_n) \leq B$ for any $n \geq \overline{n}$. Set

$$T := t_{\overline{n}};$$

write inequality (4.2.14) for $\varphi \equiv 1$ in $[-1,1]$ and

$$g_{AB}(s) = \begin{cases} 0 & \text{for } s \leq B, \\ > 0 & \text{for } s > B \end{cases}$$

where g_{AB} is non-decreasing. Using Definition 4.2.1-(ii), for any $t \geq T$, we have

$$\int_{-1}^{\xi(t)} G_{AB}(s_1(v(x,t)))dx + \int_{\xi(t)}^{1} G_{AB}(s_2(v(x,t)))dx \leq \tag{4.4.17}$$
$$\leq \int_{-1}^{\xi(T)} G_{AB}(s_1(v(x,T)))dx + \int_{\xi(T)}^{1} G_{AB}(s_2(v(x,T)))dx = 0,$$

by our choice of T and by the uniform convergence of $v(\cdot, t_n)$ to v^* in $[-1,1]$ (here G_{AB} is defined by (4.2.8) in correspondence of g_{AB}). On the other hand, observe that the non-negative function

$$G_{AB}(\lambda) := \int_0^\lambda g_{AB}(\phi(s))ds$$

is strictly positive for any $\lambda > s_2(B)$, thus inequality (4.4.17) implies

$$v(\cdot, t) \leq B \qquad \text{for any } t \geq T. \tag{4.4.18}$$

Next, for any $\rho > 0$, set

$$g_\rho(s) := \begin{cases} 0 & \text{if } \; s < B - \rho, \\ \rho^{-1/2} & \text{if } \; B - \rho \leq s \leq B\,. \end{cases}$$

Set

$$G_\rho(\lambda) := \int_0^\lambda g_\rho(\phi(s))ds$$

and consider the entropy inequalities (4.2.14) for $g = g_\rho$ and $t_2 \geq t_1 \geq T$. We obtain

$$\begin{aligned}
& \left(\int_{-1}^{\xi(t_1)} G_\rho(s_1(v(x,t_1)))\varphi(x)dx + \int_{\xi(t_1)}^{1} G_\rho(s_2(v(x,t_1)))\varphi(x)dx \right) + \\
- & \left(\int_{-1}^{\xi(t_2)} G_\rho(s_2(v(x,t_2)))\varphi(x)dx + \int_{\xi(t_2)}^{1} G_\rho(s_2(v(x,t_2)))\varphi(x)dx \right) \geq \\
\geq & \int_{t_1}^{t_2} \int_{-1}^{1} g_\rho(v(x,t))v_x(x,t)\varphi_x(x)dxdt = \\
= & -\int_{t_1}^{t_2} \int_{-1}^{1} \varphi_{xx}(x) \left(\int_0^{v(x,t)} g_\rho(s)ds \right) dxdt \qquad (4.4.19)
\end{aligned}$$

for any $\varphi \in C_c^1(-1,1)$, $\varphi \geq 0$. Concerning the right-hand side of (4.4.19), we have

$$\begin{aligned}
& \left| \int_{t_1}^{t_2} \int_{-1}^{1} \varphi_{xx}(x) \left(\int_0^{v(x,t)} g_\rho(s)ds \right) dxdt \right| = \qquad (4.4.20) \\
= & \left| \int_{t_1}^{t_2} \int_{\{v(x,t) \geq B-\rho\}} \rho^{-1/2}(v(x,t) - B + \rho)\varphi_{xx}(x)dxdt \right| \leq \\
\leq & \ \rho^{1/2} \int_{t_1}^{t_2} \int_{-1}^{1} |\varphi_{xx}(x)|dx \to 0
\end{aligned}$$

as $\rho \to 0$ (here use of (4.4.18) has been made). Next, observe that, for any $t \geq T$, there holds

$$\begin{aligned}
& \int_{-1}^{\xi(t)} G_\rho(s_1(v(x,t)))\varphi(x)dx + \int_{\xi(t)}^{1} G_\rho(s_2(v(x,t)))\varphi(x)dx = \\
= & \int_{-1}^{\xi(t)} \chi_{\{v(x,t)<B-\rho\}}(x,t) \left(\int_{s_0(B-\rho)}^{s_1(B-\rho)} \rho^{-1/2}ds \right) dx + \\
+ & \int_{-1}^{\xi(t)} \chi_{\{v(x,t)\geq B-\rho\}}(x,t) \left(\int_{s_0(B-\rho)}^{s_1(v(x,t))} \rho^{-1/2}ds \right) dx + \\
+ & \int_{\xi(t)}^{1} \chi_{\{v(x,t)\geq B-\rho\}}(x,t) \left(\int_{s_2(B-\rho)}^{s_2(v(x,t))} \rho^{-1/2}ds \right) dx; \qquad (4.4.21)
\end{aligned}$$

Since $\phi''(b) \neq 0$ (see Assumption (H_1)), it follows that

$$\begin{aligned}
& \lim_{\rho \to 0} \int_{-1}^{\xi(t)} G_\rho(s_1(v(x,t)))\varphi(x)dx + \int_{\xi(t)}^{1} G_\rho(s_2(v(x,t)))\varphi(x)dx = \\
= & \ -C \int_{-1}^{\xi(t)} [2\chi_{\{v(x,t)<B\}}(x,t) + \chi_{\{v(x,t)=B\}}(x,t)]\varphi(x)dx, \qquad (4.4.22)
\end{aligned}$$

for some $C > 0$, depending on the value $\phi''(b)$. Thus, in view of (4.4.20)-(4.4.22), taking the limit as $\rho \to 0$ in (4.4.19) gives

$$\begin{aligned} &\int_{-1}^{\xi(t_1)} [2\chi_{\{v(x,t_1)<B\}} + \chi_{\{v(x,t_1)=B\}}]\varphi(x)dx \leq \\ \leq\ &\int_{-1}^{\xi(t_2)} [2\chi_{\{v(x,t_2)<B\}} + \chi_{\{v(x,t_2)=B\}}]\varphi(x)dx, \end{aligned} \qquad (4.4.23)$$

for any $\varphi \in C_c^1(-1,1)$, $\varphi \geq 0$. Ruling out of contradiction, suppose that $\xi(t_2) < \xi(t_1)$, fix any $\overline{x} \in (\xi(t_2), \xi(t_1))$ and observe that (4.4.23) implies

$$0 < 2\chi_{\{v(x,t_1)<B\}}(\overline{x}, t_1) + \chi_{\{v(x,t_1)=B\}}(\overline{x}, t_1) \leq 0,$$

which gives an absurd. Hence, $\xi(t_2) \geq \xi(t_1)$ for any $t_2 \geq t_1 \geq T$.

(ii) The case $v^* > A$ can be treated in a similar way □

Proof of Theorem 4.2.5. Let us distinguish the cases $A < v^* < B$, $v^* = A$, $v^* = B$ and $v^* < A$, $v^* > B$.
(i) If $A < v^* < B$, in view of Lemma 4.4.3 there exists $T > 0$ such that $\xi(t_1) \leq \xi(t_2) \leq \xi(t_1)$ for any $t_2 \geq t_1 \geq T$. Hence for any $t \geq T$ the function $\xi(t)$ is constant and the claim folows.

(ii) In the case $v^* = A$ ($v^* = B$), in view of Lemma 4.4.3 there exists $T > 0$ such that the map $t \mapsto \xi(t)$ is non-decreasing (non-increasing) on (T, ∞) and again (4.2.18) holds.

(iii) If $v^* < A$, by Proposition 4.2.4-(ii) there exists $T > 0$ such that $v(\cdot, t) < A$ in $[-1, 1]$ for any $t \geq T$. Hence, in view of Definition 4.2.1-(ii), $u(\cdot, t) = s_1(v(\cdot, t))$ - namely, $\xi(t) = 1$ - for any $t \geq T$.

(iv) In the case $v^* > B$, by Proposition 4.2.4-(ii) there exists $T > 0$ such that $v(\cdot, t) > B$ in $[-1, 1]$ for any $t \geq T$. Hence, in view of Definition 4.2.1-(ii), $u(\cdot, t) = s_2(v(\cdot, t))$ - namely, $\xi(t) = -1$ - for any $t \geq T$. . □

Proof of Theorem 4.2.6. Let v^* and ξ^* be the constants given by Proposition 4.2.4 and Theorem 4.2.5, respectively. Fix any $k \in \mathbb{N}$ and consider any $\{t_n\} \subseteq B_k$. We have

$$\sup_{n\in\mathbb{N}} \int_{-1}^{1} v_x^2(x, t_n)dx \leq k, \qquad (4.4.24)$$

hence

$$\begin{aligned} |v(x_2, t_n) - v(x_1, t_n)| &\leq \left(\int_{-1}^{1} v_x^2(x, t_n)dx \right)^{1/2} |x_2 - x_1|^{1/2} \leq \\ &\leq k^{1/2} |x_2 - x_1|^{1/2}, \end{aligned} \qquad (4.4.25)$$

for any $x_1,\ x_2 \in [-1,1]$. Moreover,

$$\|v(\cdot,t_n)\|_{C([-1,1])} \leq C \tag{4.4.26}$$

(see Definition 4.2.1-(i)). Estimates (4.4.25) and (4.4.26) imply that the sequence $\{v(\cdot,t_n)\}$ is equi-continuous and uniformly bounded in $C([-1,1])$, thus there exists $\tilde{v} \in C([-1,1])$ such that, eventually passing to a subsequence, there holds

$$v(\cdot,t_n) \to \tilde{v} \qquad \text{in } C([-1,1]).$$

Let us show that

$$\tilde{v} \equiv v^* \qquad \text{in } [-1,1]. \tag{4.4.27}$$

To this purpose, we can find two sequences $\{t_n^1\},\ \{t_n^2\}$ such that

$$v(\cdot,t_n^i) \to v^* \qquad \text{in } C([-1,1]), \qquad\qquad (i=1,2)$$

and

$$t_n^1 \leq t_n \leq t_n^2, \qquad |t_n - t_n^i| \leq 1$$

for any $n \in \mathbb{N},\ i=1,2$ (here use of Proposition 4.2.4 has been made). Then, in view of inequalities (4.2.14), we obtain

$$\begin{aligned}
&\left(\int_{-1}^{1} G(u(x,t_n^1))\varphi(x)dx - \int_{-1}^{1} G(u(x,t_n))\varphi(x)dx\right) \geq \\
\geq\ &\int_{t_n^1}^{t_n}\int_{-1}^{1} g(v(x,t))v_x(x,t)\varphi_x(x)dxdt, \qquad (4.4.28)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{-1}^{1} G(u(x,t_n))\varphi(x)dx - \int_{-1}^{1} G(u(x,t_n^2))\varphi(x)dx\right) \geq \\
\geq\ &\int_{t_n}^{t_n^2}\int_{-1}^{1} g(v(x,t))v_x(x,t)\varphi_x(x)dxdt, \qquad (4.4.29)
\end{aligned}$$

for any G defined by (4.2.8) in terms of any $g \in C^1(\mathbb{R}),\ g' \geq 0$, and for any $\varphi \in C^1([-1,1]),\ \varphi \geq 0$. In view of estimate (4.2.16), there holds

$$\left|\int_{t_n}^{t_n^i}\int_{-1}^{1} v_x^2(x,t)dxdt\right| \to 0,$$

thus, passing to the limit as $n \to \infty$ in (4.4.28) and (4.4.29) gives

$$\begin{aligned}
&\int_{-1}^{\xi^*} G(s_1(v^*))\varphi(x)dx + \int_{\xi^*}^{1} G(s_2(v^*))\varphi(x)dx \leq \\
\leq\ &\int_{-1}^{\xi^*} G(s_1(\tilde{v}(x)))\varphi(x)dx + \int_{\xi^*}^{1} G(s_2(\tilde{v}(x)))\varphi(x)dx \leq \\
\leq\ &\int_{-1}^{\xi^*} G(s_1(v^*))\varphi(x)dx + \int_{\xi^*}^{1} G(s_2(v^*))\varphi(x)dx.
\end{aligned}$$

Observe that the above equality implies

$$s_1(v^*) = s_1(\tilde{v}(x)) \quad \text{for any } x \in (-1, \xi^*),$$

and

$$s_2(v^*) = s_2(\tilde{v}(x)) \quad \text{for any } x \in (\xi^*, 1).$$

Since s_1 and s_2 are strictly monotone functions, (4.4.27) follows.

(ii) Fix any $k > 0$ and any sequence $\{t_n\} \subseteq A_k$. If

$$\sup_{n\in\mathbb{N}} \int_{-1}^{1} v_x^2(x, t_n) dx < \infty$$

we can argue as in the proof of (i). Therefore suppose

$$\sup_{n\in\mathbb{N}} \int_{-1}^{1} v_x^2(x, t_n) dx = \infty.$$

In this case the sequence $\{v(\cdot, t_n)\}$ need not be relatively compact in the strong topology of $C([-1, 1])$. However, by means of Propositin 4.2.4 we can find two sequences $\{t_n^1\}$, $\{t_n^2\}$ such that

$$v(\cdot, t_n^i) \to v^* \qquad \text{in } C([-1, 1]), \qquad\qquad (i = 1, 2)$$

and

$$t_n^1 \le t_n \le t_n^2, \qquad |t_n - t_n^i| \le 1$$

for any $n \in \mathbb{N}$, $i = 1, 2,$. Arguing as above gives

$$\int_{-1}^{1} G(u(x, t_n^1))\varphi(x)dx - \int_{-1}^{1} G(u(x, t_n))\varphi(x)dx \ge$$
$$\ge \int_{t_n^1}^{t_n} \int_{-1}^{1} g(v(x, t))v_x(x, t)\varphi_x(x)dxdt, \tag{4.4.30}$$

and

$$\int_{-1}^{1} G(u(x, t_n))\varphi(x)dx - \int_{-1}^{1} G(u(x, t_n^2))\varphi(x)dx \ge$$
$$\ge \int_{t_n}^{t_n^2} \int_{-1}^{1} g(v(x, t))v_x(x, t)\varphi_x(x)dxdt, \tag{4.4.31}$$

for any $g \in C^1(\mathbb{R})$, $g' \ge 0$, and for any $\varphi \in C^1([-1, 1])$, $\varphi \ge 0$ (here G is defined by (4.2.8)). Thus, passing to the limit as $n \to \infty$ gives

$$\lim_{n\to\infty} \int_{-1}^{1} G(u(x, t_n))\varphi(x)dx = \tag{4.4.32}$$
$$= \int_{-1}^{\xi^*} G(s_1(v^*))\varphi(x)dx + \int_{\xi^*}^{1} G(s_2(v^*))\varphi(x)dx.$$

Observe that in view of Definition 4.2.1, we have

$$\int_{-1}^{1} G(u(x,t_n))\varphi(x)dx = \tag{4.4.33}$$
$$= \int_{-1}^{\xi(t_n)} G(s_1(v(x,t_n)))\varphi(x)dx + \int_{\xi(t_n)}^{1} G(s_2(v(x,t_n)))\varphi(x)dx$$

and, for any $\delta > 0$ we can assume

$$\xi^* - \delta < \xi(t_n) < \xi^* + \delta$$

for n large enough (by Theorem 4.2.5). Thus, by (4.4.32) and (4.4.33) we have

$$\lim_{n\to\infty} \int_{-1}^{\xi^*-\delta} |s_1(v(x,t_n))|^p \varphi(x)dx = \int_{-1}^{\xi^*-\delta} |s_1(v^*)|^p \varphi(x)dx$$

for any $\varphi \in C_c^1(-1, \xi^* - \delta)$ and $p > 1$ (here we have choosen $g(s) = p^{-1}|s_1|^{(p-1)}(s)$ in (4.4.32)) and

$$\lim_{n\to\infty} \int_{\xi^*+\delta}^{1} |s_2(v(x,t_n))|^p \varphi(x)dx = \int_{\xi^*+\delta}^{1} |s_2(v^*)|^p \varphi(x)dx$$

for any $\varphi \in C_c^\infty(\xi^* + \delta, 1)$ and $p > 1$ (here we have choosen $g(s) = p^{-1}|s_2|^{(p-1)}(s)$). In other words, by the arbitrariness of δ, we have proven that

$$s_1(v(\cdot,t_n)) \to s_1(v^*) \qquad \text{in } L^p(-1, \xi^*), \tag{4.4.34}$$

and

$$s_2(v(\cdot,t_n)) \to s_2(v^*) \qquad \text{in } L^p(\xi^*, 1), \tag{4.4.35}$$

As a consequence of the above convergences, we obtain

$$v(\cdot,t_n) \to v^* \qquad \text{in } L^p(-1,1),$$

for any $1 \le p < \infty$, and the claim (4.2.22) follows. □

Proof of Theorem 4.2.7. For any diverging sequence $\{t_n\}$, in view of Definition 4.2.1 we have

$$u(x,t_n) = \chi_{(-1,\xi(t_n))} s_1(v(x,t_n)) + \chi_{(\xi(t_n),1)} s_2(v(x,t_n)). \tag{4.4.36}$$

(*i*) Assume $\{t_n\} \subseteq B_k$, where B_k is the set defined by (4.2.19) for any $k \in \mathbb{N}$. Since $v(\cdot,t_n) \to v^*$ in $C([-1,1])$ by Theorem 4.2.6-(*i*) and $\xi(t_n) \to \xi^*$ by Theorem 4.2.5, taking the limit as $n \to \infty$ in (4.4.36) gives

$$u(x,t_n) \to u^*$$

for any $x \in [-1,1] \setminus \xi^*$, the function u^* being defined by (4.2.23). Moreover, if $M_{u_0} < a$ (respectively $M_{u_0} > d$) $v^* = \phi(M_{u_0})$ (see Proposition 4.2.4-(ii)) and equation (4.4.36) reduces to

$$u(x,t_n) = s_1(v(x,t_n)) \qquad (u(x,t_n) = s_2(v(x,t_n)))$$

for $n \in \mathbb{N}$ large enough (see Remark 4.2.2). Therefore $u(\cdot,t_n) \to M_{u_0}$ uniformly in $[-1,1]$ by Theorem 4.2.6-(i).

(ii) Now asume $\{t_n\} \subseteq A_k$, where A_k is the set defined by (4.2.20). In this case $v(\cdot,t_n) \to v^*$ in $L^p(-1,1)$ for any $1 \leq p < \infty$ (see Theorem 4.2.6-(ii)) and $\xi(t_n) \to \xi^*$ (see Theorem 4.2.5), hence passing to the limit in (4.4.36) gives (4.2.26) and the claim follows. □

Bibliography

[B] H. Brezis, “Analyse Fonctionnelle”, Masson Editeur, Paris, 1983.

[BBDU] G. I. Barenblatt, M. Bertsch, R. Dal Passo & M. Ughi, *A degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow*, SIAM J. Math. Anal. **24** (1993), 1414–1439.

[Be] Ph. Benilan, *Opérateurs accrétifs et semi-groupes dans les espaces L^p ($1 \leq p < \infty$)*, in Functional Analysis and Numerical Analysis, Japan Society for the Promotion of Sciences, Tokyo, 1978,

[BFG] G. Bellettini, G. Fusco & N. Guglielmi, *A concept of solution and numerical experiments for forward-backward diffusion equations*, Discrete Contin. Dyn. Syst. **16** (2006), 783–842.

[BFJ] K. Binder, H.L. Frisch & J. Jäckle, *Kinetics of phase separation in the presence of slowly relaxing structural variables*, J. Chem. Phys. **85** (1986), 1505–1512.

[BNPT] G. Bellettini, M. Novaga, M. Paolini & C. Tornese, *Convergence of discrete schemes for the Perona-Malik equation*, J. Differential Equations **245** (2008) 892–924.

[BS] (1411908) M. Brokate & J. Sprekels, *Hysteresis and phase transitions*, Appl. Math. Sci. **121** (Springer, 1996).

[BU] M. Bertsch, M. Ughi, *Positivity properties of viscosity solutions of a degenerate parabolic equation*, Nonlinear Anal. **14** (1990), 571-592.

[C] J. W. Cahn, *On spinodal decomposition*, Acta Metall. **9** (1961), 795

[D] S. Demoulini, *Young measures solutions for a nonlinear parabolic equation of forward-backward type*, SIAM J. Math. Anal. **27** (1996), 376–403

[E1] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS Reg. Conf. Ser. Math., **74**, Amer. Math. Soc., Providence (1990).

[E2] L. C. Evans, *A survey of entropy methods for partial differential equations*, Bull. Amer. Math. Soc. **41** (2004), 409–438.

[EP] L. C. Evans & M. Portilheiro, *Irreversibility and hysteresis for a forward-backward diffusion equation*, Math. Models Methods Appl. Sci. **14** (2004), 1599–1620.

[FGP] F. Fierro, R. Goglione & M. Paolini, *Numerical simulations of mean curvature flow in presence of a non convex anisotropy*, Math. Models Method Appl. Sci. **8** (1998), 573–601.

[G] M. Gobbino, *Entire solutions of the one-dimensional Perona-Malik equation*, Comm. Partial Differential Equations, **32** (2007), 719–743.

[GG] M. Ghisi & M. Gobbino, *A class of local solutions for the one-dimensional Perona-Malik equation*, preprint (2006).

[GMS] M. Giaquinta, G. Modica & J. Souček, *Cartesian Corrents in the Calculus of Variations* (Springer, 1998).

[H] K. Höllig, *Existence of infinitely many solutions for a forward backward heat equation*, Trans. Amer. Math. Soc. **278** (1983), 299–316.

[HPO] D. Horstmann, K.J. Painter & H.G. Othmer, *Aggregation under local reinforcement, from lattice to continuum*, Eur. J. Appl. Math. **15** (2004), 545–576.

[K] S. Kichenassamy, *The Perona-Malik paradox*, SIAM J. Appl. Math., **57** (1997), 1328–1342.

[KK] B. Kawohl & N. Kutev, *Maximum and comparison principle for one-dimensional anisotropic diffusion*, Math. Ann., **311** (1998), 107–123.

[LSU] O.A. Ladyženskaja, V.A. Solonnikov & N.N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic type" Trans. of Math. Mono, Providence, 1968.

[MTT] C. Mascia, A. Terracina & A. Tesei, *Evolution of stable phases in forward-backward parabolic equations*, in: *Asymptotic Analysis and Singularities* (edited by H. Kozono, T. Ogawa, K. Tanaka, Y. Tsutsumi & E. Yanagida), pp. 451-478, Advanced Studies in Pure Mathematics **47-2** (Math. Soc. Japan, 2007).

[MTT2] C. Mascia, A. Terracina & A. Tesei, *Two-phase entropy solutions of a forward-backward parabolic equation*, Archive Rational Mech. Anal. (to appear)

[Mü] S. Müller, *Variational models for microstructure and phase transitions*, in " Calculus of variations and geometric evolution problems ", Lecture Notes in Math. **1713**, Springer (1999), 85–210.

[NMS] M. Nitzberg, D. Mumford & T. Shiota, *Filtering, Segmentation and Depth*, Lecture Notes in Computer Science, Vol. 662, Springer-Verlag, Berlin (1993).

[NP] A. Novick-Cohen & R. L. Pego, *Stable patterns in a viscous diffusion equation*, Trans. Amer. Math. Soc., **324** (1991), 331–351.

[Pa] V. Padrón, *Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations*, Comm. Partial Differential Equations **23** (1998), 457–486.

[Pay] L. E. Payne, *Improperly Posed Problems in Partial Differential Equations*, Regional Conference Series in Applied Mathematics **22**, SIAM, Philadelphia (1975)

[Pl1] P. I. Plotnikov, *Passing to the limit with respect to viscosity in an equation with variable parabolicity direction*, Diff. Equ. **30** (1994), 614–622.

[Pl2] P. I. Plotnikov, *Equations with alternating direction of parabolicity and the hysteresis effect*, Russian Acad. Sci., Dokl., Math. **47** (1993), 604–608.

[Pl3] P. I. Plotnikov, *Forward-backward parabolic equations and hysteresis*, J. Math. Sci. **93** (1999), 747–766.

[Pl4] P. I. Plotnikov, *Passage to the limit over small parameter in the Cahn-Hilliard equations*, Siberian Mathematical Journal **38** (1997), 550–566.

[PM] P. Perona & J. Malik, *Scale space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Mach. Intell. **12** (1990), 629–639.

[RH] T. Roubíček & K. Hoffmann, *About the Concept of Measure-valued Solutions to Distributed Parameter Systems*, Math. Methods Appl. Sci., **18** (1995), 671–685.

[Se] D. Serre, " Systems of conservation laws, Vol. 1: Hyperbolicity, entropies, shock waves ", translated from the 1996 French original by I. N. Sneddon., Cambridge University Press, Cambridge (1999).

[Sl] M. Slemrod, *Dynamics of measure valued solutions to a backward-forward heat equation*, J. Dynam. Differential Equations **3** (1991), 1–28.

[Sm] F. Smarrazzo, *On a class of equations with variable parabolicity direction*, Discrete Contin. Dyn. Syst. **22** (2008), 729–758.

[SSW] D.G. Schaeffer, M. Shearer & T.P. Witelski, *A discrete model for an ill-posed non linear parabolic PDE*, Physica D **160** (2001), 189–221

[ST] F. Smarrazzo, A. Tesei, *Long-time Behaviour of Solutions to a Class of Forward-Backward Parabolic Equations*, preprint

[ST2] F. Smarrazzo, A. Tesei, *Degenerate Pseudoparabolic Regularization of a Forward-Backward Parabolic Equations*, preprint

[V] M. Valadier, *A course on Young measures*, Rend. Ist. Mat. Univ. Trieste **26** (1994), suppl., 349–394 (1995).

[Va] J. L. Vázquez, Smoothing and decay estimates for nonlinear parabolic equations of porous medium type, *Oxford Lecture Notes in Maths and its Applications, 33*, Oxford Univ. Press, to appear

[Z] K. Zhang, *Existence of infinitely many solutions for the one-dimensional Perona-Malik model*, Calc. Var., **26** (2006), 171-199.