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Metastable dynamics of interfaces for a class of parabolic-hyperbolic systems

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Contents

CHAPTER 1

Introduction: the phenomenon of Metastability in literature

The slow motion of internal shock layer has been widely studied for a large class of evolutive PDEs. Such phenomenon is known as *metastability*. The qualitative features of the metastable dynamics are the follows: through a transient process, a pattern of internal layers is formed from initial data over a $\mathcal{O}(1)$ time interval. However, once this pattern is formed, the subsequent motion of the internal layers is exponentially slow, converging to their asymptotic limit. As a consequence, two different time scales emerge: for short times, the solution are close to some non-stationary state; subsequently, they drifts towards the equilibrium solution with a speed rate that is exponentially small.

In other words, the equation exhibits in finite time metastable shock profiles (called *interfaces*) that persist during an exponentially (with respect to a small parameter) long time period and that move with exponentially slow speed. Such interfaces exhibit significant changes in size, shape and location as time passes.

Many fundamental partial differential equations, concerning different areas, exhibit such behavior: among others we include viscous shock problems (see [37], [38], [64], [69]), phase transition problems described by the Allen-Cahn equation, with the fundamental contributions [12], [22], and Cahn-Hilliard equation, studied in [1] and [63].

To name some of the most important ones

$$\textbf{Viscous conservation law:}\quad \begin{cases} \partial_t u = \varepsilon^2 \partial_x^2 u - \partial_x f(u) \\ u(0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R}^+ \times I, \tag{1.1}$$

$$\textbf{Allen} - \textbf{Cahn}:\quad \begin{cases} \partial_t u = \varepsilon^2 \partial_x^2 u - W'(u) \\ u(0,x) = u_0(x) \end{cases} \qquad (t,x) \in \mathbb{R}^+ \times I \tag{1.2}$$

$$\textbf{Cahn} - \textbf{Hilliard}:\quad \begin{cases} \partial_t u = -\partial_x^2(\varepsilon^2 \partial_x^2 u + W'(u)) \\ u(0,x) = u_0(x) \end{cases} \qquad (t,x) \in \mathbb{R}^+ \times I, \tag{1.3}$$

where I denotes a bounded interval of the real line. All these equations are complemented with appropriate boundary conditions.

In this Chapter we want to give an outline of some of these contributes.

1. Slow motion of a patter of internal layers for the Allen-Cahn and Cahn-Hilliard equations

We here begin by analizing the cases of the Allen-Cahn equation (1.2) and the Cahn-Hilliard equation (1.3) as main examples since viscous shock problems, one of the main topic of this thesis, will be extensively introduced in the following.

For the Allen-Cahn equation we refer to [12], where the authors study the slow evolution of patterns of internal layers for the reaction diffusion equation

$$\partial_t u = \varepsilon^2 \partial_x^2 u - W'(u), \quad x \in (0,1), \quad t > 0 \tag{1.4}$$

complemented with Neumann boundary conditions

$$\partial_x u(0,t) = 0, \qquad \partial_x u(1,t) = 0, \tag{1.5}$$

Here the function $W(u)$ is a smooth double well function such that $W(\pm 1) = W'(\pm 1) = 0$. Moreover, it is required that $W(u) > 0$ for $-1 < u < 1$.

It has been proved that, under appropriate conditions on W, any solution $u(x,t)$ to (1.4) converges to $\bar{u}(x)$ as $t \to +\infty$, where $\bar{u}(x)$ is a stationary solution (see [24, 58]). Concerning the stable equilibrium solutions to (1.4), it is known that they are constant in space, and minimize the energy functional

$$I(u) = \int_0^1 (W(u) + \frac{1}{2}\varepsilon^2 \partial_x u^2)\, dx$$

so that, for large t, solutions will be approximately constant in space. However, for small ε, the time-dependent solution reach these asymptotic states in a time interval that can be exponentially long. Hence the solution exhibits a first short time transient phase, in which it develops a pattern of transition layers, and an exponentially long time phase (proportional to $e^{c/\varepsilon}$) in which such configuration evolves extremely slow.

In order to give rigorous results on the existence of these metastable states (i.e. configurations characterized by a pattern of internal layers that persist for an extremely long time), the authors develop the following approach.

For fixed ε and N, given a configuration $\xi = (\xi_1, ..., \xi_N)$ of N layer positions, they consider a function $u^\xi(x)$ which approximates a metastable state with N transition layers. More precisely, u^ξ is well approximated by translates of Φ and its reflection, where Φ is the unique solution to

$$\varepsilon^2 \partial_x^2 \Phi - W'(\Phi) = 0, \quad \Phi(0) = 0, \quad \Phi(x) \to \pm 1 \;\text{ as }\; x \to \pm\infty \tag{1.6}$$

which have the property that the distance between layer locations is given by a fixed $l > 0$. Hence, when $x \sim \xi_j$, then $u^\xi \sim \Phi(x - \xi_j)$ or $u^\xi \sim \Phi(\xi_j - x)$, and the layer positions are well separated and bounded away from the boundary points. With such a construction, it follows that the quantity $\mathcal{F}^\varepsilon[u^\xi] := \varepsilon^2 \partial_x^2 u^\xi - W'(u^\xi)$ is very small, and it is equal to zero if $|x - \xi_j| \geq \varepsilon$ for all j.

The admissible layer positions lie in a set Ω_ρ, where $\xi_j - \xi_{j-1} > \varepsilon/\rho$, so that the set of states u^ξ forms an N-dimensional manifold $\mathcal{M} = \{u^\xi : \xi \in \Omega_\rho\}$.

To study the dynamics of solutions located near $\mathcal{M}$, the authors linearize around an element of the manifold, so that a solution $u(x,t)$ belonging to a neighborhood of $\mathcal{M}$ is written as

$$u(x,t) = u^{\xi(t)}(x) + v(x,t)$$

By substituting into (1.4), they obtain equations for $\xi(t)$ and $v(x,t)$ that show how the motion of the shock layers and the subsequent evolution of the patterns are influenced by layer interactions. Moreover the authors show that, up to a very small error, the dynamics near $\mathcal{M}$ is well described by $u^{\xi(t)}$.

Such results are also obtained by studying the linearized operator $\mathcal{L}^{\varepsilon}_{\xi}$, arisen from the linearization around u^{ξ}. Indeed, if one considers the linear operator L_* obtained by linearizing $\mathcal{F}[u]$ around and exact solution Φ to (1.6), the principal eigenvalue turns to be equal to zero. Since u^{ξ} is a well approximation of Φ, the zero eigenvalue of L_* translates into exactly N small eigenvalues for the operator $\mathcal{L}^{\varepsilon}_{\xi}$. Thanks to these spectral results, the authors obtain estimates for the perturbation v, and derive a system of ordinary differential equations for $\xi_i(t)$, $i = 1.....N$, whose study is rigorous performed in [**12**, Section 3].

To describe the metastable dynamics for the Cahn-Hilliard equation (1.3), we give a brief outline of [**1**], one of the fundamental contributes in this specific area. In this paper, the authors describe the slowly dynamics of the solutions to the Cahn-Hilliard equation

$$\partial_t u = \partial_x^2(-\varepsilon^2 \partial_x^2 u + W'(u)), \quad x \in (0,1), \quad t > 0 \tag{1.7}$$

subject to the boundary conditions

$$\partial_x u = \partial_x^3 u = 0, \quad \text{at} \quad x = 0, 1.$$

The function W is a C^4 function with exactly three critical points, $\alpha < \gamma < \beta$, with α, β local minima and γ a local maximum. Moreover

$$W \geq 0, \quad W(\alpha) = W(\beta), \quad W''(\alpha),\ W''(\beta),\ -W''(\gamma) > 0$$

Since equation (1.7) possesses a Liapunov functional $I(u)$, defined by

$$I(u) = \int_0^1 \left[\frac{\varepsilon^2}{2} \partial_x u^2 + W(u) \right] \, dx$$

it is known that, when $t \to \infty$, solutions stabilize to an equilibrium state that is a local minimizer of I with a constrain on the total mass M, that is

$$\int_0^1 u \, dx = M$$

Numerical results show that the initial datum quickly evolves into a layered structure, which than slowly evolves at an exponentially long time scale, followed again by a fast evolution at an $\mathcal{O}(1)$ time scale, to be followed by a slow motion, etc.

In [**1**], the authors rigorous show that this behavior occurs for the Chan-Hilliard equation, and the strategy used is the following:

- They construct a manifold $\mathcal{M}$ of stationary solutions.

- They study the spectrum of the linearized Cahn-Hilliard operator about this manifold, showing that there exists an exponentially small positive eigenvalue of order $e^{-1/\varepsilon}$, while the rest of the spectrum is bounded away from zero uniformly with respect to ε.

More precisely, the manifold $\mathcal{M}$ id defined as the set of translates $u^\xi(x-\xi)$ (restricted to $[0,1]$), where $u(x)$ is a two-layer equilibrium to (1.7) which has mass M. In formula

$$\bar{\mathcal{M}} = \{u^\xi(x) = u(x-\xi) : -l+\delta_0 < \xi < l-\delta_0, x \in [0,1]\}$$

where $l = (\beta - M)/2(\beta-\alpha)$ is the asymptotic distance between a layer and the boundary of the interval and δ_0 is a fixed small positive number. With these notations, the location of the layers are given by

$$x_1 = \ell + \xi, \quad x_2 = 1 - \ell + \xi$$

By linearizing around an element of the manifold, the authors prove that $\mathcal{M}$ is an invariant manifold for (1.7), and that, for ε sufficiently small, the dynamics near $\mathcal{M}$ is described by the ordinary differential equation $\dot{\xi} = b(\xi)$, where

$$b(\xi) = \mathcal{O}\left(e^{-\frac{1}{\varepsilon}}\right), \quad b(0) = 0$$

so that the speed of propagation of the layers is exponentially small in ε.

As already stressed before, this slow motion is the result of the presence of a first small eigenvalue; in fact, in [1], it is proven that, if one consider the eigenvalue problem for the linearized operator around u^ξ, then the linearized operator $\mathcal{L}^\varepsilon_\xi$ has exactly one simple positive eigenvalue

$$0 < \lambda^\varepsilon_1(\xi) = O(e^{-c/\varepsilon}\varepsilon^7)$$

while all the remaining eigenvalues are bounded away from zero uniformly in ε, i.e.

$$\lambda^\varepsilon_k(\xi) \leq -C < 0, \qquad k = 2, 3, ...$$

where C is independent on ε and ξ.

In both the contributes, the common aim is to determine equations for the parameters that represent the location of the layers, considered as functions of time; usually, such equations are determined by linearizing around a manifold of approximate steady states, and by projecting. In fact, the existence of, at least, one first small eigenvalue suggest that one needs to solve the equation in a subspace in which the linearized operator doesn't vanish.

2. Metastability for viscous shock problems

In this section we mean to give an overview on the analysis of slow dynamics for evolutive parabolic partial differential equations, with particular attention to the scalar conservation law

$$\partial_t u + \partial_x f(u) = \varepsilon\, \partial_x^2 u, \tag{1.8}$$

with the space variable x belonging to a one-dimensional interval $I = (a, b)$. The parameter $\varepsilon > 0$ is considered as a strictly positive and small parameter. Equation (1.8) is complemented with Dirichlet boundary conditions

$$u(a,t) = u_- \qquad \text{and} \qquad u(b,t) = u_+ \tag{1.9}$$

for given data $u^\pm$ to be discussed in details.

Equation (1.8) is considered as a (simplified) archetype of more complicate systems of partial differential equations arising in different fields of applied mathematics. Inspired by the equations of fluid-dynamics, the parameter ε is interpreted as a *viscosity coefficient* and the main problem is to identify and quantify its rôle in the emergence and/or disappearance of the phenomenon of metastability.

Formally, in the limit $\varepsilon \to 0^+$, the parabolic equation (1.8) reduces to a first-order quasi-linear equation of hyperbolic type

$$\partial_t u + \partial_x f(u) = 0, \tag{1.10}$$

whose standard setting is given by the *entropy formulation*, hence possessing possibly discontinuous solutions with speed of propagation s given by the Rankine–Hugoniot relation

$$s[\![u]\!] = [\![f(u)]\!] \tag{1.11}$$

(where $[\![\cdot]\!]$ denotes the jump) together with appropriate entropy conditions. In addition, the treatment of the boundary conditions (1.9) is much more delicate with respect to the parabolic case, because of the eventual appearance of boundary layers.

Concerning the function $f(u)$, let us assume

$$f''(s) \geq c_0 > 0, \qquad f'(u_+) < 0 < f'(u_-), \qquad f(u_+) = f(u_-), \tag{1.12}$$

for some positive constant c_0. The last two assumptions guarantee that the jump from the value u_- to u_+ is admissible and its speed of propagation, dictated by (1.11), is zero. In this case, equation (1.10) possesses a one-parameter family $\{U_{\text{hyp}}(\cdot;\xi)\}$ of stationary solutions satisfying the boundary conditions (1.9), given by

$$U_{\text{hyp}}(x;\xi) := u_-\chi_{(a,\xi)}(x) + u_+\chi_{(\xi,b)}(x) \tag{1.13}$$

where χ_I denotes the characteristic function of the set I. The dynamics determined by initial-value problem for (1.10)-(1.9) is very simple: it is possible to prove that every entropy solution converges *in finite time* to an element of the family $\{U_{\text{hyp}}(\cdot;\xi)\}$. Hence, at the level $\varepsilon = 0$, there are infinitely many stationary solutions, generating a "finite-time" attracting manifold for the dynamics. Note that, at the hyperbolic level, there is no way of distinguishing one element from any other in the family of steady states $\{U_{\text{hyp}}(\cdot;\xi)\}$ in term of stability properties.

For $\varepsilon > 0$, the situation is different. Apart from the well-known smoothing effect, the presence of the Laplace operator at the right hand-side of (1.8), together with the boundary conditions (1.9), has the effect of a drastic reduction of the number of stationary solutions: from infinitely many to a single stationary state (see [35]). Such solution, denoted here by $\bar{U}^\varepsilon_{\text{par}} = \bar{U}^\varepsilon_{\text{par}}(x)$, converges in the limit $\varepsilon \to 0^+$ to a specific element of the family $\{U_{\text{hyp}}(\cdot;\xi)\}$. As an example, in the case of the Burgers equation, $f(s) = \frac{1}{2}\,s^2$, $a = -\ell$ and $b = \ell$, there holds

$$\bar{U}^\varepsilon_{\text{bur}}(x) = -\kappa\,\tanh\left(\frac{\kappa\,x}{2\,\varepsilon}\right)$$

where $\kappa = \kappa(\varepsilon, \pm\ell, u_\pm)$ is implicitly defined by imposing the boundary conditions. In the limit $\varepsilon \to 0^+$, the single steady state $\bar{U}^\varepsilon_{\text{bur}}$ converges pointwise to $\bar{U}_{\text{hyp}} := U_{\text{hyp}}(\cdot\,;0)$. Therefore, the element of the one-parameter family $\{U_{\text{hyp}}(\cdot;\xi)\}$ corresponding to $\xi = 0$ exhibits a form of *structural stability* which is not shared with any other element of the same family.

Similar results are obtained for a class of general $f(u)$ that verify hypotheses (1.12); in this case the stationary solution $\bar{U}^{\varepsilon}_{\text{par}}$ converges pointwise to $U_{\text{hyp}}(\cdot;\bar{\xi})$, for some $\bar{\xi} \in I$.

A deeper understanding of the problem can be gained by analyzing the dynamical properties of (1.8) for initial data close to the equilibrium configuration by means of the linearized equation at the state $\bar{U}^{\varepsilon}_{\text{par}}$

$$u_t = \mathcal{L}_\varepsilon\, u := \varepsilon\, u_t + \big(a(x)\, u\big)_x \qquad \text{with} \qquad a(x) := -f'(\bar{U}^{\varepsilon}_{\text{par}}(x)).$$

In [35] it shown that, in the case of Burgers flux $f(s) = \frac{1}{2}\, s^2$, the eigenvalues of $\mathcal{L}_\varepsilon$, considered with homogeneous Dirichlet boundary conditions, are real and negative. Moreover, for $f(u_+) = f(u_-)$, there holds as $\varepsilon \to 0$

$$\lambda^{\varepsilon}_1 = O(e^{-1/\varepsilon}) \quad \text{and} \quad \lambda^{\varepsilon}_k < -\frac{c_0}{\varepsilon} < 0 \qquad \forall\, k \geq 2$$

for some $c_0 > 0$ independent on ε. Negativity of the eigenvalues implies that the steady state $\bar{U}^{\varepsilon}_{\text{bur}}$ is asymptotically stable with exponential rate; while the precise distribution shows that the large time behavior is described by term of the order $e^{\lambda^{\varepsilon}_1 t}$ and, as a consequence, the convergence is very slow, when ε is small.

Such is the picture relative to the behavior determined by an initial data close to the equilibrium solution $\bar{U}^{\varepsilon}_{\text{bur}}$.

The next question concerns with the dynamics generated by initial data still presenting a sharp transition from u^- to u^+, but that are localized far from the position of the steady state $\bar{U}^{\varepsilon}_{\text{bur}}$. Figure 4 represents the solution

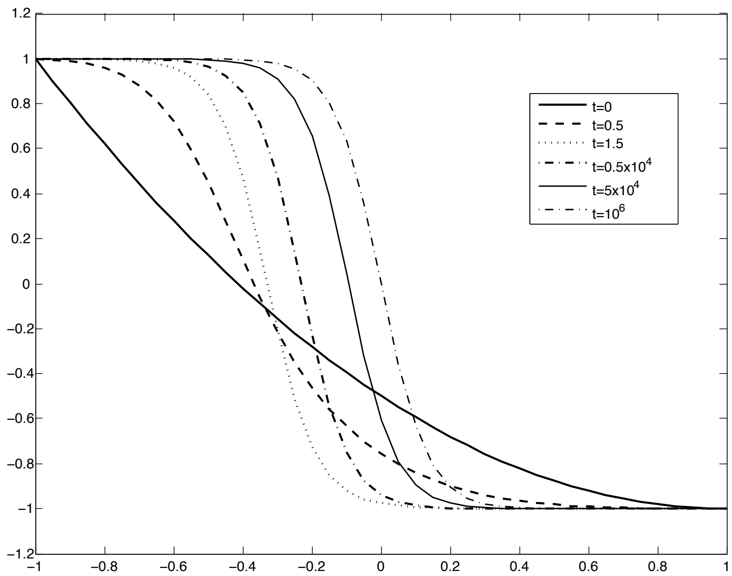


FIGURE 1. The solution to (1.8) with $f(u) = u^2/2$, $\varepsilon = 0.07$, $u_\pm = \mp 1$ and initial datum $u(x,0) = u_0(x) := \frac{1}{2}x^2 - x - \frac{1}{2}$.

to the initial-value problem for (1.8) with boundary conditions (1.9) in the interval $I = (-1, 1)$, relative to the initial datum $u(x, 0) = u_0(x) := \frac{1}{2}x^2 - x - \frac{1}{2}$.

Starting with a decreasing initial datum, a shock layer is formed in an $\mathcal{O}(1)$-timescale, so that the solution is approximately given by a translation of the (unique) stationary solution of the problem. Once such a layer is formed, it moves towards the location corresponding to the equilibrium solution and this motion is exponentially slow. As a consequence, two different time scales emerge: for short times, the solution becomes closer to a monotone transition connecting the boundary data, close to a space-translation of the single steady state; for long time, the profile slowly moves toward the equilibrium configuration.

Main contributes. We mean to give a brief outline of some of the most significant papers concerning the phenomenon of metastability for viscous shock problems.

Among others, such problem has been examined in [64], [69] and in [38], where different approaches have been considered. The former is based either on *projection method* or on *WKB expansions*. We want to give a brief outline of these results.

In [64] the authors consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \varepsilon \partial_x^2 u - \partial_x f(u), & x \in (0,1), \quad t > 0 \\ u(0,t) = u_-, \ u(1,t) = u_+ & t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.14}$$

where $u_- > 0$ and $u_+ \equiv -u_-$, so that $u_\pm = \mp u^*$ for some $u^* > 0$. Here $f(u)$ is a convex function such that $f'(0) = 0$, $uf'(u) > 0$ and $f(u^*) = f(-u^*)$.

The aim of this paper is to determine a description of the slow motion of the internal layer in the vanishing viscosity limit $\varepsilon \to 0$; as usual, the idea is to construct an approximation for the stationary solution in order to derive an equation of motion for the shock layer location $\xi(t)$.

In this contest, the authors use the *method of matched asymptotic expansions* (MMAE) to built an approximate stationary solution to (1.14). More precisely, for $\varepsilon \to 0$, the leading order MMAE solution is given by

$$u \sim u_s[\varepsilon^{-1}(x - \xi)]$$

where $\xi \in (0,1)$ represents the location of the shock layer. With the change of variable $z = (x - \xi)/\varepsilon$, the shock profile u_s turns to solve

$$\begin{cases} u_s'(z) = f(u_s(z)) - f(u^*) \quad z \in (-\infty, +\infty) \\ u_s(0) = 0 \\ u_s(z) \sim u^* - a_\pm e^{\mp \nu_\pm z}, \quad z \to \pm\infty \end{cases}$$

where the positive constants $\nu_\pm$ and $a_\pm$, describing the tail behavior of u_s, are defined by

$$\begin{aligned} \nu_\pm &= \mp f'(\mp u^*) \\ \log\left(\frac{a_\pm}{u^*}\right) &= \pm \nu_\pm \int_0^{\mp u^*} \left[\frac{1}{f(\eta) - f(u^*)} \pm \frac{1}{\nu_\pm(\eta \pm u^*)}\right] d\eta \end{aligned}$$

Notice that the MMAE solutions satisfies exactly the equation, while the boundary conditions are satisfy within exponentially small terms.

In order to determine an equation of motion for the location of the shock layer, the authors utilize the *projection method*, based on the linearization of (1.14) around the MMAE stationary solution u_s. More precisely, this method is based on the fact that the linear operator associated with the linearization around the shock profile has exactly one exponentially small eigenvalue, so that the solution to the linearized problem must have no component in the direction of the first eigenfunction.

More precisely, the authors look for a solution to (1.14) on the form $u(x,t) \sim u_s(z) + w(x,t)$, where $z = \varepsilon^{-1}[x - \xi(t)]$. Subsequently, they study the spectrum of the linearized problem, showing that, asymptotically, the principal eigenvalue of the linearized operator tends to zero as $\varepsilon \to 0$. Hence, by setting an algebraic condition in order to remove the singular part of the linearized operator, and by projection on the first component, one obtains an approximate ODE for the motion of $\xi(t)$. The main result is the following (for more details see [64, Proposition 4a])

Proposition 1.1. *For $\varepsilon \to 0$, the exponentially slow evolution of the shock layer for (1.14) is described by $u \sim u_s[\varepsilon^{-1}(x - \xi(t))]$, where $\xi(t)$ satisfies the ODE*

$$\frac{d\xi}{dt} = \frac{1}{2u^*}\left[Aa_+\nu_+ e^{-\nu_+\varepsilon^{-1}(1-\xi)} - Ba_-\nu_- e^{-\nu_-\varepsilon^{-1}\xi}\right]$$

where the constant A, B depend on $\nu_\pm$ and on the boundary conditions.

A similar approach is used in [69] to study the metastable dynamics for the generalized Burgers problem

$$\begin{cases} \partial_t u = u - u\partial_x u + \varepsilon\partial_x^2 u, & x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.15}$$

Such equation arise from the study of the dynamics of an upwardly propagating flame-front in a vertical channel; precisely, the shape $y = y(x,t)$ of the flame front interface satisfies

$$\begin{cases} \partial_t y = \frac{1}{2}\partial_x y^2 + \varepsilon\partial_x^2 y - \int_0^1 y\, dx, & x \in (0,1), \quad t > 0 \\ \partial_x y(0,t) = \partial_x y(1,t) = 0, & t > 0 \\ y(x,0) = y_0(x) \end{cases} \tag{1.16}$$

so that, by setting $u(x,t) = -\partial_x y(x,t)$, one obtains equation (1.15).

By linearizing the equation around an MMAE equilibrium solution, it turns out that the principal eigenvalue associated with such linearization is exponentially small in ε, so that a metastable behavior for the time-dependent problem occurs as a consequence of the size of the first eigenvalue.

This behavior of slow motion of internal layer is studied here by deriving an asymptotic ordinary differential equation characterizing the slow motion of the tip location of a parabolic-shaped interface, corresponding to the location of the shock layer for the function u. In this contest, numerical computations suggest that a sufficient condition for the appearance of a metastable behavior is that the initial datum satisfies

$$u_0(x) < 0 \quad \text{for} \quad x \in (0, a), \qquad \text{and} \qquad u_0(x) > 0 \quad \text{for} \quad x \in (a, 1)$$

for some $a > 0$. In this case, three different time scales emerge: a first transient phase of $\mathcal{O}(1)$ where the parabolic-shaped flame-front interface is formed; an exponentially long time phase where the tip of the parabolic flame-front drifts towards one of the walls (i.e. $x = 0$ or, equivalently, $x = 1$); finally, an $\mathcal{O}(1)$ phase where the flame-front collapses against the wall and reach its equilibrium configuration. In terms of $u(x,t)$, the first two phases are pictured in Figure 2. The fact that the time interval corresponding to the second phase is exponentially long with respect to ε, creates the illusion that the flame-front has reached some final equilibrium. However, this second phase is only a quasi-equilibrium that persists for an exponentially long time interval.

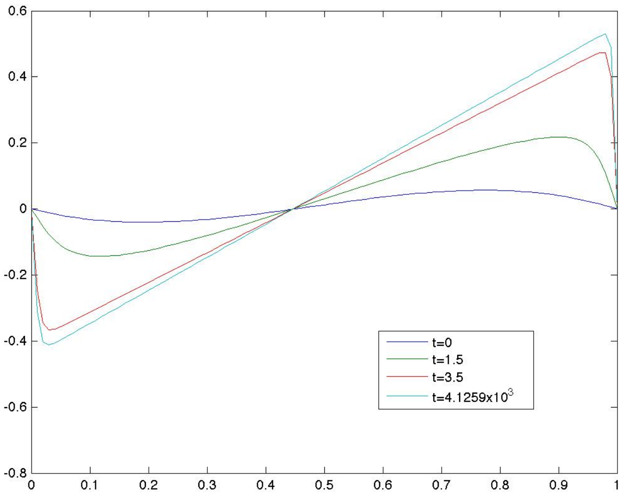


FIGURE 2. The solution to (1.15) with $f(u) = u^2/2$, $\varepsilon = 0.07$, and initial datum $u(x, 0) = x(1-x)(x-0.45)$. Notice that the zero of u, which is the tip of the flame-front interface, moves slowly towards the wall at $x = 0$.

With all of this in mind, let us briefly describe the methods used in [38], where the authors specifically consider the initial boundary value problem

$$\begin{cases} \partial_t u = \varepsilon \partial_x^2 u + u \partial_x u, & x \in (0,1), \quad t > 0 \\ u(\pm 1, t) = \pm 1, & \qquad\qquad\quad\; t > 0 \\ u(x, 0) = u_0(x) & \end{cases} \tag{1.17}$$

with

$$u_0(x) = \begin{cases} -1, & -1 \leq x \leq a \\ 1, & a < x \leq 1 \end{cases}$$

for some $a \in (-1, 1)$ and for small ε.

Here the shock layer location $\xi(t)$ is defined as the value where the solution is equal to zero, that is

$$u(\xi(t), t) \equiv 0$$

As usual, numerical computations show that the transition between the boundary values -1 and 1 occurs through a shock layer located in $\xi(t)$, whose thickness is of $\mathcal{O}(\varepsilon)$. To obtain an asymptotic equation for the shock layer location, an adapted version of the *method of matched asymptotic expansion* is used. Hence, the author introduce the stretched spacial and time variables

$$\eta = \frac{(x - \xi(t))}{2\varepsilon}, \quad s = \frac{t}{e^{1/\varepsilon}}$$

so that any solution $u(\eta, s)$ to (1.17) satisfies

$$\begin{cases} \partial_\eta^2 u + 2u\partial_\eta u = e^{-1/\varepsilon}(4\varepsilon\partial_s u - 2\dfrac{d\xi}{d\sigma}\partial_\eta u) \\ u\left(\dfrac{\pm 1 - \xi(s)}{2\varepsilon}, s\right) = \pm 1 \\ u(0, s) = 0 \end{cases} \tag{1.18}$$

Since $u(\eta, s) = \tanh \eta$ satisfies the limiting problem with $\varepsilon = 0$, the authors look for a solution to (1.19) in the form

$$u(\eta, s) = \tanh \eta + e^{-1/\varepsilon} u_1(\eta, s) + ...$$

obtaining an initial value problem for $\xi(s)$, that is

$$\frac{d\xi}{ds} = e^{-\xi/\varepsilon} - e^{\xi/\varepsilon}, \quad \xi(0) = \xi_0$$

that corresponds to the equation for the shock layer location obtained in [64].

The same techniques are used in [37], where the authors performe the same study for a generalized problem on the form

$$\begin{cases} \partial_t u = \varepsilon\partial_x(g(u)\partial_x u) + f(u)\partial_x u, & x \in (0, 1), \quad t > 0 \\ u(\pm 1, t) = \pm 1, & t > 0 \\ u(x, 0) = u_0(x) \end{cases} \tag{1.19}$$

Also in this case, under opportune hypotheses on the function g, the authors prove that, starting from an initial datum on the form

$$u_0(x) = \begin{cases} -1, & -1 \leq x \leq a \\ 1, & a < x \leq 1 \end{cases}$$

a single shock layer of thickness of $\mathcal{O}(\varepsilon)$ is formed, and its convergence to the asymptotic limit is extremely slow.

The common aim of all these papers is to determine an expression and/or an equation for the parameter ξ, considered as a function of time, describing the movement of the transition from a generic point of the interval toward the equilibrium location. In these contributes, the analysis is conducted at a formal level and validated numerically by means of comparison with significant computations.

A rigorous analysis of the solution to the Burgers' type equation

$$\begin{cases} \partial_t u = \varepsilon \partial_x^2 u + \partial_x f(u), & x \in (-1,1), \quad t > 0 \\ u(\pm 1, t) = \pm 1, & \qquad\qquad\quad\; t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.20}$$

has been performed in [18] (and generalized to the case of nonconvex flux in [20]); here the flux function f satisfies

$$f'(1) > 0, \quad f'(-1) < 0, \quad f(\pm 1) = 0 \;\text{ and }\; f(s) < 0, \; \forall \, |s| < 1$$

In order to study the existence of an exponentially slow internal shock the authors, instead of deriving an equation of motion for the location of such shock layer, obtain an asymptotic expression for its speed; the idea is to approximate the solution to (1.20) with a traveling wave $\Phi(x - V(\xi)\, t)$ that fits exactly the boundary conditions and satisfies $\Phi(\xi) = 0$.

Since there no exist traveling waves on a finite interval, the authors construct such solution by restricting a true traveling wave $\psi(x - vt)$ of the equation (1.20), defined on all $\mathbb{R}^2$, to a finite subdomain $[-1,1] \times [t_1, t_2]$, where the time interval $[t_1, t_2]$ is chosen so that the unique zero of $\psi(x - vt)$ belongs to the open interval $(-1, 1)$.

In particular it is shown that, for each $\xi \in (-1,1)$, a unique traveling wave $\Phi(x - V(\xi)\, t)$, satisfing $\Phi(\xi) = 0$ and $\Phi(\pm 1) = \pm 1$, exists. Furthermore, the authors derive a precise estimate of its exponentially slow velocity V. The approach is based on the use of such traveling wave to obtain upper and lower estimates by the maximum principle, from which rigorous asymptotic formulae for the slow velocity are obtained; precisely

$$V(\xi) = -\alpha \exp\left(-f'(1)\frac{1-\xi}{\varepsilon}\right) + \beta \exp\left(f'(-1)\frac{1+\xi}{\varepsilon}\right) + O(R_\varepsilon^2/\varepsilon)$$

where the constants α and β depends on $f(\pm 1)$, and

$$R_\varepsilon := \exp\left(-f'(1)\frac{1-\xi}{\varepsilon}\right) + \exp\left(f'(-1)\frac{1+\xi}{\varepsilon}\right)$$

Moreover, it is proven the the function $\xi \mapsto V(\xi)$ is a monotone function, which is positive for $\xi \sim -1$ and negative for $\xi \sim 1$, meaning that the wave moves to the right in the former case and to the left in the latter.

Furthermore, for the analytical study of the metastable dynamics, the equation (1.20) is linearized around the approximate traveling wave and, as usual, a study of the spectral proprieties of the linearized operator is performed. Precisely, the authors show that the first eigenvalue is exponentially small with respect to ε, and that there is a spectral gap between the principal eigenvalue and the rest of the spectrum.

Such considerations show that the convergence to the asymptotically stable state is exponentially slow in the limit $\varepsilon \to 0$.

The case of nonconvex flux, studied by the same authors in [20], is more delicate: in this case, f has several zeros between u_- and u_+, so that several internal transition layer may occur. Indeed, the authors suppose that the flux function f has to satisfy the following hypothesis

- $f \in C^\infty(\mathbb{R})$ has a finite number of zeroes in the interval $[-1,1]$, but at least two. Moreover, all zeroes are of finite order.

As a consequence, the structure of the stationary solution is the following: there exists only one stationary solution that is monotone; it is characterized by flat regions that correspond to the zeroes of f of higher order and by transition layers connecting these areas, as depicted in Figure 3.

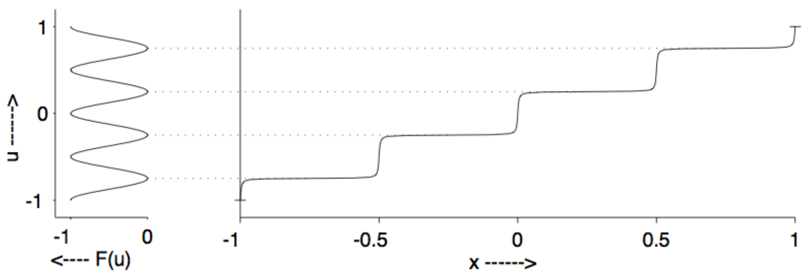


FIGURE 3. Equilibrium solution to (1.20) in the case of the non-convex flux $f(u) = -\cos^2 2\pi u$, with boundary conditions $u_\pm = \pm 1$. This figure was produced by P.P.N. De Groen and G.E. Karadzhov in [20]

In this paper the authors generalize the result obtained in [18], proving that, taking the initial datum close to a traveling wave profile, the time dependent solution moves in a small neighborhood of such profile with slow speed for a long time interval $(0, T^\varepsilon)$. In particular

$$T^\varepsilon = \mathcal{O}(\varepsilon^{-\frac{p}{p-1}})$$

where p is the maximal order of zeroes of f, so that a metastable behavior appears.

Viscous shock problems in unbounded domains. Slow motion for the viscous Burgers equation in unbounded domains has been also considered in literature.

The case of the whole real line has been examined in [32] (for subsequent contributions in the same direction, see also [31]) with emphasis on the generation of $N-$wave like structures and their evolution towards nonlinear diffusion waves. Here the authors consider the following Cauchy problem

$$\begin{cases} \partial_t u = \varepsilon \partial_x^2 u - u\partial_x u, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = u_0(x), & \qquad\quad t > 0 \end{cases} \tag{1.21}$$

The analysis is based on the use of self-similar variables, suggested by the invariance of the Burgers equation under the group of scaling transformations $(x,t,u) \mapsto (cx, c^2 t, u/c)$. This property suggests the change of variables

$$s = \log t, \quad \eta = x/\sqrt{t}, \quad w(\eta, s) = \sqrt{t}\, u(x,t)$$

so that equation (1.21) becomes

$$\partial_s w = \varepsilon \partial_\eta^2 w + \frac{1}{2}\partial_\eta(\eta w - w^2) \tag{1.22}$$

The *diffusion waves* are, by definition, the equilibrium solutions to (1.22).

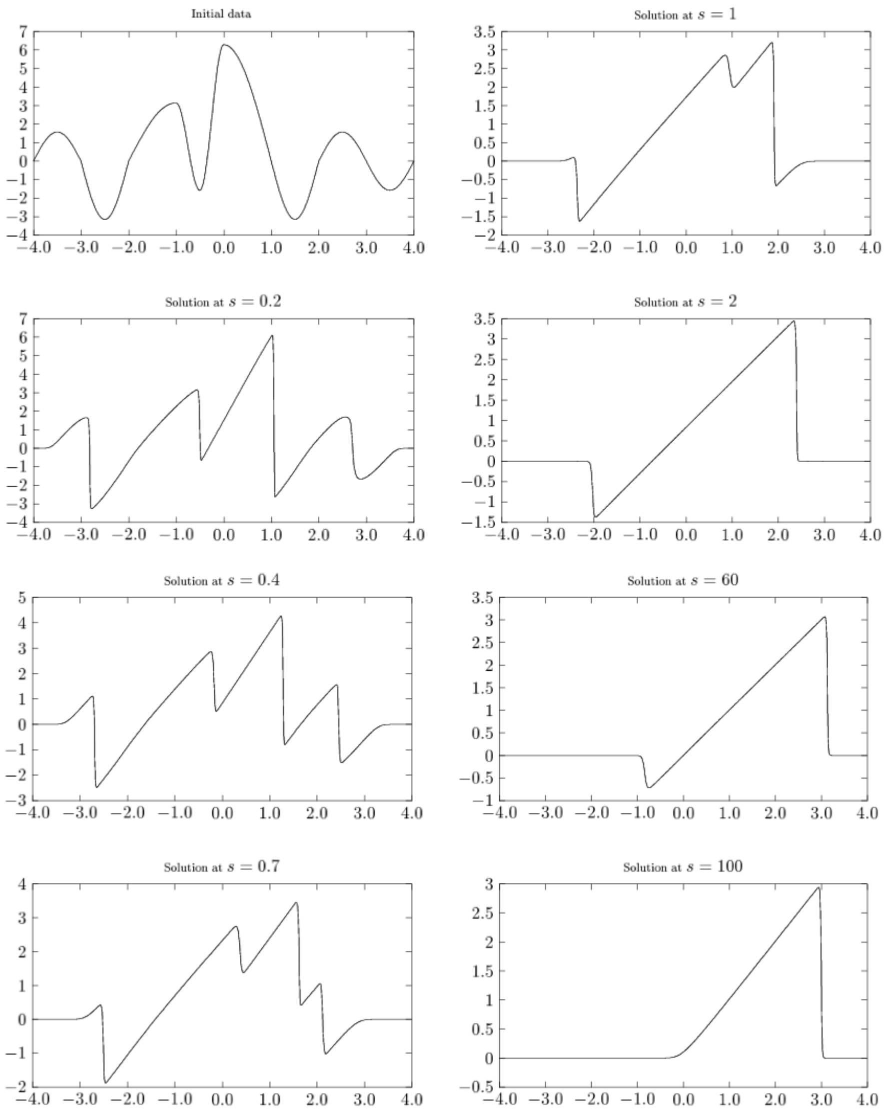


FIGURE 4. Numerical solution to (1.22), generated by the Godunov scheme produced by Y.-J. Kim and A. Tzavaras in [32].

Also in this case, numerical computations (see Figure 4) show that the evolution of the time-dependent solution can be divided into different stages of different orders in time; starting, for example, from an oscillatory initial datum w_0, the authors show that there is a first transition in which the solution develops the so called *saw-tooth profile*. In the subsequent stage, the waves interact and eventually produce an approximative $N-$wave. These first two stages are relatively quick, of order one in time, and here the diffusion act weakly, so that the dynamics is essentially the same as the unviscid equation

$$\partial_s w = \frac{1}{2}\partial_\eta(\eta w - w^2) \tag{1.23}$$

Once the latter stage is reached, it persists for an exponentially long time interval; indeed, the $N-$waves are stationary solutions for (1.23), but, because of the presence of a small diffusion, they are only approximate solutions for (1.22). This consideration leads to the last stage of the evolution of the solution: a very slow transition from an approximate $N-$wave to a a diffusion wave, the steady state of (1.22).

Roughly speaking, the solution spend a very long time near the family of approximate $N-$waves, before converging to the stable family of diffusive waves. Such a slow evolution (of order $e^{1/\varepsilon}$ in time) is a manifestation of the metastable behavior that occurs for the solutions to (1.21).

More recently, it has been shown in [10] that the slow motion for the viscous Burgers equation on the whole real line is determined by the presence of a one-dimensional center manifold of steady states for the equation in the self-similar variable (corresponding to the diffusion waves) and a relative family of one-dimensional global attractive invariant manifold (corresponding to the diffusive $N-$waves). In a short-time scale of order $\mathcal{O}(|\log\varepsilon|)$, the solution approaches one of the attractive manifolds and remains close to it in a long-time scale.

In [67], it is analyzed the case of the half-line $(0,+\infty)$ for the space variable x, with constant initial and boundary data chosen so that speed of the shock generated at $x=0$ would be stationary for the hyperbolic equation. The presence of the viscosity generates a motion of the transition layer, which is precisely identified by means of the Lambert's W function. Later, the (slow) motion of a shock wave, with zero hyperbolic speed, for the Burgers equation in the quarter plane has been considered in [47], where it is shown that the location of the wave front is of order $\ln(1+t)$; the same result has been generalized in [61] in the case of general fluxes (for other contributions to the same problem, we refer also to [49, 72]).

Conclusions. Summing up, apart for the formal expansions methods, the rigorous approaches used in the literature are largely based on typical scalar equations features. The first of these properties is the direct link between the scalar Burgers equation and the heat equation given by the Hopf–Cole transformation: $u = -2\varepsilon\,\phi^{-1}\partial_x\phi$, and the consequent invariance of equation (1.8) under the group of scaling transformations $(x,t,u)\mapsto(cx,c^2\,t,u/c)$. On the one hand, the presence of such a connection is an evident advantage, since it permits to determine optimal descriptions for the behavior under study (see [32, 47, 67]); on the other hand, to use such exceptional property makes the approach very stiff and difficult to apply to more general cases. A different "scalar hallmark" is the base of the approach considered in [18], where the authors make wide use of maximum principle and comparison arguments, taking benefit from the fact that the equation is second-order parabolic.

In order to extend the results to more general settings and specifically for systems of PDEs, it is useful to determine strategies and techniques that are more flexible, paying, if necessary, the price of a less accurate description of the dynamics. A contribution in this direction has been given in [61], where the location of the shock transition for a scalar conservation law in the quarter plane has been proved by means of weighted energy estimates, extending the result proved in [47], that used an explicit formula –determined by means of the Hopf–Cole transformation– for the Green function of the linearization at the shock profile of the Burgers equation.

3. Metastability for systems in higher space dimension

Metastable dynamics in a multispacial dimensional setting has been recently established (see, for example, [2, 3, 65]), and, at the present day, results relative to metastability in the case of systems appear to be rare. One of the main difference between the study of slow motion of internal layers in one-space dimension, and the problem of evolution of interfaces in higher dimensions is that, usually, in the former case the dynamics of the shock layer location is described by an ordinary differential equation, while in the latter it turns out that the interface evolves according to a partial differential equation.

To name some of these results, in [13] the author considers an interfacial problem for a class of reaction diffusion systems of parabolic type, that is

$$\begin{cases} \partial_t u = \varepsilon \Delta u + \dfrac{1}{\varepsilon} f(u,v), & x \in \mathbb{R}^n, t > 0 \\ \partial_t v = \Delta v + g(u,v), & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x) \end{cases} \tag{1.24}$$

The nonlinear terms in the equation are on the form

$$f(u,v) = u(1-u)(u-a) - v, \quad a \in (0,1)$$
$$g(u,v) = u - \gamma v, \quad \gamma > 0$$

System (1.24) can be interpreted as a model for the propagation of chemical waves in excitable media.

The assumptions on the smallness of the parameter ε and the bistable property of the term f are crucial in term of the appearance of a phenomenon of metastability; indeed, under these hypotheses, the evolution of internal interfaces can be divided into two consecutive phases: the first one is a short time period in which the interfaces are formed; the next one is a long time interval, where one can observe the evolution of such interfaces.

The approach used is the following. For short time (i.e. $t \ll 1$), the diffusion term $\varepsilon\Delta$ is neglected, and the dynamics of u is well approximate by the following equation

$$\partial_t u = \frac{1}{\varepsilon} f(u,\psi), \quad v \sim \psi \tag{1.25}$$

Going deeper in details, that author proves that, for short time, the solution u to (1.25) approaches two different branches, so that the whole space can be divided into two subdomains here named as *"the exited region"* $\Omega_+(t)$ and *"the rest region"* $\Omega_-(t)$, divided by a thin interfacial layer region $\Omega_0(t)$ whose thickness is of order $\mathcal{O}(\varepsilon)$. Is these regions, the solution u is given by $u \approx h_+(v)$ and $u \approx h_-(v)$ respectively, where

$$u = h_+(v), \qquad u = h_-(v)$$

represent two branches of the nullocline $\{(u,v) : f(u,v)=0\}$ of the function f.

To study the subsequent dynamics of the interfacial region $\Omega_0(t)$, that can move and deform as t evolves, the author uses a method of matched asymptotic expansion to obtain a limiting equation that well approximate equation (1.24) when $\varepsilon \to 0$: indeed, for large t, the diffusive term $\varepsilon\Delta u$ can no longer be neglected.

The hypotheses required are the following:

- The transition layer region $\Omega_0(t)$ approaches a compact hypersurface $\Gamma(t) \in \mathbb{R}^n$, called here *interface*.
- The solution $u(\cdot;t)$ has a jump discontinuity across $\Gamma(t)$. In particular

$$\begin{cases} u = h_+(t) & \text{on} \quad \Omega_+(t) \\ u = h_-(t) & \text{on} \quad \Omega_-(t) \end{cases} \qquad \mathbb{R}^n \backslash \Gamma(t) = \Omega_+(t) \cup \Omega_-(t)$$

- $\Gamma(t)$ changes smoothly in time.

Under such hypotheses, the author derives a partial differential equation for the interface $\Gamma(t)$, that shows how its motion depends on the main curvature κ of the interface itself. More precisely, the interfacial tension $\varepsilon(n-1)\kappa$ plays a fundamental role in the stability/instability of Γ (for more details see [**13**, Section 3]).

In [**28**], slow dynamics for a one-dimensional semi linear parabolic system, known as the *phase-field equations*, is studied. The goal of this paper is to show the existence of *metastable* solutions, i.e. solutions that preserve a particular structure for a large finite time that tends to infinity as the viscosity coefficient goes to zero. Slow dynamics analysis for systems of conservation laws have been considered in [**26**], basic model examples being the Navier-Stokes equations of compressible viscous heat conductive fluid and the Keyfitz-Kranzer system, arising in elasticity. The approach is based on asymptotic expansions and consists in deriving appropriate limiting equations for the leading order terms, in the case of periodic data.

In [**36**], the problem of proving convergence to a stationary solution for a system of conservation laws with viscosity is addressed, with an approach based on a detailed analysis of the linearized operator at the steady state.

Precisely, the authors consider the following problem

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u, \quad x \in (-\ell, \ell), \quad t \geq 0 \tag{1.26}$$

where $u = u(x,t)$ is a vector function with n components, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth function. The evolution in time of a solution to (1.26) involves two different processes; the first one consists on the appearance of an internal smooth shock layer. Diffusion acts weakly in this process, and the dynamics is well modeled by the corresponding inviscid equations, so that the position of the shock can be determined via the Rankine–Hugoniot relation. Moreover, the amplitude of such interface is of order $\mathcal{O}(\sqrt{\varepsilon})$. The other process involves perturbations of the shock layer, as the adjustment in shape and position of such interface. Here the diffusive effects are large, and the time scale is of order $\mathcal{O}(1/\varepsilon)$, much faster compared to the first phase, that is of $\mathcal{O}(1)$ in time.

In order to study the stability of the shock profile under *local* perturbations, (i.e. solutions starting from an initial datum close to some equilibrium configuration) the authors focuses on the fast process by considering just the space region near the shock layer. Hence, by eliminating the parameter ε from the equation thanks to the standard scaling of space and time, the authors analyze the stability of a steady state $U(x)$ by describing the dynamics of the solution of the initial boundary value problem

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 u, & x \in (-\ell, \ell), \quad t \geq 0 \\ u(x,0) = U(x) + y_0(x), & \qquad\qquad\quad t \geq 0 \\ u(\pm\ell) = U(\pm\ell) \end{cases} \tag{1.27}$$

where, as stressed, the initial datum u_0 is close to the equilibrium profile $U(x)$. The linearized problem around $U(x)$ reads

$$\begin{cases} \partial_t u + \partial_x (A(x)u) = \partial_x^2 u \\ u(x,0) = y_0(x) \\ u(\pm\ell) = 0 \end{cases} \tag{1.28}$$

where $A(x) = J(U(x))$ and $J = \partial f/\partial u$ is the Jacobian of f. By considering the corresponding eigenvalue problem

$$\lambda\varphi + \partial_x(A(x)\varphi) = \partial_x^2 \varphi, \quad |x| \leq \ell, \quad \varphi(\pm\ell) = 0 \tag{1.29}$$

the authors prove the following Theorem (see [36, Theorem 1])

Theorem 1.2. *For sufficiently large ℓ, the eigenvalue problem (1.29) has a first simple isolated eigenvalue λ_1 with corresponding eigenfunction φ_1, satisfying*

$$\lambda_1 = \mathcal{O}(e^{-\gamma(\ell-\ell_0)}), \quad |\varphi_1 - \partial_x U(x)| \leq Ce^{-\gamma\ell}$$

where $\gamma > 0$ is independent of ℓ, and ℓ_0 is a fixed, sufficiently large constant. Further, there exists a constant $\delta > 0$ independent on ℓ so that all the other eigenvalues satisfy

$$Re\lambda_k \leq -\delta, \quad k = 2, 3, ...$$

The Theorem implies that the approach to the steady profile is exponentially slow but, because of the rescaling, the dependence of such slow motion on ε doesn't appear.

A recent contribution is the reference [7], where the authors consider the Saint-Venant equations for shallow water and, precisely, the phenomenon of formation of roll-waves. The approach merges together analytical techniques and numerical results to present some intriguing properties relative to the dynamics of solitary wave pulses.

CHAPTER 2

Metastability for scalar conservation laws in a bounded domain

1. Introduction

In this chapter we wish to investigate the phenomenon of slow motion of internal shock layer for the initial-boundary value problem for a scalar conservation law with viscosity with Dirichlet boundary condition in the bounded interval $I = (-\ell, \ell)$, that is

$$\begin{cases} \partial_t u + \partial_x f(u) = \varepsilon\, \partial_x^2 u & x \in I, \quad t > 0 \\ u(\pm\ell, t) = u_\pm & t > 0 \\ u(x, 0) = u_0(x) & x \in I \end{cases} \tag{2.1}$$

for some $\varepsilon, \ell > 0$, $u_\pm \in \mathbb{R}$ and flux function f, chosen so that assumptions (1.12) hold.

Our goal is to rigorous analyze the dynamics generated by initial data that present a sharp transition from u^- to u^+, and localized far from the position of the steady state $\bar{U}^\varepsilon_{\text{par}}$, defined as the unique solution to

$$\partial_x f(\bar{U}^\varepsilon_{\text{par}}) = \varepsilon\, \partial_x^2 \bar{U}^\varepsilon_{\text{par}}, \qquad \bar{U}^\varepsilon_{\text{par}}(\pm\ell, t) = u_\pm$$

As stressed in the previous chapter, starting from such an initial datum, a shock layer is formed in an $\mathcal{O}(1)$-timescale. Once such a layer is formed, it moves towards the location corresponding to the equilibrium solution and this motion is exponentially slow.

To describe the dynamics generated by such an initial configuration and to determine a detailed description of the relation between the unviscous and the low-viscosity behavior, our strategy is:

– to build up a one-parameter family of functions $\{U^\varepsilon(\cdot;\xi)\}$ such that $U^\varepsilon(\cdot;\bar{\xi}) = \bar{U}^\varepsilon_{\text{par}}$ for some $\bar{\xi}$, and with the additional property that $U^\varepsilon(\cdot;\xi) \to U^\varepsilon_{\text{hyp}}(\cdot;\xi)$ as $\varepsilon \to 0$, in an appropriate sense;

– to describe the dynamics of the viscous scalar conservation law in a tubular neighborhood of the family $\{U^\varepsilon(\cdot;\xi)\}$.

The essence of this approach is to contribute to the definition of a flexible language, hopefully relevant to more general contexts and, mainly, in the case of systems. With this direction in mind, we follow an approach that it is strictly related with the *projection method* considered in [**12**, **64**] and we go behind the philosophy tracked in the analysis of stability of viscous shock waves by K.Zumbrun and co-authors (see [**55**, **56**, **73**]). Precisely, once a set of reference states $\{U^\varepsilon(\cdot;\xi)\}$ is chosen, we separate two distinct phases:

– to determine spectral properties of the linearized operator at such states;

– to show that appropriate assumptions on the structure of the eigenvalues of such operator together with a control on how far is the approximate state from being an exact solution imply the presence of a metastable behavior.

The main advantage in such a separation stems from the fact that, in principle, it should be possible to obtain numerical evidence of special spectral structures in cases where analytical results appear to be not achievable.

With respect to the framework of shock waves stability analysis, there are two main differences. First of all, we concentrate on the case of bounded domains and, therefore, the spectrum of the linearized operator is discrete and given by a divergent sequence of (real) values. Additionally, the reference states U^ε generically are approximate solutions, in the sense that they satisfy the steady state equation with an error that converges to zero as $\varepsilon \to 0^+$. Hence the perturbations of such states satisfy at first order a *non-homogeneous* linear equation, with forcing term, formally negligible as $\varepsilon \to 0^+$.

Our approach consists in approximating the evolution of the couple (ξ, v), where ξ denotes the parameter for the approximate manifold and represents the location of the shock layer, and v the perturbation of the profile U^ε, by a partial linearization, giving raise to a system which we call quasi-linear system. This is obtained by linearizing with respect to v and keeping the nonlinear dependence on ξ, in order to keep track of the nonlinear evolution along the approximate manifold. Our main contribution is Theorem 2.2, stating that, assuming a number of assumptions relative to the elements of the approximate manifold U^ε and the linearized operator at such states, and a coupling condition, linking the first eigenvalue of the linearized operator with the nonlinear operator evaluated at U^ε, the solution to the quasi-linearized system is described by the evolution of a reduced system where the equation for ξ is decoupled from v and hence solvable by means of the standard separation of variable method.

2. General framework

Let us consider the Cauchy problem for a general evolution equation

$$\partial_t u = \mathcal{F}^\varepsilon[u], \qquad u\big|_{t=0} = u_0 \tag{2.2}$$

where $\mathcal{F}^\varepsilon$ denote a nonlinear differential operator, depending singularly on the parameter $\varepsilon > 0$, so that the formal limiting problem $\partial_t u = \mathcal{F}^0[u]$ is of lower order. Typically, equation (2.2) is complemented with appropriate boundary conditions, appearing in the definition of an appropriate Hilbert space X such that a solution to (2.2) is a function $u : [0, +\infty) \to X$.

Denoting by $u^\varepsilon = u^\varepsilon(x,t)$ the solution of (2.2), we are interested in describing the behavior of u^ε for small ε. The basic example we have in mind is the initial-boundary value problem (2.1).

We assume to have a one-parameter family $\{U^\varepsilon(\cdot;\xi)\}$ in X, parametrized by $\xi \in I$, whose elements are approximate stationary solution to the problem, i.e. $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] \to 0$ as $\varepsilon \to 0$. Precisely, we assume that the term

$\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of the continuous functions space $C(I)$ and there exists a family of smooth positive functions $\Omega^\varepsilon = \Omega^\varepsilon(\xi)$, uniformly convergent to zero as $\varepsilon \to 0$, such that, for any $\xi \in I$, there holds

$$|\langle \psi(\cdot), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot, \xi)]\rangle| \leq \Omega^\varepsilon(\xi)\, |\psi|_{L^\infty} \qquad \forall\, \psi \in C(I).$$

The family $\{U^\varepsilon(x;\xi)\}_{\xi \in I}$ will be referred to as an approximate invariant manifold with respect to the flow determined by (2.2) in the Hilbert space X.

The dependence of Ω^ε on ε plays a fundamental rôle, since it drives the departure from the approximate invariant manifold. Roughly speaking, it measures how far is an element of the approximate manifold from being an exact stationary solution to the problem. In the specific case of a scalar conservation law, such term is *exponentially small*, meaning that it behaves as $e^{-C/\varepsilon}$, with $C > 0$.

Let us stress that, differently to the construction in [**38**] and in [**64**], where the approximate solutions satisfies exactly the equation and the boundary condition to within exponentially small terms, here we assume the elements U^ε to satisfy the boundary conditions exactly and the equation approximately.

Example 2.1. In the case of Burgers equation, i.e. $f(s) = \frac{1}{2}s^2$ and $u^\pm := \mp u_*$, for some $u_* > 0$, we consider a function obtained by matching two different steady states satisfying, respectively, the left and the right boundary condition together with the request $U(\xi) = 0$; in formulas,

$$U^\varepsilon(x;\xi) = \begin{cases} \kappa_- \tanh\left(\kappa_-(\xi - x)/2\varepsilon\right) & \text{in} \quad (-\ell, \xi) \\ \kappa_+ \tanh\left(\kappa_+(\xi - x)/2\varepsilon\right) & \text{in} \quad (\xi, \ell), \end{cases}$$

where $\kappa_\pm$ are chosen so that the boundary conditions are satisfied

$$\kappa_\pm \tanh\left(\frac{\kappa_\pm}{2\varepsilon}(\xi \mp \ell)\right) = u_\pm. \tag{2.3}$$

By direct substitution, denoting by $\delta_{x=\xi}$ the usual Dirac's delta distribution concentrated at $x = \xi$, we obtain the identity

$$\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] = [\![\partial_x U^\varepsilon]\!]_{x=\xi}\,\delta_{x=\xi}$$

in the sense of distributions. In particular, $U^\varepsilon(\cdot, \xi)$ is a stationary solution if and only if $\xi = 0$. Going further, by differentiation, we have

$$[\![\partial_x U^\varepsilon]\!]_{x=\xi} = \frac{1}{2\varepsilon}(\kappa_- - \kappa_+)(\kappa_- + \kappa_+).$$

In order to determine the behavior of Ω^ε for small ε, we need an asymptotic description of the values $\kappa_\pm$. To this aim, let us set $\kappa_\pm := \mp u_\pm(1 + h_\pm)$, so that, denoting by $\Delta_\pm := \ell \mp \xi$ the distance from ξ to $\pm\ell$. relation (2.3) becomes

$$\tanh\left(\mp\frac{u_\pm \Delta_\pm}{2\varepsilon}(1 + h_\pm)\right) = \frac{1}{1 + h_\pm}.$$

Therefore, the values $h_\pm$ are both positive and thus

$$\tanh\left(\mp\frac{u_\pm \Delta_\pm}{2\varepsilon}\right) \leq \frac{1}{1 + h_\pm}.$$

that gives the asymptotic representation

$$h_{\pm} \leq \frac{1}{\tanh(\mp u_{\pm}\Delta_{\pm}/2\varepsilon)} - 1 = \frac{2}{e^{\mp u_{\pm}\Delta_{\pm}/\varepsilon} - 1} = 2\,e^{\pm u_{\pm}\Delta_{\pm}/\varepsilon} + l.o.t.,$$

where $l.o.t.$ denotes lower order terms. Thus, we infer

$$\begin{aligned}
[\![\partial_x U^\varepsilon]\!]_{x=\xi} &= \frac{u_*^2}{2\varepsilon}(h_- - h_+)(2 + h_- + h_+) = \frac{u_*^2}{\varepsilon}(h_- - h_+) + l.o.t. \\
&= \frac{2\,u_*^2}{\varepsilon}(e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}) + l.o.t. \sim C\,\xi\,e^{-C/\varepsilon},
\end{aligned}$$

showing that the term $[\![\partial_x U^\varepsilon]\!]_{x=\xi}$ is null at $\xi = 0$ and exponentially small for $\varepsilon \to 0^+$.

Once the one-parameter family $\{U^\varepsilon(\cdot;\xi)\}$ is chosen, we write the solution to the initial value problem (2.2) as

$$u(\cdot,t) = U^\varepsilon(\cdot;\xi(t)) + v(\cdot,t)$$

with $\xi = \xi(t) \in I$ and $v = v(\cdot,t) \in X$ to be determined. Substituting, we obtain

$$\partial_t v = \mathcal{L}^\varepsilon_\xi v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot;\xi)\,\frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v,\xi] \tag{2.4}$$

where

$$\begin{aligned}
\mathcal{L}^\varepsilon_\xi v &:= d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]\,v \\
\mathcal{Q}^\varepsilon[v,\xi] &:= \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi) + v] - \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]\,v.
\end{aligned}$$

Next, let us assume that, for any ξ, the linear operator $\mathcal{L}^\varepsilon_\xi$ has an decreasing sequence of real eigenvalues $\lambda^\varepsilon_k = \lambda^\varepsilon_k(\xi)$ with $\lambda_k \to -\infty$ as $k \to +\infty$ with corresponding right eigenfunctions $\phi^\varepsilon_k = \phi^\varepsilon_k(\cdot;\xi)$. Denoting by $\psi^\varepsilon_k = \psi^\varepsilon_k(\cdot;\xi)$ the eigenfunctions of the corresponding adjoint operator $\mathcal{L}^{\varepsilon,*}_\xi$ and setting

$$v_k = v_k(\xi;t) := \langle \psi^\varepsilon_k(\cdot;\xi), v(\cdot,t)\rangle,$$

we can use the degree of freedom we still have in the choice of the couple (v,ξ) in such a way that component v_1 is identically zero, that is

$$\frac{d}{dt}\langle \psi^\varepsilon_1(\cdot;\xi(t)), v(\cdot,t)\rangle = 0 \qquad \text{and} \qquad \langle \psi^\varepsilon_1(\cdot;\xi_0), v_0(\cdot))\rangle = 0.$$

Using equation (2.4), we infer

$$\langle \psi^\varepsilon_1(\xi,\cdot), \mathcal{L}^\varepsilon_\xi v + \mathcal{F}[U^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot;\xi)\frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v,\xi]\rangle + \langle \partial_\xi \psi^\varepsilon_1(\xi,\cdot)\frac{d\xi}{dt}, v\rangle = 0$$

Since $\langle \psi^\varepsilon_1, \mathcal{L}_\xi v\rangle = \lambda_1\langle \psi^\varepsilon_1, v\rangle$, we obtain a scalar differential equation for the variable ξ, describing the reduced dynamics along the approximate manifold, that is

$$\alpha^\varepsilon(\xi,v)\frac{d\xi}{dt} = \langle \psi^\varepsilon_1(\cdot;\xi), \mathcal{F}[U^\varepsilon(\cdot;\xi)] + \mathcal{Q}^\varepsilon[v,\xi]\rangle \tag{2.5}$$

where

$$\alpha^\varepsilon_0(\xi) := \langle \psi^\varepsilon_1(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi)\rangle \quad \text{and} \quad \alpha^\varepsilon(\xi,v) := \alpha^\varepsilon_0(\xi) - \langle \partial_\xi \psi^\varepsilon_1(\cdot;\xi), v\rangle,$$

together with the condition on the initial datum ξ_0

$$\langle \psi^\varepsilon_1(\cdot;\xi_0), v_0(\cdot)\rangle = 0$$

To rewrite equation (2.5) in normal form in the regime of small v, we assume

$$|\alpha_0^\varepsilon(\xi)| = |\langle \psi_1^\varepsilon(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi)\rangle| \geq c_0 > 0 \qquad \forall\, \xi \in I.$$

for some $c_0 > 0$. Such assumption gives a (weak) restriction on the choice of the members of the family $\{U^\varepsilon\}$ asking for the manifold to be never transversal to the first eigenfunction of the corresponding linearized operator. From now on, thanks to the previous hypothesis, we can renormalize the eigefunction ψ_1^ε so that

$$\alpha_0^\varepsilon(\xi) = \langle \psi_1^\varepsilon(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi)\rangle = 1 \qquad \forall\, \varepsilon > 0,\, \xi \in I.$$

Since we are interested in the regime $v \to 0$, we expand $1/\alpha^\varepsilon$ as

$$\frac{1}{\alpha^\varepsilon(\xi, v)} = \frac{1}{\alpha_0^\varepsilon(\xi)}\left(1 + \frac{\langle \partial_\xi \psi_1^\varepsilon, v\rangle}{\alpha_0^\varepsilon(\xi)}\right) + o(|v|) = 1 + \langle \partial_\xi \psi_1^\varepsilon, v\rangle + o(|v|).$$

Inserting in (2.5), we end up with the nonlinear equation for ξ

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi)\big(1 + \langle \partial_\xi \psi_1^\varepsilon, v\rangle\big) + \rho^\varepsilon[\xi, v], \tag{2.6}$$

where

$$\theta^\varepsilon(\xi) := \langle \psi_1^\varepsilon, \mathcal{F}[U^\varepsilon]\rangle \tag{2.7}$$

$$\rho^\varepsilon[\xi, v] := \frac{1}{\alpha^\varepsilon(\xi, v)}\big(\langle \psi_1^\varepsilon, \mathcal{Q}^\varepsilon\rangle + \langle \partial_\xi \psi_1^\varepsilon, v\rangle^2\big). \tag{2.8}$$

Using (3.23), the equation (2.4) for v can be rephrased as

$$\partial_t v = H^\varepsilon(x;\xi) + (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon)v + \mathcal{R}^\varepsilon[v, \xi] \tag{2.9}$$

where

$$\begin{aligned} H^\varepsilon(x;\xi) &:= \mathcal{F}^\varepsilon[U^\varepsilon(x;\xi)] - \partial_\xi U^\varepsilon(x;\xi)\,\theta^\varepsilon(\xi), \\ \mathcal{M}_\xi^\varepsilon v &:= -\partial_\xi U^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi)\,\langle \partial_\xi \psi_1^\varepsilon, v\rangle, \\ \mathcal{R}^\varepsilon[v, \xi] &:= \mathcal{Q}^\varepsilon[v, \xi] - \partial_\xi U^\varepsilon(\cdot;\xi)\,\rho^\varepsilon[\xi, v]. \end{aligned}$$

Let us stress that, by definition, there holds

$$\langle \psi_1^\varepsilon(\cdot;\xi), H^\varepsilon(\cdot;\xi)\rangle = 0, \tag{2.10}$$

so that $H^\varepsilon(\cdot;\xi)$ is the projection of $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]$ onto the space orthogonal to $\phi_1^\varepsilon(\cdot;\xi)$.

To show how such formulas can be handled, at least formally, in concrete situations, let us analyze problem (2.1), namely

$$\mathcal{F}^\varepsilon[u] = \varepsilon\partial_x^2 u - \partial_x f(u)$$

for $\varepsilon > 0$ and f satisfying assumptions (1.12). Retracing the definitions introduced above and setting $a^\varepsilon(x;\xi) := f'(U^\varepsilon(x;\xi))$, we get the following expressions

$$\mathcal{L}_\xi^\varepsilon v := \varepsilon\partial_x^2 v - \partial_x\big(a^\varepsilon(\cdot;\xi)\,v\big) \qquad\qquad \mathcal{L}_\xi^{\varepsilon,*} v := \varepsilon\partial_x^2 v + a^\varepsilon(\cdot;\xi)\,\partial_x v$$

where the adjoint operator $\mathcal{L}_\xi^{\varepsilon,*}$ has to be considered with Dirichlet boundary conditions, and

$$\mathcal{Q}^\varepsilon[v, \xi] := -\partial_x \mathcal{N}^\varepsilon[v, \xi] = -\partial_x\Big\{f(U^\varepsilon(\cdot;\xi) + v) - f(U^\varepsilon(\cdot;\xi)) - f'(U^\varepsilon(\cdot;\xi))\,v\Big\}$$

where $\mathcal{N}^\varepsilon = o(|v|)$, so that $\mathcal{Q}^\varepsilon = o(|v|, |\partial_x v|)$. Formally, for small ε and small v, the dynamics of the parameter ξ is approximately given by

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi) + \dots,$$

with θ^ε given in (2.7). Next, we need to identify the functions ψ_1^ε and $\partial_\xi U^\varepsilon$ in the limiting regime $\varepsilon \to 0$, at least approximately. For $\varepsilon \sim 0$, the function ψ_1^ε is close to the eigenfunction of the operator $\mathcal{L}_\xi^{0,*}$ relative to the eigenvalue $\lambda = 0$, with

$$a^0(x;\xi) := u_- \chi_{(-\ell,\xi)}(x) + u_+ \chi_{(\xi,\ell)}(x)$$

Hence, we obtain the representation formula

$$\psi_1^\varepsilon(x) \sim \psi_1^0(x) := \begin{cases} (1 - e^{u_+(\ell-\xi)/\varepsilon})(1 - e^{-u_-(\ell+x)/\varepsilon}) & x < \xi, \\ (1 - e^{-u_-(\ell+\xi)/\varepsilon})(1 - e^{u_+(\ell-x)/\varepsilon}) & x > \xi, \end{cases}$$

so that $\psi_1^\varepsilon \sim 1$, provided ξ is bounded away from the boundaries $\pm\ell$. Additionally, with the approximation $U^\varepsilon(x;\xi) \sim U_{\text{hyp}}(x;\xi)$, defined in (1.13), we infer

$$\frac{U^\varepsilon(x;\xi+h) - U^\varepsilon(x;\xi)}{h} \sim -\frac{1}{h} [\![u]\!] \, \chi_{(\xi,\xi+h)}(x)$$

so that we expect $\partial_\xi U^\varepsilon$ to converge to $-[\![u]\!]\,\delta_\xi$ as $\varepsilon \to 0$ in the sense of distributions, so that $\alpha_0^\varepsilon(\xi) \sim -[\![u]\!]$. Therefore, we deduce an (approximate) expression for the function θ^ε

$$\theta^\varepsilon(\xi) \sim -\frac{1}{[\![u]\!]} \langle 1, \mathcal{F}[U^\varepsilon] \rangle,$$

that, with the choice of U^ε proposed in Example 2.1, reduces to

$$\theta^\varepsilon(\xi) \sim \frac{1}{\varepsilon} u_* \, (e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}), \tag{2.11}$$

which coincides with the corresponding formula determined in [64].

3. Analysis of the quasi-linearized system

Next, let us go back to the system (3.23)–(2.9) for the couple (ξ, v) and let us neglect the $o(v)$ order terms:

$$\begin{cases} \dfrac{d\zeta}{dt} = \theta^\varepsilon(\zeta)\big(1 + \langle \partial_\zeta \psi_1^\varepsilon, w \rangle\big), \\ \partial_t w = H^\varepsilon(\zeta) + (\mathcal{L}_\zeta^\varepsilon + \mathcal{M}_\zeta^\varepsilon) w \end{cases} \tag{2.12}$$

to be complemented with initial conditions

$$\zeta(0) = \zeta_0 \in (-\ell, \ell) \qquad \text{and} \qquad w(x,0) = w_0(x) \in L^2(I). \tag{2.13}$$

From now on, we will refer to this system as the quasi-linearization of (3.23)–(2.9). We are interested in describing the behavior of the solution to such system in the regime of small ε.

Shortly, the quasi-linearized system is determined by an appropriate combination of the term $\mathcal{F}^\varepsilon[U^\varepsilon]$, measuring how far is the function U^ε from being a stationary solution, and the linear operator $\mathcal{L}_\xi$, controlling at first order how solutions to (2.2) depart from U^ε when the latter is taken as initial datum. To state our first result, we need to precise the assumption on such terms.

H1. The family $\{U^\varepsilon(\cdot,\xi)\}$ is such that $\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of $C(I)$ and there exists a family of smooth positive functions Ω^ε such that

$$|\langle\psi(\cdot),\mathcal{F}^\varepsilon[U^\varepsilon(\cdot,\xi)]\rangle| \leq \Omega^\varepsilon(\xi)\,|\psi|_{L^\infty} \qquad \forall\,\psi\in C(I).$$

with Ω^ε converging to zero as $\varepsilon\to 0$, uniformly with respect to ξ.

H2. Let $\{\cdots<\lambda_k^\varepsilon(\xi)<\cdots<\lambda_2^\varepsilon(\xi)<\lambda_1^\varepsilon(\xi)\}$ be the sequence of eigenvalues of the linear operator $\mathcal{L}_\xi^\varepsilon$. Assume that for any $\xi\in(-\ell,\ell)$ there hold

$$\lambda_1^\varepsilon(\xi)-\lambda_2^\varepsilon(\xi)\geq C, \qquad \lambda_1^\varepsilon(\xi)<0 \qquad \lambda_k^\varepsilon(\xi)\leq -C\,k^2 \quad \text{with } k\geq 2.$$

for some constant $C>0$ independent on k, ε and ξ.

H3. Given $\xi\in I$, let $\phi_k^\varepsilon(\cdot;\xi)$ and $\psi_k^\varepsilon(\cdot;\xi)$ be a sequence of eigenfunction for the operators $\mathcal{L}_\xi^\varepsilon$ and $\mathcal{L}_\xi^{\varepsilon,*}$, respectively, normalized so that

$$\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle=1 \qquad \text{and} \qquad \langle\psi_j^\varepsilon,\phi_k^\varepsilon\rangle=\begin{cases}1 & \text{if } j=k,\\ 0 & \text{if } j\neq k,\end{cases} \tag{2.14}$$

Then we assume

$$\sum_j\langle\partial_\xi\psi_k^\varepsilon,\phi_j^\varepsilon\rangle^2=\sum_j\langle\psi_k^\varepsilon,\partial_\xi\phi_j^\varepsilon\rangle^2\leq C \qquad \forall\,k. \tag{2.15}$$

for some constant C independent on the parameter ξ.

For later use, note that, by differentiation, there holds

$$\langle\partial_\xi\psi_j^\varepsilon,\phi_k^\varepsilon\rangle+\langle\psi_j^\varepsilon,\partial_\xi\phi_k^\varepsilon\rangle=0. \tag{2.16}$$

Also, we use the notation $\Lambda_k^\varepsilon:=\sup\limits_{\xi\in I}\lambda_k^\varepsilon(\xi)$.

Theorem 2.2. *Let hypotheses* **H1-2-3** *be satisfied. Additionally, assume that*

$$\Omega^\varepsilon(\xi)\leq C|\lambda_1^\varepsilon(\xi)|, \qquad \forall\,\xi\in(-\ell,\ell) \tag{2.17}$$

for some constant $C>0$ independent on ε and ξ.

Then, denoted by (ζ,w) the solution to the initial-value problem (2.12)–(2.13), for any ε sufficiently small, there exists a time T^ε such that for any $t\leq T^\varepsilon$ the solution w can be represented as

$$w=z+R$$

where z is defined by

$$z(x,t):=\sum_{k\geq 2}w_k(0)\exp\left(\int_0^t\lambda_k^\varepsilon(\zeta(\sigma))\,d\sigma\right)\phi_k^\varepsilon(x;\zeta(t)),$$

and the remainder R satisfies the estimate

$$|R|_{L^2}\leq C\,|\Omega^\varepsilon|_{L^\infty}\left\{\exp\left(\int_0^t\lambda_1^\varepsilon(\zeta(\sigma))d\sigma\right)|w_0|_{L^2}+1\right\} \tag{2.18}$$

for some constant $C>0$.

Moreover, for w_0 sufficiently small in L^2, the final time T^ε can be chosen of the order $-C\,|\Omega^\varepsilon|_{L^\infty}^{-1}\ln|\Omega^\varepsilon|_{L^\infty}$.

The conclusion of the proof of Theorem 2.2 is based on the following version of a standard nonlinear iteration argument.

Lemma 2.3. *Let $f = f(t), g = g(t)$ and $h = h(s,t)$ be continuous functions for $t \in [0,T]$ for some $T > 0$, such that*

$$f(t) \geq 0, \quad g(t) > 0, \quad g \text{ decreasing}, \quad h(s,t) \geq 0.$$

Let $y = y(t)$ be a non-negative function satisfying the estimate

$$y(t) \leq \int_0^t \left\{ f(s)\, g(t)\, y^2(s) + h(s,t) \right\} ds$$

for any $t \leq T$. Then, if, for any $t \in [0,T]$ there holds

$$\sup_{t\in[0,T]} \int_0^t g^2(s)\, f(s)\, ds \cdot \sup_{t\in[0,T]} g^{-1}(t) \int_0^t h(s,t)\, ds < \frac{1}{4} \tag{2.19}$$

then

$$y(t) \leq 2 \sup_{\tau\in[0,t]} \int_0^\tau h(s,\tau)\, ds$$

for any $t \in [0,T]$.

Proof of Lemma 2.3. The auxiliary function $w(t) := g^{-1}(t)\, y(t)$ enjoyes the estimate

$$w(t) \leq \int_0^t \left\{ \alpha(s)\, w^2(s) + \beta(s,t) \right\} ds$$

where $\alpha(t) := f(t)\, g^2(t)$ and $\beta(s,t) = g^{-1}(t)\, h(s,t)$. Set

$$N(t) := \sup_{\tau\in[0,t]} w(\tau).$$

Then, for any $t \in [0,T]$, there holds

$$w(t) \leq \left(\int_0^t \alpha(s)\, ds \right) N^2(T) + \int_0^t \beta(s,t)\, ds$$

and, as a consequence, also

$$N(T) \leq A\, N^2(T) + B$$

where

$$A = A(T) := \sup_{t\in[0,T]} \int_0^t \alpha(s)\, ds, \qquad B = B(T) := \sup_{t\in[0,T]} \int_0^t \beta(s,t)\, ds.$$

Since $N(0) = 0$, if $1 - 4AB > 0$, then

$$N < \frac{1 - \sqrt{1 - 4AB}}{2A} = \frac{2B}{1 + \sqrt{1 - 4AB}} \leq 2B.$$

In term of y, if (2.19) holds, then

$$y(t) < 2\, g(t) \sup_{t\in[0,T]} g^{-1}(t) \int_0^t h(s,t)\, ds.$$

The final estimate follows from the monotonicity of the function g. $\square$

Lemma 2.3 gives the final step needed to prove the Theorem.

PROOF OF THEOREM 2.2. Setting

$$w(x,t)=\sum_j w_j(t)\,\phi_j^\varepsilon(x,\zeta(t)),$$

we obtain an infinite-dimensional differential system for the coefficients w_j

$$\frac{dw_k}{dt}=\lambda_k^\varepsilon(\zeta)\,w_k+\langle\psi_k^\varepsilon,F\rangle \tag{2.20}$$

where, omitting the dependencies for shortness,

$$F:=H^\varepsilon+\sum_j w_j\left\{\mathcal{M}_\zeta^\varepsilon\,\phi_j^\varepsilon-\partial_\xi\phi_j^\varepsilon\,\frac{d\zeta}{dt}\right\}=H^\varepsilon-\theta^\varepsilon\sum_j\Big(a_j+\sum_\ell b_{j\ell}\,w_\ell\Big)w_j.$$

and the coefficients a_j, b_{jk} are given by

$$a_j:=\langle\partial_\xi\psi_1^\varepsilon,\phi_j^\varepsilon\rangle\,\partial_\xi U^\varepsilon+\partial_\xi\phi_j^\varepsilon,\qquad b_{j\ell}:=\langle\partial_\xi\psi_1^\varepsilon,\phi_\ell^\varepsilon\rangle\,\partial_\xi\phi_j^\varepsilon$$

Convergence of the series is guaranteed by assumption (2.15).

By (2.16), for the coefficients a_j there hold

$$\langle\psi_k^\varepsilon,a_j\rangle=\langle\partial_\xi\psi_1^\varepsilon,\phi_j^\varepsilon\rangle\,\big(\langle\psi_k^\varepsilon,\partial_\xi U^\varepsilon\rangle-1\big),$$

so that we can also take advantage from the relation $\langle\psi_1^\varepsilon,a_j\rangle=0$ for any j. Thanks to these relations, equation (2.20) for $k=1$ simplifies to

$$\frac{dw_1}{dt}=\lambda_1^\varepsilon(\zeta)\,w_1-\theta^\varepsilon(\zeta)\sum_{\ell,j}\langle\psi_1^\varepsilon,b_{j\ell}\rangle\,w_\ell\,w_j \tag{2.21}$$

Now let us set

$$E_k(s,t):=\exp\left(\int_s^t\lambda_k^\varepsilon(\zeta(\sigma))d\sigma\right).$$

Note that, for $0\le s<t$, there hold

$$E_k(s,t)=\frac{E_k(0,t)}{E_k(0,s)}\qquad\text{and}\qquad 0\le E_k(s,t)\le e^{\Lambda_k(t-s)}.$$

From equalities (2.21) and (2.20), choosing $w_1(0)=0$, there follow

$$\begin{aligned}w_1(t)&=-\int_0^t\theta^\varepsilon(\zeta)\sum_{\ell,j}\langle\psi_1^\varepsilon,b_{j\ell}\rangle\,w_\ell\,w_j\,E_1(s,t)\,ds\\ w_k(t)&=w_k(0)\,E_k(0,t)\\&\quad+\int_0^t\Big\{\langle\psi_k^\varepsilon,H^\varepsilon\rangle-\theta^\varepsilon(\zeta)\sum_j\Big(\langle\psi_k^\varepsilon,a_j\rangle+\sum_\ell\langle\psi_k^\varepsilon,b_{j\ell}\rangle\,w_\ell\Big)w_j\Big\}E_k(s,t)\,ds,\end{aligned}$$

for $k\ge 2$. Such expressions suggest to introduce the function

$$z(x,t):=\sum_{k\ge2}w_k(0)\,E_k(0,t)\,\phi_k^\varepsilon(x;\zeta(t)),$$

which satisfies the estimate $|z|_{L^2}\le|w_0|_{L^2}e^{\Lambda_2^\varepsilon t}$.

From the representation formulas for the coefficients w_k, since

$$|\theta^\varepsilon(\zeta)|\le C\,\Omega^\varepsilon(\zeta)\qquad\text{and}\qquad|\langle\psi_k^\varepsilon,H^\varepsilon\rangle|\le C\,\Omega^\varepsilon(\zeta)\,\{1+|\langle\psi_k^\varepsilon,\partial_\xi U^\varepsilon\rangle|\}$$

for some constant $C > 0$ depending on the $L^\infty-$norm of ψ_k^ε, there holds

$$
\begin{aligned}
|w-z|^2_{L^2} &\leq C\Big(\int_0^t \Omega^\varepsilon(\zeta)\sum_j |\langle\psi_1^\varepsilon,\,\partial_\xi\phi_j^\varepsilon\rangle|\,|w_j|\,\sum_\ell |\langle\partial_\xi\psi_1^\varepsilon,\phi_\ell^\varepsilon\rangle|\,|w_\ell|\,E_1(s,t)\,ds\Big)^2\\
&\quad + C\sum_{k\geq 2}\Big(\int_0^t \Omega^\varepsilon(\zeta)\Big(1+|\langle\psi_k^\varepsilon,\partial_\xi U^\varepsilon\rangle| + |\langle\psi_k^\varepsilon,\partial_\xi U^\varepsilon\rangle|\sum_j |\langle\partial_\xi\psi_1^\varepsilon,\phi_j^\varepsilon\rangle||w_j|\\
&\quad + \sum_j |\langle\partial_\xi\psi_k^\varepsilon,\phi_j^\varepsilon\rangle||w_j| + \sum_j |\langle\psi_k^\varepsilon,\partial_\xi\phi_j^\varepsilon\rangle|\,|w_j|\sum_\ell|\langle\partial_\xi\psi_1^\varepsilon,\phi_\ell^\varepsilon\rangle|\,|w_\ell|\Big)E_k(s,t)\Big)^2\\
&\leq C\Big(\int_0^t \Omega^\varepsilon(\zeta)|w|^2_{L^2}E_1(s,t)\,ds\Big)^2 + C\sum_{k\geq 2}\Big(\int_0^t\Omega^\varepsilon(\zeta)\big(1+|w|^2_{L^2}\big)E_k(s,t)\,ds\Big)^2
\end{aligned}
$$

Since $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$, we infer

$$
\begin{aligned}
|w-z|_{L^2} &\leq C\int_0^t \Omega^\varepsilon(\zeta)|w|^2_{L^2}\,E_1(s,t)\,ds + C\sum_{k\geq 2}\int_0^t\Omega^\varepsilon(\zeta)\big(1+|w|^2_{L^2}\big)E_k(s,t)\,ds\\
&\leq C\int_0^t\Omega^\varepsilon(\zeta)\Big\{|w|^2_{L^2}\,E_1(s,t) + \big(1+|w|^2_{L^2}\big)\sum_{k\geq 2}E_k(s,t)\Big\}\,ds.
\end{aligned}
$$

The assumption on the asymptotic behavior of the eigenvalues λ_k can now be used to bound the series. Indeed, there holds

$$
\sum_{k\geq 2}E_k(s,t)\leq E_2(s,t)\sum_{k\geq 2}\frac{E_k(s,t)}{E_2(s,t)}\leq C\,(t-s)^{-1/2}\,E_2(s,t)
$$

As a consequence, for unknown w such that $|w|_{L^2}\leq M$ for some $M>0$, we infer

$$
\begin{aligned}
E_1(t,0)|w-z|_{L^2} \leq C\int_0^t \Omega^\varepsilon(\zeta)\Big\{&|w-z|^2_{L^2}\,E_1(s,0)\\
&+|z|^2_{L^2}\,E_1(s,0) + (t-s)^{-1/2}\,E_2(s,t)E_1(t,0)\Big\}\,ds
\end{aligned}
$$

Let us set

$$
N(t) := \sup_{s\in[0,t]}|w-z|_{L^2}\,E_1(s,0)
$$

Then, since $\Lambda_2^\varepsilon\leq\Lambda_1^\varepsilon$, we obtain

$$
\begin{aligned}
E_1(t,0)|w-z|_{L^2} &\leq C\int_0^t\Omega^\varepsilon(\zeta)N^2(s)\,E_1(0,s)\,ds\\
&\quad + C\int_0^t\Omega^\varepsilon(\zeta)\Big\{|w_0|^2_{L^2}e^{(2\Lambda_2^\varepsilon-\Lambda_1^\varepsilon)s} + (t-s)^{-1/2}\,E_2(s,t)E_1(t,0)\Big\}ds
\end{aligned}
$$

By assumption (2.17), $\lambda_1^\varepsilon\leq -C\Omega^\varepsilon$ for some constant $C>0$, hence

$$
\begin{aligned}
\int_0^t\Omega^\varepsilon(\zeta)N^2(s)\,E_1(0,s)\,ds &\leq \int_0^t\Omega^\varepsilon(\zeta)N^2(s)\,\exp\left(-C\int_0^s\Omega^\varepsilon(\zeta)\,d\sigma\right)\,ds\\
&\leq N^2(t)\Big\{1-\exp\left(-C\int_0^t\Omega^\varepsilon(\zeta)\,d\sigma\right)\Big\}.
\end{aligned}
$$

Moreover, there holds

$$\int_0^t e^{(2\Lambda_2^\varepsilon-\Lambda_1^\varepsilon)s}\,ds \le \int_0^t e^{\Lambda_2^\varepsilon s}\,ds = \frac{1}{\Lambda_2^\varepsilon}(e^{\Lambda_2^\varepsilon s}-1) \le \frac{1}{|\Lambda_2^\varepsilon|}$$
$$\int_0^t (t-s)^{-1/2}\,E_2(s,t)\,ds \le \int_0^t (t-s)^{-1/2}\,e^{\Lambda_2^\varepsilon(t-s)}\,ds \le \frac{1}{|\Lambda_2^\varepsilon|^{1/2}}$$

so that, recalling that Λ_2 is bounded away from 0,

$$E_1(t,0)|w-z|_{L^2} \le C\Big\{N^2(t)\Big[1-\exp\Big(-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\Big)\Big] + C|\Omega^\varepsilon|_\infty\Big(|w_0|^2_{L^2}+E_1(t,0)\Big)\Big\}$$

Next, we end up with the estimate

$$N(t) \le AN^2(t)+B \qquad \text{with} \qquad \begin{cases} A := C\left\{1-\exp\left(-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\right)\right\}, \\ B := C|\Omega^\varepsilon|_{L^\infty}\Big(|w_0|^2_{L^2}+E_1(t,0)\Big) \end{cases}$$

Hence, as soon as

$$4AB = 4C^2|\Omega^\varepsilon|_{L^\infty}\Big(|w_0|^2_{L^2}+E_1(t,0)\Big)\Big(1-\exp\Big(-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\Big)\Big) < 1 \tag{2.22}$$

there holds

$$N(t) \le \frac{2B}{1+\sqrt{4AB}} \le 2B = C\,|\Omega^\varepsilon|_{L^\infty}\Big(|w_0|^2_{L^2}+E_1(t,0)\Big)$$

that means, in term of the difference $w-z$,

$$|w-z|_{L^2} \le C\,|\Omega^\varepsilon|_{L^\infty}\Big(|w_0|^2_{L^2}E_1(0,t)+1\Big)$$

Condition (2.22) gives a constraint on the final time T^ε. Since $1-e^{-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma} < 1$, it is enough to ask

$$4C^2\,|\Omega^\varepsilon|_{L^\infty}\Big(|w_0|^2_{L^2}+E_1(t,0)\Big) < 1 \tag{2.23}$$

to assure condition (2.22) is satisfied. Constraint (2.23) can be rewritten as

$$C\,\exp\Big(-\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\Big) \le \exp\Big(-\int_0^t \lambda_1^\varepsilon(\zeta)\,d\sigma\Big) = E_1(t,0) \le \frac{C}{|\Omega^\varepsilon|_{L^\infty}} - |w_0|^2_{L^2},$$

so that we can choose T^ε of the form

$$T^\varepsilon := \frac{1}{|\Omega^\varepsilon|_{L^\infty}}\ln\Big(\frac{C}{|\Omega^\varepsilon|_{L^\infty}} - |w_0|^2_{L^2}\Big) \sim -C\,|\Omega^\varepsilon|^{-1}_{L^\infty}\ln|\Omega^\varepsilon|_{L^\infty}$$

for w_0 sufficiently small. □

As a consequence of the estimate (2.18), for $|w|_{L^2} < M$ for some $M>0$, the function ζ satisfies

$$\frac{d\zeta}{dt} = \theta^\varepsilon(\zeta)(1+r) \qquad \text{with} \quad |r| \le C\big(|w_0|_{L^2}\,e^{\Lambda_2^\varepsilon t} + |\Omega^\varepsilon|_{L^\infty}\big).$$

where the constant C depends also on M. In particular, if ε and $|w_0|_{L^2}$ are sufficiently small, the function $\zeta = \zeta(t)$ has similar decay properties with respect to the function η, solution to the reduced Cauchy problem

$$\frac{d\eta}{dt} = \theta^\varepsilon(\eta), \qquad \eta(0) = \zeta_0.$$

This preludes to the following consequence of Theorem 2.2.

Corollary 2.4. *Let hypotheses* **H1-2-3** *and* (2.17) *be satisfied. Assume also*

$$s\,\theta^\varepsilon(s) < 0 \quad \text{for any } s \in I,\ s \neq 0 \qquad \text{and} \qquad {\theta^\varepsilon}'(\bar\zeta) < 0. \tag{2.24}$$

Then, for ε and $|w_0|_{L^2}$ sufficiently small, the estimate (2.18) *holds globally in time and the solution (ζ, w) converges to $(\bar\zeta, 0)$ as $t \to +\infty$.*

Proof. Thanks to assumption **H1**, for ε and $|w_0|_{L^2}$ sufficiently small, estimate (2.18) holds. Hence, for any initial datum ζ_0, the variable $\zeta = \zeta(t)$ solves an equation of the form

$$\frac{d\zeta}{dt} = \theta^\varepsilon(\zeta)(1 + r(t)) \qquad \text{with} \quad |r(t)| \leq C(|w_0|_{L^2} e^{\Lambda_2^\varepsilon t} + |\Omega^\varepsilon|_{L^\infty}).$$

Therefore, $\zeta(t)$ converges to $\bar\zeta$ as $t \to +\infty$ and the convergence is exponential, in the sense that there exists $\beta^\varepsilon < 0$ such that

$$|\zeta(t) - \bar\zeta| \leq |\zeta_0| e^{\beta^\varepsilon t}, \quad \beta^\varepsilon \sim {\theta^\varepsilon}'(\bar\zeta) \tag{2.25}$$

for any t under consideration. Furthermore, from (2.20), we deduce

$$w_k(t) = w_k(0)\,\exp\left(\int_0^t \lambda_k^\varepsilon\,d\sigma\right) + \int_0^t \langle\psi_k^\varepsilon, F\rangle(s)\,\exp\left(\int_s^t \lambda_k^\varepsilon\,d\sigma\right)ds$$

Thus, by the Jensen's inequality, we infer the estimate

$$\begin{aligned} |w|_{L^2}^2(t) &\leq C\left\{|w_0|_{L^2}^2\,e^{2\Lambda_1^\varepsilon t} + \sum_k \left(\int_0^t \langle\psi_k^\varepsilon, F\rangle(s)\,e^{\Lambda_1^\varepsilon (t-s)}\,ds\right)^2\right\} \\ &\leq C\left\{|w_0|_{L^2}^2\,e^{2\Lambda_1^\varepsilon t} + t\int_0^t |F|_{L^2}^2(s)\,e^{2\Lambda_1^\varepsilon (t-s)}\,ds\right\} \end{aligned}$$

Let $\nu^\varepsilon > 0$ be such that $|F|_{L^2}(t) \leq C\,e^{-\nu^\varepsilon t}$; then, if $\nu^\varepsilon \neq |\Lambda_1^\varepsilon|$, there holds

$$|w|_{L^2}^2(t) \leq C\left\{|w_0|_{L^2}^2\,e^{2\Lambda_1^\varepsilon t} + t\left(e^{-2\nu^\varepsilon t} + e^{2\Lambda_1^\varepsilon t}\right)\right\}$$

showing the exponential convergence to 0 of the component w. □

Estimate (2.25) shows the exponentially slow motion of the shock layer for small ε. Precisely, the evolution of the location of the shock towards the equilibrium position is much slower as ε becomes smaller, since $\beta^\varepsilon \to 0$ as $\varepsilon \to 0$. For example, when $f(s) = s^2/2$, $\bar\zeta = 0$ and ${\theta^\varepsilon}'(0) \sim e^{-1/\varepsilon}$ (see formula (2.11)).

Let us also stress that in the regime $(\zeta, w) \sim (\bar\zeta, 0)$, a linearization at the equilibrium solution $U^\varepsilon(x;\bar\zeta)$ would furnish a more detailed description of the dynamics, since the source term due to the approximation at an approximate steady state would not be present. In fact, the description given by the quasi-linearization is meaningful in the regime far from equilibrium and its aim is to describe the slow motion around a manifold of approximate solutions.

4. Spectral analysis for the diffusion-transport operator

Our concern in the present Section is to estabilish a precise description on the location of the eigenvalues of the linearized operator, in order to show that the general procedure developed in the previous Sections is indeed applicable in the case of a scalar conservation law. The problem of determining the limiting structure of the spectrum of the type of second order differential operators we deal with has been widely considered in the literature. Among others, let us quote the approach, based on the use of Prüfer transform, used in [**12**], in the context of metastability analysis for the Allen–Cahn equation. Here, we prefer to follow the strategy implemented in [**35**], for the linearization at the steady state of the Burgers equation. In what follows, we show that the same kind of eigenvalues distribution holds in a much more general situation, the main ingredient being the resemblance of the coefficient a^ε to a step function a^0, jumping from a positive to a negative value, as $\varepsilon \to 0^+$.

Fixed $\varepsilon > 0$ and linearizing the scalar conservation law (2.1) at a given a reference profile $U^\varepsilon = U^\varepsilon(x;\xi)$, satisfying the boundary conditions $U^\varepsilon(\pm\ell;\xi) = u_\pm$, we end up with the differential linear diffusion-transport operator

$$\mathcal{L}^\varepsilon_\xi u := \varepsilon\,\partial_x^2 u - \partial_x(a^\varepsilon\,u) \qquad u(\pm\ell) = 0, \tag{2.26}$$

where $a^\varepsilon = a^\varepsilon(x;\xi) := f'(U^\varepsilon(x;\xi))$. The aim of this Section is to describe the structure of the spectrum $\sigma(\mathcal{L}^\varepsilon_\xi)$ of the operator $\mathcal{L}^\varepsilon_\xi$ for ε sufficiently small.

Given the function a^ε, let us introduce the self-adjoint operator

$$\mathcal{M}^\varepsilon_\xi v := \varepsilon^2\,\partial_x^2 v - b^\varepsilon\,v \qquad v(\pm\ell) = 0, \tag{2.27}$$

where

$$b^\varepsilon := \left(\frac{1}{2}\,a^\varepsilon\right)^2 + \frac{1}{2}\,\varepsilon\,\frac{da^\varepsilon}{dx}. \tag{2.28}$$

By omitting the dependencies from ξ for shortness, a straightforward calculation shows that if u is an eigenfunction of (2.26) relative to the eigenvalue λ, then the function $v(x)$ defined by

$$v(x) = \exp\left(-\frac{1}{2\varepsilon}\int_{x_0}^x a^\varepsilon(y)\,dy\right)u(x)$$

(with x_0 arbitrarily chosen) is an eigenfunction of the operator $\mathcal{M}^\varepsilon_\xi$ relative to the eigenvalue $\mu := \varepsilon\lambda$. Since $\mathcal{M}^\varepsilon_\xi$ is self-adjoint, we can state that the spectrum of the operator $\mathcal{L}^\varepsilon_\xi$ is composed by <u>real</u> eigenvalues. Moreover, if u is an eigenfunction of (2.26) relative to the first eigenvalue λ^ε_1, integrating in $(-\ell,\ell)$ the relation $\mathcal{L}^\varepsilon_\xi u = \lambda\,u$, we deduce the identity

$$0 = \int_{-\ell}^{\ell}(\mathcal{L}^\varepsilon_\xi - \lambda^\varepsilon_1)u\,dx = \varepsilon\left(\partial_x u(\ell) - \partial_x u(-\ell)\right) - \lambda^\varepsilon_1\int_{-\ell}^{\ell}u(x)\,dx$$

Assuming, without loss of generality, u to be strictly positive in $(-\ell,\ell)$ and normalized so that its integral in $(-\ell,\ell)$ is equal to 1, we get

$$\lambda^\varepsilon_1 = \varepsilon\left(\partial_x u(\ell) - \partial_x u(-\ell)\right) < 0$$

Hence, for any choice of the function a^ε, there holds

$$\sigma(\mathcal{L}^\varepsilon_\xi) \subset (-\infty, 0).$$

Our next aim is to show that under appropriate assumption on the behavior of the family of functions a^ε as $\varepsilon \to 0^+$, it is possible to furnish a detailed representation of the eigenvalue distributions for small ε. Specifically, we are interested in coefficients a^ε behaving, in the limit $\varepsilon \to 0^+$ as a step function of the form

$$a^0(x) := \begin{cases} a_- & x \in (-\ell, \xi), \\ a_+ & x \in (\xi, \ell), \end{cases} \tag{2.29}$$

for some $\xi \in (-\ell, \ell)$ and $a_+ < 0 < a_-$. We will show that, under appropriate assumptions making precise in which sense a^ε "resemble" a^0 for ε small, the first eigenvalue λ^ε_1 turns to be "very close" to 0 for ε small, and all of the others eigenavalues λ^ε_k, with $k \geq 2$, are such that $\varepsilon \lambda^\varepsilon_k = O(1)$ as $\varepsilon \to 0^+$.

Estimate from below for the first eigenvalue. We estimate the first eigenvalue μ^ε_1 of the operator $\mathcal{M}^\varepsilon_\xi$ by means of the inequality

$$|\mu^\varepsilon_1| \leq \frac{|\mathcal{M}^\varepsilon_\xi \psi|_{L^2}}{|\psi|_{L^2}}.$$

for smooth test function ψ such that $\psi(\pm\ell) = 0$. Let us consider as test function $\psi^\varepsilon(x) := \psi^\varepsilon_0(x) - K^\varepsilon(x)$, where

$$\psi^\varepsilon_0(x) := \exp\left(\frac{1}{2\varepsilon}\int_\xi^x a^\varepsilon(y)\,dy\right),$$
$$K^\varepsilon(x) := \frac{1}{2\ell}\big\{\psi^\varepsilon_0(-\ell)(\ell - x) + \psi^\varepsilon_0(\ell)(\ell + x)\big\}.$$

A direct calculation shows that $\mathcal{M}^\varepsilon_\xi \psi := b^\varepsilon\, K$ and, assuming the family b^ε to be uniformly bounded, we infer

$$|\mu^\varepsilon_1| \leq \frac{|b^\varepsilon\, K^\varepsilon|_{L^2}}{|\psi^\varepsilon_0 - K^\varepsilon|_{L^2}} \leq C\,\frac{|K^\varepsilon|_{L^2}}{|\psi^\varepsilon_0|_{L^2} - |K^\varepsilon|_{L^2}} = \frac{C}{|K^\varepsilon|^{-1}_{L^2}|\psi^\varepsilon_0|_{L^2} - 1}$$

as soon as $|\psi^\varepsilon_0|_{L^2} > |K^\varepsilon|_{L^2}$.

The opposite case being similar, let us assume $\psi_0(-\ell) \geq \psi_0(\ell)$. From the definition of K^ε, it follows

$$|K^\varepsilon|^2_{L^2} = \frac{2\ell}{3}\big\{\psi^2_0(\ell) + \psi_0(\ell)\psi_0(-\ell) + \psi^2_0(-\ell)\big\} \leq 2\ell\,\psi^2_0(-\ell).$$

Therefore, we deduce

$$|K^\varepsilon|^{-2}_{L^2}|\psi^\varepsilon_0|^2_{L^2} \geq 2\ell\,\psi_0^{-2}(-\ell)\int_{-\ell}^\ell |\psi^\varepsilon_0(x)|^2\,dx = 2\ell\, I^\varepsilon$$

where

$$I^\varepsilon := \int_{-\ell}^\ell \exp\left(\frac{1}{\varepsilon}\int_{-\ell}^x a^\varepsilon(y)\,dy\right)\,dx$$

Since a^ε converges to the step function a^0 as $\varepsilon \to 0^+$, it is natural to approximate the latter integral in term of the corresponding one for a^0:

$$I^\varepsilon = \int_{-\ell}^{\ell} \exp\left(\frac{1}{\varepsilon}\int_{-\ell}^{x}(a^\varepsilon - a^0)(y)\,dy\right)\exp\left(\frac{1}{\varepsilon}\int_{-\ell}^{x} a^0(y)\,dy\right)\,dx \geq e^{-|a^\varepsilon - a^0|_{L^1}/\varepsilon}\, I^0.$$

Since, for ε small,

$$\begin{aligned} I^0 &= \int_{-\ell}^{\xi} e^{a_-(x+\ell)/\varepsilon}\,dx + e^{a_-(\xi+\ell)/\varepsilon}\int_{\xi}^{\ell} e^{a_+(x-\xi)/\varepsilon}\,dx \\ &= \varepsilon\, e^{a_-(\xi+\ell)/\varepsilon}\left\{\frac{1}{a_-}\left(1 - e^{-a_-(\xi+\ell)/\varepsilon}\right) - \frac{1}{a_+}\left(1 - e^{a_+(\ell-\xi)/\varepsilon}\right)\right\} \sim \frac{[a]}{a_-a_+}\,\varepsilon\, e^{a_-(\xi+\ell)/\varepsilon}. \end{aligned}$$

the subsequent estimate holds

$$|K^\varepsilon|_{L^2}^{-2}|\psi_0^\varepsilon|_{L^2}^2 \geq 2\,\ell\, e^{-|a^\varepsilon - a^0|_{L^1}/\varepsilon}\, I^0 \geq C_1\, e^{C_2/\varepsilon}.$$

whenever $|a^\varepsilon - a^0|_{L^1} \leq c_0\varepsilon$ for some $c_0 > 0$. Thus, we deduce for the first eigenvalue μ_1^ε of the self-adjoint operator $\mathcal{M}_\xi^\varepsilon$ the estimate $|\mu_1^\varepsilon| \leq C_1\, e^{C_2/\varepsilon}$ for some positive constant C_1, C_2. As a consequence, since the spectrum $\sigma(\mathcal{L}_\xi^\varepsilon)$ coincides with $\varepsilon^{-1}\sigma(\mathcal{M}_\xi^\varepsilon)$, the next result holds.

Proposition 2.5. *Let a^ε be a family of functions satisfying the assumption:*
A0. *there exists $C > 0$, indipendent on $\varepsilon > 0$, such that*

$$|a^\varepsilon|_{L^\infty} + \varepsilon\left|\frac{da^\varepsilon}{dx}\right|_{L^\infty} \leq C$$

If there exists $\xi \in (-\ell, \ell)$, $a_+ < 0 < a_-$ and $C > 0$ for which $|a^\varepsilon - a^0|_{L^1} \leq C\varepsilon$, then there exist constants $C, c > 0$ such that $-C\, e^{-c/\varepsilon} \leq \lambda_1^\varepsilon < 0$.

Let us stress that the request $a_+ < 0 < a_-$ is essential, even if hided in the proof. If this is not the case, the term K^ε would not be small as $\varepsilon \to 0^+$ and its L^2 norm would not be bounded by the L^2-norm of ψ_0^ε. In fact, the statement in Proposition 2.5 may not hold when $a_\pm$ have the same sign, the easiest example being the case $a^\varepsilon \equiv a_+ = a_- > 0$.

The next Example gives a heuristic estimate for the first eigenvalue λ_1^ε.

Example 2.6. Given $-\alpha < 0 < \beta$ and $a_\pm \in \mathbb{R}$, let us set $I = (-\alpha, \beta)$, $[a] := a_+ - a_-$ and

$$a(x) = a_-\chi_{(-\alpha,0)}(x) + a_+\chi_{(0,\beta)}(x).$$

Given $\lambda^\varepsilon > 0$, let us look for function $u \in C(I)$, such that

$$\mathcal{L}u := \varepsilon\, u'' + (a(x)\,u)' + \lambda^\varepsilon\, u = 0, \qquad u(-\alpha) = u(\beta) = 0$$

in the sense of distributions. Since $a' = [a]\,\delta_0$, this amounts in finding two functions $u^\pm$ such that

$$\mathcal{L}_\pm u := \varepsilon\, u_\pm'' + a_\pm\, u_\pm' + \lambda^\varepsilon\, u = 0, \qquad u_-(-\alpha) = u_+(\beta) = 0$$

and the following transmission conditions are satisfied

$$u_+(0) - u_-(0) = 0 \qquad \text{and} \qquad \varepsilon\left(u_+'(0) - u_-'(0)\right) + [a]\,u_\pm(0) = 0.$$

The characteristic polinomial of $\mathcal{L}_\pm$ is $p_\pm(\mu;\lambda^\varepsilon) := \varepsilon\,\mu^2 + a_\pm\,\mu + \lambda^\varepsilon$, with roots

$$\mu_-^\pm := \frac{-a_- \pm \Delta_-}{2\varepsilon}, \qquad \mu_+^\pm := \frac{-a_+ \pm \Delta_+}{2\varepsilon}, \qquad \text{where } \Delta_\pm := \sqrt{(a_\pm)^2 - 4\,\varepsilon\,\lambda^\varepsilon}.$$

Assume $\lambda^\varepsilon < (a_\pm)^2/4\,\varepsilon$. Choosing $u_\pm$ in the form

$$u_-(x) = A_-(e^{\mu_-^+(\alpha+x)} - e^{\mu_-^-(\alpha+x)}) \quad \text{and} \quad u_+(x) = A_+(e^{-\mu_+^+(\beta-x)} - e^{-\mu_+^-(\beta-x)}).$$

Setting $\theta_-^\pm := e^{\mu_-^\pm\alpha}$ and $\theta_+^\pm := e^{-\mu_+^\pm\beta}$, there holds

$$\begin{aligned} u_-(0) &= A_-(\theta_-^+ - \theta_-^-) \qquad & u'_-(0) &= A_-(\mu_-^+\theta_-^+ - \mu_-^-\theta_-^-) \\ u_+(0) &= A_+(\theta_+^+ - \theta_+^-) \qquad & u'_+(0) &= A_+(\mu_+^+\theta_+^+ - \mu_+^-\theta_+^-). \end{aligned}$$

Therefore, the transmission conditions take the form of a linear system in $A_\pm$

$$\begin{cases} (\theta_+^+ - \theta_+^-)A_+ - (\theta_-^+ - \theta_-^-)A_- = 0, \\ \Big\{\big(2\varepsilon\,\mu_+^+ + [a]\big)\,\theta_+^+ - \big(2\varepsilon\,\mu_+^- + [a]\big)\,\theta_+^-\Big\}A_+ \\ \qquad + \Big\{\big(-2\varepsilon\,\mu_-^+ + [a]\big)\,\theta_-^+ + \big(2\varepsilon\,\mu_-^- - [a]\big)\,\theta_-^-\Big\}A_- = 0. \end{cases} \tag{2.30}$$

After some manipulation, the determinant $D = D(\lambda^\varepsilon, \varepsilon)$ of (2.30) can be rewritten as

$$D = ([a] + [\Delta])\,\theta_+^+\theta_-^+ - ([a] - \{\Delta\})\,\theta_+^-\theta_-^+ - ([a] + \{\Delta\})\,\theta_+^+\theta_-^- + ([a] - [\Delta])\,\theta_+^-\theta_-^-,$$

where $[\Delta] := \Delta_+ - \Delta_-$ and $\{\Delta\} := \Delta_+ + \Delta_-$.

Since $\sqrt{\kappa^2 - 4\,x} = |\kappa| - 2|\kappa|^{-1}\,x + o(x)$, for $\varepsilon\lambda^\varepsilon \to 0$, there hold

$$\begin{aligned} \varepsilon\ln(\theta_+^+\theta_-^+) &= |a_-|\alpha + \left(\frac{\beta}{a_+} + \frac{\alpha}{a_-}\right)\varepsilon\lambda^\varepsilon + o(\varepsilon\lambda^\varepsilon), \\ \varepsilon\ln(\theta_+^-\theta_-^+) &= (a_+\,\beta + |a_-|\alpha) - \left(\frac{\beta}{a_+} + \frac{\alpha}{|a_-|}\right)\varepsilon\lambda + o(\varepsilon\lambda), \\ \varepsilon\ln(\theta_+^+\theta_-^-) &= \left(\frac{\beta}{a_+} + \frac{\alpha}{|a_-|}\right)\varepsilon\lambda^\varepsilon + o(\varepsilon\lambda^\varepsilon), \\ \varepsilon\ln(\theta_+^-\theta_-^-) &= a_+\beta - \left(\frac{\beta}{a_+} + \frac{\alpha}{a_-}\right)\varepsilon\lambda^\varepsilon + o(\varepsilon\lambda^\varepsilon) \end{aligned}$$

$$\begin{aligned} \{\Delta\} &= \sqrt{(a_+)^2 - 4\,\varepsilon\lambda^\varepsilon} + \sqrt{(a_-)^2 - 4\,\varepsilon\lambda^\varepsilon} = [a]\left(1 + \frac{2\,\varepsilon\lambda^\varepsilon}{a_+\,a_-}\right) + o(\varepsilon\lambda^\varepsilon) \\ [\Delta] &= \sqrt{(a_+)^2 - 4\,\varepsilon\,\lambda^\varepsilon} - \sqrt{(a_-)^2 - 4\,\varepsilon\,\lambda^\varepsilon} = \{a\}\left(1 - \frac{2\,\varepsilon\lambda^\varepsilon}{a_+a_-}\right) + o(\varepsilon\lambda^\varepsilon) \end{aligned}$$

Hence, we infer

$$D \sim 2\left(a_+e^{|a_-|\alpha/\varepsilon} + \frac{\varepsilon\lambda^\varepsilon}{a_+\,a_-}e^{(a_+\,\beta+|a_-|\alpha)/\varepsilon} - a_-e^{a_+\beta/\varepsilon}\right),$$

so that $D \sim 0$ for

$$\lambda^\varepsilon \sim -\frac{a_+\,a_-}{\varepsilon\,[a]}\left(a_+e^{-a_+\beta/\varepsilon} - a_-e^{-|a_-|\alpha/\varepsilon}\right)$$

For $a_\pm = \pm u_*$, $\alpha = \ell + \xi$ and $\beta = \ell - \xi$, the above expression becomes

$$\lambda^\varepsilon \sim \frac{u_*^2}{2\varepsilon}\left(e^{-u_*(\ell-\xi)/\varepsilon} + e^{-u_*(\ell+\xi)/\varepsilon}\right) = \frac{u_*^2}{\varepsilon}\cosh(u_*\xi/\varepsilon)\, e^{-u_*\ell/\varepsilon}$$

to be compared with the expression for Ω^ε obtained in Example 2.1

$$\Omega^\varepsilon \sim \left|\frac{2\,u_*^2}{\varepsilon}(e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon})\right| = \frac{4\,u_*^2}{\varepsilon}|\sinh(u_*\,\xi/\varepsilon)|\, e^{-u_*\ell/\varepsilon}$$

so that

$$\left|\frac{\Omega^\varepsilon}{\lambda^\varepsilon}\right| \sim 4|\tanh(u_*\,\xi/\varepsilon)| \le 4$$

Let us stress that this formula shows that hypothesis (2.17) is verified heuristically for Burgers type equations.

Estimate from above for the second eigenvalue. Controlling the location of the second (and subsequent) eigenvalue needs much more care and, also, a number of additional assumption on the limiting behavior of the function a^ε as $\varepsilon \to 0^+$. Precisely, we suppose $a^\varepsilon \in C^0([-\ell, \ell])$ satisfies the following hypotheses:

A1. the function a^ε is twice differentiable at any $x \neq \xi$ and

$$\frac{da^\varepsilon}{dx}, \frac{d^2a^\varepsilon}{dx^2} < 0 < a^\varepsilon \quad \text{in } (-\ell, \xi), \qquad \text{and} \qquad a^\varepsilon, \frac{da^\varepsilon}{dx} < 0 < \frac{d^2a^\varepsilon}{dx^2} \quad \text{in } (\xi, \ell),$$

A2. for any $C > 0$ there exists $c_0 > 0$ such that, for any x satisfying $|x - \xi| \ge c_0\varepsilon$, there holds

$$|a^\varepsilon - a^0| \le C\,\varepsilon \qquad \text{and} \qquad \varepsilon\left|\frac{da^\varepsilon}{dx}\right| \le C;$$

A3. there exists the left/right first order derivatives of a^ε at ξ and

$$\liminf_{\varepsilon\to 0^+} \varepsilon\left|\frac{da^\varepsilon}{dx}(\xi\pm)\right| > 0$$

As a consequence, the function $b^\varepsilon + \varepsilon\lambda^\varepsilon$ satisfies a number of corresponding properties, listed in the next statement.

Lemma 2.7. *Let the family a^ε be such that hypotheses* A1-2-3 *are satisfied, and let $\lambda^\varepsilon < 0$ be such that*

$$\inf_{\varepsilon>0} \varepsilon\lambda^\varepsilon > -\frac{1}{4}\,\alpha_0^2 \qquad \textit{where} \;\; \alpha_0 := \min\{|a_-|, |a_+|\}. \tag{2.31}$$

Then there exist $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, the functions $b^\varepsilon + \varepsilon\lambda^\varepsilon$, with b^ε defined in (2.28), enjoy the following properties:

B1. *the function $b^\varepsilon + \varepsilon\lambda^\varepsilon$ is decreasing in $(-\ell, \xi)$ and increasing in (ξ, ℓ);*

B2. *there exist $C, c > 0$ such that, for any x with $|x - \xi| \ge c\,\varepsilon$ there holds $b^\varepsilon + \varepsilon\lambda^\varepsilon \ge C > 0$;*

B3. *there exist the left/right limits of $b^\varepsilon + \varepsilon\lambda^\varepsilon$ at ξ and*

$$\beta := \limsup_{\varepsilon\to 0^+}\big(b^\varepsilon(\xi\pm) + \varepsilon\lambda^\varepsilon\big) < 0;$$

Proof. Property B1. is an immediate consequence of assumption A1, since

$$\frac{d}{dx}\left(b^\varepsilon + \varepsilon\lambda^\varepsilon\right) = \frac{1}{4}\,a^\varepsilon\,\frac{da^\varepsilon}{dx} + \frac{1}{2}\,\varepsilon\,\frac{d^2a^\varepsilon}{dx^2}.$$

From A2, given $C > 0$, for $x \le \xi - c_0\,\varepsilon$, there holds

$$\begin{aligned} b^\varepsilon + \varepsilon\lambda^\varepsilon &\ge \frac{1}{4}(a^\varepsilon + a^0)(a^\varepsilon - a^0) - \frac{1}{2}\,\varepsilon\left|\frac{da^\varepsilon}{dx}\right| + \varepsilon\lambda^\varepsilon + \frac{1}{4}a_-^2 \\ &\ge \varepsilon\lambda^\varepsilon + \frac{1}{4}\alpha_0^2 - \frac{1}{2}\left(1 + |a^0|\,\varepsilon + \frac{1}{2}\,C\,\varepsilon^2\right)\,C \end{aligned}$$

From such inequality, by choosing $C > 0$ sufficiently small, and combining with an analogous estimate on $(\xi + c\,\varepsilon, \ell)$, property B2. follows.

For what concerns B3, we observe that, since $a(\xi) = 0$ and $\lambda \le 0$, there holds

$$\limsup_{\varepsilon\to0^+}\bigl(b^\varepsilon(\xi\pm) + \varepsilon\lambda^\varepsilon\bigr) \le \limsup_{\varepsilon\to0^+}\frac{1}{2}\,\varepsilon\,\frac{da^\varepsilon}{dx}(\xi) = -\liminf_{\varepsilon\to0^+}\,\varepsilon\left|\frac{da^\varepsilon}{dx}(\xi\pm)\right| < 0,$$

thanks to A3. □

For later reference, we denote $y_\pm^\varepsilon$ the zeros of $b^\varepsilon + \varepsilon\lambda^\varepsilon$, with $-\ell < y_-^\varepsilon < \xi < y_+^\varepsilon < \ell$. Since property B2 holds, we deduce that $|y_\pm^\varepsilon - \xi| \le c_0\,\varepsilon$.

Assume the assumption of Lemma 2.7 to hold, and let λ_2^ξ and $\mu_2^\xi = \varepsilon\,\lambda_2^\xi$ be the second eigenvalue of the operators $\mathcal{L}_\xi^\varepsilon$ and $\mathcal{M}_\xi^\varepsilon$, respectively, with corresponding eigenfunctions ϕ_2^ξ and ψ_2^ξ. Such eigenfunctions are linked together by the relation

$$\psi_2^\xi(x) = A\,\exp\left(-\frac{1}{2\varepsilon}\int_{x_*}^x a^\varepsilon(y)\,dy\right)\phi_2^\xi(x) \tag{2.32}$$

for some constants A and x_*. Since λ_2^ξ is the second eigenvalue, the functions ϕ_2^ξ and ψ_2^ξ possess a single root located at some point $x_0^\xi \in (-\ell, \ell)$. The sign properties of $b^\varepsilon + \mu_2^\xi$ described in Lemma 2.7 imply that $x_0^\xi \in (y_-^\varepsilon, y_+^\varepsilon)$. Then, ϕ_2^ξ and ψ_2^ξ restricted to the intervals $(-\ell, x_0^\xi)$ and (x_0^ξ, ℓ) are eigenfunctions relative to the first eigenvalue of the same operator considered in the corresponding intervals and with Dirichlet boundary conditions.

From now on, we drop, for shortness, the dependence on ε of $\lambda_2, \phi_2, \psi_2, x_0$, we assume, without loss of generality, $x_0 \ge \xi$ and we restrict our attention to the interval $J = (x_0, \ell)$. Integrating on J, we deduce

$$\lambda_2\int_{x_0}^\ell \phi_2\,dx = \varepsilon\left(\partial_x\phi_2(\ell) - \partial_x\phi_2(x_0)\right) < -\varepsilon\,\partial_x\phi_2(x_0)$$

having chosen ϕ_2 positive in J. Assuming ψ_2 to be given as in (2.32) with $A = 1$ and $x_* = x_0$, and normalized so that $\max\psi_2 = 1$, from the latter inequality we infer the inequality

$$|\lambda_2| > \varepsilon\,I^{-1}\,\partial_x\psi_2(x_0), \tag{2.33}$$

where

$$I := \int_{x_0}^\ell \exp\left(\frac{1}{2\varepsilon}\int_{x_0}^x a^\varepsilon(y)\,dy\right)\,dx$$

Our next aim is to deduce an estimate from above on I_ε and an estimate from below for $\partial_x\psi_2(x_0)$, in order to get a control on the size of the second eigenvalue λ_2.

From the definition of I_ε, since $x_0 \geq \xi$, it follows

$$
\begin{aligned}
I_\varepsilon &\leq e^{|a^\varepsilon-a^0|_{L^1}/2\varepsilon}\int_{x_0}^{\ell} e^{a_+(x-x_0)/2\varepsilon}\,dx = \frac{2\varepsilon}{|a_+|}\,e^{|a^\varepsilon-a^0|_{L^1}/2\varepsilon}\big(1-e^{a_+(\ell-x_0)/2\varepsilon}\big)\\
&\leq \frac{2\varepsilon}{|a_+|}\,e^{|a^\varepsilon-a^0|_{L^1}/2\varepsilon} \leq C\,\varepsilon
\end{aligned}
$$

whenever $|a^\varepsilon - a^0|_{L^1} \leq C\,\varepsilon$. Thus, estimate (2.33) provisionally becomes

$$
|\lambda_2| > C\,\frac{d\psi_2}{dx}(x_0) \tag{2.34}
$$

for some positive constant C, independent on ε.

Let x_M be such that $\psi_2(x_M) = 1$, minimum with such property. From the assumption on $b^\varepsilon + \varepsilon\,\lambda$, it follows $x_M \in (x_0, y_+)$. Then there exists $x_L \in (x_0, x_M)$ such that

$$
\frac{d\psi_2}{dx}(x_L) = \frac{1}{x_M - x_0} \geq \frac{1}{y_+ - \xi} \geq \frac{1}{c_0\varepsilon}.
$$

Since the function ψ is concave in the interval (x_0, y_+), we deduce

$$
\frac{d\psi_2}{dx}(x_0) \geq \frac{d\psi_2}{dx}(x_L) \geq \frac{1}{c_0\,\varepsilon}.
$$

Plugging into (2.34), we end up with

$$
|\lambda_2| \geq \frac{C}{\varepsilon}. \tag{2.35}
$$

for some C independent on ε.

As a consequence, we can state the next result relative to the second eigenvalue λ_2.

Proposition 2.8. *Let a^ε be a family of functions sastisfying* A1-2-3 *then there exists a constant $C > 0$ such that $\lambda_2^\xi \leq -C/\varepsilon$ for any ε sufficiently small.*

Spectral estimates. Collecting the results of Propositions 2.5 and 2.8 give a complete description for the spectrum of operator $\mathcal{L}^\varepsilon$ for small ε, under assumptions A0-1-2-3 on the family of functions a^ε.

Corollary 2.9. *Let a^ε be a family of functions satisfying the assumptions* A0-1-2-3 *for some $\xi \in (-\ell, \ell)$, $a_+ < 0 < a_-$. Then there exist constants $c, C_1, C_2 > 0$ such that*

$$
\lambda_k^\xi \leq -C_1/\varepsilon \qquad \text{and} \qquad -C_2\,e^{-c/\varepsilon} \leq \lambda_1^\xi < 0.
$$

for any $k \geq 2$.

Hypotheses A0-1-2-3 are satisfied in the case of a family of function a^ε that is a (small) perturbation of a function $\bar{a}^\varepsilon$ with the form

$$
\bar{a}^\varepsilon(x) = A_-\left(\frac{x-\xi}{\varepsilon}\right)\chi_{(-L,\xi)}(x) + A_+\left(\frac{x-\xi}{\varepsilon}\right)\chi_{(\xi,L)}(x).
$$

for some decreasing smooth bounded functions $A_\pm$, bounded together with their first and second order derivatives, and such that $A_\pm(\pm\infty) = a_\pm$ and $A'_\pm(\pm\infty) = 0$.

5. Appendix. The hyperbolic dynamics

In this Section, we concentrate on the dynamics of the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0 \tag{2.36}$$

with $x \in (-\ell, \ell)$, together with boundary condition $u(\pm\ell, t) = u_\pm$, and a given initial datum $u(x, 0) = u_0(x)$. Our aim is to give a self-contained proved of the *finite-time stabilization* of the solution under appropriate assumption on the flux function f and on the boundary values $u_\pm$. This kind of properties has been showed for the first time in [46] in the case of the Cauchy problem.

Theorem 2.10. *Assume the function f to be uniformly convex, i.e. $f''(s) \geq c_0 > 0$ for some constant c_0. If u_-, u_+ are such that $u_+ < 0 < u_-$ and $f(u_+) = f(u_-)$, then, for any $u_0 \in BV(-\ell, \ell)$, the solution u to the initial value problem (2.36), $u(\pm\ell, t) = u_\pm$, $u(x, 0) = u_0(x)$ is such that for some $T > 0$ and $\xi \in [-\ell, \ell]$, there holds*

$$u(\cdot, T) = U_{hyp}(\cdot; \xi) \qquad \text{in} \quad (-\ell, \ell)$$

where $U_{hyp}(x; \xi) := u_-\chi_{(-\ell,\xi)}(x) + u_+\chi_{(\xi,\ell)}(x)$.

To prove the statement, we use the *theory of generalized characteristics*, introduced in [15]. The convexity assumption on the flux function f guarantees that for any point $(x, t) \in (-\ell, \ell) \times (0, +\infty)$ there exist a minimal, respectively maximal, backward characteristics, which are classical characteristic curves, hence a straight lines with slope $f'(u(x - 0, t))$, resp. $f'(u(x + 0, t))$.

The boundary conditions are understood in the sense of Bardos–leRoux–Nédélec [5], meaning that the trace of the solution at the boundary is requested to take values in appropriate sets. To be precise, let $u_* \in (u_+, u_-)$ be such that $f'(u_*) = 0$ and set

$$\mathcal{R}u := \begin{cases} w & \text{if } \exists w \neq u \text{ s.t. } f(w) = f(u), \\ u_* & \text{if } u = u_*, \end{cases}$$

Then, skipping the details (see [54]), the boundary conditions $u(\pm\ell, t) = u_\pm$ translate into

$$u(-\ell + 0, t) \in (-\infty, \mathcal{R}u_-)] \cup \{u_-\}, \qquad u(\ell - 0, t) \in \{u_+\} \cup [\mathcal{R}u_+, +\infty)$$

Since $f(u_+) = f(u_-)$, there holds $\mathcal{R}u_\pm = u_\mp$, and the condition can be rewritten as

$$u(-\ell + 0, t) \in (-\infty, u_+] \cup \{u_-\}, \qquad u(\ell - 0, t) \in \{u_+\} \cup [u_-, +\infty) \tag{2.37}$$

In particular, characteristic curves entering in the domain from the left side $x = -\ell$ (respectively, from the right side $x = \ell$) possess speed $f'(u_-)$ (resp. speed $f'(u_+)$).

Now, we are ready to prove Theorem 2.10.

PROOF. Let $u = u(x,t)$ be the solution to the initial-boundary value problem under consideration with initial datum u_0. For later use, we set

$$\begin{aligned}\zeta_-(t) &:= \sup\{x \in [-\ell,\ell] \,:\, u(y,t) = u_- \quad \forall\, y \in (-\ell,x)\} \cup -\ell,\\ \zeta_+(t) &:= \inf\{x \in [-\ell,\ell] \,:\, u(y,t) = u_+ \quad \forall\, y \in (x,\ell)\} \cup \ell.\end{aligned}$$

In particular, $\zeta_- \leq \zeta_+$. We are going to show that $\zeta_-(T) = \zeta_+(T)$ for some $T > 0$.

1. There exists $T_0 > 0$ such that $u(x,t) \in [u_+, u_-]$ for any $x \in (-\ell,\ell)$.

Indeed, let $\overline{u} = \overline{u}(x,t)$ be the solution to the Riemann problem for (2.36) with initial datum

$$\bar{u}_0(x) = \begin{cases} u_- & x < -\ell,\\ \max\{u_-, \sup u_0\} & x > -\ell,\end{cases}$$

Hence, the restriction of $\bar{u}$ to $(-\ell,\ell) \times (0,\infty)$ is a super-solution to the initial boundary value problem under consideration and, by comparison principle for entropy solution, we infer $u(x,t) \leq \bar{u}(x,t)$. Since $\bar{u}(x,t) = u_-$ for any $x < f'(u_-)\,t - \ell$, there holds

$$u(x,t) \leq u_- \qquad \text{for } x \in (-\ell,\ell),\ t \geq 2\ell/f'(u_-).$$

A similar estimate from below can be obtained by considering as subsolution the restriction of $\underline{u}$ to $(-\ell,\ell) \times (0,\infty)$, where $\underline{u}$ is the solution to (2.36) with initial datum

$$\bar{u}_0(x) = \begin{cases} \min\{u_+, \inf u_0\} & x < \ell,\\ u_+ & x > \ell,\end{cases}$$

From now on, we assume that the solution u takes values in the interval $[u_-, u_+]$.

2. Assume that $-\ell < \zeta_-(t) \leq \zeta_+(t) < \ell$ for any t; then there exists $T_1 > 0$ such that $u(\zeta_-(t) + 0, t) < u_-$ and $u_+ < u(\zeta_+(t) - 0, t)$ for any $t > T_1$.

If u is continuous at $(\zeta_-(\tau), \tau)$ for some $\tau > 0$, then $u(\zeta_-(\tau) + 0, t) = u_-$. Therefore, the maximal backward characteristic from $(\zeta_-(\tau), \tau)$ is the straight line $x = \zeta_-(\tau) + f'(u_-)(t - \tau)$. For $\tau > 2L/f'(u_-)$, such curve intersects the boundary $x = -\ell$ at some $\sigma \in (0,\tau)$. By continuity, all of the maximal backward characteristics from (ξ,τ) with $\xi > \zeta_-(t)$ and sufficiently close to $\zeta_-(\tau)$ intersect the boundary $x = -\ell$ at some time $\sigma_*(\xi)$ smaller than σ and close to it. Because of the boundary conditions, this may happen if and only if $u(\xi,\tau) = u_-$. Hence, $u(x,\tau) = u_-$ for $x \in (\zeta_-(\tau), \zeta_-(\tau) + \varepsilon)$ for some $\varepsilon > 0$, in contradiction with the definition of ζ_-. Thus, continuity of u at $(\zeta_-(\tau), \tau)$ may happen only for $\tau \leq 2L/f'(u_-)$. A similar assertion holds for ζ_+.

3. There exist $T > 0$ and $\xi \in [-\ell,\ell]$ such that $u(x,t) = U_{hyp}(\cdot;\xi)$ for any $t \geq T$.

Given $\theta > 0$, let $T_\theta := 2\ell/\theta$ be such that

$$u_-^\theta := u(\zeta_-(T_\theta) + 0, T_\theta) < u_- \qquad \text{and} \qquad u_+ < u_-^\theta := u(\zeta_+(T_\theta) - 0, T_\theta).$$

Let x_-^θ be the maximal backward characteristic from $(\zeta_-(T_\theta), T_\theta)$, whose equation is $x = \zeta_-(T_\theta) + f'(u_-^\theta)(t - T_\theta)$. If x_-^θ hits the right boundary $x = \ell$ at some positive time, the solution u coincides with $U_{\text{hyp}}(x; \zeta_-(T_\theta))$. Otherwise, there holds $\zeta_-(T_\theta) - f'(u_-^\theta)T_\theta < \ell$, which gives

$$f'(u_-^\theta) > \frac{\zeta_-(T_\theta) - \ell}{T_\theta} \geq -\frac{2\ell}{T_\theta} = -\theta$$

Similarly, let x_+^θ be the maximal backward characteristic from $(\zeta_+(T_\theta), T_\theta)$, whose equation is $x = \zeta_+(T_\theta) + f'(u_+^\theta)(t - T_\theta)$. If x_+^θ does not intersect the left boundary $x = -\ell$ at some positive time, there holds $f'(u_+^\theta) < \theta$.

Hence, for any $\varepsilon > 0$, we can choose θ sufficiently large so that $u_-^\theta > u_* - \varepsilon$ and $u_+^\theta < u_* + \varepsilon$. Thus, we have

$$\frac{d\zeta_+}{dt} - \frac{d\zeta_-}{dt} < \frac{f(u_+) - f(u_* + \varepsilon)}{u_+ - u_* - \varepsilon} - \frac{f(u_-) - f(u_* - \varepsilon)}{u_- - u_* + \varepsilon}$$

which is uniformly negative for ε sufficiently small. Hence, the curves ζ_+ and ζ_- intersect at some finite positive time $T > 0$.

□

CHAPTER 3

Slow motion of internal shock layers for the Jin-Xin system in one space dimension

1. Introduction

In this Chapter we investigate the slow motion of the shock layer for the hyperbolic system with relaxation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + a^2 \partial_x \phi(u) = \dfrac{1}{\varepsilon}(f(u) - v), \quad \phi'(u) > 0 \end{cases} \tag{3.1}$$

where the space variable x belongs to a one-dimensional interval $I = (-\ell, \ell)$, $\ell > 0$. System (3.1) is complemented with appropriate boundary conditions and initial data for the couple (u, v), and it is a particular case of a class of more general hyperbolic relaxation systems of the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon^{-1} q(u,v) \end{pmatrix}$$

usually utilized to model a variety of non equilibrium processes in continuum mechanics: for example, non-thermal equilibrium gas dynamics ([39], [59]) , traffic dynamics ([4], [40], [50]) , and multiphase flows ([6], [8], [60]). Here ε is a parameter, usually small, determining relaxation time.

In the case of system (3.1), the parameter ε can be seen as a *viscosity coefficient*; we are interested in studying the behavior of the solution to (3.1) in the limit of small ε, and we want to identify the role of this parameter in the appearance and/or disappearance of phenomena of metastability.

The main example we have in mind is the initial-boundary value problem for the quasilinear Jin-Xin system, with Dirichlet boundary conditions in the bounded interval $I = (-\ell, \ell)$, that is

$$\begin{cases} \partial_t u + \partial_x v = 0 & x \in I, \ t \geq 0 \\ \partial_t v + a^2 \partial_x u = \dfrac{1}{\varepsilon}(f(u) - v) & \\ u(\pm\ell, t) = u_\pm & t \geq 0 \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \equiv f(u_0(x)) & x \in I \end{cases} \tag{3.2}$$

for some $\varepsilon, \ell, a > 0$, $u_\pm \in \mathbb{R}$ and flux function f that satisfies

$$f''(s) \geq c_0 > 0, \quad f'(u_+) < 0 < f'(u_-), \quad f(u_+) = f(u_-) \tag{3.3}$$

We stress that, once the boundary conditions for the function u are chosen, the boundary conditions for the function v are univocally determined. This model was firstly introduced in [29] as a numerical scheme approximating solutions of the hyperbolic conservation law $\partial_t u + \partial_x f(u) = 0$. System (3.2) is strictly hyperbolic, with the spectrum $\sigma(df, dg)^t$ composed by two distinct real eigenvalues $\pm a$.

In the relaxation limit ($\varepsilon \to 0^+$), system (3.2) can be approximated to leading order by

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ v = f(u) \end{cases} \tag{3.4}$$

that is

$$\partial_t u + \partial_x f(u) = 0 \tag{3.5}$$

together with $v = f(u)$, and complemented with boundary conditions

$$u(-\ell, t) = u_- \quad \text{and} \quad u(\ell, t) = u_+ \tag{3.6}$$

For the study of stationary solutions to (3.5), we recall Cahpter 2, where we construct a one-parameter family $\{U_{\text{hyp}}(\cdot;\xi)\}$ of steady states, parametrized by ξ that represents the location of the jump, and given by

$$U_{\text{hyp}}(x;\xi) = u_- \chi_{(-\ell,\xi)}(x) + u_+ \chi_{(\xi,\ell)}(x)$$

Once $U_{\text{hyp}}(\cdot;\xi)$ is chosen, the class of stationary solutions $(U_{\text{hyp}}, V_{\text{hyp}})$ for the system (3.4) is given by the relation $V_{\text{hyp}} = f(U_{\text{hyp}})$, so that

$$V_{\text{hyp}}(x;\xi) = f(u_-)\chi_{(-\ell,\xi)}(x) + f(u_+)\chi_{(\xi,\ell)}(x)$$

Moreover, every entropy solution to the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \quad v = f(u) \\ u(\pm\ell, t) = u_\pm \end{cases}$$

converges in finite time to an element of the family $\{U_{\text{hyp}}(\cdot;\xi), V_{\text{hyp}}(\cdot;\xi)\}$.

For $\varepsilon > 0$, the situation is very different. If we differentiate with respect to x the second equation of (3.2), we obtain

$$\partial_t u = \varepsilon(a^2 \partial_x^2 u - \partial_t^2 u) - \partial_x f(u) \tag{3.7}$$

Thus, stationary solutions to (3.2) solve

$$a^2 \varepsilon \partial_x^2 u = \partial_x f(u), \quad \partial_x v = 0 \tag{3.8}$$

As an example, if we consider the case of Burgers flux, i.e. $a = 1$, $f(u) = \frac{1}{2}u^2$, we can explicitly write the stationary solution for the problem (3.8)-(3.6) as

$$\bar{U}^\varepsilon_{\text{bur}}(x) = -k \tanh\left(\frac{kx}{2\varepsilon}\right), \quad \bar{V}^\varepsilon_{\text{bur}}(x) = f(k) \tag{3.9}$$

where $k = k(\varepsilon, \ell, u_\pm)$ is implicitly defined by imposing the boundary conditions.

In the limit $\varepsilon \to 0^+$, the single steady state $(\bar{U}^\varepsilon_{\text{bur}}, \bar{V}^\varepsilon_{\text{bur}})$ converges pointwise to $(U_{\text{hyp}}(\cdot;0), V_{\text{hyp}}(\cdot;0))$, while, for a class of general $f(u)$ that verify hypotheses (3.3), the stationary solution $(\bar{U}^\varepsilon, \bar{V}^\varepsilon)$ converges pointwise to $(U_{\text{hyp}}(\cdot;\bar{\xi}), V_{\text{hyp}}(\cdot;\bar{\xi}))$, for some $\bar{\xi} \in I$.

Finally, the single steady state $(\bar{U}^\varepsilon, \bar{V}^\varepsilon)$ is asymptotically stable (for more details see the spectral analysis performed in the following), i.e. starting from an initial datum close to the equilibrium configuration, the time dependent solution approaches the steady state for $t \to +\infty$.

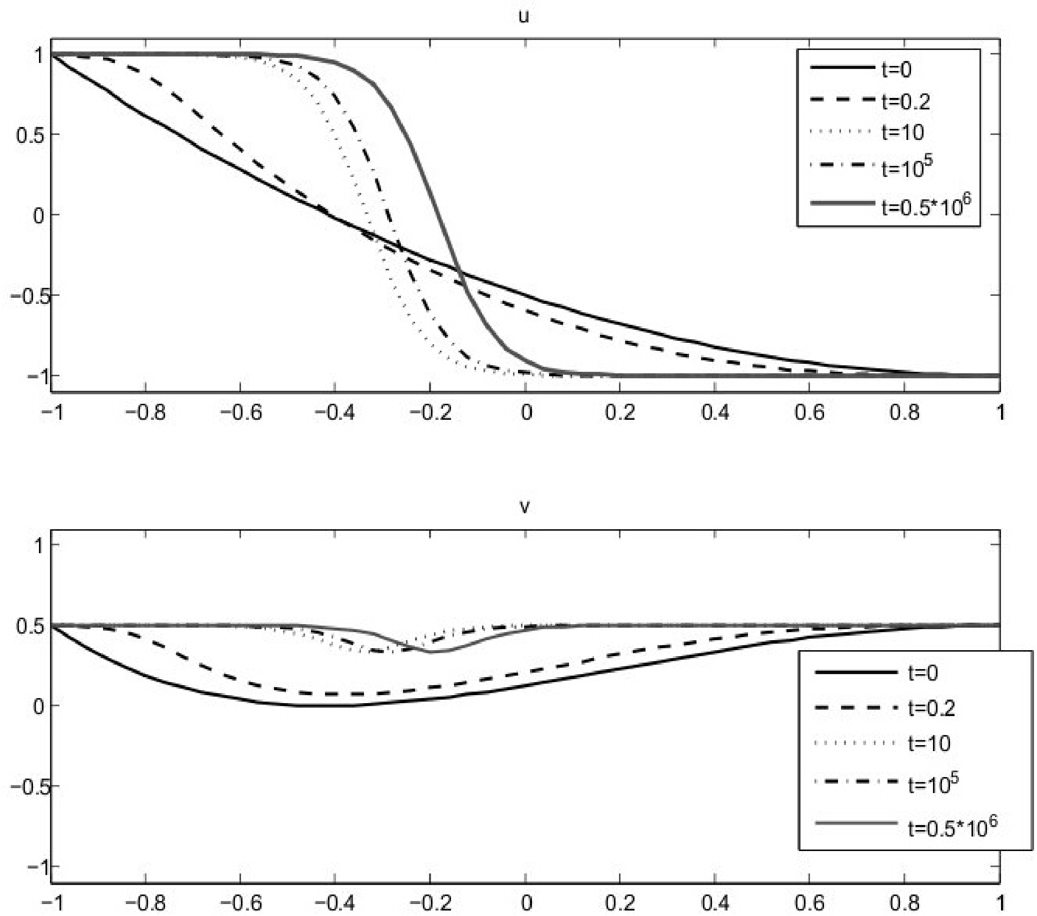


FIGURE 1. Profiles of (u, v), solutions to (3.2), with $f(u) = u^2/2$, a =1 $\varepsilon = 0.04$ and $u_\pm = \mp 1$. The initial data is given by the couple $(u_0(x), f(u_0(x)))$, with $u_0(x)$ a decreasing function connecting u_+ and u_-. Profiles at times $t = 0,\ 0.2,\ 10,\ 10^5,\ 0.5 \times 10^6$.

Next question is what happens to the dynamics generated by an initial datum localized far from the equilibrium solution $(\bar U^\varepsilon_{\text{bur}}, \bar V^\varepsilon_{\text{bur}})$. Numerical computations show that, starting, for example, with a decreasing initial datum $u_0(x)$ (see Fig.1), because of the viscosity, a shock layer is formed in a $\mathcal{O}(1)$ time scale. More precisely, the solution generated by such initial datum still presents a smooth transition from u_- to u_+, but the shock is located far away from zero, so that the solution is approximately given by a translation of the (unique) stationary solution of the problem. Once the shock layer is formed, it moves towards the equilibrium solution, and this motion is exponentially slow. Thus we have a first transient phase where the shock layer is formed, and an exponentially long time interval where the shock layer approaches the equilibrium solution.

Concerning the function v, starting with the initial datum $v_0(x) = f(u_0(x))$, we can observe that the position of the shock of u corresponds to the location the minimum value of the function v; so we have a first transient phase in which the profile of v stabilizes, and an exponentially slow phase where the value of the minimum of such profile drifts towards the value ξ that represents the location of the equilibrium solution for u.

Our aim is to describe the dynamics generated by an initial datum localized far from the equilibrium solution and to determine a detailed description of the low-viscosity behavior of the solutions.

To the best of our knowledge, the problem of the slow motion for the hyperbolic-parabolic Jin-Xin system (3.1) has been never examined before. However, system (3.2) can be reduced by differentiation to (3.7) together with the equation $\partial_t u + \partial_x v = 0$, and, as stressed, the study of stationary solutions to (3.7) is the same of that of the scalar conservation law

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u \tag{3.10}$$

together with the additional condition $\partial_x v = 0$.

Motivated by the analogies among the study of our problem and some results for the scalar conservation law (3.10), here we follow the same approach presented in Chapter 3 for the study of (2.1). Hence,

– we build-up a one parameter family of approximate steady states

$$\{\mathbf{W}^\varepsilon(x;\xi)\}_{\xi\in I} = \{U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)\}_{\xi\in I}$$

such that $(U^\varepsilon(\cdot;\bar\xi), V^\varepsilon(\cdot,\bar\xi)) := (\bar U^\varepsilon, \bar V^\varepsilon)$ for some $\bar\xi$, and with the additional propriety that $(U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot,\xi)) \to (U_{\text{hyp}}(\cdot;\xi), V_{\text{hyp}}(\cdot;\xi))$ as $\varepsilon \to 0$ in an appropriate sense. Moreover we require the error

$$\begin{pmatrix} \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon] \\ \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon] \end{pmatrix} := \begin{pmatrix} -\,\partial_x V^\varepsilon \\ -a^2\partial_x U^\varepsilon + \dfrac{1}{\varepsilon}(f(U^\varepsilon) - V^\varepsilon) \end{pmatrix}$$

to be small in ε in a sense to be specified.

– we describe the dynamics of the viscous system in a neighborhood of the family $\{U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)\}$.

Once a set of reference states $\{U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)\}$ is chosen, we determine spectral proprieties of the linearized operator around $(U^\varepsilon, V^\varepsilon)$ showing that, under a control on how far is the approximate state from being an exact stationary solution, a metastable behavior appears.

2. General Framework

Let us consider the Jin-Xin system

$$\begin{cases} \partial_t u + \partial_x v = 0 & x \in I,\ t \ge 0 \\ \partial_t v + a^2 \partial_x u = \dfrac{1}{\varepsilon}(f(u) - v) & \\ u(\pm\ell, t) = u_\pm & t \ge 0 \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \equiv f(u_0(x)) & x \in I \end{cases} \tag{3.11}$$

for some flux function f chosen so that assumptions (3.3) hold. System (3.11) can be rewritten as

$$\partial_t Z = \mathcal{F}^\varepsilon[Z], \quad Z\big|_{t=0} = Z_0 \tag{3.12}$$

where

$$Z = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \mathcal{F}^\varepsilon[Z] := \begin{pmatrix} \mathcal{P}_1^\varepsilon[Z] \\ \mathcal{P}_2^\varepsilon[Z] \end{pmatrix} = \begin{pmatrix} -\,\partial_x v \\ -a^2\partial_x u + \dfrac{1}{\varepsilon}(f(u)-v) \end{pmatrix}$$

We are interested in studying the behavior of the solution to (3.12) in the relaxation limit, i.e. $\varepsilon \to 0$. We assume that there exists a one-parameter family of functions $\{U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)\}_{\xi\in I}$ such that $(U^\varepsilon(\cdot;\bar\xi), V^\varepsilon(\cdot;\bar\xi) = (\bar U^\varepsilon, \bar V^\varepsilon)$ for some $\bar\xi \in I$. When $\xi \neq \bar\xi$, an element of this family can be seen as an approximate stationary solution to the problem, i.e. $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)] \to 0$ as $\varepsilon \to 0$ in an appropriate sense to be specified. Moreover, we require that, in the relaxation limit, $(U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)) \to (U_{\text{hyp}}(\cdot;\xi), V_{\text{hyp}}(\cdot,\xi))$. We remark that, once the one-parameter family of functions $\{U^\varepsilon(\cdot;\xi)\}$ is chosen, the couple $\{U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi)\}$ is univocally determined by the relation

$$V^\varepsilon = -a^2\varepsilon\partial_x U^\varepsilon + f(U^\varepsilon)$$

Example 3.1. In the case of Burgers flux, i.e. $f(s) = \frac{1}{2}s^2$, a stationary solution to (3.11) satisfies

$$\varepsilon\partial_x u = \frac{u^2}{2} - \frac{C^2}{2}, \quad v = \frac{C^2}{2} \tag{3.13}$$

with boundary conditions $u(\pm\ell) = \mp u^*$, for some $u^* > 0$. As in Example 2.1, an approximate solution $U^\varepsilon(x;\xi)$ to the first equation of (3.13) is obtained by matching two different steady states satisfying, respectively, the left and the right boundary conditions together with the request $U^\varepsilon|_{x=\xi} = 0$. In formula

$$U^\varepsilon(x;\xi) = \begin{cases} k_- \tanh\left(k_-(\xi - x)/2\varepsilon\right) & \text{in } (-\ell,\xi) \\ k_+ \tanh\left(k_+(\xi - x)/2\varepsilon\right) & \text{in } (\xi,\ell) \end{cases} \tag{3.14}$$

where $a = 1$, and $k_\pm$ are chosen so that the boundary conditions are satisfied

$$k_\pm \tanh\left(\frac{k_\pm}{2\varepsilon}(\xi \mp \ell)\right) = u_\pm \tag{3.15}$$

Moreover, by the condition $v = \frac{C^2}{2}$, we have

$$V^\varepsilon(x;\xi) = \begin{cases} k_-^2/2 & \text{in } (-\ell,\xi) \\ k_+^2/2 & \text{in } (\xi,\ell) \end{cases}$$

3. The linearized Problem

As already stated before, in order to describe the dynamics generated by an initial configuration localized far from the steady state $(\bar U^\varepsilon, \bar V^\varepsilon)$, we assume to have a one-parameter family

$$\mathbf{W}^\varepsilon(x;\xi(t)) := (U^\varepsilon(x;\xi(t)), V^\varepsilon(x;\xi(t)))$$

parametrized by $\xi \in I$, such that the couple $(U^\varepsilon(x;\xi(t)), V^\varepsilon(x;\xi(t)))$ is an approximate stationary solution to (3.11), in the sense that it satisfies the stationary equation up to an error that is small in ε. More precisely, we assume that there exist two family of smooth positive functions $\Omega_1^\varepsilon = \Omega_1^\varepsilon(\xi)$

and $\Omega_2^\varepsilon = \Omega_2^\varepsilon(\xi)$, uniformly convergent to zero as $\varepsilon \to 0$, such that, for any $\xi \in I$, the following estimates holds

$$\begin{aligned} |\langle \psi(\cdot), \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle| \le \Omega_1^\varepsilon(\xi)|\psi|_{L^\infty} \quad \forall \psi \in C(I) \\ |\langle \psi(\cdot), \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle| \le \Omega_2^\varepsilon(\xi)|\psi|_{L^\infty} \quad \forall \psi \in C(I) \end{aligned} \tag{3.16}$$

Once a one-parameter family $\{\mathbf{W}^\varepsilon(\cdot;\xi)\}$ satisfying (3.16) is chosen, we look for a solution to (3.11) in the form

$$\begin{cases} u(\cdot,t) = U^\varepsilon(\cdot;\xi(t)) + u^1(\cdot,t) \\ v(\cdot,t) = V^\varepsilon(\cdot;\xi(t)) + v^1(\cdot,t) \end{cases}$$

Thus we are trying to describe the dynamics in a neighborhood of the family $\{U^\varepsilon(\cdot;\xi(t)), V^\varepsilon(\cdot;\xi(t))\}$ using as coordinates the parameter ξ and a distance vector $Y = (u^1, v^1)$, determined by the difference between the solution (u,v) and an element of the approximate family. Substituting in (3.11), we obtain

$$\begin{cases} \partial_t u^1 + \partial_\xi U^\varepsilon(\cdot;\xi)\dfrac{d\xi}{dt} + \partial_x V^\varepsilon(\cdot;\xi) + \partial_x v^1 = 0 \\ \partial_t v^1 + \partial_\xi V^\varepsilon(\cdot;\xi)\dfrac{d\xi}{dt} + a^2(\partial_x U^\varepsilon(\cdot;\xi) + \partial_x u^1) = \dfrac{1}{\varepsilon}\left\{f(U^\varepsilon(\cdot;\xi) + u^1) - V^\varepsilon(\cdot;\xi) - v^1\right\} \end{cases}$$

Since $f(U^\varepsilon + u^1) = f(U^\varepsilon) + f'(U^\varepsilon)u^1 + o((u^1)^2)$, we get

$$\begin{cases} \partial_t u^1 = -\partial_x v^1 - \partial_\xi U^\varepsilon(\cdot;\xi)\dfrac{d\xi}{dt} + \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)] \\ \partial_t v^1 = -a^2\partial_x u^1 + \dfrac{1}{\varepsilon}(f'(U^\varepsilon(\cdot,\xi))u^1 - v^1) - \partial_\xi V^\varepsilon(\cdot;\xi)\dfrac{d\xi}{dt} \\ \qquad\qquad + \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)] + \mathcal{Q}^\varepsilon[u_1] \end{cases} \tag{3.17}$$

where

$$\begin{cases} \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon] := -\partial_x V^\varepsilon \\ \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon] := -a^2\partial_x U^\varepsilon + \dfrac{1}{\varepsilon}(f(U^\varepsilon) - V^\varepsilon) \\ \quad \mathcal{Q}^\varepsilon[u] := o(u) \end{cases}$$

Example 3.2. Let us recall the Example 3.1, where we construct an approximate stationary solution for the Jin-Xin system with $f(u) = u^2/2$ and $a = 1$. We compute in this specific case

$$\mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)] := -\partial_x V^\varepsilon(\cdot;\xi), \quad \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)] := -\partial_x U^\varepsilon(\cdot;\xi) + \frac{1}{\varepsilon}\left((U^\varepsilon(\cdot;\xi))^2/2 - V^\varepsilon(\cdot;\xi)\right)$$

From the explicit formula for $U^\varepsilon(x;\xi)$ given in (3.14) we get

$$-\partial_x U^\varepsilon(x;\xi) = \begin{cases} \dfrac{k_-^2}{2\varepsilon}\left[1 - \tanh^2\left(\dfrac{k_-}{2\varepsilon}(\xi - x)\right)\right] & \text{in } (-\ell,\xi) \\ \dfrac{k_+^2}{2\varepsilon}\left[1 - \tanh^2\left(\dfrac{k_+}{2\varepsilon}(\xi - x)\right)\right] & \text{in } (\xi,\ell) \end{cases}$$

so that, in this specific case, $\mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon] \equiv 0$. On the other hand, $-\partial_x V^\varepsilon(x;\xi) = \varepsilon\partial_x^2 U^\varepsilon(x;\xi) - \partial_x f(U^\varepsilon(x;\xi))$. By direct substitution, we obtain the identity

$$\mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)] = [\![\partial_x U^\varepsilon]\!]_{x=\xi}\delta_{x=\xi}$$

in the sense of distributions. We also have

$$[\![\partial_x U^\varepsilon]\!]_{x=\xi} = \frac{1}{2\varepsilon}(k_- - k_+)(k_- + k_+)$$

In order to determine the behavior of $\mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)]$ for small ε, we need an asymptotic description of the values $k_\pm$. To this aim, let us set $k_\pm := \mp u_\pm(1+h_\pm)$ and $\Delta_\pm := \ell \mp \xi$. Relation (3.15) becomes

$$\tanh\left(\mp\frac{u_\pm \Delta_\pm}{2\varepsilon}(1+h_\pm)\right) = \frac{1}{1+h_\pm}$$

Therefore, the values $h_\pm$ are both positive and then

$$\tanh\left(\mp\frac{u_\pm \Delta_\pm}{2\varepsilon}\right) \leq \frac{1}{1+h_\pm}$$

that gives the asymptotic representation

$$h_\pm \leq \frac{1}{\tanh\left(\mp u_\pm \Delta_\pm / 2\varepsilon\right)} - 1 = \frac{2}{e^{\mp u_\pm \Delta_\pm/\varepsilon} - 1} = 2e^{\pm u_\pm \Delta_\pm/\varepsilon} + \text{l.o.t} \quad (3.18)$$

where l.o.t. denotes lower order terms. Finally

$$[\![\partial_x U^\varepsilon]\!]_{x=\xi} = \frac{1}{2\varepsilon}(k_- - k_+)(k_- + k_+) = \frac{u_*^2}{\varepsilon}(h_- + h_+) + l.o.t.$$

where $u_\pm = \mp u^*$ for some $u^* > 0$, so that we end up with

$$[\![\partial_x U^\varepsilon]\!]_{x=\xi} \leq \frac{u_*^2}{\varepsilon}(e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}) + l.o.t. \quad (3.19)$$

showing that this term is exponentially small for $\varepsilon \to 0$ and is null when $\xi = 0$, that corresponds to the equilibrium location of the shock when $f(u) = u^2/2$.

In this case, if we neglect the lower order terms, we can write an asymptotic formula for Ω_1^ε, that is

$$\Omega_1^\varepsilon(\xi) \sim \frac{u_*^2}{\varepsilon}(e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}) \quad (3.20)$$

It follows that the hypothesis (3.16) is satisfied in the special case of $f(u) = u^2/2$.

We can also numerically compute the limit of the solution $(U^\varepsilon, V^\varepsilon)$ for $\varepsilon \to 0^+$. Figure 2 explicitly shows how the profile of the stationary solution depends on the value of ε. For fixed ξ, we observe that, as ε becomes smaller, the transition between u_- and u_+ becomes more sharp, while v tends to $f(u^*)\delta_{x=\xi}$, according to the fact that, in the limit $\varepsilon \to 0^+$, the solution $(U^\varepsilon(\cdot;\xi), V^\varepsilon(\cdot;\xi))$ converges to $(U_{\text{hyp}}(\cdot;\xi), V_{\text{hyp}}(\cdot;\xi))$.

Let us go back to the system (3.17). From now on, $(u^1, v^1) = (u, v)$ and

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathcal{L}_\xi^\varepsilon Y := \begin{pmatrix} -\partial_x v \\ -a^2\partial_x u + \frac{1}{\varepsilon}(f'(U^\varepsilon)u - v) \end{pmatrix}$$

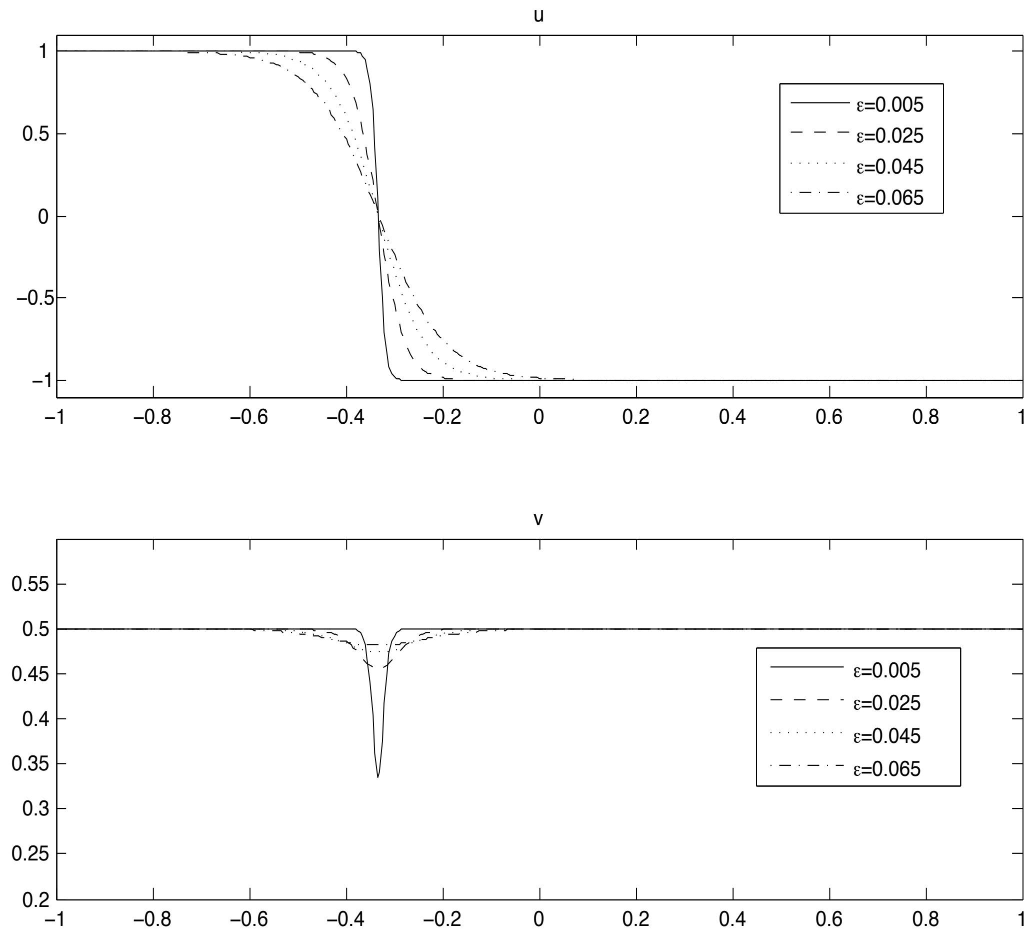


FIGURE 2. Profile of the stationary solution $(u, v) = (U^\varepsilon, V^\varepsilon)$ when $f(u) = u^2/2$. The steepening of the shock layer and the convergence to a Delta function of v as ε becames smaller are depicted.

Moreover, we introduce the following notation: if ψ, $\phi \in \mathbb{C}$, then $\langle \psi, \phi \rangle := \int_I \bar{\psi}\, \phi$, while if $\boldsymbol{\psi} = (\psi_1, \psi_2)$ and $\boldsymbol{\phi} = (\phi_1, \phi_2)$, then $\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle := \langle \psi_1, \phi_1 \rangle + \langle \psi_2, \phi_2 \rangle$.

Let us assume that, for any ξ, the linear operator $\mathcal{L}^\varepsilon_\xi$ has a sequence of eigenvalues $\lambda^\varepsilon_k = \lambda^\varepsilon_k(\xi)$ with corresponding (right) eigenfunctions $\boldsymbol{\phi}^\varepsilon_k = \boldsymbol{\phi}^\varepsilon_k(\xi, \cdot)$ (for more details see Section 3). Denoting by $\boldsymbol{\psi}^\varepsilon_k = \boldsymbol{\psi}^\varepsilon_k(\xi, \cdot)$ the eigenfunctions of the corresponding adjoint operator $\mathcal{L}^{\varepsilon^*}_\xi$ and setting $Y_k = Y_k(\xi; t) := \langle \boldsymbol{\psi}^\varepsilon_k(\cdot; \xi), Y(\cdot, t) \rangle$, we impose that the component $Y_1 = (u_1, v_1)$ is identically zero. More precisely, since we will prove that the first eigenvalue is real and tends to zero as $\varepsilon \to 0$, we need to solve the equation in a subspace in which the operator doesn't vanish. To this aim, we set an algebraic condition ensuring orthogonality between $\boldsymbol{\psi}^\varepsilon_1$ and Y, in order to remove the singular part of the operator $\mathcal{L}^\varepsilon_\xi$. Thus, denoting by Y_0 the initial datum of the perturbation, we have

$$\frac{d}{dt}\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi(t)), Y(\cdot,t)\rangle = 0 \quad \text{and} \quad \langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi_0), Y_0(\cdot)\rangle = 0 \tag{3.21}$$

so that

$$\langle \boldsymbol{\psi}_1^\varepsilon(\cdot,\xi), \partial_t Y\rangle + \langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon(\cdot,\xi)\frac{d\xi}{dt}, Y\rangle = 0$$

Since $\boldsymbol{\psi}_1^\varepsilon = (\psi_1^u, \psi_1^v)$ is the first (left) eigenfunction, there holds $\mathcal{L}_\xi^{\varepsilon,*}\boldsymbol{\psi}_1^\varepsilon = \bar{\lambda}_1^\varepsilon \boldsymbol{\psi}_1^\varepsilon$, that is

$$\begin{pmatrix} a^2\partial_x\psi_1^v + \dfrac{1}{\varepsilon}f'(U^\varepsilon(\cdot;\xi))\psi_1^v \\ \partial_x\psi_1^u - \dfrac{1}{\varepsilon}\psi_1^v \end{pmatrix} = \bar{\lambda}_1^\varepsilon \begin{pmatrix} \psi_1^u \\ \psi_1^v \end{pmatrix}$$

Hence, from (3.21) we get

$$\langle \begin{pmatrix} \partial_\xi\psi_1^u \dfrac{d\xi}{dt} \\ \partial_\xi\psi_1^v \dfrac{d\xi}{dt} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle + \langle \begin{pmatrix} \psi_1^u \\ \psi_1^v \end{pmatrix}, \mathcal{L}_\xi^\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -\partial_\xi U^\varepsilon(\cdot,\xi)\dfrac{d\xi}{dt} + \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon] \\ -\partial_\xi V^\varepsilon(\cdot,\xi)\dfrac{d\xi}{dt} + \mathcal{Q}^\varepsilon[u] + \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon] \end{pmatrix}\rangle = 0$$

Since $\langle \boldsymbol{\psi}_1^\varepsilon, \mathcal{L}_\xi^\varepsilon Y\rangle = \bar{\lambda}_1^\varepsilon\langle \boldsymbol{\psi}_1^\varepsilon, Y\rangle = 0$, we have

$$\begin{aligned} &\langle \partial_\xi\psi_1^u(\cdot,\xi)\frac{d\xi}{dt}, u\rangle + \langle \psi_1^u(\cdot;\xi), -\partial_\xi U^\varepsilon(\cdot,\xi)\frac{d\xi}{dt} + \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)]\rangle \\ &+ \langle \partial_\xi\psi_1^v(\cdot;\xi)\frac{d\xi}{dt}, v\rangle + \langle \psi_1^v(\cdot;\xi), -\partial_\xi V^\varepsilon(\cdot,\xi)\frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[u] + \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle = 0 \end{aligned}$$

and we end up with a scalar differential equation for the variable ξ, that is

$$\frac{d\xi}{dt} = \frac{\langle \psi_1^v(\cdot;\xi), \mathcal{Q}^\varepsilon[u] + \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle + \langle \psi_1^u(\cdot,\xi), \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)]\rangle}{\alpha^\varepsilon(\xi,u,v)} \tag{3.22}$$

where

$$\alpha^\varepsilon(\xi,u,v) = -\langle \partial_\xi\psi_1^u(\cdot,\xi), u\rangle - \langle \partial_\xi\psi_1^v(\cdot;\xi), v\rangle + \langle \psi_1^u(\cdot;\xi), \partial_\xi U^\varepsilon\rangle + \langle \psi_1^v(\cdot;\xi), \partial_\xi V^\varepsilon\rangle$$

Since we are interested in the regime $Y \sim 0$, the equation (3.22) is approximately solved for small Y. Thus the term $1/\alpha^\varepsilon(\xi,u,v)$ is expanded for $u, v \sim 0$, yielding

$$\frac{1}{\alpha^\varepsilon(\xi,Y)} = \frac{1}{\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle} + \frac{1}{\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle^2}\langle \partial_\xi\boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle + R_1$$

$$\begin{aligned} R_1 &= \frac{1}{\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle - \langle \partial_\xi\boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle} - \frac{1}{\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle} - \frac{\langle \partial_\xi\boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle}{\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle^2} \\ &= \frac{\langle \partial_\xi\boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle^2}{\left[\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle - \langle \partial_\xi\boldsymbol{\psi}_1^\varepsilon,(\cdot;\xi)Y\rangle\right]\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle^2} \end{aligned}$$

where

$$\langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon\rangle := \langle \psi_1^u(\cdot;\xi), \partial_\xi U^\varepsilon\rangle + \langle \psi_1^v(\cdot;\xi), \partial_\xi V^\varepsilon\rangle$$

Now, for sake of simplicity, let us call $\alpha_0^\varepsilon(\xi) := \langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), \partial_\xi\mathbf{W}^\varepsilon(\cdot,\xi)\rangle$. Thus we end up with the nonlinear equation for $\xi(t)$, which reads

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi)\left(1 + \frac{\langle \partial_\xi\boldsymbol{\psi}_1^\varepsilon, Y\rangle}{\alpha_0^\varepsilon(\xi)}\right) + \rho^\varepsilon[\xi, Y], \qquad \langle \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi_0), Y_0(\cdot)\rangle = 0 \tag{3.23}$$

where

$$\begin{cases} \theta^\varepsilon(\xi) := \dfrac{\langle \boldsymbol{\psi}_1^\varepsilon, \mathcal{F}^\varepsilon[\mathbf{W}^\varepsilon]\rangle}{\alpha_0^\varepsilon(\xi)} \\ \rho^\varepsilon[\xi, Y] := \theta_1(\xi, Y)\left(1 + \dfrac{\langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon, Y\rangle}{\alpha_0^\varepsilon(\xi)}\right) + \langle \boldsymbol{\psi}_1^\varepsilon, \mathcal{F}^\varepsilon[\mathbf{W}^\varepsilon] + \boldsymbol{\mathcal{Q}}^\varepsilon[Y]\rangle R_1 \\ R_1 := \dfrac{\langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle^2}{[\alpha_0^\varepsilon(\xi) - \langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle](\alpha_0^\varepsilon(\xi))^2} \\ \theta_1(\xi, v) := \dfrac{\langle \boldsymbol{\psi}_1^\varepsilon, \boldsymbol{\mathcal{Q}}^\varepsilon[Y]\rangle}{\alpha_0^\varepsilon(\xi)} \\ \boldsymbol{\mathcal{Q}}^\varepsilon[Y] = (0, \mathcal{Q}^\varepsilon[u]) \end{cases} \tag{3.24}$$

Equation (3.23) has to be coupled with the equation for the perturbation Y. To this end, (3.17) is rewritten in the form

$$\partial_t Y = \mathcal{L}_\xi^\varepsilon Y - \partial_\xi \mathbf{W}^\varepsilon(\cdot;\xi)\frac{d\xi}{dt} + \mathcal{F}^\varepsilon[\mathbf{W}^\varepsilon] + \boldsymbol{\mathcal{Q}}^\varepsilon[Y] \tag{3.25}$$

Using (3.23), we end up with the following equation

$$\partial_t Y = (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon)Y + H^\varepsilon(x;\xi) + \mathcal{R}^\varepsilon[Y, \xi] \tag{3.26}$$

where

$$\mathcal{M}_\xi^\varepsilon Y = \frac{1}{\alpha_0^\varepsilon(\xi)}\begin{pmatrix} -\partial_\xi U^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi)\,\langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle \\ -\partial_\xi V^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi)\,\langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon(\cdot;\xi), Y\rangle \end{pmatrix}$$

$$H(x;\xi) = \begin{pmatrix} \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot,\xi)\theta^\varepsilon(\xi) \\ \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)] - \partial_\xi V^\varepsilon(\cdot,\xi)\theta^\varepsilon(\xi) \end{pmatrix}$$

$$\mathcal{R}^\varepsilon[Y, \xi] = \begin{pmatrix} -\partial_\xi U^\varepsilon(\cdot;\xi)\,\rho^\varepsilon[\xi, Y] \\ \mathcal{Q}^\varepsilon[u] \end{pmatrix}$$

Hence we obtain the following coupled system for the shock layer location $\xi(t)$ and the perturbation Y

$$\begin{cases} \dfrac{d\xi}{dt} = \theta^\varepsilon(\xi)\left(1 + \dfrac{\langle \partial_\xi \boldsymbol{\psi}_1^\varepsilon, Y\rangle}{\alpha_0^\varepsilon(\xi)}\right) + \rho^\varepsilon[\xi, Y] \\ \partial_t Y = (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon)Y + H^\varepsilon(x;\xi) + \mathcal{R}^\varepsilon[\xi, Y] \end{cases} \tag{3.27}$$

Example 3.3. Let us consider the Jin-Xin system, for which one obtains

$$\begin{cases} \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon] = -\partial_x V^\varepsilon(\cdot;\xi) \\ \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon] = -a^2\partial_x U^\varepsilon(\cdot;\xi) + \dfrac{1}{\varepsilon}(f(U^\varepsilon(\cdot;\xi)) - V^\varepsilon(\cdot;\xi)) \end{cases}$$

For what concerns the linear operator, setting $a^\varepsilon(x;\xi) := f'(U^\varepsilon(\cdot,\xi))$, we get the following expressions

$$\mathcal{L}_\xi^\varepsilon Y := \begin{pmatrix} -\,\partial_x v \\ -a^2\partial_x u + \dfrac{1}{\varepsilon}(a^\varepsilon(\cdot;\xi)u - v) \end{pmatrix}, \qquad \mathcal{L}_\xi^{\varepsilon,*} Y := \begin{pmatrix} a^2\partial_x v \, + \, \dfrac{1}{\varepsilon}a^\varepsilon(\cdot;\xi)v \\ \partial_x u - \dfrac{1}{\varepsilon}v \end{pmatrix}$$

complemented with Dirichlet boundary conditions.

To obtain an asymptotic expression for the function $\theta^\varepsilon(\xi)$, we need to approximately compute the functions $\boldsymbol{\psi}_1^\varepsilon$ and $\partial_\xi \mathbf{W}^\varepsilon$. As usual, we refer to the case $f(u) = u^2/2$.

For $\varepsilon \sim 0$, the function $\boldsymbol{\psi}_1^\varepsilon$ is close to the eigenfunction of the operator $\mathcal{L}_\xi^{0,*}$ relative to the eigenvalue $\lambda = 0$, with

$$a^0(x;\xi) := u_-\chi_{(-\ell,\xi)}(x) + u_+\chi_{(\xi,\ell)}(x)$$

For example, in $(-\ell,\xi)$ we have

$$\begin{cases} a^2\partial_x\psi_1^{0,v} + \dfrac{u_-}{\varepsilon}\psi_1^{0,v} = 0 \\ \partial_x\psi_1^{0,u} - \dfrac{1}{\varepsilon}\psi_1^{0,v} = 0 \\ \boldsymbol{\psi}_1^0(-\ell) = 0, \quad [\![\boldsymbol{\psi}_1^0]\!]_\xi = 0 \end{cases}$$

That is $\psi_1^{0,u} = A(1 - e^{-\frac{u_-}{a^2\varepsilon}(x+x_0)})$ and $\psi_1^{0,v} = \varepsilon\partial_x\psi_1^{0,u}$. By imposing the conditions on the boundary and on the jump, and by doing the same computations in the interval (ξ,ℓ) we obtain

$$\psi_1^u(x) \sim \psi_1^{0,u}(x) = \begin{cases} (1 - e^{u_+(\ell-\xi)/a^2\varepsilon})(1 - e^{-u_-(\ell+x)/a^2\varepsilon}) & x < \xi \\ (1 - e^{-u_-(\ell+\xi)/a^2\varepsilon})(1 - e^{u_+(\ell-x)/a^2\varepsilon}) & x > \xi \end{cases}$$

$$\psi_1^v(x) \sim \psi_1^{0,v}(x) = \begin{cases} \dfrac{u_-}{a^2}(1 - e^{u_+(\ell-\xi)/a^2\varepsilon})e^{-u_-(\ell+x)/a^2\varepsilon} & x < \xi \\ -\dfrac{u_+}{a^2}(1 - e^{-u_-(\ell+\xi)/a^2\varepsilon})e^{e^{u_+(\ell-x)/a^2\varepsilon}} & x > \xi \end{cases}$$

so that $\boldsymbol{\psi}_1^\varepsilon = (\psi_1^u, \psi_1^v) \sim (1,0)$ for $\varepsilon \sim 0$. Furthermore, with the approximation $U^\varepsilon(x;\xi) \sim U_{hyp}(x;\xi)$ and $V^\varepsilon(x;\xi) \sim V_{hyp}(x)$, we have

$$\frac{U^\varepsilon(x;\xi+h) - U^\varepsilon(x;\xi)}{h} \sim -\frac{1}{h}[\![u]\!]\chi_{(\xi,\xi+h)}(x)$$
$$\frac{V^\varepsilon(x;\xi+h) - V^\varepsilon(x;\xi)}{h} \sim -\frac{1}{h}[\![f(u)]\!]\chi_{(\xi,\xi+h)}(x)$$

so that $\partial_\xi U^\varepsilon$ and $\partial_\xi V^\varepsilon$ converge to $-[\![u]\!]\delta_\xi$ and $-[\![f(u)]\!]\delta_\xi$ respectively as $\varepsilon \to 0$ in the sense of distribution. Thus, since $\langle \boldsymbol{\psi}_1^\varepsilon, \partial_\xi \mathbf{W}^\varepsilon\rangle \sim -[\![u]\!]$, we deduce an asymptotic expression for the function θ^ε

$$\theta^\varepsilon(\xi) \sim -\frac{1}{[\![u]\!]}\langle 1, \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon]\rangle$$

With the choice of $\mathbf{W}^\varepsilon = (U^\varepsilon, V^\varepsilon)$ proposed in Example 3.1, such expression becomes

$$\theta^\varepsilon(\xi) \sim \frac{u^*}{\varepsilon}(e^{-u_*(l+\xi)/\varepsilon} - e^{-u_*(l-\xi)/\varepsilon}) \tag{3.28}$$

4. Spectral analysis

In this Section we analyze the spectrum of the linearized operator $\mathcal{L}_\xi^\varepsilon$ in order to determine a precise description of the location of the eigenvalues.

We recall the expression of the operator

$$\mathcal{L}^\varepsilon_\xi Y := \begin{pmatrix} -\partial_x v \\ -a^2\partial_x u + \dfrac{1}{\varepsilon}(f'(U^\varepsilon)u - v) \end{pmatrix}$$

so that the eigenvalue problem $\mathcal{L}^\varepsilon_\xi \Phi = \lambda\Phi$ reads

$$\begin{cases} \lambda\varphi = -\partial_x\psi \\ \lambda\psi = -a^2\partial_x\varphi + \dfrac{1}{\varepsilon}(f'(U^\varepsilon)\varphi - \psi) \end{cases} \tag{3.29}$$

complemented with Dirichlet boundary conditions. Hence, by differentiating the second equation with respect to x, we obtain

$$\varepsilon a^2\partial_x^2\varphi - \partial_x(f'(U^\varepsilon)\varphi) = \lambda(1+\varepsilon\lambda)\varphi \tag{3.30}$$

Thus we are interested in studying the eigenvalue problem for the differential linear diffusion-transport operator

$$\mathcal{L}^{\varepsilon,vsc}\varphi := \varepsilon\partial_x^2\varphi - \partial_x(a^\varepsilon\varphi), \quad a^\varepsilon(x;\xi) := f'(U^\varepsilon(x;\xi)) \tag{3.31}$$

In Chapter 2, Corollary 2.9, we have already proven that, under opportune hypotheses on the behavior of the function $a^\varepsilon(x;\xi)$ in the limit $\varepsilon \to 0$, the eigenvalues of $\mathcal{L}^{\varepsilon,vsc}$ have the following distribution

$$-Ce^{-c/\varepsilon} \le \lambda_1^{vsc} < 0 \quad \text{and} \quad \lambda_k^{vsc} \le -\frac{c_0}{\varepsilon} \quad \forall\, k \ge 2$$

From (3.30) we observe that λ is an eigenvalue of $\mathcal{L}^\varepsilon_\xi$ if and only if $\lambda^{vsc} := \lambda(1+\varepsilon\lambda)$ is an eigenvalue for the operator $\mathcal{L}^{\varepsilon,vsc}$ defined in (3.31). Hence, if $\lambda = \lambda_n^{JX}$ is an eigenvalue of $\mathcal{L}^\varepsilon_\xi$, then there exists an eigenvalue λ_n^{vsc} such that

$$\varepsilon{\lambda_n^{JX}}^2 + \lambda_n^{JX} = \lambda_n^{vsc}$$

so that

$$\lambda_{n,\pm}^{JX} = -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1+4\varepsilon\lambda_n^{vsc}} \tag{3.32}$$

Hence, if $\lambda_n^{vsc} > -\frac{1}{4\varepsilon}$, then $\lambda_{n,\pm}^{JX} \in \mathbb{R}$. Moreover, since λ_n^{vsc} are negative for all $n \in \mathbb{N}$

$$\lambda_{n,+}^{JX} = \frac{2\lambda_n^{vsc}}{1+\sqrt{1+4\varepsilon\lambda_n^{vsc}}} < 0, \qquad \lambda_{n,-}^{JX} = \frac{-2\lambda_n^{vsc}}{\sqrt{1+4\varepsilon\lambda_n^{vsc}}-1} < 0 \tag{3.33}$$

Thanks to Corollary 2.9, we know that $\lambda_1^{vsc} > -\frac{1}{4\varepsilon}$ and $\lambda_1^{vsc} \sim e^{-C/\varepsilon}$ as $\varepsilon \to 0$. Thus, from (3.32) and (3.33), there exists a constant C' such that

$$-e^{-C'/\varepsilon} \le \lambda_{1,+}^{JX} < 0 \qquad \lambda_{1,-}^{JX} \le -\frac{1}{2\varepsilon}$$

Moreover, if for some $n > 1$ there exist others λ_n^{vsc} such that $\lambda_n^{vsc} > -\frac{1}{4\varepsilon}$, then they are of order $1/\varepsilon$, so that

$$\lambda_{n,\pm}^{JX} \le -C''/\varepsilon$$

On the other hand, if $\lambda_n^{vsc} < -\frac{1}{4\varepsilon}$, then $\lambda_{n,\pm}^{JX} \in \mathbb{C}$. More precisely

$$\lambda_{n,\pm}^{JX} = -\frac{1}{2\varepsilon} \pm \frac{i}{2\varepsilon}\sqrt{|1+4\varepsilon\lambda_n^{vsc}|}$$

Proposition 2.9 assures that there exists $k \geq 2$ such that $\lambda_n^{vsc} < -\frac{1}{4\varepsilon}$ for all $n \geq k$, so that $Re(\lambda_{n,\pm}^{JX})$ and $Im(\lambda_{n,\pm}^{JX})$ are terms of order $1/\varepsilon$. For example, if $k = 2$ and we take into account $\lambda_2^{vsc} < 0$, the corresponding eigenvalues for $\mathcal{L}_\xi^\varepsilon$ verifies

$$Re(\lambda_{2,\pm}^{JX}) = -\frac{1}{2\varepsilon}, \qquad Im(\lambda_{2,\pm}^{JX}) = \pm\frac{1}{2\varepsilon}\sqrt{|1 + 4\varepsilon\lambda_2^{vsc}|}$$

Moreover, for $\lambda_{3,\pm}^{JX}$, since $|\lambda_3^{vsc}| > |\lambda_2^{vsc}|$, we have

$$Re(\lambda_{3,\pm}^{JX}) = -\frac{1}{2\varepsilon} = Re(\lambda_{2,\pm}^{JX}) \qquad |Im(\lambda_{3,\pm}^{JX})| = \pm\frac{1}{2\varepsilon}\sqrt{|1 + 4\varepsilon\lambda_3^{vsc}|} > |Im(\lambda_{2,\pm}^{JX})|$$

Figure 3 shows the connection between the two spectra when $k = 2$, so that only the first two eigenvalues belong to $\mathbb{R}$.

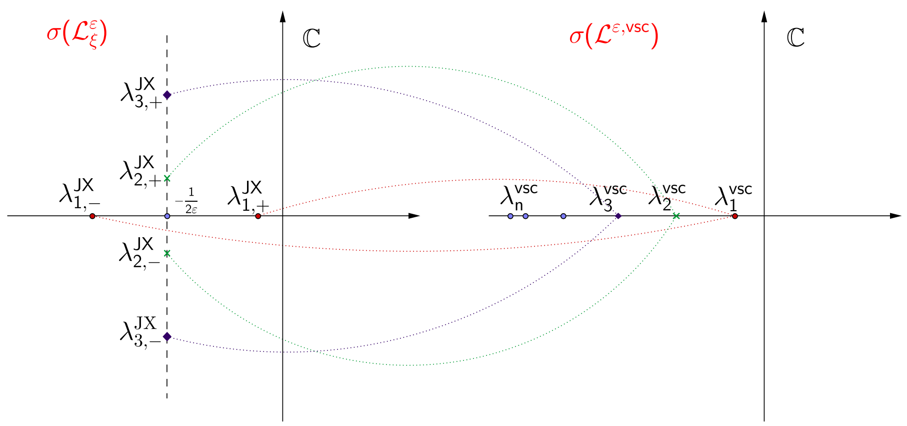


FIGURE 3. The spectra of the operators L_ξ^ε and $L^{\varepsilon,vsc}$.

Hence, the following proposition holds

Proposition 3.4. *Let a^ε be a family of functions satisfying assumptions A0-1-2-3 for some $\xi \in (\ell, \ell)$ and for some $a_+ < 0 < a_-$. Then the spectrum of the linearized operator $\mathcal{L}_\xi^\varepsilon$ can be decomposed as follow*

1. $\lambda_{1,+}^{JX} \in \mathbb{R}$ *and* $\quad -e^{-C'/\varepsilon} \leq \lambda_{1,+}^{JX} < 0$
2. $\lambda_{1,-}^{JX} \in \mathbb{R}$ *and* $\quad \lambda_{1,-}^{JX} \leq -1/\varepsilon$
3. *There exists $k \geq 0$ such that*
$$\lambda_{n,\pm}^{JX} \in \mathbb{R} \quad \text{and} \quad \lambda_{n,\pm}^{JX} \leq -C''/\varepsilon, \quad \forall\ n = 2, ..., 1 + k$$
4. $\lambda_{n,\pm}^{JX} \in \mathbb{C}$ *for all $n \geq 2 + k$ and*
$$Re(\lambda_{n,\pm}^{JX}) = -\frac{1}{2\varepsilon}, \quad Im(\lambda_{n,\pm}^{JX}) \sim \pm\frac{C}{\varepsilon}$$

Remark 3.1. In [35], Kreiss G. and Kreiss H. performed the spectral analysis for the operator

$$\mathcal{L}_\varepsilon u := \varepsilon \partial_{xx} u - \partial_x (f'(\bar{U}^\varepsilon(x))u)$$

arising from the linearization around the exact steady state $\bar{U}^\varepsilon(x)$ of

$$\partial_t u = \varepsilon \partial_{xx} u - \partial_x f(u)$$

proving that all the eigenvalues are real and negative. By using this result and our spectral analysis, if we linearize the system (3.11) around the exact stationary solution $(\bar{U}^\varepsilon, \bar{V}^\varepsilon)$, we can prove that the real part of all the eigenvalues of the linearized operator is negative, so that the steady state $(\bar{U}^\varepsilon, \bar{V}^\varepsilon)$ is asymptotically stable with exponential rate.

5. Asymptotic estimates for the first eigenvalue

In this Section we want to study the behavior in ε of the principal eigenvalue of the operator $\mathcal{L}^\varepsilon_\xi$ associated to the linearization of (3.11) around an approximate stationary solution. Since usually the metastable behavior is the result of the presence of a first small eigenvalue, our aim is to determine an asymptotic expression for $\lambda^{JX}_{1,+}$. We have already emphasized the fact that λ^{JX} is an eigenvalue of the nonlinear Jin-Xin system if and only if $\lambda^{vsc} = \lambda^{JX}(1 + \varepsilon \lambda^{JX})$ is an eigenvalue for the following operator

$$\mathcal{L}^{\varepsilon, vsc} := \varepsilon \partial_x^2 u - \partial_x (f'(U^\varepsilon(x;\xi))u) \tag{3.34}$$

where $U^\varepsilon(x;\xi(t))$ is an approximate stationary solution for the scalar conservation law

$$\begin{cases} \partial_t u = \varepsilon \partial_x^2 u - \partial_x f(u) \\ u(\pm \ell, t) = \mp u^*, \quad u(x,0) = u_0(x) \end{cases} \tag{3.35}$$

In particular

$$|\lambda^{JX}_{1,+}| = \frac{2|\lambda_1^{vsc}|}{1 + \sqrt{1 + 4\varepsilon \lambda_1^{vsc}}} \tag{3.36}$$

In Chapter 3, in the special case $f(u) = u^2/2$, $U^\varepsilon(x;\xi(t))$ is defined in (3.14). We have also proven that , for $\varepsilon \sim 0$

$$\lambda_1^{vsc}(\xi) \sim -\frac{u^{*2}}{2\varepsilon}\left[e^{-u^*\varepsilon^{-1}(\ell-\xi)} + e^{-u^*\varepsilon^{-1}(\ell+\xi)} \right]$$

so that

$$|\lambda^{JX}_{1,+}| \sim \frac{\frac{u^{*2}}{\varepsilon}\left[e^{-u^*\varepsilon^{-1}(\ell-\xi)} + e^{-u^*\varepsilon^{-1}(\ell+\xi)} \right]}{1 + \sqrt{1 - 2u^{*2}\left[e^{-u^*\varepsilon^{-1}(\ell-\xi)} + e^{-u^*\varepsilon^{-1}(\ell+\xi)} \right]}} \tag{3.37}$$

This formula shows that the principal eigenvalue of the Jin-Xin system when $f(u) = u^2/2$ is exponentially small in ε.

In order to determine an asymptotic expression of the first eigenvalue of the operator (3.34) for a general class of flux function $f(u)$, we refer to the paper of Reyna L.G. and Ward M.J., [64]; here the authors use the method of matched asymptotic expansions (MMAE) to determine an approximate stationary solution to (3.35).

Miming their approach and computing the same calculations of [64], with the appropriate changes due to the fact that the study of our equation is performed in the interval $(-\ell,\ell)$ instead of $(0,1)$, the leading order MMAE solution for $\varepsilon\to 0^+$ is given by a function $u_s(x;\xi)\sim u_s[\varepsilon^{-1}(x-\xi)]$, where $\xi\in(-\ell,\ell)$ and the shock profile $u_s(z)$ satisfies

$$\begin{cases} u_s'(z)=f[u_s(z)]-f(u^*), & -\infty<z<\infty \\ u_s(z)\sim u^*-a_-e^{\nu_- z}, & z\to-\infty \\ u_s(z)\sim -u^*+a_+e^{-\nu_+ z}, & z\to+\infty \end{cases} \tag{3.38}$$

The positive constant $\nu_\pm$ and $a_\pm$ describe the tail behavior of $u_s(z)$ and are defined by

$$\nu_\pm = \mp f'(\mp u^*)$$
$$\log\left(\frac{a_\pm}{u^*}\right)=\pm\nu_\pm\int_0^{\mp u^*}\left[\frac{1}{f(\eta)-f(u^*)}\pm\frac{1}{\nu_\pm(\eta\pm u^*)}\right]d\eta$$

In particular, when $f(u)=u^2/2$, $u_s(z)=-u^*\tanh(u^*z/2)$, according to (3.9). Notice that the MMAE solutions satisfies exactly the equation, while the boundary conditions are satisfy within exponentially small terms. Instead, the construction presented in Example 2.1 and followed here gives a function $U^\varepsilon(x;\xi)$ that verifies exactly the boundary conditions and solves approximately the stationary equation.

The eigenvalue problem associated to the linearization around u_s is given by

$$\begin{cases} L\phi\equiv\varepsilon^2\partial_{xx}\phi-V[\varepsilon^{-1}(x-\xi)]\phi=\lambda\phi \\ \phi(\pm\ell)=0 \\ V(z)=\dfrac{1}{4}(f'[u_s(z)])^2+\dfrac{1}{2}f''[u_s(z)]u_s'(z) \end{cases} \tag{3.39}$$

In [64] it is proven that the first eigenvalue of (3.39) has the following asymptotic representation (for details see [64, Formula (2.14)])

$$\lambda_1^{vsc}(\xi)\sim-\frac{1}{2u^*}\left[a_+\nu_+^2e^{-\nu_+\varepsilon^{-1}(\ell-\xi)}+a_-\nu_-^2e^{-\nu_-\varepsilon^{-1}(\ell+\xi)}\right] \tag{3.40}$$

Finally, from (3.36), we get

$$|\lambda_{1,+}^{JX}|\sim\frac{\frac{1}{u^*}\left[a_+\nu_+^2e^{-\nu_+\varepsilon^{-1}(\ell-\xi)}+a_-\nu_-^2e^{-\nu_-\varepsilon^{-1}(\ell+\xi)}\right]}{1+\sqrt{1-\frac{2\varepsilon}{u^*}\left[a_+\nu_+^2e^{-\nu_+\varepsilon^{-1}(\ell-\xi)}+a_-\nu_-^2e^{-\nu_-\varepsilon^{-1}(\ell+\xi)}\right]}} \tag{3.41}$$

This formula shows that $\lambda_{1,+}^{JX}$ is exponentially small as $\varepsilon\to 0$. We remark that, when $f(u)=u^2/2$, $a_+=a_-=2u^*$ and $\nu_+=\nu_-=u^*$, so that (3.41) is the same as (3.37).

6. The behavior of the shock layer position

Let us consider the system (3.27) for the couple (ξ,Y) and let us neglect the $o(Y)$ terms

$$\begin{cases} \dfrac{d\xi}{dt}=\theta(\xi)\left(1+\dfrac{\langle\partial_\xi\psi_1^\varepsilon,Y\rangle}{\alpha_0^\varepsilon(\xi)}\right) \\ Y_t=(\mathcal{L}_\xi^\varepsilon+\mathcal{M}_\xi^\varepsilon)Y+H^\varepsilon(x;\xi) \end{cases} \tag{3.42}$$

This system is obtained by linearizing with respect to Y and by keeping the nonlinear dependence on ξ, in order to describe the slow motion of the shock layer position far from the equilibrium location $\bar{\xi}$.

We complement the so called **quasi-linearized system** (3.42) with initial data

$$\xi(0) = \xi_0 \in (-\ell, \ell) \quad \text{and} \quad Y(x,0) = (u_0(x), v_0(x)), \quad u_0, v_0 \in L^2(-\ell, \ell) \tag{3.43}$$

The aim of this Section is to analyze the behavior of the solution to (3.42) in the limit of small ε. Subsequently, we will prove a result that characterizes the behavior of the shock layer location, proving how it moves towards the unique stationary solution with exponentially small rate.

Before stating our result, let us recall the assumptions. Let the family $\{\mathbf{W}^\varepsilon(\cdot;\xi)\}$ be such that there exists two families of smooth positive functions Ω_1^ε and Ω_2^ε such that

$$\begin{aligned} &|\langle \psi(\cdot), \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle| \leq \Omega_1^\varepsilon(\xi)|\psi|_{L^\infty} \quad \forall \psi \in C(I) \\ &|\langle \psi(\cdot), \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle| \leq \Omega_2^\varepsilon(\xi)|\psi|_{L^\infty} \quad \forall \psi \in C(I) \end{aligned} \tag{3.44}$$

We also assume that $\mathbf{W}^\varepsilon$ is asymptotically a solution, i.e. we require that

$$\lim_{\varepsilon \to 0} |\Omega_1^\varepsilon|_{L^\infty} = 0, \quad \lim_{\varepsilon \to 0} |\Omega_2^\varepsilon|_{L^\infty} = 0 \tag{3.45}$$

uniformly with respect to ξ.

Example 3.2 show that (3.44) and (3.45) are verified in the case of the quadratic flux $f(u) = u^2/2$.

For what concern the linear operator $\mathcal{L}_\xi^\varepsilon$, we have already proven that there exist two positive constants C_1, C_2 independent on ξ such that

$$\lambda_{1,+}^{JX}(\xi) - Re[\lambda_{2,\pm}^{JX}(\xi)] > C_1, \quad -e^{-C_2/\varepsilon} < \lambda_1^\varepsilon(\xi) < 0 \quad \forall \xi \in (-\ell, \ell) \tag{3.46}$$

Additionally, we assume that there exists a constant $C_3 > 0$ such that

$$\Omega_1^\varepsilon(\xi) + \Omega_2^\varepsilon(\xi) \leq C_3 |\lambda_1^\varepsilon(\xi)|, \quad \forall\, \xi \in (-\ell, \ell) \tag{3.47}$$

By comparing the asymptotic expression for $\lambda_{1,+}^{JX}$ given in (3.37) with the one for Ω_1^ε and Ω_2^ε obtained in Example 3.3, we can easily check that hypothesis (3.47) is verified for the Jin-Xin system when $f(u) = u^2/2$.

Finally, concerning the solution $Z = (z, w)^T$ to the linear problem $\partial_t Z = \mathcal{L}_\xi^\varepsilon Z$, we require that there exists $\nu > 0$ such that for all $\xi \in (-\ell, \ell)$, there exist constants C_ξ and $\bar{C}$ such that

$$|(z,w)(t)|_{L^2} \leq C_\xi |(z_0, w_0)|_{L^2} e^{-\nu t}, \quad C_\xi \leq \bar{C} \quad \forall \xi \in (-\ell, \ell) \tag{3.48}$$

Remark 3.2. The assumption that $C_\xi < \bar{C}$ for all ξ means that the estimate (3.48) holds uniformly in ξ. Since ξ belongs to a bounded interval of the real line, if we suppose that $\xi \mapsto C_{\xi(t)}$ is a continuous function, then there exists a maximum $\bar{C}$ in $[-\ell, \ell]$.

Estimates for the perturbation Y. Our first aim is to obtain an estimate for the perturbation Y. We recall that

$$\partial_t Y = (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon) Y + H^\varepsilon(x;\xi) \tag{3.49}$$

where

$$\mathcal{M}^\varepsilon_\xi Y = -\frac{1}{\alpha^\varepsilon_0(\xi)}\begin{pmatrix}\partial_\xi U^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi)\,\langle\partial_\xi\boldsymbol{\psi}^\varepsilon_1(\cdot;\xi),Y\rangle\\ \partial_\xi V^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi)\,\langle\partial_\xi\boldsymbol{\psi}^\varepsilon_1(\cdot;\xi),Y\rangle\end{pmatrix}$$

$$H^\varepsilon(x;\xi) = \begin{pmatrix}\mathcal{P}^\varepsilon_1[\mathbf{W}^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot,\xi)\theta^\varepsilon(\xi)\\ \mathcal{P}^\varepsilon_2[\mathbf{W}^\varepsilon(\cdot,\xi)] - \partial_\xi V^\varepsilon(\cdot,\xi)\theta^\varepsilon(\xi)\end{pmatrix}$$

In particular, $\mathcal{M}^\varepsilon_\xi$ is a bounded operator, such that

$$\|\mathcal{M}^\varepsilon_\xi\|_{\mathcal{L}(L^2;\mathbb{R}^2)} \le C|\theta^\varepsilon(\xi)| \le C(|\Omega^\varepsilon_1|_{L^\infty} + |\Omega^\varepsilon_2|_{L^\infty}), \quad \forall\xi\in(-\ell,\ell) \tag{3.50}$$

while

$$|H^\varepsilon|_{L^\infty} \le C_1|\Omega^\varepsilon_1|_{L^\infty} + C_2|\Omega^\varepsilon_2|_{L^\infty} \tag{3.51}$$

For the special case of $f(u) = u^2/2$, both $\mathcal{M}^\varepsilon_\xi$ and H^ε are bounded by terms that are exponentially small in ε, while, for a general class of flux functions $f(u)$ that verify (3.3), the hypotheses we required assure that all the terms in the equations for the perturbation Y are small in ε.

Theorem 3.5. *Let hypotheses* (3.44)-(3.45)-(3.47)-(3.48) *be satisfied. Then, for ε sufficiently small, there exists a time $T > 0$ such that, for all $t \le T$ the solution Y to* (3.49) *is such that*

$$|Y|_{L^2}(t) \le [C_1|\Omega^\varepsilon_1|_{L^\infty} + C_2|\Omega^\varepsilon_2|_{L^\infty}]\left(1 - e^{-\mu^\varepsilon t}\right) + e^{-\mu^\varepsilon t}|Y_0|_{L^2}$$

for some positive constants C_1, C_2 and

$$\mu^\varepsilon := \sup_\xi \lambda^{JX}_{1,+}(\xi) - C(|\Omega^\varepsilon_1|_{L^\infty} + |\Omega^\varepsilon_2|_{L^\infty}) > 0$$

Proof. Since the operator $\mathcal{L}^\varepsilon_\xi + \mathcal{M}^\varepsilon_\xi$ is a linear operator that depends on time, to obtain rigorous estimates on the solution Y, we need to use the theory of *stable families of generators*, that is a generalization of the theory of semigroups for evolution systems of the form $\partial_t u = Lu$. We will use some results of [**62**], which have been summarized in the Appendix A. More precisely, we want to show that $\mathcal{L}^\varepsilon_\xi + \mathcal{M}^\varepsilon_\xi$ is the infinitesimal generator of a C_0 semigroup $\mathcal{T}_\xi(t,s)$.

To this aim, concerning the eigenvalues of the linear operator $\mathcal{L}^\varepsilon_\xi$, we know that $\lambda^{JX}_{1,+}(\xi)$ is negative and behaves like $e^{-1/\varepsilon}$ for all $\xi\in(-\ell,\ell)$, so that $\Lambda^\varepsilon_1 := \sup_\xi \lambda^{JX}_{1,+}(\xi)$ is such that $-e^{-1/\varepsilon} \le \Lambda^\varepsilon_1 < 0$, and this estimate is independent of t. Hence, by using Definition 3.7 and Remark 3.5 (See Appendix A), we know that, for $t\in[0,T]$, $\mathcal{L}^\varepsilon_{\xi(t)}$ is the infinitesimal generator of a C_0 semigroup $\mathcal{S}_{\xi(t)}(s)$, $s > 0$. Furthermore, since (3.48) holds, we get

$$\|\mathcal{S}_{\xi(t)}(s)\| \le \bar{C}e^{-|\Lambda^\varepsilon_1|s}$$

so that the family $\{\mathcal{L}^\varepsilon_{\xi(t)}\}_{\xi(t)\in(-\ell,\ell)}$ is stable with stability constants $M = \bar{C}$ and $\omega = -|\Lambda^\varepsilon_1|$. Furthermore, since

$$\|\mathcal{M}^\varepsilon_\xi\|_{\mathcal{L}(L^2;\mathbb{R}^2)} \le C(|\Omega^\varepsilon_1|_{L^\infty} + |\Omega^\varepsilon_2|_{L^\infty}), \quad \forall\xi\in(-\ell,\ell)$$

Theorem 3.8 (see Appendix A) states that the family $\{\mathcal{L}^\varepsilon_{\xi(t)} + \mathcal{M}^\varepsilon_{\xi(t)}\}_{\xi(t)\in(-\ell,\ell)}$ is stable with $M = \bar{C}$ and $\omega = -|\Lambda^\varepsilon_1| + C(|\Omega^\varepsilon_1|_{L^\infty} + |\Omega^\varepsilon_2|_{L^\infty}) < 0$.

From Theorem 3.11 (in Appendix A), we can define $\mathcal{T}_\xi(t,s)$ as the *evolution system* of $\partial_t Y = (\mathcal{L}^\varepsilon_\xi + \mathcal{M}^\varepsilon_\xi)Y$, so that

$$Y(t) = \mathcal{T}_\xi(t,s)Y_0 + \int_s^t \mathcal{T}_\xi(t,r)H^\varepsilon(x;\xi(r))dr, \quad 0 \le s \le t \le T \tag{3.52}$$

Moreover, there holds

$$\|\mathcal{T}_\xi(t,s)\| \le \bar{C}e^{-\mu^\varepsilon(t-s)}, \qquad \mu^\varepsilon := |\Lambda^\varepsilon_1| - C(|\Omega^\varepsilon_1|_{L^\infty} + |\Omega^\varepsilon_2|_{L^\infty}) > 0$$

Finally, from the representation formula (3.52) with $s=0$, it follows

$$|Y|_{L^2}(t) \le e^{-\mu^\varepsilon t}|Y_0|_{L^2} + \sup_{\xi\in I}|H^\varepsilon(\xi)| \int_0^t e^{-\mu^\varepsilon(t-r)}\,dr, \quad 0 \le t \le T \tag{3.53}$$

so that, by using (3.51), we end up with

$$|Y|_{L^2}(t) \le \tilde{C}\left[|\Omega^\varepsilon_1|_{L^\infty} + |\Omega^\varepsilon_2|_{L^\infty}\right]\left(e^{-\mu^\varepsilon t}+1\right) + e^{-\mu^\varepsilon t}|Y_0|_{L^2} \tag{3.54}$$

□

Remark 3.3. In the special case of Burgers flux, μ^ε is going to zero exponentially as $\varepsilon \to 0$, since λ^ε_1 behaves like $e^{-1/\varepsilon}$ and from the explicit formula of Ω^ε_1 and Ω^ε_2 in Example 3.2. In the general case, assumption (3.45) assures that $\mu^\varepsilon \to 0$ as $\varepsilon \to 0$.

Remark 3.4. To apply Theorem 3.11, we need to check that the domain of $\mathcal{L}^\varepsilon_\xi + \mathcal{M}^\varepsilon_\xi$ does not depend on time, and this is true since $\mathcal{L}^\varepsilon_\xi + \mathcal{M}^\varepsilon_\xi$ depends on time through the function $U^\varepsilon(x;\xi(t))$, that does not appear in the higher order terms of the operator. More precisely, the principal part of the operator does not depend on $\xi(t)$.

Slow motion of the shock layer. An immediate consequence of the estimate (3.54) is that, for $|Y|_{L^2} < M$ for some $M > 0$, the function $\xi(t)$ satisfies

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi)(1 + r(t)) \quad \text{with} \quad |r(t)| \le C_1|\Omega^\varepsilon_1|_{L^\infty} + C_2|\Omega^\varepsilon_2|_{L^\infty} + e^{-\mu^\varepsilon t}|Y_0|_{L^2}$$

More precisely, we can prove the following

Proposition 3.6. *Let hypotheses* (3.44), (3.45), (3.47) *and* (3.48) *be satisfied. Assume also*

$$s\,\theta^\varepsilon(s) < 0 \quad \text{for any} \quad s \in I,\, s \ne 0 \qquad \text{and} \qquad {\theta^\varepsilon}'(\bar{\xi}) < 0 \tag{3.55}$$

Then, for small ε and $|Y_0|_{L^2}$, the solution ξ converges to $\bar{\xi}$ as $t \to +\infty$.

Proof. Thanks to assumption (3.45), for ε and $|Y_0|_{L^2}$ sufficiently small, estimate (3.54) holds globally in time. Hence, for any initial datum ξ_0, the location of the shock layer satisfies

$$\int_{\xi_0}^{\xi(t)} \frac{dz}{\theta^\varepsilon(z)} = \int_0^t (1 + r(s))ds \tag{3.56}$$

where

$$|r(t)| \le C_1|\Omega^\varepsilon_1|_{L^\infty} + C_2|\Omega^\varepsilon_2|_{L^\infty} + e^{-\mu^\varepsilon t}|Y_0|_{L^2}$$

More precisely, if ε and $|Y_0|_{L^2}$ are sufficiently small, the dynamics of the shock location ξ is well described by

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi) \ , \qquad \theta^\varepsilon(\xi) = \frac{\langle \psi_1^\varepsilon, \mathcal{F}[\mathbf{W}^\varepsilon]\rangle}{\langle \psi_1^\varepsilon, \partial_\xi \mathbf{W}^\varepsilon\rangle} \tag{3.57}$$

Therefore ξ converges to $\bar{\xi}$ as $t \to +\infty$, and the convergence is exponential for any t under consideration, since, by means of the standard separation of variable method, we obtain the following estimate for the shock layer location

$$|\xi(t) - \bar{\xi}| \leq |\xi_0| e^{\beta^\varepsilon t}, \quad \beta^\varepsilon \sim \theta^{\varepsilon\prime}(\bar{\xi}) \tag{3.58}$$

where $\theta^{\varepsilon\prime}(\bar{\xi}) \to 0$ as $\varepsilon \to 0$.

□

Formula (3.58) shows the slow motion of the shock layer for small ε. Precisely, the evolution of the collocation of the shock towards the equilibrium position is much slower as ε becomes smaller.

For example, when $f(u) = u^2/2$, $\bar{\xi} = 0$ and $\theta^{\varepsilon\prime}(0) \sim e^{-1/\varepsilon}$ (see formula (3.28)). We also emphasize that hypothesis (3.55) are verified in the case of the Jin-Xin system with $f(u) = u^2/2$.

The following table shows a numerical computation for the location of the shock layer for different values of the parameter ε and $f(u) = u^2/2$. The initial datum for the function u is $u_0(x) = \frac{1}{2}x^2 - x - \frac{1}{2}$. We can see that the convergence to $\bar{\xi} = 0$ is much slower as ε becomes smaller.

The numerical location of the shock layer $\xi(t)$ for different values of the parameter ε

TIME t	$\xi(t)$, $\varepsilon = 0.1$	$\xi(t)$, $\varepsilon = 0.07$	$\xi(t)$, $\varepsilon = 0.055$	$\xi(t)$, $\varepsilon = 0.04$	$\xi(t)$, $\varepsilon = 0.02$
0.2	-0.4008	-0.4020	-0.4029	-0.4040	-0.4059
1	-0.3314	-0.3345	-0.3360	-0.3374	-0.3389
10	-0.3070	-0.3263	-0.3304	-0.3320	-0.3326
10^3	-0.0103	-0.1600	-0.2562	-0.3181	-0.3325
10^4	$-1.9725e^{-12}$	-0.0084	-0.1115	-0.2531	-0.3320
$0.5 * 10^6$	$-1.9725e^{-12}$	$-2.2102e^{-11}$	$-1.5057e^{-10}$	-0.0379	-0.3099

Figure 4 shows the dynamics of the shock layer (i.e the dynamics of the solution u to (3.11)), obtained numerically. When $\varepsilon = 0.1$, the shock layer location converges to zero very fast: as we can also see from the table, when $t = 10^3$, the value of $\xi(t)$ is already very close to zero. On the other hand, when ε becomes smaller the shock layer location moves slower and it approaches the equilibrium location only for very large t. Finally, Figure 5 shows the profile of the shock layer for the flux function $f(u) = u^4/4$ that still verifies hypothesis (3.3).

7. Appendix A

In this section, we briefly review some results on the theory of evolution systems by A. Pazy [62, Chapter 5]. For more details and for the proofs of the Theorems, see [62], Theorem 2.3, Theorem 3.1, Theorem 4.2.

Let X be a Banach space. For every $0 \leq t \leq T$, let $A(t) : D(A(t)) \subset X \to X$ be a linear operator in X and let $f(t)$ be an X valued function. Let

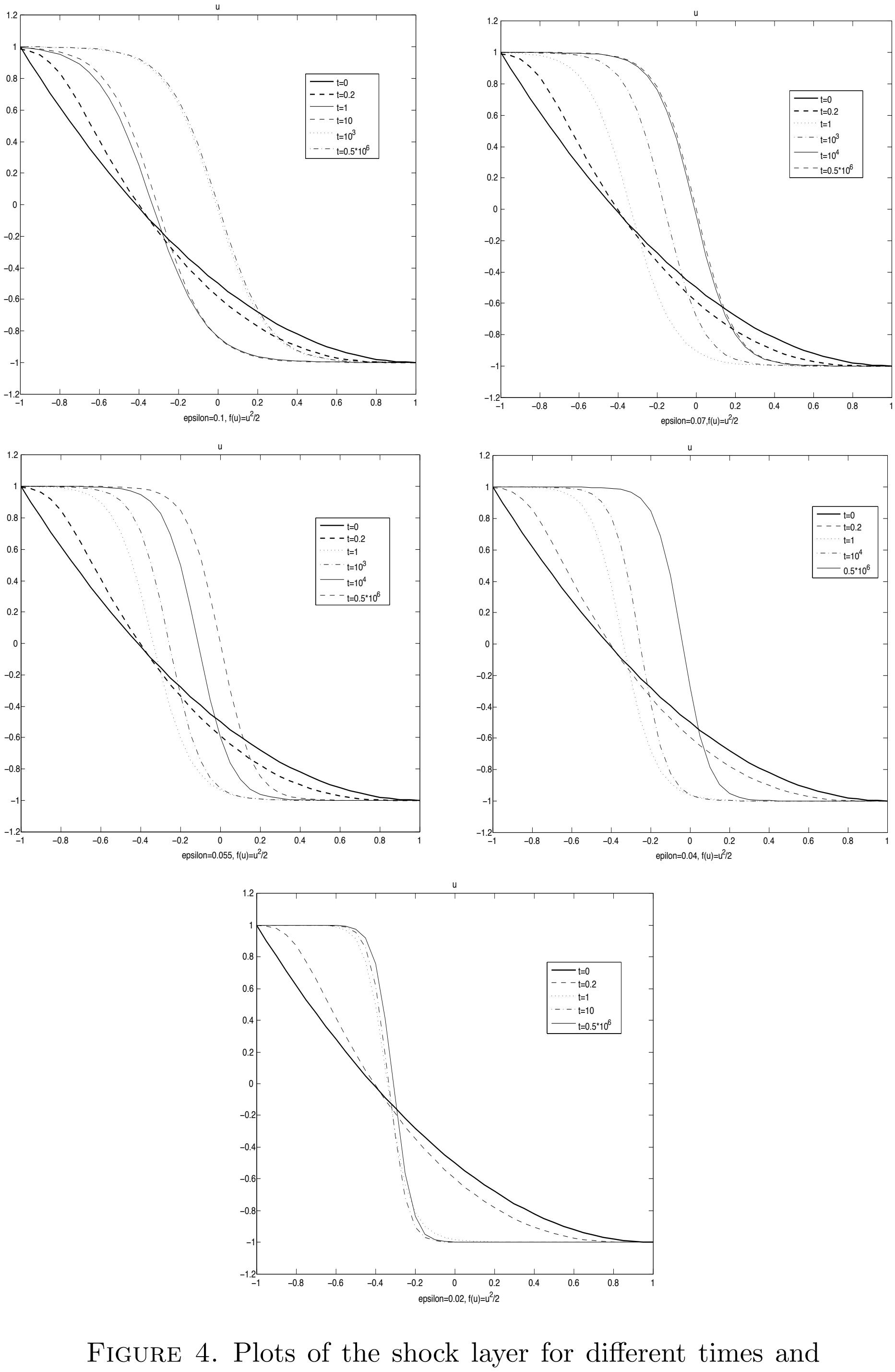


FIGURE 4. Plots of the shock layer for different times and different value of the parameter ε. Notice also the steepening of the shock layer as ε goes to zero.

us consider the initial value problem

$$\partial_t u = A(t)u + f(t), \quad u(s) = u_0 \qquad 0 \leq s \leq t \leq T \tag{3.59}$$

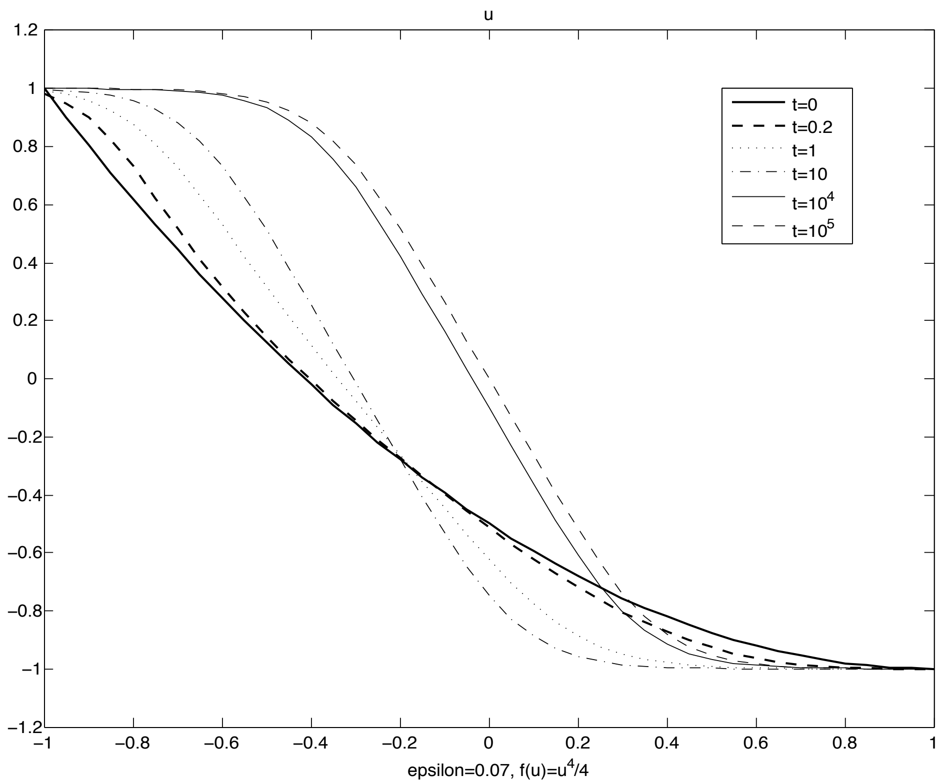


FIGURE 5. Profiles of the shock layer at different times with a convex flux function $f(u)$.

In the special case where $A(t) = A$ is independent of t, the solution to (3.59) can be represented via the formula of *variations of constants*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\ ds$$

where $T(t)$ is the C_0 semigroup generated by A. In [62] it is shown that a similar representation formula is true also when $A(t)$ depends on time.

Definition 3.7. Let X a Banach space. A family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0 semigroups on X is called stable if there are constants $M \geq 1$ and ω (called the stability constants) such that

$$(\omega, +\infty) \subset \rho(A(t)), \quad \text{for} \quad t \in [0, T]$$

and

$$\left\| \Pi_{j=1}^k R(\lambda : A(t_j)) \right\| \leq M(\lambda - \omega)^{-k},$$

for $\lambda > \omega$ and for every finite sequence $0 \leq t_1 \leq t_2, ..., t_k \leq T$, $k = 1, 2, ...$

Remark 3.5. If for $t \in [0, T]$, $A(t)$ is the infinitesimal generator of a C_0 semigroup $S_t(s)$, $s \geq 0$ satisfying $\|S_t(s)\| \leq e^{\omega s}$, then the family $\{A(t)\}_{t\in[0,T]}$ is clearly stable with constants $M = 1$ and ω.

The previous remark means that, if for every fixed $t \in [0, T]$ the operator $A(t)$ generates a C_0 semigroup $S_t(s)$, and we can find an estimate for $\|S_t(s)\|$ that is independent of t, then the whole family $\{A(t)\}_{t\in[0,T]}$ is stable in the sense of Definition 3.7.

Theorem 3.8. *Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of infinitesimal generators with stability constants M and ω. Let $B(t)$, $0 \le t \le T$ be a bounded linear operators on X. If $\|B(t)\| \le K$ for all $t \le T$, then $\{A(t) + B(t)\}$ is a stable family of infinitesimal generators with stability constants M and $\omega + MK$.*

In order to prove the existence of the the so called *evolution system* $U(t, s)$ for the initial value problem (3.59), let us introduce X and Y Banach spaces with norms $\| \ \|_X$, $\| \ \|_Y$ respectively. Moreover, let us assume that Y is a dense subspace of X and that there exists a constant C such that $\|w\|_X \le C\|w\|_Y$ for all $w \in Y$.

Definition 3.9. Let A be the infinitesimal generator of a C_0 semigroup $S(s)$, $s \ge 0$, on X. Y is called A-*admissible* if it is an invariant subspace of $S(s)$, and the restriction $\tilde{S}(s)$ of $S(s)$ to Y is a C_0 semigroup on Y. Moreover, $\tilde{A}$, is the infinitesimal generator of the semigroup $\tilde{S}(s)$ on Y, and it is called the *part of A in Y*.

Next, let us fix $t \in [0, T]$, and let $A(t)$ be the infinitesimal generator of a C_0 semigroup $S_t(s)$ on X. The following assumptions are made

(**H1**) $\{A(t)\}_{t\in[0,T]}$ is a stable family with stability constants M and ω.

(**H2**) Y is $A(t)$-admissible for $t \in [0, T]$ and the family $\{\tilde{A}(t)\}_{t\in[0,T]}$ of parts $\tilde{A}(t)$ of $A(t)$ in Y is a stable family in Y with stability constants $\tilde{M}$, $\tilde{\omega}$.

(**H3**) For $t \in [0, T]$, $Y \subset D(A(t))$, $A(t)$ is a bounded operator from Y into X and $t \to A(t)$ in continuous in the $B(X, Y)$ norm.

Remark 3.6. The assumption that the family $\{A(t)\}_{t\in[0,T]}$ satisfies (**H2**) is not always easy to check. A sufficient condition for (**H2**) which can be effectively checked in many applications states that (**H2**) holds if there is a family $\{Q(t)\}$ of isomorphism of Y onto X such that $\|Q(t)\|_{Y\to X}$ and $\|Q(t)^{-1}\|_{Y\to X}$ are uniformly bounded and $t \to Q(t)$ is of bounded variation in the $B(Y, X)$ norm.

Remark 3.7. Condition (**H3**) can be replaced by the weaker condition

(**H3**)' For $t \in [0, T]$, $Y \subset D(A(t))$ and $A(t) \in L^1([0, T]; B(Y, X))$.

Theorem 3.10. *Let $A(t)$, $0 \le t \le T$ be the infinitesimal generator of a C_0 semigroup $S_t(s)$, $s \ge 0$ on X. If the family $\{A(t)\}_{t\in[0,T]}$ satisfies the conditions (**H1**)-(**H3**), then there exists a unique evolution system $U(t, s)$, $0 \le s \le t \le T$, in X satisfying*

$$\|U(t, s)\| \le Me^{\omega(t-s)}, \quad \text{for} \quad 0 \le s \le t \le T \tag{3.60}$$

and such that the solution to (3.59) can be written as

$$u(t) = U(t, s)u_0 + \int_s^t U(t, r)f(r)\ dr \tag{3.61}$$

for all $0 \leq s \leq t \leq T$.

One special case in which the conditions of Theorem 3.10 can be easily checked is the one where the domain of the operator $D(A(t)) \equiv D$ is independent on t. In this case we can take D as the Banach space which we denote by Y, and the following Theorem holds

Theorem 3.11. *Let* $\{A(t)\}_{t\in[0,T]}$ *be a stable family of infinitesimal generators of* C_0 *semigroups on* X. *If* $D(A(t)) = D$ *is independent on* t *and for* $v \in D$, $A(t)v$ *is continuously differentiable in* X, *then there exists a unique evolution system* $U(t,s)$, $0 \leq s \leq t \leq T$, *satisfying* (3.60). *Morevoer, if* $f \in C^1([s,T],X)$, *then, for every* $u_0 \in X$, *the initial value problem* (3.59) *has a unique solution given by* (3.61).

CHAPTER 4

Existence of stationary solutions for the viscous shallow water system in a bounded interval

1. Introduction

The shallow water equations describe different situations in fluid dynamics by modeling the dynamics of a shallow compressible/incompressible fluid; typically, they are used to describe vertically average flow in two or three dimensions in terms of the horizontal velocity and depth variation.

The term *shallow* refers to the fact that, by assumption, the horizontal length of the channel L is much greater than the height H, so that that the quantity H/L can be interpreted as a small parameter. When the open channel flow has a vertical scale that is small relatively to the horizontal one, it is possible to derive simplified equations; this is the case of the Saint-Venant system, that provides a one-dimensional model of free surface water flow in a channel.

The first derivation of such system has been performed by Saint- Venant [66] in 1971; the model consists in a hyperbolic system of two partial differential equations with a structure that is the same of the system for isentropic gas-dynamics in Eulerian coordinates in the case of a pressure with a power-law form with exponent equal to 2. Since then, the simpler model for shallow water is called Saint-Venant system. These equations are widely used in practice and in literature being object of many thousands of publications devoted to the applications, the validations and to the analytical and numerical study of the solutions.

Depending on assumptions and approximations, shallow water models may also contain other terms and give raise to different type of partial differential equations. Indeed, natural modifications of the model emerge when additional physical effects are taken into account, like viscosity, friction or Coriolis forces.

The original hydraulic model of Saint-Venant is written in the form of two partial differential equations in one dimension

$$\partial_t h + \partial_x(hw) = 0, \quad \partial_t(hw) + \partial_x(hw^2 + P(h)) = \varepsilon\partial_x(\mu(h)\partial_x w) \tag{4.1}$$

Here $h(x,t), w(x,t)$ and P stand for the fluid depth, velocity and pressure respectively. The viscosity coefficient $\mu(h)$ for simplicity is assumed to be $\mu(h) = h$ and the parameter ε is the ratio $\varepsilon = \frac{H}{L}$ where H and L are two caracteristic lengths along the axis Oz and Ox respectively. As already stressed, it is assumed that $L \gg H$ (shallow water model) so that ε can be considered small.

The Saint Venant equations can be derived from the hydrostatic approximation in the Navier Stokes system (see, for example, J.F.Gerbeau and B.Perthame [23]), while the Saint Venant problem without viscosity can be viewed as a special case of the compressible Euler system (Lions, Perthame, Tadmor [44, 45]). For more details, see also [51].

There are many literatures on mathematical studies on system (4.1), which represents a class of reaction-convenction-diffusion system of the form

$$\partial_t U + \partial_x F(U) = \partial_x (B(U)(U)_x)_x$$

Global existence results and asymptotic stability of equilibrium states are obtained from Kawashima's theory of parabolic-hyperbolic systems in [30], D. Bresch, B. Desjardins, G. Métivier in [11], P.L. Lions in [43] and W. Wang in [70] for viscous model, and Dafermos (see [16]) for inviscid model. K. Zumbrun, C. Mascia, P. Howard and F. Rousset (see, for example, [52, 56, 57]) have developed a general program for proving stability of shock waves. They have been able to show that, for general classes of shock waves, the spectral stability of linearized operator implies nonlinear stability. Most of these results concern with free boundary conditions. Recently, initial-boundary value problem with $\mu = h^\alpha, (\alpha > 1/2)$ has been studied by Li, Li and Xin [41].

In this Chapter we specifically consider the initial-boundary value problem for the hyperbolic-parabolic viscous shallow water system with a general term of pressure $P(u)$, that is

$$\begin{cases} \partial_t u + \partial_x v = 0, & x \in I, \ t \geq 0 \\ \partial_t v + \partial_x \left(\dfrac{v^2}{u} + P(u) \right) = \varepsilon \partial_x \left(u \partial_x \left(\dfrac{v}{u} \right) \right) & \\ u(\pm\ell, t) = u_\pm, \quad v(\pm\ell, t) = v_\pm & t \geq 0 \end{cases} \tag{4.2}$$

where the space variable x belongs to a one-dimensional interval $I = (-\ell, \ell)$ of the real line. As usual, t is the time variable, while the parameter $\varepsilon \ll 1$ represents the viscosity intensity. Equation (4.2) is also complemented with initial datum for the couple (u, v). A primary prototype for the term of pressure $P(u)$ is given by the power law $P(u) = \kappa u^\gamma$, $\gamma \in (1, 2)$.

When $P(u) = \frac{1}{2} g u^2$, $g > 0$, system (4.2) is the usual Saint-Venant system. In this special case, the constant ε is defined as the ratio between the length and the hight of the channel. Moreover, with the change of variables $u(x,t) = h(x,t)$ and $v(x,t) = h(x,t)w(x,t)$, we obtain system (4.1).

Given an unsteady flow under steady boundary conditions, it is expected that the flow will eventually tend towards a steady state. Hence, in what follows, our concern is to prove the existence and uniqueness of a stationary solution to (4.2).

The existence of stationary solutions for Saint-Venant's type systems has been considered for a long time in the literature. To name some of these results, see [9], [21] and [52].

We point out anyway that these papers deal with the open channel case (i.e. $x \in \mathbb{R}$), in which the study of stationary solutions presents less difficulties than the case of a bounded domain where boundary conditions play an important role: indeed, in this case, we have to handle with compatibility conditions on the boundary values.

2. The unviscid problem

To begin with, let us consider system (4.2) in the small viscosity limit ($\varepsilon \to 0^+$). Formally, the solutions to (4.2) converge to the solutions of the hyperbolic system

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x \left(\dfrac{v^2}{u} + P(u) \right) = 0 \end{cases} \tag{4.3}$$

whose standard setting is given by the *entropy formulation*. Hence we primarily concentrate on the problem of determining the entropy jump conditions for the hyperbolic system under consideration (see also [51]).

Such conditions are determined by the choice of a couple entropy/entropy flux, that, in the present setting, are given by

$$\mathcal{E}(u,v) := \frac{v^2}{2u} + P(u), \qquad \mathcal{Q}(u,v) := \frac{v^3}{2u^2} + 2P(u)\frac{v}{u}$$

corresponding to the physical energy/energy flux of the system.

Given $u_\pm > 0$, $v_\pm > 0$ and $c \in R$, let (u_-, v_-) and (u_+, v_+) be an entropic discontinuity of (4.3) with speed c, that is we assume that the function

$$(U,V)(x,t) := \begin{cases} (u_-, v_-) & \text{for } x < ct \\ (u_+, v_+) & \text{for } x > ct \end{cases}$$

is a weak solution satisfying, in the sense of distributions, the entropy inequality

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{Q}}{\partial x} \leq 0 \tag{4.4}$$

The request of weak solution translates in the **Rankine-Hugoniot** conditions

$$\left[u \left(\frac{v}{u} - c \right) \right] = 0, \qquad \left[v \left(\frac{v}{u} - c \right) + P(u) \right] = 0 \tag{4.5}$$

where $[g] := g_+ - g_-$ denotes the jump of the function g. Setting $h := \frac{v}{u} - c$, equations (4.5) become

$$[uh] = 0, \qquad [vh + P(u)] = 0 \tag{4.6}$$

The entropy condition (4.4) reads as $[\mathcal{Q} - c\mathcal{E}] \leq 0$. Hence, plugging (4.6) into (4.4), one obtains

$$\left[\frac{1}{2} u h^3 + 2P(u)h \right] \leq 0 \tag{4.7}$$

By using the first equation in (4.6), we obtain a system for the quantities $h_\pm^2$:

$$u_+^2 h_+^2 - u_-^2 h_-^2 = 0, \quad u_+ h_+^2 - u_- h_-^2 = P(u_-) - P(u_+)$$

whose solutions are

$$h_+^2 = \frac{u_-}{u_+} \frac{[P(u_-) - P(u_+)]}{(u_- - u_+)} \qquad h_-^2 = \frac{u_+}{u_-} \frac{[P(u_-) - P(u_+)]}{(u_- - u_+)} \tag{4.8}$$

When we look for stationary solutions to (4.3), i.e. $c = 0$, (4.8) translates into a condition for the boundary values. Indeed, since $h = \frac{v}{u}$

$$v_+^2 = u_- u_+ \frac{[P(u_-) - P(u_+)]}{(u_- - u_+)} \qquad v_-^2 = u_+ u_- \frac{[P(u_-) - P(u_+)]}{(u_- - u_+)} \tag{4.9}$$

Example 4.1 (Power law)**.** Let us consider system (4.3) with $P(u) = \kappa u^\gamma$, $\gamma \in (1,2)$. Stationary solutions to (4.3) solve

$$v = C_1, \quad \frac{C_1^2}{u} + \kappa u^\gamma = C_2 \tag{4.10}$$

where C_1, C_2 are positive integration constants. Since C_1 is univocally determined by the boundary conditions $v(\pm\ell) = v_\pm = C_1$, the equation for u reads

$$\kappa u^{\gamma+1} - C_2 u + \bar{v}^2 = 0 \tag{4.11}$$

where $\bar{v} := v_\pm = C_1$. Once $u(\pm\ell) = u_\pm$ are imposed, there exists a solution to (4.11) that satisfies the boundary conditions if and only if there exists C_2^* such that both the equations above are satisfied

$$\kappa u_-^{\gamma+1} - C_2^* u_- + \bar{v}^2 = 0, \quad \kappa u_+^{\gamma+1} - C_2^* u_+ + \bar{v}^2 = 0$$

Hence

$$C_2^* = \kappa u_-^\gamma + \frac{\bar{v}^2}{u_-} = \kappa u_+^\gamma + \frac{\bar{v}^2}{u_+}$$

Thus we obtain

$$C_2^* = \kappa u_+^\gamma + \frac{\kappa u_-(u_+^\gamma - u_-^\gamma)}{u_+ - u_-} \tag{4.12}$$

and

$$\bar{v}^2(u_+ - u_-) = \kappa u_- u_+ (u_+^\gamma - u_-^\gamma) \tag{4.13}$$

Let us stress that equation (4.13) is exactly the condition (4.9) in the special case $P(u) = \kappa u^\gamma$. Moreover, the entropy condition (4.7) becomes

$$\left[\frac{v^3}{2u^2} + 2\kappa u^\gamma \frac{v}{u}\right] \leq 0$$

In the special case of the scalar Saint-Venant , i.e. $P(u) = \frac{1}{2} g u^2$, equation (4.10) reads

$$v = \bar{v}, \quad \frac{1}{2} g u^3 - C_2 u + \bar{v}^2 = 0 \tag{4.14}$$

where

$$\bar{v}^2 = \frac{1}{2} g u_- u_+ (u_+ + u_-), \qquad C_2 = \frac{1}{2} g (u_+^2 + u_+ u_- + u_-^2) \tag{4.15}$$

Moreover, only entropy solutions are admitted, so that, from (4.7)

$$\frac{v_+}{u_+}(u_+ - u_-) \geq 0$$

Since $v_+, u_+ > 0$, then $u_- < u_+$ so that the jump condition describes the realistic phenomenon of the **hydraulic jump** consisting in an abrupt rise of the fluid surface and a corresponding decrease of the velocity.

Equation (4.14) admits a large class of entropy solutions satisfying the boundary conditions, given by all that piecewise constants functions in the form

$$(U_{\rm SV}, V_{\rm SV})(x) = \begin{cases} (u_-, \bar v) & x \in (-\ell, x_0) \\ (u_+, \bar v) & x \in (x_0, \ell) \end{cases}$$

where x_0 is a point in the interval $I = (-\ell, \ell)$ and $u_- < u_+$.

When the term of pressure $P(u)$ is not specified, stationary solutions to (4.3) satisfy

$$v = C_1, \qquad P(u)u - C_2 u + C_1^2 = 0$$

together with the boundary conditions for the couple (u, v). Hence, since $C_1 = v_\pm := \bar v$, there exists a stationary solution to (4.3) in the form

$$(U, V)(x) = \begin{cases} (u_-, \bar v) & x \in (-\ell, x_0) \\ (u_+, \bar v) & x \in (x_0, \ell) \end{cases}$$

if and only if both (4.7) and (4.9) are satisfied.

3. Existence and uniqueness of a stationary solution for the viscous system

For $\varepsilon > 0$, stationary solutions to (4.2) solve

$$\begin{cases} \partial_x v = 0 \\ \partial_x \left(\dfrac{v^2}{u} + P(u) \right) = \varepsilon \partial_x \left(u \partial_x \left(\dfrac{v}{u} \right) \right) \\ u(\pm\ell) = u_\pm, \quad v(\pm\ell) = v_\pm \end{cases}$$

The previous system can be rewritten as

$$\begin{cases} v = C_1 \\ \dfrac{C_1^2}{u} + P(u) = -\varepsilon \dfrac{v}{u} \partial_x u + C_2 \\ u(\pm\ell) = u_\pm, \quad v(\pm\ell) = v_\pm \end{cases} \tag{4.16}$$

where C_1, C_2 are integration constants. The second equation of the system reads

$$\varepsilon C_1 \partial_x u = -P(u)u + C_2 u - C_1^2$$

In order to prove the existence of a solution to (4.16), we state a Lemma that gives a description of the function

$$f(u) = -P(u)u + C_2 u - C_1^2$$

Lemma 4.2. *Let $f(u) = -P(u)u + C_2 u - C_1^2$, with $P(u)$ such that*

$$P(0) = 0, \quad P(+\infty) = +\infty, \quad P'(u) > 0, \quad P''(u) > 0 \tag{4.17}$$

for all $u \in \mathbb{R}^+$. Then, for all C_1, there exist at least a value C_2 such that there exist two positive solutions to the equation $f(u) = 0$.

Proof. Since $f(0) = -C_1^2$ and $f(+\infty) = -\infty$, we want to prove that there exists a value u^* such that

$$f(u^*) = \max_{\mathbb{R}^+} f, \quad f(u^*) > 0$$

We have

$$f'(u) = -P(u) + C_2 - P'(u)u, \quad f''(u) \quad = -2P'(u) - P''(u)$$

Without loss of generality, we now suppose $C_2 > 0$. Since $f'(0) = C_2 > 0$, $f'(+\infty) = -\infty$ and $f''(u) < 0$, there exists u^* such that $f'(u^*) = 0$. Now we ask for $f(u^*) > 0$ and we get

$$f(u^*) = -P(u^*)u^* + C_2u^* - C_1^2 = P'(u^*){u^*}^2 - C_1^2 > 0$$

so that we get a condition on the constant C_2, that is

$$C_1 < \sqrt{P'(u^*){u^*}^2} \tag{4.18}$$

where $u^* = u^*(C_2)$ since it solves $f'(u) = 0$. □

Remark 4.1. Condition (4.18) assures that, for every choice of $C_1 > 0$, there exists a function $\phi := \phi(C_1)$ such that, for every $C_2 > \phi(C_1)$, there exist two positive solutions to the equation $f(u) = 0$.

Let us go back to the study of solutions to (4.16). First of all, let us notice that, once the boundary conditions are imposed, the solution v to (4.16) is univocally determined by $C_1 = \bar{v}$, where $\bar{v} = v_\pm$.

Concerning the equation for u , a *positive connection* between u_- and u_+ (i.e. a positive solution to $\varepsilon\bar{v}\partial_x u = f(u)$ connecting u_- and u_+) exists only if $(u_-, u_+) \subset (u_1, u_2)$, where u_1 and u_2 are the positive zeros of $f(u)$.

Thus we are interested in studying the behavior of $f(u)$ as a function of C_2, trying to describe how the distance between u_1 and u_2 changes for different values of the constant C_2.

Lemma 4.3. *Let $P(u)$ such that (4.17) holds, let $f(u) = -P(u)u + C_2u - \bar{v}^2$, and let C_2 be such that (4.18) holds, so that there exist two positive solutions $u_1 < u_2$ to the equation $f(u) = 0$. Hence, given $u_\pm > 0$, the set $\mathcal{A}$ defined as*

$$\mathcal{A} := \{C_2 > 0 : u_1 < u_- < u_+ < u_2\}$$

is such that $\mathcal{A} = [\bar{C}_2, +\infty)$, for some $\bar{C}_2 > 0$.

Proof. Since $u_1 = u_1(C_2)$ and $u_2 = u_2(C_2)$, we want to show that $f(u, C_2)$ is an increasing function as a function of C_2. This implies that, if there exists a value C_2 such that

$$u_1 < u_- < u_+ < u_2$$

than, for all $C_2' > C_2$

$$u_1' < u_- < u_+ < u_2'$$

where u_1' and u_2' are the positive zeros of $f(u, C_2')$. We have

$$f(u, C_2) - f(u, C_2') = (C_2 - C_2')u$$

so that, since $u > 0$, if $C_2' > C_2$, then $f(u, C_2') - f(u, C_2) > 0$. Thus, to prove Lemma 4.3, we only need to prove that there exists at least a value $\bar{C}_2$ such that $u_1 < u_- < u_+ < u_2$ holds.

To estimate from above u_1, we write the equation of the line through the points $u = (0, -\bar{v}^2)$ and $u = (u^*, f(u^*))$, that is

$$v = -\bar{v}^2 + \frac{f(u^*) + \bar{v}^2}{u^*} u$$

Since u^* is such that $P(u^*) = C_2 - P'(u^*)u^*$ (i.e. $f'(u^*) = 0$), we get

$$v = -\bar{v}^2 + P'(u^*)u^* u$$

Hence $u_1 < \tilde{u}$, where $\tilde{u}$ is defined as

$$-\bar{v}^2 + P'(u^*)u^*\tilde{u} = 0 \quad \Rightarrow \quad \tilde{u} = \frac{\bar{v}^2}{P'(u^*)u^*}$$

To understand the behavior of $\tilde{u}$ for $C_2 \to +\infty$ we want to compute $\lim_{C_2 \to +\infty} u^*$. To this aim, let us define $\Phi(u) = P(u) + P'(u)u$. We have

$$\Phi(0) = 0, \quad \Phi(u^*) = C_2, \quad \Phi'(u) = 2P'(u) + P''(u)u > 0, \quad \Phi(u) \geq P(u)$$

Hence, since $P(+\infty) = +\infty$, $\Phi(+\infty) = +\infty$. Moreover there exists Φ^{-1} such that

$$u^* = \Phi^{-1}(C_2) \to +\infty \;\text{ as }\; C_2 \to +\infty$$

that is

$$\lim_{C_2 \to +\infty} u^* = +\infty$$

Thus we have proved that

$$0 < u_1 < \frac{\bar{v}^2}{P'(u^*)u^*} \to 0 \;\text{ as }\; C_2 \to +\infty$$

so that u_1 remains close to zero as C_2 becomes bigger. Furthermore, we also have

$$\Phi(u^*) \geq P'(u^*)u^* \geq P'(\bar{u})u^*, \quad \exists\, \bar{u}$$

Since $\Phi(u^*) \to 0$ as $C_2 \to 0$, we get

$$\lim_{C_2 \to 0} u^* = 0$$

On the other hand we know that $u_2 > u^*$, so that $u_2 \to +\infty$ as $C_2 \to +\infty$.

Hence, if we choose $\bar{C}_2$ large enough, then the amplitude of the interval (u_1, u_2) is such that $(u_-, u_+) \subset (u_1, u_2)$.

More precisely $\bar{C}_2$ has to be bigger than $\max\{C_2^*, C_2^{**}\}$, where C_2^* and C_2^{**} are such that either $-P(u_-)u_- + C_2^* u_- - \bar{v} = 0$ or $-P(u_+)u_+ + C_2^{**}u_+ - \bar{v} = 0$, that is either u_- or u_+ solve $f(u) = 0$.

□

Now let us define the region Σ of *admissible values* C_2, i.e. the set of values C_2 such that there exists two positive solution to the equation $f(u) = 0$ and lemma 4.3 holds. Σ is determined, in the plane $\{C_1, C_2\}$ by the equations

$$C_1 < \sqrt{P'(u^*)u^{*2}}, \quad C_2 > \frac{1}{u_-}C_1^2 + P(u_-), \quad C_2 > \frac{1}{u_+}C_1^2 + P(u_+)$$

Proposition 4.4. *Σ is the* **epigraph** *of an increasing function $g : \mathbb{R} \to \mathbb{R}$, that is*

$$\Sigma := \mathbf{epi}(g) = \{(C_1, C_2) : C_1 \in \mathbb{R}, C_2 \in \mathbb{R}, C_2 \geq g(x)\} \subset \mathbb{R} \times \mathbb{R}$$

Proof. In the plane (C_1, C_2) the functions $C_2 = \frac{1}{u_\pm}C_1^2 + P(u_\pm)$ define two parabolas. Moreover, setting $\varphi(C_2) := \sqrt{P'(u^*)}u^*$, we have

$$\lim_{C_2\to 0}\varphi(C_2) = 0, \qquad \lim_{C_2\to+\infty}\varphi(C_2) = +\infty, \qquad \varphi'(C_2) > 0$$

Thus the function $C_1 = \varphi(C_2)$ is an increasing function in the plane (C_1, C_2). Hence, $\frac{dg}{dC_1} > 0$ for all $C_1 > 0$.

For example, if there no exists $\bar{C}_1$ such that either $\varphi^{-1}(\bar{C}_1) = \frac{1}{u_-}\bar{C}_1^2 + P(u_-)$ or $\varphi^{-1}(\bar{C}_1) = \frac{1}{u_+}\bar{C}_1^2 + P(u_+)$, then

$$g(C_1) = \begin{cases} \dfrac{1}{u_+}C_1^2 + P(u_+) & \text{in } (0, \mathrm{C}_1^*) \\ \dfrac{1}{u_-}C_1^2 + P(u_-) & \text{in } (\mathrm{C}_1^*, +\infty) \end{cases} \tag{4.19}$$

where C_1^* is such that $\frac{1}{u_-}(C_1^*)^2 + P(u_-) = \frac{1}{u_+}(C_1^*)^2 + P(u_+)$.

Otherwise, if there are intersections between $\varphi^{-1}(C_1)$ and both $C_2 = \frac{1}{u_\pm}C_1^2 + P(u_\pm)$, g will be piecewise defined depending on the values of such intersections.

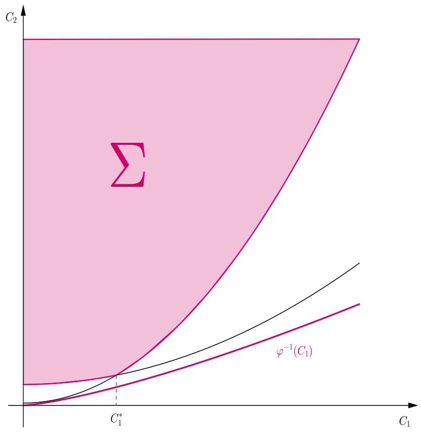


Figure 1. Plot of the region Σ of admissible values (C_1, C_2) when there are no intersections between φ^{-1} and the two parabolas. Σ is plotted in the plane (C_1, C_2), but we stress on the fact that the constant C_1 is univocally determined once the boundary values $v_\pm$ are imposed.

□

Let us go back to the problem of existence and uniqueness of a solution to (4.16); we have to prove that, once the boundary conditions are imposed , there always exists a 2ℓ-*connection* $u(x)$, i.e. a solution to

$$\varepsilon\bar{v}\partial_x u = -P(u)u + C_2 u - \bar{v}^2$$

such that $u(\pm\ell) = u_\pm$. Hence

$$2\ell = \varepsilon\bar{v}\int_{u_-}^{u_+} \frac{du}{-P(u)u + C_2 u - \bar{v}^2} := F(C_2) \tag{4.20}$$

Remark 4.2. When $\ell = +\infty$, the $+\infty$-connection is the so called *heteroclinic connection* , i.e. a solution to the stationary problem for $x \in \mathbb{R}$ tending to two fixed points for $x \to -\infty$ and $x \to +\infty$.

Concerning the function $F(C_2)$, we want to prove that there always exists a value C_2^* such that $F(C_2^*) = 2\ell$. We first notice that

$$F\big|_{\Gamma} = +\infty, \qquad \Gamma = \partial\Sigma$$

Next We can easily check that $F(C_2) \to 0$ as $C_2 \to +\infty$. Finally

$$\frac{dF}{dC_2} = -\varepsilon\bar{v}\int_{u_-}^{u_+} \frac{u}{(-P(u)u + C_2 u - \bar{v}^2)^2} du < 0$$

since $u, \bar{v} > 0$. Hence, given $\ell > 0$, there always exists C_2^* such that $F(C_2^*) = 2\ell$. Now we can prove the following

Theorem 4.5. *For any $\ell > 0$, let us consider the following problem*

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x \left(\dfrac{v^2}{u} + P(u)\right) = \varepsilon\partial_x \left(u\partial_x \left(\dfrac{v}{u}\right)\right) \\ v(\pm\ell) = v_\pm, \quad u(\pm\ell) = u_\pm \end{cases} \tag{4.21}$$

where $P(u)$ is such that (4.17) holds and $v_\pm, u_\pm > 0$ satisfies

- $v_+ = v_- = \bar{v}$
- $\bar{v}^2(u_+ - u_-) = u_- u_+(P(u_+) - P(u_-))$.
- *The entropy condition*

$$\left[\frac{v^3}{2u^2} + 2P(u)\frac{v}{u}\right] \leq 0$$

Then there exists a unique stationary solution to (4.21).

Proof. Lemma 4.3 assures that, given $u_\pm > 0$, there exists a set of values C_2 such that $u_1 < u_- < u_+ < u_2$, so that there exists a *positive connection* $u(x)$ satisfying the boundary conditions. Moreover, from the study of the function $F(C_2)$, we have proved that, fixed $C_1 \equiv \bar{v}$ and $\ell > 0$, there exists a unique value $C_2^* \in \mathring{\Sigma}$ such that

$$\varepsilon\bar{v}\int_{u_-}^{u_+} \frac{du}{-P(u)u + C_2^* u - \bar{v}^2} = 2\ell$$

Hence there exists a unique stationary solution $(\bar{U}, \bar{V})$ to (4.21), where $\bar{V} = C_1$ and $\bar{U}$ is a *positive connection* between u_- and u_+, of "length" 2ℓ. $\square$

The case of Power Law. Let us consider a term of pressure given by $P(u) = \kappa u^\gamma$, $\gamma > 1$, that is such that (4.17) holds. The stationary problem (4.16) reads

$$\begin{cases} v = C_1 \\ \varepsilon C_1 \partial_x u = -\kappa u^{\gamma+1} + C_2 u - C_1^2 \\ u(\pm\ell) = u_\pm, \quad v(\pm\ell) = v_\pm \end{cases} \tag{4.22}$$

and $f(u) = -\kappa u^{\gamma+1} + C_2 u - C_1^2$. We have

$$f'(u) = -\kappa(\gamma+1)u^\gamma + C_2 = 0 \quad \Leftrightarrow \quad u^* = \left(\frac{C_2}{\kappa(1+\gamma)}\right)^{\frac{1}{\gamma}}$$

Following the idea of the proof of Lemma 4.2, the request $f(u^*) > 0$ translates into

$$C_1 < \left(\frac{\gamma}{(1+\gamma)^{1+\frac{1}{\gamma}}}\frac{1}{\kappa^{\frac{1}{\gamma}}}\right)^{1/2} \cdot C_2^{\frac{1}{2}(1+\frac{1}{\gamma})} \tag{4.23}$$

Let us stress that condition (4.23) is exactly (4.18) in the case $P(u) = \kappa u^\gamma$. Furthermore, Lemma 4.3 is still verified and we have the following estimates for the two positive solutions to $f(u) = 0$

$$u_1 < \frac{\gamma+1}{\gamma} \cdot \frac{\bar{v}^2}{C_2}, \quad u_2 > \left(\frac{1}{\kappa(1+\gamma)}\right)^{\frac{1}{\gamma}} \cdot C_2^{\frac{1}{\gamma}}$$

where In this case, the region Σ of *admissible values* (C_1, C_2), is determined by the relation (4.23) together with $C_2 > \frac{1}{u_\pm}C_1^2 + u_\pm^\gamma$, so that Proposition 4.4 is obviously still verified.

The Saint-Venant system. An interesting case in which we can explicitly develop calculations is the Saint-Venant system, where the term of pressure $P(u)$ is given by the quadratic formula $P(u) = \frac{1}{2}gu^2$, $g > 0$.

In this case stationary solutions solve

$$v = \bar{v}, \quad \varepsilon\bar{v}\partial_x u = -\frac{1}{2}gu^3 + C_2 u - \bar{v}^2$$

where, as usual, $\bar{v} = v_- = v_+$. The condition (4.23) for the existence of two positive solution u_1 and u_2 reads $C_2^3 > 27/8gC_1^4$. Let us stress that, from the Cardano formula for the equations of third degree in the form $u^3 + pu + q = 0$, we know that there exist three real solutions if and only if

$$\frac{q^2}{4} + \frac{p^3}{27} < 0 \quad \Leftrightarrow \quad C_2^3 > \frac{27}{8}gC_1^4$$

Moreover, since $f(0) = -\bar{v}^2$ and $C_2 > 0$, we can explicitly show that $u_0 < 0 < u_1 < u_2$, where u_0 is the third (negative) root of the equation $f(u) = 0$.

Figure 2 shows an explicit plot of $f(u)$ for different choice of C_2. We can see how the first positive zero u_1 remains close to zero while the value of u_2 becomes bigger as $C_2 \to \infty$. From Figure 2 we can also see that the interval (u_-, u_+) is included or not inside (u_1, u_2), depending on the choice of C_2.

Figure 3-4 show the solution to the equation $\varepsilon\bar{v}\partial_x u = f(u)$ for different choices of the boundary values.

For the Saint-Venant problem, it is also possible to explicitly plot the region Σ, contained in the positive half-plane $\{C_1 > 0, C_2 > 0\}$. The equations for Σ reads

$$C_2 > \frac{3}{2}\sqrt[3]{g}\, C_1^{4/3}, \quad C_2 > \frac{1}{u_-}C_1^2 + \frac{1}{2}gu_-^2, \quad C_2 > \frac{1}{u_+}C_1^2 + \frac{1}{2}gu_+^2$$

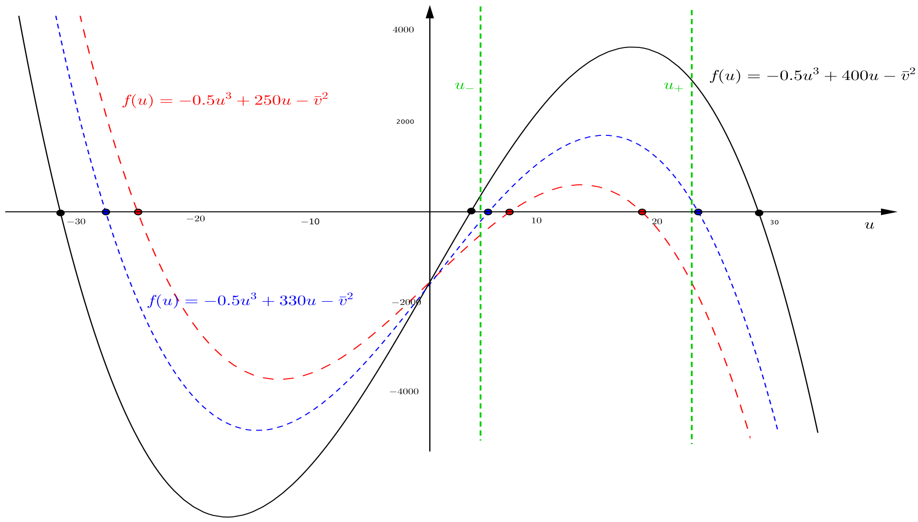


FIGURE 2. Plot of $f(u) = -\frac{1}{2}gu^3 + C_2u - \bar{v}^2$ for fixed $\bar{v}$ and multiple choice of C_2, together with $u_- = 5$ and $u_+ = 25$.

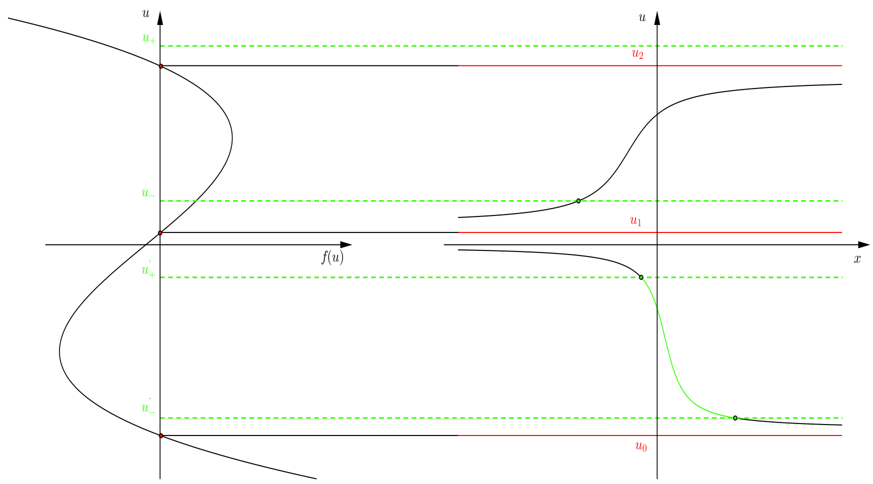


FIGURE 3. Plot of the solutions to $\varepsilon\bar{v}\partial_x u = f(u)$. The first choice for u_- and u_+ is such that $u_+ > u_2$. In the plane (x, u) we can see that the solution starting from $u(-\ell) = u_-$ can not reach u_+, since u_2 is an equilibrium solution for the equation. On the other hand, if we choose u'_-, u'_+ such that $(u'_-, u'_+) \subset (u_0, u_1)$, we have a *negative connection.*

We want to know if there exists a value $\bar{C}_1$ such that $\frac{3}{2}\sqrt[3]{g}\,\bar{C}_1^{4/3} = \frac{\bar{C}_1^2}{u_\pm} + \frac{1}{2}gu_\pm^2$. To solve the equation, let us set $\xi = \bar{C}_1^{1/3}$, and let us study

$$F(\xi) = \frac{1}{u_\pm}\xi^6 - \frac{3}{2}\sqrt[3]{g}\,\xi^4 + \frac{1}{2}gu_\pm^2$$

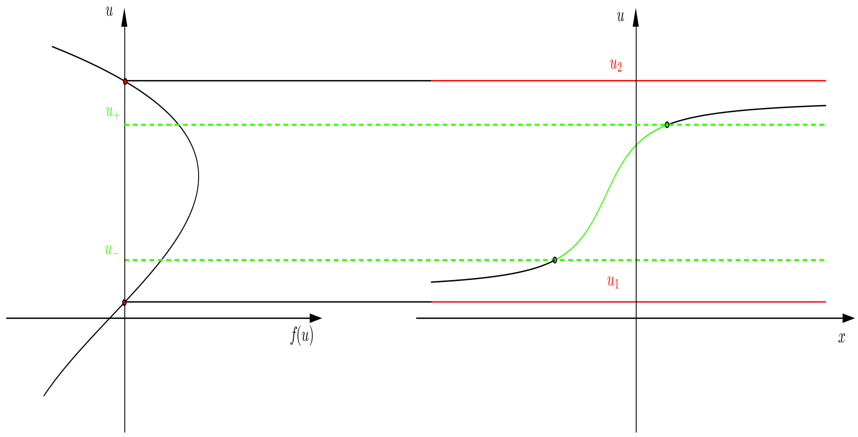


FIGURE 4. Plot of the solutions to $\varepsilon\bar{v}\partial_x u = f(u)$. In this case $(u_-, u_+) \subset (u_1, u_2)$, so that there exists a positive connection between u_- and u_+. Moreover u_1 and u_2, being zeros of the function $f(u)$, are equilibrium solutions for the equation.

We have

$$F(0) > 0, \quad F(+\infty) = +\infty, \quad F'(\xi) = \frac{6\xi^3}{u_\pm}(\xi^2 - \sqrt[3]{g}\; u_\pm)$$

Thus $F'(\xi) = 0$ if and only if $\bar{\xi}_\pm = \sqrt[6]{g}\;\sqrt{u_\pm}$. Moreover $F(\bar{\xi}_\pm) = 0$, so we can define

$$\bar{C}_{1,-} := \sqrt{g}\; u_-{}^{3/2}, \quad \bar{C}_{1,+} := \sqrt{g}\; u_+{}^{3/2}$$

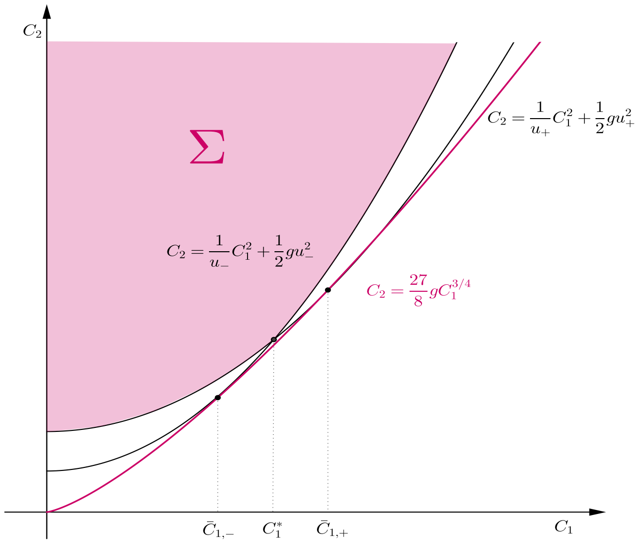
as the unique solutions to the equations $\frac{3}{2}\sqrt[3]{g}\; C_1^{4/3} = \frac{C_1^2}{u_\pm} + \frac{1}{2}gu_\pm^2$. Furthermore, if we define C_1^* such that

$$\frac{1}{u_-}C_1^{*2} + \frac{1}{2}gu_-^2 = \frac{1}{u_+}C_1^{*2} + \frac{1}{2}gu_+^2$$

we have $\bar{C}_{1,-} < C_1^* < C_{1,+}$, so that $\partial\Sigma := g(C_1)$ is defined as

$$g(C_1) = \begin{cases} \dfrac{1}{u_+}C_1^2 + \dfrac{1}{2}gu_+^2 & \text{in } (0, \mathrm{C}_1^*) \\ \dfrac{1}{u_-}C_1^2 + \dfrac{1}{2}gu_-^2 & \text{in } (\mathrm{C}_1^*, +\infty) \end{cases} \tag{4.24}$$

Figure 5 shows the region Σ when the boundary conditions are imposed. We can explicitly see that Proposition 4.4 holds. The region Σ is plotted in the plane (C_1, C_2), but we stress on the fact that the constant C_1 is univocally determined once the boundary values $v_\pm$ are imposed.



4. The Power Law System with a General Viscosity

To complete this Chapter, let us consider the more general system

$$\begin{cases} \partial_t u + \partial_x v = 0, & x \in I, t \geq 0 \\ \partial_t v_t + \partial_x \left(\dfrac{v^2}{u} + P(u) \right) = \varepsilon \partial_x \left(\nu(u) \partial_x \left(\dfrac{v}{u} \right) \right) & \\ u(\pm\ell) = u_\pm, \quad v(\pm\ell) = v_\pm & t \geq 0 \end{cases} \tag{4.25}$$

where the space variable x belongs to the interval of the real line $I = [-\ell, \ell]$. The term of pressure $P(u)$ still verifies

$$P(0) = 0, \quad P(+\infty) = +\infty, \quad P'(s), P''(s) > 0 \quad \forall s > 0 \tag{4.26}$$

while the viscosity term is such that $\nu(u) > 0$ for all $u > 0$.

We will show that, in this case as well, it is possible to prove the existence and uniqueness of a stationary solution with the techniques of the previous section. The stationary problem reads

$$\begin{cases} \partial_x v = 0 \\ \partial_x \left(\dfrac{v^2}{u} + P(u) \right) = \varepsilon \partial_x \left(\nu(u) \partial_x \left(\dfrac{v}{u} \right) \right) \\ u(\pm\ell) = u_\pm, \quad v(\pm\ell) = v_\pm \end{cases} \tag{4.27}$$

so that $v = C_1 \equiv \bar{v}$, and u has to satisfy

$$\bar{v}\varepsilon \frac{\nu(u)}{u} \partial_x u = -P(u)u + C_2 u - \bar{v}^2 \tag{4.28}$$

Let us define

$$\Phi(u) = \int_0^u \frac{\nu(s)}{s} ds$$

Since $\nu > 0$, then

$$\Phi(u) > 0, \quad \Phi'(u) = \frac{\nu(u)}{u} > 0, \quad \forall u > 0$$

where Φ' means the derivative of Φ with respect to u. Thus, equation (4.28) can be rewritten as

$$\bar{v}\varepsilon\partial_x\Phi(u) = -P(u)u + C_2u - \bar{v}^2$$

Now, let us define, as usual, $f(u) = -P(u)u + C_2u - \bar{v}^2$; with the change of variable $w = \Phi(u)$, and since $\Phi(u)$ is invertible, we have

$$\varepsilon\bar{v}\partial_x w = (f \circ \Phi^{-1})(w) \equiv g(w)$$

We want to study the function $g(w)$ as a function of w. First of all all we know that the function f is increasing for $u \in [0, u^*)$ and decreasing for $u \in (u^*, +\infty)$, where u^*, implicitly defined as

$$P(u^*) = C_2 - P'(u^*)u^*$$

is such that $f'(u^*) = 0$. Moreover , if C_2 is such that

$$f(u^*) > 0 \quad \Leftrightarrow \quad P'(u^*)u^{*2} > \bar{v}^2 \tag{4.29}$$

then there exist two positive solution to the equation $f(u) = 0$. Given $\nu(u) > 0$, since Φ is defined only for $u > 0$ and since $\Phi(u) > 0$, $\Phi'(u) > 0$, we have

$$\Phi^{-1}(w) > 0, \quad (\Phi^{-1})'(w) = \frac{1}{\Phi'(u)} > 0$$

so that also Φ^{-1} is a positive increasing function. Now, let us consider $g(w) = (f \circ \Phi^{-1})(w)$; we want to show that there still exist $w_1, w_2 > 0$ such that $g(w_1) = g(w_2) = 0$. Since $f(u_1) = f(u_2) = 0$, w_1 and w_2 has to be such that

$$\Phi^{-1}(w_1) = u_1, \quad \Phi^{-1}(w_2) = u_2 \tag{4.30}$$

Since $\Phi^{-1}(0) = 0$ and $(\Phi^{-1})' > 0$, then there exist and they are unique w_1 and w_2 such that (4.30) holds. Hence, $g(w)$ has exactly two positive zero for all the choices of $\nu(u) > 0$. Furthermore

$$g'(w) = [f(\Phi^{-1}(w))]' = f'(\Phi^{-1}(w)) \cdot (\Phi^{-1})'(w)$$

so that the sign of g' is univocally determined by the sign of f'. Therefore, if w^* is such that $\Phi^{-1}(w^*) = u^*$, then

$$g'(w^*) = 0, \quad g'(w) > 0 \text{ for } w \in [0, w^*), \quad g'(w) < 0 \text{ for } w \in (w^*, +\infty)$$

Example 4.6 (The Saint-Venant system)**.** When $P(u) = \frac{1}{2}gu^2$, the stationary equation (4.27) for u reads

$$\bar{v}\varepsilon\partial_x\Phi(u) = -\frac{1}{2}gu^3 + C_2u - \bar{v}^2$$

where $\Phi(u) = \int_0^u \frac{\nu(s)}{s}ds$. Now, let us consider a simple case where $\nu(u) = Cu^\alpha$, $\alpha > 0$, and let us plot the function $g(w) = (f \circ \Phi^{-1})(w)$. We have

$$\Phi(u) = C\int_0^u s^{\alpha-1}ds = \frac{C}{\alpha}u^\alpha, \quad \Phi^{-1}(w) = \left(\frac{\alpha}{C}u\right)^{\frac{1}{\alpha}}$$

so that

$$g(w) = -\frac{1}{2}g\left(\frac{C}{\alpha}\right)^{\frac{3}{\alpha}} w^{\frac{3}{\alpha}} + \left(\frac{C}{\alpha}\right)^{\frac{1}{\alpha}} C_2 w^{\frac{1}{\alpha}} - \bar{v}^2$$

Figure 5 shows the plot of $g(w)$ for different choice of $\nu(u)$, compared with the plot of $f(w)$ (where $\nu(u) = u$). More precisely, the red line and

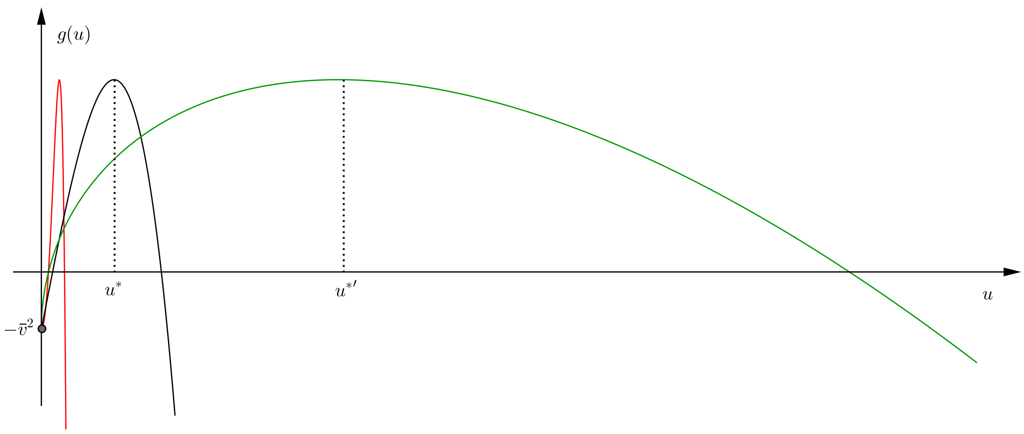


FIGURE 5. Plots of different $g(w)$ with $g = 1$, $C_2 = 400$ and $\bar{v}^2 = 1000$. The red line plots $g(w) = -\frac{1}{2}w^6 + 400w^2 - 1000$, the green line plots $g(w) = -\frac{1}{2}w^{3/2} + 400\sqrt{w} - 1000$, while the black line plots $f(w) = -\frac{1}{2}w^3 - 400w - 1000$.

the green line plot $g(w)$ with $\nu(u) = \frac{\sqrt{s}}{2}$ and $\nu(u) = 2s^2$ respectively. The picture shows, as we have proved, that the monotonicity of the function is preserved as well as the existence of two positive zeros.

5. Existence and uniqueness of a stationary solution

We are interested in studying the existence and uniqueness of the solution to the stationary problem (4.27). We have already shown that, once the boundary conditions for the function v are imposed, problem (4.27) reads

$$\begin{cases} v = \bar{v}, \quad v_\pm = \bar{v} \\ \varepsilon\bar{v}\partial_x w = g(w), \quad w(\pm\ell) = \Phi(u_\pm) \end{cases}$$

where $g(w) = (f \circ \Phi^{-1})(w)$, $f(u) = -P(u)u + C_2 u - \bar{v}^2$.

We first notice that condition (4.29) for the existence of two positive solution to the equation $f(u) = 0$, assures that also $g(w)$ has two positive zeros. Indeed

$$g'(w) = f'(\Phi^{-1}(w)) \cdot (\Phi^{-1})'(w) = \frac{f'(\Phi^{-1}(w))}{\Phi'(w)}$$

Thus $g'(w) = 0$ if and only if $w = w^*$, where w^* is such that $\Phi^{-1}(w^*) = u^*$. Furthermore, following the idea of lemma 4.2, we have

$$g(w^*) > 0 \;\Leftrightarrow\; f(\Phi^{-1}(w^*)) > 0 \;\Leftrightarrow f(u^*) > 0$$

so that we obtain (4.23). Moreover, a *positive connection* between $\Phi(u_-)$ and $\Phi(u_+)$ (i.e. a positive solution to $\varepsilon\bar{v}\partial_x w = g(w)$ connecting $\Phi(u_-)$ and $\Phi(u_+)$) exists only if $(\Phi(u_-), \Phi(u_+)) \subset (w_1, w_2)$, where w_1 and w_2 verifies $g(w_1) = g(w_2) = 0$.

Lemma 4.7. *Let $P(u)$ such that (4.26) holds, let $g(w) = (f \circ \Phi^{-1})(w)$ where $f(u) = -P(u)u + C_2 u - \bar{v}^2$, and let C_2 be such that (4.29) holds, so that* *there exist two positive solutions $w_1 < w_2$ to the equation $g(w) = 0$. Thus, given $u_\pm > 0$, the set $\mathcal{A}$ defined as*

$$\mathcal{A} := \{C_2 > 0 : w_1 < \Phi(u_-) < \Phi(u_+) < w_2\}$$

is such that $\mathcal{A} = [\bar{C}_2, +\infty)$, $\bar{C}_2 > 0$.

Proof. Following the idea of the proof of Lemma 4.3, since $g(w, C_2) = f(\phi^{-1}(w, C_2))$ and $\Phi^{-1}(w)$ is an increasing function that does not depend on C_2, then $g(w, C_2)$ is an increasing function in the variable C_2.

Thus we only need to prove that there exist a value $\bar{C}_2$ such that $w_1 < \Phi(u_-) < \Phi(u_+) < w_2$ holds. We know that $g(0) = -\bar{v}^2 < 0$ and $g'(w) > 0$ for all $w \in [0, w^*)$. Moreover

$$g(w^*) = f(\Phi^{-1}(w^*)) = f(u^*) > 0$$

so that $w_1 \in (0, w^*)$. Furthermore, we ask for

$$g\left(\frac{2\bar{v}^2}{C_2}\right) = f\left(\Phi^{-1}\left(\frac{2\bar{v}^2}{C_2}\right)\right) > 0 \tag{4.31}$$

so that $w_1 < \frac{2\bar{v}}{C_2}$. Condition (4.31) can be rewritten as

$$f\left(\Phi^{-1}\left(\frac{2\bar{v}^2}{C_2}\right)\right) > f(u_1) = 0$$

that is, since $\Phi^{-1}(w_1) = u_1$

$$f\left(\Phi^{-1}\left(\frac{2\bar{v}^2}{C_2}\right)\right) > f\left(\Phi^{-1}(w_1)\right)$$

Since f and Φ^{-1} are increasing function in the interval $[0, u^*)$ and $[0, w^*)$ respectively, we obtain a condition for the constant C_2 that is $2\bar{v}^2/C_2 > w_1$. If this condition holds, then we have

$$0 < w_1 < \frac{2\bar{v}^2}{C_2}$$

This formula shows that $w_1 \to 0$ as $C_2 \to +\infty$. On the other hand we know that $u_2 > u^*$ where u^* is such that $f(u^*) = \max_{\mathbb{R}} f$. Hence

$$\Phi^{-1}(w_2) > \Phi^{-1}(w^*) \quad \Rightarrow \quad w_2 > \Phi(u^*)$$

Since $u^* \to +\infty$ as $C_2 \to +\infty$, and since Φ is an increasing and continuous function, we know that $\Phi(u^*) \to \Phi(+\infty) = +\infty$ as $C_2 \to +\infty$. Hence $w_2 \to +\infty$ as $C_2 \to +\infty$.

$\square$

Remark 4.3. The region Σ of admissible values C_2, i.e. the set of values C_2 such that there exists two positive solution to the equation $g(w) = 0$ and Lemma 4.7 holds, is determined, in the plane $\{C_1, C_2\}$, by the equations

$$C_1^2 < P'(u^*){u^*}^2, \quad g(\Phi(u_\pm)) > 0, \quad C_2 < \frac{2{C_1}^2}{w_1}$$

Proposition 4.8. *The region Σ of admissible values is the* **epigraph** *of an increasing function $g : \mathbb{R} \to \mathbb{R}$.*

Proof. In the proof of Proposition 4.4, we have already seen that the function $\varphi(C_2) = \sqrt{P'(u^*)}u^*$ is an increasing function. Moreover, the condition $g(\Phi(u_\pm)) > 0$ is equivalent to

$$g(\Phi(u_\pm)) = (f \circ \Phi^{-1})(\Phi(u_\pm)) = f(u_\pm) > 0$$

so that we obtain the usual conditions

$$C_2 > \frac{1}{u_-}C_1^2 + P(u_-), \quad C_2 > \frac{1}{u_+}C_1^2 + P(u_+)$$

Finally, the function $\Psi(C_2) = \frac{2{C_1}^2}{w_1(C_2)}$ is such that

$$\lim_{C_2 \to +\infty} \Psi(C_2) = +\infty, \quad \Psi'(C_2) = -\frac{2{C_1}^2}{w_1^2} w_1' > 0$$

since $w_1(C_2)$ is a decreasing function. Thus the function g is an increasing function, since it is obtained by matching increasing functions.

□

The final step is to investigate the existence of a 2ℓ-*connection*, i.e. a solution to $\varepsilon\bar{v}\partial_x w = g(w)$ such that $w(\pm\ell) = \Phi(u_\pm)$. Thus

$$2\ell = \varepsilon\bar{v} \int_{\Phi(u_-)}^{\Phi(u_+)} \frac{dw}{(f \circ \Phi^{-1})(w)} := G(C_2)$$

We first noticed that $G\big|_{\partial\Sigma} = +\infty$. From the study of $G(C_2)$, we can prove that there always exists a value C_2^* such that $G(C_2^*) = 2\ell$. Indeed, we can easily see that

$$\lim_{C_2 \to +\infty} G(C_2) = 0, \qquad \frac{dG}{dC_2} < 0$$

for all $C_2 > 0$.

Hence, Theorem 4.5 can be generalized to a Shallow Water's type system (4.25), with a viscosity term $\nu(u) > 0$ for all $u > 0$.

6. Perspectives

The study of the existence of stationary solutions for a partial differential equation is usually strictly related to the study of their asymptotic stability. Indeed, it is expected that the time dependent solution will eventually tend towards the steady state.

There is a broad range of techniques to investigate such kind of problem. In our contest, the main difficulty stems from the fact that the problem under consideration is an hyperbolic system, so that the spectrum of the linearized operator around a steady state can be contained in the complex plane. However, if one succeed in proving that the real part of the eigenvalues is negative, this would be enough to prove the asymptotic stability of the equilibrium solution.

To this aim, the idea should be to linearize the original system around the steady state, and to perform a spectral analysis. The main problem is that, in this case, the stationary solution is not constant in space, so that one has to deal with a system in the form

$$\mathcal{L}Y = \lambda Y, \quad Y \in \mathbb{R}^2$$

where the linear operator $\mathcal{L}$ is a second order operator with variable coefficients

$$\mathcal{L} = A(x)\partial_x^2 + B(x)\partial_x + C(x)$$

Another possible way to deal with the problem is to try to find a Liapunov function for the original system. For example, system (4.2) admits a mathematical entropy which is also a physical energy

$$\mathcal{E}(u,v) = \frac{v^2}{2u} + P(u)$$

that satisfies the "energy inequality"

$$\partial_t \left(\frac{v^2}{2u} + P(u) \right) + \partial_x \left(\frac{v}{u} \left(\frac{v^2}{2u} + 2P(u) - \varepsilon v \partial_x \left(\frac{v}{u} \right) \right) \right) = -\varepsilon u \left[\partial_x \left(\frac{v}{u} \right) \right]^2$$

For example, in the Saint-Venant case, i.e. $P(u) = \frac{1}{2}gu^2$, a candidate Liapunov function could be

$$\bar{\mathcal{E}}(u,v) = \int_{-\ell}^{\ell} \left\{ \frac{(v-\bar{V})^2}{2u} + \frac{1}{2}g(u-\bar{U})^2 \right\}$$

The next step is to prove that its time derivative along the solutions to (4.2) is negative definite. The main difficulty here is to deal with the sign of the solutions u, v and their derivatives at the boundary values.

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