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Doctoral Thesis

PDE and Dynamical Methods to Weakly Coupled Hamilton-Jacobi Systems

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To my parents Abdallah and Zahra,
my husband Mustafa.

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Abstract

This thesis is concerned with degenerate weakly coupled systems of Hamilton-Jacobi equations, imposed on flat torus, using both PDE and dynamical methods. The PDE approach relies essentially on control and viscosity solutions tools. Our main contribution is the construction of an algorithm through which we can get a critical solution to the system as limit of monotonic sequence of subsolutions and we also adapt the algorithm to non compact setting. Moreover, we get a characterization of isolated points of the Aubry set and establish semi-concavity type estimates for critical subsolution. A crucial step in our work is to reduce our analysis from systems into either scalar Eikonal equations or discounted ones. Whereas, in the dynamical approach we use the random frame introduced in [26] to provide a cycle condition characterizing the points of Aubry set. This generalizes a property already known in the scalar case.

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Chapter 1

General Introduction

1.1 Context and background

The main purpose of the present thesis is to recover some crucial PDE and dynamical facts from the theory of Hamilton Jacobi scalar equations and generalize it to systems, using the techniques of viscosity solutions and tools in a suitable probabilistic framework which govern some switching between equations. More precisely, we are interested in a family of Hamilton-Jacobi systems of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \quad \text{in } \mathbb{T}^N \quad \text{for every } i \in \{1, 2, ..., m\}, \tag{HJα}$$

where α is a real constant, $H_1, \cdots, H_m$ are continuous Hamiltonians defined on $\mathbb{T}^N \times \mathbb{R}^N$, convex and coercive in the momentum variable, and $A := (a_{ij})$ is an $m \times m$ coupling matrix satisfying

$$a_{ij} \leq 0 \text{ for every } j \neq i,$$

and A is diagonally dominant, i,e

$$\sum_{j=1}^{m} a_{ij} \geq 0 \text{ for any } i \in \{1, 2, ..., m\}.$$

Throughout the thesis we will be interested in the case where a degeneracy condition is assumed, namely

$$\sum_{j=1}^{m} a_{ij} = 0 \text{ for any } i \in \{1, 2, ..., m\}.$$

We further assume that the matrix irreducible, meaning roughly speaking, that the coupling is non-trivial and the system cannot split into independent subsysytems.

To frame the problem in the literature, we should mention that the first papers dealing with such a kind of systems with a viscosity solution approach are [16], [21], [24]. Here the focus is on the **non-degenerate case**, where every row sum of the coupling matrix is assumed to be strictly positive. It is introduced the class of monotone systems, which contain as special instance the non–degenerate weakly coupled systems. Existence, and uniqueness of viscosity solution for such systems are established. At this level, it is also important to mention that when adding a suitable irreducibility condition on the coupling matrix, the non degenerate case is achieved even if the sum of only one row is strictly positive. In the scalar case, $m = 1$, where the matrix reduces to a scalar, the non degeneracy amounts ensures that this scalar is strictly positive. Hence the corresponding scalar equation is the discounted Hamilton–Jacobi equation which, as well known, can be uniquely solved on the whole torus, and the solution is the value function of a related control problem, see [12], [22].

In our work we are specifically interested in the **degenerate case**, which corresponds in the scalar case to a single equation, not depending on the unknown function and classified as of Eikonal type. In this scalar setting a rich theory has been developed by linking PDE facts to geometrical/dynamical properties. Representation formula for (sub) solutions have been provided through minimization of a suitable action functional, showing, among other things, the existence of an unique value of α, named a critical value, for which viscosity solutions do exist, and the existence of subsolution for any value greater or equal to the critical one. It is also proved the existence of a distinguished closed set for which the obstruction in getting subsolutions below critical value concentrates, named Aubry set. A PDE as well as dynamical characterization of the Aubry set is provided see [18]. This body of results is a part of the so-called weak KAM theory, see [17]. Several facts from weak kAM theory were generalized to the case of weakly coupled Hamilton-Jacobi systems and will be presented in what follows according to two approaches: PDE and dynamical approach.

We start with the PDE approach. It has been actually a merit of [8, 29, 27] to first realize and point out that under the aforementioned assumptions on the coupling matrix, some phenomena, already occurring in the Eikonal scalar case, also take place for systems, and can be analyzed in the spirit of the weak KAM theory. In these papers it has been proved the existence of a unique constant $\alpha \in \mathbb{R}$ for which the system (HJα) admits viscosity solutions. Such a quantity is qualified as critical. It is characterized as the minimal $\alpha \in \mathbb{R}$ for which the corresponding weakly coupled system admits viscosity subsolutions.

A significant forward step in the analysis of systems at the critical level has been recently taken by Davini-Zavidovique in [15], proving that similar to what happens in the critical scalar case, the obstruction in getting globally strict subsolution is not spread over the whole torus, but instead concentrates on a distinguished closed set named after Aubry. From a PDE point of view the Aubry set is then defined with the crucial property that the maximal critical subsolution (i.e., a subsolution to the system with α equals to the critical value) taking a given value, among admissible ones, at any fixed point of the Aubry set is indeed a critical solution.

The aforementioned admissibility refers to the fact that there is a restriction in the values that a subsolution of the system can assume at any given point. This is a further relevant property pointed out in [15], which genuinely depends on the vectorial structure of the problem and has no counterpart in the scalar case. Due to stability properties of viscosity subsolutions and the convex nature of the problem, these admissible values make up a closed convex set at any point y of the torus.

All the above results belong to the PDE side of the theory, and are solely obtained by means of PDE techniques. In the control literature, it is worth pointing out that the first contributions in the framework are of the so–called Hierarchical control, where the coupling matrix represents the hierarchy. Recently, a deep dynamical and variational approach, integrating the PDE methods, was brought in by by Mitake, Siconolfi, Tran and Yamada in [26] . This angle allowed detecting the stochastic character of the problem. The effectiveness of their approach is demonstrated by recovering some crucial facts holding in the scalar case and intertwining between PDE and dynamical aspects which is at the core of weak KAM theory. Namely, they adapted the action functional for the system and they used it to fully characterize critical and supercritical subsolutions of the system. Moreover, they used the functional to give a dynamical formulation of the property of being admissible value at a given point. By this way they also provided a representation formula for critical solutions taking a prescribed admissible value at a given point of the Aubry set. The crucial step in their analysis was to put the problem in a suitable random frame, exploiting the fact that the coupling matrix under our assumptions being generator of a semigroup of stochastic matrices.

1.2 Contributions

As pointed out in the previous section, Davini and Zavidovique proved the existence of a solution for the Hamilton-Jacobi system at the critical level. Our main achievement is the construction of an algorithm through which we can get such a critical solution starting from a given subsolution. The construction is based on control-theoretic techniques and its key tool is the reduction to discounted Hamilton-Jacobi equation, where existence and comparison results hold. We also adapt the algorithm in the non- compact setting ($\mathbb{R}^N$), to get solutions at any critical/supercritical levels. Moreover, we introduce the notion of equilibrium points for weakly coupled Hamilton-Jacobi systems and prove that any isolated point of the Aubry set is equilibrium. We also show that any critical subsolution is strictly differentiable at every isolated point of the Aubry set. This generalizes a known fact holding in the scalar Eikonal case. Following Fathi-Siconolfi in [18], we establish a semi-concavity type estimates for critical subsolutions. Namely, we prove that the superdifferential of any solution of the critical system is nonempty at any point of the torus and the same property holds true for any critical subsolution on the Aubry set. The idea of the proof is to reduce the analysis of system into scalar Eikonal equation and then exploit the regularity properties of critical subsolutions holding in the scalar case. All the above results pertain to the PDE side of the thesis and are presented in chapter 5.

The geometric/dynamical characterization of the Aubry set is so far missed and this is our primary task in chapter 4 of the thesis. Following the dynamical approach in [26], we precise the random frame of our problem, which does not introduce an abstract probability space. This makes the presentation self-contained and readable by people having no background in probability, only some basic knowledge of measure theory is in fact necessary. To provide a dynamical characterization of the Aubry set, we employ the action functional introduced in [26] in relation to the systems and we use random cycles defined on intervals with a stopping time, say τ, as right endpoint, which we call τ–cycles. However a first serious difficulty is that we need a strict positiveness of the stochastic matrix $e^{-A\tau}$. To overcome this difficulty we introduce the notion of stopping times strictly greater than a positive constant ϵ, denoted by $\tau \gg \epsilon$. This produces an ϵ advantage in the filtration which guarantee that $e^{-A\tau}$ is positive and above all is essential in proving our main result. A second issue is that we use stopping times $\tau \gg \epsilon$ to provide a new characterization of admissible values which is used to get our desired result. The main output is presented in chapter 4 in two versions, Theorems 4.4.3, 4.4.4, with the latter one, somehow more geometrically flavored, exploiting the notion of characteristic vector of a stopping time.

1.3 Organization

The thesis is organized as follows:

• chapter 2 presents the basic tools and material used through out the thesis. It also gives a brief overview of results on scalar Hamilton-Jacobi equations.

• chapter 3 collects basic results on weakly coupled Hamilton-Jacobi systems, both from a PDE and dynamical point of views.

• chapter 4 provides a dynamical/geometric characterization of the Aubry set by the behavior of action functional on cycles. This work has been submitted to Communications in Contemporary Mathematics and presented in [19].

• chapter 5 is devoted to establish further PDE properties about weakly coupled Hamilton-Jacobi systems. Namely, we prove existence results in both compact and non-compact setting. Moreover, we establish semi-concavity type estimates for subsolutions. This chapter is based on the e-print [34].

• Appendix A collects some elementary algebraic results on coupling and stochastic matrices.

• Appendix B provides a self contained proof about using Lipschitz functions as test functions for viscosity solutions.

Chapter 2

Overview of results on Hamilton-Jacobi equations

This chapter is devoted to the basic theory of Hamilton Jacobi equation of the form

$$H(x, u(x), Du(x)) = 0, \tag{HJ}$$

where the state variable x belongs to $\mathbb{R}^N$, or to an open bounded subset of $\mathbb{R}^N$, or to the torus $\mathbb{T}^N$ according to the different problems we tackle, and the Hamiltonian H is a continuous real-valued function on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$.

2.1 Viscosity solutions

2.1.1 Semidifferentials and generalized gradient

We first introduce the semidifferentials of a function which will be important in our later analysis. For a detailed treatment of all the results, we refer readers to [2], [1] and [10].

Definition 2.1.1 ***(Super and subtangent)***.

Given an USC function $u : \mathbb{R}^N \to \mathbb{R}$, we say that a C^1 function ϕ is supertangent to u at some point x_0 if $\phi(x_0) = u(x_0)$ and $\phi(x) \geq u(x)$ for every $x \in U$ where U is a neighborhood of x_0. This means that x_0 is a local maximizer of $u - \phi$.

Similarly, a C^1 function ψ is called subtangent to a LSC function u at x_0 if $\psi(x_0) = u(x_0)$ and $\psi(x) \leq u(x)$ for every $x \in V$, a neighborhood of x_0. Accordingly, x_0 is a local minimizer of $u - \psi$.

Remark 2.1.2 *One can always assume that the local minimum (resp. maximum) at* x_0 *is strict, by replacing* $\phi(x)$ *with* $\phi(x) - |x - x_0|^2$ *(resp.* $\phi(x) + |x - x_0|^2$*).*

Definition 2.1.3 ***(Super and subdifferential)****.*

Let $u : \mathbb{R}^N \to \mathbb{R}$ *be an USC function, the set of superdifferentials of* u *at a point* $x \in \mathbb{R}^N$ *is defined as*

$$D^+u(x) = \{p \in \mathbb{R}^N : p = D\phi(x);\ \phi \text{ is supertangent to } u \text{ at } x\}.$$

Similarly, if $u : \mathbb{R}^N \to \mathbb{R}$ *is a LSC function, the set of subdifferentials of* u *at a point* $x \in \mathbb{R}^N$ *is defined as*

$$D^-u(x) = \{p \in \mathbb{R}^N : p = D\psi(x);\ \psi \text{ is subtangent to } u \text{ at } x\}.$$

The superdifferential and subdifferential are called semidifferentials.
From the definition it follows that, for any $x \in \mathbb{R}^N$

$$D^+u(x) = -D^-(-u(x)).$$

Some basic properties of sub and superdifferentials are collected in the following lemma.

Lemma 2.1.4

(a) $D^+u(x)$ *and* $D^-u(x)$ *are closed convex (possibly empty) subsets of* $\mathbb{R}^N$,

(b) *if* u *is differentiable at* x*, then*

$$D^+u(x) = D^-u(x) = \{Du(x)\},$$

(c) *if for some* x *both* $D^+u(x)$ *and* $D^-u(x)$ *are nonempty, then* u *is differentiable at* x *and the above equality of sets holds,*

(d) *if* $v : \mathbb{R}^N \to \mathbb{R}$ *is a function with* $v \leq u$ *and* $v(x) = u(x)$*, then* $D^-v(x) \subset D^-u(x)$ *and* $D^+v(x) \supset D^+u(x)$*.*

Now we introduce some weak differential for locally Lipschitz continuous functions called Generalized Gradient (Clarke's gradient), which is one of the most important concepts in nonsmooth analysis. We first recall that any locally Lipschtiz continuous function is almost everywhere differentiable with locally bounded gradient, in force of Rademacher Theorem. Then we have the following

Definition 2.1.5 ***(Generalized gradient)***.

Let $u : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschtiz continuous. The generalized gradient of u, denoted by ∂u, is defined for every x via the formula

$$\partial u(x) = co\{p = \lim_n Du(x_n), u \text{ is differentiable at } x_n, \lim_n x_n = x\},$$

where $co(A)$ denotes the convex hull of A.

Some basic properties of generalized gradient are collected in the following proposition.

Proposition 2.1.6 *Let $u : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz continuous. The following hold*

a) *if $\partial u(x) = \{p\}$, a singleton, then u is strictly differentiable at x, in the sense that u is differentiable and Du is continuous at x. In this case, $Du(x) = p$.*

b) *let $x_n \in \mathbb{R}^N$ and $p_n \in \mathbb{R}^N$ be sequences such that $p_n \in \partial u(x_n)$ for any $n \in \mathbb{N}$. Suppose that $x_n \to x$ and $p_n \to p$, as $n \to \infty$. Then one has $p \in \partial u(x)$, that is the set-valued map $\partial u(\cdot)$ is closed.*

Let us point out that the presence of the convex hull in the definition of generalized gradient is essential to keep, in the nonsmooth setting, the usual variational property, as fully stated in the next result.

Proposition 2.1.7 *Let u be locally Lipschitz continuous then $0 \in \partial u(x_0)$ at any local maximum or minimum x_0 of u.*

Now we directly deduce from the definition of generalized gradient:

Proposition 2.1.8 *Let u and ϕ be locally Lipschitz-continuous and C^1 functions respectively, then*

$$\partial(u - \phi)(x) = \partial u(x) - D\phi(x) \quad \text{for every } x.$$

We derive from Proposition 2.1.7 and Proposition 2.1.8

Corollary 2.1.9 *Let u be locally Lipschitz continuous then*

$$D^+u(x) \cup D^-u(x) \subset \partial u(x) \quad \text{for any } x.$$

Notice that, in contrast to what happens for the sub and superdifferential defined before which can be empty, $\partial u(x) \neq \emptyset$ for any x.

Remark 2.1.10 *In addition to the properties presented in Lemma 2.1.4, the sub and superdifferentials of a locally Lipschitz continuous function are bounded in* $\mathbb{R}^N$

Remark 2.1.11 *For a locally Lipschitz-continuous function u, we have the relation*

$$u(y) - u(x) = \int_0^T \frac{d}{dt} u(\xi(t))dt$$

for any x, y, any absolutely continuous curve ξ with $\xi(0) = x, \xi(T) = y$.
Under the above assumptions on u and ξ, in fact, it can be proved that the composition $u \circ \xi$ inherits absolute continuity and so $\frac{d}{dt}u(\xi(t))$ do exist for a.e. t. However, one should be careful when writing down the apparently equivalent formula

$$u(y) - u(x) = \int_0^T Du(\xi(t)) \cdot \dot{\xi}(t)dt,$$

since the relation $\frac{d}{dt}u(\xi(t)) = Du(\xi(t)) \cdot \dot{\xi}(t)$, on which it is based, is surely true if both u and ξ are differentiable, and thus is indeed the case for ξ at a.e. t. But such regularity is not guaranteed for u at any $\xi(t)$ for the reason that the support of the curve has vanishing $N-$dimensional Lebesgue measure. However one can prove the next result on the a.e. derivative of u along an absolute continuous curve, see for instance Lemma 1.4 *in [14]:*

Lemma 2.1.12 *Let $\xi : (-\infty, 0] \to \mathbb{T}^N$ be an absolutely continuous curve. Let s be such that $t \mapsto u\big(\xi(t)\big)$ and $t \mapsto \xi(t)$ are both differentiable at s. Then*

$$\frac{d}{dt} u(\xi(t))\Big|_{t=s} = p \cdot \dot{\xi}(s) \quad \textit{for some } p \in \partial u(\xi(s)).$$

2.1.2 Definitions and basic properties

Now we have all the ingredients to present the notion of viscosity solution. The notion of viscosity solution was first introduced by Crandall and Lions in 1982 and it works very well for many first- and second-order nonlinear PDEs, and satisfies properties of existence, uniqueness and stability. Moreover, this notion selects in a suitable sense the optimal almost everywhere solution.

Definition 2.1.13 ***(Viscosity solutions)*** *.*
An USC function $u : \mathbb{R}^N \to \mathbb{R}$ is called a viscosity subsolution of (HJ) if

$$H(x, u(x), p) \leq 0 \quad \textit{for any } x \in \mathbb{R}^N, \ \textit{any } p \in D^+u(x).$$

Similarly, a LSC function u is a viscosity supersolution of (HJ) if

$$H(x, u(x), p) \geq 0 \quad \textit{for any } x \in \mathbb{R}^N, \ \textit{any } p \in D^-u(x).$$

Finally, a continuous function u is a viscosity solution of (HJ) if u is both a viscosity sub and supersolution.

We state for latter use the next result which is an enlargement of the class of viscosity test functions, weakening the standard requirement of C^1-regularity.

Proposition 2.1.14 *Let u be an USC subsolution (resp. LSC supersolution) to (HJ), and ϕ a Lipschitz-continuous supertangent (resp. subtangent) to u at some point x. Then*

$$H(x, u(x), p) \leq 0 \, (resp. \geq 0) \quad \textit{for some } p \in \partial\phi(x).$$

A self-contained proof of this proposition is proposed in the Appendix B, by exploiting some facts arising in convex analysis.

The following proposition explains the local character of the notion of viscosity solution and its consistency with the classical pointwise definition.

Proposition 2.1.15 (a) *If $u \in C(\Omega)$ is a viscosity solution of (HJ) in Ω, then u is a viscosity solution of (HJ) in Ω', for any open subset $\Omega' \subset \Omega$;*

(b) *$u \in C^1(\mathbb{R}^N)$, u is a classical solution of (HJ) if and only if it is a viscosity solution of (HJ).*

Remark 2.1.16 *A striking fact to be stressed here is that viscosity solutions are not preserved by change of sign in the equation. Indeed, since any local maximum of $u - \phi$ is a local minimum of $-u - (-\phi)$, hence u is viscosity subsolution of (HJ) if and only if $-u$ is a viscosity supersolution of $-H(x, u, Du) = 0$ in Ω. Similarly, u is viscosity supersolution of (HJ) if and only if $-u$ is a viscosity subsolution of $-H(x, u, Du) = 0$.*

We now remark that the convexity of the Hamiltonian in the gradient variable allows us to prove some additional results in the theory of viscosity solutions of Hamilton-Jacobi equations. Namely, we can show that Lipschitz continuous *viscosity subsolution* and *almost everywhere subsolution* are equivalent as made precise in the next proposition.

Proposition 2.1.17 *Assume for each $x \in \mathbb{R}^N$, $r \in \mathbb{R}$, the Hamiltonian $H(x,r,p)$ is convex in p, and $u : \mathbb{R}^N \to \mathbb{R}^N$ be a locally Lipschitz continuous function. Then, the following statements are equivalent:*

a) *u is a viscosity subsolution to* (HJ) *in $\mathbb{R}^N$;*

b) *u is an almost everywhere subsolution to* (HJ) *in $\mathbb{R}^N$;*

c) *$H(x,u(x),p) \leq 0$, for any $x \in \mathbb{R}^N$ and any $p \in \partial u(x)$.*

Proof. (a) $\Rightarrow$ (b): This is evident because if u is as in the statement, then

$$Du(x) \in D^+u(x) \quad \text{at any } x \text{ where } u \text{ is differentiable,}$$

accordingly, if u is subsolution in the viscosity sense then it is also *a.e.* subsolution.

(b) $\Rightarrow$ (c): Assume u be an almost everywhere subsolution to (HJ) and take any point $x \in \mathbb{R}^N$ and let $p \in \partial u(x)$. By the very definition of generalized gradient,

$$p = \sum_i \lambda_i p_i,$$

where

$$\lambda_i \geq 0, \ \sum_i \lambda_i = 1, \ p_i = \lim_n Du(x_n^i), \ \lim_n x_n^i = x, \quad \text{for any } i.$$

We derive exploiting the continuity of H

$$H(x,u(x),p_i) = \lim_n H(x_n^i, u(x_n^i), Du(x_n^i)) \leq 0 \quad \text{for any } i.$$

By the convexity assumption on H, we get

$$H(x,u(x),p) = H(x,u(x),\sum_i \lambda_i p_i) \leq \sum_i \lambda_i H(x,u(x),p_i) \leq 0.$$

(c) $\Rightarrow$ (a): The implication directly comes from the fact that $D^+u(x) \subset \partial u(x)$ for any $x \in \mathbb{R}^N$, and taking into account the definition of viscosity subsolution.

If we ask some more on the differential structure of u, we can state a more general result without the assumption of convexity for H.

Proposition 2.1.18 *Let u be a locally Lipschitz continuous function in $\mathbb{R}^N$.*

a) *If $D^+u(x) = \partial u(x),\ \forall x \in \mathbb{R}^N$, then u is a viscosity supersolution to* (HJ) *if and only if it is an almost everywhere supersolution;*

b) *If $D^-u(x) = \partial u(x),\ \forall x \in \mathbb{R}^N$, then u is a viscosity subsolution to* (HJ) *if and only if it is an almost everywhere subsolution.*

It is important to notice that a viscosity subsolution is not necessarily locally Lipschitz continuous. However, this will be the case if the Hamiltonian $H(x, r, p)$ is assumed to satisfy the coercive condition in p, i.e.,

$$H(x, r, p) \longrightarrow +\infty \quad \text{as } |p| \to +\infty, \text{ uniformly in } x \text{ and in } r. \tag{2.1}$$

Proposition 2.1.19 *Assume that the Hamitonian H satisfies the coercivity condition* (2.1)*. If $u \in BC(\mathbb{R}^N)$ is a viscosity subsolution of* (HJ)*, then u is Lipschitz continuous in $\mathbb{R}^N$.*

Proof. Fix $x_0 \in \mathbb{R}^N$. Consider the function

$$\phi(x) = u(x_0) + C|x - x_0|,$$

where $C > 0$ is a suitable constant to be chosen later. The boundedness of u implies the existence of $y \in \mathbb{R}^N$ such that y is a maximizer of $u - \phi$ on $\mathbb{R}^N$, we claim that $y \equiv x_0$. Indeed if $y \neq x_0$, then the function ϕ is differentiable at y and since u is a viscosity subsolution of (HJ), we get

$$H\big(y, u(y), D\phi(y)\big) = H\Big(y, u(y), C\frac{y - x_0}{|y - x_0|}\Big) \leq 0.$$

For C sufficiently large, the above inequality is in contradiction to the coercivity condition (2.1) and hence $y \equiv x_0$. Therefore

$$u(x) - u(x_0) \leq C|x - x_0|, \quad \text{for every } x \in \mathbb{T}^N.$$

By interchanging the roles of x and x_0, we get $u \in Lip(\mathbb{R}^N)$. □

One of the most useful properties of viscosity solution is stability which allows us to pass to limits even when the Hamilton-Jacobi equation is fully nonlinear. The following stability result plays an important role in viscosity solution theory.

Theorem 2.1.20 ***(Stability Result)***. *Suppose that (u_n) is a sequence of continuous viscosity solutions of*

$$H_n(x, u(x), Du(x)) = 0, \quad x \in \mathbb{R}^N$$

that converges locally uniformly to a function $u : \mathbb{R}^N \to \mathbb{R}$. Suppose moreover that $H_n \to H$ locally uniformly, then u is a viscosity solution of

$$H(x, u(x), Du(x) = 0, \quad x \in \mathbb{R}^N.$$

The validity of all stability arguments is due to the next elementary fact which is very useful in many situations in viscosity theory. It is termed as stability for minimizers/maximizers (see Lemma 2.4 in [1]).

Lemma 2.1.21 *Let $v \in C(\Omega)$. Suppose that v has a strict local maximum (or minimum) at a point $x_0 \in \Omega$. If $v_n \in C(\Omega)$ converges locally uniformly to v in Ω, then there exist a sequence $\{x_n\}$ such that $x_n \to x_0$, and v_n has a local maximum (or minimum) at x_n.*

We now present the nice behavior of viscosity solutions w.r.t. the operations of infimum and supremum.

Proposition 2.1.22 *Let $\mathcal{F}$ be a family of viscosity subsolutions (resp. supersolutions) of (HJ) and cosider:*

$$u(x) = \sup_{v \in \mathcal{F}} v(x) \quad (resp.\ u(x) = \inf_{v \in \mathcal{F}} v(x)).$$

Assume also that u is USC (resp. LSC), then u is a viscosity subsolution (resp. supersolution) of (HJ).

Remark 2.1.23 *Note that in general the supremum of viscosity supersolutions (resp. the infimum of viscosity subsolutions) is not a viscosity supersolution (resp. viscosity subsolution). Moreover even if we consider a family of continuous viscosity subsolution (resp. continuous viscosity supersolutions), it is not necessary that the supremum (infimum) will be USC (LSC) as desired. Nevertheless, without the correct semi continuity needed, the result is still true but one has to use the theory of discontinuous viscosity solution, which means to deal with the lower and the upper semicontinuous envelopes.*

2.1.3 Some comparison results

In this section we address the problem of comparison and uniqueness of viscosity solutions which is a major issue in the theory. In what follows we select some comparison results between viscosity sub and supersolutions in the case of bounded domain Ω and on the whole space $\mathbb{R}^N$. As a simple remark, each comparison result produces a uniqueness theorem for the associated Dirichlet problem.

We start with comparison results for Hamilton-Jacobi equations of the form

$$\lambda u + H(x, Du(x)) = 0, \tag{2.2}$$

where λ is a positive scalar, the state variable x belongs to $\mathbb{R}^N$ or to an open bounded subset Ω of $\mathbb{R}^N$ and the continuous Hamiltonian H is now strictly increasing with respect to u.

We remark that, in what follows, by modulus we mean nondecreasing function from $[0, +\infty)$ to $[0, +\infty)$ vanishing and continuous at 0.

Theorem 2.1.24 *Let Ω be a bounded open subset of $\mathbb{R}^N$. Assume that u, $v \in C(\bar{\Omega})$ are, respectively, viscosity sub- and supersolution of (2.2) such that*

$$u \leq v \quad \text{on } \partial\Omega.$$

Assume also that H satisfies

$$|H(x,p) - H(y,p)| \leq \omega_1(|x-y|(1+|p|)) \quad \text{for } x,\ y \in \Omega,\ p \in \mathbb{R}^N, \tag{2.3}$$

where ω_1 is a modulus, then

$$u \leq v \quad \text{on } \bar{\Omega}$$

We now provide a global comparison result (case of $\Omega = \mathbb{R}^N$) in the space of bounded continuous functions on $\mathbb{R}^N$:

Theorem 2.1.25 *Assume that u, v are, respectively, bounded continuous viscosity sub- and supersolution of (2.2) such that that H satisfies (2.3) in $\mathbb{R}^N \times \mathbb{R}^N$ and also*

$$|H(x,p) - H(x,q)| \leq \omega_2(|p-q|) \quad \text{for all } x,\ p,\ q \in \mathbb{R}^N, \tag{2.4}$$

where ω_2 is a modulus, then

$$u \leq v \quad \text{on } \mathbb{R}^N.$$

A relevant example of Hamiltonian satisfying (2.3), (2.4) is given by so–called Bellman Hamiltonian

$$H(x,p) = \max_{a\in A}\{-p\cdot f(x,a) - \ell(x,a)\} \tag{2.5}$$

where A is a compact subset of some $\mathbb{R}^M$, with M possibly different from N, $f:\mathbb{R}^N\times A :\to \mathbb{R}^N$ is Lipschitz-continuous in x, uniformly with respect to a, namely

$$|f(x,a) - f(y,a)| \le L_f\,|x-y| \text{ for any } x,\ y,\ a, \text{ some } L_f > 0.$$

Finally, the function $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ is bounded and continuous in both arguments. Notice that H is convex in p, being the maximum of linear functions.
Equation (2.2) with H being of Bellman type is called Hamilton–Jacobi–Bellman equation. We mention, without entering in many details, that it is related to the following infinite horizon control problem

$$\inf_{\alpha\in\mathcal{A}} \int_0^{+\infty} e^{-\lambda t}\ell(y(t,x_0,\alpha),\alpha)\,dt, \tag{2.6}$$

where $\mathcal{A} = L^\infty(0,+\infty;A)$, i.e. the space of measurable functions defined in $[0,+\infty)$ and taking values in A, the boundedness condition being just a consequence of the fact that A has been assumed to be compact. We consider the trajectory solution of the Cauchy problem

$$\begin{cases} y'(t) = f(y(t),\alpha(t)) \\ y(0) = x_0 \end{cases} \tag{2.7}$$

Notice that under our assumptions such a problem actually admits, for any given $\alpha(.)\in\mathcal{A}$, unique solution defined in the whole $[0,+\infty)$, that will be denoted in what follows by

$$t\mapsto y(t,x,\alpha).$$

In optimal control problems the set A is called the *control set* and the entities making up $\mathcal{A}$, are called *controls*. The previously introduced dynamics is qualified as *controlled*, while ℓ is the *running cost* and the constant λ plays the role of a *discount factor*. Finally, the functional appearing in (2.6) takes the name of *payoff*. See [1] for a general treatment of this topics using the *Dynamic Programming Principle* .

The relationship of the above model with (2.2), with H of Bellman-type, is given by the *value function* v associating to any initial point x the infimum of the payoff, namely

$$v(x) = \inf_{\alpha\in\mathcal{A}} \int_0^{+\infty} e^{-\lambda t}\ell(y(t,x,\alpha),\alpha)\,dt.$$

Indeed under the above assumptions we have

Proposition 2.1.26 *The value function v is a viscosity solution of (2.2) on $\mathbb{R}^N$, with H of Bellman type.*

We proceed by presenting a comparison and uniqueness result for Hamilton-Jacobi equations of the form

$$H(x, Du) = 0, \quad x \in \Omega, \tag{2.8}$$

where Ω is bounded open subset of $\mathbb{R}^N$ and H is real valued Hamiltonian continuous on both variables. Here the convexity of H with respect to the p variable plays a key role in getting the result.

Theorem 2.1.27 *Assume that u, $v \in C(\bar{\Omega})$ are, respectively, viscosity sub- and supersolution of (2.8) such that $u \leq v$ on $\partial\Omega$. Assume also that H satisfies*

- *$p \to H(x, p)$ is convex and coercive on $\mathbb{R}^N$ for every $x \in \Omega$,*
- *there exists a strict subsolution of (2.8) in Ω, namely there exists $\phi \in C(\bar{\Omega})$ with $H(x, D\phi) \leq -\delta$ in the viscosity sense in Ω for some positive δ.*

Then $u \leq v$ in Ω.

Proof. Due to the coercivity, u is locally Lipschitz–continuous in Ω. The argument is by contradiction, we assume the minimum of $v - u$ in $\overline{\Omega}$ is strictly negative, which implies that all minimizers of $v - u$ must be in Ω, by the assumption $u \leq v$ on $\partial\Omega$.

Next we construct a sequence of strict subsolutions, say u_n, uniformly converging to u in $\overline{\Omega}$. For this we essentially exploit the existence of a strict global subsolution ϕ. Given a strictly increasing sequence λ_n of positive numbers converging to 1, we define

$$u_n = \lambda_n \, u + (1 - \lambda_n) \, \phi.$$

It is easy to see, exploiting the convex character of the Hamiltonian, that the u_n are indeed locally Lipschitz-continuous strict subsolutions of (2.8) converging uniformly to u. This implies that any sequence x_n of minimizers of $v - u_n$ in $\overline{\Omega}$ converges, up to subsequences, to a minimizer of $v - u$ which we know to be in Ω. Hence $x_n \in \Omega$ for n large enough.

The function u_n is therefore a Lipschitz–continuous subtangent to v at x_n, then by Proposition 2.1.14

$$H(x_n, p) \geq 0 \qquad \text{for some } p \in \partial u_n(x_n). \tag{2.9}$$

Passing to the limit, and taking into account the continuity properties of H and Generalized Gradient, we get

$$H(x,p) \geq 0 \qquad \text{for some } p \in \partial u_n(x), \tag{2.10}$$

where x is a minimizer of $v - u$ in Ω.

On the other side, owing to convexity and being strict subsolution, u_n satisfies

$$H(x,p) < 0 \qquad \text{for all } p \in \partial u_n(x). \tag{2.11}$$

The relations in (2.10), (2.11) are in contradiction. We therefore conclude that there is some minimizer of $v - u$ on $\partial\Omega$, which in turn implies $v \geq u$ in $\overline{\Omega}$, as desired.

Remark 2.1.28 *The core of the above argument is to have a strict subsolution which is subtangent to a given supersolution. So if we consider the Eikonal equation (2.8) imposed on flat torus $\mathbb{T}^N$ and we assume the existence of a strict subsolution and solution at some level a, then the same argument holds and we get a contradiction, being all minimizers interior. Therefore, we derive that it is impossible to have a solution and strict subsolution simultaneously. In this case, we understand the existence of a unique level, qualified as critical, to get a solution and below which we don't have subsolutions. This is explained precisely in section 2.2.*

2.2 Scalar eikonal equation

In this section we study a family of Hamilton-Jacobi equations of eikonal type in the compact setting, in particular on the flat torus $\mathbb{T}^N$. Our aim is to collect some basic materials from Weak KAM theory for convex, coercive Hamiltonians. For more details, this topic has been extensively studied by Fathi and Siconolfi in [18], .

We consider the family of Hamilton-Jacobi equations

$$H(x, Du(x)) = a \quad \text{on } \mathbb{T}^N, \tag{HJa}$$

with a real parameter. We assume that the Hamiltonian $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions:

(H1) $(x,p) \mapsto H(x,p)$ is continuous for every $x \in \mathbb{T}^N$;

(H2) $p \mapsto H(x,p)$ is convex for every $x \in \mathbb{R}^N$;

(H3) $p \mapsto H(x,p)$ is coercive, uniformly in x, namely

$$\lim_{|p|\to+\infty} H(x,p) = +\infty;$$

Thanks to coercivity, all viscosity subsolutions of (HJa) are Lipschitz continuous. Moreover, by the convexity assumption, there is complete equivalence between the notions of viscosity subsolution and a.e. subsolution. Therefore a function u is a subsolution of (HJa) if and only if $H(x,p) \leq a$ for every x and $p \in \partial u(x)$, see Proposition 2.1.17.

Moreover, for any fixed $a \in \mathbb{R}$, the set of viscosity subsolutions of (HJa) is equi-Lipschitz with a common Lipschitz constant κ_a given by

$$\kappa_a = \sup\{|p| : \ H(x,p) \leq a\}.$$

In the qualitative analysis of the family of Hamilton-Jacobi equations (HJa), it is well known, as already pointed out, that a special value of a is relevant, qualified as critical and denoted in the remainder by c. It is characterized from a PDE viewpoint, by the property that it is the unique value for which the corresponding equation can be solved in the viscosity sense on the whole torus. The uniqueness of the critical value is due to the following fact:

Proposition 2.2.1 *If there exists a subsolution u of* (HJa) *with $a = a_1$ and a supersolution w of* (HJa) *with $a = a_2$, then $a_1 \geq a_2$.*

From this result, we see that the unique value of a for which there can be a solution of (HJa) is

$$c = \inf\{a : \ \text{(HJa) has a subsolution}\}.$$

By the Ascoli-Arzelà Theorem and the stability of the notion of viscosity subsolution, it is easily seen that such an infimum is attained, meaning that there are subsolutions also at the critical level. Therefore we focus our attention on the critical equation

$$H(x, Du(x)) = c. \tag{HJc}$$

The existence of solution at critical level is obtained through the study of the critical subsolutions given, for every fixed $y \in \mathbb{T}^N$, by

$$\sup\{u(.) : \ \text{u is subsolution of (HJc) with } u(y) = 0\}$$

and by relating it to a cover of the torus with open balls.
Next we need to provide an integral representation formula for subsolutions/solutions. For this purpose, we define for any x the a–sublevel

$$Z(x) = \{p \mid H(x,p) \le a\}$$

It is a patent consequence of the convexity and coercivity assumptions that $Z(x)$ is convex and compact for any x. Moreover, we directly derive, from the continuity of the Hamiltonian, the following continuity properties for the set–valued function Z

$$x_n \to x,\, p_n \in Z(x_n),\, p_n \to p \Rightarrow p \in Z(x). \tag{2.12}$$

$$p \in Z(x),\, x_n \to x \Rightarrow \exists p_n \in Z(x_n) \mid p_n \to p \tag{2.13}$$

We proceed considering the *support function*

$$\sigma(x,q) = \max\{p \cdot q \mid p \in Z(x)\} \tag{2.14}$$

we see that it is positively homogeneous in p and, in addition convex because of the convexity of $Z(x)$. This implies subadditivity, namely

$$\sigma(x, p_1 + p_2) \le \sigma(x,p_1) + \sigma(x,p_2) \quad \text{for any } x,\, p_1,\, p_2.$$

It moreover comes from (2.12), (2.13) that

$$x \mapsto \sigma(x,p) \qquad \text{is continuous for any fixed } p.$$

Following the metric method which has revealed to be a powerful tool for the analysis of Hamilton-Jacobi equations, we carry out the study of properties of subsolutions of (HJa) by means of semidistances S_a defined on $\mathbb{T}^N \times \mathbb{T}^N$, as follows:

$$S_a(x,y) = \inf\left\{\int_0^1 \sigma(\xi(s),\dot{\xi}(s))ds : \xi \in B_{x,y}\right\}, \tag{2.15}$$

where

$$B_{x,y} = \{\xi : [0,1] \to \mathbb{T}^N \text{ is Lipschitz-continuous with } \xi(0) = x \text{ and } \xi(1) = y)\}.$$

We now recall some basic properties of the semidistance S_a and then provide a class of fundamental (sub) solutions to (HJa). The mentioned results are taken from [18, 33], for more details.

Since the juxtaposition of trajectories in $B_{x,z}$ and $B_{z,y}$ respectively give, up to a change of parameter, a curve in $B_{x,y}$, one can establish:

$$S_a(x,y) \le S_a(x,z) + S_a(z,y) \quad \text{for all } x,\ y,\ z \in \mathbb{T}^N. \tag{2.16}$$

Moreover, the semi distance S_a satisfies, for every x, $y \in \mathbb{T}^N$, the following inequality:

$$S_a(x,y) \le R_a|x-y| \quad \text{for some positive constant } R_a. \tag{2.17}$$

The semidistance S_a plays a crucial role in the representation formulae for (sub) solutions of (HJa) as

Proposition 2.2.2 *Given $a \ge c$, we have*

(i) The function $S_a(y,.)$ is a solution on $\mathbb{T}^N \setminus \{y\}$ and subsolution on $\mathbb{T}^N$ of (HJa) *, for every $y \in \mathbb{T}^N$. In addition*

$$S_a(y,x) = \max\{u(x) : \ u \text{ subsolution of (HJa) with } u(y) = 0\}$$

(ii) A function u is subsolution of (HJa) *if and only if*

$$u(x) - u(y) \le S_a(y,x) \quad \text{for all } x, y \in \mathbb{T}^N. \tag{2.18}$$

Based on this proposition we next present the relation between viscosity solutions and a.e solution:

Remark 2.2.3 *Indeed any lipschitz continuous function which is a viscosity solution of* (HJa) *is also a.e solution. However the converse, in general, is not true. The above proposition tells us that at any supercritical level we have an infinite number of a.e solutions which are not viscosity solutions, given by the intrinsic distance $S_a(y,.)$. This shows, since we have an equivalence between viscosity subsolutions and a.e subsolutions, that in the qualitative analysis of* (HJa) *the viscosity techniques enter to select supersolutions.*

In the analysis of the behavior of critical subsolutions, a special role is played by the so-called Aubry set, denoted by $\mathcal{A}_e$, defined as the collection of points $y \in \mathbb{T}^N$ such that

$$\inf\left\{\int_0^1 \sigma_c(\xi(s),\dot\xi(s))ds : \xi \in B_{y,y}, \ell(\xi) \ge \delta\right\} = 0 \quad \text{for some } \delta > 0,$$

where $\ell(\xi)$ indicates the length of the curve ξ.
The set $\mathcal{A}_e$ is nonempty and closed in the torus.

We recall that a critical subsolution of (HJc) is called strict in an open set B if

$$H(x, Du(x) \le c - \delta, \quad \text{for some } \delta > 0.$$

In our setting and as mentioned previously it is impossible to get a solution and strict subsolution simultaneously (Remark 2.1.28). However, the obstruction in getting global strict subsolution is not spread over the whole torus but instead concentrates on the Aubry set. We now provide a pde characterization of $\mathcal{A}_e$.

Proposition 2.2.4 .

(i) $y \in \mathcal{A}_e$ *if and only if* $S_c(y, .)$ *is a solution of* (HJc).

(i) Any point y *with* $\min_p H(y, p) = c$ *belongs to* $\mathcal{A}_e$.

(ii) $y \notin \mathcal{A}_e$ *if and only if there exists a critical subsolution which is strict in some neighborhood of* y.

It is well known that the existence of a strict subsolution is a sufficient and necessary condition to get comparison and uniqueness result for Hamilton-Jacobi equations of Eikonal type, Theorem 2.1.27. However, in our setting such a condition is only satisfied outside the Aubry set. Therefore the Aubry set behaves as a sort of interior boundary where, to have uniqueness, the value of the solution has to be prescribed on $\mathcal{A}_e$. More precisely, the next theorem shows in particular that $\mathcal{A}_e$ is a uniqueness set for the equation (HJc).

Proposition 2.2.5 *(i) If* u *and* v *are a subsolution and a supersolution of* (HJc), *respectively, and* $u \le v$ *on* $\mathcal{A}_e$, *then* $u \le v$ *on* $\mathbb{T}^N$. *In particular, if two solutions of* (HJc) *coincide oh* $\mathcal{A}_e$, *then they coincide on* $\mathbb{T}^N$.

(ii) If u_0 *is a continuous function defined on a closed set* C *such that*

$$u_0(x) - u_0(y) \le S_a(y, x) \quad \text{for every } x, y \in C, \tag{2.19}$$

then the function

$$u := \min_{y \in C} \{u_0(y) + S_a(y, .)\} \tag{2.20}$$

is the maximal subsolution of (HJa) *in* $\mathbb{T}^N$ *equaling* u_0 *on* C, *and a solution of* (HJa) *on* $\mathbb{T}^N \setminus C$ *as well.*

(iii) If we further set $C = \mathcal{A}_e$ *and* $a = c$ *in (2.20), then* u *is the unique critical solution of* (HJc) *equal to* u_0 *on* $\mathcal{A}_e$.

We end up this section by stating the regularity properties of critical subsolutions. For this purpose we assume additionally

$$(x,p) \mapsto H(x,p) \quad \text{is locally Lipschtiz–continuous ,}$$

$$p \mapsto H(x,p) \quad \text{is strictly convex for every } x.$$

Then we have the following semiconcavity-type estimates for $S_c(y,.)$.

Proposition 2.2.6 *Assume either $x \neq y$ or $x = y \in \mathcal{A}_e$ then*

$$\partial S_c(y,x) = D^+ S_c(y,x).$$

Exploiting this proposition, we deduce the following regularity result for critical subsolutions.

Theorem 2.2.7 .

(i) For $y \in \mathbb{T}^{\mathbb{N}}$, the function $S_c(y,.)$ is strictly differentiable at every point of $\mathcal{A}_e$.

(ii) Any critical subsolution w is differentiable on $\mathcal{A}_e$ and $Dw(y) = DS_c(y,y)$ for every $y \in \mathcal{A}_e$.

Now it is important to mention that the above presented results, concerning intrinsic distance and representation formulas, are still true in the whole space $\mathbb{R}^N$. However, there are some facts that only hold in the non-compact setting, namely in $\mathbb{R}^N$.

The first remark is that the assumptions $(H1) - (H3)$ don't guarantee that the critical value is finite. To clarify this, we consider the following example of simple Eikonal equation:

$$|Du| = f(x) - a \quad \text{on } \mathbb{R}^N,$$

where f is the potential function such that $f(x) \to -\infty$ as $|x| \to +\infty$. In this case the set of subsolutions at any level a is empty, and hence the critical value is infinite ($c = +\infty$).

Unlike the compact case where the critical value c is characterized by the property of being the unique value such that (HJa) has solutions in $\mathbb{T}^N$, in the non-compact case ($\mathbb{R}^N$) a solution does exist at the critical value as well as at any supercritical level. Namely, we have

Proposition 2.2.8 *Assume c is finite. Then for any $a \geq c$, the equation* (HJa) *has solutions in* $\mathbb{R}^N$.

Proof. Given $a \geq c$ and $x \in \mathbb{R}^m$. Let y_n be a sequence in $\mathbb{R}^m$ going to infinity, and we set

$$u_n(x) := S_a(y_n, x),$$

then, by Proposition 2.2.2, u_n is a sequence of subsolutions of (HJa) in $\mathbb{R}^m$ and solutions in $\mathbb{R}^m \setminus \{y_n\}$.
Define, for each $n \in \mathbb{N}$,

$$\begin{aligned} \tilde{u}_n(x) &:= u_n(x) - u_n(0) \\ &= S_a(y_n, x) - S_a(y_n, 0). \end{aligned}$$

Clearly, $\tilde{u}_n$ are also subsolutions of (HJa) in $\mathbb{R}^m$ and solutions in $\mathbb{R}^m \setminus \{y_n\}$, and moreover

$$\tilde{u}_n(0) = 0, \quad \text{for every } n \in \mathbb{N}.$$

We next prove that the sequence $\tilde{u}_n$ is equi-Lipschitz continuous and locally equi-bounded in $\mathbb{R}^m$. Indeed, for any $n \in \mathbb{N}$ and $x, z \in \mathbb{R}^m$, one has

$$\begin{aligned} \tilde{u}_n(x) - \tilde{u}_n(z) &= S_a(y_n, x) - S_a(y_n, z) \\ &\leq S_a(z, x) \\ &\leq R_a|x - z|, \end{aligned}$$

for some $R_a > 0$, where the inequalities follows from the triangle inequality of semidistance S_a.
By exchanging the roles of x and z, we get

$$|\tilde{u}_n(x) - \tilde{u}_n(z)| \leq R_a|x - z|, \quad \text{for any } x, z \in \mathbb{R}^m, \text{ any } n \in \mathbb{N},$$

which shows the equi-Lipschitz continuity of $\tilde{u}_n$ in $\mathbb{R}^m$. The locally equi-boundedness character of $\tilde{u}_n$ is easily obtained from the following estimate

$$|\tilde{u}_n(x)| = |\tilde{u}_n(x) - \tilde{u}_n(0)| \leq R_a|x|, \quad \text{for every } x \in \mathbb{R}^m,\ n \in \mathbb{N}.$$

Up to subsequences, by Ascoli-Arzelà theorem, there exists a continuous function u_0 such that

$$\tilde{u}_n \to u_0, \qquad \text{locally uniformly in } \mathbb{R}^m.$$

In the rest of the proof we verify that u_0 is a solution of (HJa) in $\mathbb{R}^m$. Clearly, u_0 is a subsolution of (HJa) in $\mathbb{R}^m$ by using subsolution property of $\tilde{u}_n$ and basic stability property of viscosity solution theory. We are left to prove the supersolution property of u_0 in $\mathbb{R}^m$. To this end, we take any $x_0 \in \mathbb{R}^m$ and φ a C^1 strict subtangent to u_0 at x_0, i.e x_0 is the unique minimizer of $u_0 - \varphi$ in a suitable closed ball B centered at x_0. By Lemma 2.1.21, there exists a sequence x_n of minimizers of $u_n - \varphi$ in B converges to x_0. We now remark that, for n large enough, $x_n \neq y_n$ and x_n is in the interior of B, hence by the supersolution property of $\tilde{u}_n$ in $\mathbb{R}^m \setminus \{y_n\}$, one has

$$H\big(x_n, D\varphi(x_n)\big) \geq a.$$

Passing to the limit and by the continuity of H and φ being C^1, we get

$$H(x_0, D\varphi(x_0)) \geq a.$$

This yields u_0 is supersolution to (HJa) in $\mathbb{R}^m$.

□

Assume c is finite and $a \geq c$. We may couple equation (HJa) with Dirichlet boundary condition

$$\begin{cases} H(x, Du) & = a \quad \text{in } \Omega \\ \quad u & = g \quad \text{on } \partial\Omega \end{cases} \tag{2.21}$$

where Ω is bounded open set of $\mathbb{R}^N$ and g a given function on $\partial\Omega$. We derive from aforementioned results the following representation formula:

Theorem 2.2.9 *Assume the boundary datum to satisfy*

$$g(y) - g(x) \leq S_a(x, y) \qquad \textit{for any } x,\ y \textit{ in } \partial\Omega$$

then the function

$$v(x) = \inf\{S_a(y, x) + g(y) \mid y \in \partial\Omega\}$$

is subsolution to (2.21) *in* $\mathbb{R}^N$*, solution in* $\mathbb{R}^N \setminus \partial\Omega$ *and agrees with* g *on* $\partial\Omega$.
Moreover if $a > c$*, then* v *is the unique solution to (2.21).*

Chapter 3

Weakly coupled systems of Hamilton–Jacobi equations

3.1 Introduction

This chapter deals with weakly coupled Hamilton–Jacobi system of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, 2, ..., m\}.$$

Here $\mathbf{u} = (u_1, \cdots, u_m)$ is the vector–valued unknown function, Du_i is the gradient of u_i, α is a real number, H_i are mutually unrelated Hamiltonians enjoying standard properties, see Section 3.2, and A is the so called $m \times m$ coupling matrix, which constitutes the relevant item in the problem. The hypothesis taken on the coupling matrix A correspond to suitable monotonicity properties of the equation with respect to the entries u_j, see Remark 3.2.3. They are complemented by a degeneracy condition requiring all rows of A sums to 0 and any non–diagonal entry is nonpositive, yielding that $-A$ is a generator of semigroup of stochastic matrices.

In the PDE literature, such weakly coupled systems have been studied as a particular instance of monotone systems; see [16, 21, 24]. More recently, they have been considered in connection with homogenization problems [8, 28] and for the long-time behavior of the associated evolutionary system [9, 27, 29, 30]. These works are a generalization of results established in the case of a single equation, see [6, 13, 25, 32].

Note that we must assume an irreducibility condition on the coupling matrix to solve our system, see Section 3.2 for the precise definition. Under these assumptions, it is established in [8, 29, 27] that there is a unique value for α, termed as critical, such that the corresponding system admits viscosity solutions. Critical solutions are, instead, not unique even up to the addition of a common constant to all the components u_i. More recently, a qualitative analysis on the critical weakly coupled system, investigating the non–uniqueness phenomena taking place at the critical level, has been taken by Davini-Zavidovique in [15]. The core of their analysis is discovering that, at the critical level, a very rigid object appears, characterized as the region where the obstructions of strict critical subsolutions concentrates and named after Aubry in analogy with the scalar case. They explore the properties of this Aubry set, in particular to show that it is a uniqueness set for the critical system i.e., two critical solutions that coincide on $\mathcal{A}$ do coincide on the whole torus and they highlight some rigidity phenomena taking place on it. All these results pertain to the PDE side of the theory, and are solely obtained by means of PDE techniques, and they are presented in section 3.4.

A new approach has been taken recently by Mitake, Siconolfi, Tran and Yamada in [26] covering the dynamical part of the problem and recovering some crucial facts holding in the scalar case. A crucial step in their analysis is to put the problem in a suitable random frame, which avoid introducing an abstract probability space and just work with concrete path spaces, see sections 3.5.1 and 3.5.2. Their main achievement is demonstrated by the definition of family of related action functionals containing the Lagrangians obtained by duality from the Hamiltonians of the system. They use them to characterize, by means of a suitable estimate, all the subsolutions of the system, and to explicitly represent some subsolutions enjoying an additional maximality property, see section 3.5.3.

In this chapter we give a brief overview of the existing PDE and dynamical results and show the link between the two approaches which is the core of Weak KAM theory. We also provide comparison results for non–degenerate systems.

3.2 Setting of the problem

Through out the thesis, we are specifically interested in degenerate weakly coupled system of Hamilton-Jacobi equations of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \quad \text{in } \mathbb{T}^N \quad \text{for every } i \in \{1, 2, ..., m\}, \tag{HJα}$$

where α is a real constant and $H_1, H_2, ..., H_m$ are Hamiltonians satisfying the following set of assumptions:

(H6) $H_i : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous for every $i \in \{1, 2, ..., m\}$;

(H7) $p \mapsto H_i(x, p)$ is convex for every $x \in \mathbb{T}^N$, and $i \in \{1, 2, ..., m\}$;

(H8) H_i is coercive in the momentum variable for every $i \in \{1, 2, ..., m\}$.

We denote by $A := (a_{ij})$ an $m \times m$ coupling matrix satisfying the following hypotheses:

(A1) $a_{ij} \leq 0$ for every $i \neq j$;

(A2) A is degenerate i.e

$$\sum_{j=1}^{m} a_{ij} = 0 \text{ for any } i \in \{1, 2, ..., m\};$$

(A3) The matrix A is irreducible, i.e for every $W \subsetneq \{1, 2, ..., m\}$ there exists $i \in W$ and $j \notin W$ such that $a_{ij} < 0$. Roughly speaking this means that the system cannot split into independent subsysytems.

As made precise in Appendix A, the conditions (A_1), (A_2), (A_3) implies that the matrix A is singular with rank $m - 1$ and kernel spanned by $\mathbb{1}$, namely the vector with all components equal to 1, moreover $Im(A)$ cannot contain vectors with strictly positive or negative components. This in particular implies $Im(A) \cap ker(A) = \{0\}$.

The notion of viscosity (sub/super) solution can be easily adapted to systems as (HJα).

Definition 3.2.1 *We say that a continuous function* $\mathbf{u} : \mathbb{T}^N \to \mathbb{R}^m$ *is a viscosity subsolution of (HJα) if for every* $(x, i) \in \mathbb{T}^N \times \{1, 2, ..., m\}$, *we have*

$$H_i(x, p) + \sum_{j=1}^{m} a_{ij} u_j(x) \leq \alpha \quad \text{for every } p \in D^+ u_i(x).$$

Symmetrically, $\mathbf{u}$ *is a viscosity supersolution of (HJα) if for every* $(x, i) \in \mathbb{T}^N \times \{1, 2, ..., m\}$, *we have*

$$H_i(x, p) + \sum_{j=1}^{m} a_{ij} u_j(x) \geq \alpha \quad \text{for every } p \in D^- u_i(x).$$

Finally, if $\mathbf{u}$ *is both a viscosity sub and supersolution, then it is called a viscosity solution.*

Remark 3.2.2 *One could wonder why we are considering systems with the same constant appearing in the right–hand side of any equation, while a more natural condition should be to have instead a vector of* $\mathbb{R}^m$*, say* $\boldsymbol{\alpha}$*, with possibly different components. Indeed the choice of taking the same scalar* α *on the right hand–side of the system is not a restriction. In fact, if we write the vector* $\alpha\,\mathbb{1}$ *as* $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ *with* $\boldsymbol{\alpha}_1 = \alpha\mathbb{1} \in ker(A)$*,* $\boldsymbol{\alpha}_2 \in \mathrm{I}m(A)$*, where this form is uniquely determined because* $\mathrm{I}m(A) \cap \ker(A) = \{0\}$*, and pick* $\boldsymbol{b}$ *with* $A\,\boldsymbol{b} = -\boldsymbol{\alpha}_2$*, then* u *is a (super/sub) solution to (HJ*α*) if and only if* $u + b$ *satisfies the same properties for the system obtained from (HJ*α*) by replacing in the right hand side* $\alpha\,\mathbb{1}$ *by* $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$*.*

Remark 3.2.3 *The weakly coupled system (HJ*α*) is a particular type of monotone system, i.e. a system of the form*

$$F_i(x, \mathbf{u}(x), Du_i) = 0 \quad in\ \mathbb{T}^N,$$

where suitable monotonicity conditions with respect to the u_j- *variables are assumed on the function* F_i*, see [8], [16], [20], [21], [24]. In our case the conditions assumed on the coupling matrix* A *imply that each* F_i *is strictly increasing in* u_i *and decreasing in* u_j *for every* $j \neq i$ *and our system (HJ*α*) is expressed as*

$$F_i(x, \mathbf{u}(x), Du_i) = H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) - \alpha \quad in\ \mathbb{T}^N \quad for\ every\ i \in \{1, 2, ..., m\}. \tag{3.1}$$

3.3 Comparison results

In this section we will present two comparison results based on the method of doubling of variables. The basic difference is that the first comparison result requires irreducibility assumption on the matrix A and non degeneracy in at least one row while the second one does not need the matrix A to be irreducible but instead requires a non degeneracy in every row .

We begin with the first comparison result.

Theorem 3.3.1 *Let A satisfies (A1), (A3) and*

$$\sum_{j=1}^{m} a_{kj} > 0 \text{ for some } k \in \{1, 2, ..., n\}. \tag{3.2}$$

Let $\mathbf{u}$, $\mathbf{v} : \mathbb{T}^N \to \mathbb{R}^m$ be continuous sub and supersolution of the system (HJα) respectively, then

$$\mathbf{u}(x) \leq \mathbf{v}(x) \quad \text{for every } x \in \mathbb{T}^N.$$

Proof. The proof is done by contradiction. Assume that there exists a point $x \in \mathbb{T}^N$ such that $u_i(x) > v_i(x)$ for some $i \in \{1, 2, ..., m\}$. We then consider:

$$M = \max_{1 \leq i \leq m} \max_{\mathbb{T}^N}(u_i - v_i) > 0.$$

Let $x_0 \in \mathbb{T}^N$ be the point where this maximum is attained. Define

$$W = \{i \in \{1, 2, ..., m\}; u_i(x_0) - v_i(x_0) = M\}.$$

The key idea is to insert penalization terms and use the method of doubling the variables. For every $\varepsilon > 0$, we define the function $\phi_\varepsilon : \mathbb{T}^N \times \mathbb{T}^N \to \mathbb{R}$ by

$$\phi_\varepsilon(x, y) = u_i(x) - v_i(y) - \frac{|x - y|^2}{2\varepsilon^2} - \frac{|x - x_0|^2}{2}$$

This clearly admits a maximum at some point $(x_\varepsilon, y_\varepsilon) \in \mathbb{T}^N \times \mathbb{T}^N$. By a standard argument in the theory of viscosity solution, see for instance [2], the following properties hold:

$$x_\varepsilon, y_\varepsilon \to x_0, \quad \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

Also,

$$p_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} \in D^- v_i(y_\varepsilon) \quad \text{and } q_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} - (x_\varepsilon - x_0) \in D^+ u_i(x_\varepsilon). \tag{3.4}$$

Due to the lipschitz character of u_i, we deduce that the vectors q_ε are equibounded and hence converges, up to a subsequence, to some vector $p \in \mathbb{R}^N$.

So, in view of (3.3) and (3.4), we get

$$p_\varepsilon, q_\varepsilon \to p \quad \text{as } \varepsilon \to 0.$$

From the definition of viscosity sub and supersolution we obtain

$$H_i(x_\varepsilon, q_\varepsilon) + \sum_{j=1}^{m} a_{ij} u_j(x_\varepsilon) \leq a,$$

$$H_i(y_\varepsilon, p_\varepsilon) + \sum_{j=1}^{m} a_{ij} v_j(y_\varepsilon) \geq a.$$

By subtracting the above second inequality from the first one and passing to the limit for $\varepsilon \to 0$, we get

$$\sum_{j=1}^{m} a_{ij}(u_j(x_0) - v_j(x_0)) \leq 0 \quad \text{for every } i \in W. \tag{3.5}$$

If $W = \{1, 2, ..., m\}$, then by inequality (3.5), we get

$$M \sum_{j=1}^{m} a_{ij} \leq 0 \quad \text{for every } i \in \{1, 2, ..., m\},$$

which is a contradiction to assumption (3.2).
If $W \neq \{1, 2, ..., m\}$, by irreducibility character of A, there exists $i \in W$ and $k \notin W$ such that $a_{ik} < 0$. From (3.5) and the assumptions $a_{ij} \leq 0$ for every $i \neq j$ and $M > 0$, we obtain

$$\begin{aligned} M a_{ii} &\leq \sum_{j \neq i} |a_{ij}|(u_j(x_0) - v_j(x_0)) \\ &\leq M \sum_{j \neq i} |a_{ij}| \\ &\leq M a_{ii}. \end{aligned}$$

Hence these inequalities are equalities and so $u_k(x_0) - v_k(x_0) = M$, which yields to a contradiction since $k \notin W$. □

In contrast to the previous result, in the degenerate case where each row sums to zero, we don't have a comparison between sub and super solution. Because arguing as in the proof of Theorem 3.3.1, we get $W = \{i \in \{1, 2, ..., m\}; u_i(x_0) - v_i(x_0) = M\} = \{1, 2, ..., m\}$. So, we have the next remarkable property of weakly coupled systems (see Theorem 2.3 in [15]):

Proposition 3.3.2 *Assume that u, v are respectively a continuous sub and supersolution of the system (HJα) for some $\alpha \in \mathbb{R}$. Let $x_0 \in \mathbb{T}^N$ be such that*

$$u_i(x_0) - v_i(x_0) = M = \max_{1 \leq i \leq m} \max_{\mathbb{T}^N} (u_i(x) - v_i(x)) \quad \text{for some } i \in \{1, 2, ..., m\}.$$

Then $\mathbf{u}(x_0) - \mathbf{v}(x_0) = M\mathbb{1}$.

We now proceed to show a comparison result for (HJα) in the non-degenerate case under the following hypotheses:

(A4) $a_{ij} \le 0$ for every $i \neq j$ and $\sum_{j=1}^{m} a_{ij} \ge \lambda > 0$ for every $i \in \{1, 2, ..., m\}$.

Theorem 3.3.3 . *Assume (A4) holds. Let* $\mathbf{u}$, $\mathbf{v} : \mathbb{T}^N \to \mathbb{R}^m$ *be respectively, continuous sub and supersolution of the weakly coupled system (3.1). Then*

$$\mathbf{u}(x) \le \mathbf{v}(x) \quad \text{for every } x \in \mathbb{T}^N.$$

Proof. Set

$$M = \max_{1 \le i \le m} \max_{\mathbb{T}^N} (u_i - v_i).$$

We want to show that $M \le 0$. Assume by contradiction that $M > 0$ and pick a point $x_0 \in \mathbb{T}^N$ where such a maximum is attained. Define

$$W = \{i \in \{1, 2, ..., m\}; u_i(x_0) - v_i(x_0) = M\}.$$

Arguing as in the proof of Theorem 3.3.1, we get

$$\sum_{j=1}^{m} a_{ij}(u_j(x_0) - v_j(x_0)) \le 0 \quad \text{for every } i \in W, \tag{3.6}$$

that is, since $i \in W$,

$$M a_{ii} \le \sum_{j \neq i} |a_{ij}|(u_j(x_0) - v_j(x_0)) \le M \sum_{j \neq i} |a_{ij}|.$$

This implies that

$$M\,\lambda < M \sum_{j=1}^{m} a_{ij} \le 0,$$

which is contradiction to λ, $M\ > 0$. □

3.4 PDE approach to Hamilton-Jacobi systems

This section is devoted to study degenerate Hamilton–Jacobi systems from a PDE point of view. We review some main results, taken from Davini-Zavidovique [15], and prove new properties for latter use.

We first define the critical value β, in analogous to the scalar case, as

$$\beta = \inf\{\alpha |\ (\text{HJ}\alpha) \text{ admits subsolution }\}.$$

Indeed the infimum in the definition of β is actually a minimum, see Proposition 2.9 in [15]. We are then interested in the critical weakly coupled system

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \beta \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, 2, ..., m\}. \tag{HJβ}$$

Moreover we have the next characterization of the critical value

Proposition 3.4.1 *The critical system is the unique in the one–parameter family (HJα), $\alpha \in \mathbb{R}$, for which there are solutions.*

We now state two basic propositions which are analogous to scalar Hamilton–Jacobi equations. Due to coercivity we have

Proposition 3.4.2 *The family of all subsolutions to (HJα) is equi-Lipschitz continuous with Lipschitz constant denoted by ℓ_α.*

Due to the convexity of the Hamiltonians H_i, the following equivalences also hold:

Proposition 3.4.3 *Let $\mathbf{u} : \mathbb{T}^N \to \mathbb{R}^m$ be Lipschitz continuous function. Then the following facts are equivalent for every $(x, i) \in \mathbb{T}^N \times \{1, 2, ..., m\}$:*

(i) $H_i(x, p) + \sum_{j=1}^m a_{ij} u_j(x) \leq \alpha$ *for every* $p \in D^+ u_i(x)$;

(ii) $H_i(x, Du_i(x)) + \sum_{j=1}^m a_{ij} u_j(x) \leq \alpha$ *for a.e.* $x \in \mathbb{T}^N$;

(iii) $H_i(x, p) + \sum_{j=1}^m a_{ij} u_j(x) \leq \alpha$ *for every* $p \in \partial u_i(x)$.

An adaptation of the pull-up method used in the scalar version of the theory gives:

Proposition 3.4.4 *The maximal subsolution in the family of subsolutions to (HJα) taking the same value at a given point y is solution to (HJα) in $\mathbb{T}^N \setminus \{y\}$.*

Next we start our qualitative analysis on the critical weakly coupled system. Similar to what happens in the scalar case, the obstruction in getting subsolutions of the system below the critical value is not spread over the torus, but concentrated instead on a distinguished closed nonempty subset of $\mathbb{T}^N$, named Aubry set and denoted by $\mathcal{A}$. Following [15], we give the definition of the Aubry set from the PDE point of view.

Definition 3.4.5 *A point y belongs to the Aubry set $\mathcal{A}$ if any maximal critical subsolution taking a given value at y is a solution to (HJβ).*

More specifically, there cannot be any critical subsolution which is, in addition, locally strict at a point in $\mathcal{A}$, in the sense of subsequent definition.

Definition 3.4.6 *Given a critical subsolution $\mathbf{u}$ of (HJβ). We say that u_i is strict at $y \in \mathbb{T}^N$ if there exists a neighborhood U of y and $\delta > 0$ such that*

$$H_i(x, Du_i(x)) + \sum_{j=1}^{m} a_{ij} u_j(x) \leq \beta - \delta \quad \text{for a.e. } x \in U.$$

Moreover, we say that $\mathbf{u}$ is strict at y if u_i is strict at y for every $i \in \{1, 2, ..., m\}$.

An useful criterion to check the latter property is the following

Lemma 3.4.7 *Let $y \in \mathbb{T}^N$ and $\mathbf{u}$ be a subsolution of (HJβ). The $i-$th component of $\mathbf{u}$ is locally strict at y if and only if*

$$H_i(y, p) + \sum_{j=1}^{m} a_{ij} u_j(y) < \beta \quad \text{for any } p \in \partial u_i(y).$$

The mathematical formulation of the obstruction property is contained in the following Theorem, which is an similar characterization to that in scalar eikonal case:

Theorem 3.4.8 *A point $y \notin \mathcal{A}$ if and only if for any index $i \in \{1, \cdots, m\}$ there exists a critical subsolution $\mathbf{u}$ with u_i locally strict at y.*

We proceed by stating a global version of the previous proposition, based on covering argument. Namely we have

Theorem 3.4.9 *There exists a subsolution which is strict in $\mathbb{T}^N \setminus \mathcal{A}$.*

A converse of this result is the next proposition which shows that the i-th component of any critical subsolution fulfills the supersolution test on $\mathcal{A}$.

Proposition 3.4.10 *Let $y \in \mathcal{A}$. Then, for every $i \in \{1, 2, ..., m\}$ and $\mathbf{u}$ a subsolution of (HJβ),*

$$H_i(y, p) + \sum_{j=1}^{m} a_{ij} u_j(y) = 0 \quad \text{for every } p \in D^- u_i(y).$$

An interesting fact pointed out in [15] is that there is a restriction on the values that a subsolution to (HJα) can attain at a given point. This is a property due to the vectorial structure of the problem and has no counterpart in the scalar case. They refer to it as rigidity phenomenon. We have

Theorem 3.4.11 *Let $y \in \mathcal{A}$ and $\mathbf{u}, \mathbf{v}$ be two subsolutions of (HJβ), then*

$$\mathbf{u}(y) - \mathbf{v}(y) = k\mathbf{1}, \quad k \in \mathbb{R} \tag{3.7}$$

For $\alpha \geq \beta$, we define for $x \in \mathbb{T}^N$

$$F_\alpha(x) = \{\mathbf{b} \in \mathbb{R}^m \mid \exists\, \mathbf{u} \text{ subsolution to (HJ}\alpha\text{) with } \mathbf{u}(x) = \mathbf{b}\}. \tag{3.8}$$

It is clear that

$$\mathbf{b} \in F_\alpha(x) \;\Rightarrow\; \mathbf{b} + \lambda\,\mathbf{1} \in F_\alpha(x) \qquad \text{for any } \lambda \in \mathbb{R},$$

where $\mathbf{1}$ is the vector of $\mathbb{R}^m$ with all the components equal to 1. This is in a sense equivalent of adding a constant to a subsolution in the scalar case. It is also apparent from the stability properties of subsolutions and the convex character of the Hamiltonians, that F_α is closed and convex at any x.

The rigidity phenomenon becomes severe on $\mathcal{A}$ where the admissible values make up a one–dimensional set on $\mathcal{A}$, as made precise by the following proposition:

Proposition 3.4.12 *An element y belongs to the Aubry set if and only if*

$$F_\beta(y) = \{\mathbf{b} + \lambda\,\mathbf{1} \mid \lambda \in \mathbb{R}\}$$

where $\mathbf{b}$ is some vector in $\mathbb{R}^m$ depending on y, and $\mathbf{1}$ is the vector of $\mathbb{R}^m$ with all the components equal to 1.

To complete the picture we proved recently in [19] that $F_\beta(y)$ possesses a non empty interior outside $\mathcal{A}$, which is characterized as follows:

Proposition 3.4.13 *Given $y \notin \mathcal{A}$, the interior of $F_\beta(y)$ is nonempty, and $\mathbf{b} \in \mathbb{R}^m$ is an internal point of $F_\beta(y)$ if and only if there is a critical subsolution $\mathbf{u}$ locally strict at y with $\mathbf{u}(y) = \mathbf{b}$.*

Proof. The values $\mathbf{b}$ corresponding to critical subsolutions locally strict at y make up a nonempty set in force of Theorem 3.4.8, it is in addition convex by the convex character of the system. We will denote it by $\widetilde{F}_\beta(y)$.

Let $\mathbf{b} \in \widetilde{F}_\beta(y)$, we claim that there exists $\nu_0 > 0$ with

$$\mathbf{b} + \nu\, \mathbf{e}_i \in \widetilde{F}_\beta(y) \qquad \text{for any } i,\ \nu_0 > \nu > 0. \tag{3.9}$$

We denote by $\mathbf{u}$ the locally strict critical subsolution with $\mathbf{u}(y) = \mathbf{b}$, then there exists $0 < \epsilon < 1$ and $\delta > 0$ such that

$$H_i(x, Du_i(x)) + \sum_{j=1}^{m} a_{ij} u_j(x) \leq \beta - 2\,\delta \quad \text{for any } i, \text{ a.e. } x \in B(y, \epsilon). \tag{3.10}$$

We fix i and assume

$$\eta(\epsilon) + a_{ii}\,\frac{\epsilon^2}{2} < \delta \qquad \text{for any } i. \tag{3.11}$$

where η is a continuity modulus for $(x, p) \mapsto H_i(x, p)$ in $\mathbb{T}^N \times B(0, \ell_\beta + 1)$ and ℓ_β is a Lipschitz constant for all critical subsolutions, see Proposition 3.4.2.

We define $\mathbf{w} : \mathbb{T}^N \to \mathbb{R}^M$ via

$$w_j(x) = \begin{cases} u_j(x) & \text{if } j \neq i \\ \max\{\phi(x), u_i(x)\} & \text{if } j = i \end{cases}$$

where

$$\phi(x) := u_i(x) - \frac{1}{2}\,|y - x|^2 + \frac{\epsilon^2}{2}$$

Notice that

$$w_i = \phi > u_i \quad \text{in} \quad B(y, \epsilon) \qquad \text{and} \qquad w_i = u_i \quad \text{outside } B(y, \epsilon). \tag{3.12}$$

By (3.10), (3.11) and the assumptions on the coupling matrix, we have for any i and a.e. $x \in B(y, \epsilon)$

$$\begin{aligned} & H_i(x, Dw_i(x)) + \sum_j a_{ij}\, w_j(x) \\ = \;& H_i(x, Du_i(x) + (y - x)) + \sum_{j \neq i} a_{ij}\, u_j(x) + a_{ii}\, \phi(x) \\ \leq \;& H_i(x, Du_i(x)) + \eta(\epsilon) + \sum_j a_{ij}\, u_j(x) + a_{ii}\, \frac{\epsilon^2}{2} \\ \leq \;& \beta - 2\,\delta + \delta = \beta - \delta \end{aligned}$$

Further, for $j \neq i$ and for a.e. $x \in B(y, \epsilon)$, we have

$$\begin{aligned} & H_j(x, Dw_j(x)) + \sum_k a_{jk} w_k(x) \\ = \; & H_j(x, Du_j(x)) + \sum_k a_{jk} u_j(x) + a_{ik} \left(-\frac{1}{2} |y - x|^2 + \frac{\epsilon^2}{2} \right) \\ \leq \; & \beta - 2\,\delta, \end{aligned}$$

where the last inequality is due to the fact that $a_{ki} \leq 0$. The previous computations and (3.12) show that $\mathbf{w}$ is a critical subsolution locally strict at y, and this property is inherited by

$$\lambda\, \mathbf{w} + (1 - \lambda)\, \mathbf{u}$$

for any $\lambda \in [0, 1]$. We therefore prove (3.9) setting $\nu_0 = \frac{\epsilon^2}{2}$.

Taking into account that $\mathbf{b} + \lambda\, \mathbf{1} \in \widetilde{F}_\beta(y)$ for any $\lambda \in \mathbb{R}$ and that the vectors $\mathbf{e}_i$, $i = 1, \cdots, m$, and $-\mathbf{1}$ are affinely independent, we derive from (3.9) and $\widetilde{F}_\beta(y)$ being convex, that $\mathbf{b}$ is an internal point of $\widetilde{F}_\beta(y)$ and consequently that $\widetilde{F}_\beta(y)$ is an open set. Finally it is also dense in $F_\beta(y)$ because if $\mathbf{v}$ is any critical subsolution and $\mathbf{u}$ is in addition locally strict at y then any convex combination of $\mathbf{u}$ and $\mathbf{w}$ is locally strict and

$$\lambda\, \mathbf{u}(y) + (1 - \lambda)\, \mathbf{v}(y) \to \mathbf{v}(y) \qquad \text{as } \lambda \to 0.$$

The property of being open, convex and dense in $F_\beta(y)$ implies that $\widetilde{F}_\beta(y)$ must coincide with the interior of $F_\beta(y)$, as claimed.

□

3.5 Dynamical approach to Hamilton-Jacobi systems

The aim of this section is to specify the random setting of our problem and recall the existing dynamical results.

3.5.1 Path spaces

We refer readers to [7] for the material presented in this section without the proof.

The term càdlàg is used to indicate a function which is continuous on the right and has left limit. We denote by $\mathcal{D} := \mathcal{D}(0, +\infty; \{1, \cdots, m\})$ and $\mathcal{D}(0, +\infty; \mathbb{R}^N)$ the spaces of càdlàg paths defined in $[0, +\infty)$ with values in $\{1, \cdots, m\}$ and $\mathbb{R}^N$, respectively. For any

$t > 0$, we also indicate by $\mathcal{D}(0, t; \{1, \cdots, m\})$ the space of càdlàg paths defined in $[0, t]$ with values in $\{1, \cdots, m\}$. It can be proved that

$$\text{Any càdlàg path has at most countably many discontinuities.} \tag{3.13}$$

$$\text{Any càdlàg path is locally (in time) bounded.} \tag{3.14}$$

To any finite increasing sequence of times $t_1, \cdots, t_k$, with $k \in \mathbb{N}$, and indices $j_1, \cdots, j_k$ in $\{1, \cdots, m\}$ we associate with a cylinder defined as

$$\mathcal{C}(t_1, \cdots, t_k; j_1, \cdots, j_k) = \{\omega \mid \omega(t_1) = j_1, \cdots, \omega(t_k) = j_k\} \subset \mathcal{D}.$$

We denote by $\mathcal{D}_i$ cylinders of type $\mathcal{C}(0; i)$ for any $i \in \{1, \cdots, m\}$.

We call multi-cylinders the sets made up by finite unions of mutually disjoint cylinders.

The space $\mathcal{D}$ of càdlàg paths is endowed with the σ–algebra $\mathcal{F}$ spanned by cylinders of the type $\mathcal{C}(s; i)$, for $s \geq 0$ and $i \in \{1, \cdots, m\}$. A natural related filtration $\mathcal{F}_t$ is obtained by picking, as generating sets, the cylinders $\mathcal{C}(t_1, \cdots, t_k; j_1, \cdots, j_k)$ with $t_k \leq t$, for any fixed $t \geq 0$.

We can perform same construction in $\mathcal{D}(0, +\infty; \mathbb{R}^N)$, and in this case the σ–algebra, denoted by $\mathcal{F}'_t$, is spanned by the sets

$$\{\xi \in \mathcal{D}(0, +\infty; \mathbb{R}^N) \mid \xi(s) \in E\} \tag{3.15}$$

for $s \geq 0$ and E varying in the Borel σ–algebra related to the natural topology of $\mathbb{R}^N$. A related filtration is given by the increasing family of σ–algebras $\mathcal{F}'_t$ spanned by cylinders in (3.15) with $s \leq t$.

Both $\mathcal{D}$ and $\mathcal{D}(0, +\infty; \mathbb{R}^N)$ can be endowed with a metric, named after Skorohod, which makes them Polish spaces, namely complete and separable. Above σ–algebras are the corresponding Borel σ–algebras.

A consequence of the previous definitions is that $\mathcal{F}$ is the minimal σ–algebra for which the evaluation maps

$$t \mapsto \omega(t) \qquad t \in [0, +\infty)$$

are measurable and the same holds true for the σ–algebra in $D(0, +\infty; \mathbb{R}^N)$ with respect to the evaluation maps

$$\xi \mapsto \xi(t).$$

A map $\Xi : \mathcal{D} \to D(0,+\infty;\mathbb{R}^N)$ (resp $\phi : \mathcal{D} \to \mathcal{D}$) is accordingly measurable if and only if the maps $\omega \mapsto \Xi(\omega)(t)$ from $\mathcal{D}$ to $\mathbb{R}^N$ (resp., $\omega \mapsto \phi(\omega)(t)$ from $\mathcal{D}$ to $\{1,\cdots,m\}$) are measurable for any t.

The convergence induced by Skorohod metric can be defined, say in $\mathcal{D}(0,+\infty;\mathbb{R}^N)$ to fix ideas, requiring that there exists a sequence f_n of strictly increasing continuous functions from $[0,+\infty]$ onto itself (then $f_n(0) = 0$ for any n) such that

$$\begin{aligned} f_n(s) &\to s \quad \text{uniformly in } [0,+\infty] \\ \xi_n(f_n(s)) &\to \xi(s) \quad \text{uniformly in } [0,+\infty]. \end{aligned}$$

We infer from the previous definition that

$$\xi_n \to \xi \ \text{ in the Skorohod sense } \Rightarrow \xi_n(t) \to \xi(t) \ \text{ at any continuity point of } \xi. \tag{3.16}$$

which in particular implies

$$\xi_n \to \xi \ \text{ in the Skorohod sense } \Rightarrow \xi_n(0) \to \xi(0) \tag{3.17}$$

We moreover have

Any sequence convergent in the Skorohod sense is locally uniformly bounded. (3.18)

We consider the measurable shift flow ϕ_h on $\mathcal{D}$, for $h \geq 0$, defined by

$$\phi_h(\omega)(s) = \omega(s+h) \qquad \text{for any } s \in [0,+\infty),\ \omega \in \mathcal{D}.$$

Notice that ϕ_h is not in general continuous since the fact that $\omega_n \to \omega$ in the Skorohod metric does not in general implies that $\phi_h(\omega_n)(0) = \omega_n(h) \to \phi_h(\omega)(0) = \omega(h)$, unless of course h is a continuity point for ω, and so does not in turn implies, by (3.17), that $\phi_h(\omega_n)$ converges to $\phi_h(\omega)$.

Proposition 3.5.1 *Given nonnegative constants h, t, we have*

$$\phi_h^{-1}(\mathcal{F}_t) \subset \mathcal{F}_{t+h}.$$

Proof. For any $t_1 \geq 0$, $j_1 \in \{1,\cdots,m\}$ we have

$$\phi_h^{-1}(\mathcal{C}(t_1;j_1)) = \mathcal{C}(t_1+h,j_1).$$

The assertion thus comes from the fact that $\mathcal{F}_t$ is spanned by cylinders of the form $\mathcal{C}(t_1; j_1)$, with $t_1 \leq t$, and in this case $\mathcal{C}(t_1 + h; j_1) \in \mathcal{F}_{t+h}$. □

We also consider that space $\mathcal{C}(0, +\infty; \mathbb{T}^N)$ of continuous paths defined in $[0, +\infty)$ taking values in $\mathbb{T}^N$. It is endowed with a metric giving it the structure of a Polish space, which induces the local uniform convergence.

We define a map

$$\mathcal{I} : \mathcal{D}(0, +\infty; \mathbb{R}^N) \to \mathcal{C}(0, +\infty; \mathbb{R}^N)$$

via

$$\mathcal{I}(\xi)(t) = \left(\int_0^t \xi \, ds \right).$$

Proposition 3.5.2 *The map $\mathcal{I}(\cdot)$ is continuous.*

3.5.2 Random Setting

The material of this section is taken from [26]. We are going to define a family of probability measures on $(\mathcal{D}, \mathcal{F})$. We start from a preliminary result. Taking into account that $\mathcal{F}$, $\mathcal{F}_t$ are generated by cylinders, we get by the Approximation Theorem for Measures, see [23, Theorem 1.65].

Proposition 3.5.3 *Let μ be a finite measure on $\mathcal{F}$. For any $E \in \mathcal{F}$, there is a sequence E_n of multi–cylinders in $\mathcal{F}$ with*

$$\lim_n \mu(E_n \triangle E) = 0,$$

where $\triangle$ stands for the symmetric difference.

As a consequence we see that two finite measures on $\mathcal{D}$ coinciding on the family of cylinders, are actually equal.

Given a probability vector $\mathbf{a}$ in $\mathbb{R}^m$, namely with nonnegative components summing to 1, we define for any cylinder $\mathcal{C}(t_1, \cdots, t_k; j_1, \cdots, j_k)$ a nonnegative function μ_a

$$\mu_a(\mathcal{C}(t_1, \cdots, t_k; j_1, \cdots, j_k)) = \left(\mathbf{a}\, e^{-At_1}\right)_{j_1} \prod_{l=2}^{k} \left(e^{-(t_l - t_{l-1})A}\right)_{j_{l-1}\, j_l}. \tag{3.19}$$

We then exploit that e^{-At} is stochastic to uniquely extend $\mu_{\mathbf{a}}$, through Daniell–Kolmogorov Theorem, to a probability measure $\mathbb{P}_{\mathbf{a}}$ on $(\mathcal{D}, \mathcal{F})$, see for instance [37, Theorem 1.2]. Hence, in view of (3.19), we have

Proposition 3.5.4 *The map*

$$\boldsymbol{a} \to \mathbb{P}_{\mathbf{a}}$$

is injective, linear and continuous from $\mathcal{S} \subset \mathbb{R}^m$ to the space of probability measures on $\mathcal{D}$ endowed with the weak convergence.
Consequently, the measures $\mathbb{P}_{\mathbf{a}}$ are spanned by $\mathbb{P}_i := \mathbb{P}_{\mathbf{e_i}}$, for $i \in \{1, \cdots, m\}$, and

$$\mathbb{P}_{\mathbf{a}} = \sum_{i=1}^{m} a_i \, \mathbb{P}_i. \tag{3.20}$$

Since by (3.19) the measures $\mathbb{P}_i$ are supported in $\mathcal{D}_i \in \mathcal{F}_0$, we also deduce

$$\mathbb{P}_{\mathbf{a}}(A) = \sum_{i=1}^{m} a_i \, \mathbb{P}_i(A \cap \mathcal{D}_i) \qquad \text{for any } A \in \mathcal{F},$$

and

$$a_i = \mathbb{P}_{\mathbf{a}}(\mathcal{D}_i) \qquad \text{for any } i \in \{1, \cdots, m\}.$$

By (3.19) we also get that $\mathbb{P}_i$ are supported in $\mathcal{D}_i := \mathcal{C}(0; i)$.

We denote by $\mathbb{E}_{\mathbf{a}}$ the expectation operators relative to $\mathbb{P}_{\mathbf{a}}$, and we put $\mathbb{E}_i$ instead of $\mathbb{E}_{\mathbf{e_i}}$.

We say that some property holds almost surely, a.s. for short, if it is valid up to $\mathbb{P}_{\mathbf{a}}-$null set for all $\mathbf{a} > 0$. We state the next property for later use

Lemma 3.5.5 *Let f, $\boldsymbol{a}$ be a real random variable and a positive probability vector, respectively. If*

$$\int_E f \, d\mathbb{P}_{\boldsymbol{a}} = 0 \qquad \textit{for any } E \in \mathcal{F}$$

then $f = 0$ a.s.

Definition 3.5.6 *A random variable is a measurable map from $(\mathcal{D}, \mathcal{F})$ to a Polish space endowed with the Borel σ–algebra. A simple random variable is the one that takes on finitely many values.*

We consider the push–forward of the probability measure $\mathbb{P}_{\mathbf{a}}$, for any $\mathbf{a} \in \mathcal{S}$, through the flow ϕ_h on $\mathcal{D}$. For a cylinder $C := \mathcal{C}(t_1, \cdots, t_k; j_1, \cdots, j_k)$, we have for any $\mathbf{a} \in \mathcal{S}$

$$\begin{aligned} \phi_h \# \mathbb{P}_{\mathbf{a}}(C) &= \mathbb{P}_{\mathbf{a}}\{\omega \mid \phi_h(\omega) \in C\} = \mathbb{P}_{\mathbf{a}}(\mathcal{C}(t_1 + h, \cdots, t_k + h; j_1, \cdots, j_k)) \\ &= \left(\mathbf{a}\, e^{-(t_1+h)A}\right)_{j_1} \prod_{l=2}^{k-1} \left(e^{-A(t_l - t_{l-1})}\right)_{j_l \, j_{l-1}} = \mathbb{P}_{\mathbf{a}\, e^{-Ah}}(C), \end{aligned}$$

which implies

$$\phi_h \# \mathbb{P}_{\mathbf{a}}(E) = \mathbb{P}_{\mathbf{a}\,e^{-Ah}}(E) \qquad \text{for any } E \in \mathcal{F}.$$

We therefore establish:

Proposition 3.5.7 *For any $\boldsymbol{a} \in \mathcal{S}$, $h \geq 0$,*

$$\phi_h \# \mathbb{P}_{\boldsymbol{a}} = \mathbb{P}_{\boldsymbol{a}\,e^{-hA}}.$$

Accordingly, for any measurable function $f : \mathcal{D} \to \mathbb{R}$, we have by the change of variable formula

$$\mathbb{E}_{\mathbf{a}} f(\phi_h) = \int_{\mathcal{D}} f(\phi_h(\omega))\, d\mathbb{P}_{\mathbf{a}} = \int_{\mathcal{D}} f(\omega)\, d\phi_h \# \mathbb{P}_{\mathbf{a}} = \mathbb{E}_{\mathbf{a}\,e^{-Ah}} f. \tag{3.21}$$

Given $t > 0$,the evaluation maps

$$\omega \mapsto \omega(t)$$

are random variables taking values in $\{1, \cdots, m\}$.

The push–forward of $\mathbb{P}_{\mathbf{a}}$ through $\omega(t)$ is a probability measure on indices. More precisely, we have by (3.19)

$$\omega(t) \# \mathbb{P}_{\mathbf{a}}(i) = \mathbb{P}_{\mathbf{a}}(\{\omega \mid \omega(t) = i\}) = \big(\mathbf{a}\, e^{-At}\big)_i$$

for any index $i \in \{1, \cdots, m\}$, so that

$$\omega(t) \# \mathbb{P}_{\mathbf{a}} = \mathbf{a}\, e^{-At}. \tag{3.22}$$

Moreover for $\mathbf{b} = (b_1, \cdots, b_m) \in \mathbb{R}^m$, we have

$$\mathbb{E}_{\mathbf{a}} b_{\omega(t)} = \mathbf{a}\, e^{-At} \cdot \mathbf{b}.$$

Formula (3.22) can be partially recovered for measures of the type $\mathbb{P}_{\mathbf{a}} \llcorner E$ which means $\mathbb{P}_{\mathbf{a}}$ is restricted to E, where E is any set in $\mathcal{F}$.

Lemma 3.5.8 *For a given $\boldsymbol{a} \in \mathcal{S}$, $E \in \mathcal{F}_t$ for some $t \geq 0$, we have*

$$\omega(s) \# (\mathbb{P}_{\boldsymbol{a}} \llcorner E) = \big(\omega(t) \# (\mathbb{P}_{\boldsymbol{a}} \llcorner E)\big)\, e^{-A(s-t)} \qquad \text{for any } s \geq t.$$

Definition 3.5.9 *A stopping time, adapted to $\mathcal{F}_t$, is a nonnegative random variable τ satisfying*

$$\{\tau \leq t\} \in \mathcal{F}_t \qquad \text{for any } t,$$

which also implies $\{\tau < t\}$, $\{\tau = t\} \in \mathcal{F}_t$.

We will repeatedly use the following non increasing approximation of a bounded random variable τ by simple stopping times. We set

$$\tau_n = \sum_j \frac{j}{2^n} \mathbb{I}(\{\tau \in [(j-1)/2^n, j/2^n)\}), \tag{3.23}$$

where $\mathbb{I}(\cdot)$ stands for the *indicator function* of the set at the argument, namely the function equal 1 at any element of the set and 0 in the complement. We have for any j, n

$$\{\tau_n = j/2^n\} = \{\tau < j/2^n\} \cap \{\tau \geq (j-1)/2^n\} \in \mathcal{F}_{j/2^n},$$

moreover the sum in (3.23) is finite, being τ bounded. Hence τ_n are simple stopping times and letting n go to infinity we get:

Proposition 3.5.10 *Given a bounded stopping time τ, the τ_n, defined as in (3.23), make up a sequence of simple stopping times with*

$$\tau_n \geq \tau, \quad \tau_n \to \tau \quad \textit{uniformly in } \mathcal{D}.$$

Admissible controls: We call control any random variable Ξ taking values in $\mathcal{D}(0,+\infty;\mathbb{R}^N)$ such that

(i) it is locally bounded (in time), i.e. for any $t > 0$ there is $M > 0$ with

$$\sup_{[0,t]} |\Xi(t)| < M \qquad \text{a.s.}$$

(ii) it is nonanticipating, i.e. for any $t > 0$

$$\omega_1 = \omega_2 \text{ in } [0,t] \;\; \Rightarrow \;\; \Xi(\omega_1) = \Xi(\omega_2) \text{ in } [0,t].$$

Remark 3.5.11 *The second condition is equivalent to require Ξ to be adapted to the filtration $\mathcal{F}_t$ which means that the map*

$$\omega \mapsto \Xi(\omega)(t)$$

from $\mathcal{D}$ to $\mathbb{R}^N$ is measurable with respect $\mathcal{F}_t$ and the Borel σ–algebra on $\mathbb{R}^N$. This in turn implies that Ξ is in addition progressively measurable, namely, for any t the map

$$(\omega, s) \mapsto \Xi(\omega)(s)$$

from $\mathcal{D} \times [0,t]$ to $\mathbb{R}^N$ is measurable with respect to the σ–algebras $\mathcal{F}_t \times \mathcal{B}[0,t]$ and $\mathcal{B}$, where $\mathcal{B}[0,t]$ and $\mathcal{B}$ denote the family of Borel sets of $[0,t]$ and $\mathbb{R}^N$, respectively.

We denote by $\mathcal{K}$ the class of admissible controls.
Given a bounded stopping time τ and a pair x, y of elements of $\mathbb{T}^N$, we set

$$\mathcal{K}(\tau, y-x) = \{\Xi \in \mathcal{K} \mid \mathcal{I}(\Xi)(\tau) = y - x \,\text{a.s.}\},$$

where the symbol $-$ refers to the structure of additive group on $\mathbb{T}^N$ induced by the projection of $\mathbb{R}^N$ onto $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. The controls belonging to $\mathcal{K}(\tau, 0)$ are called τ–cycles.

3.5.3 Main dynamical results

Here we recall some dynamical facts about critcal/supercritical subsolutions, admissible set and Aubry set. We refer to [26] for proofs and more details on the results stated.

Given $\alpha \geq \beta$ and an initial point $x \in \mathbb{T}^N$. The action functional adapted to the system is

$$\mathbb{E}_{\mathbf{a}}\left[\int_0^\tau L_{\omega(s)}(x + \mathcal{I}(\Xi)(s), -\Xi(s)) + \alpha\, ds\right],$$

where $\mathbf{a}$ is any probability vector of $\mathbb{R}^m$, τ a bounded stopping time and Ξ a control.

The action functional is used to characterize all the subsolutions of the system, by means of suitable estimate. We have

Theorem 3.5.12 *A function $\mathbf{u} : \mathbb{T}^N \to \mathbb{R}^m$ is a subsolution of (HJα), for any $\alpha \geq \beta$, if and only if*

$$\mathbb{E}_{\mathbf{a}}\big[u_{\omega(0)}(x) - u_{\omega(\tau)}(y)\big] \leq \mathbb{E}_{\mathbf{a}}\left[\int_0^\tau L_{\omega(s)}(x + \mathcal{I}(\Xi)(s), -\Xi(s)) + \alpha\, ds\right],$$

for any pair of points x, y in $\mathbb{T}^N$, $\mathbf{a} \in \mathcal{S}$, any bounded stopping time τ and $\Xi \in \mathcal{K}(\tau, y-x)$.

Given y in $\mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}^m$, we define

$$v_i(x) = \inf\, \mathbb{E}_i\left[\int_0^\tau L_\omega((x + \mathcal{I}(\Xi), -\Xi) + \alpha\, ds + b_{\omega(\tau)}\right] \tag{3.24}$$

for any $i \in \{1, \cdots, m\}$, $x \in \mathbb{T}^N$, where the infimum is taken with respect to any bounded stopping times τ and $\Xi \in \mathcal{K}(\tau, y-x)$. We have

Theorem 3.5.13 *The function $\mathbf{v}$ defined by* (3.24) *is subsolution to (HJα).*

The action functional is also used to give geometric formulation of the set $\mathcal{F}_\alpha(y)$ as following:

Theorem 3.5.14 *For $y \in \mathbb{T}^N$, $\boldsymbol{b} \in \mathcal{F}_\alpha(y)$ if and only if*

$$\mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi)(s), -\Xi(s)) + \alpha \, ds - b_i + b_{\omega(\tau)} \right] \geq 0, \tag{3.25}$$

for any $i \in \{1, \cdots, m\}$, bounded stopping times τ and τ–cycles Ξ.

Proof. We denote by $\mathbf{v}$ the function defined in (3.24). If (3.25) holds, we get

$$\mathbf{v}(y) \geq \mathbf{b}.$$

Moreover, by taking the stopping time $\tau \equiv 0$ and the control $\Xi \equiv 0$ in (3.24), we also get

$$\mathbf{v}(y) \leq \mathbf{b}.$$

Then $\mathbf{v}(y) = \mathbf{b}$ with $\mathbf{v}$ being subsolution to (HJα), inview of Theorem 3.5.13, which proves $\mathbf{b} \in \mathcal{F}_\alpha(y)$.

Conversely, if there is a subsolution u of (HJα) with $u(y) = \mathbf{b}$ then (3.25) is a direct consequence of Theorem 3.5.12. □

We also have the following explicit representation of critical and supercritical subsolutions enjoying an additional maximality property. More precisely, we have

Theorem 3.5.15 *Assume $\mathbf{b} \in F_\alpha(y)$, then*

(i) $\mathbf{v}(y) = \mathbf{b}$*;*

(ii) $\mathbf{v}$ *is the maximal subsolution to* (HJα) *taking the value* $\mathbf{b}$ *at* y*;*

(iii) *If $\alpha = \beta$ and $y \in \mathcal{A}$ then* $\mathbf{v}$ *is a critical solution.*

Proof. Item **(i)** has already been proved in Theorem 3.5.14. If $\mathbf{u}$ is a subsolution to (HJα) with $\mathbf{u}(y) = \mathbf{b}$, then by Theorem 3.5.12 we get

$$u_i(y) \leq \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(x + \mathcal{I}(\Xi)(s), -\Xi(s)) + \alpha \, ds + b_{\omega(\tau)} \right]$$

for any $i \in \{1, \cdots, m\}$, bounded stopping time τ and τ–cycle Ξ. This shows

$$\mathbf{v} \geq \mathbf{u}.$$

Item **(iii)** directly comes from the definition of the Aubry set.

We also state the next convergence result:

Lemma 3.5.16 *Given a control Ξ^0, a bounded stopping time τ^0 and $\mathbf{b} \in \mathbb{R}^m$, let Ξ_n, τ_n be sequences of controls and bounded stopping times, respectively, with*

$$\begin{aligned} \Xi_n &\to \Xi^0 && \textit{a.s. with respect to Skorohod metric} \\ \tau_n &\to \tau^0 && \textit{uniformly in } \mathcal{D} \\ \tau_n &\geq \tau^0 && \textit{a.s. for any } n. \end{aligned}$$

Assume in addition that the Ξ_n are equibounded locally in time, then

$$\mathbb{E}_i \left(\int_0^{\tau_n} L_\omega(x + \mathcal{I}(\Xi_n), -\Xi_n)\, ds - b_i + b_{\omega(\tau_n)} \right)$$

converges in $\mathbb{R}$ to

$$\mathbb{E}_i \left(\int_0^{\tau^0} L_\omega(x + \mathcal{I}(\Xi^0), -\Xi^0)\, ds - b_i + b_{\omega(\tau)} \right)$$

for any $i \in \{1, \cdots, m\}$.

Chapter 4

Cycle characterization of the Aubry set for weakly coupled Hamilton-Jacobi systems

4.1 Introduction

The aim of this chapter is to provide a dynamical characterization of the Aubry set, associated to degenerate weakly coupled Hamilton-Jacobi systems:

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \quad \text{in } \mathbb{T}^N \quad \text{for every } i \in \{1, \cdots, m\}, \tag{HJα}$$

with $m \geq 2$ and α varying in $\mathbb{R}$. Here $\mathbf{u} = (u_1, \cdots, u_m)$ is the unknown function, A is an $m \times m$ matrix, the so–called coupling matrix, and $H_1, \cdots, ..., H_m$ are Hamiltonians. The H_i satisfy the following set of assumptions for all $i \in \{1, \cdots, m\}$:

(H1) $H_i : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous;

(H2) $p \mapsto H_i(x, p)$ is convex for every $x \in \mathbb{T}^N$;

(H3) $p \mapsto H_i(x, p)$ is superlinear for every $x \in \mathbb{T}^N$.

The superlinearity condition (H3) allows to define the corresponding Lagrangians through the Fenchel transform, namely

$$L_i(x, q) = \max_{p \in \mathbb{R}^N} \{p \cdot q - H_i(x, p)\} \quad \text{for any } i.$$

The coupling matrix $A = (a_{ij})$ satisfies:

(A1) $a_{ij} \leq 0$ for every $i \neq j$;

(A2) $\sum_{j=1}^{m} a_{ij} = 0$ for any $i \in \{1, \cdots, m\}$;

(A3) it is irreducible, i.e for every $W \subsetneq \{1, 2, ..., m\}$ there exists $i \in W$ and $j \notin W$ such that $a_{ij} < 0$.

We remark that the assumptions (A1) and (A2) on the coupling matrix are equivalent to e^{-At} being a stochastic matrix for any $t \geq 0$ and due to irreducibility we get e^{-At} is positive for any $t > 0$, as made precise in Appendix A.2.

As pointed out previously, we are interested in the critical system:

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \beta \quad \text{in } \mathbb{T}^N \quad \text{for every } i \in \{1, \cdots, m\}, \tag{HJβ}$$

where β is the critical value defined as :

$$\beta = \inf\{\alpha \in \mathbb{R} \mid (\text{HJ}\alpha) \text{ admits subsolutions}\}.$$

We recall that the obstruction to the existence of subsolutions below the critical value is not spread indistinctly on the torus, but instead concentrated on the Aubry set. However, so far, no geometrical/dynamical description of $\mathcal{A}$ is available, and the aim of our investigation is precisely to mend this gap.

To deepen knowledge of the Aubry set seems important for the understanding of the interplay between equational and dynamical facts in the study of the system, which is at the core of an adapted weak KAM theory. This will hopefully allow to attack some open problem in the field, the most relevant being the existence of regular subsolutions. Another related application, at least when the Hamiltonians are of Tonelli type, is in the analysis of random evolutions associated to weakly coupled systems, see [14].

To this purpose, we take advantage of the action functional introduced in section 3.5.3 in relation to the systems. We also make a crucial use of the characterization of admissible values through the action functional computed on random cycles, see Theorem 3.5.14.

The starting point is the cycle characterization of the Aubry set holding in the scalar case, see [18]. It asserts that a point is in the Aubry set if and only there exists, for some ϵ positive, a sequence of cycles based on it, and defined in $[0, t]$ with $t > \epsilon$, on which the action functional is infinitesimal. Of course the role of the lower bound ϵ is crucial, otherwise the property should be trivially true for any element of the torus.

To generalize it in the context of systems, we need using random cycles defined on intervals with a stopping time, say τ, as right endpoint. We call it τ–cycles, see section 3.5.2. This makes the adaptation of the ϵ–condition quite painful. To perform the task, we use the notion of stopping time strictly greater than ϵ, $\tau \gg \epsilon$, see Definition 4.2.5, which seems rather natural but that we were not able to find in the literature. We therefore present in Section 4.2 some related basic results. We, in particular, prove that the exponential of the coupling matrix related to a $\tau \gg \epsilon$ is strictly positive, see Proposition 4.2.6. This property will be repeatedly used throughout the chapter.

We moreover provide a strengthened version of the aforementioned Theorem 3.5.14, roughly speaking showing that the τ–cycles with $\tau \gg \epsilon$ are enough to characterize admissible values for critical subsolutions, see Theorem 4.4.1. This result is in turn based on a cycle iteration technique we explain in Section 4.3.

The main output is presented in two versions, see Theorems 4.4.3, 4.4.4, with the latter one, somehow more geometrically flavored, exploiting the notion of characteristic vector of a stopping time, see Definition 4.2.1.

This chapter is based on the submitted paper [19] and organized as follows: Section 4.2 is devoted to illustrate some properties of stopping times and the related shift flows. Section 4.3 is about the cycle iteration technique. In section 4.4 we give the main results.

4.2 Properties of stopping times

In this section, we provide more properties of stopping times, in addition to those presented in Section 3.5.2. We also prove properties of shift flow via stopping times.

Given a stopping time τ, then similar to the deterministic case, the push–forward of $\mathbb{P}_{\mathbf{a}}$

through $\omega(\tau)$ is a probability measure on indices $\{1, \cdots, m\}$, which can be identified with an element of $\mathcal{S}$. Then

$$\mathbf{a} \mapsto \omega(\tau)\#\mathbb{P}_{\mathbf{a}},$$

defines a map from the simplex of probability vectors $\mathcal{S}$ to $\mathcal{S}$ which is, in addition, linear. Hence, thanks to Proposition A.2.3, it can be represented by a stochastic matrix which we denote by $e^{-A\tau}$ acting on the right, i.e.

$$\mathbf{a}\, e^{-A\tau} = \omega(\tau)\#\mathbb{P}_{\mathbf{a}} \qquad \text{for any } \mathbf{a} \in \mathcal{S}. \tag{4.1}$$

Definition 4.2.1 *We say that $\boldsymbol{a} \in \mathcal{S}$ is a* characteristic vector *of τ if it is an eigenvector of $e^{-A\tau}$ corresponding to the eigenvalue 1, namely $\mathbf{a} = \mathbf{a}\, e^{-A\tau}$.*

Remark 4.2.2 *According to Proposition A.2.4, any stopping time possesses a characteristic vector $\boldsymbol{a}$, and*

$$\mathbb{E}_{\boldsymbol{a}} b_{\omega(\tau)} = \boldsymbol{a}\, e^{-A\tau} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} \qquad \text{for every } \boldsymbol{b} \in \mathbb{R}^m.$$

According to the remark above, Theorem 3.5.12 takes a simpler form if we just consider expectation operators $\mathbb{E}_{\mathbf{a}}$ with $\mathbf{a}$ characteristic vector. This result will play a key role in Lemma 4.4.5.

Corollary 4.2.3 *A function $\mathbf{u}$ is a subsolution to (HJα) if and only if*

$$\mathbf{a} \cdot \big(\mathbf{u}(x) - \mathbf{u}(y)\big) \le \mathbb{E}_{\mathbf{a}} \left(\int_0^\tau L_{\omega(s)}(x + \mathcal{I}(\Xi)(s), -\Xi(s)) + \beta\, ds \right). \tag{4.2}$$

for any $i \in \{1, \cdots, m\}$, bounded stopping times τ, $\mathbf{a}$ characteristic vector of τ, and $\Xi \in \mathcal{K}(\tau, y - x)$.

Lemma 4.2.4 *Take τ_n as in (3.23). Then*

$$e^{-A\tau_n} \to e^{-A\tau} \qquad \text{as } n \text{ goes to infinity.}$$

Proof. Let $\mathbf{a} \in \mathcal{S}$, $\mathbf{b} \in \mathbb{R}^m$. Being ω right-continuous and $\tau_n \ge \tau$, we get $\omega(\tau_n) \to \omega(\tau)$ for any $\omega \in \mathcal{D}$, and consequently

$$b_{\omega(\tau_n)} \to b_{\omega(\tau)}.$$

This implies, taking into account (4.1)

$$(\mathbf{a}\, e^{-A\tau_n}) \cdot \mathbf{b} = \mathbb{E}_{\mathbf{a}} b_{\omega(\tau_n)} \to \mathbb{E}_{\mathbf{a}} b_{\omega(\tau)} = (\mathbf{a}\, e^{-A\tau}) \cdot \mathbf{b},$$

and yields the assertion. □

Definition 4.2.5 *Given any positive constant ϵ, we say that τ is strongly greater than ϵ, written mathematically as $\tau \gg \epsilon$, to mean that $\tau - \epsilon$ is still a stopping time, or equivalently*

$$\tau \geq \epsilon \ a.s. \quad \text{and} \quad \{\tau \leq t\} \in \mathcal{F}_{t-\epsilon} \quad \text{for any } t \geq \epsilon. \tag{4.3}$$

Moreover for $i \in \{1, \cdots, m\}$, we say

$$\tau \gg \epsilon \text{ in } \mathcal{D}_i$$

to mean

$$\tau \geq \epsilon \ a.s. \text{ in } \mathcal{D}_i \quad \text{and} \quad \{\tau \leq t\} \cap \mathcal{D}_i \in \mathcal{F}_{t-\epsilon} \quad \text{for any } t \geq \epsilon. \tag{4.4}$$

Proposition 4.2.6 *Let $\epsilon > 0$, $i \in \{1, \cdots, m\}$. Then for every $\tau \gg \epsilon$ in $\mathcal{D}_i$, there exists a positive constant ρ, solely depending on ϵ and on the coupling matrix, such that*

$$\left(e^{-A\tau}\right)_{ij} > \rho \qquad j \in \{1, \cdots, m\}. \tag{4.5}$$

Proof. We approximate τ by a sequence of simple stopping times τ_n with $\tau_n \geq \tau$, as indicated in Proposition 3.5.10. For a fixed n, we then have

$$\tau_n = \sum_j \frac{j}{2^n} \mathbb{I}(\{\tau \in [(j-1)/2^n, j/2^n)\}).$$

By the assumption on τ, the set $F_j := \{\tau \in [(j-1)/2^n, j/2^n)\} \cap \mathcal{D}_i$ belongs to $\mathcal{F}_{\frac{j}{2^n}-\epsilon}$. By applying Lemma 3.5.8, we therefore get

$$\begin{aligned} \mathbf{e}_i \, e^{-A\tau_n} &= \omega(\tau_n)\#\mathbb{P}_i = \sum_j \omega(j/2^n)\#(\mathbb{P}_i \llcorner F_j) = \left(\sum_j \omega(j/2^n - \epsilon)\#(\mathbb{P}_i \llcorner F_j) \right) e^{-A\epsilon} \\ &= \left(\omega(\tau_n - \epsilon)\#\mathbb{P}_i\right) e^{-A\epsilon} \end{aligned}$$

Owing to $\omega(\tau_n - \epsilon)\#\mathbb{P}_i \in \mathcal{S}$, we deduce

$$\mathbf{e}_i \, e^{-A\tau_n} \in \{\mathbf{b} \, e^{-A\epsilon} \mid \mathbf{b} \in \mathcal{S}\},$$

we have in addition $e^{-A\tau_n} \to e^{-A\tau}$ by Lemma 4.2.4, and consequently

$$\mathbf{e}_i \, e^{-A\tau} \in \{\mathbf{b} \, e^{-A\epsilon} \mid \mathbf{b} \in \mathcal{S}\}.$$

This set is compact, and contained in the relative interior of $\mathcal{S}$ because $e^{-A\epsilon}$ is positive by Proposition A.2.7. Since the component of $\mathbf{e}_i \, e^{-A\tau}$ make up the i–th row of $e^{-A\tau}$, we immediately derive the assertion. □

According to the previous proposition and Proposition A.2.5, the the characteristic vector of a $\tau \gg \epsilon$, for some $\epsilon > 0$, is unique and positive.

Remark 4.2.7 *Take $\tau \gg \epsilon$ and denote by ρ -the positive constant satisfying* (4.5) *for any i, j, according to Proposition 4.2.6. Then, since $e^{-A\tau}$ is a stochastic matrix, we have*

$$\left(e^{-A\tau}\right)_{ij} = 1 - \sum_{k \neq j} \left(e^{-A\tau}\right)_{ik} \leq 1 - (m-1)\rho \leq 1 - \rho.$$

Remark 4.2.8 *Let τ, ρ be as in the previous remark. If* **a** *is the characteristic vector of τ then by the Proposition 4.2.6 and Proposition A.2.5* **a** *is unique and positive. Moreover for any i we have*

$$a_i = \sum_j a_j \left(e^{-A\tau}\right)_{ji} > \rho.$$

We proceed by establishing some properties for the flow on $\mathcal{D}$ when the shift is given by a stochastic time τ. For any stopping time τ, we consider the shift flow ϕ_τ on $\mathcal{D}$ defined by :

$$\begin{array}{rcl} \phi_\tau : \mathcal{D} & \to & \mathcal{D} \\ \omega & \mapsto & \omega(\cdot + \tau(\omega)). \end{array}$$

We derive

Lemma 4.2.9 *Assume that τ_n is a sequence of stopping times converging to τ uniformly in $\mathcal{D}$, then*

$$\phi_{\tau_n} \to \phi_\tau \qquad \text{as } n \to +\infty$$

pointwise in $\mathcal{D}$, with respect to the Skorohod convergence, see Appendix 3.5.1 for the definition.

Proof. We fix $\omega \in \mathcal{D}$, we set

$$g_n(t) = t + \tau(\omega) - \tau_n(\omega) \qquad \text{for any } n,\, t \geq 0.$$

We have for any t

$$\phi_\tau(\omega)(t) = \omega(t + \tau_n(\omega) + (\tau(\omega) - \tau_n(\omega))) = \phi_{\tau_n}(\omega)(g_n(t)).$$

This yields the asserted convergence because the g_n are a sequence of strictly increasing functions uniformly converging to the identity. □

Proposition 4.2.10 *The shift flow $\phi_\tau : \mathcal{D} \to \mathcal{D}$ is measurable.*

Proof. If τ is a simple stopping time, say of the form $\tau = \sum_k t_k \,\mathbb{I}(E_k)$, then

$$\phi_\tau(\omega) = \sum_k \phi_{t_k}(\omega)\,\mathbb{I}(E_k)(\omega)$$

and the assertion follows being ϕ_{t_k} measurable for any k, $\mathbb{I}(E_k)$ measurable. If τ is not simple then, by Proposition 3.23, there exists a sequence of simple stopping times τ_n converging to τ uniformly in $\mathcal{D}$, this implies that ϕ_τ is measurable as well, as pointwise limit of measurable maps, in force of Lemma 4.2.9. □

We now define the probability measure $\phi_\tau \# \mathbb{P}_{\mathbf{a}}$, for $\mathbf{a} \in \mathcal{S}$. The following result generalizes Proposition 3.5.7 to shifts given for stopping times. It will be used in Theorem 4.3.2 and in Lemma 4.4.2.

Theorem 4.2.11 *Let* $\mathbf{a}$ *be a probability vector, then*

$$\phi_\tau \# \mathbb{P}_{\mathbf{a}} = \mathbb{P}_{\mathbf{a}\, e^{-A\tau}}.$$

We need the following preliminary result:

Lemma 4.2.12 *Let* $\mathbf{a}$*,* t*,* E *be a vector in* $\mathcal{S}$*, a positive deterministic time and a set in* $\mathcal{F}_t$*, respectively, then*

$$\phi_t \# (\mathbb{P}_{\mathbf{a}} \llcorner E) = \mathbb{P}_{\mathbf{a}}(E)\,\mathbb{P}_{\mathbf{b}} \qquad \textit{for some } \mathbf{b} \in \mathcal{S}.$$

Proof.

We first assume E to be a cylinder, namely

$$E = \mathcal{C}(t_1, \cdots, t_k; j_1, \cdots, j_k)$$

for some times and indices, notice that the condition $E \in \mathcal{F}_t$ implies $t_k \le t$. We fix $i \in \{1, \cdots, m\}$ and consider a cylinder $C \subset \mathcal{D}_i$, namely

$$C = \mathcal{C}(0, s_2, \cdots, s_m; i, i_2, \cdots, i_m)$$

for some choice of times and indices. We set

$$F = \{\omega \mid \phi_t(\omega) \in C\} \cap E,$$

then

$$F = \mathcal{C}(t_1, \cdots, t_k, t, t + s_2, \cdots, t + s_m; j_1, \cdots, j_k, i, i_2, \cdots, i_m).$$

We have

$$
\begin{aligned}
\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(C) &= \mathbb{P}_{\mathbf{a}}(F) \\
&= \left(\mathbf{a}\, e^{-At_1}\right)_{j_1} \prod_{l=2}^{k} \left(e^{-A(t_l-t_{l-1})}\right)_{j_{l-1}\, j_l} \left(e^{-A(t-t_k)}\right)_{j_k\, i} \prod_{r=2}^{m} \left(e^{-A(s_r-s_{r-1})}\right)_{i_{r-1}\, i_r} \\
&= \mathbb{P}_{\mathbf{a}}(E) \left(e^{-A(t-t_k)}\right)_{j_k\, i} \prod_{r=2}^{m} \left(e^{-A(s_r-s_{r-1})}\right)_{i_{r-1}\, i_r},
\end{aligned}
$$

we also have

$$\mathbb{P}_i(C) = \prod_{r=2}^{m} \left(e^{-A(s_r-s_{r-1})}\right)_{i_{r-1}\, i_r},$$

and we consequently get the relation

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(C) = \mathbb{P}_{\mathbf{a}}(E)\, \mu_i\, \mathbb{P}_i(C)$$

with

$$\mu_i = \left(e^{-A(t-t_k)}\right)_{j_k\, i} \tag{4.6}$$

just depending on E and i. If C is any cylinder, we write

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(C) = \sum_i \phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(C \cap \mathcal{D}_i) = \mathbb{P}_{\mathbf{a}}(E) \sum_i \mu_i\, \mathbb{P}_i(C) \tag{4.7}$$

where the μ_i are defined as in (4.6). Taking into account that $\mu_i \geq 0$ for any i and $\sum_i \mu_i = 1$, $\mathbf{b} := \sum_i \mu_i\, \mathbf{e}_i \in \mathcal{S}$, we derive from (4.7)

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(C) = \mathbb{P}_{\mathbf{a}}(E)\, \mathbb{P}_{\mathbf{b}}(C).$$

This in turn implies, taking into account (3.20)

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E) = \mathbb{P}_{\mathbf{a}}(E)\, \mathbb{P}_{\mathbf{b}} \tag{4.8}$$

showing the assertion in the case where E is a cylinder. If instead E is a multi–cylinder, namely $E = \cup_j E_j$ with E_j mutually disjoint cylinders then by the previous step

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E) = \sum_j \phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E_j) = \sum_j \mathbb{P}_{\mathbf{a}}(E_j)\, \mathbb{P}_{\mathbf{b}_j}$$

which again implies (4.8) with

$$\mathbf{b} = \sum_j \frac{\mathbb{P}_{\mathbf{a}}(E_j)}{\mathbb{P}_{\mathbf{a}}(E)}\, \mathbf{b}_j.$$

Finally, for a general E, we know from Proposition 3.5.3 that there is a sequence of multi–cylinders E_n with

$$\lim_n \mathbb{P}_{\mathbf{a}}(E_n \triangle E) = 0. \tag{4.9}$$

Given $F \in \mathcal{F}$, we set

$$C = \{\omega \mid \phi_t(\omega) \in F\},$$

we have

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E_n)(F) = \mathbb{P}_{\mathbf{a}}(C \cap E_n) \leq \mathbb{P}_{\mathbf{a}}\big((C \cap E) \cup (E \triangle E_n)\big) = \phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(F) + \mathbb{P}_{\mathbf{a}}(E \triangle E_n)$$

and similarly

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(F) \leq \phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E_n)(F) + \mathbb{P}_{\mathbf{a}}(E \triangle E_n).$$

We deduce in force of (4.9)

$$\lim_n \phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E_n)(F) = \phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)(F)$$

which in turn implies that $\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E_n)$ weakly converges to $\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E)$. Since, by the previous step in the proof

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E_n) = \mathbb{P}_{\mathbf{a}}(E_n)\, \mathbb{P}_{\mathbf{b}_n} \qquad \text{for some } \mathbf{b}_n \in \mathcal{S}$$

we derive from Proposition 3.5.4 and (4.9)

$$\phi_t\#(\mathbb{P}_{\mathbf{a}} \llcorner E) = \mathbb{P}_{\mathbf{a}}(E)\, \mathbb{P}_{\mathbf{b}} \qquad \text{with } \mathbf{b} = \lim_n \mathbf{b}_n.$$

This conclude the proof. □

Proof. (*of the Theorem 4.2.11*)

We first show that

$$\phi_\tau\#\mathbb{P}_{\mathbf{a}} = \mathbb{P}_{\mathbf{b}} \qquad \text{for a suitable } \mathbf{b} \in \mathcal{S}. \tag{4.10}$$

If $\tau = \sum_k t_k\, \mathbb{I}(E_k)$ is simple then by Lemma 4.2.12

$$\phi_\tau\#\mathbb{P}_{\mathbf{a}} = \sum_k \phi_{t_k}\#(\mathbb{P}_{\mathbf{a}} \llcorner E_k) = \sum_k \mathbb{P}_{\mathbf{a}}(E_k)\, \mathbb{P}_{\mathbf{b}_k}$$

for some $b_k \in \mathcal{S}$, and we deduce (4.10) with $\mathbf{b} = \sum_k \mathbb{P}_{\mathbf{a}}(E_k)\, \mathbf{b}_k$.

Given a general stopping time τ, we approximate it by a sequence of simple stopping times τ_n, and, exploiting the previous step, we consider $\mathbf{b}_n \in \mathcal{S}$ with

$$\phi_{\tau_n}\#\mathbb{P}_{\mathbf{a}} = \mathbb{P}_{\mathbf{b}_n}.$$

We know from Lemma 4.2.9 that

$$\phi_{\tau_n}(\omega) \to \phi_\tau(\omega) \qquad \text{for any } \omega \text{ in the Skorohod sense,}$$

and we derive via Dominate Convergence Theorem

$$\mathbb{E}_{\mathbf{a}} f(\phi_{\tau_n}) \to \mathbb{E}_{\mathbf{a}} f(\phi_\tau)$$

for any bounded measurable function $f : \mathcal{D} \to \mathbb{R}$. Using change of variable formula (3.21) we get

$$\int_{\mathcal{D}} f \, d\phi_{\tau_n} \# \mathbb{P}_{\mathbf{a}} \to \int_{\mathcal{D}} f \, d\phi_\tau \# \, \mathbb{P}_{\mathbf{a}}$$

or equivalently

$$\mathbb{P}_{\mathbf{b}_n} = \phi_{\tau_n} \# \mathbb{P}_{\mathbf{a}} \to \phi_\tau \# \mathbb{P}_{\mathbf{a}}$$

in the sense of weak convergence of measures. This in turn implies by the continuity property stated in Proposition 3.5.4 that $\mathbf{b}_n$ is convergent in $\mathbb{R}^m$ and

$$\mathbb{P}_{\mathbf{b}_n} \to \mathbb{P}_{\mathbf{b}} \qquad \text{with } \mathbf{b} = \lim_n \mathbf{b}_n$$

which shows (4.10). We can compute the components of $\mathbf{b}$ via

$$b_i = \mathbb{P}_{\mathbf{a}}\{\omega \mid \phi_\tau(\omega) \in \mathcal{D}_i\} = \mathbb{P}_{\mathbf{a}}\{\omega \mid \omega(\tau(\omega)) = i\} = \big(\omega(\tau) \# \mathbb{P}_{\mathbf{a}}\big)_i = \big(\mathbf{a}\, e^{-A\tau}\big)_i.$$

This concludes the proof. □

4.3 Cycle iteration

It is immediate that we can construct a sequence of (deterministic) cycles going through a given closed curve any number of times. We aim at generalizing this iterative procedure in the random setting we are working with, starting from τ^0–cycle, for some stopping time τ^0. In this case the construction is more involved and requires some detail.

Let τ^0, Ξ^0 be a simple stopping time and a τ^0–cycle, respectively, we recursively define for $j \geq 0$

$$\tau^{j+1}(\omega) = \tau^0(\omega) + \tau^j(\phi_{\tau^0}(\omega)) \tag{4.11}$$

and

$$\Xi^{j+1}(\omega)(s) = \begin{cases} \Xi^j(\omega)(s), & \text{for } s \in [0, \tau^j(\omega)) \\ \Xi^0(\phi_{\tau^j}(\omega))(s - \tau^j(\omega)) & \text{for } s \in [\tau^j(\omega), +\infty). \end{cases} \tag{4.12}$$

We will prove below that the Ξ^j make up the sequence of iterated random cycles we are looking for. A first step is:

Proposition 4.3.1 *The τ^j, defined by (4.11), are simple stopping times for all j. If, in addition, $\tau^0 \gg \delta$ in $\mathcal{D}_i$, for some $i \in \{1, \cdots, m\}$, $\delta > 0$, then $\tau^j \gg \delta$ in $\mathcal{D}_i$.*

Proof. We argue by induction on j. The property is true for $j = 0$, assume to know that τ^j is a simple stopping time, then by Proposition 4.2.10 τ^{j+1} is a random variable, as sum and composition of measurable maps, taking nonnegative values. Assume

$$\tau^0 = \sum_{l=1}^{m_0} s_l \, \mathbb{I}(F_l) \tag{4.13}$$

$$\tau^j = \sum_{k=1}^{m_j} t_k \, \mathbb{I}(E_k) \tag{4.14}$$

then the sets

$$F_l \cap \{\omega \mid \phi_{\tau^0}(\omega) \in E_k\} \qquad l = 1, \cdots, m_0, \; k = 1, \cdots, m_j$$

are mutually disjoint and their union is the whole $\mathcal{D}$. Moreover if

$$\omega \in F_l \cap \{\omega \mid \phi_{\tau^0}(\omega) \in E_k\}$$

then

$$\tau^{j+1}(\omega) = \tau^0(\omega) + \tau^j(\phi_{\tau^0}(\omega)) = s_l + t_k,$$

which shows that τ^{j+1} is simple. Since τ^0, τ^j are stopping time then $F_l \in \mathcal{F}_{s_l}$ and $E_k \in \mathcal{F}_{t_k}$. By Proposition 3.5.1

$$F_l \cap \{\omega \mid \phi_{\tau^0}(\omega) \in E_k\} \in \mathcal{F}_{s_l + t_k},$$

which shows that τ^{j+1} is a stopping time.
Moreover if $\tau^0 \gg \delta$ in $\mathcal{D}_i$ then $F_l \cap \mathcal{D}_i \in \mathcal{F}_{s_l - \delta}$ and consequently

$$F_l \cap \mathcal{D}_i \cap \{\omega \mid \phi_{\tau^0}(\omega) \in E_k\} \in \mathcal{F}_{s_l + t_k - \delta},$$

which shows that $\tau^{j+1} \gg \delta$ in $\mathcal{D}_i$. □

The main result of the section is

Theorem 4.3.2 *The Ξ^j, as defined in* (4.12)*, are τ^j–cycles for all j.*

A lemma is preliminary.

Lemma 4.3.3 *For any j, ω*

$$\tau^{j+1}(\omega) = \tau^j(\omega) + \tau^0(\phi_{\tau^j}(\omega)).$$

Proof. Given $j \geq 1$, we preliminarily write

$$\begin{aligned}\phi_{\tau^{j-1}}(\phi_{\tau^0}(\omega))(s) &= \phi_{\tau^0}(\omega)(s + \tau^{j-1}(\phi_{\tau^0}(\omega)) \\ &= \omega(s + \tau^0(\omega) + \tau^{j-1}(\phi_{\tau^0}(\omega)) = \omega(s + \tau^j(\omega)) = \phi_{\tau^j}(\omega)(s)\end{aligned}$$

which gives

$$\phi_{\tau^{j-1}} \circ \phi_{\tau^0} = \phi_{\tau^j}. \tag{4.15}$$

We proceed arguing by induction on j. The formula in the statement is true for $j = 0$. We proceed showing that it is true for $j+1$ provided it holds for $j \geq 0$. We have, taking into account (4.15)

$$\begin{aligned}\tau^{j+1}(\omega) &= \tau^0(\omega) + \tau^j(\phi_{\tau^0}(\omega)) = \tau^0(\omega) + \tau^{j-1}(\phi_{\tau^0}(\omega)) + \tau^0(\phi_{\tau^{j-1}}(\phi_{\tau^0}(\omega))) \\ &= \tau^j(\omega) + \tau^0(\phi_{\tau^{j-1}}(\phi_{\tau^0}(\omega))) = \tau^j(\omega) + \tau^0(\phi_{\tau^j}(\omega))\end{aligned}$$

as asserted. □

Proof.(of Theorem 4.3.2)

The property is true for $j = 0$, then we argue by induction on j. We exploit the principle that Ξ^j is a control if and only the maps $\omega \mapsto \Xi^j(\omega)(s)$ from $\mathcal{D}$ to $\mathbb{R}^N$ are $\mathcal{F}_s$–measurable for all s. Given s and a a Borel set B of $\mathbb{R}^N$, we therefore need to show

$$\{\omega \mid \Xi^{j+1}(\omega)(s) \in B\} \in \mathcal{F}_s, \tag{4.16}$$

knowing that Ξ^0, Ξ^j are controls, the first by assumption and the latter by inductive step. We set

$$E = \{\tau^j > s\},$$

then we have by the very definition of Ξ^{j+1}

$$\{\omega \mid \Xi^{j+1}(s) \in B\} = F_1 \bigcup F_2 \tag{4.17}$$

with

$$\begin{aligned}F_1 &= \{\omega \mid \Xi^j(s) \in B\} \cap E \\ F_2 &= \{\omega \mid \Xi^0(\phi_{\tau^j}(\omega))(s - \tau^j(\omega)) \in B\} \setminus E.\end{aligned}$$

We know that

$$F_1 \in \mathcal{F}_s. \tag{4.18}$$

because τ^j is a stopping time and Ξ^j a control. Assume now τ^j to be of the form (4.13), then $E_k \setminus E = E_k$ or $E_k \setminus E = \emptyset$ according on whether $t_k \leq s$ or $t_k > s$ and so

$$F_2 = \bigcup_{t_k \leq s} \{\omega \mid \Xi^0(\phi_{t_k}(\omega))(s - t_k) \in B\} \cap E_k$$

Consequently, if $t_k \leq t$, $\Xi^{j+1}(s)$ is represented in E_k by the composition of the following maps

$$\omega \xrightarrow{\psi_1} \phi_{t_k}(\omega) \xrightarrow{\psi_2} \Xi^0(\phi_{t_k}(\omega)) \xrightarrow{\psi_3} \Xi^0(\phi_{t_k}(\omega))(s - t_k).$$

By the very definition of the σ–algebra $\mathcal{F}'_t$, $\psi_3^{-1}(\mathcal{B}) \subset \mathcal{F}'_{s-t_k}$, moreover, being Ξ^0 adapted implies that $\psi_2^{-1}(\mathcal{F}'_{s-t_k}) \subset \mathcal{F}_{s-t_k}$ and finally $\psi_1^{-1}(\mathcal{F}_{s-t_k}) \subset \mathcal{F}_s$ by Proposition 3.5.1. We deduce, taking also into account that $E_k \in \mathcal{F}_{t_k} \subset \mathcal{F}_s$, that if $t_k \leq s$ then

$$\{\omega \mid \Xi^{j+1}(s) \in B\} \cap E_k = \{\omega \mid \Xi^0(\phi_{t_k}(\omega))(s - t_k) \in B\} \cap E_k \in \mathcal{F}_s,$$

and consequently F_2, being the union of sets in $\mathcal{F}_s$, belongs to $\mathcal{F}_s$ as well. By combining this information with (4.17), (4.18), we prove (4.16) and conclude that Ξ^{j+1} is a control. To show that Ξ^{j+1} is a τ^{j+1}–cycle, we use the very definition of τ^{j+1}, Ξ^{j+1} and write for any ω

$$\int_0^{\tau^{j+1}(\omega)} \Xi^{j+1}(\omega)\, ds = I(\omega) + J(\omega) \tag{4.19}$$

with

$$\begin{aligned} I(\omega) &= \int_0^{\tau^j(\omega)} \Xi^j(\omega)\, ds \\ J(\omega) &= \int_{\tau^j(\omega)}^{\tau^{j+1}(\omega)} \Xi^0(\phi_{\tau^j}(\omega))(s - \tau^j(\omega))\, ds \end{aligned}$$

Due to Ξ^j being a τ^j–cycle, we have

$$I(\omega) = 0 \quad \text{a.s.} \tag{4.20}$$

We change the variable in J, setting $t = s - \tau^j(\omega)$, and exploit Lemma 4.3.3 to get

$$J(\omega) = \int_0^{\tau^0(\phi_{\tau^j}(\omega))} \Xi^0(\phi_{\tau^j}(\omega))(t)\, dt. \tag{4.21}$$

Let E be any set in $\mathcal{F}$ and $\mathbf{a}$ a positive probability vector. We integrate $J(\omega)$ over E with respect to $\mathbb{P}_{\mathbf{a}}$ using (4.21), replace $\phi_{\tau^j}(\omega))$ with ω by means of change of variable formula, and exploit Theorem 4.2.11. We obtain

$$\int_E J(\omega)\, d\mathbb{P}_{\mathbf{a}} = \int_{\phi_{\tau^j}(E)} \left(\int_0^{\tau^0(\omega))} \Xi^0(\omega)(t)\, dt \right) d\mathbb{P}_{\mathbf{a}e^{-A\tau^j}}. \tag{4.22}$$

Due to Ξ_0 being a τ^0–cycle

$$\int_0^{\tau^0(\omega)} \Xi^0(\omega)(t)\, dt = 0 \qquad \text{a.s,}$$

and therefore the integral in the right hand–side of (4.22) is vanishing and so

$$\int_E J(\omega)\, d\mathbb{P}_{\mathbf{a}} = 0.$$

Since E has been arbitrarily chosen in $\mathcal{F}$ and $\mathbf{a} > 0$, we deduce in force of Lemma 3.5.5

$$J(\omega) = 0 \qquad \text{a.s.}$$

This information combined with (4.19), (4.20) shows that Ξ^{j+1} is a τ^{j+1}–cycle and conclude the proof.

□

4.4 Dynamical properties of the Aubry set

In this section we give the main results of this chapter on the cycle characterization of the Aubry set. As explained in the Introduction, a key step is to establish a strengthened version of Theorem 3.5.14, which is based on the cycle iteration technique presented in Section 4.3.

Theorem 4.4.1 *Given $\epsilon > 0$, $\alpha \geq \beta$, and $y \in \mathbb{T}^N$, $\mathbf{b} \in \mathcal{F}_\alpha(y)$ if and only if*

$$\mathbb{E}_k \left(\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi)(s), -\Xi(s)) + \beta\, ds - b_k + b_{\omega(\tau)} \right) \geq 0, \tag{4.23}$$

for any $k \in \{1, \cdots, m\}$, $\tau \gg \epsilon$ bounded stopping times and τ–cycles Ξ.

We break the argument in two parts. The first one is presented in a preliminary lemma.

Lemma 4.4.2 *Let $i \in \{1, \cdots, m\}$, $\mathbf{b} \in \mathbb{R}^m$, $\delta > 0$, assume τ^0 to be a simple stopping time vanishing outside $\mathcal{D}_i$, with $\tau^0 \gg \delta$ in $\mathcal{D}_i$ satisfying*

$$\mathbb{E}_i \left(\int_0^{\tau^0} L_{\omega(s)}(y + \mathcal{I}(\Xi^0)(s), -\Xi^0(s)) + \beta\, ds - b_i + b_{\omega(\tau^0)} \right) =: -\mu < 0.$$

Then for any $j \in \mathbb{N}$

$$\mathbb{E}_i \left(\int_0^{\tau^j} L_{\omega(s)}(y + \mathcal{I}(\Xi^j)(s), -\Xi^j(s)) + \beta\, ds - b_i + b_{\omega(\tau^j)} \right) < -\mu\, (1 + \rho\, j), \tag{4.24}$$

where τ^j, Ξ^j are as in (4.11), (4.12), respectively, and ρ is the positive constant, provided by Proposition 4.2.6, with

$$\left(e^{-A\tau}\right)_{ik} > \rho \quad \text{for any } \tau \gg \delta \text{ in } \mathcal{D}_i,\ k = 1, \cdots, m.$$

Proof. We denote by I_j the expectation in the left hand–side of (4.24) and argue by induction on j. Formula (4.24) is true for $j = 0$, and we assume by inductive step that it holds for some $j \geq 1$. Taking into account that

$$\Xi^{j+1}(\omega)(s) = \Xi^j(\omega)(s) \qquad \text{in } [0, \tau^j(\omega)) \text{ for any } \omega,$$

we get by applying the inductive step

$$I_{j+1} = I_j + K_j \leq -\mu\,(1 + \rho\, j) + K_j \tag{4.25}$$

with

$$K_j = \mathbb{E}_i \left(\int_{\tau^j}^{\tau^{j+1}} L_{\omega(s)}(y + \mathcal{I}(\Xi^{j+1})(s), -\Xi^{j+1}(s)) + \beta\, ds - b_{\omega(\tau^j)} + b_{\omega(\tau^{j+1})} \right).$$

We further get by applying Lemma 4.3.3 and the very definition of Ξ^{j+1}

$$K_j = \mathbb{E}_i\big(W(\omega)\big) + \mathbb{E}_i(-b_{\omega(\tau^j)} + b_{\omega(\tau^j + \tau^0(\phi_{\tau^j}))}) \tag{4.26}$$

where

$$W(\omega) = \int_{\tau^j}^{\tau^j + \tau^0(\phi_{\tau^j})} L_{\omega(s)}(y + \mathcal{I}(\Xi^0(\phi_{\tau^j}))(s - \tau^j), -\Xi^0(\phi_{\tau^j})(s - \tau^j)) + \beta\, ds.$$

We fix ω and set $t = s - \tau^j(\omega)$, we have

$$W(\omega) = \int_0^{\tau^0(\phi_{\tau^j}(\omega))} L_{\phi_{\tau^j}(\omega)(t)}(y + \mathcal{I}(\Xi^0(\phi_{\tau^j}(\omega)))(t), -\Xi^0(\phi_{\tau^j}(\omega))(t)) + \beta\, dt$$

By using the above relation and change of variable formula(from $\phi_{\tau^j}(\omega)$ to ω), and Theorem 4.2.11, we obtain

$$\mathbb{E}_i W(\omega) = \mathbb{E}_{\mathbf{e}_i e^{-A\tau^j}} \left(\int_0^{\tau^0} L_\omega(y + \mathcal{I}(\Xi^0(\omega)), -\Xi^0(\omega)) + \beta\, ds \right) \tag{4.27}$$

We also have by applying the same change of variable

$$\begin{aligned}
\mathbb{E}_i \left(-b_{\omega(\tau^j)} + b_{\omega(\tau^j + \tau^0(\phi_{\tau^j}))} \right) &= \mathbb{E}_i \left(-b_{\phi_{\tau^j}(\omega)(0)} + b_{\phi_{\tau^j}(\omega)(\tau^0)} \right) \\
&= \mathbb{E}_{\mathbf{e}_i e^{-A\tau^j}} \left(-b_{\omega(0)} + b_{\omega(\tau^0)} \right)
\end{aligned}$$

By using the above relation, (4.26), (4.27) and the fact that τ^0 vanishes outside $\mathcal{D}_i$, we obtain

$$\begin{aligned}
K_j &= \mathbb{E}_{\mathbf{e}_i e^{-A\tau^j}} \left(\int_0^{\tau^0} L_{\omega(s)}(y + \mathcal{I}(\Xi^0)(s), -\Xi^0(s)) + \beta\, ds - b_i + b_{\omega(\tau^0)} \right) \\
&= \left(e^{-A\tau^j} \right)_{ii} \mathbb{E}_i \left(\int_0^{\tau^0} L_{\omega(s)}(y + \mathcal{I}(\Xi^0)(s), -\Xi^0(s)) + \beta\, ds - b_i + b_{\omega(\tau^0)} \right) \\
&< -\rho\, \mu
\end{aligned}$$

By plugging this relation in (4.25), we end up with

$$I_{j+1} \le -\mu\,(1+\rho\,(j+1))$$

proving (4.24).

□

Proof. (of Theorem 4.4.1) .
The first implication is direct by Theorem 3.5.14.
Conversely, if $\mathbf{b} \notin \mathcal{F}_\beta(y)$ then there exists, by Theorem 3.5.14, $i \in \{1, \cdots, m\}$, bounded stopping time τ^0 and τ^0–cycle Ξ^0 such that

$$\mathbb{E}_i\left(\int_0^{\tau^0} L_{\omega(s)}(y+\mathcal{I}(\Xi^0)(s), -\Xi^0(s)) + \beta\, ds - b_i + b_{\omega(\tau^0)}\right) =: -\mu < 0. \tag{4.28}$$

We can also assume $\tau^0 = 0$ outside $\mathcal{D}_i$ without affecting (4.28). We set

$$\widetilde{\Xi}(\omega)(s) = \begin{cases} \Xi^0(\omega)(s), & \text{for } \omega \in \mathcal{D},\ s \in [0, \tau^0(\omega)) \\ 0, & \text{for } \omega \in \mathcal{D},\ s \in [\tau^0(\omega), +\infty). \end{cases}$$

We claim that $\widetilde{\Xi}$ is still a τ^0–cycle; the unique property requiring some detail is actually the nonanticipating character. We take $\omega_1 = \omega_2$ in $[0,t]$, for some positive t, and consider two possible cases:

<u>**Case 1:**</u> If $s := \tau^0(\omega_1) \le t$ then

$$\omega_1 \in A := \{\omega \mid \tau^0(\omega) = s\} \in \mathcal{F}_s \subseteq \mathcal{F}_t,$$

which yields $\omega_2 \in A$ and hence $\tau^0(\omega_1) = \tau^0(\omega_2) = s$.
In this case

$$\begin{aligned} &\widetilde{\Xi}(\omega_1) = \Xi^0(\omega_1) = \Xi^0(\omega_2) = \widetilde{\Xi}(\omega_2) \qquad \text{in } [0,s], \\ &\widetilde{\Xi}(\omega_1) = \widetilde{\Xi}(\omega_2) = 0 \qquad \text{in } [s,t], \end{aligned}$$

<u>**Case 2:**</u> If $\tau^0(\omega_1) > t$, then

$$\omega_1 \in \{\omega \mid \tau^0(\omega) > t\} \in \mathcal{F}_t,$$

which implies that ω_2 belongs to the above set and consequently $\tau^0(\omega_2) > t$. Therefore

$$\widetilde{\Xi}(\omega_1) = \Xi^0(\omega_1) = \Xi^0(\omega_2) = \widetilde{\Xi}(\omega_2) \qquad \text{in } [0,t].$$

This shows the claim. Therefore we can assume that the Ξ^0 appearing in (4.28) vanishes when $t \geq \tau^0(\omega)$ for any ω.

We know from Proposition 3.5.10 that there is a nonincreasing sequence τ'_n of simple stopping times with

$$\tau'_n \to \tau^0 \quad \text{uniformly in } \mathcal{D}.$$

We define

$$\tau_n = \begin{cases} \tau'_n + \frac{1}{n} & \text{in } \mathcal{D}_i \\ 0 & \text{in } \mathcal{D}_k \text{ for } k \neq i \end{cases}$$

The τ_n are simple stopping times, moreover, since τ^0 is vanishing outside $\mathcal{D}_i$ and $\frac{1}{n} \to 0$, we get

$$\tau_n \geq \tau^0 \quad \text{and} \quad \tau_n \to \tau^0 \quad \text{uniformly in } \mathcal{D},$$

in addition

$$\{\tau_n \leq t\} \cap \mathcal{D}_i = \{\tau'_n + 1/n \leq t\} \cap \mathcal{D}_i \in \mathcal{F}_{t-1/n} \quad \text{for } t \geq \tfrac{1}{n}$$

which shows that

$$\tau_n \gg \frac{1}{n} \quad \text{in } \mathcal{D}_i.$$

It is clear that $\widetilde{\Xi}$ belongs to $\mathcal{K}(\tau_n, 0)$, we further have

$$\left| \int_0^{\tau_n} L_{\omega(s)}(y + \mathcal{I}(\Xi^0), -\Xi^0)\, ds - \int_0^{\tau^0} L_{\omega(s)}(y + \mathcal{I}(\Xi^0), -\Xi^0)\, ds \right| \leq \int_{\tau^0}^{\tau_n} |L_{\omega(s)}(y, 0)|\, ds.$$

Owing to the boundedness property of the integrand and that $\tau_n \to \tau^0$, the right hand-side of the above formula becomes infinitesimal, as n goes to infinity. Therefore the strict negative inequality in (4.28) is maintained replacing τ^0 by τ_n, for a suitable n.

Hence we can assume, without loss of generality, that τ^0 appearing in (4.28) satisfies the assumptions of Lemma 4.4.2 for a suitable $\delta > 0$. Let τ^j, Ξ^j be as in (4.11), (4.12), we define for any j

$$\widetilde{\tau}^j = \tau^j + \epsilon$$

and

$$\widetilde{\Xi}^j(\omega)(s) = \begin{cases} \Xi^j(\omega)(s) & \text{for } s \in [0, \tau^j(\omega)) \\ 0 & \text{for } s \in [\tau^j(\omega), \widetilde{\tau}^j(\omega)) \end{cases}$$

the $\widetilde{\tau}^j$ are apparently stopping times with $\widetilde{\tau}^j \gg \epsilon$, and that $\widetilde{\Xi}^j$ are $\widetilde{\tau}^j$–cycles.

We have

$$\mathbb{E}_i \left(\int_0^{\widetilde{\tau}^j} L_{\omega(s)}(y + \mathcal{I}(\widetilde{\Xi}^j)(s), -\widetilde{\Xi}^j(s)) + \beta\, ds - b_i + b_{\omega(\widetilde{\tau}^j)} \right) = U_j + V_j$$

with

$$\begin{aligned} U_j &= \mathbb{E}_i\left(\int_0^{\tau^j} L_{\omega(s)}(y+\mathcal{I}(\Xi^j)(s), -\Xi^j(s)) + \beta\, ds - b_i + b_{\omega(\tau^j)}\right) \\ V_j &= \mathbb{E}_i\left(\int_{\tau^j}^{\tau^j+\epsilon} L_{\omega(s)}(y,0) + \beta\, ds - b_{\omega(\tau^j)} + b_{\omega(\tau^j+\epsilon)}\right) \end{aligned}$$

The term U_j diverges negatively as $j \to +\infty$ by Lemma 4.4.2, while V_j stays bounded, which implies

$$\mathbb{E}_i\left(\int_0^{\widetilde{\tau}^j} L_{\omega(s)}(y+\mathcal{I}(\widetilde{\Xi}^j)(s), -\widetilde{\Xi}^j(s)) + \beta\, ds - b_i + b_{\omega(\widetilde{\tau}^j)}\right) < 0$$

for j large. Taking into account that $\widetilde{\tau}^j \gg \epsilon$ and Theorem 3.5.14, the last inequality shows that stopping times strongly greater than ϵ and corresponding cycles based at y are sufficient to characterize values $\mathbf{b} \notin \mathcal{F}_\beta(y)$. This concludes the proof.

□

Next we state and prove the first main theorem.

Theorem 4.4.3 *Given $\epsilon > 0$, $y \in \mathbb{T}^N$, $\mathbf{b} \in \mathbb{R}^m$, we consider*

$$\inf \mathbb{E}_i\left[\int_0^{\tau} L_{\omega(s)}(y+\mathcal{I}(\Xi), -\Xi) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau)}\right] \tag{4.29}$$

where the infimum is taken with respect to any bounded stopping times $\tau \gg \epsilon$ and τ–cycles Ξ. The following properties are equivalent:

(i) $y \in \mathcal{A}$

(ii) the infimum in (4.29) *is zero for any index i, any $\mathbf{b} \in F_\beta(y)$*

(iii) the infimum in (4.29) *is zero for some i, any $\mathbf{b} \in F_\beta(y)$.*

The assumption that the stopping times involved in the infimum are strongly greater than a positive constant is essentially used for proving (iii) ⇒ (i), while in the implication (i) ⇒ (ii) it is exploited the characterization of admissible values provided in Theorem 4.4.1.

Proof.

We start proving the implication (i) ⇒ (ii).

Let $y \in \mathcal{A}$, assume to the contrary that

$$\inf_{\tau \gg \epsilon} \mathbb{E}_i\left[\int_0^{\tau} L_{\omega(s)}(y+\mathcal{I}(\Xi), -\Xi) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau)}\right] \neq 0 \quad \text{for some } i \text{ and } \mathbf{b} \in F_\beta(y).$$

We deduce from Theorem 3.5.14 that

$$\inf_{\tau \gg \epsilon} \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] > 0 \tag{4.30}$$

for such i, $\mathbf{b}$. We claim that $\mathbf{b} + \nu \, \mathbf{e}_i \in \mathcal{F}_\beta(y)$ for any positive ν less than the infimum in (4.30) denoted by η. Taking into account (4.30) and $e^{-A\tau}$ being stochastic, we have for any stopping time $\tau \gg \epsilon$ and τ–cycle Ξ

$$\begin{aligned} & \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - (\mathbf{b} + \nu \, \mathbf{e}_i)_{\omega(0)} + (\mathbf{b} + \nu \, \mathbf{e}_i)_{\omega(\tau)} \right] \\ = \; & \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] - \nu + \nu \, \mathbf{e}_i \, e^{-A\tau} \cdot \mathbf{e}_i \\ \geq \; & \eta - \nu \geq 0. \end{aligned}$$

We further get for $j \neq i$ in force of Theorem 3.5.14,

$$\begin{aligned} & \mathbb{E}_j \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - (\mathbf{b} + \nu \, \mathbf{e}_i)_{\omega(0)} + (\mathbf{b} + \nu \, \mathbf{e}_i)_{\omega(\tau)} \right] \\ = \; & \mathbb{E}_j \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] - \nu \, \mathbf{e}_j \cdot \mathbf{e}_i + \nu \, \mathbf{e}_j \, e^{-A\tau} \cdot \mathbf{e}_i \\ \geq \; & 0. \end{aligned}$$

Combining the information from the above computations with Theorem 4.4.1, we get the claim, reaching a contradiction with y being in $\mathcal{A}$ via Proposition 3.4.12.

It is trivial that (ii) implies (iii). We complete the proof showing that (iii) implies (i). Let us assume that (4.29) is vanishing for some i and any $\mathbf{b} \in F_\beta(y)$.

For any positive constant ν, select $\delta > 0$ satisfying $\delta - \rho \, \nu < 0$, where $\rho > 0$ is given by Proposition 4.2.6. Notice that we can invoke Proposition 4.2.6 because we are working with stopping times strongly greater than ϵ. We fix $\mathbf{b} \in F_\beta(y)$ and deduce from (4.29) being zero that there exist a bounded stopping time $\tau \gg \epsilon$, and a τ–cycle Ξ with

$$\mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] < \delta. \tag{4.31}$$

Taking into account Remark 4.2.7 and (4.31), we have

$$\begin{aligned} & \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - (\mathbf{b} + \nu \, \mathbf{e}_i)_{\omega(0)} + (\mathbf{b} + \nu \, \mathbf{e}_i)_{\omega(\tau)} \right] \\ = \; & \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta \, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] - \nu + \nu \, \mathbf{e}_i \, e^{-A\tau} \cdot \mathbf{e}_i \\ < \; & \delta - \rho \, \nu \\ < \; & 0 \end{aligned}$$

which proves that $\mathbf{b} + \nu\,\mathbf{e}_i \notin \mathcal{F}_\beta(y)$, in view of Theorem 3.5.14. This proves that $\mathbf{b}$, arbitrarily taken in $F_\beta(y)$, is not an internal point, and consequently that the interior of $F_\beta(y)$ must be empty. This in turn implies that $y \in \mathcal{A}$ in force of Proposition 3.4.13. $\square$

Using expectation operators related to characteristic vectors of stopping times, we have a more geometric formulation of the cycle characterization provided in Theorem 4.4.3, without any reference to admissible values for critical subsolutions. This is our second main result.

Theorem 4.4.4 *Given $\epsilon > 0$, $y \in \mathcal{A}$ if and only if*

$$\inf \mathbb{E}_{\mathbf{a}} \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds \right] = 0 \tag{4.32}$$

where the infimum is taken with respect to any bounded stopping times $\tau \gg \epsilon$, τ–cycles Ξ and $\mathbf{a}$ characteristic vector of τ.

Theorem 4.4.4 comes from Theorem 4.4.3 and the following

Lemma 4.4.5 *Given $\epsilon > 0$, $y \in \mathcal{A}$ and $\mathbf{b} \in F_\beta(y)$, let us consider*

$$\inf \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] \tag{4.33}$$

$$\inf \mathbb{E}_{\mathbf{a}} \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds \right] \tag{4.34}$$

where both the infima are taken with respect to any bounded stopping times $\tau \gg \epsilon$, τ–cycles Ξ , and in (4.34) $\mathbf{a}$ *is a characteristic vector of τ.*

Then (4.34) *vanishes if and only if* (4.33) *vanishes for any $i \in \{1, \cdots, m\}$.*

Proof. Let us assume that (4.33) vanishes for any i, then for any $\delta > 0$, any i we find a $\tau_i \gg \epsilon$ and τ_i–cycles Ξ_i with

$$\mathbb{E}_i \left[\int_0^{\tau_i} L_{\omega(s)}(y + \mathcal{I}(\Xi_i), -\Xi_i) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau_i)} \right] < \delta$$

We define a new stopping time $\tau \gg \epsilon$ and a τ–cycle Ξ setting

$$\begin{aligned} \tau &= \tau_i \qquad \text{on } \mathcal{D}_i \\ \Xi &= \Xi_i \qquad \text{on } \mathcal{D}_i \end{aligned}$$

then we get

$$\mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] < \delta \qquad \text{for any } i.$$

Taking a characteristic vector $\mathbf{a} = (a_1, \cdots, a_m)$ of τ, and making convex combinations in the previous formula with coefficients a_i, we get taking into account Remark 4.2.2

$$\delta > \mathbb{E}_{\mathbf{a}} \left[\int_0^\tau L_\omega(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds \right] - \mathbf{a} \cdot \mathbf{b} + (\mathbf{a}\, e^{-A\tau}) \cdot \mathbf{b} = \mathbb{E}_{\mathbf{a}} \left[\int_0^\tau L_\omega(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds \right].$$

Since we know that the infimum in (4.34) is greater that or equal to 0 thanks to Corollary 4.2.3 with $x = y$, the above inequality implies that it must be 0, as claimed.

Conversely assume that (4.34) is equal to 0, then for any $\delta > 0$ there is a stopping time $\tau \gg \epsilon$ with characteristic vector $\mathbf{a}$, and a τ–cycle Ξ with

$$\mathbb{E}_{\mathbf{a}} \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds \right] < \delta$$

Taking into account that

$$\sum_i a_i\, \mathbb{E}_i [b_{\omega(0)} - b_{\omega(\tau)}] = \mathbf{a} \cdot \mathbf{b} - (\mathbf{a}\, e^{-A\tau}) \cdot \mathbf{b} = 0$$

for any $\mathbf{b} \in F_\beta(y)$, we derive

$$\sum_i a_i\, \mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] < \delta.$$

From Remark 4.2.8 and the fact that the expectations in the above inequality must be nonnegative because of Theorem 3.5.14, we deduce

$$\mathbb{E}_i \left[\int_0^\tau L_{\omega(s)}(y + \mathcal{I}(\Xi), -\Xi) + \beta\, ds - b_{\omega(0)} + b_{\omega(\tau)} \right] < \frac{\delta}{\rho} \qquad \text{for any } i,$$

where ρ is the constant appearing in Proposition 4.2.6. This implies that the infima in (4.33) must vanish for any i.

□

Chapter 5

Scalar reduction techniques for weakly coupled Hamilton-Jacobi systems

5.1 Introduction

This chapter is devoted to study the degenerate weakly coupled system, introduced previously, from a PDE point of view. Namely, we deal with a family of systems of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \cdots, m\}, \tag{HJα}$$

with $m \geq 2$ and α varying in $\mathbb{R}$. Here $\mathbf{u} = (u_1, \cdots, u_m)$ is the unknown function, A is an $m \times m$ matrix, the so–called coupling matrix, and $H_1, \cdots, ..., H_m$ are Hamiltonians. The H_i satisfy the following set of assumptions for all $i \in \{1, \cdots, m\}$:

(H1) $H_i : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous;

(H2) $p \mapsto H_i(x, p)$ is convex for every $x \in \mathbb{T}^N$;

(H3) $p \mapsto H_i(x, p)$ is coercive for every $x \in \mathbb{T}^N$.

The coupling matrix $A = (a_{ij})$ satisfies:

(A1) $a_{ij} \leq 0$ for every $i \neq j$;

(A2) $\sum_{j=1}^{m} a_{ij} = 0$ for any $i \in \{1, \cdots, m\}$;

(A3) it is irreducible, i.e for every $W \subsetneq \{1, 2, ..., m\}$ there exists $i \in W$ and $j \notin W$ such that $a_{ij} < 0$.

We recall that under our assumptions there exists a unique $\alpha \in \mathbb{R}$ for which (HJα) admits viscosity solution, denoted by β and termed critical value. We then focus on the critical weakly coupled system:

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \beta \quad \text{in } \mathbb{T}^N \quad \text{for every } i \in \{1, \cdots, m\}, \tag{HJβ}$$

This chapter is centered on a method, which seems new, to tackle this kind of problems: namely the scalar reduction technique mentioned in the title. It simply consists in associating to the system a family of scalar discounted equations. These are roughly speaking obtained by picking one of the equations in the system, and freezing all the components of a given critical subsolution except the one corresponding to the index of the selected equation. It becomes the unknown of the discounted equation. This approach advantageously allows exploiting the wide knowledge of this kind of equations to gather information on the system. We in particular use that a comparison principle holds for discounted equations, the solutions can be represented as infima of integral functionals, and the fact that corresponding optimal trajectories do exist.

Our achievements are as follows: we provide a constructive algorithm for getting a critical solution by suitably modifying an initial critical subsolution outside the Aubry set. The solution is obtained as uniform limit of a monotonic sequence of subsolutions. The procedure can be useful for numerical approximation of a critical solution and of the Aubry set as well, see Remark 5.3.2 for more details. We also adapt the construction in the non-compact setting ($\mathbb{R}^N$) in order to get a solution on the whole space at any supercritical level, in analogous to scalar case.

We moreover give a characterization of isolated points of the Aubry, adapting the notion of equilibrium point to systems. This enables us to also show the strict differentiability of any critical subsolution on such points. The final outcome is about a semiconcavity property for critical subsolutions on the Aubry set, and on the whole torus for solutions. We more precisely prove that the superdifferential is nonempty. These results are clearly related to an open problem about the differentiability of critical subsolutions on the Aubry set. They can viewed as a partial positive answer to the regularity issue. We hope they will be useful to fully crack the problem.

The results of this chapter are obtained in [34] and it is organized as follows. In Section 5.2 we introduce the family of scalar discounted equations associated to the critical system. In Section 5.3 we provide an algorithm to get a critical solution starting from any subsolution. Section 5.4 is devoted to construct solutions of our system at any critical-supercritical level, in non compact setting. In Section 5.5 we study, through the scalar reduction technique, the nature of isolated points of $\mathcal{A}$ and prove semiconcavity properties for critical subsolutions.

5.2 Scalar reduction

In this section we associate to the critical system some discounted scalar equations. Using these equations, we will thereafter write an algorithm to construct a solution of the critical system by suitably modifying outside $\mathcal{A}$ a given critical subsolution.

We denote by $\mathbf{w} = (w_1, \cdots, w_m)$ the initial subsolution of (HJβ) and freeze all its components except one obtaining for a given $i \in \{1, \cdots, m\}$, the following discounted equation:

$$a_{ii}v(x) + H_i(x, Dv) + \sum_{j \neq i} a_{ij}w_j(x) - \beta = 0, \quad \text{in } \mathbb{T}^N. \tag{5.1}$$

For simplicity we set $f_i(x) = -\sum_{j \neq i} a_{ij}w_j(x) + \beta$, for every i.

The discounted equation satisfies a comparison principle. This is well known, however the proof in our setting is simplified to some extent by exploiting the compactness of the ambient space plus the coercivity of the Hamiltonian with respect to the momentum variable. This straightforwardly implies that all subsolutions are Lipschitz–continuous and allows using Proposition 2.1.14. We provide the argument for reader's convenience.

Theorem 5.2.1 *If u, v an USC continuous subsolution and a LSC supersolution of (5.1), respectively, then $u \leq v$ in $\mathbb{T}^N$.*

Proof. We recall that u is Lipschitz continuous. Let x_0 be a point in $\mathbb{T}^N$ where $v - u$ attains its minimum and assume, for purposes of contradiction, that

$$v(x_0) - u(x_0) < 0. \tag{5.2}$$

The function u is therefore a Lipschitz continuous subtangent to v at x_0 and hence we have

$$a_{ii}v(x_0) + H_i(x_0, p) - f_i(x_0) \geq 0 \quad \text{for some } p \in \partial u(x_0).$$

But u is a viscosity subsolution of (5.1), then

$$a_{ii}u(x_0) + H_i(x_0, p) - f_i(x_0) \leq 0 \quad \text{for all } p \in \partial u(x_0).$$

Subtracting the above two inequalities, we get

$$a_{ii}(v(x_0) - u(x_0)) \geq 0,$$

which contradicts (5.2). We therefore conclude that $\min_{\mathbb{T}^N}(v - u) \geq 0$, which in turn implies $v \geq u$ in $\mathbb{T}^N$, as desired. □

Due to the fact that any subsolution of (HJβ) is ℓ_β–Lipschitz continuous, we may modify the H_i's outside the compact set $\{(x, p) : |p| \leq \ell_\beta\}$, to obtain a new Hamiltonian which is still continuous and convex, and in addition satisfies superlinearity condition, for every i. Since the sublevels contained in $B(0, \ell_\beta)$ are not affected, the subsolutions of the system obtained by replacing the H_i's in (HJβ) by the new Hamiltonians are the same as the original one.

In the remainder of the paper, we will therefore assume without any loss of generality

(H'2) H_i is superlinear in p for any $i \in \{1, 2, ..., m\}$,

We can thus associate to any H_i a Lagrangian function L_i through the Fenchel transform, i.e.

$$L_i(x, q) = \sup_{p \in \mathbb{R}^N} \{pq - H_i(x, p)\}.$$

The function L_i is continuous on $\mathbb{T}^N \times \mathbb{R}^N$, convex and superlinear in q. Moreover, for every $(x, p), (x, q) \in \mathbb{T}^N \times \mathbb{R}^N$, we have the following inequality:

$$p.q \leq L_i(x, q) + H_i(x, p). \tag{5.3}$$

This is called the Fenchel inequality. For more details about the properties of L, we refer readers to [10], [17].

The equation (5.1) can be interpreted as the Hamilton–Jacobi–Bellman equation of a control problem with the Lagrangian L_i as cost and a_{ii} as discount factor. The control is given by the velocities which are in principle unbounded, but this is somehow compensated by the coercive character of the Hamiltonian. The corresponding value function $v^i : \mathbb{T}^N \to \mathbb{R}$ is defined by

$$v^i(x) = \inf_{\gamma} \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot{\gamma}(s)) + f_i(\gamma(s)) \right) ds, \tag{5.4}$$

where the infimum is taken over all absolutely continuous curves $\gamma :]-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = x$.

We now state for latter use a sub-optimality condition satisfied by subsolutions of (5.1).

Lemma 5.2.2 *If u is a continuous subsolution of (5.1), then*

$$u(\gamma(0)) - e^{-a_{ii}t} u(\gamma(-t)) \leq \int_{-t}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot{\gamma}(s)) + f_i(\gamma(s))) \right) ds$$

for every $t > 0$ and every $\gamma : [-t, 0] \to \mathbb{T}^N$ absolutely continuous curve.

Proof. If u be a subsolution of (5.1), then it is is Lipschitz continuous. By convexity, we get

$$a_{ii}u(x) + H_i(x, p) - f_i(x) \leq 0 \quad \text{for every } p \in \partial u(x). \tag{5.5}$$

Let $\gamma \in AC([-t, 0]; \mathbb{T}^N)$. Since u is Lipschitz continuous and γ is absolutely continuous, we have

$$\begin{aligned} u(\gamma(0)) - e^{-a_{ii}t} u(\gamma(-t)) &= \int_{-t}^{0} \frac{d}{ds} \left(e^{a_{ii}s} u(\gamma(s)) \right) ds \\ &= \int_{-t}^{0} e^{a_{ii}s} \left(a_{ii} u(\gamma(s)) + p(s) \cdot \dot{\gamma}(s) \right) ds \quad a.e. \text{ for some } p(s) \in \partial u(\gamma(s)). \end{aligned}$$

Exploiting the Fenchel inequality (5.3) and (5.5), we get

$$u(\gamma(0)) - e^{-a_{ii}t} u(\gamma(-t)) \leq \int_{-t}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot{\gamma}(s)) + f_i(\gamma(s)) \right) ds.$$

□

The next main result holds:

Theorem 5.2.3 *[12, Appendix 2] The discounted value function v^i is the unique continuous viscosity solution of (5.1).*
Moreover, for every $x \in \mathbb{T}^N$ there exists a curve $\gamma : (-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = x$ such that

$$v^i(x) = \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot{\gamma}(s)) + f_i(\gamma(s)) \right) ds. \tag{5.6}$$

Corollary 5.2.4 *We have*

$$w_i \le v^i \text{ in } \mathbb{T}^N \quad \text{and} \quad w_i = v^i \text{ in } \mathcal{A}. \tag{5.7}$$

Proof. Since w_i is a viscosity subsolution of (5.1), we derive from Theorem 5.2.1 the inequality in (5.7). Taking into account that $a_{ij} \le 0$ for every $i \ne j$, we further derive that the vector valued function obtained from w by replacing w_i with v^i and keeping all other components unaffected, is still a subsolution of (HJβ). The equality in (5.7) then comes from the rigidity phenomenon in $\mathcal{A}$, see Theorem 3.4.11. □

5.3 Algorithm

In this section we will construct a monotonic sequence of critical subsolutions $(\mathbf{v}_n)$ which converges, up to a subsequence, to a solution of (HJβ).

step 1: Construction of the sequence $(\mathbf{v}_n)$.

Let $\mathbf{w} = \mathbf{v}_0 = (v_0^1, v_0^2, ..., v_0^m)$ be any subsolution of (HJβ).
The first element $\mathbf{v}_1 = (v_1^1, v_1^2, ..., v_1^m)$ of $(\mathbf{v}_n)$ is defined component by component as follows :

For $k = 1 \cdots m$, v_1^k is the solution of the discounted equation

$$H_k(x, Du) + a_{kk}u + \sum_{j<k} a_{kj} v_1^j(x) + \sum_{j>k} a_{kj} v_0^j(x) = \beta,$$

where the possible empty sums in the above formula are counted as 0. By construction, see the proof of Corollary 5.2.4, $\mathbf{v}_1$ is a critical subsolution of the system. In addition, using Corollary 5.2.4, we get

$$\mathbf{w} = \mathbf{v}_0 \le \mathbf{v}_1 \quad \text{and} \quad \mathbf{w} = \mathbf{v}_0 = \mathbf{v}_1 \text{ on } \mathcal{A}.$$

We iterate the above procedure to construct $\mathbf{v}_n$, for any $n \in \mathbb{N}$, $n > 1$, starting from $\mathbf{v}_{n-1}$.
We get that any element $\mathbf{v}_n$ is a critical subsolution of the system and

$$\mathbf{v}_{n-1} \le \mathbf{v}_n \qquad \text{for any } n \tag{5.8}$$

$$\text{all the } \mathbf{v}_n \text{ coincide on } \mathcal{A}. \tag{5.9}$$

step 2: Convergence of the sequence (v_n).

We exploit Proposition 3.4.2 to infer that the functions $(\mathbf{v}_n)$ are equi-Lipschitz. Moreover, all the $(\mathbf{v}_n)$'s take a fixed value on the Aubry set, see (5.9), so they are equibounded as well. We derive by Ascoli-Arzelà Theorem that $(\mathbf{v}_n)$ converge uniformly, up to subsequences, and we in turn deduce convergence of the whole sequence because of its monotonicity, see (5.8).

step 3: proving the limit is a critical solution .

We denote by $\mathbf{V} = (V_1, V_2, ..., V_m)$ the uniform limit of $\mathbf{v}_n$.

Given $k \in \{1, \cdots, m\}$, we have, by construction, that v_n^k is the solution of

$$F_n^k(x, u, Du) := H_k(x, Du) + a_{kk}u + \sum_{j<k} a_{kj}v_n^j(x) + \sum_{j>k} a_{kj}v_{n-1}^j(x) = \beta.$$

The Hamiltonians F_n^k converge uniformly in $\mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N$, as $n \to +\infty$, to

$$F^k(x, u, p) := H_k(x, p) + a_{kk}u + \sum_{j \neq k} a_{kj}V_j(x).$$

Consequently, by basic stability properties in viscosity solutions theory, V_k is solution to the limit equation

$$F^k(x, u, Du) = H_k(x, Du) + a_{kk}u + \sum_{j \neq k} a_{kj}V_j(x) = \beta.$$

We conclude that the limit $\mathbf{V} = (V_1, V_2, ..., V_m)$ is solution of (HJβ), as it was claimed.

Remark 5.3.1 . *The above algorithm implies that the trace of any critical subsolution on the Aubry set can be uniquely extended to the whole torus in such a way that the output is a critical solution, where the uniqueness is due to the monotonicity of the sequence. This is consistent with [15, Theorem 5.5].*

Remark 5.3.2 *1. As already pointed out in the Introduction, the above algorithm can have a numerical interest to compute critical solutions of the system via the analysis of a sequence of scalar discounted equations. The latter problem has been extensively studied and well tested numerical codes are available. It is clearly required the* *knowledge of a critical subsolution as starting point, but this is easier than the determination of a solution. In addition, since the initial subsolution is not affected on $\mathcal{A}$ at any step of the procedure, the algorithm can be useful to get an approximation of the Aubry set itself.*

2. *In principle we could apply the algorithm also starting from any supercritical subsolution. What happens is that the sequence we construct is not any more anchored at the Aubry set, and we get in the end a sequence of functions positively diverging at any point. We believe that the rate of divergence could be exploited to estimate how far the supercritical value we have chosen is from β, but we do not investigate any further this issue in the present paper.*

5.4 Weakly coupled Hamilton-Jacobi systems in the non-compact setting

In this section we will provide two methods for constructing a solution to the weakly coupled system (HJα) at any critical/supercritcal level in the non compact setting, $\mathbb{R}^N$. The first method is a generalization of the scalar case while the second one is an adaptation of the algorithm introduced in section 5.3.

We start by stating a priori estimate for the subsolutions of the weakly coupled system (HJα). This estimate is used in the subsequent results to recover equiboundedness of a sequence of subsoultions. The result is well known in the compact case, where Davini-Zavidovique provided in [15] a uniform estimate on the difference between components of a given subsolution. Here we follow the same strategy they followed, with a local argument, to get:

Proposition 5.4.1 *Let K be a compact subset of $\mathbb{R}^N$ and $\mathbf{u} : \mathbb{R}^N \to \mathbb{R}^m$ be continuous subsolution of the system (HJα). Then there exists a constant C_k, depending on α, H_i, K and the coupling matrix A, such that*

$$|u_i(x) - u_j(x)| \leq C_k \quad \text{for every } x \in K, \text{ and } i, j \in \{1, \cdots, m\}$$

Proof. We first set

$$\mu_i = \min_{(x,p)\in K\times\mathbb{R}^N} H_i(x,p) \quad \text{for every } i \in \{1, \cdots, m\}, \quad \mu = \min_i \mu_i.$$

For every $i \in \{1, \cdots, m\}$, the following inequalities hold in the viscosity sense:

$$\mu + \sum_{j=1}^{m} a_{ij} u_j(x) \leq H_i(x, Du_i(x)) + \sum_{j=1}^{m} a_{ij} u_j(x) \leq \alpha \quad \text{on } K,$$

yielding

$$\sum_{j=1}^{m} a_{ij} u_j(x) \leq \alpha - \mu \quad \text{for every } x \in K. \tag{5.10}$$

Let us now fix $x \in K$ and assume, without any loss of generality,

$$u_1(x) \leq u_2(x) \leq \cdots, \leq u_m(x). \tag{5.11}$$

We then define

$$a_\star = \min_{1 \leq i \leq m} a_{ii} \quad \text{and } a^\star = \max_{1 \leq i,j \leq m} |a_{ij}|.$$

First notice that, by subtracting $\sum_{j=1}^{m} a_{mj} u_m(x) = 0$ from both sides of equation (5.10) with $i = m$, one gets

$$\sum_{j \neq m} -a_{mj} \left(u_m(x) - u_j(x) \right) \leq \alpha - \mu,$$

implying

$$\left(u_m(x) - \max_{j \neq m} u_j(x) \right) \sum_{j \neq m} -a_{mj} \leq \alpha - \mu.$$

By exploiting (5.11) and the degenerate character of the matrix A we get

$$0 \leq u_m(x) - u_{m-1}(x) \leq \frac{\alpha - \mu}{a_{mm}} \leq \frac{\alpha - \mu}{a_\star}. \tag{5.12}$$

This proves the assertion when $m = 2$. To prove it in the general case, we argue by induction: we assume the result is true for m and we prove it for $m + 1$. To prove this, we rewrite equation (5.10) as

$$\sum_{j=1}^{m-1} a_{ij} u_j(x) + (a_{im} + a_{im+1}) \, u_m(x) + a_{im+1} \left(u_{m+1}(x) - u_m(x) \right) \leq \alpha - \mu,$$

then we exploit (5.12) to get

$$\sum_{j=1}^{m-1} a_{ij} u_j(x) + (a_{im} + a_{im+1}) \, u_m(x) \leq (\alpha - \mu) \left(1 + \frac{a^\star}{a_\star} \right) \tag{5.13}$$

for every $i \in \{1, \cdots, m+1\}$. The irreducible character of the matrix A applied to the set $\mathcal{I} = \{m, m+1\}$ implies that

$$a_{im} + a_{im+1} > 0$$

for either $i = m$ or $i = m + 1$, let us say $i = m$. The assertion then follows by applying the induction hypothesis to the system given by (5.13) with i varying in $\{1, \cdots, m\}$, the corresponding coupling matrix being still irreducible and degenerate. □

Next we adapt to system case the method of constructing a solution at supercritical levels in non-periodic setting. We have

Proposition 5.4.2 *Assume β is finite. Then there exists a solution to (HJα) for every $\alpha \geq \beta$.*

Proof. Let K be a compact subset of $\mathbb{R}^N$ and y_n be a sequence in $\mathbb{R}^N$ tending to infinity. For every $\alpha \geq \beta$ and $n \in \mathbb{N}$, we define the functions $\mathbf{v}_n$ as the maximal subsolutions to (HJα) taking the same value at y_n. Then, in view of Proposition 3.4.2, we infer that the functions $\mathbf{v}_n$ are locally equi- Lipschitz. Up to subtracting a vector of the form $v_n^1(0)\mathbb{1}$ to each $\mathbf{v}_n$, we can furthermore assume that $v_n^1(0) = 0$ for every $n \in \mathbb{N}$, yielding

$$|v_n^1(x)| \leq L_k \quad \text{for some } L_k \in \mathbb{R}, \text{ for every } x \in K,$$

by the equi-Lipschitz character of the sequence.
Moreover,

$$|v_n^j(x) - v_n^1(x)| \leq C_k \quad \text{for every } x \in K, j \in \{1, \cdots, m\}, \text{ and } n \in \mathbb{N}$$

yielding

$$|v_n^j(x)| \leq C_k + L_k \quad \text{for every } x \in K, j \in \{1, \cdots, m\}, \text{ and } n \in \mathbb{N},$$

which implies that $\mathbf{v}_n$ are locally equibounded in $\mathbb{R}^N$.
Furthermore, in view of Proposition 3.4.4, $\mathbf{v}_n$ are subsolutions to (HJα) in $\mathbb{R}^N$ and solutions in $\mathbb{R}^N \setminus \{y_n\}$ for every n.
By Ascoli-Arzela theorem, up to subsequences, there exists a continuous function v such that

$$\mathbf{v}_n \to \mathbf{v} \quad \text{locally uniformly in } \mathbb{R}^N.$$

Next we will show that $\mathbf{v}$ is a solution to (HJα) in $\mathbb{R}^N$. Clearly, $\mathbf{v}$ is a subsolution of (HJα) in $\mathbb{R}^N$ by stability property of viscosity subsolution.
We proceed by proving the supersolution property of $\mathbf{v}$ in $\mathbb{R}^N$. Let $x_0 \in \mathbb{R}^N$ and ϕ_i be a C^1 strict subtangent to v_i at x_0, i.e x_0 is the unique minimizer of $v_i - \phi_i$ in a suitable closed ball B centered at x_0.
Let x_n be a sequence of minimizers of $v_n^i - \phi_i$ in B. Then, by the stability of minimizers

under local uniform convergence, $x_n \to x_0$ as $n \to \infty$. Notice that $x_n \neq y_n$ for n large enough, hence by the supersolution property of $\mathbf{v}_n$ in $\mathbb{R}^N \setminus \{y_n\}$, we have

$$H_i(x_n, D\phi_i(x_n)) + \sum_{j=1}^{m} a_{ij} v_n^j(x_n) \geq \alpha.$$

Passing to the limit as $n \to \infty$, and by the continuity of H and ϕ_i being C^1, we get

$$H_i(x_0, D\phi_i(x_0)) + \sum_{j=1}^{m} a_{ij} v_j(x_0) \geq \alpha.$$

This proves that $\mathbf{v}$ is a supersolution of (HJα) in $\mathbb{R}^N$. □

In what coming, we will adapt the Algorithm in Section 5.3 to construct a solution to the weakly coupled system (HJα) at any critical/supercritcal level in $\mathbb{R}^N$.

For this purpose we start by a subsolution $\mathbf{w} = \mathbf{v_0}$, and consider the same discounted equation as before, see (5.1), but on balls and assigning $\mathbf{w} = \mathbf{v_0}$ as Dirichlet boundary datum. Namely,

$$\begin{cases} a_{ii} v(x) + H_i(x, Dv(x)) + \sum_{j \neq i} a_{ij} w_j(x) - \alpha = 0, & B_R \\ v = w_i, & \partial B_R \end{cases} \tag{BVP}$$

where B_R is any ball of radius R and $\mathbf{w} : \mathbb{R}^N \to \mathbb{R}^m$ is a subsolution of the system (HJα). For simplicity we set $f_i(x) = -\sum_{j \neq i} a_{ij} w_j(x) + \beta$, for every i

We first define the value function $v^i : B_R \to \mathbb{R}$ by

$$v^i(x) = \inf_{\gamma} \int_{-\tau}^{0} e^{a_{ii} s} \left(L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} w_j(\gamma(s)) + \alpha \right) ds + e^{-a_{ii}\tau} w_i(\gamma(-\tau)), \tag{5.14}$$

where the infimum is taken over all absolutely continuous curves $\gamma :]-\infty, 0] \to \mathbb{R}^N$ with $\gamma(0) = x$, and τ is the last entrance time to B_R given by:

$$\tau = \inf\{t \geq 0 : \gamma(-t) \notin B_R\}.$$

It is worth mentioning that the theory requires regularity conditions on the boundary as well as some conditions on admissible trajectories to guarantee the existence and uniqueness of viscosity solution for such boundary value problems. However in our case, being the problem posed on balls and due to coercivity which means roughly speaking any direction is admissible, all the necessary and sufficient conditions are automatically satisfied. Hence, the next main result holds:

Theorem 5.4.3 *([5, Theorem 2.1]) The value function (5.14) is continuous on B_R and it is the unique viscosity solution of (BVP). Moreover a Strong Comparison Result holds.*

Next we show that the value function defined by (5.14) attains the value w at the boundary continuously. For this purpose we investigate the value function with state space constraint. It is defined on $\overline{B_R}$ by the infinite horizon:

$$V_c^i(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} w_j(\gamma(s)) + \alpha \right) ds, \tag{5.15}$$

where the infimum here is taken only with respect to absolutely continuous curves γ living in $\overline{B_R}$. The condition that the trajectories lie in $\overline{B_R}$ constitutes the state space constraint. We aim at providing a characterization of the constrained value function (5.15) in relation with the discounted equation

$$a_{ii}v(x) + H_i(x, Dv(x)) + \sum_{j \neq i} a_{ij} w_j(x) - \alpha = 0 \quad \text{on } \overline{B_R}. \tag{5.16}$$

Next we give a more restrictive definition of viscosity solution which takes into account the presence of constraint.

Definition 5.4.4 *(**Constrained Viscosity Solution**). Let u be a continuous function on $\overline{B_R}$. We say that u is constrained viscosity solution of (5.16) in $\overline{B_R}$ if:*

1. *u is viscosity subsolution in B_R.*
2. *u is constrained viscosity supersolution in $\overline{B_R}$.*

Equivalently,

1. *For every $\phi \in C^1(\mathbb{R}^{\mathbb{N}})$ and $x_0 \in B_R$ such that $(u - \phi)$ has a relative maximum at x_0, we have*
$$a_{ii}u(x_0) + H_i(x, D\phi(x_0)) + \sum_{j \neq i} a_{ij} w_j(x_0) - \alpha \leq 0.$$
2. *For every $\phi \in C^1(\overline{B_R})$ and $x_0 \in \overline{B_R}$ such that $(u - \phi)(x_0)$ is a relative minimum for $(u - \phi)$ in $\overline{B_R}$, we have*

$$a_{ii}u(x_0) + H_i(x, D\phi(x_0)) + \sum_{j \neq i} a_{ij} w_j(x_0) - \alpha \leq 0.$$

In other words, compared to the classical definition of viscosity solutions, we haven't changed the requirement for supertangents (that tests subsolutions), but we have enlarged the class of subtangents (that tests supersolutions).

The next two results hold:

Theorem 5.4.5 ***(Comparison Result)****. Let u, $v \in C(\overline{B_R})$ such that*

1. *u is viscosity subsolution of (5.16) in B_R.*
2. *v is constrained viscosity supersolution of (5.16) in $\overline{B_R}$.*

Then, $u \leq v$.

Theorem 5.4.6 *The value function V_c^i is continuous on $\overline{B_R}$. Moreover, it is the unique constrained viscosity solution of (5.16) in $\overline{B_R}$.*

The above asserted continuity is due to the property that at each boundary point there is always a trajectory pointing directly inside B_R. Furthermore owing to the comparison result and the fact that w_i is a viscosity subsolution of (5.16), we get

$$V_c^i \geq w_i \quad \text{on } \overline{B_R} \tag{5.17}$$

This theory of state-space constraint problems is due to Soner. For a detailed treatment of the theory and the proofs of the mentioned results we refer readers to [35, 36].

Now we prove the next preliminary proposition:

Proposition 5.4.7 *The value function (5.14) is continuous up to the boundary.*

Proof. We start by proving the upper semicontinuity i.e

$$\limsup_{y \to x,\, x \in \partial B_R} v^i(y) \leq w_i(x). \tag{5.18}$$

Let $\gamma : [-d(x,y), 0] \to B_R$ be the geodesic joining x to y parameterized by the arc-length. For every absolutely continuous curve $\nu : (-\infty, 0] \to \mathbb{R}^N$ with $\nu(0) = \gamma(-d(x,y)) = x$, we define a curve $\xi : (-\infty, -d(x,y)] \to \mathbb{R}^N$ by setting $\xi(s) = \nu(s + d(x,y))$. We now define an absolutely continuous curve $\eta : (-\infty, 0] \to \mathbb{R}^N$ as follows

$$\eta(s) = \begin{cases} \gamma(s) & : \quad s \in [-d(x,y), 0] \\ \xi(s) & : \quad s \in (-\infty, -d(x,y)]. \end{cases}$$

Let $C := \max\{ L_i(x,v) | \, x \in B_R \, , \, \|v\| \leq 1\}$, then from the definition of v^i we get :

$$\begin{aligned} v^i(y) &\leq \int_{-d(x,y)}^{0} e^{a_{ii}s}(L_i(\eta(s),\dot{\eta}(s)) + f_i(\eta(s)))ds + e^{-a_{ii}\,d(x,y)}\,w_i(\eta(-d(x,y)) \\ &= \int_{-d(x,y)}^{0} e^{a_{ii}s}(L_i(\gamma(s),\dot{\gamma}(s)) + f_i(\gamma(s)))ds + e^{-a_{ii}\,d(x,y)}\,w_i(x) \\ &\leq (C + \|f_i\|_{L^\infty(\overline{B_R})}) \left(\frac{1 - e^{-a_{ii}d(x,y)}}{a_{ii}} \right) + e^{-a_{ii}\,d(x,y)}\,w_i(x). \end{aligned}$$

Taking lim sup in the last inequality we get (5.18) as desired.

Next we prove the lower semi continuity. Let us fix a point x on the boundary and a sequence x_n converging to it. We claim that

$$\liminf_{x_n \to x} v^i(x_n) \geq w_i(x). \tag{5.19}$$

Let $\gamma_n : (-\infty, 0] \to \mathbb{R}^N$ with $\gamma_n(0) = x_n$ be an $\frac{1}{n}-$optimal curve for $v^i(x_n)$ and τ_n be the corresponding entrance times, i.e.

$$v^i(x_n) + \frac{1}{n} \geq \int_{-\tau_n}^{0} e^{a_{ii}s}(L_i(\gamma_n(s),\dot{\gamma_n}(s)) + f_i(\gamma_n(s)))ds + e^{-a_{ii}\,\tau_n}\,w_i(\gamma_n(-\tau^n)). \tag{5.20}$$

Then two cases are possible:

<u>Case 1:</u> If $\tau_n \to +\infty$, we define the family of absolutely continuous curves $\xi_n : (-\infty, 0] \to \mathbb{R}^N$ as follows:

$$\xi_n(s) = \begin{cases} \gamma_n(s) & : \quad s \in [-\tau_n, 0] \\ \gamma_n(-\tau_n) & : \quad s \in (-\infty, -\tau_n]. \end{cases}$$

We have, by the definition of constraint value function

$$V_c^i(x_n) \leq \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\xi_n(s),\dot{\xi_n}(s)) + f_i(\xi_n(s)) \right) ds = I_n + J_n \tag{5.21}$$

with

$$I_n = \int_{-\infty}^{-\tau_n} e^{a_{ii}s}(L_i(\gamma_n(-\tau_n),0) + f_i(\gamma_n(-\tau_n)))ds$$

$$J_n = \int_{-\tau_n}^{0} e^{a_{ii}s}(L_i(\gamma_n(s),\dot{\gamma_n}(s)) + f_i(\gamma_n(s)))ds.$$

The term I_n vanishes as $n \to +\infty$ by the boundedness of the integrand, while, in view of (5.20), J_n satisfies

$$\liminf_{n\to+\infty} J_n \leq \liminf_{n\to+\infty} \left(v^i(x_n) + \frac{1}{n} + e^{-a_{ii}\,\tau_n} \|w_i\|_{L^\infty(\overline{B}_R)} \right) = \liminf_{x_n \to x} v^i(x_n). \tag{5.22}$$

Taking lim inf in (5.21) and taking into account the continuity up to the boundary of the constraint value function and (5.22) , we get

$$V_c^i(x) \leq \liminf_{x_n\to x} v^i(x_n).$$

Therefore, by the maximality of constraint value function and w_i being subsolution of (5.16), we have

$$w_i(x) \leq V_c^i(x) \leq \liminf_{x_n\to x} v^i(x_n)$$

as desired.

Case 2: If τ_n is bounded, then it converges up to subsequence to some τ. Moreover, there exists a sequence of boundary points y_n such that $\gamma_n(-\tau_n) = y_n$.

Fix $\epsilon > 0$ and let $\zeta_n : [-\tau - \epsilon, 0] \to \mathbb{R}^N$ as follows:

$$\zeta_n(s) = \begin{cases} \gamma_n(s) & \text{for} \quad s \in [-\tau_n, 0] \\ \gamma_n(-\tau_n) & \text{for} \quad s \in [-\tau - \epsilon, -\tau_n] \end{cases}$$

We have

$$\int_{-\tau-\epsilon}^{0} e^{a_{ii}s}\, L_i(\zeta_n(s), \dot{\zeta}_n(s))ds = K_n + W_n \tag{5.23}$$

with

$$\begin{aligned} K_n &= \int_{-\tau-\epsilon}^{-\tau_n} e^{a_{ii}s}\, L_i(y_n, 0)ds \\ &\leq \frac{2\,\max_{z\in\partial B_R} L_i(z,0)}{a_{ii}} \end{aligned}$$

and

$$\begin{aligned} W_n &= \int_{-\tau_n}^{0} e^{a_{ii}s}\, L_i(\gamma_n(s), \dot{\gamma_n}(s)ds \\ &\le v^i(x_n) + \frac{1}{n} - \int_{-\tau_n}^{0} e^{a_{ii}s}\, f_i(\gamma_n(s))ds - e^{-a_{ii}\,\tau_n}\, w_i(y_n) \\ &\le \|v^i\|_{L^\infty(\overline{B}_R)} + \frac{2\,\|f_i\|_{L^\infty(\overline{B}_R)}}{a_{ii}} + \|w_i\|_{L^\infty(\partial B_R)} \end{aligned}$$

Combining the above information, we get

$$\sup_n \int_{-\tau-\epsilon}^{0} e^{a_{ii}s}\, L_i(\zeta_n(s), \dot{\zeta_n}(s))ds < +\infty.$$

According to Dunford–Pettis type theorem, see for instance [12, Theorem 6.4], the curves ζ_n uniformly converge, up to subsequences, to an absolutely continuous curve $\zeta : [-\tau - \epsilon, 0] \to \mathbb{R}^N$ with $\zeta(0) = x$ and satisfying

$$\int_{-\tau-\epsilon}^{0} e^{a_{ii}s}\, L_i(\zeta(s), \dot{\zeta}(s))ds \le \liminf_{n\to+\infty} \int_{-\tau-\epsilon}^{0} e^{a_{ii}s}\, L_i(\zeta_n(s), \dot{\zeta_n}(s))ds \tag{5.24}$$

Passing to lim inf in (5.23) and taking into account (5.24), we get

$$\int_{-\tau-\epsilon}^{0} e^{a_{ii}s} L_i(\zeta(s), \dot{\zeta}(s))ds \le \liminf_{x_n\to x} v^i(x_n) + \int_{-\tau-\epsilon}^{-\tau} e^{a_{ii}s} L_i(\zeta(-\tau), 0)ds - \int_{-\tau}^{0} e^{a_{ii}s} f_i(\zeta(s))ds - e^{-a_{ii}\tau} w_i(\zeta(-\tau)). \tag{5.25}$$

Assume to the contrary that (5.19) doesn't hold, then in view of (5.25) and that $\zeta(0) = x$, we obtain

$$w_i(\zeta(0)) - e^{-a_{ii}\tau} w_i(\zeta(-\tau)) > \int_{-\tau-\epsilon}^{0} e^{a_{ii}s} L_i(\zeta(s), \dot{\zeta}(s))ds - \int_{-\tau-\epsilon}^{-\tau} e^{a_{ii}s} L_i(\zeta(-\tau), 0)ds + \int_{-\tau}^{0} e^{a_{ii}s} f_i(\zeta(s))ds.$$

Hence

$$w_i(\zeta(0)) - e^{-a_{ii}\tau} w_i(\zeta(-\tau)) > \int_{-\tau}^{0} e^{a_{ii}s} \Big(L_i(\zeta(s), \dot{\zeta}(s)) + f_i(\zeta(s)) \Big)\, ds,$$

which contradicts the sub-optimality principle for subsolutions (Lemma 5.2.2). This proves (5.19) as desired. □

Next we aim at providing a solution to (HJα) in $\mathbb{R}^N$, at any supercritical level. As a first step we will construct a solution to (HJα) in B_R, as a limit function of a sequence of subsolutions (v_n). To do that, we follow the algorithm presented in Section 5.3, with the Dirichlet problem (BVP) replacing the discounted equation in each step. More precisely, we start with a subsolution $\mathbf{w} = (w_1, \cdots, w_m) = \mathbf{v}_0 = (v_0^1, \cdots, v_0^m)$ of (HJα) and then we define the first term $\mathbf{v}_1 = (v_1^1, \cdots, v_1^m)$ of (v_n) as follows :

Let v_1^k be the solution of

$$\begin{cases} H_k(x, Du) + a_{kk}u(x) + \sum_{j<k} a_{kj}v_1^j(x) + \sum_{j>k} a_{kj}v_0^j(x) - \alpha = 0, & B_R \\ v_1^k = v_0^k, & \partial B_R \end{cases} \tag{5.26}$$

Applying Theorem 5.4.3, we get

$$\mathbf{w} = \mathbf{v}_0 \leq \mathbf{v}_1 \quad \text{and} \quad \mathbf{w} = \mathbf{v}_0 = \mathbf{v}_1 \text{ on } \partial B_R.$$

Similarly we can get the second term $\mathbf{v}_2$ but now starting with the subsolution $\mathbf{v}_1 = (v_1^1, \cdots, v_1^m)$ instead of $\mathbf{w}$, and again we have

$$\mathbf{v}_2 \leq \mathbf{v}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{v}_1 \text{ on } \partial B_R.$$

Repeating the same construction for $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \cdots$, we get

$$\mathbf{w} \leq \mathbf{v}_1 \leq \mathbf{v}_2 \leq \mathbf{v}_3 \leq \cdots \text{ on } B_R$$

and

$$\mathbf{w} = \mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \cdots \text{ on } \partial B_R.$$

According to the construction we have produced a monotonic sequence (v_n) of subsolutions in B_R taking the value $\mathbf{w}$ on the boundary, and hence equi-Lipschitz and equibounded. Applying Ascoli-Theorem in B_R and exactly same stability argument we performed before , we get a solution of (HJα) in B_R assuming a fixed value on the boundary which is the starting subsolution w.

Next, let (R_n) be a sequence of radii going to infinity and v_n be sequence of solutions to (HJα) in B_{R_n}, resulting from performing the Algorithm in B_{R_n} for every n according to the previous step.
We now define another sequence of functions $u_n : \mathbb{R}^N \to \mathbb{R}^m$ as following

$$\mathbf{u}_n(x) = \begin{cases} \mathbf{v}_n(x) & \text{if } x \in B_{R_n} \\ \mathbf{w}(x) & \text{otherwise} \end{cases}$$

Clearly, $\mathbf{u}_n$ are solutions to (HJα) in B_{R_n} and subsolutions in $\mathbb{R}^N$. Following the same argument of Proposition 5.4.2, we infer that the functions $\mathbf{u}_n$ are locally equi-Lipschitz and locally equibounded. Up to subsequences, by Ascoli-Arzela Theorem, we get

$$\mathbf{u}_n \to \mathbf{u} \quad \text{locally uniformly in } \mathbb{R}^N.$$

Now we will show that $\mathbf{u}$ is solution of (HJα) in $\mathbb{R}^N$. Clearly, $\mathbf{u}$ is a subsolution to (HJα) in $\mathbb{R}^N$ by using basic stability property of viscosity solution theory.
We are left to prove the supersolution property of u. To this end we take any $x \in \mathbb{R}^N$, then $x \in B_{R_n}$ for n large enough, hence by the supersolution property of u_n in B_{R_n}, one has

$$H_i(x, Du_n^i(x)) + \sum_{j=1}^m a_{ij} u_n^j(x) \geq \alpha.$$

Passing to the limit we get

$$H_i(x, Du_i(x)) + \sum_{j=1}^m a_{ij} u_j(x) \geq \alpha,$$

which proves the supersolution property of $\mathbf{u}$ in $\mathbb{R}^N$.

5.5 Applications of the scalar reduction

True to title, we describe in this section two applications of the scalar reduction method. Namely, we provide a characterization of the isolated points of $\mathcal{A}$ and establish some semiconcavity properties of critical subsolutions to the system. To do that, we need a strengthened form of Theorem 5.2.3 for points belonging to $\mathcal{A}$.

Proposition 5.5.1 *Let $y \in \mathcal{A}$ and $\mathbf{u}$ be a subsolution of (HJβ). Then for every $i \in \{1, \cdots, m\}$ there exists a curve $\gamma : (-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = y$ such that*

$$u_i(y) = \int_{-\infty}^0 e^{a_{ii}s} \left(L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} u_j(\gamma(s)) + \beta \right) ds,$$

and $\gamma(t) \in \mathcal{A}$ for every t.

For the proof we need a preliminary lemma

Lemma 5.5.2 *Let $\mathbf{u}$ be a subsolution of (HJβ) strict outside $\mathcal{A}$. Then for every $y \in \mathcal{A}$, there exists a critical solution $\mathbf{v}$ such that*

$$\mathbf{u}(y) = \mathbf{v}(y) \quad \textit{and} \quad u < v \textit{ on } \mathbb{T}^N \setminus \mathcal{A}.$$

Proof. Given $y \in \mathcal{A}$, we consider the maximal critical subsolution $\mathbf{v}$ taking the value $\mathbf{u}(y)$ at y. Then, by the very definition of Aubry set, $\mathbf{v}$ is a critical solution, and $\mathbf{v} \geq u$. If this inequality were not strict at some $x_0 \in \mathbb{T}^N \setminus \mathcal{A}$, then $u_i(x_0) = v_i(x_0)$ for some index i, and consequently u_i should be subtangent to v_i at x_0, and hence by Proposition 2.1.14

$$\beta \leq H_i(x_0, p) + \sum_{j=1}^{m} a_{ij} v_j(x_0) \leq H_i(x_0, p) + \sum_{j=1}^{m} a_{ij} u_j(x_0)$$

for some $p \in \partial u_i(x_0)$. This contradicts u_i being locally strict at x_0, in view of Lemma 3.4.7.

□

Proof. of the Proposition 5.5.1 We consider a critical subsolution $\mathbf{w}$ to the system strict outside $\mathcal{A}$, see Theorem 3.4.9. It is not restrictive, by adding a suitable constant, to assume $u_i(y) = w_i(y)$, where $\mathbf{u}$ is the subsolution appearing in the statement. This in turn implies by the rigidity property on the Aubry set, see Theorem 3.4.11, $\mathbf{u}(y) = \mathbf{w}(y)$. We in addition denote by $\bar{u}$ the maximal subsolution taking the value $\mathbf{w}(y) = \mathbf{u}(y)$ at y. It is a critical solution to the system in view of Theorem 3.4.8 and, according to Lemma 5.5.2, we also have

$$w < \bar{u} \quad \text{on } \mathbb{T}^N \setminus \mathcal{A}. \tag{5.27}$$

Now, let $\underline{v}$ be the solution of the discounted equation

$$H_i(x, Dv) + a_{ii} v(x) + \sum_{j \neq i} a_{ij} w_j(x) = \beta,$$

and $\bar{v}$ the solution of

$$H_i(x, Dv) + a_{ii} v(x) + \sum_{j \neq i} a_{ij} \bar{u}_j(x) = \beta.$$

We deduce from Corollary 5.2.4

$$\bar{v}(y) = \underline{v}(y) = \bar{u}_i(y) = u_i(y) = w_i(y). \tag{5.28}$$

There exists, in force of Theorem 5.2.3, a curve $\gamma : (-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = y$ such that

$$\bar{u}_i(y) = \int_{-\infty}^{0} e^{a_{ii} s} \left(L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} \bar{u}_j(\gamma(s)) + \beta \right) ds.$$

Assume, for purposes of contradiction, that the support of γ is not contained in $\mathcal{A}$ then, taking into account (5.27), that $a_{ij} < 0$ for $i \neq j$ by (A1) plus irreducibility of A, we get

$$\begin{aligned} w_i(y) &\leq \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} w_j(\gamma(s)) + \beta \right) ds \\ &< \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} \bar u_j(\gamma(s)) + \beta \right) ds \\ &= \bar u_i(y), \end{aligned}$$

which is impossible in view of (5.28). By the maximality property of $\bar u$, we also have

$$\begin{aligned} u_i(y) &\leq \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} u_j(\gamma(s)) + \beta \right) ds \\ &\leq \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} \bar u_j(\gamma(s)) + \beta \right) ds \\ &= \bar u_i(y), \end{aligned}$$

which proves that γ is also optimal for $u_i(y)$ and concludes the proof. □

5.5.1 Equilibria of weakly coupled system

In this section we provide a characterization of the isolated points of $\mathcal{A}$. To this aim, we introduce the notion of equilibrium points of the weakly coupled system, which we compactly write in the form

$$\mathbf{H}(x, Du) + Au = \beta \mathbf{1}, \tag{5.29}$$

where the Hamiltonian $\mathbf{H} : \mathbb{T}^N \times \mathbb{R}^{mN}$ has the separated variable form

$$\mathbf{H}(x, p_1, \cdots, p_m) = (H_1(x, p_1), \cdots, H_m(x, p_m)).$$

We consider the equilibrium distribution $\mathbf{o} \in \mathbb{R}^m$ which is uniquely identified by the following conditions:

1) $\mathbf{o}\, A = 0$

2) $\mathbf{o} \cdot \mathbf{1} = 1$

It is an immediate consequence of the condition $\mathrm{Im}\, A \cap \mathbb{R}^m_+ = \{0\}$ plus $\dim \mathrm{Im} A = m - 1$, that all the vectors orthogonal to $\mathrm{Im}\, A$ have either strictly positive or strictly negative components. Consequently $\mathbf{o}$ is a probability vector, i.e. all its components are nonnegative and sum up to 1.

Multiplying the system (5.29) by $\mathbf{o}$ we get that all subsolutions $\mathbf{u}$ satisfy

$$\mathbf{o} \cdot \mathbf{H}(x, p_1, \cdots, p_m) \le \beta \qquad \text{for any } x,\, p_i \in \partial u_i(x). \tag{5.30}$$

For $x \in \mathbb{T}^N$, we set $\min_p \mathbf{H}(x,p) := \left(\min_p H_1(x,p), \cdots, \min_p H_m(x,p) \right)$.

Then we deduce from (5.30) that

$$\mathbf{o} \cdot \min_p \mathbf{H}(x,p) \le \beta \qquad \text{for any } x.$$

We call a point x equilibrium if

$$\mathbf{o} \cdot \min_p \mathbf{H}(x,p) = \beta.$$

We see from the above definition that if x is an equilibrium and $\mathbf{u}$ a critical subsolution then for any i and $q \in \partial u_i(x)$ we have

$$H_i(x,q) = \min_p H_i(x,p) \tag{5.31}$$

$$\beta = H_i(x,q) + \sum_{j=1}^{m} a_{ij}\, u_j(x) \tag{5.32}$$

This implies that any subsolution also satisfies the supersolution property at an equilibrium point, and we deduce from Proposition 3.4.4 and Theorem 3.4.8 that the set of equilibria is contained in the Aubry set.

The next proposition is a partial converse of this fact, it provides a characterization of the isolated points of Aubry set, which is a generalization of the scalar case.

Proposition 5.5.3 *Any isolated point of the Aubry set is an equilibrium.*

Proof.

Let x be an isolated point of Aubry set and $\mathbf{u}$ be a critical subsolution of the system. Then, in view of Proposition 5.5.1, there exists a curve γ with $\gamma(0) = x$ such that

$$u_i(x) = \int_{-\infty}^{0} e^{a_{ii}s} \left(L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \ne i} a_{ij} u_j(\gamma(s)) + \beta \right) ds, \qquad \text{for every } i \in \{1, \cdots, m\}$$

and the support of γ is contained in $\mathcal{A}$. Exploiting the fact that x is isolated, we get $\gamma(t) \equiv x$ for every t and hence

$$u_i(x) = \frac{1}{a_{ii}} \left(L_i(x,0) - \sum_{j \ne i} a_{ij} u_j(x) + \beta \right), \qquad \text{for every } i \in \{1, \cdots, m\}.$$

Then

$$\begin{aligned} A\mathbf{u}(x) &= (L_1(x,0),\cdots,L_m(x,0))+\beta\mathbf{1} \\ &= -\min_p \mathbf{H}(x,p)+\beta\mathbf{1}. \end{aligned}$$

Multiplying by $\mathbf{o}$ and taking into account that $\mathbf{o}$ is a probability vector orthogonal to $\mathrm{Im}(A)$, we get

$$\mathbf{o}\cdot\min_p \mathbf{H}(x,p)=\beta$$

as desired. □

Assuming the strict convexity assumption (H4), we get a regularity result.

Proposition 5.5.4 *Under the additional assumption (H4), any critical subsolution is strictly differentiable at every isolated point of $\mathcal{A}$.*

Proof. Let x_0 be an isolated point of $\mathcal{A}$ and $\mathbf{u}$ be a subsolution of (HJβ). Then for every $i\in\{1,\cdots,m\}$, we have

$$H_i(x_0,p_i)+\sum_{j=1}^m a_{ij}u_j(x_0)\le\beta \quad \text{for every } p_i\in\partial u_i(x_0).$$

This implies

$$\sum_{i=1}^m o_i H_i(x_0,p_i)\le\beta.$$

Taking into account that x_0 is equilibrium we deduce from the above inequality that

$$\beta=\sum_{i=1}^m o_i\min_p H_i(x_0,p)\le\sum_{i=1}^m o_i H_i(x_0,p_i)\le\beta,$$

which in turn gives that

$$H_i(x_0,p_i)=\min_p H_i(x_0,p), \qquad \text{for every } p_i\in\partial u_i(x_0),\ i\in\{1,\cdots,m\}.$$

Due to H_i being strictly convex, the above minimum is unique and hence $\partial u_i(x_0)$ reduces to a singleton. This implies strict differentiability of $\mathbf{u}$ at x_0. □

5.5.2 Semiconcavity-type estimates for critical subsolutions

In this section we study a family of Eikonal equations derived from the critical system. The main information we gather through this approach, under the additional assumptions

(H4),(H5), is that the superdifferential of any critical solution of (HJβ) is nonempty at every point of the torus. The same property holds true for any critical subsolution on the Aubry set.

We start by stating and proving a consequence of Theorem 5.2.3.

Proposition 5.5.5 *Let* $\mathbf{u}$*,* x*,* i *be a critical solution to the system, a point in* $\mathbb{T}^N$ *and an index in* $\{1, \cdots, m\}$*, respectively. There is a curve* γ *defined in* $(-\infty, 0]$ *such that* $\gamma(0) = x$ *and*

$$\frac{d}{dt} u_i(\gamma(t)) = L_i(\gamma(t), \dot\gamma(t)) - \sum_j a_{ij}\, u_j(\gamma(t)) + \beta \qquad \text{for a.e. } t \in (-\infty, 0).$$

Proof.

Taking into account that u_i is the solution of the discounted equation (5.1) with u_j in place of w_j, we know by Theorem 5.2.3 that there is an optimal curve γ defined in $(-\infty, 0]$ with $\gamma(0) = x$ such that

$$u_i(x) = \int_{-\infty}^{0} e^{a_{ii} s} \left[L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} u_j(\gamma(s)) + \beta \right] ds. \tag{5.33}$$

We claim that γ also satisfies the statement of the proposition. We define

$$g(t) = e^{a_{ii} t} u_i(\gamma(t)) \qquad \text{for } t \in (-\infty, 0),$$

accordingly

$$\frac{d}{dt}\, g(t) = a_{ii}\, e^{a_{ii} t} u_i(\gamma(t)) + e^{a_{ii} t}\, p(t) \cdot \dot\gamma(t) \tag{5.34}$$

for a.e. t, where $p(t)$ is a suitable element of $\partial u_i(\gamma(t))$ satisfying $\frac{d}{dt} u_i(\gamma(t)) = p(t) \cdot \dot\gamma(t)$ for a.e. t, see Lemma 2.1.12. We further get taking into account that $\mathbf{u}$ is a solution to the critical system

$$p(t) \cdot \dot\gamma(t) \leq H_i(\gamma(t), p(t)) + L_i(\gamma(t), \dot\gamma(t)) \leq - \sum_j a_{ij} u_j(\gamma(t)) + \beta + L_i(\gamma(t), \dot\gamma(t)). \tag{5.35}$$

We derive from (5.33), (5.34), (5.35)

$$\begin{aligned} u_i(x) &= \lim_{t \to -\infty} g(0) - g(t) = \int_{-\infty}^{0} \frac{d}{dt}\, g(t)\, dt \\ &\leq \int_{-\infty}^{0} a_{ii}\, e^{a_{ii} t} u_i(\gamma(t)) - e^{a_{ii} t} \left(\sum_j a_{ij} u_j(\gamma(t)) - \beta - L_i(\gamma(t), \dot\gamma(t)) \right) dt \\ &= \int_{-\infty}^{0} e^{a_{ii} t} \left[L_i(\gamma(t), \dot\gamma(t)) - \sum_{j \neq i} a_{ij}\, u_j(\gamma(t)) + \beta \right] dt = u_i(x). \end{aligned}$$

This in turn implies

$$\frac{d}{dt}u_i(\gamma(t)) = p(t)\cdot\dot\gamma(t) = -\sum_j a_{ij}u_j(\gamma(t)) + \beta + L_i(\gamma(t),\dot\gamma(t)) \qquad \text{for a.e. } t,$$

as it was to be proved. □

In the case where the point x belongs in addition to $\mathcal{A}$, we get, thanks to Proposition 5.5.1, a strengthened form of the previous assertion.

Corollary 5.5.6 *The statement of Proposition 5.5.5 holds true for any critical subsolution* $\mathbf{u}$*, provided* $x \in \mathcal{A}$*. The curve* γ *is in addition contained in* $\mathcal{A}$*.*

Proof. If $\mathbf{u}$ is any critical subsolution, we know from Proposition 5.5.1 that there is an optimal curve γ for $u_i(x)$ which is in addition contained in $\mathcal{A}$. We then prove that γ satisfies the assertion arguing as in Proposition 5.5.5.

We recognize that the integrand appearing in the statement of Proposition 5.5.5 is nothing but the Lagrangian associated through Fenchel transform to the Hamiltonian

$$H_i^{\mathbf{u}}(x,p) = H_i(x,p) + \sum_{j=1}^m a_{ij}u_j(x). \tag{5.36}$$

Given a critical subsolution $\mathbf{u}$ to the system, we therefore consider the Eikonal equation

$$H_i^{\mathbf{u}}(x,Dv) = \beta \qquad \text{in } \mathbb{T}^N, \tag{5.37}$$

and denote by $\sigma_i^{\mathbf{u}}$, $S_i^{\mathbf{u}}$ the corresponding support function and intrinsic distance, respectively, given by suitably adapting (2.14) and (2.15). Since u_i is a subsolution to (5.37), it is clear that the critical value of $H_i^{\mathbf{u}}$ is less than or equal to β. We in addition have:

Proposition 5.5.7 *The critical value of* $H_i^{\mathbf{u}}(x,p)$ *is equal to* β*, for any critical subsolution* $\mathbf{u}$ *to the system, any index* $i \in \{1,\cdots,m\}$*. In addition the limit points, as* $t \to -\infty$*, of any curve satisfying the statement of Proposition 5.5.5/Corollary 5.5.6 belong to the corresponding Aubry set.*

Proof. We fix $\mathbf{u}$ and i. Let us consider $x \in \mathcal{A}$ and an optimal curve γ as in the statement with $\gamma(0) = x$. We denote by y a limit point of γ as $t \to -\infty$. If the set of such limit points reduces to y, then there is a sequence $t_n \to -\infty$ with

$$\frac{d}{dt}u_i(\gamma(t_n)) = L_i(\gamma(t_n),\dot\gamma(t_n)) - \sum_j a_{ij}\,u_j(\gamma(t_n)) + \beta \quad \text{and} \quad \dot\gamma(t_n) \to 0,$$

therefore

$$0 = \lim_{t_n \to -\infty} \frac{d}{dt} u_i(\gamma(t_n)) = \lim_{t_n \to -\infty} L_i(\gamma(t_n), \dot\gamma(t_n)) - \sum_j a_{ij}\, u_j(\gamma(t_n)) + \beta.$$

By continuity of L_i, u_j we deduce

$$L_i(y,0) - \sum_j a_{ij}\, u_j(y) = -\beta$$

or equivalently $\min_p H_i^u(y,p) = \beta$. Since we know that β is supercritical for $H_i^{\mathbf{u}}$, this implies that β is actually the critical value of $H_i^{\mathbf{u}}$ and y belongs to the corresponding Aubry set, by Proposition 2.2.4.

If instead the limit set of γ, as $t \to -\infty$, is not a singleton, then we find $\gamma(t_n)$ converging to y such that the curves $\gamma_n := \gamma\big|_{[t_n, t_{n+1}]}$ possess Euclidean length bounded from below by a positive constant. We have

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \sigma_i^{\mathbf{u}}(\gamma(s), \dot\gamma(s))\, ds \quad &\leq \quad \int_{t_n}^{t_{n+1}} \left[L_i(\gamma(s), \dot\gamma(s)) - \sum_j a_{ij}\, u_j(\gamma(s)) + \beta \right] ds \\ &= \quad u_i(\gamma(t_{n+1})) - u_i(\gamma(t_n)). \end{aligned}$$

We deduce using (ii) of Proposition 2.2.2 that the leftmost inequality in the above formula must actually be an equality. This shows that the intrinsic length $\int_{t_n}^{t_{n+1}} \sigma_i^{\mathbf{u}}(\gamma(s), \dot\gamma(s))\, ds$ is infinitesimal as $n \to +\infty$.

We construct a sequence of cycles η_n based on y by concatenating the segment linking y to $\gamma(t_n)$, γ_n and the segment linking $\gamma(t_{n+1})$ to y. We find that the intrinsic lengths of such cycles are infinitesimal, as $n \to +\infty$, while the Euclidean lengths stay bounded from below by a positive constant. Taking into account the very definition of Aubry set for scalar Eikonal equations, we derive also in this case that β is the critical value of H_i^u, and y belongs to the corresponding Aubry set. This concludes the proof. □

We denote by $\mathcal{A}_i^{\mathbf{u}}$ the Aubry set associated with $H_i^{\mathbf{u}}$ at the critical level β, for $i \in \{1, \cdots, m\}$.

Proposition 5.5.8 *We have that*

$$\mathcal{A}_i^{\mathbf{u}} \cap \mathcal{A} \neq \emptyset \qquad \textit{for any susbsolution } \mathbf{u} \textit{ to } (\mathrm{HJ}\beta), \textit{ any } i. \tag{5.38}$$

If, in addition, $\mathbf{u}$ *is strict on* $\mathbb{T}^N \setminus \mathcal{A}$*, then*

$$\mathcal{A}_i^{\mathbf{u}} \subseteq \mathcal{A} \quad \textit{for every } i \in \{1, \cdots, m\} \tag{5.39}$$

Proof. Formula (5.38) is a direct consequence of Corollary 5.5.6 and Proposition 5.5.7. To show (5.39), let us consider $y \notin \mathcal{A}$, then u_i is locally strict at y for every $i \in \{1, \cdots, m\}$. Hence, there exists an open neighborhood W of y and $\delta > 0$ such that

$$H_i(x, Du_i(x)) + \sum_{j=1}^{m} a_{ij} u_j(x) < -\delta + \beta \quad \text{for a.e. } x \in W, \quad \text{for every } i \in \{1, \cdots, m\}.$$

Therefore, u_i is a critical subsolution of (5.37) which is locally strict at y and consequently $y \notin \mathcal{A}_i^{\mathbf{u}}$.

□

To establish the final result, we will also need the following additional requirements for $i \in \{1, 2, ..., m\}$:

(H4) $p \mapsto H_i(x, p)$ is strictly convex for every $x \in \mathbb{T}^N$;

(H5) $(x, p) \mapsto H_i(x, p)$ is locally Lipschitz continuous in $\mathbb{T}^N \times \mathbb{R}^N$.

Note that, due to the Lipschitz character of any subsolution to the system, the Hamiltonians H_i^u are locally Lipschitz–continuous in $\mathbb{T}^N \times \mathbb{R}^N$, for any subsolution $\mathbf{u}$ of (HJβ), any index i.

In this setting we obtain:

Theorem 5.5.9 *We assume(H4), (H5). If* $\mathbf{u}$ *is a critical subsolution of (HJβ), then*

$$D^+ u_i(x) \neq \emptyset \qquad \textit{for every } i \in \{1, 2, ..., m\},\ x \in \mathcal{A}.$$

If, in addition, $\mathbf{u}$ *is a solution to (HJβ) then the above property holds true for any* $x \in \mathbb{T}^N$.

Proof.

First assume $\mathbf{u}$ to be subsolution of (HJβ). If $x_0 \in \mathcal{A}_i^{\mathbf{u}}$ then u_i is differentiable at x_0, according to Theorem 2.2.7. This proves the assertion. If instead $x_0 \in \mathcal{A} \setminus \mathcal{A}_i^{\mathbf{u}}$, then we derive from the proof of Proposition 5.5.7 that

$$u_i(x_0) \geq \min_{y \in \mathcal{A}_i^{\mathbf{u}}} \{u_i(y) + S_i^{\mathbf{u}}(y, x_0)\}. \tag{5.40}$$

By Proposition 2.2.5, the function on the right hand–side of the above formula is the maximal subsolution to (5.37) with trace u_i on $\mathcal{A}_i^{\mathbf{u}}$, this implies that equality must prevail in (5.40). There is then an element $y_0 \in \mathcal{A}_i^{\mathbf{u}}$ such that

$$u_i(x_0) = u_i(y_0) + S_i^{\mathbf{u}}(y_0, x_0).$$

Hence $u_i(y_0) + S_i^{\mathbf{u}}(y_0, .)$ is supertangent to u_i at x_0, and so by Proposition 2.2.6 the superdifferential of u_i is nonempty at x_0, as it was claimed. If $\mathbf{u}$ is in addition solution of (HJβ), the same argument of above gives that $D^+u_i(x_0) \neq \emptyset$ at any $x_0 \in \mathbb{T}^N$. This concludes the proof. $\square$

Appendix A

Linear Algebra

A.1 Coupling matrix

Here we briefly present some elementary linear algebraic results concerning coupling matrices.

Definition A.1.1 *Let $A := (a_{ij})$ be a $m \times m$ matrix. We say that A is a coupling matrix if it satisfies the following conditions:*

(A1) $a_{ij} \leq 0$ *for every* $i \neq j$;

(A2) *A is diagonal dominant, namely* $\sum_{j=1}^{m} a_{ij} \geq 0$ *for any* $i \in \{1, 2, ..., m\}$;
It is additionally termed degenerate if

$$\sum_{j=1}^{m} a_{ij} = 0 \text{ for any } i \in \{1, 2, ..., m\};$$

Moreover, A is said to be irreducible if for every $W \subsetneq \{1, 2, ..., m\}$ there exists $i \in W$ and $j \notin W$ such that $a_{ij} < 0$.

When a coupling matrix is also irreducible, an additional information can be derived on its diagonal elements. We have

Proposition A.1.2 *Let A be an $m \times m$ irreducible matrix satisfying $(A1)$ and $(A2)$. Then $a_{ii} > 0$ for every $i \in \{1, 2, ..., m\}$.*

Proof.

It is clear, due to the coupling assumptions, that $a_{ii} \geq 0$. Indeed, if $a_{kk} = 0$ for some $k \in \{1, 2, ..., m\}$, then condition $(A2)$ would imply $a_{kj} = 0$ for every $j \in \{1, \cdots, m\}$, which contradicts the irreducibility character of the matrix A. $\square$

The following property also holds:

Proposition A.1.3 *If A satisfies $(A1)$ and $(A2)$, then $A = sI - B$ for some $s > 0$ and a non negative matrix $B = (b_{ij})_{1 \leq i,j \leq m}$, with $s \geq \rho(B)$ and $\rho(B)$ the spectral radius of B.*

Proof. We define $s := \max_{1 \leq i \leq m} a_{ii} > 0$ and the matrix $B = (b_{ij})$ with $b_{ii} = s - a_{ii}$ and $b_{ij} = -a_{ij}$ for $j \neq i$, then $A = sI - B$. By Perron-Frobenius theorem for nonnegative irreducible matrices, we deduce that the spectral radius $\rho(B)$ of B is a positive eigenvalue of B and there exists a positive eigenvector q such that $Bq = \rho(B)q$ and hence $Aq = (s - \rho(B))q$. Let $q_k = \max_{1 \leq i \leq m} q_i > 0$. Taking into account $(A2)$, we get

$$(s - \rho(B))q_k = \sum_{j=1}^{m} a_{kj} q_j \geq \sum_{j=1}^{m} a_{kj} q_k \geq 0,$$

and therefore $s \geq \rho(B)$. $\square$

We also have the following invertibility criterion:

Proposition A.1.4 *Let A be an $m \times m$ irreducible coupling matrix, then*

(i) $\mathrm{Ker}(A) \subseteq$ *span* $\{(1, \cdots, 1\}$;

(ii) $\mathrm{Ker}(A) =$ *span* $\{(1, \cdots, 1\}$ *if and only if A is degenerate.*

In particular, A is invertible if and only if

$$\sum_{j=1}^{m} a_{ij} > 0 \text{ for some } i \in \{1, 2, ..., m\}.$$

Proof. We first remark that, by assumptions $(A1)$ and $(A2)$,

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}| \quad \text{for every } i \in \{1, \cdots, m\}.$$

We first prove (i). Let $v = (v_1, \cdots, v_m) \in \mathrm{Ker}(A)$ and set

$$I = \{i \in \{1, \cdots, m\}; v_i = \max\{v_1, \cdots, v_m\}\}.$$

We claim that $I = \{1, \cdots, m\}$. If this were not the case then, by the irreducibility assumption on A, there would exist $i \in I$ and $k \notin I$ such that $a_{ik} \neq 0$. Since $Av = 0$, we would get

$$a_{ii}v_i = \sum_{j \neq i} v_j |a_{ij}| \leq v_i \sum_{j \neq i} |a_{ij}| \leq a_{ii}v_i.$$

Then these inequalities must be equalities., Hence

$$v_i|a_{ij}| = v_j|a_{ij}| \quad \text{for every } j \neq i,$$

in particular $v_k = v_i = \max\{v_1, \cdots, v_m\}$, yielding k belongs to I, which is a contradiction. The remainder of the statement trivially follows from item (i). □

The next proposition shows an obstruction in being in the image of a degenerate coupling matrix.

Proposition A.1.5 *Let A be an $m \times m$ degenerate coupling matrix. If $a = (a_1, \cdots, a_m)$ satisfies $a_i > 0$ for every $i \in \{1, \cdots, m\}$, then $a \notin \mathrm{Im}(A)$.*

Proof. Let us assume to the contrary that $a \in \mathrm{I}m(A)$ i.e there exists $v = (v_1, \cdots, v_m)$ such that

$$Av = a.$$

Let $v_k = \min\{v_1, \cdots, v_m\}$, then

$$a_k = \sum_{j=1}^{m} a_{kj}v_j \leq \sum_{j=1}^{m} a_{kj}v_k = 0,$$

in contradiction with the hypothesis $a_k > 0$.

A.2 Stochastic matrices

In this appendix we collect some elementary results concerning stochastic matrices. The results stated are taken from [31], [11] for more details.

We denote by $\mathcal{S} \subset \mathbb{R}^m$ the simplex of probability vectors of $\mathbb{R}^m$, namely with nonnegative components summing to 1.

Definition A.2.1 *A positive matrix M is a matrix for which all the entries are positive, and we write $M > 0$.*

Definition A.2.2 *A right stochastic matrix is a matrix of nonnegative entries with each row summing to* 1.

Proposition A.2.3 *A matrix B is stochastic if and only if*

$$\boldsymbol{a}\,B \in \mathcal{S} \ \text{ whenever } \boldsymbol{a} \in \mathcal{S}. \tag{A.1}$$

Proof. B is stochastic if and only if each one of its rows is a probability vector, i.e.

$$e_i\,B \in \mathcal{S} \quad \text{for every } i,$$

which in turn is equivalent to (A.1).

□

By Perron-Frobenius theorem for nonnegative matrices, we have

Proposition A.2.4 *Let B be a stochastic matrix, then its maximal eigenvalue is* 1 *and there is a corresponding left eigenvector in $\mathcal{S}$.*

By Perron-Frobenius theorem for positive matrices, we have

Proposition A.2.5 *Let B be a positive stochastic matrix, then its maximal eigenvalue is* 1 *and is simple. In addition, there exists a unique positive corresponding left eigenvector which is an element of $\mathcal{S}$.*

We now remark that the coupling matrix, under the above assumptions, spans a semi group of stochastic matrices. We have the following

Proposition A.2.6 *Given a matrix A and $t \geq 0$. Assume (A1) and (A2) hold, then e^{-At} is stochastic.*

Proof. If A satisfies (A1) and (A2), then, $I - \frac{tA}{n}$ is stochastic for n suitably large and $t > 0$. Hence $\left(I - \frac{tA}{n}\right)^n$ is stochastic because the product of stochastic matrices is still stochastic, and

$$e^{-tA} = \lim_{n\to\infty}\left(I - \frac{tA}{n}\right)^n$$

is also stochastic since stochastic matrices make up a compact subset in the space of square matrices.

□

Moreover, the irreducibility condition (A3) allows to derive the next result, see Theorem 3.2.1 in [31].

Proposition A.2.7 *Let A be the coupling matrix of the system satisfying (A1), (A2), (A3) then e^{-At} is positive for any $t > 0$.*

Appendix B

Proof of Proposition 2.1.14

This Appendix is devoted to give a proof of Proposition 2.1.14 of Section 2.1.2. For this purpose we introduce some basic tools from convex and non smooth analysis.

We first start with two special classes of Lipschitz functions: semiconvex and semiconcave functions. For a comprehensive survey on these functions, we refer readers to [10].

Definition B.0.8 *A function u defined in an open subset Ω of $\mathbb{R}^N$ is said* semiconvex *if one of the following equivalent conditions is valid, for some $\alpha \geq 0$*

$u(x) + \alpha|x|^2$ *is a convex function;*

$u(x) + \alpha|x - x_0|^2$ *is a convex function for some $x_0 \in \Omega$;*

$u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y) + \alpha\lambda(1-\lambda)|x-y|^2$ *for any x, $y \in \Omega$.*

We refer to α as a semiconvexity constant for u.

A function u is said semiconcave *if one of the following equivalent conditions hold for some $\alpha \geq 0$*

$u(x) - \alpha|x|^2$ *is concave ;*

$u(x) - \alpha|x - x_0|^2$ *is concave for some x_0;*

$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y) - \alpha\lambda(1-\lambda)|x-y|^2$ *for any x, y.*

The constant α refers to semiconcavity constant for u.

Semiconvex and semiconcave functions inherit the regularity properties of convex and concave functions, in particular they are twice differentiable almost everywhere, in force of Alexandrov Theorem.

With the term *paraboloid*, we mean a function of the form

$$x \mapsto \beta + p \cdot (x - x_0) + \alpha \, |x - x_0|^2$$

for some α, β in $\mathbb{R}$, p, x_0 in $\mathbb{R}^N$. The constant α, which can have any sign, is called *opening* of the paraboloid.

The next proposition holds:

Proposition B.0.9 *Let u be semiconvex (resp. semiconcave) then $\partial u(x) = D^- u(x)$ (resp. $\partial u(x) = D^+ u(x)$) for all x. Moreover, there exists a paraboloid globally subtangent (resp. supertangent) to u at any point.*

As a consequence of the above result, we can write a semiconvex function u at any point x_0 as

$$u(x_0) = \sup_{x, p \in \partial u(x)} (u(x) + p(x - x_0) - \alpha |x - x_0|^2),$$

namely as the sup envelope of a family of paraboloid with fixed opening. Next we will perform an inverse construction: starting from any upper semicontinuous (resp. lower semicontinuous) function we will define semiconvex (resp semiconcave) functions through sup (resp. inf) envelope of suitable classes of paraboloids.

We proceed by the relevant definitions

Definition B.0.10 *Given an usc function u bounded from above in $\mathbb{R}^N$ and ϵ, the ϵ–sup convolution of u is given by*

$$u^\varepsilon(x) = \max_{y \in \mathbb{R}^N} \left(u(y) - \frac{1}{2\varepsilon} |y - x|^2 \right). \tag{B.1}$$

Similarly, for a lsc function v bounded from below in $\mathbb{R}^N$ and ϵ, the ϵ–inf convolution of v is given by

$$v_\varepsilon(x) = \min_{y \in \mathbb{R}^N} \left(v(y) + \frac{1}{2\varepsilon} |y - x|^2 \right). \tag{B.2}$$

It is apparent that maxima and minima in the previous definitions do exist in force of semicontinuity and boundedness assumptions on u, v. We also clearly have

- $u^\epsilon \geq u$, $v_\epsilon \leq v$ for any ε.
- $u^\epsilon(x)$ is nonincreasing and $v_\epsilon(x)$ nondecreasing with respect to ε for any fixed x.

From now on we give definitions and statement of results mainly for sup– convolutions. By slightly adapting them, we get the corresponding entities and facts for inf– convolutions.

Definition B.0.11 *We say that y_0 is u^ε–*optimal *for a given x_0 if*

$$u^\varepsilon(x_0) = u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2$$

Proposition B.0.12 *For any usc (resp. lsc) function u bounded from above (resp. below), any $\epsilon > 0$ the sup (resp. inf)– convolution is semiconvex (resp. semiconcave) with semiconvexity (resp. semiconcavity) constant $\frac{1}{2\epsilon}$.*

Proof. We just prove the part of the statement about sup convolution. We will show that $u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2$ is convex. We compute

$$\begin{aligned} u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2 &= \sup_{y\in\mathbb{R}^N}\{u(x) - \frac{1}{2\varepsilon}|y-x|^2 + \frac{1}{2\varepsilon}|x|^2\} \\ &= \sup_{y\in\mathbb{R}^N}\{u(y) - \frac{1}{2\varepsilon}|y|^2 + \frac{1}{\varepsilon}(y\cdot x)\} \end{aligned}$$

Then $u^\epsilon(x)+\frac{1}{2\varepsilon}|x|^2$ can be written as the supremum of linear function, and so it is convex, as claimed. □

We proceed inquiring about first order properties of sup–convolutions.

Proposition B.0.13 *Fix $\epsilon > 0$, then, for any x_0*

$$\partial u^\varepsilon(x_0) = co\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \;\; u^\varepsilon\text{–optimal for } x_0\right\},$$

consequently u^ε is (strictly) differentiable at x_0 if and only if it admits an unique u^ε–optimal point y_0, and then $Du^\varepsilon(x_0) = \frac{y_0 - x_0}{\epsilon}$.

We preliminarily show a continuity property for u^ϵ–optimal points.

Lemma B.0.14 *Let x_n be a sequence convergent to some x_0. If, for any n, y_n is u^ϵ–optimal for x_n and $y_n \to y_0$, then y_0 is u^ϵ–optimal for x_0 and $u(y_n) \to u(y_0)$.*

Proof. We have

$$u^\varepsilon(x_n) = u(y_n) - \frac{1}{2\varepsilon}|x_n - y_n|^2,$$

for any n. Passing at the limit, and taking into account that u is usc, we obtain:

$$u^\varepsilon(x_0) \le u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2.$$

By the very definition of sup–convolution, the inequality in the previous formula must indeed be an equality, which shows that y_0 is optimal, as desired, and, in addition the claimed convergence of $u(y_n)$ to $u(y_0)$. □

Proof of Proposition B.0.13 Let us fix x_0 and take y_0 u^ϵ–optimal for it, then, by the very definition of u^ϵ, the quadratic function

$$x \mapsto u(y_0) - \frac{1}{2\varepsilon}|x - y_0|^2$$

is subtangent to u^ϵ at x_0. This shows

$$\frac{y_0 - x_0}{\epsilon} \in D^- u^\varepsilon(x_0) = \partial u^\varepsilon(x_0)$$

and, since the generalized gradient is convex valued

$$\partial u^\varepsilon(x_0) \supset co\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \;\; u^\varepsilon\text{–optimal for } x_0\right\}. \tag{B.3}$$

Now, take a sequence x_n, where u^ϵ differentiable, with $x_n \to x_0$ and $Du^\varepsilon(x_n)$ converges, then

$$Du^\varepsilon(x_n) = \frac{1}{\varepsilon}(y_n - x_n) \quad \text{for any } n, \tag{B.4}$$

where y_n is u^ϵ–optimal for x_n. By Lemma B.0.14 and (B.4)

$$\lim_n Du^\epsilon(x_n) = \frac{y_0 - x_0}{\epsilon}$$

for some y_0 optimal for x_0. Keeping in mind the definition of generalized gradient, we deduce

$$\partial u^\varepsilon(x_0) \subset co\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \;\; u^\varepsilon\text{–optimal for } x_0\right\},$$

which, together with (B.3), yields the assertion. □

We end by giving additional properties of sub/inf convolutions in the case where the initial function is Lipschitz–continuous.

Proposition B.0.15 *Let u, ϵ be a Lipschitz continuous function and a positive constant, respectively. Let x be any point and y u^ϵ–optimal for x then*

i. $|x - y| \le O(\epsilon)$ *($O(\cdot)$ is the Landau symbol).*

ii. $\partial u^\epsilon(x) \cap D^+u(y) \neq \emptyset$.

iii. $u^\epsilon \to u$ *uniformly as* $\epsilon \to 0$.

Proof. We have

$$u(x) \le u^\varepsilon(x) = u(y) - \frac{1}{2\varepsilon}|x-y|^2.$$

We derive

$$|x-y|^2 \le 2\,\varepsilon(u(y)-u(x)),$$

and exploiting Lipschitz–continuity of u, we get

$$|x-y| \le 2\,\epsilon\, L$$

where L is a Lipschitz constant for u, which gives item **i.**.
Using last inequality we further get

$$u^\epsilon(x) - u(x) \le u(y) - u(x) \le 2\, L^2\, \epsilon$$

which shows the asserted uniform convergence.
Finally, from the relation

$$u^\epsilon(x) = u(y) - \frac{1}{2\epsilon}\,|x-y|^2 = \max_z\{u(z) - \frac{1}{2\epsilon}\,|x-z|^2\}$$

we derive, in view of Proposition 2.1.7, $\frac{y-x}{\epsilon} \in D^+u(y)$ which in turn gives item **iii.** taking into account Proposition B.0.13.

□

We exploit the monotonicity of u^ϵ to prove the next proposition:

Proposition B.0.16 *Let* Ω *be a bounded open set,* u *and* w *be a Lipschitz-continuous function and lsc on* $\overline{\Omega}$*, respectively. If* x_ϵ *is a sequence of maximizers of* $u^\epsilon - w$ *in* $\overline{\Omega}$*, then any of its limit points is a maximizer of* $u - w$*. In addition, the corresponding maximum values* M_ϵ *converge to* $M_0 := \max_{\overline{\Omega}} u - w$.

Proof. The sequence u^ϵ decreases with respect to ϵ, and $u^\epsilon \ge u$, for any ϵ. Therefore, by monotonicity, $M_\epsilon := \max_{\overline{\Omega}} u^\epsilon - w$ does converge, as $\epsilon \to 0$, and

$$\lim M_\epsilon \ge M_0 := \max_{\overline{\Omega}} u - w.$$

We can assume, without loosing generality, that $x_\epsilon \to x_0$, for some $x_0 \in \overline{\Omega}$, then, taking into account that u^ϵ converges uniformly to u in view of Proposition B.0.15 and that w is lsc, we get

$$M_0 \le \lim M_\epsilon = \limsup u^\epsilon(x_\epsilon) - w(x_\epsilon) \le u(x_0) - w(x_0) \le M_0.$$

This shows, at the same time, that x_0 is a maximizer of $u - w$ in $\overline{\Omega}$ and that $M_\epsilon \to M_0$, as asserted. □

The next proposition about viscosity test functions holds:

Proposition B.0.17 *Let u be an usc subsolution (resp. lsc supersolution) of $H(x, u(x), Du(x)) = 0$, and ψ a semiconcave (resp. semiconvex) supertangent (resp. subtangent) to u at some point x. Then*

$$H(x, u(x), p) \leq 0 \text{ (resp. } \geq 0) \qquad \text{for all } p \in \partial\psi(x)$$

Proof. We just prove the case subsolution with semiconcave supertangent. Since any supertangent to ψ at x is also supertangent to u at the same point, we get

$$\partial\psi(x) = D^+\psi(x) \subset D^+u(x),$$

which shows the assertion. □

Now we have all the ingredients to prove our proposition:

Proof of Proposition 2.1.14 We treat the subsolution case. Being the argument local, we can take, without loosing generality, ψ bounded. Up to adding a quadratic term $y \to |y - x|^2$, we can assume ψ to be strict supertangent, and so x to be the unique maximizer of $u - \psi$ in a suitable closed ball B centered at x.

From this uniqueness property we deduce, taking into account Proposition B.0.16, that any sequence x_ϵ of maximizers of $u - \psi_\epsilon$ in B converges to x, where ψ_ϵ denotes the inf–convolution. Hence x_ϵ is in the interior of B for ϵ sufficiently small, and then for such ϵ, ψ_ϵ is supertangent to u at x_ϵ so that

$$H(x_\epsilon, u(x_\epsilon), p_\epsilon) \leq 0 \qquad \text{for any } p_\epsilon \in \partial\psi_\epsilon(x_\epsilon) \tag{B.5}$$

in force of Proposition B.0.17. Further

$$u(x_\epsilon) - \psi_\epsilon(x_\epsilon) = \max_B u - \psi_\epsilon \to \max_B u - \psi = u(x) - \psi(x),$$

which, implies, bearing in mind that $\lim \psi_\epsilon(x_\epsilon) = \psi(x)$ by Proposition B.0.15

$$\lim u(x_\epsilon) = u(x) \tag{B.6}$$

We also know, by Proposition B.0.15, that for any y_ϵ ψ_ϵ–optimal for x_ϵ

$$\partial\psi_\epsilon(x_\epsilon) \cap \partial\psi(y_\epsilon) \neq \emptyset. \tag{B.7}$$

Taking into account that $y_\epsilon \to x$, as ϵ goes to 0 by Proposition B.0.15, exploiting (B.6), (B.7) plus the continuity properties of H and generalized gradients, see Proposition 2.1.6, we find $q_\epsilon \in \partial\psi_\epsilon(x_\epsilon) \cap \partial\psi(y_\epsilon)$ suth that

$$\begin{aligned} q_\epsilon &\to p \in \partial\psi(x) \\ H(x_\epsilon, u(x_\epsilon), q_\epsilon) &\to H(x, u(x), p) \end{aligned}$$

then, thanks to (B.5)

$$H(x, u(x), p) \leq 0 \quad \text{and} \quad p \in \partial\psi(x)$$

as claimed.

Bibliography

[1] M. Bardi and I. Capuzzo Dolcetta, *Optimal control theory and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston (1997), doi:10.1007/978-0-8176-4755-1.

[2] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Mathématiques & Applications (Berlin) 17, Springer-Verlag, Paris, 1994.

[3] G. Barles and B. Perthame, *Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations*, Appl. Math. Optim. 21 (1990), no. 1, pp. 21–44, doi:10.1007/bf01445155.

[4] G. Barles and E. Rouy, *A strong comparison result for the Bellman equation arising in stochastic exit time control problems and its applications*, Communications in Partial Differential Equations, 23 (1998), no. 11-12, pp. 552–562, doi:10.1080/03605309808821409.

[5] G. Barles and E. Rouy, *Deterministic and Stochastic Exit Time Control Problems and Hamilton-Jacobi-Bellman Equations with Generalized Dirichlet Boundary Conditions*, Université de Tours, Faculté des Sciences et Techniques, Parc de Grandmont Tours, France.

[6] G. Barles and P. E. Souganidis, *On the large time behavior of solutions of Hamilton–Jacobi equations*, SIAM J. Math. Anal., 31 (2000), no. 4, pp. 935–939, doi:10.1137/s0036141099350869.

[7] P. Billingsley, *Convergence of probability measures*, John Wiley, New York (1999), doi:10.1002/9780470316962.

[8] F. Camilli, O. Ley, and P. Loreti, *Homogenization of monotone systems of Hamilton- Jacobi equations*, ESAIM Control Optim. Calc. Var., 16 (2010), no. 1, pp. 58–76, doi:10.1051/cocv:2008061.

[9] F. Camilli, O. Ley, P. Loreti, and V. D. Nguyen, *Large time behavior of weakly coupled systems of first-order Hamilton-Jacobi equations*, NoDEA Nonlinear Differential Equations and Applications, 19 (2012), no. 6, pp. 719–749, doi:10.1007/s00030-011-0149-7.

[10] P.M Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations and optimal control*, Prog. Nonlinear Differential Equations and Their Applications 58 (2004), Birkhäuser, Boston, doi:10.1007/b138356 .

[11] Carl D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia (2000), doi:10.1137/1.9780898719512.

[12] A. Davini, A. Fathi, R. Iturriaga, and M. Zavidovique, *Convergence of the solutions of the discounted equation*, Inventiones mathematicae, 206 (2016), no. 1, pp. 29–55, doi:10.1007/s00222-016-0648-6.

[13] A. Davini and A. Siconolfi, *A generalized dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations*, SIAM J. Math. Anal., 38 (2006), no. 2, pp. 478–502, doi:10.1137/050621955.

[14] A. Davini, A. Siconolfi, and M. Zavidovique, *Random Lax–Oleinik semigroups for Hamilton–Jacobi systems* , arXiv:1608.01836.

[15] A. Davini and M. Zavidovique, *Aubry sets for weakly coupled systems of Hamilton-Jacobi equations*, SIAM Journal on Mathematical Analysis, 46 (2014), no. 5, pp. 3361–3389, doi:10.1137/120899960.

[16] H. Engler and S. M. Lenhart, *Viscosity solutions for weakly coupled systems of Hamilton- Jacobi equations*,Proceedings of the London Mathematical Soceity, s3-63 (1991), no. 1, pp. 212–240, doi:10.1112/plms/s3-63.1.212.

[17] A. Fathi, *Weak KAM Theorem in Lagrangian Dynamics*, www.crm.sns.it/media/person/1235/fathi.pdf.s.

[18] A. Fathi and A. Siconolfi, *PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians*, Calc, Var. Partial Differential Equations, 22 (2005), no. 2, pp.185–228, doi:10.1007/s00526-004-0271-z.

[19] H. Ibrahim, A. Siconolfi, and S. Zabad, *Cycle Chacterization of the Aubry Set for Weakly Coupled Hamilton-Jacobi Systems*, arXiv: 1604.08012.

[20] H. Ishii, *Perron's method for monotone systems of second-order elliptic partial differential equations*, Differential Integral Equations, 5 (1992), no. 1, pp. 1–24.

[21] H. Ishii and S. Koike, *Viscosity solutions for monotone systems of second-order elliptic PDEs*, Communications in Partial Differential Equations, 16 (1991), no. 6-7, pp. 1095–1128, doi:10.1080/03605309108820791.

[22] R. Iturriaga and H. SÁNCHEZ-MORGADO *Limit of the infinite horizon discounted Hamilton-Jacobi equation*, Discrete and Continuous Dynamical Systems, 15 (2011), no. 3, pp. 623–635, doi:10.3934/dcdsb.2011.15.623.

[23] A. Klenke, *Probability Theory*, Springer, Berlin (2008), doi:10.1007/978-1-84800-048-3.

[24] S. M. Lenhart, *Viscosity solutions for weakly coupled systems of first-order partial differential equations*, J. Math. Anal. Appl., 131 (1988), no. 1, pp. 180–193, doi:10.1016/0022-247x(88)90199-0.

[25] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan, *Homogenization of Hamilton–Jacobi Equation*, preprint, 1987.

[26] H.Mitake, A.Siconolfi, H.V. Tran, and N. Yamada, *A Lagrangian approach to weakly coupled Hamilton-Jacobi systems*, SIAM J. MATH. ANAL. 48 (2016), no. 2, pp. 821–846, doi:10.1137/15M1010841.

[27] H. Mitake and Hung V. Tran, *A dynamical approach to the large-time behavior of solutions to weakly coupled systems of Hamilton–Jacobi equations*,J. Math. Pures Appl. 101 (2014), no. 1, pp 76–93, doi:10.1016/j.matpur.2013.05.004.

[28] H. Mitake and Hung V. Tran, *Homogenization of Weakly Coupled Systems of Hamilton–Jacobi Equations with Fast Switching Rates*, Arch. Ration. Mech. Anal., 211 (2014), no. 3, pp 733–769, doi:10.1007/s00205-013-0685-x.

[29] H. Mitake and H. V. Tran, *Remarks on the large time behavior of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton-Jacobi equations*, Asymptot. Anal., 77(2012), no. 1-2, pp. 43–70, doi: 10.3233/ASY-2011-1071.

[30] V. Nguyen, *Some results on the large–time behavior of weakly coupled systems of first–order HamiltonŰJacobi equations*, J. Evolution Equations, 14(2013), no. 2, pp. 299–331, doi:10.1007/s00028-013-0214-2.

[31] James R. Norris, *Markov chains*, Cambridge University Press, Cambridge (1997), doi:10.1017/CBO9780511810633.

[32] F. Rezakhanlou and J. E. Tarver, *Homogenization for stochastic Hamilton–Jacobi equations*, Arch. Rational Mech. Anal., 151 (2000), no. 4, pp. 277–309, doi:10.1007/s002050050198.

[33] A. Siconolfi, *Metric character of Hamilton-Jacobi equations*, Trans. A.M.S. 355 (2003), no. 5, pp. 1987–2009, doi:10.2307/1194990 .

[34] A. Siconolfi and S. Zabad, *Scalar reduction techniques for weakly coupled Hamilton–Jacobi systems*, arXiv: 1701.08390.

[35] H.M Soner, *Optimal control with the state-space constraint I*, SIAM J. Control and Optimization, 24(1986), no. 3, pp. 552–562, doi:10.1137/0324032 .

[36] H.M Soner, *Optimal control with the state-space constraint II*, SIAM J. Control and Optimization, 24(1986), no. 6, pp. 1110–1122, doi:10.1137/0324067 .

[37] J. Swart, A. Winter *Markov processes: theory and examples*, Mimeo, 2013, https://www.uni-due.de/ hm0110/Markovprocesses/sw20.pdf.