Local rings of minimal length

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Abstract

This paper deals with local rings $R$ possessing an $m$-canonical ideal $\omega$, $R \subseteq \omega$. In particular those rings such that the length $l_R(\omega/R)$ is as short as possible are studied. The same notion for one-dimensional local Cohen-Macaulay rings was introduced and studied with the name of Almost Gorenstein. Some necessary conditions, that become also sufficient under additional hypotheses, are given and examples are provided also in the non Noetherian case. The case when the maximal ideal of $R$ is stable is also studied.

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1 Introduction

The study of domains possessing an $m$-canonical ideal was first introduced in [6]. We adopt their terminology which derives from the study of canonical ideals for one-dimensional local Cohen-Macaulay rings. In fact, if $R$ is a one-dimensional Cohen-Macaulay ring with certain properties (in particular a one-dimensional analytically unramified ring or a one-dimensional analytically irreducible domain) then $R$ has an $m$-canonical ideal. On the other hand if a Noetherian ring has an $m$-canonical ideal, it is necessarily one-dimensional. However an $m$-canonical ideal exists also in other cases: when $R$ is a valuation domain (in this case the maximal ideal is $m$-canonical) or a PVD of valuation overring $(V,m)$ with $[V/m : R/m] < \infty$. It exists, more generally, for suitable subrings of valuation domains (cf. [2, Theorems 2.14, 2.15, 2.16]). By [2, Theorem 2.1] an integrally closed local domain with a finitely generated $m$-canonical ideal is a valuation domain with principal maximal ideal.
The concept of $m$-canonical ideal is relevant in the local case. In fact, for a local ring, the irreducibility property of its $m$-canonical ideal has several consequences. These facts are given in Section 2.

Section 3 deals with local rings having a finitely generated $m$-canonical ideal. So the results of this section hold both for some classes of domains and for one-dimensional Noetherian rings. If the $m$-canonical ideal is principal, we have the class of local rings such that each regular ideal is divisorial, see [3] and references therein.

In this paper local rings $R$ with an $m$-canonical ideal $\omega$, $R \subseteq \omega$, such that the length $l_R(\omega/R)$ is as short as possible are considered. The same notion for one-dimensional local Cohen-Macaulay rings was studied in [1] with the name of Almost Gorenstein and recently used in [8]. Some necessary conditions (that become also sufficient under additional hypotheses) are given for such rings and examples are also provided in the non Noetherian case. As a corollary, in the Noetherian case, we obtain a new characterization of Almost Gorenstein rings. Finally, the case when the maximal ideal of $R$ is stable is studied.

Elements of a ring $R$ that are not zero divisors are called regular. A regular ideal of $R$ is one that contains a regular element. The results are stated for a local ring $R$ of regular maximal ideal $m$ and of total ring of fractions $Q$, $R \neq Q$. The ring $R$ is also assumed to be a Marot ring (cf. [5]), i.e. each regular ideal is assumed to be generated by its set of regular elements. As usual, the notation $I : J$ means \{x \in Q | xJ \subseteq I\}, for $I, J$ regular fractional ideals of $R$. An overring $T$ of $R$ is a ring $T, R \subseteq T \subseteq Q$, and the length of an $R$-module $M$ is denoted by $l_R(M)$.

2 The $m$-canonical ideal

Let $(R, m)$ be a local ring (i.e. a ring with a unique regular maximal ideal) of maximal ideal $m$ and residue field $k = R/m$. Recall that an $m$-canonical ideal is a regular fractional ideal $\omega$ of $R$ such that $\omega : (\omega : I) = I$, for all regular fractional ideals $I$ of $R$. It is well known that for a local ring $R$ possessing an $m$-canonical ideal several facts hold. We collect them in a proposition and will make use of them several times in the paper.

Proposition 2.1 Let $R$ be a local ring having an $m$-canonical ideal $\omega$. Then:

1. If $\omega$ and $\omega'$ are two $m$-canonical ideals of $R$, then $\omega' = x\omega$, for some regular element $x \in Q$.
2. $\omega : \omega = R$.
3. If $J \subseteq I$ are regular fractional ideals of $R$, then $l_R(I/J) = l_R((\omega : J)/(\omega : I))$.
4. A regular ideal $\tilde{\omega}$ of $R$ is $m$-canonical if and only if it is an irreducible regular fractional ideal, i.e. is not the intersection of any set of regular fractional ideals properly containing $\tilde{\omega}$.
5. If $\omega$ is finitely generated, the minimal number of generators of $\omega$ equals $l_R((R : m)/R)$. 

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Proof. In the following references the proofs are given for integral domains, but all of them work also in the ring case.

(1): cf. [6, Proposition 4.2].

(2): cf. [6, Lemma 2.2 (a)].

(3): cf. e.g. [2, Lemma 2.3].

(4): An $m$-canonical ideal is irreducible by [6, Lemma 4.1]. Conversely, if $\tilde{\omega}$ is an irreducible regular ideal, then $\tilde{\omega} = \omega : (\omega : \tilde{\omega}) = \bigcap_{\omega \subseteq x\omega} x\omega$ implies that $\tilde{\omega}$ equals one of the component of the intersection, i.e. $\tilde{\omega} = x\omega$ is also an $m$-canonical ideal by (1) (cf. [2, page 92]).

(5): By Nakayama's lemma the minimal number of generators of $\omega$ equals $l_R(\omega/\omega m) = l_R((\omega : \omega m)/(\omega : \omega)) = l_R((R : \omega m)/R) = l_R((R : m)/R)$ (cf. e.g. [2, Corollary 2.5]).

Note that, if an $m$-canonical ideal $\omega$ exists, and $x \in \omega$, $x$ non-zerodivisor, then $R \subseteq x^{-1}\omega$, so, by Proposition 2.1(1), we can always assume that $\omega$ is a fractional ideal containing the ring $R$.

A particular class of local rings having an $m$-canonical ideal is that of those rings having a principal $m$-canonical ideal. If the $m$-canonical ideal exists and is a principal regular ideal, then, by Proposition 2.1 (1), we can assume it to be $R$ itself. Thus each regular ideal of $R$ is divisorial. Following [3] we call such a ring a divisorial ring. It is well known that the condition $l_R((R : m)/R) = 1$ is not sufficient in general to get a divisorial ring (cf.[3]), however in our case this is true and very simple:

**Proposition 2.2** Let $(R, m)$ be a local ring having an $m$-canonical ideal. Then $l_R((R : m)/R) = 1$ if and only if $R$ is a divisorial ring.

**Proof.** As in the proof of Proposition 2.1 (5), $l_R((R : m)/R) = 1$ if and only if $l_R(\omega/\omega m) = 1$, i.e. if and only if $\omega$ is principal or equivalently if and only if $R$ is a divisorial ring.

We obtain that, applying Proposition 2.1, each regular fractional ideal turns out to be an intersection of irreducible regular fractional ideals. More precisely:

**Proposition 2.3** Suppose that the local ring $R$ has an $m$-canonical ideal $\omega$.

a) Let $I$ be a finitely generated regular fractional ideal of $R$ and let $\{i_j\}_{j=1}^h$ be a minimal set of generators of $I$, with $i_j$ regular for each $j$, $1 \leq j \leq h$. Then 

$$\omega : I = \bigcap_{j=1}^h i_j^{-1}\omega$$

and the intersection is irredundant.

b) Each regular fractional ideal of $R$ is an intersection of irreducible regular fractional ideals.
Proof. a) \( I = \sum_{j=1}^{h} i_j R = \sum_{j=1}^{h} i_j (\omega : \omega) = \sum_{j=1}^{h} (\omega : i_j^{-1} \omega) \subseteq \omega : \cap_{j=1}^{h} i_j^{-1} \omega. \)

Hence \( \cap_{j=1}^{h} i_j^{-1} \omega \subseteq \omega : I. \) On the other hand, for each \( j, \) \( (\omega : I) \subseteq i_j^{-1} \omega, \)
thus \( \omega : I = \sum_{j=1}^{h} i_j^{-1} \omega. \) Moreover the intersection is irredundant because if \( \cap_{j \neq l} i_j^{-1} \omega \subseteq i_l^{-1} \omega \) then

\[
i_l R \subseteq \omega : \cap_{j \neq l} i_j^{-1} \omega = \sum_{j \neq l} (\omega : i_j^{-1} \omega) = \sum_{j \neq l} i_j R
\]

(cf. [6, Lemma 2.2 (e)] for the first equality), a contradiction with the minimality of the set of generators of \( I. \) By Proposition 2.1 (1), each component is an irreducible fractional ideal.

b) The argument used in a) holds also when the regular ideal \( I \) is not finitely generated. Moreover the existence of an \( m \)-canonical ideal assures that each regular ideal \( J \) of \( R \) is of the form \( \omega : I, \) in fact \( J = \omega : (\omega : J). \)

Remark If the ring \( R \) is not local, but has an \( m \)-canonical ideal \( \omega, \) we can say in the same way that each regular fractional ideal of \( R \) is an intersection of fractional ideals of the form \( x\omega. \) In fact what is needed in the proof of Proposition 2.3 is that \( \omega : \omega = R \) and \( \omega : \cap_{j} I_j = \sum_{j} (\omega : I_j) \) and both these facts hold without any local hypothesis, cf. [6, Lemma 2.2, (a) and (e)].

In particular Proposition 2.3 shows the following fact that was already shown in [2, Proposition 2.6] for domains.

**Corollary 2.4** Suppose that the local ring \( R \) has an \( m \)-canonical ideal \( \omega. \) If \( \omega \) is minimally generated by \( t \) elements, then the ring \( R, \) and each principal regular ideal of \( R \) as well, is an irredundant intersection of \( t \) irreducible regular fractional ideals.

Remark. Recall that the C-M type of a local Cohen Macaulay ring \( (A, m) \) is the dimension of the \( k = A/m \)-vector space \( \text{Socle}(A/(a_1, \ldots, a_d)) \), where \( \{a_1, \ldots, a_d\} \) is a system of parameters of \( A \) (cf. e.g. [9, Chapter VI, Definition 3.18]). In particular, if \( A \) is one-dimensional and \( a \) is a non-zerodivisor of \( A, \) then \( \text{dim}_k(\text{Socle}(A/a)) = l_{A/Aa}((aA : m)/aA) = l_A((A : m)/A) = \text{dim}_k((A : m)/A) \) and the C-M type \( t \) turns out to be the number of components of an irredundant intersection of a principal regular ideal of \( A \) in irreducible integral ideals. Thus, recalling Proposition 2.1 and and Corollary 2.4, the C-M type is at the same time, in the Noetherian one-dimensional case, the number of components of an irredundant intersection of a principal regular ideal in irreducible fractional ideals and in irreducible integral ideals. We would like to stress that, as Corollary 2.4 shows, the first notion is meaningful also in a non Noetherian ring possessing a finitely generated \( m \)-canonical ideal. Note that the two notions are really different, in fact an integral ideal which is irreducible as fractional ideal is also irreducible as integral ideal, but the converse does not hold and there are in general many non-isomorphic irreducible integral ideals as the example below shows. On the contrary, as we saw, there is essentially only one irreducible
regular fractional ideal, the $m$-canonical ideal $\omega$ and all those of the form $x_\omega$, $x \in Q$, $x$ non-zero divisor, isomorphic to $\omega$ as $R$-modules.

**Example** If $k$ is a field, $R = k[[t^3, t^8, t^{10}]]$ is a one-dimensional analytically irreducible ring of value semigroup $S = v(R) = \{0, 3, 6, 8, \rightarrow\}$. By [7] a fractional ideal with value set $\{0, 2, 3, 5, 6, 8, \rightarrow\}$ is canonical. For example such is $\omega = k + t^2k + t^3k + t^5k + t^6k + t^8k[t]\}$. So each irreducible fractional ideal of $R$ is of the form $x_\omega$, for $x \in k((t))^*$. On the other hand, consider the partial order on $S$ given by

$$s_1 \leq s_2 \iff s_1 + s_3 = s_2,$$

for some $s_3 \in S$ and, if $s \in S$, set $B(s) = \{z \in S \mid z \leq s\}$. It turns out (cf. [10]) that $I = S \backslash B(s)$ is an irreducible integral semigroup ideal of $S$ because each integral semigroup ideal containing $I$ contains $s$. It follows that $I = \sum_{i \in I} t^i k$ is an irreducible integral ideal of $R$ not necessarily of the form $x_\omega$. For example, for $s = 12$, we get $I = t^8k + t^{10}k + t^{11}k + t^{13}k[t]\}.$

## 3 Rings of minimal length

For all the section we assume that $(R, m)$ is a local ring with residue field $k$, which has an $m$-canonical ideal $\omega$ minimally generated by $t$ elements. As we saw, we can suppose $R \subseteq \omega$.

By Nakayama’s lemma $t = l_R(\omega/\omega m)$. Thus, by Proposition 2.1 (5), $l_R((R : m)/R) = t > 0$, so $R \subseteq R : m$ and the ideal $m$ is divisorial.

Since $R \subseteq \omega$ we have $m \subseteq Rm \subseteq \omega m$ and

$$l_R(\omega/R) = l_R(\omega/m) - 1 \geq l_R(\omega/\omega m) - 1 = t - 1.$$

We define $R$ of **minimal length** if, up to the multiplication by a regular element of $Q$, $\omega$ can be chosen such that

$$l_R(\omega/R) = t - 1$$

or equivalently such that

$$m = \omega m.$$

In case $R$ is a one-dimensional analytically unramified Noetherian ring, according with the terminology of [1], such a ring is an **Almost Gorenstein** ring.

**Examples** Besides the several examples of one-dimensional Noetherian rings satisfying the definition above, cf. [1], we have:

(a) If $(R, m)$ is a pseudovaluation domain of valuation overring $(V, m)$ with $2 \leq [V/m : R/m] < \infty$, then by [2, Theorem 2.14] each fractional ideal $\omega$ of $R$ such that $R \subseteq \omega \subseteq V$ and $l_R(V/\omega) = 1$ is an $m$-canonical ideal of $R$, so in this case $l_R(\omega/R) = l_R(V/R) - 1 = l_R((R : m)/R) - 1 = t - 1$ and $R$ is of minimal length.

(b) Let $V = k[[X]] + Y k((X))[[Y]]$, where $k = \mathbb{Q}(\sqrt{2})$ and $X$ and $Y$ indeterminates over $k$. Then $V$ is a two-dimensional valuation domain with principal
maximal ideal \( X \). By [2, Theorem 2.15] an \( m \)-canonical ideal of the subring \( R = \mathbb{Q} + X^2V \) is \( \omega = \mathbb{Q}(\sqrt{2}) + X(\mathbb{Q} + \sqrt{2}Q) + X^2V \), so that also in this case \( l_R(\omega/R) = l_R(V/R) - 1 = l_R(R : m/R) - 1 = t - 1 \) and \( R \) is of minimal length.

**Proposition 3.1** Let \( R \subseteq \omega \subseteq T \), where \( T \) is an overring of \( R \) with \( l_R(T/R) < \infty \). Then, if \( C_T = (R : T) \),

(a) \( l_R(R/C_T) + t - 1 \leq l_R(T/R) \)

(b) If \( l_R(R/C_T) + t - 1 = l_R(T/R) \)

then \( R \) is of minimal length and \( \omega m = m \).

**Proof.** We always have

\[
l_R(T/\omega) = l_R((\omega : \omega)/(\omega : T)) = l_R(R/(\omega : \omega T)) = l_R(R/((\omega : \omega) : T)) = l_R(R/C_T)
\]

and \( l_R(T/R) = l_R(T/\omega) + l_R(\omega/R) = l_R(R/C_T) + l_R(\omega/R) \). Since \( t - 1 \leq l_R(\omega/R) \) we have \( l_R(R/C_T) + t - 1 \leq l_R(T/R) \). If the last inequality is an equality then \( t - 1 = l_R(\omega/R) \), so \( R \) is of minimal length and \( \omega m = m \).

If \( t = 1 \), i.e. if the \( m \)-canonical ideal is principal, equivalently if \( R \) is a divisorial ring, \( R \) is trivially of minimal length. If this happens, in the Noetherian one-dimensional case we have Gorenstein rings.

In [4, Theorem 1.2], it is proved that a local nonvaluation domain \( D \) of maximal ideal \( M \) is divisorial if and only if \((M : M)\) is a two-generated \( D \)-module and \( M \) is an \( m \)-canonical ideal of \((M : M)\). The following result shows that this last condition is necessary more generally for rings of minimal length.

**Theorem 3.2** Consider the following:

(1) \( R \) is of minimal length.

(2) Each regular ideal of \((m : m)\) is divisorial as ideal of \( R \).

(3) \( m \) is an \( m \)-canonical ideal of \((m : m)\).

Then (1) \( \Rightarrow \) (2) \( \iff \) (3).

**Proof.** We already observed that in our hypotheses \( m \) is a divisorial ideal of \( R \). We consider first the trivial case when \( m = xR \) is a principal ideal. In this case \( m : m = R \) and \( R : m = x^{-1}R \), so that \( t = l_R((R : m)/R) = l_R(x^{-1}R/R) = l_R(R/xR) = l_R(R/m) = 1 \) and all conditions (1), (2), (3) trivially hold.

Suppose now that \( m \) is not principal, i.e. that \( m : m = R : m \) is a proper overring of \( R \).

(1) \( \Rightarrow \) (2). Since \( R \) is of minimal length, we can choose an \( m \)-canonical ideal \( \omega \) of \( R \) such that \( m = \omega m \), so that \( \omega \subseteq m : m \). Let \( I \) be a regular ideal of \((m : m)\). Observe that \( \omega I \subseteq (m : m)I \subseteq I \). On the other hand \( I \not\subseteq \omega I \), because \( 1 \in \omega \), thus \( I = \omega I \). Now, if \( I \) is not a divisorial ideal of \( R \), we have

\[
I \not\subseteq R : (R : I) \subseteq \omega : (R : I)
\]

Hence \( R : I \not\subseteq \omega : I = \omega : \omega I = (\omega : \omega) : I = R : I \), a contradiction.
(2) ⇒ (3). We have to prove that

\[ m : (m : I) \subseteq I \]

for each regular ideal I of m : m. First note that m : I = R : I. Indeed if m : I ⊊ R : I, then zI ⊊ R and zI ⊊ m for some non-zerodivisor z ∈ Q. It follows that zI = R, thus I = z^{-1}R. But R is not an ideal of m : m (we are supposing that (m : m) is strictly bigger than R) and we get a contradiction.

Now m : (m : I) ⊊ R : (m : I) = R : (R : I) = I and we finish.

(3) ⇒ (2). For what we observed above, we suppose R ⊊ m : m. By the Remark after Proposition 2.3 each regular ideal of I of m : m is an intersection of fractional ideals of the form zm, because m is an m-canonical ideal of m : m.

Since m is divisorial in R, also I, as intersection of divisorial ideals, is divisorial in R.

The implication (1) ⇒ (2) of Theorem 3.2 can be reversed, adding further hypotheses. We need for that some terminology and a lemma.

If (A, p) is a local ring and U is a ring containing A, we call the extension A ⊆ U residually rational if p ⊊ R(U), where R(U) is the Jacobson radical of U and, for each maximal ideal n of U, the residue fields A/p and U/n are isomorphic.

**Lemma 3.3** Let (A, p) be a local ring and A ⊆ U a residually rational extension of rings. If M is a U-module of finite length, then l_U(M) = l_A(M).

**Proof.** Let M = M_0 ⊇ M_1 ⊇ · · · ⊇ M_n = 0 be a composition series for the U-module M. It is enough to show that each M_i/M_{i+1} has length 1 also as A-module. So we can suppose M is a U-module of length 1. In this case M ≃ U/n as U-module, for some maximal ideal n of U. Since n ∩ A = p, M as A-module is annihilated by p, hence also l_A(M) = 1.

**Proposition 3.4** Suppose that (R, m) ⊊ m : m is a residually rational extension of rings and that there exists an overring T of R such that R ⊊ ω ⊊ T, m : m ⊊ T and l_R(T/R) < ∞. Then in Theorem 3.2 (2) ⇒ (1). Moreover \( \omega m = m \).

**Proof.** Again we can assume R ⊊ m : m. Let C_T = R : T and U = m : m. We claim that

\[ l_R(T/U) \leq l_R(m/C_T) \]  \hspace{1cm} (*)

Since l_R(T/R) < ∞, we have that M = T/U is a U-module of finite length and by Lemma 3.3 l_R(M) = l_U(M). Let

\[ U = B_0 ⊇ B_1 ⊇ · · · ⊇ B_h = T \]

be a sequence of U-modules which realizes the length l_R(M) = l_U(M). By hypothesis (2) each B_i is a regular fractional divisorial ideal of R. With the duals we get a sequence of regular ideals of R between m and C_T,

\[ m = R : (R : m) ⊊ R : B_1 ⊊ · · · ⊊ R : T = C_T. \]
If $R : B_{i-1} = R : B_i$, then $B_{i-1} = (B_{i-1})_v = (B_i)_v = B_i$, a contradiction, hence the sequence is strictly decreasing and the inequality $(\ast)$ is proved. Thus $l_R(T/R) = l_R(T/U) + l_R(U/R) \leq l_R(m/C_T) + t = l_R(R/C_T) + t - 1$. Since the opposite inequality always holds (cf. Proposition 3.1, (a)), we have an equality and by Proposition 3.1, (b) $R$ is of minimal length and $\omega m = m$.

Recall that a one-dimensional analytically unramified ring is a one-dimensional Noetherian local ring $R$ such that $l_R(R/R) < \infty$, where $\bar{R}$ is the integral closure of $R$. It is called residually rational when the extension $\bar{R} \subseteq \bar{R}$ is residually rational. The following is a characterization of a one-dimensional analytically unramified residually rational ring of minimal length, i.e. Almost Gorenstein.

**Corollary 3.5** If $R$ is a one-dimensional analytically unramified and residually rational ring, the conditions (1), (2) and (3) of Theorem 3.2 are equivalent.

**Proof.** If $R$ is analytically unramified, an $m$-canonical ideal $\omega$ can be chosen between $R$ and $\bar{R}$. In fact since $\omega$ is a fractional ideal, for some $d \in R$, $d$ non-zero divisor, it is $d \omega \subseteq R$. Since $\bar{R}$ is a finite product of DVR’s, we have that $d \omega \bar{R}$ is a principal ideal, $d \omega \bar{R} = x \bar{R}$, for some $x \in d \omega$, so that $R \subseteq x^{-1} d \omega \subseteq \bar{R}$. Moreover $m : m \subseteq \bar{R}$ because $m$ is finitely generated. Finally $l_R(R/R) < \infty$ by definition of analytically unramified. Hence all the hypotheses mentioned in Proposition 3.4 are verified, taking $T = \bar{R}$.

Corollary 3.5 is not true if the ring is not residually rational as the following example shows.

**Example.** An example of a one-dimensional analytically irreducible ring such that conditions (2) and (3) of Theorem 3.2 hold, but condition (1) does not hold.

Let $k \subset L \subset K$ be fields, with $[L : k] = 2$, $[K : L] = 2$ and let $W$ be a $k$-subspace of $K$ dimension 3, $L \subset W \subset K$. Take for example $k = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{2})$, $K = \mathbb{Q}(\sqrt{2})$, $W = \mathbb{Q} + \sqrt{2} \mathbb{Q} + \sqrt{2}^3 \mathbb{Q}$. Consider the following subring of the ring of formal power series ring $K[[t]]$:

$$R = k + t^2 L + t^4 L + t^6 W + t^8 W + t^7 K[[t]]$$

of maximal ideal $m = t^2 L + t^4 L + t^6 W + t^8 W + t^7 K[[t]]$. We have that $U = m : m = L + t^2 L + t^6 W + t^8 W + t^7 K[[t]]$. In fact $\bar{U} = K[[t]]$ and $l_U(U/U) = 10$ (where the computation of the length is easily done according with [11, Proposition 11]). Thus an $m$-canonical ideal of $U$ is $U$ itself, $U = t^{-2} m$. It follows that conditions (3) and (2) of Theorem 3.2 are satisfied. An $m$-canonical ideal of $R$ is $\omega = k + t k + t^2 L + t^4 K + t^4 L + t^5 K + t^6 W + t^7 K[[t]]$ (cf. [11, Theorem 5]). It is $R \subseteq \omega \subseteq \bar{R}$, but $\omega \nsubseteq m : m$, so $\omega m \nsubseteq m$. For each $x \in Q$, if $R \subseteq x \omega$ then $x \omega \nsubseteq m : m$, so $R$ is not of minimal length.

We say that $m$ is a stable ideal if it is principal in the overring $m : m$.  

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Corollary 3.6 If $R$ is of minimal length and $m$ is stable, then the overring $m : m$ is a divisorial ring.

Proof. Let $U = m : m$. Since $m$ is stable, $m = xU$, that is $U = x^{-1}m$, for some $x \in Q$. For each regular ideal $I$ of $U$, we have

$$U : (U : I) = x^{-1}m : (x^{-1}m : I) = m : (m : I)$$

By Theorem 3.2 (1) $\Rightarrow$ (3), we have that $m$ is an $m$-canonical ideal of $U$ and so $U : (U : I) = m : (m : I) = I$ and $U$ is a divisorial ring.

With some additional hypotheses we have also the converse.

Corollary 3.7 If $m : m$ is a divisorial ring and the hypotheses of Proposition 3.4 are verified, then $R$ is of minimal length. If moreover one of the following two conditions hold:

(a) $m : m$ is local
(b) $mT$ is a principal ideal in the overring $T$

then $m$ is a stable ideal of $R$.

Proof. The statement is trivial if $m$ is a principal ideal of $R$. So, as in the proof of Theorem 3.2, we can suppose $m : m = R : m$. Each regular ideal of $U = m : m$ is divisorial as ideal of $U = R : m$, so it is divisorial as ideal of $R$. Thus condition (2) of Theorem 3.2 is verified and by Proposition 3.4 we get $R$ of minimal length and $\omega m = m$.

Now consider the second part of the statement. Condition (2) of Theorem 3.2 implies condition (3), thus $m$ is an $m$-canonical ideal of $U$. On the other hand $U$ is a divisorial ring, thus, in case $m : m$ is local (hypothesis (a)), by Proposition 2.1 (1), $m = xU$, for some $x \in Q$ and $m$ is stable.

Suppose now that condition (b) of the statement holds, i.e. that $mT = zT$, for some $z \in m$. We claim that $zU = m$. Since $zU \subseteq m$, it is enough to prove that $l_R(m/C_T) = l_R(zU/C_T)$, where $C_T = R : T$.

Since $U = R : m = m : m$ is a divisorial ring, $l_U(U/U : T) = l_U(T/U)$ and, by Lemma 3.3

$$l_R(U/U : T) = l_R(T/U).$$

The length on the left is equal to $l_R(zU/z(U : T))$ and, since

$$z(U : T) = z((R : m) : T) = zR : mT = zR : zT = R : T$$

we have on the left $l_R(zU/C_T)$.

We claim that the length on the right is $l_R(m/C_T)$. We know that $R$ is of minimal length, i.e. $l_R(\omega/R) = t - 1$, and $\omega m = m$, so that $R \subseteq \omega \subseteq U \subseteq T$.

Since $l_R(\omega/R) = l_R((R : m)/R) = t$ we have $l_R(U/\omega) = 1$ and $l_R(T/U) = l_R(T/\omega) - 1 = l_R(R/C_T) - 1 = l_R(m/C_T)$.

Thus $zU = m$, i.e. $m$ is principal in $U = m : m$ and so it is a stable ideal.

As a particular case of Corollary 3.7 we reobtain Proposition 25 in [1]. This result has been recently used by S. L. Kleiman and R. Vidal Martins to study the canonical model of a singular curve, cf. [8].
Corollary 3.8  [1, Proposition 25] Let $(R, m)$ be a one-dimensional analytically unramified residually rational ring. Then the following conditions are equivalent:

(1) $R$ is Almost Gorenstein of maximal embedding dimension.
(2) $m : m$ is a Gorenstein ring.

Proof. (1) $\Rightarrow$ (2). It is Corollary 3.6, in fact Almost Gorenstein means of minimal length and of maximal embedding dimension is equivalent to say $m$ stable.

(2) $\Rightarrow$ (1). It is enough to observe that if $(R, m)$ is a one-dimensional analytically unramified residually rational ring the hypotheses of Proposition 3.4 are verified with $T = \bar{R}$. So Corollary 3.7 can be applied with the hypothesis (b) because $T = \bar{R}$ is a direct product of DVR’s in this case.

Remark Note that the Example after Corollary 3.5 shows how also Corollary 3.8 holds only if the ring is residually rational. With the Noetherian language the ring $(R, m)$ of that example is not Almost Gorenstein, but $m : m$ is Gorenstein.

References


