Probability ACSAI 2023-24

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Week 2

Exercise 1. A company consists of 25 people. Among these people, a president and a secretary have to be selected.

1) In how many different ways can the president and secretary be selected?

2) If the president and secretary are chosen uniformly at random, what is the probability that a given person is selected for either of the positions?

Exercise 2. Count the number of anagrams (even with no meaning) of the words: RICE, PASTA and POTATOES.

Exercise 3. Let S be a set of cardinality n. Count the number of subsets of S of cardinality k (with $k = 0, \dots, n$).

Exercise 4. To pass an exam, students have to provide answers to (exactly) 10 out of the 13 proposed questions.

1) In how many ways can the 10 questions be selected?

2) Assuming that the first two questions are compulsory, in how many ways can the 10 questions be selected?

3) Assuming that students are required to answer either the first question or the second one, but not both questions, in how many ways can the 10 questions be selected?

Exercise 5. Bob and Alice go out with 5 friends. They start the evening at a bar. In front of the bar counter there are 7 empty stools in a row, and each person chooses one stool at random. What is the probability that Bob and Alice sit next to each other? After the bar they head to a restaurant, where they are given a round table with 7 chairs. Each person chooses one chair uniformly at random. What is the probability that Bob and Alice sit next to each other?

Exercise 6. 5 cards are randomly picked from a deck of 52 cards. Compute the probability of getting:

1) poker;

2) colour;

3) full;

4) double pair (but not full);

5) tris (but not poker nor full).

Exercise 7. In a Computer Science department, rooms I and II can host 50 students each, while room III can host 100 students. In order to attend the Probability course in those rooms, 200 students are divided into 3 groups (of 50, 50 and 100 students).

1) In how many ways can these groups be formed, and in how many ways can they be assigned to the different rooms?

Alice and Bob would like to attend the course together, but they would like to avoid being in the same class as Will.

2) Compute the probability that their wish comes true.

Exercise 8. Let Ω be a finite, non-empty set and let $H\colon \Omega \to \mathbb{R}$ denote a given function. For each $\beta \geq 0$, define the probability $\mathbb{P}_\beta$ on Ω by setting, for every $\omega \in \Omega$,

$$\mathbb{P}_\beta(\{\omega\}) = \frac{e^{-\beta H(\omega)}}{Z_\beta}$$

where Z_β is a positive real number (recall that on finite sample spaces the probability measure is uniquely determined by its values on the single outcomes $\omega \in \Omega$). Define further

$$m := \min_{\omega\in\Omega} H(\omega) \qquad E_m := \{\omega \in \Omega \,:\, H(\omega) = m\} = H^{-1}(\{m\})$$

1) Write Z_β in terms of β (and H).

2) Check that if $\beta = 0$ then $\mathbb{P}_\beta$ is the uniform probability on Ω.

3) Check that

$$\lim_{\beta\to+\infty} \mathbb{P}_\beta(E_m) = 1$$

(in other words, check that $\mathbb{P}_\beta$ concentrates on the minima of H as $\beta \to +\infty$).

4) Compute $\lim_{\beta\to+\infty} \mathbb{P}_\beta(\{\omega\})$ for each $\omega \in \Omega$.