Probability ACSAI 2023-24
L. Bertini and V. Silvestri

Week 3

Exercise 1. A lecture course is attended by 9 students. The lecturer writes 3 exam paper, and each paper is assigned to 3 students.

1) In how many ways can the 9 students be paired with the 3 exam papers?

2) If instead the lecturer prepares 9 different exam papers, in how many ways can the 9 students be paired with the 9 exam papers?

Consider the same class of 9 students.

3) In how many ways can the students be partitioned into 3 groups, each made of 3 students?

4) In how many ways can the students be partitioned into 3 groups, one made of 5 students and 2 made of 2 students?

5) In how many ways can the students be partitioned into 9 groups, each made of exactly one student?

Exercise 2. An Italian deck of cards is made of 40 cards of 4 different seeds, numbered from 1 (ace) to 10.

In a card game called "tresette" there are 4 players. Each player is given 10 cards. We say that a player got a "napoletana" if he/she gets an ace, a 2 and a 3 of the same seed.

You are sitting at the table, and are given your 10 cards.

1) Compute the probability that you get a "napoletana" of a given seed.

2) Compute the probability that you get a "napoletana" of two different seeds.

3) Compute the probability that you get at least one "napoletana".

Exercise 3. Bob must take the Maths exam. The pool of exercises consists of 50 differential equations exercises, 30 geometry exercises and 10 statistics exercises. Bob has no knowledge of these subjects, so he decides to memorise 20 differential equations exercises, 10 geometry exercises and 5 statistics exercises. During the Maths exam, Bob only solves the exercises that he has memorised.

1) If the Maths teacher prepares the exam by randomly choosing, in the pool of exercises, 4 geometry exercises, what is the probability that Bob solves all the 4 exercises?

Assume instead that the Maths teacher prepares the exam by randomly choosing, in the pool of exercises, 5 differential equations exercises, 4 geometry exercises and 1 statistics exercise.

2) How many different exam papers can the teacher prepare? (Exam papers containing the same exercises in different order are considered identical)

3) What is the probability that Bob solves all 10 exercises?

4) What is the probability that Bob solves 3 differential equations exercises, 2 geometry exercises and 1 statistics exercise?

Exercise 4. Prove the inclusion/exclusion principle: if $A_1, \dots, A_n$ are arbitrary events, then

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{P}\Big(A_{i_1} \cap \cdots \cap A_{i_k}\Big).$$

Exercise 5. Let S, S' be finite sets, with $|S| = n$ and $|S'| = k$. Answer the following questions for any $n, k \in \mathbb{N}$.

1) How many functions from S to S' are there?

2) How many strictly increasing functions from S to S' are there?

3) How many injective functions from S to S' are there?

4) How many bijective functions from S to S' are there?

5) (*) How many non-decreasing functions from S to S' are there? (Hint: build a bijection with the set of injective functions from S to a larger set than S')

6) (*) How many surjective functions from S to S' are there? (Hint: use the inclusion/exclusion principle)

Exercise 6. Alice (A), Bob (B) and Caleb (C) play a tournament with the following rules. In the first round A and B play one against the other. The winner then plays against C, and if heshe wins this game then he/she wins the tournament. If, instead, C wins the game, then C plays against the loser of the first round, and so on and so forth. The first player who wins two consecutive rounds wins the tournament.

It is known that A,B and C have the same skills, so each game is equally likely to be won by any of the two players.

1) Is any of the players favoured by the tournament rules?

2) Compute the probability that the tournament ends after n rounds, $n \ge 2$.

3) Compute the winning probability for A,B and C.

4) Can the tournament last forever?