Probability ACSAI 2023-24

L. Bertini and V. Silvestri

Week 4

Exercise 1. Let A, B be events with $\mathbb{P}(A) = 0.3$, $\mathbb{P}(A \cup B) = 0.5$ and $\mathbb{P}(B) = p$. Find the value of p in the following cases:

1) A and B are disjoint,

2) A and B are independent,

3) A is a subset of B.

Exercise 2. Let A, B, C be three independent events. Prove that the following events are independent.

1) $A^{\complement}, B, C$,

2) $A^{\complement}, B^{\complement}, C$,

3) $A^{\complement}, B^{\complement}, C^{\complement}$.

Exercise 3. Two standard dice are rolled.

1) Show that the event "the sum of the dice is 9" is not independent of the outcome of the first die.

2) Show that the event "the sum of the dice is 7" is independent of the outcome of the first die.

3) Give an intuitive explanation of why the above two cases are different.

Exercise 4. When 3 horses compete in a race, their winning probabilities are $0.3, 0.5, 0.2$ respectively. They compete in 3 consecutive races. Compute the probability of the following events:

1) the same horse wins all races,

2) each horse wins exactly one race.

Exercise 5. A rocket hits its target with probability $1/3$.

1) If 3 rockets are fired, what is the probability that at least one of them hits the target?

2) Find the minimum number of rockets which need to be fired in order to guarantee that the probability that at least one of them hits the target is at least 90%.

Exercise 6. An urn contains three coins: the first coin is fair, and has head (H) on one side and tail (T) on the other side, the second coin has H on both sides and the third coind has T on both sides. A coin is chosen at random from the urn, and it is tossed without looking at which one it is.

1) Compute the probability that the coin toss gives H.

2) Given that the coin toss gave H, compute the probability that on the other side of the coin there is T.

Exercise 7. A binary signal is transmitted via a channel. Due to the background noise, it may happen that when 0 is transmitted 1 is received, and similarly it may happen that when 1 is transmitted 0 is received. Assume that:

- the probability that 0 is received correctly is 0.94;
- the probability that 1 is received correctly is 0.91.

A single bit is transmitted, which is 0 with probability 0.45 and 1 with probability 0.55. Compute:

1) the probability of receiving 1;

2) the probability of receiving 0;

3) the probability that 1 was transmitted, given that 1 was received;

4) the probability that 0 was transmitted, given that 0 was received;

5) the probability that the signal is wrongly received (i.e. the probability that the received signal is different from the transmitted signal).

Exercise 8.

1) Let $B, N, n \in \mathbb{N}$ with $B, N \geq n$. Prove, via a probabilistic interpretation, the identity:

$$\sum_{k=0}^{n} \binom{B}{k} \binom{N}{n-k} = \binom{N+B}{n}.$$

2) Alice and Bob toss a fair coin n times each. Compute the probability that they obtain the same number of heads.