Probability ACSAI 2023-24
L. Bertini and V. Silvestri

Week 5

Exercise 1. Alice and Bob each toss a biased coin: Alice's coin gives head with probability 1/3, while Bob's coin gives head with probability 1/4.

1) Compute the probability that both Alice and Bob get head.

2) Compute the probability that they get exactly one head.

3) Knowing that the coin tosses resulted in one head and one tail, compute the probability that Alice got head.

Exercise 2. It is known that twins can be identical, in which case they are necessarily of the same sex, or non-identical, in which case they are of the same sex in 50% of the cases. Let p denote the probability that the twins are identical.

1) Compute, as a function of p, the probability that two twins are identical, knowing that they are of the same sex.

2) Compute, as a function of p, the probability that two twins are not of the same sex.

Exercise 3. In a factory three machines A, B, and C make respectively 40%, 10%, and 50% of the produced items. The respective percentages of faulty items are 2%, 3% and 4%. Pick an item at random.

1) Compute the probability that the item is faulty.

2) Knowing that the item is faulty, compute the probability that it was produced by machine A, B or C.

Exercise 4. For $n \in \mathbb{N}$ and $p \in (0,1)$ consider the binomial distribution (number of heads in n biased coin tosses)

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, \dots, n.$$

Prove that $P(k)$ is increasing for $k \le \bar{k}$, with $\bar{k} = \bar{k}(n,p)$ to be determined, and decreasing for $k > \bar{k}$.

Exercise 5. A biased coin is given, with bias p unknown and to be determined. The Maximum Likelihood Estimator $\hat{p}$ for p is defined by requiring that $\hat{p}$ maximises the probability of the observed event. Compute the Maximum Likelihood Estimator $\hat{p}$ for the following two experiments:

1) Toss the coin 200 times and obtain 67 heads.

2) The first head is observed on the fifth coin toss.

Determine the Maximum Likelihood Estimator $\hat{p}$ in the following more general cases: (i) toss the coin n times and obtain k heads; (ii) the first head is observed on the h-th coin toss.

Exercise 6. Let S be a set of cardinality n. Pick two subsets of S at random. Compute the probability that the first set is a subset of the second set.

Exercise 7. Three roads connect the houses A, B and C in such a way that from each house one can get to any other house with a direct path. Due to bad weather, the roads may be (temporarily) closed. Let $p_{AB} \in (0, 1)$ (respectively p_{BC}, p_{AC}) denote the probability that the road linking A and B (respectively B and C, A and C) is open. You can assume that each road is open or closed independently of the state of the other roads. You are at house A.

1) Compute the probability that you can get to house C.

2) Someone told you that it is not possible to get to house C due to bad weather. Compute the probability that you can get to house B.

Now suppose that between A and B there are 3 direct paths, each one open with probability q independently of the others.

3) Compute again the above probabilities, without redoing the computations from scratch.

Exercise 8. A biased coin, with probability of head $p \in (0, 1)$, is tossed repeatedly. Given $a, b \geq 1$, compute the probability that the coin gives a heads before b tails.