Probability ACSAI 2023-24
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Week 6

Exercise 1. (coupon collector) Your goal is to collect n coupons for your album. What is the probability that you will do so by buying k coupons, $k \geq n$? You may use the uniform probability measure on the k-ple of coupons you buy. [Hint: use the inclusion/exclusion principle]

Exercise 2. An urn contains a red ball and a green ball. One ball is picked at random from the urn, its colour is observed, and the ball is placed back in the urn together with a new ball of the same colour. This procedure is repeated two more times. Let R_i, for $i = 1, 2, 3$, denote the event "the i-th picked ball is red".

1) Compute $P(R_1|R_2)$.
2) Compute $P(R_3|R_2)$.
3) Compute $P(R_1|R_3)$.

Exercise 3. Write a random word made of 10 characters by choosing a character uniformly at random 10 times independently, from an alphabet of 26 characters. Let X be the random variable that counts the number of A's in the resulting word. What is the distribution of X? Compute $E(X)$, that is the average number of A's in a random word of length 10. How about repeating the experiment with a word of length N?

Exercise 4. A fair 6-faced die is tossed, and let X denote the observed value.

1) Compute the probability distribution of X.
2) Compute the expected value of X.
3) Compute the variance of X.

Answer the above questions in the case of an n-faced die, $n \in \mathbb{N}$.

Exercise 5. Toss two fair 6-faced dice, and let X denote the minimum between the observed values.

1) Compute the probability distribution of X.
2) Compute the expected value of X.

Exercise 6. A box contains 10 transistors, of which 3 are broken. You check one transistor at a time (without replacement) until you find a broken one. Compute the expected value of the number of checked transistors.

Exercise 7. Show that if a random variable $X \geq 0$ takes integer values, then

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k).$$

Exercise 8. Consider a multiple choice exam with the following rules. There are a total of 10 questions, and for each question there are 4 possible answers, of which exactly one is correct. The evaluation algorithms is as follows: each correct answer gets $+3$ marks, and each wrong answer gets -1 mark. Alice did not study, so she answers all 10 questions at random.

1) Compute the probability that Alice passes the exam (i.e. she scores at least 18/30).

2) Compute Alice's expected final grade.

3) Compute the variance of Alice's final grade.