Probability ACSAI 2023-24
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Week 7

Exercise 1. A referendum is called in a population of n individuals, all of them having the right to vote. Each individual will vote with probability $1/2$, independently of the others. Moreover, if the individual votes, he/she votes YES with probability $1/2$, independently of the others.

1) Compute the probability that a given individual goes to vote and votes YES.
2) Compute the probability that the number of YES votes is k, for $k = 0 \dots, n$.
3) Knowing that there have been exactly k YES votes, compute the probability that exactly m individuals voted, for $m = k, \dots, n$.

Exercise 2. Let X_1, X_2 be independent random variables, uniformly distributed in $\{1, \dots, n\}$.

1) Compute the probability distribution of $X_1 + X_2$.
2) Compute the expected value of $X_1 + X_2$.
3) Compute the variance of $X_1 + X_2$.

Exercise 3. (Independence of random variables) Let X and Y be random variables.

1) Prove that if X is a degenerate random variable, that is $X = c$ for some $c \in \mathbb{R}$, then X and Y are independent.
2) Prove that if X and Y are binary, that is $\big|\mathrm{Im}(X)\big| = \big|\mathrm{Im}(Y)\big| = 2$, then the random variables X and Y are independent if and only if $\mathrm{cov}(X, Y) = 0$.
3) Give an example of two random variables X and Y such that $\mathrm{cov}(X, Y) = 0$ but X and Y are not independent.

Exercise 4. (Hypergeometric random variable) Consider an urn with a white balls and b black balls. You pick k balls sequentially without replacement ($k \le a + b$). Let X_i, $i = 1, \dots, k$ be the random variable taking value 1 if the i-th ball is white and 0 if it is black. Let, moreover, X denote the total number of white balls picked.

1) Compute the distribution of X.
2) Compute the expected value of X.
(You should do both the direct calculation using the distribution of X, and the calculation using the expectations of the X_i's.)
3) Compute the covariance between X_i and X_j, $i, j = 1, \dots, k$.
4) Compute the variance of X.
(You should do both the direct calculation using the distribution of X, and the calculation using that $X = \sum_{i=1}^{k} X_i$ and the previous question.)

Exercise 5.(A LIMIT THEOREM FOR THE HYPERGEOMETRIC DISTRIBUTION) For $a, b, k \in \mathbb{N}$, consider the *hypergeometric distribution*

$$P_{a,b,k}(h) = \frac{\binom{a}{h}\binom{b}{k-h}}{\binom{a+b}{k}}, \qquad h = 0, \dots, k.$$

1) Compute the limit of $P_{a,b,k}$ as $a, b \to \infty$ with $a/(a+b) \to p \in (0,1)$ (k is fixed).

2) Discuss a probabilistic interpretation of the result.

Exercise 6. (ALTERNATIVE PROOF OF THE INCLUSION/EXCLUSION PRINCIPLE) For an event A let $\mathbf{1}_A$ denote the random variable which takes value 1 if $\omega \in A$ and value 0 if $\omega \in A^c$.

1) Let A and B be events. Check that $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$ and that $\mathbf{1}_{A\cap B} = \mathbf{1}_A \mathbf{1}_B$.

2) Let $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$. Convince yourself of the binomial identity:

$$\prod_{i=1}^{n}(a_i + b_i) = \sum_{I \subset \{1,\dots,n\}} \prod_{i \in I} a_i \prod_{j \in I^c} b_j.$$

3) Use the previous points together with the properties of the expectation to prove the inclusion/exclusion principle.