Probability ACSAI 2023-24
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Week 8

**Exercise 1.** A fair die is tossed repeatedly, until 5 or 6 is obtained. Let $T$ denote the number of tosses, and let $X$ denote the number seen on the die in the last toss.

*1)* Compute $\mathbb{P}(T = 3, X = 5)$.
*2)* Compute the probability distribution of $T$.
*3)* Compute the probability distribution of $X$.
*4)* Are $T$ and $X$ independent random variables? Explain.

**Exercise 2.** How many times should you roll – on average – a fair die in order to see all faces?
Hint: Use geometric random variables to avoid computations.

**Exercise 3.** The breaking time of component $C_i$ is given by a random variable $T_i$, $i = 1, \dots, k$. Assume that the random variables $T_1, \dots, T_k$ are independent, and that $T_i \sim \text{Geom}(p)$ with $p \in (0, 1)$, $i = 1, \dots, k$.

*1)* Find the probability distribution of the breaking time of the circuit $C_{\text{ser}}$ obtained by organising in series all the components $C_1$, ..., $C_k$.
*2)* Find the probability distribution of the breaking time of the circuit $C_{\text{par}}$ obtained by organising in parallel all the components $C_1$, ..., $C_k$.

**Exercise 4.** In a Bernoulli scheme with head probability $p \in (0, 1)$ let $X$ denote the random variable counting the number of consecutive outcomes identical to the first one: that is, $X = 1$ if the first coin toss gives head and the second gives tail or the first gives tail and the second head, $X = 2$ if two heads followed by a tail or two tails followed by a head, and so on.

*1)* Find the probability distribution of $X$.
*2)* Compute the expected value of $X$.
*3)* Compute the variance of $X$.

**Exercise 5.** (Coupon collector continued) Consider an album with $n$ coupons. In order to complete the album, you buy one coupon per day (uniformly chosen among all possible coupons, independently of the other days).

*1)* Show that the expected value of the number of days needed to complete the album is given by
$$K_n = n\Big[1 + \frac{1}{2} + \cdots + \frac{1}{n}\Big].$$
*2)* Prove the asymptotic relation
$$K_n = n\big[\log n + o(\log n)\big].$$

Hint: Formalise the problem in terms of geometric random variables.

**Exercise 6.** Assume that – on average – 2% of the population is left-handed. Given a sample of 100 individuals, use the Poisson approximation for Binomial random variables to compute the probability that at least 3 individuals are left-handed.

**Exercise 7.** A biased coin with head probability $p \in (0,1)$ is tossed a random number of times (indipendently of the results of the coin tosses) with Poisson distribution of parameter $\lambda > 0$. Find the probability distribution of the total number of heads and tails obtained, and prove that these two random variables are independent.