

Probability ACSAI 2023-24
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WEEK 9

Exercise 1. (CONFIDENCE INTERVALS) Consider a biased coin with head probability p unknown. In order to estimate p , the coin is tossed n times and p is estimated via S_n/n , where S_n is the number of heads in the n coin tosses. Given $\delta > 0$, determine how large should n be in order for the probability that $|S_n/n - p| < \delta$ to be at least 95%.

Exercise 2. From a collection of 7 batteries, of which 3 are new, 2 are used but working and 2 are broken, 3 batteries are chosen at random. Let X and Y denote respectively the number of new batteries and used working batteries, among the ones chosen.

- 1) Determine the joint distribution of (X, Y) and the marginal distributions of X and Y .
- 2) Compute $\text{cov}(X, Y)$. Are the random variables X and Y independent?
- 3) The three chosen batteries are inserted in a machine which works if none of them is broken. Determine the probability that the machine works.

Exercise 3. A die has a blue face, two red faces and three green faces. The die is rolled twice. Let R denote the number of times a red face is obtained, and G denote the number of times a green face is obtained.

- 1) Construct the table of the joint distribution of (R, G) .
- 2) Determine the distribution of $Z = \max\{R, G\}$ and compute $\mathbb{E}(Z)$ and $\mathbb{V}(Z)$.

Exercise 4. The electrical components produced in a factory are defective, independently from each other, with probability p , and they are functioning with probability $1 - p$, where $p \in (0, 1)$. The components are subject to a quality check with the following procedure: each component, independently of the others, is checked with probability α and it is not checked with probability $1 - \alpha$, where $\alpha \in (0, 1)$. A component which is found defective is discarded, while the others are put on sale. Suppose that the factory produced n components.

- 1) Compute the distribution of the number of components which are discarded after the quality check procedure.
- 2) Knowing that k components have been discarded after the quality check procedure, for $k = 0, 1, \dots, n$, compute the distribution of the number of defective components among the $n - k$ components which are put on sale.
- 3) Let X denote the number of components which are put on sale, and let Z denote the number of defective components which are put on sale. Compute $\mathbb{E}(Z/X|X)$ and $\mathbb{E}(Z/X)$.

Exercise 5. Let X, Y be two independent Bernoulli random variables with parameter p . Define

$$Z = X(1 - Y) \quad \text{e} \quad W = 1 - XY.$$

- 1) What is the joint distribution of (Z, W) ?
- 2) What are the marginal distributions of Z and W ?
- 3) For which values of p are the random variables Z and W independent?

Exercise 6. A and B play the following game: A writes 1 or 2 on a piece of paper, and B guesses the number written by A. If A wrote $i \in \{1, 2\}$ and B guesses right, then A pays i euros to B. If, on the other hand, B guesses wrong, then B pays 0.75 euros to A.

Assume that B uses a random strategy, guessing 1 with probability p and 2 with probability $1 - p$.

- 1) Knowing that A wrote 1, determine the expected gain of B.
- 2) Knowing that A wrote 2, determine the expected gain of B.
- 3) Determine the value of p which maximises the minimum between the two expected gains above.

Suppose that A uses a random strategy, writing 1 with probability q and 2 with probability $1 - q$.

- 4) Knowing that B guessed 1, determine the expected loss of A.
- 5) Knowing that B guessed 2, determine the expected loss of A.
- 6) Determine the value of q which minimises the maximum between the two expected gains above.

Compare the answers to 3) and 6).

Exercise 7. Let X, Y be two random variables. For $y \in \text{Im}(Y)$ let $\mathbb{E}(X|y)$ denote the expected value of X conditional on the event $\{Y = y\}$ (that is, the expected value with respect to the conditional distribution of X given the event $\{Y = y\}$). Prove that

$$\mathbb{E}(X) = \sum_{y \in \text{Im}(Y)} \mathbb{P}(Y = y) \mathbb{E}(X|y).$$