Probability ACSAI 2023-24
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Week 10

Exercise 1. Let X be a continuous random variable with $X \geq 0$. Show that

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\, dx.$$

Exercise 2. Let X_i, $i = 1, 2$ be independent uniform random variables in $[0, 1]$.

1) Compute the distribution (i.e. the probability density function) of $X_1 + X_2$.

2) Compute the distribution (i.e. the probability density function) of $\max\{X_1, X_2\}$.

3) Compute the distribution (i.e. the probability density function) of $\min\{X_1, X_2\}$.

Exercise 3. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{6}(x + k) & \text{if } x \in [0, k] \\ 0 & \text{otherwise.} \end{cases}$$

1) Compute the value of $k \geq 0$.

2) Compute $\mathbb{P}(1 \leq X \leq 2)$.

Exercise 4. Let U be a uniform random variable in $[0, 1]$, and let V be a random variable independent of U, uniformly distributed in $[-1, 1]$.

1) Compute the distribution (i.e. the probability density function) of V^2.

2) Compute the distribution (i.e. the probability density function) of $\log(1/U)$.

3) Compute $\mathbb{P}(U \leq V)$.

Exercise 5. Let X be a Gaussian random variable with mean 2 and variance 25. Provide answers to the following questions by using the Gaussian integral tables.

1) Compute $\mathbb{P}(|X - 2| \geq 7)$.

2) Compute $\mathbb{P}(0 \leq X \leq 7)$.

3) Determine α such that $\mathbb{P}(X \geq \alpha) \leq 0.1$.

Exercise 6. The steel spheres produced by ACME must have a diameter of 5 mm. Nevertheless, spheres of diameter between 4 mm and 6 mm are acceptable. Assume that the diameters of the steel spheres are independent Gaussian random variables with mean 5 mm and variance $0.25\,\text{mm}^2$.

1) What percentage of the produced spheres fails to respect the tolerance limits?

2) You can adjust the production procedure by changing the variance of the diameters. Determine the maximum value of the variance such that the percentage of the spheres that fail to respect the tolerance limits is less than 1%.

Exercise 7. In order to transmit a bit from a source A to a receiver B via a pair of electrical wires, one applies a potential difference of $+2\,\mathrm{V}$ for the value 1 and $-2\,\mathrm{V}$ for the value 0. Due to electromagnetic disturbance, if A applies $\mu = \pm 2\,\mathrm{V}$, B reads $X = \mu + Z$, where Z rrepresents the noise, described by a Gaussian random variable of mean 0 and variance $1\,\mathrm{V}^2$. After reading X, B registers the message with the following rule: if $X \geq 0.5\,\mathrm{V}$ then B registers 1, while if $X < 0.5\,\mathrm{V}$ then B registers 0.

1) If A sends 0, compute the probability that B registers 1.

2) If A sends 1, compute the probability that B registers 0.

Suppose now that A sends 0 or 1 with equal probability.

3) Compute the probability that B registers 1.

4) If B has registered 1, compute the probability that the message registered coincides with the message sent.

Exercise 8. Two fair dice are rolled 300 times. Let X denote the number of rolls at which the pair $(1, 1)$ is obtained.

1) Compute $\mathbb{E}(X)$ and $\mathbb{V}(X)$.

2) Using the Gaussian approximation, compute the probability of obtaining $(1, 1)$ more than 10 times.

Now consider the case where the two dice are rolled n times.

3) Using the Gaussian approximation, determine how large should n be so that the probability of obtaining $(1, 1)$ at least 10 times exceeds $1/2$.