SMALL NOISE ASYMPTOTIC OF THE GALLAVOTTI-COHEN FUNCTIONAL FOR DIFFUSION PROCESSES

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Abstract. We consider, for a diffusion process in $\mathbb{R}^n$, the Gallavotti-Cohen functional, defined as the empirical power dissipated in a time interval by the non-conservative part of the drift. We prove a large deviation principle in the limit in which the noise vanishes and the time interval diverges. The corresponding rate functional, which satisfies the fluctuation theorem, is expressed in terms of a variational problem on the classical Freidlin-Wentzell functional.

1. Introduction

The so-called Gallavotti-Cohen functional and the associated fluctuation theorem have been originally introduced in the context of chaotic deterministic dynamical systems [13]. Subsequently, they have been extended to stochastic systems, originally in [18] and in more generality in [20, 21]. Since then, this topic has become a basic issue in nonequilibrium statistical mechanics, see e.g. [4, 6, 10, 16, 22, 23, 29, 30, 32, 33], and it has even been the object of true experiments [7].

We here consider the simple setting of a diffusion process in $\mathbb{R}^n$ with constant diffusion coefficient, defined as the solution of the stochastic differential equation

$$d\xi_t = c(\xi_t)dt + \sqrt{\varepsilon}\, d\beta_t \tag{1.1}$$

where β is a standard n-dimensional Brownian motion, c is a smooth vector field on $\mathbb{R}^n$, and $\varepsilon > 0$ is the diffusion coefficient. Under suitable assumptions on the drift c, there exists a unique stationary distribution μ_ε having a strictly positive smooth density $\varrho_\varepsilon : \mathbb{R}^n \to (0, +\infty)$ with respect to Lebesgue measure. We recall that the corresponding stationary process, obtained by choosing in (1.1) the initial condition with law μ_ε, is reversible if and only if the drift is conservative, that is $c = -(1/2)\nabla V$ for some $V : \mathbb{R}^n \to \mathbb{R}$. In this case $\varrho_\varepsilon(x) = C_\varepsilon \exp\{-\varepsilon^{-1}V(x)\}$. When c is not conservative, the forward and backward time evolutions of the stationary process have different laws, and the stationary distribution is not - in general - explicitly known.

The Gallavotti-Cohen functional is a random variable on the canonical path space $C(\mathbb{R}_+; \mathbb{R}^n)$ expressed in terms of the Radon-Nikodym derivative of the law of the stationary process with respect to its time reversal. The standard definition is the following. Let $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$, a probability measure on $C\big(\mathbb{R}_+; \mathbb{R}^n\big)$, be the law of the stationary process. By stationarity, $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$ can be extended to a probability measure on $C\big(\mathbb{R}; \mathbb{R}^n\big)$. Denoting with θ the time reversal, we set $\mathcal{P}^{\varepsilon,*}_{\mu_\varepsilon} = \mathcal{P}^\varepsilon_{\mu_\varepsilon} \circ \theta^{-1}$. Given $T > 0$, the Gallavotti-Cohen functional can then be defined as the random variable

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on $C(\mathbb{R}_+;\mathbb{R}^n)$ which is $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$ a.s. given by

$$\widetilde{W}_T = -\frac{\varepsilon}{T}\log\frac{d\mathcal{P}^{\varepsilon,*}_{\mu_\varepsilon,T}}{d\mathcal{P}^{\varepsilon}_{\mu_\varepsilon,T}} \tag{1.2}$$

where the subscript T denotes the measures induced by the restriction to the time interval $[0,T]$ and we used a normalization in which $\widetilde{W}_T$ is finite as $\varepsilon\to 0$ and $T\to\infty$. Of course, $\mathcal{P}^{\varepsilon,*}_{\mu_\varepsilon}=\mathcal{P}^{\varepsilon}_{\mu_\varepsilon}$, i.e. $\widetilde{W}_T\equiv 0$, if and only if the stationary process is reversible. By denoting with $\mathcal{E}^\varepsilon_{\mu_\varepsilon}$ the expectation with respect to $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$, from Jensen inequality it follows $\mathcal{E}^\varepsilon_{\mu_\varepsilon}\big(\widetilde{W}_T\big)\ge 0$. In fact, $\mathcal{E}^\varepsilon_{\mu_\varepsilon}\big(\widetilde{W}_T\big)$ is proportional to the relative entropy of $\mathcal{P}^{\varepsilon}_{\mu_\varepsilon,T}$ with respect to $\mathcal{P}^{\varepsilon,*}_{\mu_\varepsilon,T}$, thus providing a natural measure of irreversibility.

By denoting with X_t the canonical coordinates on $C(\mathbb{R}_+;\mathbb{R}^n)$, an informal computation [20] based on Girsanov formula shows that $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$ a.s.

$$\widetilde{W}_T = \frac{2}{T}\int_0^T \langle c(X_t),\circ dX_t\rangle - \frac{\varepsilon}{T}\log\frac{\varrho_\varepsilon(X_T)}{\varrho_\varepsilon(X_0)} \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$ and $\circ dX_t$ denotes the Stratonovich integral with respect to X_t. Once again, note that $\widetilde{W}_T$ vanishes if and only if $c=-(1/2)\nabla V$. By writing the entropy balance, the random variable $\widetilde{W}_T$ can finally be interpreted as the empirical production of the Gibbs entropy [20].

The content of the so-called *fluctuation theorem* is the following. Fix the diffusion coefficient $\varepsilon>0$ and look for the large deviations asymptotic of the family of probability measures on $\mathbb{R}$ given by $\{\mathcal{P}^\varepsilon_{\mu_\varepsilon}\circ(\widetilde{W}_T)^{-1}\}_{T>0}$ as $T\to\infty$. Suppose they satisfy a large deviation principle that we informally write as

$$\mathcal{P}^\varepsilon_{\mu_\varepsilon}\big(W_T\approx q\big)\asymp\exp\big\{-TR^\varepsilon(q)\big\} \tag{1.4}$$

then the odd part of the rate function R^ε is linear. More precisely, with our choice of the normalization, $R^\varepsilon(q)-R^\varepsilon(-q)=-\varepsilon^{-1}q$. This means that ratio between the probability of the events $\{W_T\approx q\}$ and $\{W_T\approx -q\}$ becomes fixed, independently of the model, in the limit $T\to\infty$. Provided (1.4) holds, a simple argument based on the definition (1.2) and time reversal shows that the rate function R^ε satisfies the fluctuation theorem. Indeed, from the very definition of $\mathcal{P}^{\varepsilon,*}_{\mu_\varepsilon}$, the statement of the fluctuation theorem can be written as a true identity even for finite T, see again [20].

On the other hand, in the present setting of a diffusion process with non compact state space, i.e. $\mathbb{R}^n$, it is not really so clear that (1.4) holds. The main issue is the possible unboundedness of c and the necessary unboundedness of $\log\varrho_\varepsilon$ in the decomposition (1.3). In the case of a compact state space, the standard procedure is the following [20]. Modify the definition of the Gallavotti-Cohen functional by dropping from (1.3) the boundary term $\log\varrho_\varepsilon(X_T)/\varrho_\varepsilon(X_0)$, which should not matter in the limit $T\to\infty$. In other words, let $\widehat{W}_T$ be the random variable on $C(\mathbb{R}_+;\mathbb{R}^n)$ which is $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$ a.s. defined by

$$\widehat{W}_T = \frac{2}{T}\int_0^T \langle c(X_t),\circ dX_t\rangle, \tag{1.5}$$

so that $\widehat{W}_T$ can be interpreted as the empirical power dissipated by the vector field c in the time interval $[0,T]$.

Using Girsanov theorem, Feynman-Kac formula, and Perron-Frobenius theorem, see [20], we informally deduce that for each $\lambda \in \mathbb{R}$

$$\lim_{T\to\infty} \frac{1}{T} \log \mathcal{E}^{\varepsilon}_{\mu_\varepsilon} \Big(\exp\big\{\lambda T W_T\big\}\Big) = e^{\varepsilon}(\lambda)$$

where $e^{\varepsilon}(\lambda)$ is the eigenvalue with maximal real part (which is real) of the differential operator $A_{\varepsilon,\lambda}$ on $\mathbb{R}^n$ given by

$$\begin{aligned} A_{\varepsilon,\lambda} f(x) =& \frac{\varepsilon}{2}\Delta f(x) + (1+2\lambda\varepsilon)\big\langle c(x), \nabla f(x)\big\rangle \\ &+ 2\lambda(1+\lambda\varepsilon)\big\langle c(x), c(x)\big\rangle\, f(x) + \lambda\varepsilon \nabla\cdot c(x)\, f(x) \end{aligned} \tag{1.6}$$

where Δf is the Laplacian of f, ∇f is the gradient of f, and $\nabla\cdot c$ is the divergence of c. From Hölder inequality if follows that e^{ε} is a convex function; provided it is also smooth, by using the Gärtner-Ellis theorem, we deduce that the sequence of probability measures $\{\mathcal{P}^{\varepsilon}_{\mu_\varepsilon} \circ (\widehat{W}_T)^{-1}\}_{T>0}$ satisfies a large deviation principle as $T\to\infty$ with convex rate function given by the Legendre transform of e^{ε}. The fluctuation theorem then follows from the identity $e^{\varepsilon}(\lambda) = e^{\varepsilon}(-\varepsilon^{-1}-\lambda)$ which can be easily checked. Since the definition of $\widetilde{W}^{\varepsilon}_T$ in (1.3) involves explicitly the density ϱ_ε, which in general is not known, the definition of $\widehat{W}^{\varepsilon}_T$ in (1.5) is more concrete and somewhat more appealing.

In the present setting of a non compact space state, if the vector field c is unbounded, as it is typically the case, the maximal eigenvalue $e^{\varepsilon}(\lambda)$ becomes infinite. We therefore need to follow a different route. In this paper, we assume the vector field c admits the decomposition $c = -(1/2)\nabla V + b$ which is orthogonal in the sense that for each $x\in\mathbb{R}^n$ we have $\langle \nabla V(x), b(x)\rangle = 0$. We also assume that the potential $V(x)$ is super-linear as $|x|\to\infty$ and that b is a nonconservative smooth bounded vector field with bounded derivatives. In this setting, we *define* the Gallavotti-Cohen functional as the random variable on $C(\mathbb{R}_+;\mathbb{R}^n)$ which is $\mathcal{P}^{\varepsilon}_{\mu_\varepsilon}$ a.s. defined by

$$W_T = \frac{2}{T}\int_0^T \langle b(X_t), \circ dX_t\rangle \tag{1.7}$$

namely, as the empirical power dissipated by the nonconservative part of the vector field c. If the (deterministic) dynamical system $\dot{x} = c(x)$ admits a unique equilibrium solution O which is globally attractive, by classical Freidlin-Wentzell results [12, Thm. 4.3.1], we have $\varrho_\varepsilon(x) \asymp \exp\{-\varepsilon^{-1}[V(x)-V(O)]\}$. Therefore, definition (1.7) is close to (1.3) with the obvious advantage that we cancelled the unboundedness of $\log\varrho_\varepsilon$ with the conservative part of the drift c. In view of the assumptions on the nonconservative part b of the drift, as here shown, it is possible to carry out the analysis outlined below (1.5) and prove a large deviation principle for the sequence $\{\mathcal{P}^{\varepsilon}_{\mu_\varepsilon}\circ(W_T)^{-1}\}_{T>0}$. The corresponding rate function R^{ε}, which is strictly convex, satisfies the fluctuation theorem.

Having clarified the definition of the Gallavotti-Cohen functional in a non compact state space, we next discuss the main topic of the present paper, which is the analysis of the large deviation properties of the sequence $\{\mathcal{P}^{\varepsilon}_{\mu_\varepsilon}\circ(W_T)^{-1}\}$ in the joint limit $\varepsilon\to 0$ and $T\to\infty$. The motivation for such analysis is the following. While the fluctuation theorem establish a general, *model independent*, symmetry property of the rate function, the rate function itself, which appears to be experimentally accessible, may encode other, *model dependent*, relevant properties of the system which may be best revealed by the small noise limit. For instance, as here argued, even if for fixed $\varepsilon > 0$ the rate function R^ε is strictly convex, it may happen that phase transitions occur in the small noise limit.

As suggested in [19], one possibility is to consider the limit $\varepsilon \to 0$ *after* the limit $T \to \infty$ and discuss the asymptotic of the rate function R^ε. Since the latter is obtained as the Legendre transform of the maximal eigenvalue of a differential operator similar to $A_{\varepsilon,\lambda}$ in (1.6), this immediately becomes a problem in semiclassical limits. We here instead look at the asymptotic in the opposite order of the limits namely, we consider *first* the small noise limit $\varepsilon \to 0$ and *then* the long time limit $T \to \infty$. If the vector field c has a unique globally attractive attractor, we expect that the order in which these limits are taken does not matter. However, as here explicitly shown, this is not the case if there are several attractors. From a physical viewpoint, the relevant order of the limits depends on the details of the experimental setting; the analysis here performed is applicable as long as the noise is small and the power dissipated by the drift is measured over a time scale which is much shorter than the (possible) metastable time scales. From a mathematical viewpoint, the asymptotic here considered amounts to analyze the variational convergence, as $T \to \infty$, of the sequence of rate functions describing the large deviations asymptotic of the sequence $\{\mathcal{P}^\varepsilon_{\mu_\varepsilon} \circ (W_T)^{-1}\}_{\varepsilon>0}$ for a fixed $T > 0$. We mention that this order of the asymptotics is analogous to the one discussed in the context of the fluctuations of the empirical current in stochastic lattice gases [3,9], where the small noise limit corresponds to the limit of infinitely many particles.

Our results are informally stated as follows. We show that the sequence of probability measures $\{\mathcal{P}^\varepsilon_{\mu_\varepsilon} \circ (W_T)^{-1}\}_{\varepsilon>0,T>0}$ satisfies the large deviation principle

$$\mathcal{P}^\varepsilon_{\mu_\varepsilon}\big(W_T \approx q\big) \asymp \exp\big\{ - T\varepsilon^{-1} s(q)\big\}$$

as we let first $\varepsilon \to 0$ and then $T \to \infty$. The associated convex rate function $s : \mathbb{R} \to \mathbb{R}_+$ is given by the following variational problem on the classical Freidlin-Wentzell rate functional

$$s(q) = \inf_{T>0} \inf \Big\{ \frac{1}{2T} \int_0^T dt \,\big|\dot{X}_t - c(X_t)\big|^2 \,,\ X \,:\, X_0 = X_T \,,\ \frac{2}{T} \int_0^T dt\, \langle b(X_t), \dot{X}_t\rangle = q \Big\}$$

and satisfies the fluctuation theorem $s(q) - s(-q) = -q$. In words, $s(q)$ is obtained by minimizing the averaged Freidlin-Wentzell functional over all closed paths for which the power dissipated by the vector field b equals q. We remark that, differently from what happens in the classical problem of the exit from a domain, the cost is measured here per unit of time and the constraint depends on the whole path. By constructing a simple example, we also show that the function s is not - in general - strictly convex or continuously differentiable.

2. Notation and results

We denote by $\langle \cdot, \cdot \rangle$ the canonical inner product in $\mathbb{R}^n$ and by $|\cdot|$ the associated Euclidean norm. The gradient and the divergence in $\mathbb{R}^n$ are respectively denoted by ∇ and $\nabla\cdot$. Fix a function $V : \mathbb{R}^n \to \mathbb{R}$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$. We assume they satisfy the following conditions:

(A) $V \in C^2(\mathbb{R}^n)$ and $\displaystyle\lim_{|x|\to\infty} \frac{\big\langle \nabla V(x), x\big\rangle}{|x|} = +\infty;$

(**B**) the vector field b is not conservative and $b \in C^1_{\mathrm{b}}(\mathbb{R}^n;\mathbb{R}^n)$, namely b is a continuously differentiable bounded vector field with bounded derivatives;

(**C**) for each $x \in \mathbb{R}^n$ $\quad \langle \nabla V(x), b(x)\rangle = 0$.

Observe that the assumption (**A**) implies a superlinear growth of the potential V, i.e.

$$\lim_{|x|\to\infty} \frac{V(x)}{|x|} = +\infty.$$

In particular, $\int dx \, \exp\{-\varepsilon^{-1} V(x)\} < \infty$. We shall denote by $c : \mathbb{R}^n \to \mathbb{R}^n$ the continuously differentiable vector field defined by

$$c(x) := -\frac{1}{2}\nabla V(x) + b(x). \tag{2.1}$$

Fix a standard filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ carrying an n-dimensional Brownian motion β. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. Given $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we consider the stochastic differential equation

$$\begin{cases} d\xi^x_t = c(\xi^x_t)dt + \sqrt{\varepsilon}\, d\beta_t, \\ \xi^x_0 = x. \end{cases} \tag{2.2}$$

Assumptions (**A**) and (**B**), together with definition (2.1), yield

$$\lim_{|x|\to\infty} \big\langle c(x), x \big\rangle = -\infty.$$

By standard criteria, see e.g. Theorems III.4.1 and III.5.1 in [15], we deduce the existence and uniqueness of a unique non-exploding strong solution to (2.2) as well as the existence of a unique stationary probability measure μ_ε for the Markov family $\{\xi^x, x \in \mathbb{R}^n\}$. We consider the canonical path space $C(\mathbb{R}_+;\mathbb{R}^n)$ endowed with the topology of uniform convergence in compacts and the associated Borel σ-algebra. The canonical coordinate on $C(\mathbb{R}_+;\mathbb{R}^n)$ is denoted by X_t. We denote by $\mathcal{P}^\varepsilon_x$, a probability measure on $C(\mathbb{R}_+;\mathbb{R}^n)$, the law of the process $\{\xi^x_t,\, t \in \mathbb{R}_+\}$. Given a Borel probability measure ν on $\mathbb{R}^n$ we set $\mathcal{P}^\varepsilon_\nu := \int d\nu(x)\mathcal{P}^\varepsilon_x$ and denote by $\mathcal{E}^\varepsilon_\nu$ the corresponding expectation. In particular, $\mathcal{P}^\varepsilon_{\mu_\varepsilon}$ is the law of the stationary process associated to (2.2).

As discussed in the introduction, given $\varepsilon, T > 0$ and a Borel probability measure ν on $\mathbb{R}^n$, we let the *Gallavotti-Cohen functional* W_T be the real random variable on $C(\mathbb{R}_+;\mathbb{R}^n)$ which is $\mathcal{P}^\varepsilon_\nu$ a.s. defined by

$$W_T := \frac{2}{T}\int_0^T \big\langle b(X_t), \circ dX_t \big\rangle = \frac{2}{T}\int_0^T \langle b(X_t), dX_t\rangle + \frac{\varepsilon}{T}\int_0^T dt\, \nabla\cdot b\,(X_t). \tag{2.3}$$

Here, $\circ dX_t$ denotes the Stratonovich integral with respect to the semimartingale X_t. In the second equality we rewrote W_T in terms of the Ito integral by using (2.2) and the orthogonality condition (**C**).

The main purpose of this paper is to analyze the large deviations behavior of the family of Borel probability measures on $\mathbb{R}$ given by $\{\mathcal{P}^\varepsilon_\nu \circ (W_T)^{-1}\}_{\varepsilon>0, T>0}$ in the joint limit $\varepsilon \to 0$ and $T \to \infty$. We first discuss briefly the typical behavior. Fix $\varepsilon > 0$ and the initial distribution ν. By (2.3) and the ergodicity of the process ξ^x, see e.g. [15, Thm. IV.5.1], in the limit $T \to \infty$ we have $\mathcal{P}^\varepsilon_\nu$ a.s.

$$\lim_{T\to\infty} W_T = \int d\mu_\varepsilon(x)\, \big[2\big|b(x)\big|^2 + \varepsilon\, \nabla\cdot b(x)\big].$$

If the dynamical system $\dot{x} = c(x)$ admits a unique equilibrium solution O which is globally attractive, μ_ε converges weakly to δ_O as $\varepsilon \to 0$. Hence, as $b(O) = 0$,

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} W_T = 0.$$

In the presence of metastable states, e.g. when c has several critical points (and no other attractors), the sequence $\{\mu_\varepsilon\}$ concentrates on the critical points of c corresponding to the deepest minima of V and the above statement still holds. Observe that this limiting behavior is independent on the initial distribution ν. On the other hand, if we denote by $\bar{x}_t$ the solution to $\dot{\bar{x}} = c(\bar{x})$ with initial condition $\bar{x}_0 = x$ and choose $\nu = \delta_x$, for $T > 0$ fixed we have

$$\lim_{\varepsilon \to 0} W_T = \frac{2}{T} \int_0^T dt \, \big| b(\bar{x}_t) \big|^2,$$

where the limit is in probability with respect to $\mathcal{P}_x^\varepsilon$. If the dynamical system $\dot{x} = c(x)$ admits a unique equilibrium solution which is globally attractive, in the limit as $T \to \infty$ we obtain the same result as before. On the other hand, if c has not a unique attractor, there are initial conditions $x \in \mathbb{R}^n$ such that the limiting behavior of W_T is different. Thus - in general - the order in which the limits $\varepsilon \to 0$ and $T \to \infty$ are taken is relevant.

In view of definition (2.3) and our assumptions on V and b, for $\varepsilon > 0$ fixed the large deviation principle for the sequence of probability measures $\{\mathcal{P}_{\mu_\varepsilon}^\varepsilon \circ (W_T)^{-1}\}_{T>0}$ can be proven along the same lines outlined in the introduction.

Theorem 2.1. *Fix $\varepsilon > 0$ and assume the function $V : \mathbb{R}^n \to \mathbb{R}$ also satisfies*

$$\lim_{|x| \to \infty} \Big[\big| \nabla V(x) \big|^2 - 2\Delta V(x) \Big] = +\infty. \tag{2.4}$$

For each $\lambda \in \mathbb{R}$ the limit

$$e^\varepsilon(\lambda) := \lim_{T \to \infty} \frac{1}{T} \log \mathcal{E}_{\mu_\varepsilon}^\varepsilon \Big(\exp \big\{ \lambda T W_T \big\} \Big)$$

exists and defines a convex real analytic function $e^\varepsilon : \mathbb{R} \to \mathbb{R}$. Let R^ε be the Legendre transform of e^ε, i.e. $R^\varepsilon(q) := \sup_\lambda \big\{ \lambda q - e^\varepsilon(\lambda) \big\}$. Then the family of Borel probability measures on $\mathbb{R}$ given by $\{\mathcal{P}_{\mu_\varepsilon}^\varepsilon \circ (W_T)^{-1}\}_{T>0}$ satisfies a large deviation principle with speed T and essentially strictly convex rate function R^ε. Namely, for each closed set $\mathcal{C} \subset \mathbb{R}$ and each open set $\mathcal{O} \subset \mathbb{R}$ we have

$$\varlimsup_{T \to \infty} \frac{1}{T} \log \mathcal{P}_{\mu_\varepsilon}^\varepsilon \big(W_T \in \mathcal{C} \big) \le - \inf_{q \in \mathcal{C}} R^\varepsilon(q),$$
$$\varliminf_{T \to \infty} \frac{1}{T} \log \mathcal{P}_{\mu_\varepsilon}^\varepsilon \big(W_T \in \mathcal{O} \big) \ge - \inf_{q \in \mathcal{O}} R^\varepsilon(q).$$

Finally, R^ε satisfies the fluctuation theorem $R^\varepsilon(q) - R^\varepsilon(-q) = -\varepsilon^{-1} q$.

Remark. The large deviation principle stated above still holds, with the same rate function, if the stationary measure μ_ε is replaced by a probability measure ν_ε such that $\nu_\varepsilon(dx) = f_\varepsilon(x) dx$ and $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}_+$ is a probability density satisfying $\int dx \, f_\varepsilon(x)^2 \exp\{\varepsilon^{-1} V(x)\} < \infty$.

Our next result is the large deviations asymptotic for the family of probabilities $\{\mathcal{P}_x^\varepsilon \circ (W_T)^{-1}\}_{\varepsilon>0, T>0}$ as we let *first* $\varepsilon \to 0$ and *then* $T \to \infty$. As discussed in the introduction, if the limits are taken in this order it is possible to express the rate function as the solution of a suitable variational problem. Before stating the result, we define the relevant rate function. Given $T > 0$, let

$$H_T := \Big\{ \varphi \in AC\big([0,T];\mathbb{R}^n\big) \;:\; \int_0^T dt\, \big|\dot\varphi_t\big|^2 < \infty \Big\}$$

where $AC\big([0,T];\mathbb{R}^n\big)$ denotes the set of absolutely continuous paths from $[0,T]$ to $\mathbb{R}^n$. Furthermore, given $x \in \mathbb{R}^n$, we set

$$H_T^x := \big\{ \varphi \in H_T \;:\; \varphi_0 = x \big\}.$$

For $x \in \mathbb{R}^n$ and $T > 0$ we let $I_T^x : C\big([0,T];\mathbb{R}^n\big) \to [0,+\infty]$ be the Freidlin-Wentzell rate functional associated to (2.2) namely,

$$I_T^x(\varphi) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T dt\, \big|\dot\varphi_t - c(\varphi_t)\big|^2 & \text{if } \varphi \in H_T^x, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

For $T > 0$ we let $\mathcal{L}_T : H_T \to \mathbb{R}$ be the power dissipated by the nonconservative vector field b namely,

$$\mathcal{L}_T(\varphi) := \frac{2}{T}\int_0^T dt\, \langle b(\varphi_t), \dot\varphi_t \rangle. \tag{2.6}$$

For $x, y \in \mathbb{R}^n$ and $q \in \mathbb{R}$ we then introduce the following subsets of $C\big([0,T];\mathbb{R}^n\big)$

$$\begin{aligned} \mathcal{A}_T^x(q) &:= \big\{ \varphi \in H_T^x \;:\; \mathcal{L}_T(\varphi) = q \big\} \\ \mathcal{A}_T^{xy}(q) &:= \big\{ \varphi \in H_T^x \;:\; \varphi_T = y\,,\; \mathcal{L}_T(\varphi) = q \big\} \end{aligned} \tag{2.7}$$

and set

$$S_T^{xy}(q) := \inf \big\{ I_T^x(\varphi)\,,\; \varphi \in \mathcal{A}_T^{xy}(q) \big\}. \tag{2.8}$$

Given $x \in \mathbb{R}^n$, we finally define the function $s^x : \mathbb{R} \to \mathbb{R}_+$ by

$$s^x(q) := \inf_{T>0} \frac{1}{T} S_T^{xx}(q). \tag{2.9}$$

We show below, see Theorem 4.1, that the function s^x is in fact independent of x. In the sequel, we therefore denote it simply by s.

Theorem 2.2. *The family of Borel probability measures on $\mathbb{R}$ given by $\{\mathcal{P}_x^\varepsilon \circ (W_T)^{-1}\}_{\varepsilon>0,T>0}$ satisfies, as we let first $\varepsilon \to 0$ and then $T \to \infty$, a large deviation principle, uniform with respect to x in compact subsets of $\mathbb{R}^n$, with speed $\varepsilon^{-1}T$ and convex rate function s. Namely, for each nonempty compact $K \subset \mathbb{R}^n$, each closed set $\mathcal{C} \subset \mathbb{R}$, and each open set $\mathcal{O} \subset \mathbb{R}$ we have*

$$\varlimsup_{T\to\infty}\, \varlimsup_{\varepsilon\to 0}\, \sup_{x\in K} \frac{\varepsilon}{T} \log \mathcal{P}_x^\varepsilon\big(W_T \in \mathcal{C}\big) \le -\inf_{q\in\mathcal{C}} s(q),$$

$$\varliminf_{T\to\infty}\, \varliminf_{\varepsilon\to 0}\, \inf_{x\in K} \frac{\varepsilon}{T} \log \mathcal{P}_x^\varepsilon\big(W_T \in \mathcal{O}\big) \ge -\inf_{q\in\mathcal{O}} s(q).$$

Finally, s satisfies the fluctuation theorem $s(q) - s(-q) = -q$.

Remark. The above theorem, together with the *exponential tightness* of the family $\{\mu_\varepsilon\}_{\varepsilon>0}$, implies the large deviation principle, with the same rate function, for the sequence $\{\mathcal{P}_{\mu_\varepsilon}^\varepsilon \circ (W_T)^{-1}\}_{\varepsilon>0,T>0}$.

The rest of the paper is organized as follows. The proof of Theorem 2.1 is detailed in Appendix A. In Section 3 we discuss the large deviation principle for the family $\{\mathcal{P}_x^\varepsilon \circ (W_T)^{-1}\}$ when T is fixed and $\varepsilon \to 0$. By analyzing, in Section 4, the variational convergence of the associated rate functions as $T \to \infty$ we then complete the proof of Theorem 2.2. Finally, in Section 5 we give a simple example of a vector field c for which s is not strictly convex.

3. Small noise asymptotic for a fixed time interval

We start by proving the so-called *exponential tightness* of the family of probability measures $\{\mathcal{P}_x^\varepsilon \circ (W_T)^{-1}\}_{\varepsilon>0,T>0}$ as $\varepsilon \to 0$ and $T \to \infty$. We emphasize that this result holds independently of the order of the limits.

Lemma 3.1. *We have*

$$\lim_{\ell\to\infty} \varlimsup_{\substack{\varepsilon\to 0\\ T\to\infty}} \sup_{x\in\mathbb{R}^n} \frac{\varepsilon}{T} \log \mathcal{P}_x^\varepsilon\big(|W_T| > \ell\big) = -\infty.$$

Proof. We denote by M the martingale part in the representation of W_T given in (2.3), i.e.

$$M_t := \int_0^t \big\langle b(X_s), dX_s - c(X_s)ds\big\rangle = \int_0^t \big\langle b(X_s), dX_s - b(X_s)ds\big\rangle$$

which is indeed a martingale with respect to the probability $\mathcal{P}_x^\varepsilon$. In view of assumption (**B**), $|b|^2$ and $\nabla\cdot b$ are bounded real functions. It is thus enough to show

$$\lim_{\ell\to\infty} \varlimsup_{\substack{\varepsilon\to 0\\ T\to\infty}} \sup_{x\in\mathbb{R}^n} \frac{\varepsilon}{T} \log \mathcal{P}_x^\varepsilon\big(|M_T| > \ell\, T\big) = -\infty.$$

Again by the boundedness of b, the process M is a continuous martingale whose quadratic variation admits the bound $[M]_t \le \varepsilon B t$ with $B := \sup_x |b(x)|^2$. By applying the so-called Bernstein exponential inequality for martingales, see e.g. [28, IV.3.16], we get

$$\mathcal{P}_x^\varepsilon\Big(|M_T| > \ell\, T\Big) \le 2\exp\Big\{-\frac{\ell^2\, T}{2\,\varepsilon B}\Big\}$$

which concludes the proof. □

In the remaining part of this section we prove the large deviation principle for the family $\{\mathcal{P}_x^\varepsilon \circ (W_T)^{-1}\}_{\varepsilon>0}$ for a fixed $T>0$.

Theorem 3.2. *Fix $x\in\mathbb{R}^n$ and $T>0$. Recall (2.7), (2.8) and let $S_T^x : \mathbb{R}\to[0,+\infty]$ be defined by*

$$S_T^x(q) := \inf_{y\in\mathbb{R}^n} S_T^{xy}(q) = \inf\big\{I_T^x(\varphi)\,,\ \varphi\in\mathcal{A}_T^x(q)\big\}. \tag{3.1}$$

Let $\{x_\varepsilon\}\subset\mathbb{R}^n$ be a sequence converging to x. Then the family of Borel probability measures on $\mathbb{R}$ given by $\{\mathcal{P}_{x_\varepsilon}^\varepsilon \circ (W_T)^{-1}\}_{\varepsilon>0}$ satisfies a large deviation principle with speed ε^{-1} and rate function S_T^x, i.e. for each closed set $\mathcal{C}\subset\mathbb{R}$ and each open set $\mathcal{O}\subset\mathbb{R}$ we have

$$\varlimsup_{\varepsilon\to 0}\ \varepsilon\log\mathcal{P}_{x_\varepsilon}^\varepsilon\big(W_T\in\mathcal{C}\big) \le -\inf_{q\in\mathcal{C}} S_T^x(q),$$
$$\varliminf_{\varepsilon\to 0}\ \varepsilon\log\mathcal{P}_{x_\varepsilon}^\varepsilon\big(W_T\in\mathcal{O}\big) \ge -\inf_{q\in\mathcal{O}} S_T^x(q).$$

Since the random variable W_T is not a continuous function on the path space $C(\mathbb{R}_+;\mathbb{R}^n)$, this result cannot be obtained directly from the usual Freidlin-Wentzell estimates [12] by contraction principle and some work is needed. We proceed by adding W_t as a $n+1$-th coordinate to the underlying diffusion process ξ_t^x. We then exploit the Freidlin-Wentzell theory for this extended (degenerate) diffusion process and finally project on the coordinate we are interested in.

Proof. Consider the map from $C(\mathbb{R}_+;\mathbb{R}^n)$ to $C(\mathbb{R}_+;\mathbb{R}^{n+1})$ which is $\mathcal{P}^\varepsilon_x$ a.s. defined by $X_t \mapsto (X_t, tW_t)$ and denote by $\mathcal{Q}^\varepsilon_x$ the push forward of $\mathcal{P}^\varepsilon_x$. The probability measure $\mathcal{Q}^\varepsilon_x$ on $C(\mathbb{R}_+;\mathbb{R}^{n+1})$ can be realized as the distribution of the $\mathbb{R}^{n+1}$-valued diffusion process η^x which is the unique strong solution of the stochastic differential equation

$$\begin{cases} d\eta^x_t = \big[c_0(\eta^x_t) + \varepsilon\, c_1(\eta^x_t)\big]dt + \sqrt{\varepsilon}\,\sigma(\eta^x_t)d\beta_t\ , \\ \eta^x_0 = (x,0). \end{cases}$$

Denoting by $y=(x,z)$ the coordinates in $\mathbb{R}^{n+1}$ with $x\in\mathbb{R}^n$ and $z\in\mathbb{R}$, the vector fields $c_0, c_1 : \mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ are given by

$$c_0(y) = \begin{pmatrix} c(x) \\ 2|b(x)|^2 \end{pmatrix} \qquad\qquad c_1(y) = \begin{pmatrix} 0 \\ \nabla\cdot c(x) \end{pmatrix}$$

while σ is the map from $\mathbb{R}^{n+1}$ to the set of $(n+1)\times n$ matrices given by

$$\sigma(y) = \begin{pmatrix} \mathbb{1}_{n\times n} \\ 2b(x) \end{pmatrix}$$

where $\mathbb{1}_{n\times n}$ stands for the identity operator on $\mathbb{R}^n$. The corresponding diffusion coefficient is the $(n+1)\times(n+1)$ matrix given by

$$a(y) := \sigma(y)\sigma(y)^{\mathsf{T}} = \begin{pmatrix} \mathbb{1}_{n\times n} & 2b(x) \\ & \\ 2b(x)^{\mathsf{T}} & 4|b(x)|^2 \end{pmatrix}.$$

Observe that $a(y)$ is singular for any $y\in\mathbb{R}^{n+1}$. More precisely, $\operatorname{Ker} a(y)$ is the one dimensional subspace spanned by the eigenvector $v_0(y) := (2b(x), -1)$ corresponding to the zero eigenvalue of $a(y)$.

Let $\mathcal{Q}^\varepsilon_{x,T}$ be the restriction of the probability $\mathcal{Q}^\varepsilon_x$ to the time interval $[0,T]$. In order to obtain the large deviation asymptotic of the family $\{\mathcal{Q}^\varepsilon_{x_\varepsilon,T}\}_{\varepsilon>0}$ as $\varepsilon\to 0$, we need an extension of the classical Freidlin-Wentzell results to the case in which the drift depends on ε and the diffusion matrix is degenerate. Such a generalization is proven in [1, Thm. III.2.13] and gives the following. Given $y\in\mathbb{R}^{n+1}$, let $a_{\mathcal{V}}(y)$ be the restriction of the linear operator $a(y)$ to $\mathcal{V}(y) := \big(\operatorname{Ker} a(y)\big)^\perp$ and denote by $a_{\mathcal{V}}^{-1}$ be its inverse. Then, as $\varepsilon\to 0$, the family of probability measures on $C([0,T];\mathbb{R}^{n+1})$ given by $\{\mathcal{Q}^\varepsilon_{x_\varepsilon,T}\}_{\varepsilon>0}$ satisfies a large deviation principle with speed ε^{-1} and rate function $J^x_T : C([0,T],\mathbb{R}^{n+1}) \to [0,+\infty]$ given by

$$J^x_T(\psi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T dt\,\big\langle \dot\psi_t - c_0(\psi_t)\,,\, a_{\mathcal{V}}^{-1}(\psi_t)\,[\dot\psi_t - c_0(\psi_t)]\big\rangle & \text{if } \psi\in\widetilde{H}^x_T \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\widetilde{H}^x_T := \Big\{\psi\in AC([0,T];\mathbb{R}^{n+1}) \,:\, \int_0^T dt\,\big|\dot\psi_t\big|^2 < \infty\,,$$
$$\psi_0 = (x,0)\,,\ \text{ and } \dot\psi_t - c_0(\psi_t)\in\mathcal{V}(\psi_t) \text{ for a.e. } t\in[0,T]\Big\}.$$

Fix $y = (x, z) \in \mathbb{R}^n \times \mathbb{R}$ and observe that a vector ζ in $\mathbb{R}^{n+1}$ belongs to $\mathcal{V}(y)$ if and only if $\zeta = \big(\xi, 2\langle b(x), \xi\rangle\big)$ for some vector ξ in $\mathbb{R}^n$. Therefore, recalling (2.6),

$$\widetilde{H}_T^x = \Big\{(\varphi, \chi) \in C([0, T]; \mathbb{R}^n \times \mathbb{R}) \,:\, \varphi \in H_T^x\,,\ \chi_t = t\mathcal{L}_t(\varphi)\Big\}.$$

An elementary computation shows that if $\zeta = \big(\xi, 2\langle b(x), \xi\rangle\big)$ for some ξ in $\mathbb{R}^n$ then $\langle \zeta, a_{\mathcal{V}}^{-1}(y)\zeta\rangle = |\xi|^2$. Consider the projection $\pi : C([0, T]; \mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}$ given by $\pi(\varphi, \chi) = \chi_T/T$. The previous statement and the last displayed equation imply

$$S_T^x(q) = \inf\big\{J_T^x(\varphi, \chi)\,,\ \pi(\varphi, \chi) = q\big\}.$$

The proof is then concluded by contraction principle. □

4. Long time asymptotic of the rate function

In this section we conclude the proof of Theorem 2.2 by showing that the sequence of rate functions $\{S_T^x/T\}_{T>0}$ converges to s as $T \to \infty$. As the relevant quantities involved in the large deviations asymptotic are the minima of S_T^x, the appropriate notion of convergence is the so-called Γ-convergence. We recall its definition and basic properties, see e.g. [5]. Let $\mathcal{X}$ be a complete separable metric space (simply $\mathbb{R}$ in our application); a sequence of functions $F_n : \mathcal{X} \to [0, +\infty]$ is said to Γ-*converge* to $F : \mathcal{X} \to [0, +\infty]$ iff the two following conditions are satisfied for each $x \in \mathcal{X}$. For any sequence $x_n \to x$ we have $\varliminf_n F_n(x_n) \geq F(x)$ (Γ-*liminf inequality*). There exists a sequence $x_n \to x$ such that $\varlimsup_n F_n(x_n) \leq F(x)$ (Γ-*limsup inequality*). It is easy to show that Γ-converges of $\{F_n\}$, together with its *equi-coercivity*, implies the convergence of the minima of F_n. It is also easy to check that the Γ-convergence of a sequence $\{F_n\}$ implies a lower bound on the infimum over a compact set and an upper bound of the infimum over an open set, see e.g. [5, Prop. 1.18]. Hence, in view of the exponential tightness in Lemma 3.1 and Theorem 3.2, the proof of Theorem 2.2 is a consequence of the following result.

Theorem 4.1. *The following statements hold.*

(i) *The function s^x, as defined in (2.9), is independent of x and convex. Furthermore, for each $x \in \mathbb{R}^n$ and $q \in \mathbb{R}$*

$$s(q) = \lim_{T\to\infty} \frac{S_T^{xx}(q)}{T}. \tag{4.1}$$

Finally, s satisfies the fluctuation theorem $s(q) - s(-q) = -q$.

(ii) *Recall (3.1) and let $\{x_T\} \subset \mathbb{R}^n$ be a bounded sequence. For $T > 0$ let $s_T : \mathbb{R} \to [0, +\infty)$ be the function defined by $s_T(q) := T^{-1}\, S_T^{x_T}(q)$. Then, as $T \to \infty$, the sequence of real functions $\{s_T\}_{T>0}$ Γ-converges to s.*

Postponing the proof of the above statements, we first show that they imply Theorem 2.2.

Proof of Theorem 2.2. We start by showing the large deviations upper bound. Fix $T, \varepsilon > 0$. In view of the representation used in Section 3, the map $\mathbb{R}^n \ni x \mapsto \mathcal{P}_x^\varepsilon \circ (W_T)^{-1}$ is continuous with respect to the weak topology of probability measures on $\mathbb{R}$. Given a closed set $\mathcal{C} \subset \mathbb{R}$, the map $\mathbb{R}^n \ni x \mapsto \mathcal{P}_x^\varepsilon\big(W_T \in \mathcal{C}\big) \in \mathbb{R}$ is therefore upper semicontinuous. Hence, given a nonempty compact $K \subset \mathbb{R}^n$, there exists a sequence $\{x_{T,\varepsilon}\} \subset K$ such that

$$\sup_{x\in K} \mathcal{P}_x^\varepsilon\big(W_T \in \mathcal{C}\big) = \mathcal{P}_{x_{T,\varepsilon}}^\varepsilon\big(W_T \in \mathcal{C}\big).$$

By taking, if necessary, a subsequence we may assume that $\{x_{T,\varepsilon}\}_{\varepsilon>0}$ converges to some $x_T \in K$. The large deviations upper bound in Theorem 3.2 now yields

$$\varlimsup_{\varepsilon\to 0} \sup_{x\in K} \frac{\varepsilon}{T} \log \mathcal{P}^\varepsilon_x\big(W_T \in \mathcal{C}\big) \le - \inf_{q\in\mathcal{C}} \frac{1}{T} S^{x_T}_T(q).$$

In view of the exponential tightness proven in Lemma 3.1, we may assume that $\mathcal{C}$ is a compact subset of $\mathbb{R}$. Since $\{x_T\} \subset K$, by item (ii) in Theorem 4.1 and [5, Prop. 1.18] we deduce

$$\varliminf_{T\to\infty} \inf_{q\in\mathcal{C}} \frac{1}{T} S^{x_T}_T(q) = \varliminf_{T\to\infty} \inf_{q\in\mathcal{C}} s_T(q) \ge \inf_{q\in\mathcal{C}} s(q),$$

which concludes the proof the large deviations upper bound.

The proof of the large deviations lower bound is analogous and the other statements follow directly from item (i) in Theorem 4.1. □

In order to prove Theorem 4.1, we start by the following topological lemma.

Lemma 4.2. *Fix $q \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. Then:*

(i) *for each $T > 0$ the set $\mathcal{A}^{xy}_T(q)$, as defined in (2.7), is not empty;*
(ii) *there exist reals $T_0, C \in (0,\infty)$, depending on q and x, y, such that for any $T \ge T_0$*

$$S^{xy}_T(q) \le C\,T.$$

Proof. The idea of the proof is quite simple. Since the vector field b is not conservative, there exists a closed path for which the power dissipated by b does not vanish. We thus only need to go from x to such closed path, repeat it an appropriate number of times, and then go to y.

Fix $q \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. For suitable constants $T_0, C > 0$, given $T \ge T_0$ we shall exhibit a path $\varphi \in \mathcal{A}^{xy}_T(q)$ such that $I^x_T(\varphi) \le C\,T$. Since the work done by the vector field b along the path φ, i.e. $\int_0^T dt\, \langle b(\varphi_t), \dot\varphi_t\rangle$, is invariant with respect to reparameterizations of φ, this will prove both the statements of the lemma.

Since the vector field b is not conservative, there exists a point in $\mathbb{R}^n$, say z, and a closed path $\tilde\xi \in H^z_1$ such that $\tilde\xi_0 = \tilde\xi_1 = z$ and $\mathcal{L}_1(\tilde\xi) = \bar q \neq 0$. By denoting with $\hat\xi$ the time reversal of $\tilde\xi$ we also have $\mathcal{L}_1(\hat\xi) = -\bar q$. By the continuity of the functional $\mathcal{L}_1$ on H^z_1 and the fact that $\mathbb{R}^n$ is simply connected, for each $p \in [-|\bar q|, |\bar q|]$ there exists a path $\xi \in H^z_1$ such that $\xi_0 = \xi_1 = z$ and $\mathcal{L}_1(\xi) = p$. Given $\lambda > 0$ we let $\xi^\lambda \in \mathcal{A}^{zz}_{\lambda^{-1}}(\lambda p)$ be defined by $\xi^\lambda_t := \xi_{\lambda t}$ and extend it by periodicity to a function defined on $\mathbb{R}$. Given $u, v \in \mathbb{R}^n$, let $\zeta^{uv}_t := u + t(v-u)$, $t \in [0,1]$ and $\ell_{uv} := \mathcal{L}_1(\zeta^{uv})$. Given a positive integer N and $T_1 \ge 0$ we then define the path φ, going from x to y in the time interval $[0, 2+T_1+N\lambda^{-1}]$, by

$$\varphi_t := \begin{cases} \zeta^{xz}_t & t \in [0,1) \\ \xi^\lambda_{t-1} & t \in [1, 1+N\lambda^{-1}) \\ \zeta^{zy}_{t-(1+N\lambda^{-1})} & t \in [1+N\lambda^{-1}, 2+N\lambda^{-1}) \\ y & t \in [2+N\lambda^{-1}, 2+T_1+N\lambda^{-1}]. \end{cases} \tag{4.2}$$

Then, by construction,

$$\mathcal{L}_{2+T_1+N\lambda^{-1}}(\varphi) = \frac{1}{2+T_1+N\lambda^{-1}} \big(\ell_{xz} + \ell_{zy} + Np\big)$$

and

$$I^x_{2+T_1+N\lambda^{-1}}(\varphi) = I^x_1(\zeta^{xz}) + N I^z_{\lambda^{-1}}(\xi^\lambda) + I^z_1(\zeta^{zy}) + \frac{T_1}{2}\,|c(y)|^2.$$

We now choose $\lambda = \lambda(q) > 0$ such that $|q|\lambda^{-1} \le (1/2)|\bar{q}|$, $T_1 \in [0, \lambda^{-1})$, and let $T_0 = T_0(x, y, q) > 2 + \lambda^{-1}$ be such that

$$\Big|\frac{(2+T_1)q - \ell_{xz} - \ell_{zy}}{\lambda(T_0 - 2) - 1}\Big| \le \frac{1}{2}\,|\bar{q}|.$$

For $T \ge T_0$ we next choose $N = [\lambda(T-2)]$, where $[\cdot]$ denotes the integer part, $T_1 = (T-2) - \lambda^{-1}[\lambda(T-2)]$, and

$$p = \lambda^{-1} q + \frac{(2+T_1)q - \ell_{xz} - \ell_{zy}}{N}.$$

Note that $T_1 \in [0, \lambda^{-1})$ and $p \in [-|\bar{q}|, |\bar{q}|]$ by the previous choices. As it is simple to check, the path φ above constructed then satisfies $\mathcal{L}_T(\varphi) = q$ and the bound $I^x_T(\varphi) \le CT$ for some $C = C(x, y, q)$ independent of $T \ge T_0$. □

Proof of Theorem 4.1, item (i). Fix $x \in \mathbb{R}^n$, $q \in \mathbb{R}$ and $T_1, T_2 > 0$. From Lemma 4.2 and the goodness of the rate function I^x_T it follows there exist $\varphi^i \in \mathcal{A}^{xx}_{T_i}(q)$ such that $S^{xx}_{T_i}(q) = I^x_{T_i}(\varphi^i)$, $i = 1, 2$. By considering the path φ_t, $t \in [0, T_1 + T_2]$ given by

$$\varphi_t := 1\!\!1_{[0,T_1)}(t)\,\varphi^1_t + 1\!\!1_{[T_1,T_1+T_2]}(t)\,\varphi^2_{t-T_1}$$

we deduce that the sequence $\{S^{xx}_T(q)\}_{T>0}$ is subadditive, i.e.

$$S^{xx}_{T_1+T_2}(q) \le S^{xx}_{T_1}(q) + S^{xx}_{T_2}(q).$$

Recalling (2.9), the subadditivity just proven implies (4.1). By using the existence of the limit and again Lemma 4.2, it is now simple to show that s^x does not depend on x and that it is convex.

To prove the fluctuation theorem, observe that in view of (2.6), (2.8), and the orthogonality condition (**C**)

$$\frac{1}{T}\,S^{xx}_T(q) = \inf\Big\{\frac{1}{2T}\int_0^T dt\,\Big[\big|\dot{\varphi}_t\big|^2 + \big|c(\varphi_t)\big|^2\Big] - \frac{1}{2}q\,,\ \varphi \in \mathcal{A}^{xx}_T(q)\Big\}.$$

In particular, if φ is a minimizer for the right hand side above, then the path φ^* defined by $\varphi^*_t := \varphi_{T-t}$ is a minimizer for the analogous problem with q replaced by $-q$. Hence

$$\frac{1}{T}\,S^{xx}_T(q) - \frac{1}{T}\,S^{xx}_T(-q) = -q$$

and the statement follows by taking the limit $T \to \infty$. □

Proof of Theorem 4.1, item (ii), *Γ-limsup inequality.* Fix a sequence $T_n \to \infty$ and $q \in \mathbb{R}$. We need to show there exists a sequence $q_n \to q$ such that $\overline{\lim}_n\, s_{T_n}(q_n) \le s(q)$. We claim it is enough to choose the constant sequence $q_n = q$. Indeed, letting $x_n := x_{T_n}$, from the very definition (3.1) of S^x_T it follows

$$\overline{\lim_n}\, s_{T_n}(q) = \overline{\lim_n}\,\frac{1}{T_n} S^{x_n}_{T_n}(q) \le \overline{\lim_n}\,\frac{1}{T_n} S^{x_n x_n}_{T_n}(q). \tag{4.3}$$

Since $\{x_n\}$ is a bounded sequence, from Lemma 4.2 and (4.1) it easily follows that

$$\lim_n \frac{1}{T_n} S^{x_n x_n}_{T_n}(q) = s(q) \tag{4.4}$$

which concludes the proof. □

In order to prove the Γ-liminf inequality in Theorem 4.1, we need to show that in the inequality in (4.3) we did not loose much. On the other hand, if we let $\varphi^{T,x}$ be a minimizer for the variational problem on the right hand side of (3.1), there is no reason for $\varphi^{T,x}_T$ to be equal to x. In the next proposition, we show that we can extend $\varphi^{T,x}$ to a path ψ defined on a the longer time interval $[0, T+\tau]$ in such way that $\psi_{T+\tau} = x$ and the loss in the inequality in (4.3) becomes negligible as $T \to \infty$.

Proposition 4.3. *Fix $q \in \mathbb{R}$, a bounded sequence $\{x_n\} \subset \mathbb{R}^n$, and sequences $T_n \to \infty$, $q_n \to q$. Let $\varphi^n \in \mathcal{A}^{x_n}_{T_n}(q_n)$ be such that $S^{x_n}_{T_n}(q_n) = I^{x_n}_{T_n}(\varphi^n)$ and set $y_n := \varphi^n_{T_n}$. There exist sequences $\tau_n \to \infty$ and $\gamma^n \in \mathcal{A}^{y_n x_n}_{\tau_n}(\widehat{q}_n)$, where $\widehat{q}_n := q + (q - q_n)T_n/\tau_n$, such that*

$$\lim_n \frac{\tau_n}{T_n} = 0 \qquad\qquad \lim_n \frac{1}{T_n} I^{y_n}_{\tau_n}(\gamma^n) = 0. \tag{4.5}$$

Assuming the above proposition, we conclude the proof of Theorem 4.1.

Proof of Theorem 4.1, item (ii), *Γ-liminf inequality.* Fix $q \in \mathbb{R}$, a sequence $T_n \to \infty$, and a sequence $q_n \to q$. We need to show that $\underline{\lim}_n\, s_{T_n}(q_n) \geq s(q)$. We define $x_n := x_{T_n}$ so that $s_{T_n}(q_n) = T_n^{-1} S^{x_n}_{T_n}(q_n)$.

Let $\varphi^n \in \mathcal{A}^{x_n}_{T_n}(q_n)$, $\{y_n\} \subset \mathbb{R}^n$, $\tau_n \to \infty$, $\{\widehat{q}_n\} \subset \mathbb{R}$, and $\gamma^n \in \mathcal{A}^{y_n x_n}_{\tau_n}(\widehat{q}_n)$ be as in the statement of Proposition 4.3. Define the path $\psi^n \in H^{x_n}_{T_n+\tau_n}$ by

$$\psi^n_t := 1\!\!1_{[0,T_n)}(t)\, \varphi^n_t + 1\!\!1_{[T_n, T_n+\tau_n]}(t)\, \gamma^n_{t-T_n}.$$

From the definition of $\widehat{q}_n$ it follows that $\psi^n \in \mathcal{A}^{x_n x_n}_{T_n+\tau_n}(q)$. Since

$$I^{x_n}_{T_n+\tau_n}(\psi^n) = I^{x_n}_{T_n}(\varphi^n) + I^{y_n}_{\tau_n}(\gamma^n),$$

we have

$$S^{x_n}_{T_n}(q_n) = I^{x_n}_{T_n+\tau_n}(\psi^n) - I^{y_n}_{\tau_n}(\gamma^n) \geq S^{x_n x_n}_{T_n+\tau_n}(q) - I^{y_n}_{\tau_n}(\gamma^n).$$

Divide the previous inequality by T_n and take the liminf as $n \to \infty$. Since $\{x_n\}$ is bounded, the proof is achieved by using (4.4) and (4.5). □

The two following lemmata are used in the proof of Proposition 4.3. In the first one we show that the endpoint of the minimizer for the variational problem on the right hand side of (3.1) is at most at distance $O(T)$ from the initial point x. In the second one we then show we can get back to a compact independent of T in a time which is $o(T)$.

Lemma 4.4. *Let $\{y_n\} \subset \mathbb{R}^n$ be as in the statement of Proposition 4.3. Then there exists a constant $C > 0$ such that $|y_n| \leq C(T_n + 1)$ for any $n \geq 1$.*

Proof. Let $\{x_n\} \subset \mathbb{R}^n$, $T_n \to \infty$, $q_n \to q$ and $\varphi^n \in A^{x_n}_{T_n}(q_n)$ be as in the statement of Proposition 4.3. Since $\{x_n\}$ is bounded, by Lemma 4.2 there exists a constant $C_1 > 0$ independent of n such that for any n large enough $I^{x_n}_{T_n}(\varphi^n) \leq C_1 T_n$. Hence, by expanding the square in the definition (2.5) of $I^{x_n}_T$,

$$\begin{aligned} C_1 \;&\geq\; \frac{1}{T_n}\int_0^{T_n} dt \left[\frac{1}{2}|\dot\varphi^n_t|^2 - \big\langle b(\varphi^n_t), \dot\varphi^n_t\big\rangle + \frac{1}{2}\big\langle \nabla V(\varphi^n_t), \dot\varphi^n_t\big\rangle\right] \\ &=\; \frac{1}{2T_n}\int_0^{T_n} dt\, |\dot\varphi^n_t|^2 - \frac{q_n}{2} + \frac{V(y_n) - V(x_n)}{2T_n}. \end{aligned}$$

Since $\{x_n\}$ is bounded and $\lim_{|x|\to\infty} V(x) = +\infty$, from the above bound it follows there exists a constant $C_2 > 0$ independent of n such that $\int_0^{T_n} dt\, |\dot\varphi^n_t|^2 \leq C_2 T_n$. This yields $|y_n - x_n| \leq C_3 T_n$ for some $C_3 > 0$ and concludes the proof. □

Lemma 4.5. *Let $\bar{x}_t^y$, $t \in \mathbb{R}_+$, be the solution to $\dot{\bar{x}} = c(\bar{x})$ with initial condition $\bar{x}_0 = y$. There exists a compact $K \subset \mathbb{R}^n$ independent of y such that the following statement holds. Denote by $\sigma_K(y) := \inf\{t \geq 0 \,:\, \bar{x}_t^y \in K\}$ the hitting time of K; then*

$$\lim_{|y|\to\infty} \frac{\sigma_K(y)}{|y|} = 0.$$

Proof. By assumption (**A**) there exists a Lipschitz function $\ell : \mathbb{R}_+ \to \mathbb{R}$ such that $\ell(r) \to +\infty$ as $r \to \infty$ and for any $x \in \mathbb{R}^n$ $\big\langle \nabla V(x), x \big\rangle \geq 2\,\ell(|x|)\,|x|$. Set $B := \sup_x |b(x)|$ and let $R_0 \in \mathbb{R}_+$ be such that $\ell(r) \geq B + 1$ for any $r \geq R_0$. We claim the lemma holds with K given by the ball of radius R_0. Indeed, recalling (2.1) and using a standard comparison argument, if $|y| > R_0$ we have that $|\bar{x}_t^y| \leq r_t$ where r_t solves the Cauchy problem

$$\begin{cases} \dot{r} = -\ell(r) + B, \\ r_0 = |y|. \end{cases}$$

Hence,

$$\lim_{|y|\to\infty} \frac{\sigma_K(y)}{|y|} \leq \lim_{|y|\to\infty} \frac{1}{|y|} \int_{R_0}^{|y|} dr\, \frac{1}{\ell(r) - B} = 0$$

since $\ell(r) \to \infty$ as $r \to \infty$. □

Proof of Proposition 4.3. Let $K \subset \mathbb{R}^n$ be as in Lemma 4.5 and denote by $\sigma_n \geq 0$ the hitting time of K for the path $\bar{x}$ with initial condition y_n. Observe that, by Lemmata 4.4 and 4.5, $\lim_n \sigma_n/T_n = 0$. Set $z_n := \bar{x}_{\sigma_n} \in K$ and let $q_n^{(1)} := 2\sigma_n^{-1} \int_0^{\sigma_n} dt\, \big|b(\bar{x}_t)\big|^2$ be the power dissipated by vector field b along the path $\bar{x}_t$, $t \in [0, \sigma_n]$. Note that $q_n^{(1)}$ is bounded by the boundedness of b.

Choose a sequence $\tilde{\sigma}_n \to \infty$ such that $\tilde{\sigma}_n/T_n \to 0$, $\sup_n \sigma_n/\tilde{\sigma}_n < \infty$, and $\sup_n (q - q_n) T_n/(\sigma_n + \tilde{\sigma}_n) < \infty$. Set $\tau_n := \sigma_n + \tilde{\sigma}_n$ and $q_n^{(2)} := \tilde{\sigma}_n^{-1}\big(\tau_n \widehat{q}_n - \sigma_n q_n^{(1)}\big)$. Observe that $q_n^{(2)}$ is bounded by the boundedness of $q_n^{(1)}$ and the choice of $\tilde{\sigma}_n$. Let ψ_t^n, $t \in [0, \tilde{\sigma}_n]$ be the path constructed in Lemma 4.2, see in particular (4.2), with x replaced by z_n, y replaced by x_n, q replaced by $q_n^{(2)}$, and T replaced by $\tilde{\sigma}_n$.

Finally, define the path γ^n going form y_n to x_n in the time interval $[0, \tau_n]$ by $\gamma_t^n := \mathbb{1}_{[0,\sigma_n)}(t)\, \bar{x}_t + \mathbb{1}_{[\sigma_n,\tau_n]}(t)\, \psi_{t-\sigma_n}^n$. Then, by construction,

$$\mathcal{L}_{\tau_n}(\gamma^n) = \frac{1}{\tau_n}\big[\sigma_n q_n^{(1)} + \tilde{\sigma}_n \mathcal{L}_{\tilde{\sigma}_n}(\psi^n)\big]$$

and

$$I_{\tau_n}^{y_n}(\gamma^n) = 0 + I_1^{z_n}(\zeta^{z_n z}) + N I_{\lambda^{-1}}^{z}(\xi^\lambda) + I_1^z(\zeta^{z x_n}) + \frac{T_1}{2}\,|c(x_n)|^2.$$

Since $\{z_n\} \subset K$ and $\{x_n\}$, $\{q_n^{(2)}\}$ are bounded, we conclude the proof by choosing sequences $\{\lambda_n\} \subset \mathbb{R}_+$, $\{N_n\} \subset \mathbb{N}$, and $\{p_n\} \subset [-|\bar{q}|, |\bar{q}|]$ as in Lemma 4.2. Note in particular that with such choices $\mathcal{L}_{\tilde{\sigma}_n}(\psi^n) = q_n^{(2)}$, λ_n is bounded, and $\lim_n N_n/T_n = 0$. □

5. An example with not strictly convex rate function

We here given a simple example a two-dimensional vector field c such that the rate function s is not strictly convex. Let $U : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function with two local minima at 0 and $R_0 > 0$ and super-linear growth as $r \to \infty$. Set

$V(x) := U(|x|)$. Let also the smooth vector field $b : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $b(x) = A(|x|)x^\perp = A(|x|)\,(x^2, -x^1)$ where $A : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function with compact support in $(0, +\infty)$ such that $A(R_0) > 0$. Set finally $c(x) := -(1/2)\nabla V(x) + b(x)$. Recalling s has been defined in (2.9), we claim that in this case it is not strictly convex.

Consider the dynamical system

$$\dot{x} = -\frac{1}{2}\nabla V(x) + b(x).$$

By the above choices, it has the equilibrium solution $x = 0$ and the periodic solution $\bar{x}(t) = R_0\big(\cos(\omega t), \sin(\omega t)\big)$ where $\omega = A(R_0)/R_0$. Let

$$\bar{q} := \frac{2\omega}{2\pi}\int_0^{2\pi/\omega} dt\,\big\langle b\big(\bar{x}(t)\big), \dot{\bar{x}}(t)\big\rangle$$

be the power dissipated along the periodic solution $\bar{x}$. Note that $\bar{q} > 0$. By choosing the test paths $\varphi = 0$ and $\varphi = \bar{x}$ we deduce $s(0) = s(\bar{q}) = 0$. Therefore, by the positivity and convexity of s, we have $s(q) = 0$ for any $q \in [0, \bar{q}]$. Moreover, the fluctuation theorem $s(q) - s(-q) = -q$ implies that $s(q) = -q$ for $q \in [-\bar{q}, 0]$.

Appendix A. Large deviations principle as $T \to \infty$

As the diffusion coefficient ε is here kept fixed, to simplify the notation we set $\varepsilon = 1$ and drop the dependence from ε from the notation. We also assume the arbitrary constant in the definition of V has been chosen so that $\exp\{-V(x)\}$ is a probability density in $\mathbb{R}^n$ and denote by $\wp$ the probability given by $d\wp(x) = \exp\{-V(x)\}\,dx$. Let $\mathcal{H}$ be the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n; d\wp)$; we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm in $\mathcal{H}$. Hereafter, we assume that the function $V : \mathbb{R}^n \to \mathbb{R}$ satisfies (2.4) without further mention.

Fix a Borel probability measure ν on $\mathbb{R}^n$ and consider the moment generating function of the random variable tW_t with respect to the probability $\mathcal{P}_\nu$. Recalling (2.3) and the assumption (**B**), a straightforward application of the Girsanov formula yields the following representation. For each $\lambda \in \mathbb{R}$

$$\mathcal{E}_\nu\Big(\exp\big\{\lambda t W_t\big\}\Big) = \int d\nu(x)\,\mathbb{E}\,\exp\Big\{\int_0^t ds\,U_\lambda\big(\xi_s^{\lambda,x}\big)\Big\} \tag{A.1}$$

where, for $x \in \mathbb{R}^n$, the process $\xi^{\lambda,x}$ is the solution to the stochastic differential equation

$$\begin{cases} d\xi_t^{\lambda,x} = \Big[-\dfrac{1}{2}\nabla V\big(\xi_t^{\lambda,x}\big) + (1+2\lambda)\,b\big(\xi_t^{\lambda,x}\big)\Big]dt + d\beta_t \\ \xi_0^{\lambda,x} = x \end{cases} \tag{A.2}$$

and $U_\lambda : \mathbb{R}^n \to \mathbb{R}$ is the function defined by

$$U_\lambda(x) := 2\lambda(1+\lambda)|b(x)|^2 + \lambda\nabla\cdot b(x). \tag{A.3}$$

For $\lambda \in \mathbb{C}$, let A°_λ be operator on $\mathcal{H}$ densely defined on $C_0^\infty(\mathbb{R}^n)$, the set of smooth functions with compact support, by

$$A^\circ_\lambda f(x) := \frac{1}{2}\Delta f(x) - \frac{1}{2}\langle\nabla V(x), \nabla f(x)\rangle + (1+2\lambda)\langle b(x), \nabla f(x)\rangle + U_\lambda(x) f(x). \tag{A.4}$$

We denote by A_λ the closure of A°_λ; its domain $\mathcal{D}_\lambda$ is given by

$$\mathcal{D}_\lambda = \big\{f \in \mathcal{H} \,:\, \exists\{f_n\} \subset C_0^\infty(\mathbb{R}^n) \text{ s.t. } f_n \to f \text{ in } \mathcal{H} \text{ and } A^\circ_\lambda f_n \text{ converges in } \mathcal{H}\big\}.$$

The adjoint of A_λ in $\mathcal{H}$ is denoted by A'_λ.

Lemma A.1. *The set $\mathcal{D}_\lambda$ is independent on λ and coincides with the domain of A'_λ for any $\lambda \in \mathbb{C}$. Moreover, A_λ generates a compact strongly continuous semigroup T_t^λ on $\mathcal{H}$. Finally, if $\lambda \in \mathbb{R}$ then the semigroup T_t^λ is positive and irreducible and for $f \in \mathcal{H}$ the representation*

$$T_t^\lambda f\,(x) = \mathbb{E}\Big(\exp\Big\{ \int_0^t ds\, U_\lambda(\xi_s^{\lambda,x}) \Big\}\, f\big(\xi_t^{\lambda,x}\big)\Big) \tag{A.5}$$

holds.

Before proving this result, we clarify the terminology used. The semigroup T_t^λ is *positive* (positivity preserving in the terminology of [27]) if $f \ge 0$ implies $T_t^\lambda f \ge 0$ for any $t \ge 0$. Given $f \in \mathcal{H}$, we write $f > 0$ iff $f \ge 0$ and $f \ne 0$. We write $f \gg 0$ iff $f(x) > 0$ $\wp$ a.e. Then, see [25, Def. C-III.3.1], the positive semigroup T_t^λ is *irreducible* (ergodic in the terminology of [27]) if, given $f > 0$ and $g > 0$ in $\mathcal{H}$, there exists $t_0 \ge 0$ such that $\big(g, T_{t_0}^\lambda f\big) > 0$. An equivalent characterization of irreducibility is the following: for sufficiently large z in the resolvent set of the generator A_λ the resolvent $R_z = (z - A_\lambda)^{-1}$ is positivity improving, i.e. $f > 0$ implies $R_z f \gg 0$.

Proof. Fix $\lambda \in \mathbb{C}$. On $C_0^\infty(\mathbb{R}^n)$ define the operators

$$\begin{aligned} H_0^\circ f(x) &= \frac{1}{2}\Delta f(x) - \frac{1}{2}\langle \nabla V(x), \nabla f(x)\rangle \\ H_1^\circ f(x) &= (1+2\lambda)\,\langle b(x), \nabla f(x)\rangle + U_\lambda(x) f(x) \end{aligned}$$

so that $A_\lambda^\circ = H_0^\circ + H_1^\circ$.

Step 1. The operator H_0° with domain $C_0^\infty(\mathbb{R}^n)$ is essentially self-adjoint in $\mathcal{H}$ and its closure has compact resolvent.

To complete this step, it is enough to consider the well-known *ground state transformation* and apply standard criteria for Schrödinger operators. Here the details. Let $U : \mathcal{H} \to L^2(\mathbb{R}^n; dx)$ be the isometry defined by $Uf = \exp\{-V/2\} f$. By an explicit computation

$$U H_0^\circ U^{-1} = -\Big[-\frac{1}{2}\Delta + \frac{1}{8}|\nabla V|^2 - \frac{1}{4}\Delta V \Big]$$

Recalling the assumption (2.4), the step now follows from [27, Thm X.29] and [27, Thm XIII.67]. We denote by $(H_0, \mathcal{D})$ the closure of H_0°. Observe that

$$(H_0 f, f) = -\frac{1}{2}\|\nabla f\|^2 \tag{A.6}$$

which implies that H_0 is bounded from above by zero.

Step 2. The operator H_1° is H_0°-bounded with H_0°-bound equal to 0, i.e. for each $\gamma > 0$ there exists a constant C_γ, also depending on λ, such that for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|H_1^\circ f\| \le \gamma \|H_0^\circ f\| + C_\gamma \|f\|. \tag{A.7}$$

Observe that this implies that H_1° can be uniquely extended to $\mathcal{D}$. We denote this extension by $(H_1, \mathcal{D})$. Moreover the bound (A.7) holds, with H_0° and H_1° replaced by H_0 and H_1, for any $f \in \mathcal{D}$.

To prove (A.7), let $g \in \mathcal{H}$, $f \in C_0^\infty(\mathbb{R}^n)$. By assumption (**B**) and Cauchy-Schwarz inequality, for any $\gamma > 0$ we have

$$\begin{aligned}\big(g, H_1^\circ f\big) &\leq (1 + 2|\lambda|) \int d\wp(x)\, |b(x)|\ |g(x)|\ |\nabla f(x)| + C\|g\|\,\|f\| \\ &\leq (1 + 2|\lambda|) \Big[\frac{\gamma}{2}\|\nabla f\|^2 + \frac{1}{2\gamma} \sup_x |b(x)|^2\, \|g\|^2\Big] + C\|g\|\|f\|\end{aligned}$$

for some constant C depending on λ and the vector field b. Thanks to (A.6), by taking the supremum over g with $\|g\| = 1$ and redefining γ, we deduce

$$\|H_1^\circ f\| \leq \gamma\|f\|\|H_0^\circ f\| + C_\gamma(1 + \|f\|)$$

for some constant $C_\gamma \in (0, +\infty)$. Replacing above f with $f/\|f\|$ the step follows.

Step 3. The operator $(H_1, \mathcal{D})$ is $(H_0, \mathcal{D})$-compact.

We need to show that for some z (hence for all z in the resolvent set of H_0, which by (A.6) is not empty) $H_1 R_{0,z}$ is compact, where $R_{0,z}$ denotes the resolvent of the operator $(H_0, \mathcal{D})$. Fix a bounded sequence $\{f_n\} \subset \mathcal{H}$. By Step 1 there exists a subsequence, still denoted by f_n, such that $\{R_{0,z} f_n\}$ converges. By linearity, we can assume that $R_{0,z} f_n \to 0$; we shall show that $H_1 R_{0,z} f_n \to 0$ as well. Indeed, by Step 2 for any $\gamma > 0$ we have

$$\|H_1 R_{0,z} f_n\| \leq \gamma\|H_0 R_z(H_0) f_n\| + C_\gamma\|R_z(H_0) f_n\|.$$

Since for each z in the resolvent set of H_0 the operator $H_0 R_z(H_0)$ is bounded, the result follows by taking first the limit $n \to \infty$ and then $\gamma \downarrow 0$.

Step 4. For every $\lambda \in \mathbb{C}$, $\mathcal{D}_\lambda = \mathcal{D}$ and A_λ generates a compact semigroup on $\mathcal{H}$.

The first statement follows from [17, Thm. IV.1.11]. Moreover, since by (A.6) H_0 generates a strongly continuous holomorphic semigroup on $\mathcal{H}$, the same is true for A_λ by [27, Thm. X.54] and Step 2. One can easily check that for z sufficiently large the resolvent R_z of A_λ satisfies

$$R_z = R_{0,z} + R_z H_1 R_{0,z}$$

hence, by Steps 1 and 3, the operator A_λ has compact resolvent. Since a holomorphic strongly continuous semigroup whose generator has compact resolvent is always compact [11, Thm. II.4.29] this concludes the step.

Step 5. The operator A'_λ has domain $\mathcal{D}$.

Observe that trivially the domain of A'_λ contains $\mathcal{D}$. Define on C_0^∞ the formal adjoint H_1^* of H_1 in $\mathcal{H}$. In view of assumption (**C**), an elementary computation shows that

$$H_1^* f(x) = -(1 + 2\bar{\lambda})\langle b(x), \nabla f(x)\rangle + \overline{U_\lambda(x)} f(x)$$

where $\bar{z}$ is the complex conjugate of z. Notice that Step 2 holds also with H_1° replaced by H_1^*. As in Step 3, we can then conclude that the adjoint H_1' of H_1 is $(H_0, \mathcal{D})$-compact. By [2] it follows that $A'_\lambda = (H_0, \mathcal{D}) + H_1$, so the domain of A'_λ can not be larger than $\mathcal{D}$.

Step 6. If $\lambda \in \mathbb{R}$, then the semigroup $T_t^\lambda := \exp\{tA_\lambda\}$ is positive, irreducible and satisfies (A.5).

In view of the assumptions (**B**), the function U_λ defined in (A.3) is continuous and bounded, hence (A.5) for $f \in C_0^\infty(\mathbb{R}^n)$ follows from the classical Feynman-Kac formula. By monotone approximation it holds for every $f \in \mathcal{H}$. In particular T_t^λ is positive. To prove irreducibility we show that the resolvent R_z of A_λ is positivity improving for z sufficiently large. Given $f \in \mathcal{H}$, $f > 0$ we can find a function

$g \in C_0^\infty(\mathbb{R}^n)$ such that $0 < g \le f$. For sufficiently large z in the resolvent of A_λ, we have that $h := R_z g \ge 0$ since the semigroup is positive. Furthermore $A_\lambda h - zh = -g < 0$. By elliptic regularity $h \in C^2(\mathbb{R}^n)$, so $h \ne 0$ and for large enough z the maximum principle [26, Thm. II.6] gives $h \gg 0$. Again by positivity of the semigroup $R_z f \ge R_z g = h \gg 0$, which gives the irreducibility. □

Given $\lambda \in \mathbb{R}$, let $e(\lambda) := \sup\{\mathrm{Re}\, z,\, z \in \mathrm{spec}\, A_\lambda\}$ be the *spectral bound* of the operator A_λ. We recall, see [25], that $e(\lambda)$ is *strictly dominant* iff there exists $\delta > 0$ such that $\mathrm{Re}\, z \le e(\lambda) - \delta$ for any $z \in \mathrm{spec}\, A_\lambda \setminus \{e(\lambda)\}$. We recall also that an eigenvalue is *simple* iff it is a pole of order 1 of the resolvent.

Lemma A.2. *Fix $\lambda \in \mathbb{R}$. Then $e(\lambda)$ is strictly dominant and a simple eigenvalue of A_λ and of A'_λ. Moreover, there exist corresponding eigenvectors Ψ_λ and Ψ'_λ of A_λ and A'_λ such that $\Psi_\lambda, \Psi'_\lambda \gg 0$ and $(\Psi_\lambda, \Psi'_\lambda) = 1$. In particular, the projection given by $P_\lambda = (\Psi'_\lambda, \cdot)\Psi_\lambda$ is positivity improving and there exist constants $\delta, C > 0$ such that for any $t \ge 0$*

$$\left\| T_t^\lambda - e^{te(\lambda)}\, P_\lambda \right\| \le C\, e^{t[e(\lambda)-\delta]}$$

where $\|\cdot\|$ denotes the operator norm.

Proof. The lemma follows from standard spectral theory of positive semigroups, see [25, Thm. C-III.1.1, Prop. C-III.3.5] and [11, Thm. V.3.7]. Notice that, by compactness and irreducibility of the semigroup, one can exclude that the spectrum is empty. This follows essentially from a result of de Pagter, see [25, Thm. C-III.3.7]. □

Remark. If $\lambda = 0$ the semigroup T_t^λ is Markovian, i.e. $T_t^0 1 = 1$. In this case $e(0) = 0$ and $\Psi_0 = 1$. It is also simple to check that the stationary measure μ of the solution to (2.2) for $\varepsilon = 1$, which coincides with (A.2) for $\lambda = 0$, is given by $d\mu = Z^{-1}\, \Psi'_0 d\wp$, where $Z = \int d\wp(x)\, \Psi'_0(x)$ and Ψ'_0 is a strictly positive eigenvector of A'_0 corresponding to the eigenvalue $e(0) = 0$.

Lemma A.3. *The map $\mathbb{R} \ni \lambda \mapsto e(\lambda)$ is real analytic.*

Proof. In view of Lemma A.1, it is straightforward to check that the family of linear operators $\{A_\lambda\}_{\lambda \in \mathbb{C}}$ is a *holomorphic family of type (A)* in the sense of [17, VII § 2]. Fix $\lambda_0 \in \mathbb{R}$. By the Kato-Rellich theorem [27, Thm. XII.8] there is an analytic function $\lambda \mapsto a(\lambda) \in \mathbb{C}$ such that $a(\lambda_0) = e(\lambda_0)$ and $a(\lambda)$ is the only eigenvalue of A_λ near $e(\lambda)$. From the definition of $e(\lambda)$ and the analyticity of $a(\lambda)$ it easily follows that for $\lambda \in \mathbb{R}$ we have $a(\lambda) = e(\lambda)$, which proves the lemma. □

Proof of Theorem 2.1. The representation (A.1) and Lemma A.1 yield

$$\mathcal{E}_\mu\Big(\exp\big\{\lambda t W_t\big\}\Big) = \int d\mu(x)\, T_t^\lambda 1\,(x) \tag{A.8}$$

where 1 denotes the function in $\mathcal{H}$ identically equal to one. Using Lemma A.2 and the remark following it we get

$$\lim_{t\to\infty} \frac{1}{t} \log \mathcal{E}_\mu \exp\big\{\lambda t W_t\big\} = \lim_{t\to\infty} \frac{1}{t} \log\Big[\, Z^{-1}\big(\Psi'_0, T_t^\lambda 1\big)\,\Big] = e(\lambda). \tag{A.9}$$

An application of Hölder inequality yields the convexity of e. Moreover, in view of Lemma A.3, the function e is real analytic. The large deviation principle for the family $\{\mathcal{P}_\mu \circ (W_T)^{-1}\}_{T>0}$ now follows from the Gärtner-Ellis theorem, see e.g. [8, Thm. 2.3.6]. Finally, by the smoothness of e and [31, Thm. 26.3], R is essentially strictly convex.

Recall that A'_λ denotes the adjoint in $\mathcal{H}$ of the operator A_λ. By using assumption (**C**), an elementary computation shows that for $\lambda \in \mathbb{R}$ the restriction to $C_0^\infty(\mathbb{R}^n)$ of the operator A'_λ is given by

$$\begin{aligned} A'_\lambda f(x) &= \frac{1}{2}\Delta f(x) - \frac{1}{2}\langle \nabla V(x), \nabla f(x)\rangle - (1+2\lambda)\langle b(x), \nabla f(x)\rangle \\ &\quad + \big[2\lambda(1+\lambda)|b(x)|^2 - (1+\lambda)\nabla\cdot b(x)\big] f(x). \end{aligned}$$

In particular, by Lemma A.1, $A'_\lambda = A_{-1-\lambda}$. Lemma A.2 then gives $e(\lambda) = e(-1-\lambda)$ and the fluctuation theorem $R(q) - R(-q) = -q$ follows by taking Legendre transforms. □

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References

[1] Azencott R.; *Grandes déviations et applications.* Lectures Notes in Mathematics 774, 1–176. Springer-Verlag, Berlin 1980.

[2] Beals R.W.; *A note on the adjoint of a perturbed operator.* Bull. Amer. Math. Soc. **70**, 314–315 (1964).

[3] Bertini L., De Sole A., Gabrielli G., Jona-Lasinio G., Landim C.; *Large deviations of the empirical current in interacting particle systems.* Theory Probab. Appl. **51**, 2–27 (2007).

[4] Bodineau, T., Lefevere, R.; *Large deviations of lattice Hamiltonian dynamics coupled to stochastic thermostats.* J. Stat. Phys. **133**, 1–27 (2008).

[5] Braides A.; *Γ-convergence for beginners.* Oxford University Press, Oxford 2002.

[6] Carmona P.; *Existence and uniqueness of an invariant measure for a chain of oscillators in contact with two heat baths.* Stochastic Process. Appl. **117**, 1076–1092 (2007).

[7] Ciliberto S., Laroche C.; *An experimental test of the Gallavotti-Cohen fluctuation theorem.* J. Physique IV, **8**, 215–219 (1998).

[8] Dembo A., Zeitouni O.; *Large deviations techniques and applications.* Second edition. Springer-Verlag, New York 1998.

[9] Derrida B.; *Non equilibrium steady states: fluctuations and large deviations of the density and of the current.* J. Stat. Mech. Theory Exp. **7**, P07023 (2007).

[10] Eckmann J.-P., Hairer M.; *Spectral properties of hypoelliptic operators.* Comm. Math. Phys. **235**, 233-253 (2003).

[11] Engel K.J., Nagel R.; *One-Parameter semigroups for linear evolution equations.* Springer-Verlag, New York 2000.

[12] Freidlin M.I., Wentzell A.D.; *Random perturbations of dynamical systems.* Springer-Verlag, Berlin 1998.

[13] Gallavotti G., Cohen E.G.D.; *Dynamical ensembles in stationary states.* J. Statist. Phys. **80**, 931–970 (1995).

[14] Hairer M.; *How hot can a heat bath get?* Comm. Math. Phys. **292**, 131–177 (2009).

[15] Hasminski R.Z.; *Stochastic stability of differential equations.* Sijthoff and Noordhoff, Alphen aan den Rijn 1980.

[16] Jiang D.-Q., Qian M., Qian M.-P.; *Mathematical theory of nonequilibrium steady states. On the frontier of probability and dynamical systems.* Lecture Notes in Mathematics, 1833. Springer-Verlag, Berlin, 2004.

[17] Kato T.; *Perturbation theory for linear operators.* Springer-Verlag, Berlin 1995.

[18] Kurchan J.; *Fluctuation theorem for stochastic dynamics.* J. Phys. A: Math. Gen. **31**, 3719–3729 (1998).
[19] Kurchan J.; *Gallavotti-Cohen theorem, chaotic hypothesis and the zero-noise limit.* J. Stat. Phys. **128**, 1307–1320 (2007).
[20] Lebowitz J.L., Spohn H.; *A Gallavotti-Cohen-type symmetry in the large deviation functional for stochastic dynamics.* J. Statist. Phys. **95**, 333–365 (1999).
[21] Maes C.; *The fluctuation Theorem as a Gibbs Property.* J. Statist. Phys. **95**, 367–392 (1999).
[22] Maes C., Redig F., Van Moffaert A.; *On the definition of entropy production, via examples.* J. Math. Phys. **41**, 1528–1554 (2000).
[23] Maes C., Netočný K.; *Time reversal and entropy.* J. Statist. Phys. **110**, 269–310 (2003).
[24] Maes C., Netočný K., Verschuere M.; *Heat conduction networks.* J. Statist. Phys. **111**, 1219–1244 (2003).
[25] Nagel R. (ed.); *One-parameter semigroups of positive operators.* Lectures Notes in Mathematics 1184. Springer-Verlag, Berlin 1986.
[26] Protter M.H., Weinberger H.F.; *Maximum principles in differential equations.* Springer-Verlag, New York 1984.
[27] Reed M., Simon B.; *Methods of modern mathematical physics. Vol. II and IV.* Academic Press, San Diego 1978.
[28] Revuz D., Yor M.; *Continuous martingales and Brownian motion.* Springer-Verlag, Berlin 1991.
[29] Rey-Bellet L., Thomas L.E.; *Asymptotic behavior of thermal nonequilibrium steady states for a driven chain of anharmonic oscillators.* Comm. Math. Phys. **215**, 1–24 (2000).
[30] Rey-Bellet L., Thomas L.E.; *Fluctuations of the entropy production in anharmonic chains.* Ann. Henri Poincaré **3**, 483–502 (2002).
[31] Rockafellar R.T.; *Convex analysis.* Princeton University Press, Princeton 1970.
[32] Wu, L.; *Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems.* Stochastic Process. Appl. **91**, 205–238 (2001).
[33] van Zon, R., Cohen, E.G.D.; *Non-equilibrium thermodynamics and fluctuations. News and expectations in thermostatistics.* Phys. A **340**, 66–75 (2004).

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