SIAM J. MATH. ANAL.
Vol. 44, No. 2, pp. 926–945

BOUNDARY EFFECTS IN THE GRADIENT THEORY OF PHASE TRANSITIONS*

LORENZO BERTINI†, PAOLO BUTTÀ†, AND ADRIANA GARRONI†

Abstract. We consider the van der Waals' free energy functional, with scaling parameter ε, in the plane domain $\mathbb{R}_+ \times \mathbb{R}_+$, with inhomogeneous Dirichlet boundary conditions. We impose the two stable phases on the horizontal boundaries $\mathbb{R}_+ \times \{0\}$ and $\mathbb{R}_+ \times \{+\infty\}$, and free boundary conditions on $\{+\infty\} \times \mathbb{R}_+$. Finally, the datum on $\{0\} \times \mathbb{R}_+$ is chosen in such a way that the interface between the pure phases is pinned at some point $(0, y)$. We show that there exists a critical scaling, $y = y_\varepsilon$, such that, as $\varepsilon \to 0$, the competing effects of repulsion from the boundary and penalization of gradients play a role in determining the optimal shape of the (properly rescaled) interface. This result is achieved by means of an asymptotic development of the free energy functional. As a consequence, such analysis is not restricted to minimizers but also encodes the asymptotic probability of fluctuations.

Key words. gradient theory of phase transitions, development by Γ-convergence, boundary layers

AMS subject classifications. 82B26, 49J45, 35J20, 82B24

DOI. 10.1137/110821469

1. Introduction. The van der Waals' theory of phase transitions [9, 14] is based on the functional

$$E(u) = \int \Big[\big|\nabla u\big|^2 + V(u)\Big]\,\mathrm{d}r\,, \tag{1.1}$$

where the scalar field $u = u(r)$, $r \in \mathbb{R}^d$, represents the local order parameter and $V(u)$ is a smooth, symmetric, double well potential whose minimum value, chosen to be zero, is attained at $u_\pm$; we also assume $V''(u_\pm) > 0$. By introducing a scaling parameter $\varepsilon > 0$, which is interpreted as the ratio between the microscopic and the macroscopic scales, a most relevant issue is the asymptotic behavior of the sequence of functionals

$$E_\varepsilon(u) = \int \Big[\varepsilon\big|\nabla u\big|^2 + \frac{1}{\varepsilon}V(u)\Big]\,\mathrm{d}r\,, \tag{1.2}$$

in the sharp interface limit $\varepsilon \to 0$. This was first analyzed in [12] and extensively studied afterwards; see [1] for a review. The limiting functional turns out to be finite only if u is a function of bounded variation taking values in $\{u_-, u_+\}$. For u in this set, the limiting functional is furthermore given by $C_V\,\mathcal{H}^{d-1}(\mathcal{S}_u)$, where $\mathcal{S}_u$ denotes the jump set of u and $\mathcal{H}^{d-1}(\mathcal{S}_u)$ is its $(d-1)$-dimensional Hausdorff measure. The surface energy density $C_V > 0$ is finally given by

$$C_V = \int_{u_-}^{u_+} 2\sqrt{V(a)}\,\mathrm{d}a\,. \tag{1.3}$$

*Received by the editors June 28, 2011; accepted for publication (in revised form) January 30, 2012; published electronically April 3, 2012.
http://www.siam.org/journals/sima/44-2/82146.html

†Dipartimento di Matematica, SAPIENZA Università di Roma, P.le Aldo Moro 5, 00185 Roma, Italy (bertini@mat.uniroma1.it, butta@mat.uniroma1.it, garroni@mat.uniroma1.it).

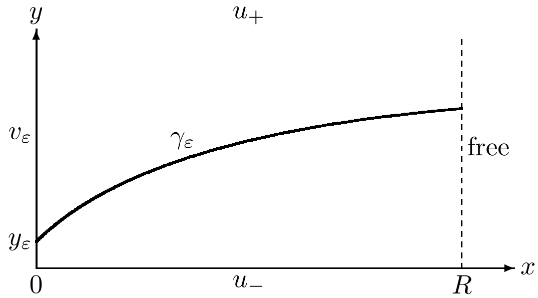

FIG. 1.1. *The domain Ω^R and the corresponding boundary conditions. The zero of v_ε is y_ε, and γ_ε is the interface.*

For any given limiting configuration an optimal sequence can be constructed by making the transition from the value u_- to the value u_+ in the direction ν orthogonal to the interface with a one-dimensional profile $\overline{m}(\frac{r\cdot\nu}{\varepsilon})$. Here $\overline{m}$, the so-called instanton, is the minimizer of the one-dimensional van der Waals' energy (1.1) with boundary conditions $u_\pm$ at $\pm\infty$, satisfying $\overline{m}(0) = 0$.

As proved in [11], when E_ε is considered together with Dirichlet boundary conditions, the latter contribute to the limiting functional with a term taking into account the discrepancy between the pure phases $\{u_-, u_+\}$ in the interior of the domain and the prescribed boundary data. We can regard this term as the cost associated with an interface localized at the boundary. In particular, when the boundary data take values in the pure phases $\{u_-, u_+\}$, this cost coincides with the one in the bulk.

Consider a geometry in which the minimizer of the limiting functional is obtained with an interface localized at the boundary. Of course, when ε is small but strictly positive, the minimizer of E_ε is smooth and the transition between the pure phases takes place in a thin layer close to the boundary. The purpose of the present paper is a detailed description, in the two-dimensional case, of such a boundary effect by means of an asymptotic development of E_ε. In particular, such analysis is not restricted to minimizers but also encodes the asymptotic probability of fluctuations.

We consider the following geometry; see Figure 1. As basic domain we choose $\Omega^R = (0, R) \times (0, +\infty)$, $R > 0$, and denote by x and y the horizontal and the vertical coordinates. We impose the phase u_- on $(0, R)\times\{0\}$, the phase u_+ on $(0, R)\times\{+\infty\}$, and free boundary conditions on $\{R\} \times (0, +\infty)$. Finally, the trace on $\{0\} \times (0, +\infty)$ is given by a suitable (e.g., monotone) continuous function $v_\varepsilon\colon [0, +\infty) \to [u_-, u_+]$ satisfying $v_\varepsilon(0) = u_-$, $v_\varepsilon(+\infty) = u_+$.

We denote by $E_\varepsilon(\cdot, \Omega^R)$ the functional in (1.2) on the domain Ω^R with these boundary conditions and let u^*_ε be a minimizer of $E_\varepsilon(\cdot, \Omega^R)$. Assume that v_ε has a unique zero at y_ε and let γ_ε be the zero level set of u^*_ε. Observe that γ_ε is a subset of the closure of Ω^R. We shall refer to it as the *interface*, and in the following heuristic discussion we assume that it is the graph of some function on $[0, R]$, still denoted by γ_ε. The boundary condition on $\{0\} \times (0, +\infty)$ pins the interface at the point $(0, y_\varepsilon)$, i.e., $\gamma_\varepsilon(0) = y_\varepsilon$. We assume that y_ε converges to zero as $\varepsilon \to 0$. The result in [11] then implies that the interface approaches the interval $[0, R] \times \{0\}$ in the limit $\varepsilon \to 0$, i.e., $\gamma_\varepsilon \to 0$. Our aim is a detailed analysis of this convergence, which includes the correction for finite ε due to the boundary condition. There are two competing effects. The boundary datum u_- on $(0, R) \times \{0\}$ effectively repels γ_ε; indeed, in order to minimize the energy along the one-dimensional sections $\{x\} \times (0, +\infty)$, $x \in (0, R)$, the zero of $u_\varepsilon^*(x, \cdot)$ should be as large as possible. On the other hand, the convergence of γ_ε to the flat interface penalizes the gradient of γ_ε. We show that there exists a critical scaling for y_ε such that γ_ε, properly rescaled, converges to a nontrivial profile for which both effects play a role.

In the spirit of the developments by Γ-convergence [3, 6, 7], we introduce the excess energy

$$\widetilde{E}_\varepsilon(u, \Omega^R) = K_\varepsilon \big[E_\varepsilon(u, \Omega^R) - C_V R \big] \tag{1.4}$$

and look for a sequence $K_\varepsilon \to +\infty$ for which $\widetilde{E}_\varepsilon$ has a nontrivial limit. In order to complete this program, however, we need to properly rescale the variables. The identification of the correct scaling is based on the following ansatz, which is suggested by the construction of the optimal sequence in the sharp interface limit of E_ε. The interface γ_ε satisfies $\gamma_\varepsilon(x) \approx y_\varepsilon + \varepsilon\, \phi(x)$, and on each vertical section the function $u_\varepsilon^*(x, \cdot)$ minimizes the corresponding energy with the constraint $u_\varepsilon^*(x, \gamma_\varepsilon(x)) = 0$. We thus perform the change of variable $y \mapsto (y - y_\varepsilon)/\varepsilon$, getting

$$\begin{aligned} F_{\varepsilon,R}(u) = K_\varepsilon \Bigg\{ \varepsilon^2 \int_0^R \int_{-\frac{y_\varepsilon}{\varepsilon}}^{+\infty} (\partial_x u)^2 \mathrm{d}y\, \mathrm{d}x \\ + \int_0^R \left[\int_{-\frac{y_\varepsilon}{\varepsilon}}^{+\infty} \Big((\partial_y u)^2 + V(u) \Big) \mathrm{d}y - C_V \right] \mathrm{d}x \Bigg\}. \end{aligned} \tag{1.5}$$

The above expression suggests that in order to appreciate the variations in the horizontal direction we have to choose $K_\varepsilon = \varepsilon^{-2}$. Moreover, the analysis of the one-dimensional case in [5] implies that the second term on the right-hand side of (1.5) is of the order $\exp\{-\beta_V \varepsilon^{-1} y_\varepsilon\}$, where $\beta_V = \sqrt{2V''(u_+)}$. We therefore conclude that the critical scaling for y_ε is given by $y_\varepsilon \approx \frac{2}{\beta_V} \varepsilon \log \varepsilon^{-1}$.

The functional $F_{\varepsilon,R}$ in (1.5) also makes sense when $R = +\infty$, i.e., Ω^R is the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$. In this case we denote it simply by F_ε. The asymptotic analysis of $F_{\varepsilon,R}$ for any $R > 0$ is then formulated directly in terms of F_ε with a local topology in the horizontal variable. In this paper we prove a Γ-convergence result for F_ε; referring to the next section for the precise statement, here we discuss informally our results.

If $y_\varepsilon \gg \frac{2}{\beta_V} \varepsilon \log \varepsilon^{-1}$, we show that the repulsion due to the boundary is not seen in the limiting functional (see Remark 2.2 for the precise statement). In particular, the rescaled profile corresponding to the minimizer u_ε^* is asymptotically flat, i.e., $\phi = 0$. In the critical scaling $y_\varepsilon = \frac{2}{\beta_V} \varepsilon \log \varepsilon^{-1}$, we prove that the Γ-limit of the functionals F_ε is finite only on functions u of the form $u(x, y) = \overline{m}(y - \phi(x))$, with $\phi(0) = 0$. On these functions the limiting functional is furthermore given by

$$\int_0^{+\infty} \left[\frac{C_V}{2}\, \phi'(x)^2 + B_V\, \mathrm{e}^{-\beta_V \phi(x)} \right] \mathrm{d}x \tag{1.6}$$

for a suitable constant $B_V > 0$ that can be computed explicitly. As a consequence of this Γ-convergence result, we deduce the sharp asymptotic, as $\varepsilon \to 0$ and $R \to +\infty$, for the minimizer u_ε^* of the original functional $E_\varepsilon(\cdot, \Omega^R)$. Namely,

$$u_\varepsilon^*(x, y) \approx \overline{m}\left(\frac{y - \gamma_\varepsilon(x)}{\varepsilon} \right), \qquad \gamma_\varepsilon(x) \approx y_\varepsilon + \varepsilon\, \phi^*(x)\,, \tag{1.7}$$

where ϕ^* is the minimizer of the energy (1.6) with the boundary condition $\phi(0) = 0$. The function ϕ^* can be computed explicitly. Indeed, as follows by simple calculations,

$$\phi^*(x) = \frac{2}{\beta_V} \log\left(1 + \beta_V \sqrt{\frac{B_V}{2C_V}}\, x\right). \tag{1.8}$$

In the case $y_\varepsilon \ll \frac{2}{\beta_V}\varepsilon \log \varepsilon^{-1}$, when x is close to zero, the repulsion from the boundary is much stronger than the penalization on the gradients of γ_ε. This implies that for each fixed x close to zero we have $\gamma_\varepsilon(x) - y_\varepsilon \gg \varepsilon$. On the other hand, if x is such that $\gamma_\varepsilon(x) \approx \frac{2}{\beta_V}\varepsilon \log \varepsilon^{-1}$, we are back in the situation described by the critical scaling. We therefore expect, but do not prove here, that in this regime the asymptotic expression of the interface γ_ε for x bounded away from 0 still has the form $\gamma_\varepsilon(x) \approx \frac{2}{\beta_V}\varepsilon \log \varepsilon^{-1} + \varepsilon\, \phi(x)$ for some function ϕ satisfying $\phi(0) = -\infty$.

We conclude with a few remarks on the relationship of the problem considered here with some (microscopic) statistical mechanics models. In the context of short-range, Ising-like models, the statistical properties of an interface above a wall have been studied mostly for the so-called *effective interface models*; see [10] for a review. These models are obtained by assuming that the interface can be described as the graph of some function $\phi\colon \Lambda \to \mathbb{R}_+$, where Λ is a finite subset of the lattice $\mathbb{Z}^{d-1}$. One then introduces a Gibbs measure on the set of the interface configurations, with a short-range energy term penalizing the gradients of ϕ, and analyzes the asymptotic behavior of this measure as Λ invades $\mathbb{Z}^{d-1}$. While the energy is minimized by an interface localized at the wall, i.e., $\phi = 0$, the presence of the fluctuations induces a repulsion; e.g., the expected value of ϕ diverges as $\Lambda \uparrow \mathbb{Z}^{d-1}$. This effect is referred to as *entropic repulsion*: for the interface it is more convenient to have some room to fluctuate rather than to minimize the energy.

The asymptotic (1.7) does not reflect an entropic repulsion effect. In the case of the van der Waals' functional, the repulsion from the wall is in fact due to an energetic effect induced by the boundary conditions. The case of long-range, Kac-like models is, on the other hand, much closer to the problem considered here. Indeed, on a suitable mesoscopic scale the behavior of those models is well described by a free energy functional which, although nonlocal, has features similar to (1.2); see [13]. In particular, the corresponding sharp interface limit has been analyzed in [2], where it is shown that the Γ-limit of the free energy functional is proportional to the perimeter of the interface between the pure phases, the proportionality constant identifying the surface tension. As far as we know, the asymptotic behavior of an interface close to a wall has not been analyzed in detail for systems with Kac-type interactions, but it seems reasonable that the effects discussed here are also relevant in such a situation.

2. The main result.

For the sake of concreteness, we restrict the analysis to the paradigmatic case of the symmetric double well potential; i.e., we choose

$$V(u) = \left(u^2 - 1\right)^2, \tag{2.1}$$

which attains its minimum at $u_\pm = \pm 1$. With this choice, the instanton $\overline{m}$ is given by $\overline{m}(y) = \tanh y$, and elementary computations show $C_V = \frac{8}{3}$, $\beta_V = 4$, and $B_V = 16$ [5, Appendix A].

As reference domain we choose the quadrant of $\mathbb{R}^2$ given by $\Omega_\ell = (0, +\infty) \times (-\ell, +\infty)$, $\ell > 0$. The parameter ℓ has been introduced in such a way that the zero of the trace on $\{0\} \times (-\ell, +\infty)$ approaches zero as $\varepsilon \to 0$. Accordingly, the asymptotic expansion of the functional will be discussed in the fine tuning $\ell = \frac{1}{2}\log\varepsilon^{-1} + O(1)$, which corresponds to the critical scaling discussed in the introduction.

Given $m\colon (-\ell,+\infty) \to \mathbb{R}$, we introduce the one-dimensional functional

$$\mathcal{F}_\ell(m) := \int_{-\ell}^{+\infty} \Big[(m')^2 + V(m)\Big]\,\mathrm{d}y\,. \tag{2.2}$$

With this notation, we can rewrite the functional in (1.5) for $R = +\infty$ as

$$F_\varepsilon(u) := \int_0^{+\infty}\int_{-\ell_\varepsilon}^{+\infty} (\partial_x u)^2\,\mathrm{d}y\,\mathrm{d}x + \varepsilon^{-2}\int_0^{+\infty}\bigg[\mathcal{F}_{\ell_\varepsilon}(u(x,\cdot)) - \frac{8}{3}\bigg]\,\mathrm{d}x\,, \tag{2.3}$$

where $\ell_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We will study the asymptotic behavior, in terms of Γ-convergence, of F_ε subject to the following boundary conditions:

$$\begin{cases} u(0,y) = w_\varepsilon(y), & y \in [-\ell_\varepsilon,+\infty)\,,\\ u(x,-\ell_\varepsilon) = -1, & x \in (0,+\infty)\,,\\ u(x,\cdot) - 1 \in L^2((-\ell_\varepsilon,+\infty)), & x \in (0,+\infty)\,, \end{cases} \tag{2.4}$$

for a suitable continuous function $w_\varepsilon\colon [-\ell_\varepsilon,+\infty) \to \mathbb{R}$ with $w_\varepsilon(-\ell_\varepsilon) = -1$. We shall regard u as a function on $[0,+\infty)\times\mathbb{R}$ by setting $u = -1$ on $[0,+\infty)\times(-\infty,-\ell_\varepsilon)$. Accordingly, w_ε is also regarded as a function in $\mathbb{R}$ by setting $w_\varepsilon = -1$ on $(-\infty,-\ell_\varepsilon)$.

We set $\chi(x,y) = \chi(y) := \mathrm{sgn}(y)$ and define the affine space

$$X := \{u \,:\, u - \chi \in L^2((0,R)\times\mathbb{R}) \text{ for any } R > 0\}\,,$$

endowed with the metric

$$d_X(u,v) = \sum_n 2^{-n}\left(1 \wedge \|u-v\|_{L^2((0,n)\times\mathbb{R})}\right).$$

We shall then regard F_ε as a functional on X, which takes value $+\infty$ whenever u does not satisfy the boundary conditions (2.4) or u is not identically equal to -1 on $[0,+\infty)\times(-\infty,-\ell_\varepsilon)$. Note that $F_\varepsilon(u) < +\infty$ implies $u \in H^1((0,R)\times(-L,L))$ for any $R, L > 0$, and therefore the condition $u(0,\cdot) = w_\varepsilon$ can be understood in terms of traces. It turns out that the Γ-limit of F_ε depends on

$$\alpha := \lim_{\varepsilon\to 0}\bigg[\ell_\varepsilon - \frac{1}{2}\log\varepsilon^{-1}\bigg].$$

First we introduce the limiting functional. Given $\alpha \in \mathbb{R}$, we let $\mathcal{G}^\alpha : C([0,+\infty)) \to [0,+\infty]$ be the lower semicontinuous—with respect to the uniform convergence on compact subsets of $[0,+\infty)$—functional defined by

$$\mathcal{G}^\alpha(\phi) = \int_0^{+\infty}\bigg[\frac{4}{3}\phi'(x)^2 + 16\,e^{-4\alpha}\,\mathrm{e}^{-4\phi(x)}\bigg]\,\mathrm{d}x\,. \tag{2.5}$$

Recall that $\overline{m}$ is the minimizer of the one-dimensional van der Waals' energy (1.1) with boundary conditions ± 1 at $\pm\infty$ satisfying $\overline{m}(0) = 0$, and denote by $\overline{m}_z$ its translation by $z \in \mathbb{R}$, i.e., $\overline{m}_z(y) := \overline{m}(y-z)$ for $y \in \mathbb{R}$. We then let $F^\alpha\colon X \to [0,+\infty]$ be defined by

$$F^\alpha(u) := \begin{cases} \mathcal{G}^\alpha(\phi) & \text{if } u = \overline{m}_\phi \text{ for some } \phi \in C([0,+\infty)), \text{ with } \phi(0) = 0,\\ +\infty & \text{otherwise,} \end{cases} \tag{2.6}$$

where we understand that $\overline{m}_\phi(x,y) = \overline{m}_{\phi(x)}(y)$.

THEOREM 2.1. *Assume* $\lim_{\varepsilon\to 0}\left[\ell_\varepsilon - \frac{1}{2}\log\varepsilon^{-1}\right] = \alpha$,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(\mathcal{F}_{\ell_\varepsilon}(w_\varepsilon) - \frac{8}{3}\right) = 0, \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} w_\varepsilon(0) = 0\,. \tag{2.7}$$

The following statements hold.

(Compactness). *If a sequence* u_ε *satisfies* $\limsup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, *then for any* $R > 0$,

$$\lim_{\varepsilon\to 0}\left\|u_\varepsilon - \overline{m}_{\phi_\varepsilon}\right\|^2_{L^2((0,R)\times\mathbb{R})} = 0 \tag{2.8}$$

for some sequence ϕ_ε *precompact in* $C([0,+\infty))$, *satisfying* $\phi_\varepsilon(0)\to 0$ *as* $\varepsilon\to 0$. *In particular, the sequence* F_ε *is equicoercive in* X.

(Γ-convergence). *The sequence* F_ε Γ*-converges to* F^α *as* $\varepsilon\to 0$*; i.e., for any* $\phi\in C([0,+\infty))$ *with* $\phi(0) = 0$, *we have the following:*

(i) (Γ-liminf). *If* $u_\varepsilon \to \overline{m}_\phi$ *in* X, *then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \mathcal{G}^\alpha(\phi).$$

(ii) (Γ-limsup). *There exists* $u_\varepsilon \to \overline{m}_\phi$ *in* X *such that*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \mathcal{G}^\alpha(\phi).$$

Remark 2.2. The above result also holds in the case $\alpha = +\infty$. More precisely, if ℓ_ε satisfies $\lim_{\varepsilon\to 0}\left[\ell_\varepsilon - \frac{1}{2}\log\varepsilon^{-1}\right] = +\infty$ and w_ε satisfies (2.7), the statements in Theorem 2.1 hold true with $\mathcal{G}^\alpha(\phi)$ replaced by $\frac{4}{3}\int_0^{+\infty}|\phi'|^2\mathrm{d}x$. This is due to the fact that the relevant estimates needed in the proof of Theorem 2.1 are uniform with respect to $\alpha\in[\alpha_0,+\infty)$, $\alpha_0\in\mathbb{R}$.

By standard properties of Γ-convergence (see, e.g., [6, Theorem 1.21]), the above results imply the convergence of the minimizers of F_ε to the (unique) minimizer of the corresponding limiting functionals. In particular, in the critical scaling, the repulsion of the boundary conditions on $\partial\Omega_{\ell_\varepsilon}$ competes with the tendency of being flat, and the minimizers of F_ε converge in X to $\overline{m}_{\phi^*_\alpha}$, where $\phi^*_\alpha(x) = \frac{1}{2}\log\left(1 + \mathrm{e}^{-2\alpha}\,4\sqrt{3}\,x\right)$ is the minimizer of $\mathcal{G}^\alpha$ with the boundary condition $\phi(0) = 0$. On the other hand, when $\ell_\varepsilon \gg \frac{1}{2}\log\varepsilon^{-1}$, the repulsion of the boundary conditions on $\partial\Omega_{\ell_\varepsilon}$ is not felt, and the optimal interface is flat. This is consistent with the fact that as $\alpha\to+\infty$ we have $\phi^*_\alpha(x)\to 0$, $x\geq 0$. To illustrate the difficulties in analyzing the asymptotic behavior of F_ε in the subcritical case $\ell_\varepsilon - \frac{1}{2}\log\varepsilon^{-1}\to-\infty$, it is instructive to discuss the asymptotics of $\mathcal{G}^\alpha$ as $\alpha\to-\infty$. We first observe that in this case $\phi^*_\alpha(x) + \alpha \to \frac{1}{2}\log\left(4\sqrt{3}\,x\right)$, $x > 0$. However, the limit of $\phi^*_\alpha + \alpha$ cannot be characterized in variational terms as a minimizer of a suitable limiting functional for $\mathcal{G}^\alpha$. Indeed, the amount of energy of ϕ^*_α stored in any neighborhood of zero diverges as $\alpha\to-\infty$, while the energy stored in any interval not containing zero remains finite and strictly positive.

The rest of the paper is organized in the following way. Section 3 and the appendix are devoted to a detailed study of the asymptotic expansion by Γ-convergence of the one-dimensional functional $\mathcal{F}_\ell$ in (2.2) as $\ell\to+\infty$. Such analysis is a preliminary tool for the proof of Theorem 2.1, which is the content of section 4 (compactness) and section 5 (Γ-convergence).

3. One-dimensional problem. In this section we analyze the one-dimensional functional $\mathcal{F}_\ell$ defined in (2.2). The development by Γ-convergence of $\mathcal{F}_\ell$ as $\ell \to +\infty$ is studied in [5]. Here we prove quantitative estimates related to that asymptotic expansion. Hereafter we denote $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ by L^2 and H^1, respectively.

Recalling $\chi(y) = \mathrm{sgn}(y)$, we set $\mathcal{X} := \{m\colon m-\chi \in L^2\}$, which we consider endowed with the strong L^2-topology. Given $\ell > 0$, we let $\mathcal{X}_\ell \subset \mathcal{X}$ be the closed subspace defined by

$$\mathcal{X}_\ell = \{m \in \mathcal{X} \ :\ m(y) = -1 \ \text{ if } y \in (-\infty, -\ell)\}\,. \tag{3.1}$$

We then regard $\mathcal{F}_\ell$ as a functional on $\mathcal{X}$ which takes value $+\infty$ whenever $m \notin \mathcal{X}_\ell$. It is simple to show that the sequence of functionals $\mathcal{F}_\ell$ Γ-converges to the functional $\mathcal{F}\colon \mathcal{X} \to [0, +\infty]$, defined by

$$\mathcal{F}(m) := \int_{-\infty}^{+\infty} \Big[(m')^2 + V(m)\Big]\, \mathrm{d}y. \tag{3.2}$$

By the well-known Modica–Mortola trick [12],

$$\min \mathcal{F} = C_V = \frac{8}{3}, \qquad \arg\min \mathcal{F} = \{\overline{m}_z,\ z \in \mathbb{R}\}\,. \tag{3.3}$$

Given $z \in (-\ell, +\infty)$, we define

$$m_z^\ell = \arg\min \big\{\mathcal{F}_\ell(m)\colon\ m \in \mathcal{X}_\ell,\ m(z) = 0\big\}\,, \tag{3.4}$$

observing that the minimizer is unique. We introduce the one-dimensional manifold $\mathcal{M}^\ell := \{m_z^\ell\colon z \in (-\ell, +\infty)\}$ in $\mathcal{X}$. Sometimes, we will use the notation $m_z^\ell(\cdot) = m^\ell(\cdot, z)$. If $y > z$, then $m_z^\ell(y) = \overline{m}_z(y)$. Moreover, for $y \in (-\ell, z)$, m_z^ℓ coincides with the (unique) solution to the following boundary value problem:

$$\begin{cases} -2m'' + V'(m) = 0 & \text{in } (-\ell, z)\,, \\ m(-\ell) = -1\,,\ m(z) = 0\,. \end{cases} \tag{3.5}$$

We next state, referring the reader to the appendix for the proof, sharp estimates concerning m_z^ℓ and its convergence to $\overline{m}_z$.

PROPOSITION 3.1. *There exists a constant A such that, for any $\ell > 0$ and $z \in \mathbb{R}$ satisfying $\ell + z \geq 1$,*

$$\sup_{y \in (-\ell, z)} |m_z^\ell(y) - \overline{m}_z(y)| \leq A\mathrm{e}^{-2(\ell+z)}, \tag{3.6}$$

$$\sup_{y \in (-\ell, z)} |\partial_z m_z^\ell(y) + \overline{m}_z'(y)| \leq A\mathrm{e}^{-2(\ell+z)}, \tag{3.7}$$

$$\sup_{y \in (-\ell, z)} |\partial_{zz} m_z^\ell(y) - \overline{m}_z''(y)| \leq A\mathrm{e}^{-2(\ell+z)}, \tag{3.8}$$

$$\big[(m_z^\ell)'\big](z) + \big[\partial_z m_z^\ell\big](z) = 0, \qquad \big|\big[(m_z^\ell)'\big](z)\big| \leq A\mathrm{e}^{-4(\ell+z)}, \tag{3.9}$$

where $[f](z)$ denotes the jump of the function f at z. Moreover, for any ℓ, z_1, and z_2 such that $(z_1 + \ell) \wedge (z_2 + \ell) \geq 1$,

$$\frac{1}{A}(|z_1 - z_2|^2 \wedge |z_1 - z_2|) \leq \|m_{z_1}^\ell - m_{z_2}^\ell\|_{L^2}^2 \leq A(|z_1 - z_2|^2 \wedge |z_1 - z_2|)\,. \tag{3.10}$$

Remark 3.2. Since $m_z^\ell(y) = \bar{m}_z(y)$ for $y > z$ and $m_z^\ell(y) = -1$ for $y < -\ell$, the above bounds and (3.5) yield that m_z^ℓ converges to $\bar{m}_z$ in H^2. In particular,

$$\lim_{\ell\to+\infty} \int_{-\ell}^{+\infty} (\partial_z m_z^\ell)^2 \,\mathrm{d}y = \int_{-\infty}^{+\infty} \bar{m}_z'(y)^2 \,\mathrm{d}y = \frac{4}{3}\,, \tag{3.11}$$

uniformly with respect to $z \in [\bar{z}_\ell, +\infty)$ with $\ell + \bar{z}_\ell \to +\infty$.

The following lemma is proved in [5, Lemma A.1]. It is the key ingredient in studying the development by Γ-convergence of the functionals $\mathcal{F}_\ell$.

LEMMA 3.3. *Let $z \in \mathbb{R}$ and let z_ℓ be a sequence converging to z. Then*

$$\lim_{\ell\to+\infty} \mathrm{e}^{4\ell}\Big[\mathcal{F}_\ell(m_{z_\ell}^\ell) - \frac{8}{3}\Big] = 16\,\mathrm{e}^{-4z},$$

and, given $\bar{z} \in \mathbb{R}$, this limit is uniform for $z \in [\bar{z}, +\infty)$.

The notion of center for functions in $\mathcal{X}_\ell$, introduced in [8], will play an important role in our analysis.

DEFINITION 3.4. *Given $m \in \mathcal{X}_\ell$, we say that $\zeta \in (-\ell, +\infty)$ is a* center *of m if*

$$\zeta \in \arg\min\left\{\|m - m_z^\ell\|_{L^2}^2 \colon z \in (-\ell, +\infty)\right\}.$$

In particular, the function m_ζ^ℓ is an L^2-projection of m on the manifold $\mathcal{M}^\ell$.

Referring to [4] for a dynamical interpretation of the above definition, we simply note that if ζ is a center of m, then the following orthogonality condition holds:

$$\int_{-\ell}^{+\infty} \big[m(y) - m_\zeta^\ell(y)\big]\, \partial_z m_\zeta^\ell(y) \,\mathrm{d}y = 0\,, \tag{3.12}$$

where $\partial_z m_\zeta^\ell(y) = \partial_z m_z^\ell(y)\big|_{z=\zeta}$. We next introduce a suitable neighborhood of the manifold $\mathcal{M}^\ell$, which takes into account the boundary conditions (3.1). More precisely, given $\delta > 0$ and $k > 0$, we set

$$\mathcal{T}^\ell(\delta, k) := \left\{m \in \mathcal{X}_\ell \colon \exists\, z \in (-\ell + k, +\infty) \text{ such that } \|m - m_z^\ell\|_{H^1} < \delta\right\}. \tag{3.13}$$

The following result shows, in particular, that if m is such that $\mathcal{F}_\ell(m)$ is close to its minimum, then m is close to the manifold $\mathcal{M}^\ell$.

THEOREM 3.5. *The following statements hold.*

(i) *For each $\delta > 0$ and $\kappa > 0$ there exist $\eta > 0$ and $\ell_0 > 0$ such that if $\mathcal{F}_\ell(m) - \frac{8}{3} < \eta$ for some $\ell \geq \ell_0$, then $m \in \mathcal{T}^\ell(\delta, \kappa)$.*

(ii) *There exist constants ℓ_1, δ_1, κ_1, and C_1 such that, for all $\ell \geq \ell_1$, $\delta \leq \delta_1$, and $\kappa \geq \kappa_1$, if $m \in \mathcal{T}^\ell(\delta, \kappa)$, then the center ζ of m is unique and satisfies $\zeta > -\ell + \kappa - 2\delta$, and*

$$\|m - m_\zeta^\ell\|_{H^1}^2 \leq C_1 \Big[\mathcal{F}_\ell(m) - \mathcal{F}_\ell(m_\zeta^\ell) + \mathrm{e}^{-4(\ell+\zeta)} \|m - m_\zeta^\ell\|_{H^1}\Big]. \tag{3.14}$$

(iii) *For each $\bar{z} \in \mathbb{R}$ there exist two positive constants C_2 and ℓ_2 such that, for any $\ell > \ell_2$ and $z \in [\bar{z}, +\infty)$,*

$$\mathcal{F}_\ell(m) - \mathcal{F}_\ell(m_z^\ell) \leq C_2 \left(\|m - m_z^\ell\|_{H^1}^2 + \|m - m_z^\ell\|_{H^1}^4 + \mathrm{e}^{-4\ell}\|m - m_z^\ell\|_{H^1}\right)$$

for all $m \in \mathcal{X}_\ell$.

We emphasize that while in statement (ii) of the above theorem ζ denotes the center of m, in statement (iii) z is arbitrary.

Remark 3.6. As a consequence of (3.14) and Lemma 3.3, there exist constants ℓ_0, δ, κ, and C_0 such that, for all $\ell \geq \ell_0$, if $m \in \mathcal{T}^\ell(\delta, \kappa)$, then the center ζ of m is unique and

$$\|m - m_\zeta^\ell\|_{H^1}^2 + \mathrm{e}^{-4(\ell+\zeta)} \leq C_0 \left[\mathcal{F}_\ell(m) - \frac{8}{3}\right]. \tag{3.15}$$

Proof of Theorem 3.5. The proof is split into separate arguments. In what follows we denote by C a strictly positive constant, independent of ℓ and ζ, whose numerical value may change from line to line.

Proof of statement (i), *step* 1. Here we prove that for each $\delta > 0$ there exist $\eta > 0$ and $\ell_0 > 0$ such that if $\mathcal{F}_\ell(m) - \frac{8}{3} < \eta$ for some $\ell \geq \ell_0$, then $\mathrm{dist}_{H^1}(m, \mathcal{M}^\ell) < \delta$. We argue by contradiction and assume that there exist $\delta_0 > 0$ and a sequence m_ℓ with

$$\liminf_{\ell \to +\infty} \mathrm{dist}_{H^1}(m_\ell, \mathcal{M}^\ell) \geq \delta_0 \tag{3.16}$$

such that

$$\limsup_{\ell \to +\infty} \mathcal{F}_\ell(m_\ell) \leq \frac{8}{3}\,. \tag{3.17}$$

Note that by (3.17) the function m_ℓ satisfies the boundary conditions (3.1). We set $z_\ell = \inf\{y \colon m_\ell(y) = 0\}$ and define $\widetilde{m}_\ell(y) = m_\ell(y + z_\ell)$, so that $\widetilde{m}_\ell(0) = 0$. The boundedness of the energy implies that $\widetilde{m}_\ell$ converges, up to a subsequence, to some continuous function m_0, uniformly in compact subsets of $[0, +\infty)$. We will show that $m_0 = \bar{m}$ and that $\widetilde{m}_\ell - \bar{m}$ actually converges to zero in H^1.

Given $\sigma > 0$, we set

$$\begin{aligned} a_\ell &= \sup\{y < z_\ell \;:\; m_\ell(y) > -1 + \sigma\}\,, \\ b_\ell &= \inf\{y > z_\ell \;:\; m_\ell(y) < 1 - \sigma\}\,. \end{aligned}$$

The boundedness of $\int_{-\infty}^{+\infty} V(m_\ell)\, \mathrm{d}y$ implies that $b_\ell - a_\ell \leq C_\sigma$ for some constant C_σ independent of ℓ. This guarantees that the energy of $\widetilde{m}_\ell$ does not escape to infinity. More precisely, using the Modica–Mortola trick,

$$\begin{aligned} \int_{-C_\sigma}^{C_\sigma} \left[|\widetilde{m}_\ell'|^2 + V(\widetilde{m}_\ell)\right]\, \mathrm{d}y &\geq 2 \int_{a_\ell - z_\ell}^{b_\ell - z_\ell} |\widetilde{m}_\ell'| \sqrt{V(\widetilde{m}_\ell)}\, \mathrm{d}y \\ &= 2 \int_{-1+\sigma}^{1-\sigma} \sqrt{V(m)}\, \mathrm{d}m = \frac{8}{3} - 4\sigma^2 \left(1 - \tfrac{\sigma}{3}\right), \end{aligned}$$

we deduce, taking into account (3.17), that

$$\lim_{\ell \to +\infty} \mathcal{F}_\ell(m_\ell) = \frac{8}{3} \tag{3.18}$$

and thence

$$\lim_{\sigma \to 0}\, \limsup_{\ell \to +\infty} \int_{[-C_\sigma, C_\sigma]^\complement} \left[|\widetilde{m}_\ell'|^2 + V(\widetilde{m}_\ell)\right]\, \mathrm{d}y = 0\,. \tag{3.19}$$

Therefore, up to a subsequence,

$$\lim_{\ell\to+\infty}\int_{-\infty}^{+\infty} V(\widetilde{m}_\ell)\,\mathrm{d}y = \int_{-\infty}^{+\infty} V(m_0)\,\mathrm{d}y\,,$$

so that

$$\lim_{\ell\to+\infty}\int_{-\infty}^{+\infty} |\widetilde{m}'_\ell|^2\,\mathrm{d}y = \int_{-\infty}^{+\infty} |m'_0|^2\,\mathrm{d}y\,. \tag{3.20}$$

In particular, by (3.18), $\mathcal{F}(m_0) = \frac{8}{3}$. Since $m_0(0) = 0$, by the uniqueness up to translations of the minimizer of $\mathcal{F}$ (recall (3.3)), $m_0 = \overline{m}$. Now, using (3.19) and the definition of a_ℓ and b_ℓ, we get the convergence of $\widetilde{m}_\ell$ to $\overline{m}$ in L^2. This, together with (3.20) and Remark 3.2, contradicts (3.16) and then concludes the proof of the step.

Proof of statement (i), *step* 2. Here we conclude the proof. Again we argue by contradiction and assume that there exist $\delta, \kappa > 0$ and a sequence $m_\ell \notin \mathcal{T}^\ell(\delta,\kappa)$ such that $\mathcal{F}_\ell(m_\ell) \to \frac{8}{3}$. By step 1 it is enough to consider the case when there exists a sequence $z_\ell \in (-\ell, -\ell+\kappa)$ such that $\|m_\ell - m^\ell_{z_\ell}\|_{H^1} < \delta$. This yields $\|m_\ell - m^\ell_{z_\ell}\|_{L^\infty} < C\delta$, and hence, if z'_ℓ is any zero of m_ℓ, then $|m_{z_\ell}(z'_\ell)| < C\delta$. By Proposition 3.1, this implies $|z_\ell - z'_\ell| < C\delta$. On the other hand, it is easy to see that

$$\min\{\mathcal{F}(m)\colon m(a) = -1\,,\ m(b) = 0\} > \frac{4}{3} \qquad \forall\, a, b\colon -\infty < a < b < +\infty\,.$$

Therefore,

$$\liminf_{\ell\to+\infty} \mathcal{F}_\ell(m_\ell) \ge \min\{\mathcal{F}(m)\colon m(0) = -1,\ m(\kappa + C\delta) = 0\} + \frac{4}{3} > \frac{8}{3}\,.$$

This is a contradiction and concludes the proof of statement (i).

Proof of statement (ii). The uniqueness of the center, for δ_1 small enough and κ_1 large enough, is stated in [4, Proposition 3.1]. The proof follows by a standard implicit function argument [8]. That proposition also guarantees that the center ζ satisfies the bound $\zeta > -\ell + \kappa - 2\delta$.

Let $m \in \mathcal{T}^\ell(\delta,\kappa)$ and let $\zeta \ge -\ell + \kappa - 2\delta$ be the unique center of m. Recalling that $m(y) - m^\ell_\zeta(y) = 0$ for $y \in (-\infty, -\ell]$, we decompose

$$\mathcal{F}_\ell(m) = \mathcal{F}_\ell(m^\ell_\zeta) + I^1_\ell + I^2_\ell + I^3_\ell\,,$$

where

$$I^1_\ell = \int_{-\infty}^{+\infty} \left[2\partial_y m^\ell_\zeta \partial_y(m - m^\ell_\zeta) + V'(m^\ell_\zeta)(m - m^\ell_\zeta)\right]\,\mathrm{d}y\,,$$

$$I^2_\ell = \int_{-\infty}^{+\infty} \left[(\partial_y(m - m^\ell_\zeta))^2 + \frac{1}{2}V''(m^\ell_\zeta)(m - m^\ell_\zeta)^2\right]\,\mathrm{d}y\,,$$

$$I^3_\ell = \int_{-\infty}^{+\infty} \left[\frac{1}{6}V'''(m^\ell_\zeta)(m - m^\ell_\zeta)^3 + \frac{1}{24}V''''(m^\ell_\zeta)(m - m^\ell_\zeta)^4\right]\,\mathrm{d}y\,.$$

The proof will be achieved by analyzing in detail the quadratic form in I^2_ℓ and showing that it can be bounded from below by $\|m - m^\ell_\zeta\|^2_{H^1}$, while the other two terms will be bounded in absolute value.

We first estimate I_ℓ^1. By integration by parts, using (3.5) and (3.9) we get

$$\begin{aligned} |I_\ell^1| &= 2\left|(m(\zeta)-m_\zeta^\ell(\zeta))\left[\partial_y m_\zeta^\ell\right](\zeta)\right| \\ &\le C\mathrm{e}^{-4(\ell+\zeta)}|m(\zeta)-m_\zeta^\ell(\zeta)| \le C\mathrm{e}^{-4(\ell+\zeta)}\|m-m_\zeta^\ell\|_{H^1}\,, \end{aligned} \tag{3.21}$$

where we have used the Sobolev embedding. As for the term I_ℓ^1, the application of the Sobolev embedding yields

$$|I_\ell^3| \le C\left(\|m-m_\zeta^\ell\|_{H^1}^3 + \|m-m_\zeta^\ell\|_{H^1}^4\right). \tag{3.22}$$

Finally, it remains to estimate I_ℓ^2. We will show that

$$I_\ell^2 \ge \frac{1}{C}\|m-m_\zeta^\ell\|_{H^1}^2\,. \tag{3.23}$$

We denote by $\mathcal{H}_\zeta^\ell$ the Schrödinger operator on $L^2((-\ell,+\infty))$ defined as

$$\mathcal{H}_\zeta^\ell = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V''(\overline{m}_\zeta)\,,$$

with domain $H^2((-\ell,+\infty))\cap H_0^1((-\ell,+\infty))$. In what follows, we shall regard the space $L^2((-\ell,+\infty))$ as a subset of L^2 by setting, for every function $\psi\in L^2((-\ell,+\infty))$, $\psi(y)=0$ if $y\in(-\infty,-\ell]$. Let us also set $\varphi=\varphi_{\ell,\zeta}=m-m_\zeta^\ell$. With this notation we rewrite I_ℓ^2 as the quadratic form

$$I_\ell^2 = \langle\varphi,\mathcal{H}_\zeta^\ell\varphi\rangle_{L^2} + \langle\varphi,(V''(m_\zeta^\ell)-V''(\overline{m}_\zeta))\varphi\rangle_{L^2}\,, \tag{3.24}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the L^2-inner product. By (3.6), the second term on the right-hand side of the above equality is bounded in absolute value by $C\mathrm{e}^{-2(\ell+\zeta)}\|\varphi\|_{L^2}^2$.

It remains to estimate the first term on the right-hand side of (3.24). As shown in [4, Theorem 3.2] the first eigenvalue $\lambda_\zeta^\ell>0$ of the operator $\mathcal{H}_\zeta^\ell$ is exponentially small as $\ell\to+\infty$, while the remaining part of the spectrum is bounded away from zero uniformly in ℓ and ζ (since $\ell+\zeta>\kappa_1-2\delta_1$). We denote by Ψ_ζ^ℓ the eigenfunction corresponding to the eigenvalue λ_ζ^ℓ. From these results it follows that there exists a constant $g_1>0$, independent of ℓ and ζ, such that for any $\psi\in L^2((-\ell,+\infty))$, $\psi\perp\Psi_\zeta^\ell$, i.e., satisfying $\langle\psi,\Psi_\zeta^\ell\rangle_{L^2}=0$,

$$\langle\psi,\mathcal{H}_\zeta^\ell\psi\rangle_{L^2} \ge g_1\langle\psi,\psi\rangle_{L^2}\,. \tag{3.25}$$

We next improve the above bound with the H^1-norm. More precisely, we prove that there exists a constant $\bar g_1>0$ independent of ℓ and ζ, such that

$$J_{\ell,\zeta} := \inf_{\psi\perp\Psi_\zeta^\ell}\frac{\langle\psi,\mathcal{H}_\zeta^\ell\psi\rangle_{L^2}}{\|\psi\|_{H^1}^2} \ge \bar g_1\,. \tag{3.26}$$

Since $J_{\ell,\zeta}=J_{\ell+\zeta,0}$ it is enough to show that

$$\liminf_{\ell\to+\infty}\inf_{\psi\perp\Psi_0^\ell}\frac{\langle\psi,\mathcal{H}_0^\ell\psi\rangle_{L^2}}{\|\psi\|_{H^1}^2} > 0\,. \tag{3.27}$$

We argue by contradiction. If (3.27) does not hold, there exists a sequence ψ_ℓ with $\|\psi_\ell\|_{H^1}=1$ and $\psi_\ell\perp\Psi_0^\ell$ such that

$$\langle\psi_\ell,\mathcal{H}_0^\ell\psi_\ell\rangle_{L^2} = \int_{-\ell}^{+\infty}\left(|\psi_\ell'|^2 + V''(\overline{m})\psi_\ell^2\right)\mathrm{d}y \to 0\,.$$

By (3.25) we necessarily have $\psi_\ell \to 0$ in L^2. In view of the boundedness of $V''(\overline{m})$ the formula above gives the required contradiction.

By writing $\varphi = \langle \varphi, \Psi^\ell_\zeta \rangle_{L^2} \Psi^\ell_\zeta + \varphi^\perp$, from (3.26) and Young's inequality we have, for each $\gamma > 0$,

$$\begin{aligned}\langle \varphi, \mathcal{H}^\ell_\zeta \varphi \rangle_{L^2} &= \langle \varphi, \Psi^\ell_\zeta \rangle^2_{L^2} \lambda^\ell_\zeta + \langle \varphi^\perp, \mathcal{H}^\ell_\zeta \varphi^\perp \rangle_{L^2} \\ &\geq \bar{g}_1 \left(\|\varphi\|^2_{H^1} - 2\langle \varphi, \Psi^\ell_\zeta \rangle_{L^2} \langle \varphi^\perp, \Psi^\ell_\zeta \rangle_{H^1} - \langle \varphi, \Psi^\ell_\zeta \rangle^2_{L^2} \|\Psi^\ell_\zeta\|^2_{H^1} \right) \\ &\geq \bar{g}_1 \left(\|\varphi\|^2_{H^1} - \gamma \langle \varphi^\perp, \Psi^\ell_\zeta \rangle^2_{H^1} - \langle \varphi, \Psi^\ell_\zeta \rangle^2_{L^2} (\|\Psi^\ell_\zeta\|^2_{H^1} + \gamma^{-1}) \right).\end{aligned} \tag{3.28}$$

Since $\mathcal{H}^\ell_z \Psi^\ell_\zeta = \lambda^\ell_z \Psi^\ell_\zeta$, we easily deduce that $\|\Psi^\ell_\zeta\|^2_{H^1}$ is bounded uniformly in ℓ and ζ. Moreover, by Schwarz's inequality and the orthogonality between $\varphi^\perp$ and Ψ^ℓ_ζ, choosing γ small enough in (3.28), we obtain

$$\langle \varphi, \mathcal{H}^\ell_\zeta \varphi \rangle_{L^2} \geq C \left(\|\varphi\|^2_{H^1} - \langle \varphi, \Psi^\ell_\zeta \rangle^2_{L^2} \right). \tag{3.29}$$

Using (3.12), we get

$$\begin{aligned}|\langle \varphi, \Psi^\ell_\zeta \rangle_{L^2}| &= \left| \left\langle \varphi, \Psi^\ell_\zeta + \frac{\partial_z m^\ell_\zeta}{\|\partial_z m^\ell_\zeta\|_{L^2}} \right\rangle_{L^2} \right| \\ &\leq \|\varphi\|_{H^1} \left(\left\| \Psi^\ell_\zeta - \frac{\overline{m}'_\zeta}{\|\overline{m}'_\zeta\|_{L^2}} \right\|_{L^2} + \left\| \frac{\overline{m}'_\zeta}{\|\overline{m}'_\zeta\|_{L^2}} + \frac{\partial_z m^\ell_\zeta}{\|\partial_z m^\ell_\zeta\|_{L^2}} \right\|_{L^2} \right).\end{aligned}$$

In order to bound the right-hand side, we first claim that

$$\left\| \Psi^\ell_\zeta - \frac{\overline{m}'_\zeta}{\|\overline{m}'_\zeta\|_{L^2}} \right\|_{L^2} \leq C \mathrm{e}^{-2(\ell+\zeta)}\,. \tag{3.30}$$

Indeed, a slightly weaker estimate is stated in [4, Theorem 3.2]. However, it is straightforward to modify the argument of the proof to get (3.30); see in particular [4, page 336]. The second term on the right-hand side can be easily estimated using (3.7) which gives, taking into account that $m^\ell_\zeta(y) = \overline{m}_\zeta(y)$ if $y > \zeta$ and that $\ell + \zeta > \kappa_1 - 2\delta_1$,

$$\left\| \frac{\overline{m}'_\zeta}{\|\overline{m}'_\zeta\|_{L^2}} + \frac{\partial_z m^\ell_\zeta}{\|\partial_z m^\ell_\zeta\|_{L^2}} \right\|_{L^2} \leq C \kappa_1 \mathrm{e}^{-2\kappa_1}\,.$$

In conclusion, choosing κ_1 large enough, the previous bounds, together with (3.29), give (3.23), which completes the proof of statement (ii).

Proof of statement (iii). We notice that in the proof of statement (ii) the estimates of the terms I^1_ℓ and I^3_ℓ do not require ζ to be the center of m, while I^2_ℓ can be easily estimated from above by the H^1-norm of $m - m^\ell_\zeta$. ∎

4. Compactness. We are now ready to analyze the two-dimensional functional. In this section we prove the compactness statement in Theorem 2.1. Let us consider a sequence u_ε in X such that $F_\varepsilon(u_\varepsilon) \leq C_3$, namely

$$\int_0^{+\infty} \int_{-\ell_\varepsilon}^{+\infty} (\partial_x u_\varepsilon)^2 \, \mathrm{d}y \, \mathrm{d}x + \int_0^{+\infty} \varepsilon^{-2} \left[\mathcal{F}_{\ell_\varepsilon}(u_\varepsilon(x,\cdot)) - \frac{8}{3} \right] \mathrm{d}x \leq C_3\,, \tag{4.1}$$

where $\ell_\varepsilon - \frac{1}{2} \log \varepsilon^{-1} \to \alpha$ and u_ε satisfies the boundary conditions (2.4) for some w_ε such that (2.7) holds.

Remark 4.1. By Schwarz's inequality and the bound (4.1), for any x_1, x_2 in $[0, +\infty)$,

$$\|u_\varepsilon(x_1, \cdot) - u_\varepsilon(x_2, \cdot)\|^2_{L^2} = \int_{-\ell_\varepsilon}^{+\infty} \left(\int_{x_1}^{x_2} \partial_x u_\varepsilon(x, y) \mathrm{d}x \right)^2 \mathrm{d}y \le C_3 |x_1 - x_2| \,. \tag{4.2}$$

Given a sequence $M_\varepsilon \to +\infty$ such that $M_\varepsilon \varepsilon^2 \to 0$ as $\varepsilon \to 0$, we define the set of *good* points in $(0, +\infty)$ as

$$B_\varepsilon = \left\{ x \in (0, +\infty) \,:\, \mathcal{F}_{\ell_\varepsilon}(u_\varepsilon(x, \cdot)) - \frac{8}{3} \le M_\varepsilon \varepsilon^2 \right\} . \tag{4.3}$$

The bound (4.1) yields $|B_\varepsilon^{\complement}| \le C_3 / M_\varepsilon$ (here $|B|$ is the Lebesgue measure of the Borel set $B \subset \mathbb{R}$). Moreover, since the bound (4.2) guarantees that the map $x \mapsto u_\varepsilon(x, \cdot)$ is continuous from $(0, +\infty)$ to $\mathcal{X}$ (see the previous section for the definition of $\mathcal{X}$), the lower semicontinuity of $\mathcal{F}_\ell$ on $\mathcal{X}$ implies that the map $x \mapsto \mathcal{F}_{\ell_\varepsilon}(u_\varepsilon(x, \cdot))$ is lower semicontinuous and hence the set B_ε is closed.

We now show how to construct the sequence ϕ_ε. Recalling the assumption (2.7) on the boundary datum, Theorem 3.5 implies that if ε is small enough and $x \in B_\varepsilon \cup \{0\}$, then there exists a unique center of $u_\varepsilon(x, \cdot)$, which we denote by $\phi_\varepsilon(x)$. Let us note that the function ϕ_ε is measurable on B_ε. This can be easily deduced by the continuity in the uniform topology of the map which to each function in the set $\mathcal{T}^\ell(\delta, \kappa)$ associates its center (see [8, Proposition 3.2]) and the measurability of the map $B_\varepsilon \ni x \mapsto u(x, \cdot)$ with respect to the Borel σ-algebra associated to the uniform topology.

Since $\ell_\varepsilon - \frac{1}{2} \log \varepsilon^{-1} \to \alpha$, in view of (3.15), there exists a constant $C_4 > 0$, depending on α, such that the following bounds hold:

$$\begin{aligned} &\phi_\varepsilon(x) \ge -\frac{1}{4} \log(C_4 M_\varepsilon) \\ &\|u_\varepsilon(x, \cdot) - m^{\ell_\varepsilon}(\cdot, \phi_\varepsilon(x))\|^2_{H^1} \le C_0 M_\varepsilon \varepsilon^2 \end{aligned} \qquad \forall\, x \in B_\varepsilon \tag{4.4}$$

and

$$\lim_{\varepsilon \to 0} \phi_\varepsilon(0) = 0 \,, \qquad \|u_\varepsilon(0, \cdot) - m^{\ell_\varepsilon}(\cdot, \phi_\varepsilon(0))\|^2_{H^1} \le \varepsilon\, \eta_\varepsilon \,, \tag{4.5}$$

where, in view of (2.7),

$$\eta_\varepsilon := \varepsilon^{-1} \left(\mathcal{F}_{\ell_\varepsilon}(w_\varepsilon) - \frac{8}{3} \right) \to 0 \,. \tag{4.6}$$

Since $B_\varepsilon^{\complement}$ is a countable union of disjoint open intervals, we extend ϕ_ε to a function on $[0, +\infty)$ by defining it in each interval of $B_\varepsilon^{\complement}$ as the affine interpolation of the values of ϕ_ε at the endpoints.

The compactness stated in Theorem 2.1 is a consequence of the following two lemmas. Indeed, Lemma 4.2 yields the precompactness of ϕ_ε in the uniform topology, while Lemma 4.3 together with (3.6) implies (2.8).

LEMMA 4.2. *Let ϕ_ε be defined as above. Then there exists a positive constant C_5 such that, for any $x_1, x_2 \in [0, +\infty)$,*

$$|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)| \wedge |\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)|^2 \le C_5 \left(|x_1 - x_2| + \frac{1}{M_\varepsilon} + M_\varepsilon \varepsilon^2 + \varepsilon \eta_\varepsilon \right), \tag{4.7}$$

where M_ε is the sequence in (4.3) *and η_ε is the sequence defined in* (4.6).

Proof. Since ϕ_ε is affine outside B_ε and $|B_\varepsilon^{\complement}| \le C_3/M_\varepsilon$, it is enough to prove that there exists $C_6 > 0$ such that for any $x_1, x_2 \in B_\varepsilon \cup \{0\}$,

$$|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)| \wedge |\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)|^2 \le C_6\big(|x_1 - x_2| + M_\varepsilon \varepsilon^2 + \varepsilon\eta_\varepsilon\big)\,. \tag{4.8}$$

If $x_1, x_2 \in B_\varepsilon \cup \{0\}$, the bound (3.10) implies

$$\begin{aligned} &|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)| \wedge |\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)|^2 \\ &\qquad \le A\|m^{\ell_\varepsilon}(\cdot, \phi_\varepsilon(x_1)) - m^{\ell_\varepsilon}(\cdot, \phi_\varepsilon(x_2))\|_{L^2}^2 \\ &\qquad \le 2A\big(\|u_\varepsilon(x_1,\cdot) - u_\varepsilon(x_2,\cdot)\|_{L^2}^2 + \|\widetilde{u}_\varepsilon(x_1,\cdot) - \widetilde{u}_\varepsilon(x_2,\cdot)\|_{L^2}^2\big), \end{aligned} \tag{4.9}$$

where $\widetilde{u}_\varepsilon(x,y) := u_\varepsilon(x,y) - m^{\ell_\varepsilon}(y, \phi_\varepsilon(x))$. By using (4.2), (4.4), and (4.5) the bound (4.8) follows. ∎

Recall that $m_z^\ell(\cdot) \equiv m^\ell(\cdot, z)$ is defined in (3.4).

LEMMA 4.3. *Let u_ε be a sequence satisfying the bound* (4.1)*, let ϕ_ε be defined as above, and set $\widetilde{u}_\varepsilon(x,y) := u_\varepsilon(x,y) - m^{\ell_\varepsilon}(y, \phi_\varepsilon(x))$, $(x,y) \in [0,+\infty) \times \mathbb{R}$. For each $R > 0$ the sequence $\widetilde{u}_\varepsilon$ converges to* 0*, as $\varepsilon \to 0$, in $L^2((0,R) \times \mathbb{R})$.*

Proof. The estimate (4.4) trivially implies that, for any $R > 0$,

$$\lim_{\varepsilon\to 0} \int_{[0,R)\cap B_\varepsilon} \int_{-\infty}^{+\infty} |\widetilde{u}_\varepsilon|^2 \, \mathrm{d}y \, \mathrm{d}x = 0\,. \tag{4.10}$$

By (4.2), (3.10), and Lemma 4.2, for a suitable constant $C_7 > 0$ we get, for any x_1, x_2 in $[0,+\infty)$,

$$\|\widetilde{u}_\varepsilon(x_1,\cdot) - \widetilde{u}_\varepsilon(x_2,\cdot)\|_{L^2}^2 \le C_7\Big(|x_1 - x_2| + \frac{1}{M_\varepsilon} + M_\varepsilon\varepsilon^2 + \varepsilon\eta_\varepsilon\Big).$$

This, together with (4.10) and the fact that $|B_\varepsilon^{\complement}| \le C_3/M_\varepsilon$, concludes the proof. ∎

5. Γ-convergence. In this section we conclude the proof of the main result by proving the Γ-convergence of the functionals F_ε.

Proof of Theorem 2.1: Γ-liminf. The formal statement of the Γ-liminf inequality is the following. For each $u \in X$ and each sequence u_ε converging to u in X, it holds that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge F^\alpha(u)$. In view of the compactness result, the Γ-liminf is achieved once we show that, for each $\phi \in C([0,+\infty))$ and each sequence u_ε converging to $\overline{m}_\phi$ in X,

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge \mathcal{G}^\alpha(\phi)\,. \tag{5.1}$$

Fix $\phi \in C([0,+\infty))$ and a sequence u_ε converging to $\overline{m}_\phi$. Without loss of generality we can assume that $F_\varepsilon(u_\varepsilon) \le C_8$. Therefore, in view of Lemmas 4.2 and 4.3, by extracting, if necessary, a subsequence, there exists a sequence ϕ_ε converging to ϕ in $C([0,+\infty))$ such that $u_\varepsilon = m^{\ell_\varepsilon}(\cdot, \phi_\varepsilon) + \widetilde{u}_\varepsilon$, with $\widetilde{u}_\varepsilon$ converging to zero in $L^2((0,R)\times\mathbb{R})$ for any $R > 0$. Let B_ε be the set of good points as defined in (4.3). Then

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\ge \int_0^{+\infty}\int_{-\infty}^{+\infty} (\partial_x u_\varepsilon)^2 \, \mathrm{d}y \, \mathrm{d}x + \int_{B_\varepsilon} \varepsilon^{-2}\left[\mathcal{F}_{\ell_\varepsilon}(u_\varepsilon(x,\cdot)) - \frac{8}{3}\right] \mathrm{d}x \\ &= \int_0^{+\infty}\int_{-\infty}^{+\infty} (\partial_x u_\varepsilon)^2 \, \mathrm{d}y \, \mathrm{d}x + \int_{B_\varepsilon} \varepsilon^{-2}\left[\mathcal{F}_{\ell_\varepsilon}(m^{\ell_\varepsilon}(\cdot, \phi_\varepsilon(x))) - \frac{8}{3}\right] \mathrm{d}x + \mathcal{R}_\varepsilon\,. \end{aligned} \tag{5.2}$$

Since, for each $R > 0$, $u_\varepsilon \to \overline{m}_\phi$ in $L^2((0,R)\times\mathbb{R})$, the lower semicontinuity of the map $u \mapsto \|\partial_x u\|^2_{L^2((0,R)\times\mathbb{R})}$ with respect to the L^2-convergence gives

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_0^R\int_{-\infty}^{+\infty}(\partial_x u_\varepsilon(x,y))^2\,\mathrm{d}x\,\mathrm{d}y \\ \geq \int_0^R\int_{-\infty}^{+\infty}(\partial_x \overline{m}_{\phi(x)}(y))^2\,\mathrm{d}y\,\mathrm{d}x = \frac{4}{3}\int_0^R \phi'(x)^2\,\mathrm{d}x\,.\end{aligned} \tag{5.3}$$

The estimate of the second term on the right-hand side of (5.2) is a direct consequence of Lemma 3.3. Indeed, since $\varepsilon^{-2}e^{-4\ell_\varepsilon} \to e^{-4\alpha}$, by Fatou's lemma and the fact that $|B_\varepsilon^{\complement}| \to 0$, we get

$$\liminf_{\varepsilon\to 0}\int_{B_\varepsilon\cap(0,R)}\varepsilon^{-2}\Big[\mathcal{F}_{\ell_\varepsilon}(m^{\ell_\varepsilon}(\cdot,\phi_\varepsilon(x))) - \frac{8}{3}\Big]\,\mathrm{d}x \geq 16\,e^{-4\alpha}\int_0^R \mathrm{e}^{-4\phi(x)}\,\mathrm{d}x \tag{5.4}$$

for any $R > 0$.

Finally, we need to estimate $\mathcal{R}_\varepsilon$ as defined in (5.2), i.e.,

$$\mathcal{R}_\varepsilon = \int_{B_\varepsilon}\varepsilon^{-2}\left[\mathcal{F}_{\ell_\varepsilon}(u_\varepsilon(x,\cdot)) - \mathcal{F}_{\ell_\varepsilon}(m^{\ell_\varepsilon}(\cdot,\phi_\varepsilon(x)))\right]\,\mathrm{d}x\,.$$

By (4.4), for any $x \in B_\varepsilon$,

$$\mathrm{e}^{-4\phi_\varepsilon(x)}\|\widetilde{u}_\varepsilon(x,\cdot)\|_{H^1} \leq C_4 C_0^{\frac{1}{2}} M_\varepsilon^{\frac{3}{2}}\varepsilon\,.$$

Thus, if we further choose M_ε such that $M_\varepsilon^3\varepsilon^2 \to 0$ as $\varepsilon \to 0$, from (3.14) we get

$$\liminf_{\varepsilon\to 0}\mathcal{R}_\varepsilon \geq \frac{1}{C_1}\lim_{\varepsilon\to 0}\int_{B_\varepsilon}\varepsilon^{-2}\|\widetilde{u}_\varepsilon(x,\cdot)\|^2_{H^1}\,\mathrm{d}x\,. \tag{5.5}$$

The bound (5.1) follows by (5.2), (5.3), (5.4), and (5.5). □

We note that the previous arguments show that if the energy of the sequence u_ε converges to $\mathcal{G}^\alpha(\phi)$, then $u_\varepsilon(x,\cdot)$, $x \in B_\varepsilon$, is actually close in $H^1(\mathbb{R})$ topology to the "right" one-dimensional profile, with an explicit control on the norm. The precise statement is given in the following remark.

Remark 5.1. Take a sequence u_ε with $F_\varepsilon(u_\varepsilon) \leq C_3$ and decompose u_ε as $u_\varepsilon = m^{\ell_\varepsilon}(\cdot,\phi_\varepsilon) + \widetilde{u}_\varepsilon$, where ϕ_ε is the sequence constructed in section 4. If $u_\varepsilon \to \overline{m}_\phi$ for some $\phi \in C([0,+\infty))$ and satisfies

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \mathcal{G}^\alpha(\phi) < +\infty\,,$$

then from (5.5) we easily deduce that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon}\varepsilon^{-2}\|\widetilde{u}_\varepsilon(x,\cdot)\|^2_{H^1}\,\mathrm{d}x = 0\,.$$

Proof of Theorem 2.1: Γ-limsup. We now show that for any function $u \in X$ of the form $u = \overline{m}_\phi$, with $\phi \in C([0,+\infty))$, we can construct a sequence $\bar{u}_\varepsilon$ such that

$$\lim_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon) = \mathcal{G}^\alpha(\phi)\,. \tag{5.6}$$

We observe that for each $\phi \in C([0,+\infty))$ such that $\phi(0)=0$ and $\mathcal{G}^\alpha(\phi)<+\infty$, we can find a sequence ϕ_n, with $\operatorname{supp}\phi_n \subset (n^{-1},+\infty)$ and $\phi_n \geq -n$, converging to ϕ and satisfying $\lim_n \mathcal{G}^\alpha(\phi_n)=\mathcal{G}^\alpha(\phi)$ (e.g., $\phi_n(x)=\phi(x-n^{-1})\vee(-n)$ if $x\geq n^{-1}$). By standard properties of the Γ-limsup (see, e.g., [6, Remark 1.29]), it is therefore enough to construct the recovery sequence for $\phi\in C([0,+\infty))$ bounded from below and with $\operatorname{supp}\phi\subset(\delta,+\infty)$, $\delta>0$.

Let ζ_ε be a center of the boundary condition w_ε. In view of (2.7) and Theorem 3.5, ζ_ε is in fact the unique center of w_ε. Moreover, by (3.6), the real sequence ζ_ε converges to zero as $\varepsilon\to 0$. By redefining ℓ_ε we can thus assume, and do so now, that $\zeta_\varepsilon=0$.

We claim that the following sequence does the job:

$$\bar{u}_\varepsilon(x,y) := \begin{cases} m^{\ell_\varepsilon}(y,\phi(x)) & \text{if } (x,y)\in[\varepsilon,+\infty)\times\mathbb{R}\,,\\ m^{\ell_\varepsilon}\big(y,0\big)+\frac{\varepsilon-x}{\varepsilon}\widetilde{w}_\varepsilon(y) & \text{if } (x,y)\in[0,\varepsilon)\times\mathbb{R}\,, \end{cases} \tag{5.7}$$

where $\widetilde{w}_\varepsilon := w_\varepsilon - m_0^{\ell_\varepsilon}$.

In what follows we use the notation $F_\varepsilon(\cdot,A)$ for the localization of the functional F_ε on the set $A\subset(0,+\infty)\times\mathbb{R}$. Since $m_z^\ell = m_0^{\ell+z}$, by Lemma 3.3 it follows that for each $\bar{z}\in\mathbb{R}$ there exists $\bar{\ell}>0$ such that

$$\mathrm{e}^{-4\ell}\left[\mathcal{F}_\ell(m_z^\ell)-\frac{8}{3}\right]\leq 17\,\mathrm{e}^{-4z} \qquad \forall\, z\in[\bar{z},+\infty),\quad \forall\,\ell>\bar{\ell}\,.$$

Since we assumed ϕ to be bounded from below, by Lemma 3.3, dominated convergence, and (3.11), we deduce

$$\lim_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon,(\delta,+\infty)\times(-\ell_\varepsilon,+\infty)) = \int_\delta^{+\infty}\left[\frac{4}{3}\phi'(x)^2+16\,e^{-4\alpha}\,\mathrm{e}^{-4\phi(x)}\right]\mathrm{d}x\,. \tag{5.8}$$

We now show that

$$\lim_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon,(0,\delta)\times(-\ell_\varepsilon,+\infty)) = 16\,\mathrm{e}^{-4\alpha}\,\delta\,. \tag{5.9}$$

Since $\operatorname{supp}\phi\subset(\delta,+\infty)$, (5.6) is a straightforward consequence of (5.8) and (5.9).

To conclude, we are left with the proof of (5.9). As follows from (2.7) and (3.14), $\lim_{\varepsilon\to 0}\varepsilon^{-1}\|\widetilde{w}_\varepsilon\|_{H^1}^2=0$ and therefore

$$\lim_{\varepsilon\to 0}\int_0^\delta\int_{-\ell_\varepsilon}^{+\infty}|\partial_x\bar{u}_\varepsilon|^2\,\mathrm{d}y\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_0^\varepsilon\int_{-\ell_\varepsilon}^{+\infty}\varepsilon^{-2}|\widetilde{w}_\varepsilon(y)|^2\,\mathrm{d}y\,\mathrm{d}x = 0\,.$$

On the other hand, since $\phi(x)=0$ for $x\in[0,\delta]$, $\bar{u}_\varepsilon(x,\cdot)=m_0^{\ell_\varepsilon}(\cdot)$ in (ε,δ); then

$$\begin{aligned}&\int_0^\delta \varepsilon^{-2}\Big[\mathcal{F}_\varepsilon(\bar{u}_\varepsilon(x,\cdot))-\frac{8}{3}\Big]\,\mathrm{d}x\\ &\quad = \int_0^\delta\varepsilon^{-2}\Big[\mathcal{F}_\varepsilon(m_0^{\ell_\varepsilon})-\frac{8}{3}\Big]\,\mathrm{d}x + \int_0^\varepsilon\varepsilon^{-2}\Big[\mathcal{F}_\varepsilon(\bar{u}_\varepsilon(x,\cdot))-\mathcal{F}_\varepsilon(m_0^{\ell_\varepsilon})\Big]\,\mathrm{d}x\,.\end{aligned}$$

As $\ell_\varepsilon-\frac{1}{2}\log\varepsilon^{-1}\to\alpha$, by Lemma 3.3,

$$\limsup_{\varepsilon\to 0}\int_0^\delta\varepsilon^{-2}\Big[\mathcal{F}_\varepsilon(m_0^{\ell_\varepsilon})-\frac{8}{3}\Big]\,\mathrm{d}x = 16\,\mathrm{e}^{-4\alpha}\,\delta\,.$$

As noted before, $\lim_{\varepsilon\to 0}\varepsilon^{-1}\|\widetilde{w}_\varepsilon\|_{H^1}^2 = 0$; therefore by Theorem 3.5(iii),

$$\lim_{\varepsilon\to 0}\int_0^\varepsilon \varepsilon^{-2}\Big[\mathcal{F}_\varepsilon(\bar{u}_\varepsilon(x,\cdot)) - \mathcal{F}_\varepsilon(m_0^{\ell_\varepsilon})\Big]\,\mathrm{d}x = 0,$$

which completes the proof of (5.9). ∎

Appendix. Sharp estimates on the constrained minimizer. In this appendix we prove the sharp estimates concerning m_z^ℓ and its convergence to $\overline{m}_z$. We regard the boundary value problem (3.5) as a one-dimensional Newtonian system with potential $-V$ and mass equal to 2. Accordingly, the space variable y is interpreted as the time and denoted by t.

Proof of Proposition 3.1. Given $T > 0$, we denote by $m_T(t)$, $t \in [-T, 0]$, the solution to the boundary value problem

$$\begin{cases} -2m'' + V'(m) = 0 & \text{in } (-T,0)\,, \\ m(-T) = -1\,,\ m(0) = 0\,. \end{cases} \tag{A.1}$$

Integrating (A.1) by using the conservation of the Newtonian energy, we get that $m_T(t)$ is the strictly increasing function on $[-T, 0]$ such that

$$-t = \int_{m_T(t)}^0 \frac{\mathrm{d}a}{\sqrt{V(a)+E_T}} \qquad \forall\, t \in [-T,0]\,, \tag{A.2}$$

where E_T is implicitly defined by the condition

$$T = \int_{-1}^0 \frac{\mathrm{d}a}{\sqrt{V(a)+E_T}}\,. \tag{A.3}$$

In what follows we denote by C a strictly positive constant, independent of T, whose numerical value may change from line to line. By [5, Lemma A.1],

$$\lim_{T\to\infty} \mathrm{e}^{4T} E_T = 64 \tag{A.4}$$

and

$$\sup_{t\in(-T,0)} \big|m_T(t) - \overline{m}(t)\big| \le C\mathrm{e}^{-2T} \qquad \forall\, T \ge 1\,. \tag{A.5}$$

We now observe that, for any $y \in (-\ell, z)$,

$$\begin{aligned} m_z^\ell(y) &= m_{\ell+z}(y-z)\,, \\ \partial_z m_z^\ell(y) &= \partial_T m_{\ell+z}(y-z) - m'_{\ell+z}(y-z)\,, \\ \partial_{zz} m_z^\ell(y) &= \partial_{TT} m_{\ell+z}(y-z) - 2\partial_T m'_{\ell+z}(y-z) + m''_{\ell+z}(y-z)\,. \end{aligned} \tag{A.6}$$

The bound (3.6) follows by (A.5). We next show that

$$\sup_{t\in(-T,0)} \big|m'_T(t) - \overline{m}'(t)\big| \le C\mathrm{e}^{-2T} \qquad \forall\, T \ge 1\,, \tag{A.7}$$

$$\sup_{t\in(-T,0)} \big|m''_T(t) - \overline{m}''(t)\big| \le C\mathrm{e}^{-2T} \qquad \forall\, T \ge 1\,, \tag{A.8}$$

$$\sup_{t\in(-T,0)} \big\{\big|\partial_T m_T(t)\big| + \big|\partial_T m'_T(t)\big| + \big|\partial_{TT} m_T(t)\big|\big\} \le C\mathrm{e}^{-2T} \qquad \forall\, T \ge 1\,, \tag{A.9}$$

which imply the estimates (3.7) and (3.8).

Proof of (A.7). Since $\overline{m}'(t) = \sqrt{V(\overline{m}(t))}$, $m_T'(t) = \sqrt{V(m_T(t)) + E_T}$, and $\overline{m}(0) = m_T(0) = 0$, we have

$$-1 < m_T(t) < \overline{m}(t) < 0 \qquad \forall\, t \in (-T, 0)\,.$$

Hence, for any $t \in (-T, 0)$,

$$\text{(A.10)} \quad \big|m_T'(t) - \overline{m}'(t)\big| \le \sqrt{E_T} + \frac{V(\overline{m}(t)) - V(m_T(t))}{\sqrt{V(\overline{m}(t))}} \le \sqrt{E_T} + 4\big|m_T(t) - \overline{m}(t)\big|\,,$$

where, in the last inequality, we used that, by the explicit expression (2.1) of V, $V(b) - V(a) \le 4\sqrt{V(b)}(b - a)$ for $-1 < a < b < 0$. The bound (A.7) now follows by (A.4), (A.5), and (A.10).

Proof of (A.8). Since $m_T''(t) - \overline{m}''(t) = V'(m_T(t)) - V'(\overline{m}(t))$ and $|V''(a)| \le 16$ for $-1 < a < 0$, the bound (A.8) is an immediate consequence of (A.5).

Proof of (A.9). Taking the derivatives with respect to the variable T in the identities (A.2), (A.3) and $m_T'(t) = \sqrt{V(m_T(t)) + E_T}$, we compute

$$\text{(A.11)} \quad \begin{aligned} E_T' &:= \frac{\mathrm{d}E_T}{\mathrm{d}T} = -2\left[\int_{-1}^{0} \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{3/2}}\right]^{-1}, \\ E_T'' &:= \frac{\mathrm{d}^2 E_T}{\mathrm{d}T^2} = -\frac{3}{4}(E_T')^3 \int_{-1}^{0} \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{5/2}}\,, \\ \partial_T m_T(t) &= -\frac{E_T'}{2}\sqrt{V(m_T(t)) + E_T} \int_{m_T(t)}^{0} \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{3/2}}\,, \\ \partial_{TT} m_T(t) &= \left[\frac{1}{2}\frac{V'(m_T(t))\,\partial_T m_T(t) + 2E_T'}{V(m_T(t)) + E_T} + \frac{E_T''}{E_T'}\right]\partial_T m_T(t) \\ &\quad + \frac{3}{4}(E_T')^2\sqrt{V(m_T(t)) + E_T} \int_{m_T(t)}^{0} \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{5/2}}\,, \\ \partial_T m_T'(t) &= \frac{1}{2}\frac{V'(m_T(t))\,\partial_T m_T(t) + E_T'}{\sqrt{V(m_T(t)) + E_T}}\,. \end{aligned}$$

By the change of variable $b = 1 + a$ it is straightforward to check that, for any integer $n \ge 1$,

$$\int_0^1 \frac{\mathrm{d}b}{\big[4b^2 + E_T\big]^{n/2}} \le \int_{-1}^{0} \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{n/2}} \le \int_0^1 \frac{\mathrm{d}b}{\big[b^2 + E_T\big]^{n/2}}\,.$$

Therefore,

$$\text{(A.12)} \qquad \frac{g_n(E_T)}{C} \le \int_{-1}^{0} \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{n/2}} \le C\, g_n(E_T) \qquad \forall\, T \ge 1\,,$$

where

$$g_n(E_T) = \begin{cases} |\log E_T| & \text{if } n = 1\,, \\ E_T^{(1-n)/2} & \text{if } n = 3, 5\,. \end{cases}$$

By (A.12) and (A.11) it follows that

$$\frac{E_T}{C} \le |E_T'| \le C\, E_T, \qquad |E_T''| \le C\, E_T \qquad \forall\, T \ge 1\,. \tag{A.13}$$

Since $V(a) \ge V(m_T(t))$ for $a \in [m_T(t)), 0]$, using (A.12) and (A.13), from (A.11) we get

$$\big|\partial_T m_T(t)\big| \le \frac{|E_T'|}{2} \int_{m_T(t)}^0 \frac{\mathrm{d}a}{\big[V(a) + E_T\big]^{1/2}} \le C\, E_T |\log E_T|\,. \tag{A.14}$$

Analogously, also using the explicit form (2.1) of V,

$$\big|\partial_{TT} m_T(t)\big| \le \left[\frac{2\big|\partial_T m_T(t)\big| + |E_T'|}{E_T} + \frac{|E_T''|}{|E_T'|}\right] \big|\partial_T m_T(t)\big| + C\frac{|E_T'|^2}{E_T} \le C E_T\,. \tag{A.15}$$

Finally,

$$\big|\partial_T m_T'(t)\big| \le \frac{1}{2}\frac{4\big|\partial_T m_T(t)\big| + |E_T'|}{\sqrt{E_T}} \le C\,\sqrt{E_T}\,. \tag{A.16}$$

The bound (A.9) now follows by (A.4), (A.14), (A.15), and (A.16).

Proof of (3.9). Recall that $m_z^\ell(y) = \overline{m}_z(y)$ for $y \ge z$, whence $(m_z^\ell)'(y) + \partial_z m_z^\ell(y) = 0$ for $y > z$. Therefore, by (A.6) and (A.11),

$$\big[(m_z^\ell)'\big](z) + \big[\partial_z m_z^\ell\big](z) = \lim_{y\uparrow z} \big\{(m_z^\ell)'(y) + \partial_z m_z^\ell(y)\big\} = \lim_{y\uparrow z} \partial_T m_T(y-z)\Big|_{T=\ell+z} = 0\,.$$

On the other hand, since $\overline{m}'(0) = 1$,

$$\big|\big[(m_z^\ell)'\big](z)\big| = \Big|1 - \sqrt{V(m_{\ell+z}(0)) + E_{\ell+z}}\Big| = \Big|1 - \sqrt{1 + E_{\ell+z}}\Big| \le E_{\ell+z}\,.$$

By (A.4), this concludes the proof of (3.9).

Proof of (3.10). Without loss of generality we assume $z_1 < z_2$. Since

$$m_{z_2}^\ell(y) < 0 < m_{z_1}^\ell(y) = \overline{m}_{z_1}(y) \qquad \forall\, y \in (z_1, z_2),$$

we have

$$\begin{aligned} \|m_{z_1}^\ell - m_{z_2}^\ell\|_{L^2}^2 &\ge \int_{z_2}^{+\infty} |\overline{m}_{z_1}(y) - \overline{m}_{z_2}(y)|^2\, \mathrm{d}y \\ &= \int_0^{+\infty} |\overline{m}(y+z) - \overline{m}(y)|^2\, \mathrm{d}y =: G(z)\,, \end{aligned} \tag{A.17}$$

with $z = z_2 - z_1$. By differentiating,

$$G'(z) = 2\int_0^{+\infty} \overline{m}'(y+z)(\overline{m}(y+z) - \overline{m}(y))\, \mathrm{d}y\,,$$

whence $G'(0) = 0$ and, since $\overline{m}$ is strictly increasing, G' is strictly increasing. Moreover,

$$G''(0) = 2\int_0^{+\infty} |\overline{m}'(y)|^2\, \mathrm{d}y = \frac{4}{3}\,.$$

The above properties of the function G imply $G(z) \geq C\, z^2 \wedge z$ for any $z \geq 0$. In view of (A.17), this yields the lower bound of the estimate (3.10).

To prove the upper bound we analyze separately the cases $z_2 - z_1 \leq 1$ and $z_2 - z_1 > 1$. In the first case, we use Schwarz's inequality and (3.8) to write

$$\begin{aligned}\|m^\ell_{z_1} - m^\ell_{z_2}\|^2_{L^2} &= \left\| \int_{z_1}^{z_2} \partial_z m^\ell_z \,\mathrm{d}z \right\|^2_{L^2} \leq (z_2 - z_1) \int_{z_1}^{z_2} \left\| \partial_z m^\ell_z \right\|^2_{L^2} \mathrm{d}z \\ &\leq \left(2\|\overline{m}'\|^2_{L^2} + 2A^2(\ell + z_2)\mathrm{e}^{-4(\ell+z_1)} \right) (z_2 - z_1)^2 .\end{aligned}$$

In the second case, recalling the definition of $m^\ell_{z_i}$ outside $(-\ell, z_i)$ and using (3.6), we have

$$\begin{aligned}\|m^\ell_{z_1} - m^\ell_{z_2}\|^2_{L^2} &\leq \int_{-\ell}^{+\infty} 2\big[\overline{m}_{z_1}(y) - \overline{m}_{z_2}(y)\big]^2 \mathrm{d}y + 8A^2(\ell + z_2)\mathrm{e}^{-4(\ell+z_1)} \\ &\leq C\,(z_2 - z_1) .\end{aligned}$$

The proposition is thus proved. ∎

REFERENCES

[1] G. Alberti, *Variational models for phase transitions, an approach via* Γ*-convergence*, in Calculus of Variations and Partial Differential Equations (Pisa, 1996), Springer, Berlin, 2000, pp. 95–114.
[2] G. Alberti, G. Bellettini, M. Cassandro, and E. Presutti, *Surface tension in Ising systems with Kac potentials*, J. Statist. Phys., 82 (1996), pp. 743–796.
[3] G. Anzellotti and S. Baldo, *Asymptotic development by* Γ*-convergence*, Appl. Math. Optim., 27 (1993), pp. 105–123.
[4] L. Bertini, S. Brassesco, and P. Buttà, *Soft and hard wall in a stochastic reaction diffusion equation*, Arch. Ration. Mech. Anal., 190 (2008), pp. 307–345.
[5] L. Bertini, S. Brassesco, and P. Buttà, *Dobrushin states in the* ϕ^4_1 *model*, Arch. Ration. Mech. Anal., 190 (2008), pp. 477–516.
[6] A. Braides, Γ*-Convergence for Beginners*, Oxford Lecture Ser. Math. Appl. 22, Oxford University Press, Oxford, UK, 2002.
[7] A. Braides and L. Truskinovsky, *Asymptotic expansions by* Γ*-convergence*, Contin. Mech. Thermodyn., 20 (2008), pp. 21–62.
[8] S. Brassesco, A. De Masi, and E. Presutti, *Brownian fluctuations of the interface in the* $d = 1$ *Ginzburg–Landau equation with noise*, Ann. Inst. H. Poincaré Probab. Statist., 31 (1995), pp. 81–118.
[9] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system.* I. *Interfacial free energy*, J. Chem. Phys., 28 (1958), pp. 258–267.
[10] T. Funaki, *Stochastic interface models*, in Lectures on Probability Theory and Statistics, Lecture Notes in Math. 1869, Springer, Berlin, 2005, pp. 103–274.
[11] L. Modica, *Gradient theory of phase transitions with boundary contact energy*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 4 (1987), pp. 487–512.
[12] L. Modica and S. Mortola, *Un esempio di* Γ*-convergenza*, Boll. Un. Mat. Ital. B (5), 14 (1977), pp. 285–299.
[13] E. Presutti, *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*, Springer-Verlag, Berlin, 2009.
[14] J. D. van der Waals, *The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density*, J. Statist. Phys., 20 (1979), pp. 200–244.