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On Large Deviations of Interface Motions for Statistical Mechanics Models

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Abstract. We discuss the sharp interface limit of the action functional associated with either the Glauber dynamics for Ising systems with Kac potentials or the Glauber+Kawasaki process. The corresponding limiting functionals, for which we provide explicit formulae of the mobility and transport coefficients, describe the large deviation asymptotics with respect to the mean curvature flow.

1. Introduction

Consider the dynamical evolution, with non-conserved order parameter, of a system undergoing a first-order phase transition. A basic paradigm of statistical mechanics is that the corresponding macroscopic behavior is described by the motion by curvature of the interfaces separating the two stable phases. For lattice systems with short-range interaction, the lattice symmetries are still felt on the macroscopic scale and the resulting evolution is an anisotropic motion by curvature. For values of the temperature below the roughening transition, the Wulff shape is not strictly convex and the corresponding evolution is crystalline; i.e., it generates facets [40,41]. On the other hand, for long-range interactions, the resulting interface evolution is described by the (isotropic) motion by mean curvature. We refer to [26] for a recent overview on stochastic interface evolutions.

In principle, the macroscopic evolution of the interfaces should be derived from a microscopic Glauber-like dynamics and the corresponding transport coefficients characterized in terms of the microscopic interaction and the jump rates. While there is plenty of numerical evidence that this is indeed the case, the analytical results are few and the derivation of motion by curvature, say for the Ising model with Glauber dynamics at positive low temperature, remains a most challenging issue. For short-range interactions, the only available

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results are in fact at zero temperature [15,34,39]. In the case of long-range interactions, or more precisely for Ising model with Kac potentials, the motion by mean curvature has been derived in [20,31]. The key feature of this model is the presence of a parameter, the interaction range, that allows to achieve this derivation in two separate steps. Firstly, the evolution of the empirical magnetization in the Lebowitz–Penrose limit is examined. One shows that its limiting behavior is described by a non-local evolution equation. Secondly, it is shown that under a diffusive rescaling of space and time, this evolution leads to motion by mean curvature. This second step is quite similar to the analogous derivation starting from the Allen–Cahn equation [3,28]. Another model with similar features is the Glauber+Kawasaki process, for which the derivation of motion by mean curvature has been achieved by the same procedure [13,30].

The present purpose is to describe, in the sense of large deviations theory, the probability of deviations from the motion by curvature. Postponing the connection with the microscopic dynamics, let us first discuss this topic purely from a phenomenological point of view in the setting introduced in [39]. On a scale large compared to the microscopic length scale, we can represent the interface between the two pure phases as a surface Γ of codimension one embedded in $\mathbb{R}^d$. The typical evolution of Γ can then be deduced by free energy considerations. We denote by τ the surface tension. In general, τ depends on the local orientation of the surface, i.e., on the local normal $\widehat{n}$ at Γ. The surface free energy is then given by

$$F = \int_\Gamma \mathrm{d}\sigma\, \tau(\widehat{n}), \tag{1.1}$$

where $\mathrm{d}\sigma$ is the surface measure. Observe that in the isotropic case τ is constant and F becomes proportional to the perimeter of Γ. Phenomenologically, it is postulated that the interface velocity along the local normal, denoted by v, is given by

$$v = -\mu \frac{\delta F}{\delta \Gamma}, \tag{1.2}$$

where the *mobility* μ may depend on the local orientation on the surface. As shown in [39], for short-range interactions, the mobility μ can be computed from the microscopic dynamics, either by a Green–Kubo formula obtained via a linear response argument, or by looking at the fluctuations of the empirical order parameter.

Let $\widetilde{\tau}$ be the 1-homogeneous extension of τ to a function on $\mathbb{R}^d$, and introduce the stiffness matrix $A(\widehat{n})$ as the Hessian of $\widetilde{\tau}$ at $\widehat{n}$ (so that $A(\widehat{n})\widehat{n} = 0$). For $x \in \Gamma$, we define

$$\kappa_A(x) := \tau(\widehat{n}(x))^{-1} \sum_{i=1}^{d-1} \langle e_i(x), A(\widehat{n}(x)) e_i(x)\rangle \kappa_i(x),$$

where $\kappa_i(x)$ are the principal curvatures and $e_i(x)$ are the corresponding principal curvature directions of Γ at x. Then, (1.2) reads

$$v = \theta \kappa_A,$$

where the *transport coefficient* θ is given by the Einstein relation,

$$\theta = \mu\tau. \tag{1.3}$$

In the isotropic case, τ and μ are constant, $A(\widehat{n}) = \tau \mathbb{I}$ on the subspace orthogonal to $\widehat{n}$, hence $\kappa_A = \kappa$, the mean curvature of Γ.

Referring to [39] for the analysis of (small) Gaussian fluctuations, we next introduce the rate function describing the asymptotics of the probability of large deviations around the motion by mean curvature. To this end, fix a time interval $[0, T]$ and a path $\Gamma(t)$, $t \in [0, T]$. On a basis of a Gaussian assumption on the noise and a fluctuation dissipation relation, the rate function ought to be given by

$$S_{\mathrm{ac}}(\Gamma) = \int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma \, \frac{(v - \theta\kappa_A)^2}{4\mu}. \tag{1.4}$$

This functional should catch the asymptotics of the probability of smooth paths, and we next discuss its extension to more general paths. As shown in [33] in the context of the Allen–Cahn equation, the path $t \mapsto \Gamma(t)$ need not be continuous since nucleation might occur at some intermediate times. In such cases, the appropriate rate function reads,

$$S(\Gamma) = S_{\mathrm{ac}}(\Gamma) + S_{\mathrm{nucl}}(\Gamma), \tag{1.5}$$

where S_{nucl} measures, according to (1.1), the free energy cost of the interfaces nucleated in the time interval $[0, T]$. As we discuss in Sect. 4, S_{nucl} can be recovered from S_{ac} by approximating nucleation events with continuous paths. Moreover, interfaces need to be counted with their multiplicity and are not necessarily smooth, even away from the nucleation times. Suitable weak definitions of the curvature and velocity are thus needed. This is accomplished by using tools of geometric measure theory, and we refer to [36] for the proper definition of the functional S in the case of non-smooth interfaces in the isotropic case. Finally, it cannot be excluded that the map $t \mapsto \Gamma(t)$ has a singular continuous (e.g., Cantor) part, which does not affect the cost functional constructed in [36]. A variational definition of S which takes into account also such singular continuous part is provided in [6], and its corresponding zero level set is given by the mean curvature flow according to the Brakke's formulation [9].

The rate functional S should describe the large deviation asymptotics of microscopic stochastic dynamics that leads to the motion by curvature of the interfaces. The corresponding analysis has been carried out mostly for the Allen–Cahn evolution. In particular, the functional (1.5) has been identified by considering the sharp interface limit of the natural action functional associated with the Allen–Cahn equation, initially in [33] and in greater detail in [36]. A stochastic Allen–Cahn equation has been considered in [6], where the large deviation upper bound with rate function S is proven. Observe that, as discussed in [39], the Allen–Cahn evolution exhibits a trivial transport coefficient, $\mu = 1/\tau$, so that $\theta = 1$ regardless of the shape of the double-well potential. The case of Glauber dynamics for Ising systems with Kac potentials, in the one-dimensional case, has been considered in [7,8]. In this work, the asymptotic probability of a displacement of an interface in a given finite time is evaluated.

Here, we discuss, in the case of smooth interfaces, the derivation of the rate function S in the isotropic case, by considering either the Glauber dynamics for Ising systems with Kac potentials or the Glauber+Kawasaki process. For these models, the large deviation asymptotics, respectively, in the Lebowitz–Penrose and in the continuum limit, has been derived in [16] and in [29]. We thus analyze the sharp interface limit of the corresponding action functionals, deducing the rate functional (1.4) and providing explicit formulae for the mobility coefficients. While the basic strategy is analogous to the one in [33], the non-local character of the action functionals requires a more clever choice of the optimizing sequences. More precisely, in order to obtain the right transport coefficient, we need to introduce a corrector in the ansatz for the recovery sequences and solve a variational problem to identify the optimal choice. In the case of the Ising model with Kac potentials, the mobility derived here agrees with that derived in [14] by a linear response argument, thus validating the fluctuation dissipation assumption. The computation of the mobility for the Glauber+Kawasaki process appears instead novel and provides a dynamical characterization of the surface tension. Note indeed that, as the invariant measure of this process is not explicitly known, a static characterization according to the guidelines of equilibrium statistical mechanics is not feasible.

It would be interesting to extend the results of the present paper on the rate function S to the case of general interfaces, possibly exhibiting nucleation events. In analogy with the results in [36] for the Allen–Cahn equation, a key step should be to describe in both the models considered here the asymptotic behavior of sequences φ_ε with equibounded action [see Eqs. (2.41) and (3.14)]. In the case of Ising–Kac, a compactness property is expected, in analogy with the result in [1] for time-independent sequences with equibounded free energy [see (2.18) below], yielding paths of sharp interfaces in the limit $\varepsilon \to 0$ with uniformly bounded perimeter. However, it is unclear how to associate with such configurations φ_ε corresponding paths of generalized surfaces $t \mapsto \Gamma_\varepsilon(t)$ (as varifolds in the sense of geometric measure theory) with suitable uniform curvature and velocity bounds. As a consequence, we are not able to deduce curvature and velocity bounds on the limiting interfaces. Moreover, it remains to be proven that the well-prepared sequences φ_ε here considered [see (2.37) and (3.19)] actually describe the typical asymptotic behavior of configurations assuming uniform boundedness of the action functionals.

2. Glauber Dynamics with Kac Potentials

In this section, we analyze the sharp interface limit of the action functional in the context of the Glauber dynamics for Ising systems with Kac potentials.

2.1. Microscopic Model and Its Mean Field Limit

Let $\mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d$ be the torus of side $L \geq 1$ in $\mathbb{R}^d$; when $L = 1$, we drop it from the notation, i.e., $\mathbb{T}^d = \mathbb{T}_1^d$. We denote by r, r' the elements of $\mathbb{T}_L^d$

and by $\mathrm{d}r$ the Haar measure on $\mathbb{T}_L^d$. Given a smooth nonnegative function $j\colon \mathbb{R}_+ \to \mathbb{R}_+$, supported in $[0, \frac{1}{2}]$ and such that $\int_{\mathbb{R}^d} \mathrm{d}z\ j(|z|) = 1$, we let $J\colon \mathbb{T}_L^d \to \mathbb{R}_+$ be the probability density defined by $J(r) = j(|r|)$. In the sequel, $J * f(r) := \int_{\mathbb{T}_L^d} \mathrm{d}r'\, J(r-r')f(r')$ is the standard convolution on $\mathbb{T}_L^d$.

Given $L > 0$, and $\gamma > 0$ such that $\gamma^{-1}L \in \mathbb{N}$, let $\mathbb{T}_{L,\gamma}^d := (\gamma\mathbb{Z}/L\mathbb{Z})^d$ be the discrete approximation of $\mathbb{T}_L^d$ with lattice spacing γ. The microscopic configuration space is $\Omega_{L,\gamma} := \{-1, 1\}^{\mathbb{T}_{L,\gamma}^d}$. The microscopic energy is the function $H_{L,\gamma}\colon \Omega_{L,\gamma} \to \mathbb{R}$ defined by

$$H_{L,\gamma}(\sigma) = -\frac{1}{2} \sum_{i,j \in \mathbb{T}_{L,\gamma}^d} \gamma^d J(i-j)\sigma(i)\sigma(j).$$

Given the inverse temperature $\beta > 0$, the corresponding Gibbs measure $\mu_{L,\gamma}^\beta$ is the probability on $\Omega_{L,\gamma}$ defined by

$$\mu_{L,\gamma}^\beta(\sigma) = \frac{1}{Z_{L,\gamma}^\beta} \exp\big\{ -\beta H_{L,\gamma}(\sigma)\big\}, \tag{2.1}$$

where $Z_{L,\gamma}^\beta$ is the partition function.

Lebowitz–Penrose Limit. We consider the supercritical case $\beta > 1$ and define the spontaneous magnetization m_β as the strictly positive solution of the Curie–Weiss equation, that is

$$m_\beta = \tanh(\beta m_\beta), \quad m_\beta > 0. \tag{2.2}$$

Denoting by $\mathcal{M}(\mathbb{T}_L^d)$ the space of bounded measures on the torus $\mathbb{T}_L^d$, equipped with the weak* topology, we define the *empirical magnetization* as the map $M^\gamma\colon \Omega_{L,\gamma} \to \mathcal{M}(\mathbb{T}_L^d)$ given by

$$M^\gamma(\sigma) = \gamma^d \sum_{i \in \mathbb{T}_{L,\gamma}^d} \sigma(i)\, \delta_i.$$

As proven in [23], in the Lebowitz–Penrose limit $\gamma \to 0$ the excess free energy functional for the Gibbs measures (2.1) is given by the functional $F_L\colon L^\infty(\mathbb{T}_L^d; [-1,1]) \to [0,\infty)$ defined by

$$\begin{aligned} F_L(m) = &\int \mathrm{d}r\ [f_\beta(m) - f_\beta(m_\beta)] \\ &+ \frac{1}{4}\int \mathrm{d}r \int \mathrm{d}r'\ J(r-r')[m(r) - m(r')]^2, \end{aligned} \tag{2.3}$$

where

$$f_\beta(m) = -\frac{m^2}{2} + \beta^{-1}\imath(m), \quad \imath(m) = \frac{1+m}{2}\log\frac{1+m}{2} + \frac{1-m}{2}\log\frac{1-m}{2}. \tag{2.4}$$

Observe that, since $\pm m_\beta$ are the minimizers of f_β, the functional F_L vanishes on the pure phases $\pm m_\beta$. The probabilistic content of this statement is that the family $\{\mu_{L,\gamma}^\gamma \circ (M^\gamma)^{-1}\}_{\gamma>0}$ of probabilities on $\mathcal{M}(\mathbb{T}_L^d)$ satisfies a large deviation principle with speed $\beta^{-1}\gamma^d$ and rate function $\mathcal{F}_L$ given by $\mathcal{F}_L(\nu) = F_L(m)$ if $\nu = m\,\mathrm{d}r$ for some $m \in L^\infty(\mathbb{T}_L^d; [-1, 1])$ and $+\infty$ otherwise.

Glauber–Kac Dynamics. The Glauber dynamics with Kac potentials is a continuous-time Markov chain on the state space $\Omega_{L,\gamma}$, reversible with respect to the Gibbs measure (2.1). It is defined by assigning the rates at which the value of the spin σ at site i is flipped. The corresponding generator $\mathbb{L}_\gamma$ is the operator acting on functions on $\Omega_{L,\gamma}$ as

$$\mathbb{L}_\gamma f(\sigma) = \sum_{i\in\mathbb{T}_{L,\gamma}^d} c(i, M^\gamma(\sigma))\mathrm{e}^{-\beta J * M^\gamma(\sigma)(i)\sigma(i)}[f(\sigma^i) - f(\sigma)], \tag{2.5}$$

where σ^i denotes the configuration obtained from σ by flipping its value at site i and $c\colon \mathbb{T}_L^d \times \mathcal{M}(\mathbb{T}_L^d) \to (0, +\infty)$ is a continuous function satisfying $c(r, \nu) = c(r, \nu - \nu(\{r\})\delta_r)$, which implies the detailed balance condition, namely, that $\mathbb{L}_\gamma$ is self-adjoint with respect to the Gibbs measure (2.1).

In order to perform the sharp interface limit, we restrict to a special class of rates. More precisely, we assume that

$$c(r, \nu) \equiv c(\nu)(r) = a(K * \nu(r)), \quad r \in \mathbb{T}_L^d, \tag{2.6}$$

where $a\colon \mathbb{R} \to (0, +\infty)$ is a Lipschitz function and K is a smooth radial function on $\mathbb{R}^d$ with support in the ball of radius $\frac{1}{2}$ and satisfying $K(0) = 0$; i.e., $K(r) = k(|r|)$ for a smooth nonnegative function $k\colon \mathbb{R}_+ \to \mathbb{R}_+$ with support in $[0, \frac{1}{2}]$. A standard choice, see [20], is

$$c(i, M^\gamma(\sigma)) = \frac{1}{2\cosh\left\{\beta\sum_{j\neq i}\gamma^d J(i-j)\sigma(j)\right\}} \tag{2.7}$$

that, provided $J(0) = 0$, corresponds to $c(r, \nu) = (2\cosh\{\beta J * \nu(r)\})^{-1}$.

Mean Field Evolution Equation. As proven in [20], in the Lebowitz–Penrose limit (mesoscopic limit) the empirical magnetization under the Glauber dynamics becomes absolutely continuous and its density m evolves according to the non-local equation,

$$\frac{\partial m}{\partial t} = -2c(m)\sqrt{1-m^2}\,\sinh(\operatorname{arctanh} m - \beta J * m). \tag{2.8}$$

We notice that expanding the sinh, Eq. (2.8) reads,

$$\frac{\partial m}{\partial t} = 2c(m)\cosh(\beta J * m)(\tanh(\beta J * m) - m). \tag{2.9}$$

In particular, with the choice (2.7), the mean field evolution becomes,

$$\frac{\partial m}{\partial t} = \tanh(\beta J * m) - m. \tag{2.10}$$

The stationary solutions to (2.8) do not depend on the particular choice of the rates. In particular, since we are assuming $\beta > 1$, recalling (2.2), the spatially homogeneous stationary solutions are $m = \pm m_\beta$ that are stable and $m = 0$, which is unstable.

Action Functional. The large deviation asymptotics for the empirical magnetization under the Glauber dynamics for an Ising spin system with Kac potentials has been analyzed in [16]. We next recall the associated rate function.

Let $B_1(L)$ be the unit ball in $L^\infty(\mathbb{T}_L^d)$ equipped with the (metrizable) weak* topology. For $T > 0$, we then let $C([0,T]; B_1(L))$ be the set of $B_1(L)$-valued continuous functions equipped with the induced uniform distance. Let finally $C_*([0,T]; B_1(L))$ be the subset of functions φ in $C([0,T]; B_1(L))$ such that there exists $\psi \in L^1([0,T] \times \mathbb{T}_L^d)$ for which

$$\varphi(t,r) - \varphi(0,r) = \int_0^t \psi(s,r)\,\mathrm{d}s \quad r\text{ - a.e.} \quad \forall t \in [0,T].$$

Clearly, ψ is unique and will be denoted by $\dot\varphi$. We define the functional $I_{T,L} \colon C([0,T]; B_1(L)) \to [0,\infty]$ by

$$I_{T,L}(\varphi) = \begin{cases} \displaystyle\int_0^T \mathrm{d}t \int \mathrm{d}r\, \mathcal{L}(\varphi(t,\cdot), \dot\varphi(t,\cdot)) & \text{if } \varphi \in C_*([0,T]; B_1(L)), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.11}$$

where, given measurable functions $u \colon \mathbb{T}_L^d \to [-1,1]$ and $v \colon \mathbb{T}_L^d \to \mathbb{R}$,

$$\begin{aligned} \mathcal{L}(u,v) = {} & \frac{v}{2\beta} \log \frac{\frac{v}{2c(u)} + \sqrt{1 - u^2 + \frac{v^2}{4c(u)^2}}}{1-u} - \frac{v}{2} J * u \\ & + \frac{c(u)}{\beta} \left(\cosh(\beta J * u) - u \sinh(\beta J * u) - \sqrt{1 - u^2 + \frac{v^2}{4c(u)^2}} \right). \end{aligned} \tag{2.12}$$

Under suitable assumptions on the initial conditions, in [16] it is proven that the empirical magnetization sampled according to the Glauber dynamics, regarded as a random variable taking values in the Skorokhod space $D([0,T]; \mathcal{M}(\mathbb{T}_L^d))$, satisfies a large deviation principle with speed $\beta^{-1}\gamma^d$ and rate function $\mathcal{I}_{T,L}$ given by $\mathcal{I}_{T,L}(\nu) = I_{T,L}(\varphi)$ if $\nu_t = \varphi_t\,\mathrm{d}r$ for some $\varphi \in C_*([0,T]; B_1(L))$ and $+\infty$ otherwise.

For our purposes, by noticing that, as $\iota'(m) = \operatorname{arctanh} m$, the functional derivative (L^2-gradient) of F_L is given by

$$\frac{\delta F_L}{\delta m} = \beta^{-1} \operatorname{arctanh} m - J * m, \tag{2.13}$$

we rewrite the Lagrangian $\mathcal{L}$ in (2.12) in the form,

$$\begin{aligned} \mathcal{L}(u,v) = {} & \frac{v}{2\beta} \left(\operatorname{arctanh} u - \beta J * u + \operatorname{arcsinh} \frac{v}{2c(u)\sqrt{1-u^2}} \right) \\ & + \frac{c(u)}{\beta} \sqrt{1-u^2} \left(\cosh\left(\beta J * u - \operatorname{arctanh} u\right) - \sqrt{1 + \frac{v^2}{4c(u)^2(1-u^2)}} \right) \end{aligned}$$

$$= \frac{v}{2}\left(\frac{\delta F_L}{\delta u} + \frac{1}{\beta}\operatorname{arcsinh}\frac{v}{2c(u)\sqrt{1-u^2}}\right)$$
$$+ \frac{c(u)}{\beta}\sqrt{1-u^2}\left(\cosh\left(\beta\frac{\delta F_L}{\delta u}\right) - \sqrt{1+\tfrac{v^2}{4c(u)^2(1-u^2)}}\right).$$

Accordingly, the action functional becomes,

$$\begin{aligned} I_{T,L}(\varphi) &= \frac{1}{2}\big[F_L(\varphi(T)) - F_L(\varphi(0))\big] \\ &\quad + \int_0^T \mathrm{d}t \int \mathrm{d}r \,\left[\frac{\dot\varphi}{2\beta}\operatorname{arcsinh} W(\varphi,\dot\varphi) \right. \\ &\quad \left. -\frac{c(\varphi)}{\beta}\sqrt{1-\varphi^2}\left(\sqrt{1+W(\varphi,\dot\varphi)^2}-1\right)\right] \\ &\quad + \int_0^T \mathrm{d}t \int \mathrm{d}r \,\frac{c(\varphi)}{\beta}\sqrt{1-\varphi^2}\left(\cosh\left(\beta\frac{\delta F_L}{\delta\varphi}\right)-1\right), \end{aligned} \tag{2.14}$$

where

$$W(\varphi,\dot\varphi) = \frac{\dot\varphi}{2c(\varphi)\sqrt{1-\varphi^2}}. \tag{2.15}$$

It is worthwhile to remark that the above representation of the action functional reflects a Legendre duality. More precisely, for $\alpha > 0$ let $G(\cdot;\alpha)$ and $G^*(\cdot;\alpha)$ be the Legendre pair of real convex even functions,

$$G(q;\alpha) := \alpha(\cosh q - 1), \quad G^*(p;\alpha) = p\operatorname{arcsinh}(p/\alpha) - \sqrt{\alpha^2+p^2} + \alpha, \tag{2.16}$$

so that $qp + G(q;\alpha) + G^*(p;\alpha) \geq 0$ with equality if and only if $p = -\alpha\sinh q$. Then, (2.14) can be rewritten as

$$\begin{aligned} I_{T,L}(\varphi) &= \frac{1}{2}\big[F_L(\varphi(T)) - F_L(\varphi(0))\big] \\ &\quad + \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}r \,\Big[G\big(\beta\tfrac{\delta F_L}{\delta\varphi};\alpha(\varphi)\big) + G^*\big(\beta^{-1}\dot\varphi;\alpha(\varphi)\big)\Big], \end{aligned} \tag{2.17}$$

where $\alpha(\varphi) = 2\beta^{-1}c(\varphi)\sqrt{1-\varphi^2}$. From this representation, we easily conclude that the solution m to the mean field Eq. (2.8) is characterized by $I_{T,L}(m) = 0$, or equivalently $I_{T,L}(m) \leq 0$. The last inequality provides the following gradient flow formulation: m is a solution to (2.8) if and only if, for any $t \in [0,T]$,

$$F_L(m(t)) + \int_0^t \mathrm{d}s \int \mathrm{d}r \,\Big[G\big(\beta\tfrac{\delta F_L}{\delta m};\alpha(m)\big) + G^*\big(\beta^{-1}\dot m;\alpha(m)\big)\Big] \leq F_L(m(0)).$$

2.2. Sharp Interface Limit

A natural and physically relevant question is to investigate the limiting behavior of the Ising–Kac model in the sharp interface limit, in which the interface between the two stable phases $\pm m_\beta$ is described by surfaces of codimension one.

Excess Free Energy and Surface Tension. We set $\varepsilon = L^{-1}$ and rescale the space variable $r \in \mathbb{T}_L^d$ by setting $r = \varepsilon^{-1}x$ with $x \in \mathbb{T}^d$. We then introduce the rescaled excess free energy renormalized with a factor L^{d-1}. We namely define $F^\varepsilon \colon L^\infty(\mathbb{T}^d; [-1,1]) \to [0,\infty)$ by $F^\varepsilon(m) := \varepsilon^{d-1} F_{\varepsilon^{-1}}(m(\varepsilon^{-1}\cdot))$, i.e.,

$$
\begin{aligned}
F^\varepsilon(m) = & \int \mathrm{d}x\, \frac{f_\beta(m) - f_\beta(m_\beta)}{\varepsilon} \\
& + \frac{\varepsilon}{4} \int \mathrm{d}x \int \mathrm{d}y\; J_\varepsilon(x-y) \left[\frac{m(x) - m(y)}{\varepsilon}\right]^2,
\end{aligned}
\tag{2.18}
$$

where $J_\varepsilon(z) := \varepsilon^{-d} J(\varepsilon^{-1}z)$. The asymptotics of the excess free energy functional (2.18) has been discussed in [1,2], where it is proven that the limiting functional is finite only if m takes the values $\pm m_\beta$, and in this case its value is proportional to the perimeter of the jump set of m. The proportionality factor defines the *surface tension* of the Ising–Kac model, which is denoted by τ and will be characterized below. This result has been extended to the anisotropic case, i.e., when J is not radial; then, the surface tension τ is no longer constant but a convex function of the orientation [1,5].

The surface tension is the excess free energy cost per unit area of the transition between the two stable phases. The characterization of τ reduces to a one-dimensional computation in the direction normal to the interface. We introduce the *instanton* $\bar{m}(\xi)$, $\xi \in \mathbb{R}$, as the optimal magnetization profile of such a transition, that is, $\bar{m}$ is solution to

$$
\bar{m}(\xi) = \tanh \beta \widetilde{J} * \bar{m}(\xi), \quad \bar{m}(0) = 0, \quad \lim_{\xi \to \pm\infty} \bar{m}(\xi) = \pm m_\beta, \tag{2.19}
$$

where, recalling $J(r) = j(|r|)$,

$$
\widetilde{J}(\xi) = \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta\; j\big(\sqrt{\xi^2 + |\eta|^2}\big). \tag{2.20}
$$

Then, $\tau = \mathcal{F}(\bar{m})$, where $\mathcal{F}$ is the free energy functional on $\mathbb{R}$,

$$
\mathcal{F}(m) = \int \mathrm{d}\xi\; [f_\beta(m) - f_\beta(m_\beta)] + \frac{1}{4} \int \mathrm{d}\xi \int \mathrm{d}\xi'\; \widetilde{J}(\xi - \xi')[m(\xi) - m(\xi')]^2. \tag{2.21}
$$

It can be shown [14] that

$$
\tau = \int \mathrm{d}\xi\; \bar{m}'(\xi) \int \mathrm{d}\xi' \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta\; j\big(\sqrt{(\xi - \xi')^2 + |\eta|^2}\big)\; \bar{m}'(\xi')\, \frac{\eta_1^2}{2}. \tag{2.22}
$$

For later purpose, we recall the main properties of the instanton, see [18,21,22]. It is an odd and strictly increasing function which converges exponentially fast to its asymptotes. More precisely, $\bar{m}'(\xi) > 0$ and there are $a, c, \delta > 0$ such that, for any $\xi \ge 0$,

$$
\begin{aligned}
& \big|\bar{m}(\xi) - (m_\beta - a\mathrm{e}^{-\alpha\xi})\big| + \big|\bar{m}'(\xi) - a\alpha\mathrm{e}^{-\alpha\xi})\big| \\
& \quad + \big|\bar{m}''(\xi) - a\alpha^2\mathrm{e}^{-\alpha\xi})\big| \le c\mathrm{e}^{-(\alpha+\delta)\xi},
\end{aligned}
\tag{2.23}
$$

where α is the unique positive solution to the equation

$$\beta(1-m_\beta^2)\int \mathrm{d}\xi\ \widetilde{J}(\xi)\mathrm{e}^{-\alpha\xi} = 1. \tag{2.24}$$

Motion by Mean Curvature. Concerning the dynamical behavior, the sharp interface limit of the non-local evolution equation has been analyzed in [19, 20, 31], with the special choice of c as in (2.7). To describe these results, let m be the solution to (2.10) and define, according to a diffusive rescaling of space and time, $m^\varepsilon \colon \mathbb{R}_+ \times \mathbb{T}^d \to [-1,1]$ by $m^\varepsilon(t,x) = m(\varepsilon^{-2}t, \varepsilon^{-1}x)$, which solves

$$\frac{\partial m^\varepsilon}{\partial t} = \varepsilon^{-2}\big(\tanh(\beta J_\varepsilon * m^\varepsilon) - m^\varepsilon\big). \tag{2.25}$$

In order to describe the limiting behavior of m^ε, we briefly recall the notion of classical mean curvature flow. Given a C^1 family of oriented smooth surfaces $\Gamma = \{\Gamma(t)\}_{t\ge 0}$, with $\Gamma(t) = \partial\Omega(t)$ for some open $\Omega(t) \subset \mathbb{T}^d$, we denote by $n_t = n_{\Gamma(t)}$ the inward normal of $\Gamma(t)$, by $v_t \colon \Gamma(t) \to \mathbb{R}$ the normal velocity of Γ at time t. Finally, we set $\kappa_t = \kappa_{\Gamma(t)}$, where $\kappa_{\Gamma(t)} \colon \Gamma(t) \to \mathbb{R}$ is the mean curvature of $\Gamma(t)$. Then, given $\theta > 0$, Γ evolves according to the mean curvature flow with transport coefficient $\theta > 0$ if

$$v_t = \theta\kappa_t, \quad t \ge 0. \tag{2.26}$$

Given a mean curvature flow as above, assuming that the initial datum for (2.10) satisfies $m^\varepsilon(0,\cdot) \to m_\beta 1\!\!1_{\Omega(0)} - m_\beta 1\!\!1_{\Omega(0)^\complement}$, then $m^\varepsilon(t,\cdot) \to m_\beta 1\!\!1_{\Omega(t)} - m_\beta 1\!\!1_{\Omega(t)^\complement}$ for any $t > 0$. The actual value of θ obtained in [19, 31] will be discussed later.

In [20, 31], the convergence to the mean curvature flow is proven also starting directly from the microscopic Glauber dynamics. More precisely, letting $M^{\gamma,\varepsilon}$ be the diffusively rescaled empirical magnetization, it is shown that if $\varepsilon = |\log\gamma|^{-1}$ then $M^{\gamma,\varepsilon}$ satisfies a law of large numbers as $\gamma \to 0$, with limiting evolution being given by the mean curvature flow.

Transport Coefficients and Einstein Relation. The value of the transport coefficient θ, for arbitrary $c(m)$ of the form (2.6), can be inferred by using a linear response argument along the guidelines in [39]. Consider the non-local mean field Eq. (2.9) on $\mathbb{R}^d$ with external field h, that is,

$$\frac{\partial m}{\partial t} = 2c(m)\cosh(\beta(J*m+h))[\tanh(\beta(J*m+h)) - m].$$

In view of (2.6) and recalling that J and K are radial, solutions to the above equation with planar symmetry along a fixed direction $\widehat{n}$ have the form $m(t,\eta) = \widetilde{m}(\eta\cdot\widehat{n}, t)$ with $\widetilde{m}(\xi,t)$, $\xi \in \mathbb{R}$, solution to

$$\frac{\partial\widetilde{m}}{\partial t} = 2a(\widetilde{K}*\widetilde{m})\cosh(\beta(\widetilde{J}*\widetilde{m}+h))[\tanh(\beta(\widetilde{J}*\widetilde{m}+h)) - \widetilde{m}], \tag{2.27}$$

where $\widetilde{J}$ is defined in (2.20) and, analogously, recalling $K(r) = k(|r|)$,

$$\widetilde{K}(\xi) = \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta\ k\big(\sqrt{\xi^2+|\eta|^2}\big). \tag{2.28}$$

In particular, if we look for a traveling wave solution along $\widehat{n}$, i.e., a solution of the form $m(t,\eta) = q_h(\eta\cdot\widehat{n} - v(h)t)$, we deduce that q_h and the front velocity $v(h)$ do not depend on the direction $\widehat{n}$ and solve (in the case of (2.10) with h small, their existence is proven in [18])

$$-v(h)q_h' = 2a(\widetilde{K} * q_h)\cosh(\beta(\widetilde{J} * q_h + h))[\tanh(\beta(\widetilde{J} * q_h + h)) - q_h]. \quad (2.29)$$

In order to compute the linear response to the external field, we expand,

$$v(h) = v_1 h + O(h^2), \qquad q_h = \bar{m} + h\psi + O(h^2),$$

where $\bar{m}$ is the instanton which solves (2.29) with $h = 0$ and $v(0) = 0$, see (2.19). In the sequel, we set

$$\bar{a}(\xi) := a(\widetilde{K} * \bar{m}(\xi)), \quad \xi \in \mathbb{R}. \quad (2.30)$$

By (2.29), at the first order in h, we obtain the following identity,

$$-v_1\bar{m}' = \frac{2\bar{a}}{\sqrt{1-\bar{m}^2}}\big[-\psi + (1-\bar{m}^2)\beta\widetilde{J} * \psi + \beta(1-\bar{m}^2)\big],$$

where we used that $\cosh(\beta\widetilde{J} * \bar{m}) = 1/\sqrt{1-\tanh^2(\beta\widetilde{J} * \bar{m})} = 1/\sqrt{1-\bar{m}^2}$. We multiply both sides of the above equation by $\bar{m}'/(2\bar{a}\sqrt{1-\bar{m}^2})$ and then integrate; using that $\bar{m}' = (1-\bar{m}^2)\beta\widetilde{J} * \bar{m}'$, we obtain

$$v_1 = -2N\beta m_\beta,$$

where

$$N = \left[\int \mathrm{d}\xi\, \frac{(\bar{m}')^2}{2\bar{a}\sqrt{1-\bar{m}^2}}\right]^{-1}. \quad (2.31)$$

But, by the definition of the (macroscopic) mobility μ, see [39], it must be $v(h) = -2m_\beta\mu h + O(h^2)$. We conclude that

$$\mu = N\beta = \beta\left[\int \mathrm{d}\xi\, \frac{(\bar{m}')^2}{2\bar{a}\sqrt{1-\bar{m}^2}}\right]^{-1}. \quad (2.32)$$

We finally remark that in the case (2.7) we have $2\bar{a} = \sqrt{1-\bar{m}^2}$, so that $N = \left[\int \mathrm{d}\xi\, \frac{(\bar{m}')^2}{1-\bar{m}^2}\right]^{-1}$ in this case.

Sharp Interface Limit of the Action Functional. The main purpose of the section is to discuss the sharp interface limit of the action functional. To this end, we perform a diffusive rescaling of space and time with parameter $\varepsilon = L^{-1}$ and normalize the resulting action with a factor L^{d-1}. Namely, given $T > 0$, we define $S_\varepsilon\colon C([0,T]; B_1) \to [0,\infty]$ [here B_1 is a short notation for the unit ball $B_1(1)$ in $L^\infty(\mathbb{T}^d)$] by

$$S_\varepsilon(\varphi) = \varepsilon^{d-1} I_{\varepsilon^{-2}T,\varepsilon^{-1}}(\varphi(\varepsilon^2\cdot,\varepsilon\cdot)) = \varepsilon^{-1}\int_0^T \mathrm{d}t \int \mathrm{d}x\, \mathcal{L}_\varepsilon(\varphi(t,\cdot),\dot{\varphi}(t,\cdot)), \quad (2.33)$$

where, given measurable functions $u\colon \mathbb{T}^d \to [-1,1]$, $v\colon \mathbb{T}^d \to \mathbb{R}$ and recalling $J_\varepsilon(\cdot) := \varepsilon^{-d} J(\cdot/\varepsilon)$,

$$\begin{aligned}\mathcal{L}_\varepsilon(u,v) &= \frac{v}{2\beta}\log\frac{\dfrac{\varepsilon^2 v}{2c_\varepsilon(u)} + \sqrt{1-u^2+\left(\dfrac{\varepsilon^2 v}{2c_\varepsilon(u)}\right)^2}}{1-u} - \frac{v}{2}J_\varepsilon * u\\ &\quad + \frac{c_\varepsilon(u)}{\beta\varepsilon^2}\left(\cosh(\beta J_\varepsilon * u) - u\sinh(\beta J_\varepsilon * u) - \sqrt{1-u^2+\left(\frac{\varepsilon^2 v}{2c_\varepsilon(u)}\right)^2}\right),\end{aligned} \tag{2.34}$$

with, recalling (2.6) and letting $K_\varepsilon(\cdot) := \varepsilon^{-d}K(\cdot/\varepsilon)$,

$$c_\varepsilon(u) := a(K_\varepsilon * u). \tag{2.35}$$

We consider a C^1 family of oriented smooth surfaces $\Gamma = \{\Gamma(t)\}_{t\in[0,T]}$, with $\Gamma(t) = \partial\Omega(t)$ for some open $\Omega(t) \subset \mathbb{T}^d$. As before, we denote by $n_t = n_{\Gamma(t)}$ the inward normal of $\Gamma(t)$, by $v_t\colon \Gamma(t) \to \mathbb{R}$ the normal velocity of Γ at time t and by κ_t the mean curvature of $\Gamma(t)$. Letting $\widetilde{d}(\cdot,\Gamma(t))$ be the signed distance from $\Gamma(t)$, i.e., $\widetilde{d}(\cdot,\Gamma(t)) := \operatorname{dist}(\cdot,\Omega(t)^{\complement}) - \operatorname{dist}(\cdot,\Omega(t))$, we denote by $d(\cdot,\Gamma(t))$ a regularized version of $\widetilde{d}(\cdot,\Gamma(t))$ such that they coincide on a neighborhood of $\Gamma(t)$.

For such families of surfaces, the corresponding action functional is

$$S_{\mathrm{ac}}(\Gamma) = \frac{1}{4\mu}\int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma\ (v_t - \theta\kappa_t)^2, \tag{2.36}$$

with μ as given in (2.32) and $\theta = \mu\tau$ with τ as defined in (2.22). As next stated, it describes the sharp interface limit of the rescaled action functional associated with the Glauber dynamics for an Ising system with Kac potentials.

Theorem 2.1. *Let $\Gamma = \{\Gamma(t)\}_{t\in[0,T]}$ as before and consider a sequence $\{\varphi_\varepsilon\} \subset C([0,T];B_1)$ converging to $m_\beta 1\!\!1_{\Omega(\cdot)} - m_\beta 1\!\!1_{\Omega(\cdot)^\complement}$ of the form*

$$\varphi_\varepsilon(t,x) = \bar{m}\left(\frac{d(x,\Gamma(t))}{\varepsilon} + \varepsilon Q\left(t,x,\frac{d(x,\Gamma(t))}{\varepsilon}\right)\right) + \varepsilon R_\varepsilon(t,x), \tag{2.37}$$

where $\bar{m}$ is the instanton, $Q\colon [0,T]\times\mathbb{T}^d\times\mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} \frac{\big|Q(t,x,\xi)\big| + \big|\partial_t Q(t,x,\xi)\big| + \big|\partial_\xi Q(t,x,\xi)\big|}{1+|\xi|} < +\infty, \tag{2.38}$$

and $R_\varepsilon\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}$ is a smooth function.

(a) *If $\|R_\varepsilon\|_\infty + \|\partial_t R_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$ then, for any Q,*

$$\liminf_{\varepsilon\to 0} S_\varepsilon(\varphi_\varepsilon) \ge S_{\mathrm{ac}}(\Gamma).$$

(b) *There exist Q^* such that choosing $Q = Q^*$ and $R_\varepsilon = 0$, we have*

$$\lim_{\varepsilon\to 0} S_\varepsilon(\varphi_\varepsilon) = S_{\mathrm{ac}}(\Gamma).$$

From a physical viewpoint, the main content of this result is the identification of the transport coefficients in the limiting rate function S. As expected, the mobility μ, initially introduced via a linear response argument, coincides with the variance of fluctuations around the motion by mean curvature. The mechanism behind this identification is an averaging property, common to homogenization problems. At the mathematical level, this is achieved by the introduction (in the spirit of [31]) of the corrector Q in the ansatz (2.37): The transport coefficients are then identified by solving an optimization problem on Q. As mentioned in Introduction, this issue does not appear in the Allen–Cahn case, in which the introduction of correctors is not needed.

From a technical viewpoint, the previous results hint at Γ-convergence of the sequence of functionals S_ε to S_{ac}. Before discussing the missing steps, we review the state of affairs in the case of the Allen–Cahn action functional. The variational convergence of the rescaled action functional has been first studied in [33], and Theorem 2.1 is the analog of Proposition 2.2 there. Using tools from geometric measure theory and relying on the static results in [38], a Γ-liminf inequality has been proven in [36]. By exploiting the argument in [6], such inequality can be improved to account for a possible singular continuous part in the limiting paths. For smooth paths, a Γ-limsup inequality has been proven in [33], by constructing suitable recovery sequences. To achieve the full Γ-convergence, the missing step is a density theorem showing that arbitrary paths of finite action can be approximated by smooth paths. We emphasize that this issue concerns the limiting functional and not the approximating sequence.

For the Ising–Kac model, the main difficulty in obtaining a Γ-liminf inequality consists in showing that sequences φ_ε satisfying $S_\varepsilon(\varphi_\varepsilon) \le C$ are necessarily of the form given by (2.37) for suitable (not necessarily smooth) path Γ and some Q and R_ε. In the Allen–Cahn case, this structure is deduced as a consequence of the vanishing property of the discrepancy measures, but here we have no clue on how to handle this issue since we have no analog of discrepancy measures. Concerning the Γ-limsup inequality, statement (b) provides the construction of a recovery sequence when the limiting path Γ is smooth without nucleations. Combining this statement with the argument presented in Sect. 4, it is also possible to construct a recovery sequence for piecewise C^1 paths.

A natural further step is the analysis of the large deviation properties of the empirical magnetization for the underlying microscopic dynamics in the joint limit $\gamma \to 0$ and $\varepsilon \to 0$, for instance when $\varepsilon = |\log \gamma|^{-1}$. For the stochastic Allen–Cahn equation, the large deviations upper bound with rate function S is proven in [6] by constructing suitable exponential martingales. This strategy seems applicable also to the Ising–Kac model, but requires, as a crucial step, the Γ-convergence lower bound discussed above.

2.3. Proof of Theorem 2.1

To carry out the proof, we shall need the following results on the linearization of the non-local evolution. Consider Eq. (2.27) for $h = 0$; by (2.30) and using again the identity $\cosh(\beta\widetilde{J} * \bar{m}) = 1/\sqrt{1-\bar{m}^2}$, the linearization around the instanton gives rise to the linear operator,

$$L\psi = \frac{2\bar{a}}{\sqrt{1-\bar{m}^2}}(-\psi + (1-\bar{m}^2)\beta\widetilde{J} * \psi). \tag{2.39}$$

We regard it as an operator on $L^2(\mathbb{R};\nu)$, where

$$\nu(\mathrm{d}\xi) = \frac{\mathrm{d}\xi}{2\bar{a}(\xi)\sqrt{1-\bar{m}^2(\xi)}}. \tag{2.40}$$

Following [21], we observe that L is bounded, symmetric and negative semi-definite, with 0 a simple eigenvalue and $\bar{m}'$ the corresponding eigenvector. In fact, using again that $\bar{m}' = (1-\bar{m}^2)\beta\widetilde{J} * \bar{m}'$, it is easy to check that $L\bar{m}' = 0$ and that

$$\int \nu(\mathrm{d}\xi)\ \psi(\xi)L\psi(\xi) = -\frac{\beta}{2}\int \mathrm{d}\xi \int \mathrm{d}\xi'\ \widetilde{J}(\xi-\xi')\bar{m}'(\xi)\bar{m}'(\xi')\left[\frac{\psi}{\bar{m}'}(\xi) - \frac{\psi}{\bar{m}'}(\xi')\right]^2.$$

As $\widetilde{J}(0) > 0$ and $\widetilde{J}$ is continuous, we infer that the integral on the right-hand side is zero if and only if $\psi/\bar{m}'$ is constant. An application of Weyl's theorem shows that L has the gap property, i.e., that 0 is an isolated eigenvalue. A similar result holds also in L^∞. This is done in [18] for the case (2.7), and the extension to the general case is straightforward.

For expository reasons, we prove the statements in reverse order.

Proof of (b). Recalling (2.14), the decomposition (2.16) and (2.17), we rewrite the rescaled action functional (2.33) as

$$S_\varepsilon(\varphi) = S_\varepsilon^{(1)}(\varphi) + S_\varepsilon^{(2)}(\varphi) + S_\varepsilon^{(3)}(\varphi), \tag{2.41}$$

where

$$\begin{aligned}
S_\varepsilon^{(1)}(\varphi) &= \frac{1}{2}\big[F^\varepsilon(\varphi(T)) - F^\varepsilon(\varphi(0))\big],\\
S_\varepsilon^{(2)}(\varphi) &= \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}x\ G^*((\beta\varepsilon)^{-1}\dot{\varphi};\alpha_\varepsilon(\varphi)),\\
S_\varepsilon^{(3)}(\varphi) &= \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}x\ G\big(\varepsilon\beta\tfrac{\delta F^\varepsilon}{\delta m}(\varphi);\alpha_\varepsilon(\varphi)\big),
\end{aligned}$$

with F^ε as in (2.18) and

$$\alpha_\varepsilon(\varphi) = 2\frac{c_\varepsilon(\varphi)}{\beta\varepsilon^3}\sqrt{1-\varphi^2}, \quad \varepsilon\beta\frac{\delta F^\varepsilon}{\delta m}(\varphi) = \operatorname{arctanh}\varphi - \beta J_\varepsilon * \varphi. \tag{2.42}$$

In the sequel, we choose $\varphi = \varphi_\varepsilon$ as in (2.37), with $R_\varepsilon = 0$ and Q to be determined later, and analyze separately the contribution of the three terms in (2.41).

(1) As proven in [37], the free energy F^ε Γ-converges to $\tau\mathrm{Per}(\cdot)$, where $\mathrm{Per}(\cdot)$ is the perimeter functional. Moreover, for any choice of the corrector Q and $t \in [0,T]$, the function $\varphi_\varepsilon(t,\cdot)$ is a recovery sequence. Hence,

$$\lim_{\varepsilon\to 0} S_\varepsilon^{(1)}(\varphi_\varepsilon) = \frac{\tau}{2}\left[\mathrm{Per}(\Omega(T)) - \mathrm{Per}(\Omega(0)\right] = -\frac{\tau}{2}\int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma\ \kappa_t v_t,$$

where in the last equality we used that $-\int_{\Gamma(t)} \mathrm{d}\sigma\, \kappa_t v_t$ is the time derivative of $\mathrm{Per}(\Omega(t))$. By (1.3), we thus have,

$$\lim_{\varepsilon\to 0} S_\varepsilon^{(1)}(\varphi_\varepsilon) = -\frac{1}{\mu}\int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma\, \frac{\theta\kappa_t v_t}{2}. \tag{2.43}$$

(2) We first notice that, by Taylor expansion, $G^*(p,\alpha) = \alpha\big[\frac{1}{2}\big(\frac{p}{\alpha}\big)^2 + O\big(\big(\frac{p}{\alpha}\big)^4\big)\big]$. By (2.38),

$$\dot\varphi_\varepsilon(x,t) = -\tfrac{\partial_t d(x,\Gamma(t))}{\varepsilon}\, \bar m'\left(\tfrac{d(x,\Gamma(t))}{\varepsilon}(1+O(\varepsilon))\right)\left(1+\left(1+\tfrac{|d(x,\Gamma(t))|}{\varepsilon}\right)O(\varepsilon)\right). \tag{2.44}$$

As $\bar m'(\xi)$ converges exponentially fast to zero as $|\xi|\to\infty$, see (2.23), and in view of (2.42), the integrand appearing in $S_\varepsilon^{(2)}$ is smaller than any power of ε if $|d(x,\Gamma(t))| > C\varepsilon(\log\varepsilon)^2$. Therefore, we can restrict the domain of integration in a small neighborhood of Γ_t. In view of the expansion of G^*, using the co-area formula, we then get,

$$\lim_{\varepsilon\to 0} S_\varepsilon^{(2)}(\varphi_\varepsilon) = \lim_{\varepsilon\to 0}\int_0^T \mathrm{d}t\int_{\mathcal{C}_\varepsilon}\mathrm{d}s\int_{d=s}\mathrm{d}\sigma\,\varepsilon^{-1}\frac{\bar m'(s/\varepsilon)^2}{2c_\varepsilon(\varphi_\varepsilon)\sqrt{1-\bar m(s/\varepsilon)^2}}\,\frac{(\partial_t d)^2}{4\beta}, \tag{2.45}$$

where $\mathcal{C}_\varepsilon := [-C\varepsilon(\log\varepsilon)^2, C\varepsilon(\log\varepsilon)^2]$, $\mathrm{d}\sigma$ is the surface measure on the level set of the distance function d, and, by (2.35), $c_\varepsilon(\varphi_\varepsilon) = a(K_\varepsilon * \varphi_\varepsilon)$. To compute the asymptotic behavior of $c_\varepsilon(\varphi_\varepsilon)$, we choose an orthonormal frame with origin in the orthogonal projection $x_{\Gamma(t)}$ of x on $\Gamma(t)$ and the first direction e_0 along the normal to $\Gamma(t)$ at $x_{\Gamma(t)}$. If $d(x,\Gamma(t)) = s$, then $x = s\ \mathrm{e}_0$ and therefore, using (2.28),

$$\begin{aligned}
&K_\varepsilon * \varphi_\varepsilon(x)\\
&\quad= \int \mathrm{d}y\,\varepsilon^{-d} k\left(\varepsilon^{-1}\sqrt{(s-y\cdot\mathrm{e}_0)^2 + |y-(y\cdot\mathrm{e}_0)\mathrm{e}_0|^2}\right)\bar m\left(\frac{y\cdot\mathrm{e}_0}{\varepsilon}\right) + O(\varepsilon)\\
&\quad= \int \mathrm{d}\xi'\ \widetilde K\left(\frac{s}{\varepsilon}-\xi'\right)\bar m(\xi') + O(\varepsilon) = \widetilde K * \bar m\left(\frac{s}{\varepsilon}\right) + O(\varepsilon).
\end{aligned}$$

We conclude that recalling the definition of $\bar a$ in (2.30),

$$\begin{aligned}
\lim_{\varepsilon\to 0} S_\varepsilon^{(2)}(\varphi_\varepsilon) &= \int_0^T \mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\,\frac{v_t^2}{4\beta}\int \mathrm{d}\xi\,\frac{\bar m'(\xi)^2}{2\bar a(\xi)\sqrt{1-\bar m(\xi)^2}}\\
&= \frac{1}{\mu}\int_0^T \mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\,\frac{v_t^2}{4},
\end{aligned} \tag{2.46}$$

where we used that $-\partial_t d(\cdot,\Gamma(t)) = v_t$ on $\Gamma(t)$, and (2.31) and (2.32) in the last identity.

(3) We are left with the limit of $S_\varepsilon^{(3)}(\varphi_\varepsilon)$. This is the point where the corrector Q plays a role and has to be chosen appropriately. As $\operatorname{arctanh} \bar m = \beta \widetilde J * \bar m$,

$$\varepsilon\beta \frac{\delta F^\varepsilon}{\delta m}(\varphi_\varepsilon)(x,t) = \beta \int \mathrm{d}\xi' \; \widetilde J \left(\frac{d(x,\Gamma(t))}{\varepsilon} + \varepsilon Q\left(t,x,\frac{d(x,\Gamma(t))}{\varepsilon}\right) - \xi'\right) \bar m(\xi')$$
$$- \beta \int \mathrm{d}y \; J_\varepsilon(x,y)\; \bar m \left(\frac{d(y,\Gamma(t))}{\varepsilon} + \varepsilon Q\left(t,x,\frac{d(y,\Gamma(t))}{\varepsilon}\right)\right).$$

Since $\bar m(\xi)$ converges exponentially fast to $\pm m_\beta$ as $\xi \to \pm\infty$, see (2.23), the above expression is smaller than any power of ε if $|d(x,\Gamma(t))| > C\varepsilon(\log\varepsilon)^2$. Therefore, as $G(q,\alpha) = \alpha\big[\frac12 q^2 + O(q^4)\big]$, restricting the domain of integration, and using the previous computation for the limit of $c_\varepsilon(\varphi_\varepsilon)$, we obtain,

$$\lim_{\varepsilon\to 0} S_\varepsilon^{(3)}(\varphi_\varepsilon)$$
$$= \lim_{\varepsilon\to 0} \frac{1}{2\beta\varepsilon^3} \int_0^T \mathrm{d}t \int_{\mathcal{C}_\varepsilon} \mathrm{d}s \int_{d=s} \mathrm{d}\sigma \; \bar a\left(\frac{s}{\varepsilon}\right) \sqrt{1 - \bar m\left(\frac{s}{\varepsilon}\right)^2} \; \left(\varepsilon\beta \frac{\delta F^\varepsilon}{\delta m}(\varphi_\varepsilon)\right)^2.$$

To compute $\varepsilon\beta\frac{\delta F^\varepsilon}{\delta m}(\varphi_\varepsilon)$, we choose an orthonormal frame with origin in the orthogonal projection $x_{\Gamma(t)}$ of x on $\Gamma(t)$, the first direction e_0 along the normal to $\Gamma(t)$ and the remaining directions $\{\mathrm{e}_1,\dots,\mathrm{e}_{d-1}\}$ along the principal curvature directions of $\Gamma(t)$. In this way, if $d(x,\Gamma(t)) = s$ with $|s| \le C\varepsilon(\log\varepsilon)^2$ and $|x-y| \le \varepsilon$, we have,

$$x = s\,\mathrm{e}_0, \qquad d(y,\Gamma(t)) = y\cdot \mathrm{e}_0 - \sum_{i=1}^{d-1} \kappa_t^{(i)} \frac{(y\cdot\mathrm{e}_i)^2}{2} + o(\varepsilon^2),$$

where $\kappa_t^{(i)}$ are the principal curvatures of $\Gamma(t)$ at $x_{\Gamma(t)}$; in particular, the mean curvature reads $\kappa_t = \sum_{i=1}^{d-1}\kappa_t^{(i)}$. Therefore, if $d(x,\Gamma(t)) = s$,

$$\beta \int \mathrm{d}\xi' \; \widetilde J \left(\frac{d(x,\Gamma(t))}{\varepsilon} + \varepsilon Q\left(t,x,\frac{d(x,\Gamma(t))}{\varepsilon}\right) - \xi'\right) \bar m(\xi')$$
$$= \beta \int \mathrm{d}\xi' \; \widetilde J\left(\frac{s}{\varepsilon} - \xi'\right) \left[\bar m(\xi') + \varepsilon Q\left(t,x,\frac{s}{\varepsilon}\right) \bar m'(\xi')\right] + o(\varepsilon)$$

and

$$\beta \int \mathrm{d}y \; J_\varepsilon(x,y)\; \bar m \left(\frac{d(y,\Gamma(t))}{\varepsilon} + \varepsilon Q\left(t,y,\frac{d(y,\Gamma(t))}{\varepsilon}\right)\right)$$
$$= \beta \int \mathrm{d}y \; \varepsilon^{-d} j\left(\varepsilon^{-1}\sqrt{(s-y\cdot\mathrm{e}_0)^2 + |y-(y\cdot\mathrm{e}_0)\mathrm{e}_0|^2}\right) \left[\bar m\left(\frac{y\cdot\mathrm{e}_0}{\varepsilon}\right)\right.$$
$$\left. + \varepsilon Q\left(t,x,\frac{y\cdot\mathrm{e}_0}{\varepsilon}\right) \bar m'\left(\frac{y\cdot\mathrm{e}_0}{\varepsilon}\right) - \sum_{i=1}^{d-1} \kappa_t^{(i)} \frac{(y\cdot\mathrm{e}_i)^2}{2}\; \bar m'\left(\frac{y\cdot\mathrm{e}_0}{\varepsilon}\right)\right] + o(\varepsilon)$$
$$= \beta \int \mathrm{d}\xi' \; \widetilde J\left(\frac{s}{\varepsilon} - \xi'\right) \left[\bar m(\xi') + \varepsilon Q(t,x,\xi')\bar m'(\xi')\right]$$
$$- \varepsilon\beta\kappa_t \int \mathrm{d}\xi' \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta \; j\left(\sqrt{\left(\frac{s}{\varepsilon} - \xi'\right)^2 + |\eta|^2}\right) \bar m'(\xi')\; \frac{\eta_1^2}{2} + o(\varepsilon).$$

We now choose $Q(t,x,\xi) = Q^*(t,x,\xi) := \mathcal{K}(t,x)\bar{Q}(\xi)$, where $\mathcal{K}\colon [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is any smooth function satisfying $\mathcal{K}(t,x) = \kappa_t(x)$ for all $x \in \Gamma(t)$, while $\bar{Q}\colon \mathbb{R} \to \mathbb{R}$ is a suitable a smooth function, to be fixed later and satisfying

$$\sup_{\xi\in\mathbb{R}} \frac{|\bar{Q}(\xi)| + |\bar{Q}'(\xi)|}{1+|\xi|} < +\infty. \tag{2.47}$$

Therefore, under this assumption,

$$\varepsilon\beta \frac{\delta F^\varepsilon}{\delta m}(\varphi_\varepsilon) = \varepsilon\beta\mathcal{K}(t,x) \int \mathrm{d}\xi'\, \widetilde{J}\left(\frac{s}{\varepsilon} - \xi'\right) \bar{m}'(\xi') \left[\bar{Q}\left(\frac{s}{\varepsilon}\right) - \bar{Q}(\xi')\right] \\ + \varepsilon\beta\kappa_t \int \mathrm{d}\xi' \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta\, j\left(\sqrt{\left(\frac{s}{\varepsilon} - \xi'\right)^2 + |\eta|^2}\right) \bar{m}'(\xi')\, \frac{\eta_1^2}{2} + o(\varepsilon). \tag{2.48}$$

Inserting this expansion in the approximated expression for $S_\varepsilon^{(3)}(\varphi_\varepsilon)$, we obtain

$$\lim_{\varepsilon\to 0} S_\varepsilon^{(3)}(\varphi_\varepsilon) = \int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma\; A_{\bar{Q}}(\kappa_t), \tag{2.49}$$

where

$$A_{\bar{Q}}(\kappa_t) := \frac{\kappa_t^2}{2\beta} \int \mathrm{d}\xi\; \bar{a}\sqrt{1-\bar{m}^2}\; \left[\beta\widetilde{J} * (\bar{m}'\bar{Q}) - \beta(\widetilde{J} * \bar{m}')\bar{Q} - \beta f\right]^2,$$

with

$$f(\xi) = \int \mathrm{d}\xi' \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta\, j\left(\sqrt{(\xi-\xi')^2 + |\eta|^2}\right) \bar{m}'(\xi')\, \frac{\eta_1^2}{2}. \tag{2.50}$$

Recalling the definitions (2.39), (2.40), and using $\bar{m}' = (1-\bar{m}^2)\beta\widetilde{J} * \bar{m}'$, we get,

$$A_{\bar{Q}}(\kappa_t) = \frac{\kappa_t^2}{4\beta} \int \nu(\mathrm{d}\xi)\; \left[L(\bar{m}'\bar{Q}) - H\right]^2,$$

where

$$H := \beta\, 2\bar{a}\sqrt{1-\bar{m}^2}\, f. \tag{2.51}$$

By (2.31) and (2.40),

$$\frac{\int \nu(\mathrm{d}\xi)\; \bar{m}'(\xi) H(\xi)}{\int \nu(\mathrm{d}\xi)\; (\bar{m}')^2} = N\beta \int \mathrm{d}\xi\; \bar{m}'(\xi) f(\xi) = N\beta\tau = \theta,$$

where in the last equalities, we used that, by (2.22), $\tau = \int \mathrm{d}\xi\; \bar{m}'(\xi) f(\xi)$, and relations (1.3) and (2.32). It follows that the component of H orthogonal to $\bar{m}'$ in $L^2(\mathbb{R};\nu)$ is

$$\widehat{H} = H - \theta\bar{m}'. \tag{2.52}$$

Therefore, by the symmetry of L and $L\bar{m}' = 0$,

$$A_{\bar{Q}}(\kappa_t) = \frac{(\theta\kappa_t)^2}{4\mu} + \frac{1}{4\beta} \int \nu(\mathrm{d}\xi)\; \left[L(\bar{m}'\bar{Q}) - \widehat{H}\right]^2.$$

The corrector $\bar{Q}$ is now determined by minimizing the above expression. More precisely, $\bar{Q}$ is the solution to the equation $L(\bar{m}'\bar{Q}) = \widehat{H}$ which satisfies (2.47) and $\bar{Q}(0) = 0$, whose existence and uniqueness is the content of Lemma A.1 in "Appendix A." In view of (2.49), with this choice of Q, we get

$$\lim_{\varepsilon \to 0} S_\varepsilon^{(3)}(\varphi_\varepsilon) = \frac{1}{\mu} \int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma \, \frac{(\theta \kappa_t)^2}{4}. \tag{2.53}$$

The statement (b) of the theorem follows from (2.43), (2.46) and (2.53). *Proof of* (a). By Legendre duality, $\mathcal{L}_\varepsilon(u, v) = \sup_p \{pv - \mathcal{H}_\varepsilon(u, p)\}$, where, given measurable functions $u \colon \mathbb{T}^d \to [-1, 1]$ and $\eta \colon \mathbb{T}^d \to \mathbb{R}$,

$$\begin{aligned} \mathcal{H}_\varepsilon(u, \eta) = \varepsilon^{-2} \frac{c_\varepsilon(u)}{\beta} \Big[& \cosh(\beta J_\varepsilon * u + 2\beta\eta) - \cosh(\beta J_\varepsilon * u) \\ & - u \sinh(\beta J_\varepsilon * u + 2\beta\eta) + u \sinh(\beta J_\varepsilon * u) \Big]. \end{aligned} \tag{2.54}$$

Whence, letting φ_ε be as in (2.37), for each $g = g(t, x)$,

$$\begin{aligned} S_\varepsilon(\varphi_\varepsilon) \geq \varepsilon^{-1} \int_0^T \mathrm{d}t \int \mathrm{d}x \, \Big\{ & \dot{\varphi}_\varepsilon g - \varepsilon^{-2} \frac{c_\varepsilon(\varphi_\varepsilon)}{\beta} \Big[\cosh(\beta J_\varepsilon * \varphi_\varepsilon + 2\beta g) \\ & - \cosh(\beta J_\varepsilon * \varphi_\varepsilon) - \varphi_\varepsilon \sinh(\beta J_\varepsilon * \varphi_\varepsilon + 2\beta g) + \varphi_\varepsilon \sinh(\beta J_\varepsilon * \varphi_\varepsilon) \Big] \Big\} \\ =: & \, \Lambda_\varepsilon(\varphi_\varepsilon, g). \end{aligned}$$

Given a fixed smooth function $p = p(t, x)$, we choose (recall N is defined in (2.31))

$$g(t, x) = g_\varepsilon(t, x) = \varepsilon N p(t, x) \left[\frac{\bar{m}'(s/\varepsilon)}{2\bar{a}(s/\varepsilon)\sqrt{1 - \bar{m}(s/\varepsilon)^2}} \right]_{s = d(x, \Gamma(t))}$$

and compute the limit of $\Lambda_\varepsilon(\varphi_\varepsilon, g_\varepsilon)$ as $\varepsilon \to 0$. By second-order Taylor expansion of $\mathcal{H}_\varepsilon(u, \cdot)$ and observing the remainder are equibounded and converge to zero point-wise as $\varepsilon \to 0$, we have

$$\Lambda_\varepsilon(\varphi_\varepsilon, g_\varepsilon) = \Lambda_\varepsilon^{(1)}(\varphi_\varepsilon, g_\varepsilon) + \Lambda_\varepsilon^{(2)}(\varphi_\varepsilon, g_\varepsilon) + \Lambda_\varepsilon^{(3)}(\varphi_\varepsilon, g_\varepsilon) + o(1),$$

where

$$\begin{aligned} \Lambda_\varepsilon^{(1)}(\varphi_\varepsilon, g_\varepsilon) &= \int_0^T \mathrm{d}t \int \mathrm{d}x \, \varepsilon^{-1} \dot{\varphi}_\varepsilon g_\varepsilon, \\ \Lambda_\varepsilon^{(2)}(\varphi_\varepsilon, g_\varepsilon) &= \int_0^T \mathrm{d}t \int \mathrm{d}x \, \frac{c_\varepsilon(\varphi_\varepsilon)}{\varepsilon^3} \cosh(\beta J_\varepsilon * \varphi_\varepsilon) \big[\varphi_\varepsilon - \tanh(\beta J_\varepsilon * \varphi_\varepsilon)\big] 2 g_\varepsilon \\ &= \int_0^T \mathrm{d}t \int \mathrm{d}x \, \frac{c_\varepsilon(\varphi_\varepsilon)}{\varepsilon^3} \sqrt{1 - \varphi_\varepsilon^2} \sinh\left(\varepsilon\beta \frac{\delta F^\varepsilon}{\delta m}(\varphi_\varepsilon) \right) 2 g_\varepsilon, \\ \Lambda_\varepsilon^{(3)}(\varphi_\varepsilon, g_\varepsilon) &= -\int_0^T \mathrm{d}t \int \mathrm{d}x \, \frac{c_\varepsilon(\varphi_\varepsilon)}{\varepsilon^3} \cosh(\beta J_\varepsilon * \varphi_\varepsilon) \big[1 - \varphi_\varepsilon \tanh(\beta J_\varepsilon * \varphi_\varepsilon)\big] 2\beta g_\varepsilon^2. \end{aligned}$$

By (2.38) and the assumptions on R_ε, the expansion (2.44) holds with an extra additive $o(\varepsilon)$ due to the presence of R_ε. Therefore, as g_ε is equibounded and recalling (2.31), the same reasoning leading to (2.46) gives,

$$\lim_{\varepsilon\to 0}\Lambda_\varepsilon^{(1)}(\varphi_\varepsilon,g_\varepsilon)=\lim_{\varepsilon\to 0}\int_0^T \mathrm{d}t\int_{\mathcal{C}_\varepsilon}\mathrm{d}\sigma\;\varepsilon^{-1}\;\frac{-\bar m'(s/\varepsilon)^2 Np\,\partial_t d}{2\bar a(s/\varepsilon)\sqrt{1-\bar m(s/\varepsilon)^2}}$$
$$=-\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;v_t p.$$

Concerning $\Lambda_\varepsilon^{(2)}$ and $\Lambda_\varepsilon^{(3)}$, we observe that as $g_\varepsilon=O(\varepsilon)\bar m'(d/\varepsilon)$ and the dependence on φ_ε of the integrands is locally Lipschitz, the contribution due to R_ε is $o(1)$ as $\varepsilon\to 0$ and therefore can be neglected.

Noticing that (2.48) holds true here with Q in place of $\mathcal{K}\bar Q$ and recalling (2.50), we have

$$\lim_{\varepsilon\to 0}\Lambda_\varepsilon^{(2)}(\varphi_\varepsilon,g_\varepsilon)=\lim_{\varepsilon\to 0}\int_0^T\mathrm{d}t\int_{\mathcal{C}_\varepsilon}\mathrm{d}s\int_{d=s}\mathrm{d}\sigma\;\;Np$$
$$\times\,\varepsilon^{-1}\Big\{\bar m'\big[-\beta\widetilde J*(\bar m'Q)+\beta(\widetilde J*\bar m')Q+\beta\mathcal{K}f\big]\Big\}\left(\frac{s}{\varepsilon}\right)$$
$$=\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;Np\int\nu(\mathrm{d}\xi)\;\bar m'\big[\beta\kappa_t\,2\bar a\sqrt{1-\bar m^2}\,f-L(\bar m'Q)\big]$$
$$=\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;\theta\kappa_t p,$$

where in the last identity we used that $\int\nu(\mathrm{d}\xi)\;\bar m'L(\bar m'Q)=0$. Finally, as $\cosh(\beta J_\varepsilon*\varphi_\varepsilon)\big[1-\varphi_\varepsilon\tanh(\beta J_\varepsilon*\varphi_\varepsilon)\big]=\sqrt{1-\bar m(s/\varepsilon)^2}+o(1)$,

$$\lim_{\varepsilon\to 0}\Lambda_\varepsilon^{(3)}(\varphi_\varepsilon,g_\varepsilon)=\lim_{\varepsilon\to 0}\int_0^T\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s\int_{d=s}\mathrm{d}\sigma\;\;\varepsilon^{-1}\beta N^2p^2\frac{-\bar m'(s/\varepsilon)^2}{2\bar a(s/\varepsilon)\sqrt{1-\bar m(s/\varepsilon)^2}}$$
$$=\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;(-\mu p^2).$$

We conclude that, for any function $p=p(x,t)$,

$$\varliminf_{\varepsilon\to 0}S_\varepsilon(\varphi_\varepsilon)\ge\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;(-v_tp+\theta\kappa_tp-\mu p^2),$$

whence, by optimizing over p,

$$\varliminf_{\varepsilon\to 0}S_\varepsilon(\varphi_\varepsilon)\ge\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;\sup_p\,(-v_tp+\theta\kappa_tp-\mu p^2)$$
$$=\frac{1}{4\mu}\int_0^T\mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\;(v_t-\theta\kappa_t)^2.$$

The statement (a) of the theorem is thus proven. □

3. Glauber+Kawasaki Process

In this section, we analyze the sharp interface limit of the action functional in the context of the Glauber+Kawasaki process.

3.1. Motivation

The so-called Glauber+Kawasaki process is a simple stochastic model describing a chemical reaction among two species together with their diffusion. Recall that $\mathbb{T}_L^d$ denotes the d-dimensional torus of side $L \in \mathbb{N}$ and, given an integer $N \geq 1$, in this section we let $\mathbb{T}_{L,N}^d := (N^{-1}\mathbb{Z}/L\mathbb{Z})^d$ be the discrete approximation of $\mathbb{T}_L^d$ with lattice spacing $1/N$. Set also $\Omega_{N,L} := \{0,1\}^{\mathbb{T}_{L,N}^d}$, if $\eta \in \Omega_{N,L}$ we regard its value at the site $i \in \mathbb{T}_{L,N}^d$, that can be either zero or one, as representing the species occupying i. The Glauber+Kawasaki process is a continuous-time Markov chain on the state space $\Omega_{N,L}$, whose dynamics is obtained superimposing two elementary mechanisms, respectively, modeling the reaction (Glauber) and the diffusion (Kawasaki). Namely, the generator of the chain is

$$\mathcal{L}_N := \mathcal{L}_{\mathrm{G}} + N^2 \mathcal{L}_{\mathrm{K}}. \tag{3.1}$$

Let c, a strictly positive local function of the configuration, be the rate of the reaction; then, given $f\colon \Omega_{N,L} \to \mathbb{R}$,

$$\mathcal{L}_{\mathrm{G}} f\,(\eta) := \sum_{i \in \mathbb{T}_{L,N}^d} c(\tau_i \eta)\big[f(\eta^i) - f(\eta)\big],$$

where τ_i is the translation, i.e., $(\tau_i \eta)_j := \eta_{j-i}$, and η^i is the configuration obtained from η by flipping the occupation number at i. The Kawasaki dynamics is instead defined by the generator,

$$\mathcal{L}_{\mathrm{K}} f\,(\eta) := \frac{1}{2} \sum_{\{i,j\}} \big[f(\eta^{i,j}) - f(\eta)\big],$$

where the sum runs over the (unordered) nearest neighbors pairs $\{i,j\} \subset \mathbb{T}_{L,N}^d$ and $\eta^{i,j}$ is the configuration obtained from η by exchanging the occupation numbers at the sites i and j. Note that in (3.1), according to a diffusive rescaling, the Kawasaki dynamics has been speeded up by N^2. Let $\mathcal{M}_+(\mathbb{T}_L^d)$ be the set of positive measures on $\mathbb{T}_L^d$ and define the *empirical density* as the map $\pi^N\colon \Omega_{L,N} \to \mathcal{M}_+(\mathbb{T}_L^d)$ given by

$$\pi^N(\eta) = \frac{1}{N^d} \sum_{i \in \mathbb{T}_{L,N}^d} \eta_i \delta_i.$$

Assuming that the initial datum $\eta(0)$ for the Glauber+Kawasaki process is well prepared, in the sense that $\pi^N(\eta(0)) \to u_0(x)\mathrm{d}x$ for some Borel function $u_0\colon \mathbb{T}_L \to [0,1]$, in [17] it is proven that $\pi^N(\eta(t)) \to u(t,x)\mathrm{d}x$ in probability, where $u\colon [0,\infty) \times \mathbb{T}_L^d \to [0,1]$ solves the reaction diffusion equation,

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + B(u) - D(u), \\ u(0) = u_0. \end{cases} \tag{3.2}$$

The reaction term is described by the coefficients $B, D\colon [0,1] \to [0,+\infty)$ that can be obtained from the microscopic rate $c\colon \Omega_{L,N} \to (0,\infty)$ according to the following procedure. For $\rho \in [0,1]$, let ν_ρ be the Bernoulli measure with parameter ρ, namely the product probability on $\Omega_{L,N}$ with marginals $\nu_\rho(\eta_i = 1) = \rho$. Then,

$$B(\rho) = \nu_\rho\big((1-\eta_0)c\big), \qquad D(\rho) = \nu_\rho\big(\eta_0 c\big).$$

Observe that, as c is a strictly positive local function, B and D are strictly positive polynomials in $(0,1)$, while $B(1) = 0$ and $D(0) = 0$.

The hydrodynamic Eq. (3.2) describes the typical behavior of the Glauber+Kawasaki process in the diffusive scaling limit. On the other hand, the statistics of the fluctuations cannot be described simply by adding a Gaussian noise to (3.2). In fact, as made precise by large deviation theory, the Poissonian nature of the underlying Glauber dynamics is still felt in the diffusive limit. A main motivation for analyzing the large deviations properties of the empirical density is the following. Since the Glauber+Kawasaki process is irreducible, by a general criterion for Markov chains, there exists a unique stationary probability $\mu_{L,N}$ on $\Omega_{L,N}$. As the dynamics does not satisfy the detailed balance condition, $\mu_{L,N}$ cannot be written in a closed form (with the exception of the special choices discussed in [27]) and, as shown in [4], it exhibits long-range correlations. According to the general philosophy concerning thermodynamics, we are not really interested in all details of the probability $\mu_{L,N}$, but rather in the statistics of the empirical density in the limit $N \to \infty$. It is therefore natural to introduce the sequence of probabilities $\{\wp_{L,N}\}_{N\in\mathbb{N}}$ on $\mathcal{M}_+(\mathbb{T}_L^d)$ defined by $\wp_{L,N} := \mu_{L,N} \circ (\pi^N)^{-1}$ and look for its asymptotic behavior as $N \to \infty$. Let $W\colon [0,1] \to \mathbb{R}$ be such that $B - D = -W'$. If W has a unique minimizer, it is natural to expect that the sequence $\{\wp_{L,N}\}_{N\in\mathbb{N}}$ converges to the stationary solution of (3.2) corresponding to the minimizer of W. Indeed, in the one-dimensional case, this is proven in [10] when W has a single well, and in [11] when W has a double well.

A finer description of the asymptotics of $\{\wp_{L,N}\}_{N\in\mathbb{N}}$ can be achieved by looking at its large deviations. This means obtaining an estimate of the form

$$\wp_{L,N}\big(\pi \sim u\,\mathrm{d}x\big) \asymp \exp\{-N^d F_L(u)\},$$

for a suitable functional F_L on the set of densities $u\colon \mathbb{T}_L^d \to [0,1]$. Here, F_L plays the same role as the Cahn–Hilliard functional in the gradient theory of phase transition or the Lebowitz–Penrose functional (2.3), with the minor inconvenience that it is not known.

According to the Freidlin–Wentzell theory for diffusions on $\mathbb{R}^n$, see [25], the functional F_L can be characterized in terms of a dynamical problem. To this end, fix $T > 0$, a sequence of initial configurations $\eta^N(0)$, and consider the large deviations asymptotics for the empirical measure in the time window $[0,T]$. Under the assumption that B and D are concave, this large deviation principle has been proven in [12,29,35] in one dimension (however, the result can be extended to higher dimension), the corresponding rate function, denoted by $I_{T,L}$, will be recalled later. As proven in [25, Chap. 6] for diffusions on $\mathbb{R}^n$ and in [24] for the present setting [in one dimension and with additional hypotheses on the coefficients B and D implying a complete characterization of the stationary solutions to (3.2)], the functional F_L is the *quasi-potential* associated with the dynamical rate functional $I_{T,L}$. This means that F_L can be obtained from $I_{T,L}$ by solving a suitable variational/combinatorial problem whose details are here omitted. Using this approach, in [24], the cluster points of $\{\wp_{L,N}\}$ are shown to be supported on stationary solutions to (3.2) associated with the minimizers of W.

We consider here the case of a bistable reaction term. Recalling W satisfies $B - D = -W'$, we namely assume that W has a twofold degenerate quadratic minimum. In other words, there exist $0 < \rho_- < \rho_+ < 1$ such that for $\rho \neq \rho_\pm$ we have $W(\rho) > W(\rho_-) = W(\rho_+)$ and $W''(\rho_-), W''(\rho_+) > 0$. In this situation, the probability $\mu_{L,N}$ describes the phase coexistence of the two stable phases, like a Gibbs measure undergoing a first-order phase transition. Our purpose is to characterize the corresponding surface tension τ. By definition, τ measures the cost of a transition between the two stable phases $\rho_\pm$.

As in the case of Ising–Kac model, the surface tension is identified by considering the sharp interface limit. By setting $\varepsilon = L^{-1}$, as far as the dynamical behavior is concerned, under diffusive rescaling the joint limit $N \to \infty$ and $\varepsilon \to 0$ (with $\varepsilon \gg N^{-1}$) of the empirical measure has been analyzed in [13,30]. More precisely, it is there proven that the limiting dynamics is described by the motion by mean curvature of the interface separating the stable phases. This is done by using classical methods in [13] and using the level-set method [30].

In order to analyze the asymptotic behavior of the probability $\wp_{L,N}$, let us introduce the family of functionals F^ε on the set of densities $u\colon \mathbb{T}^d \to [0,1]$ defined by

$$F^\varepsilon(u) := \varepsilon^{d-1} F_{\varepsilon^{-1}}\big(u(\varepsilon\cdot)\big).$$

We are next going to argue, but not rigorously prove, that as $\varepsilon \to 0$ the sequence F^ε converges to a functional F that is finite only for functions $u \in BV\big(\mathbb{T}^d; \{\rho_-;\rho_+\}\big)$, and for such u is proportional to the (measure theoretic) perimeter of the jump set of u. This means that $F(u) = \tau\mathcal{H}^{d-1}(S_u)$, where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on $\mathbb{T}^d$ and S_u denotes the jump set of u. The constant $\tau > 0$ is then identified with the surface tension for the Glauber+Kawasaki processes, and it will be characterized in terms of the solution to a one-dimensional ODE.

As the quasi-potential F_L is not directly accessible, we shall consider the sharp interface limit of the dynamical rate function $I_{T,L}$. More precisely, let S_ε be the functional on the set of paths $\phi\colon [0,T] \times \mathbb{T}^d \to [0,1]$ defined by $S_\varepsilon(\phi) := \varepsilon^{d-1} I_{\varepsilon^{-2}T,\varepsilon^{-1}}\big(\phi(\varepsilon^2\cdot, \varepsilon\cdot)\big)$. In Theorem 3.1, we prove that for suitable sequences ϕ_ε converging to

$$\phi(t,x) = \begin{cases} \rho_+ & \text{if } x \in \Omega(t), \\ \rho_- & \text{if } x \notin \bar\Omega(t), \end{cases}$$

for some open $\Omega(t) \subset \mathbb{T}^d$ with smooth boundary,

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) = \frac{1}{2}\tau \int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma \, \left(v_t - \frac{1}{2}\kappa_t\right)^2, \tag{3.3}$$

where $\Gamma(t) = \partial\Omega(t)$, $\mathrm{d}\sigma$ is the surface measure on $\Gamma(t)$, v_t is the normal velocity of $\Gamma(t)$, κ_t is its mean curvature and τ is a positive constant.

Observe now that the limiting dynamical rate function in (3.3) measures, in L^2 sense, the deviations with respect to the motion by mean curvature $v_t = \frac{1}{2}\kappa_t$. Since this evolution is—informally—the gradient flow of (one half of) the perimeter, we deduce that the quasi-potential associated with the limiting dynamical rate function is proportional to the perimeter and we identify the proportionality constant with τ, see [25, Thm. 4.3.1] for a proof of this statement in the context of diffusions in $\mathbb{R}^n$.

3.2. Preliminaries

Let $\bar{u}$ be the instanton (standing wave) associated with the hydrodynamic Eq. (3.2) in dimension one, namely the solution to

$$\frac{1}{2}\bar{u}'' + B(\bar{u}) - D(\bar{u}) = 0, \quad \bar{u}(\pm\infty) = \rho_\pm, \quad \bar{u}(0) = \frac{\rho_+ + \rho_-}{2}. \tag{3.4}$$

Clearly, $\bar{u}'(\xi) > 0$ and it can be easily shown that

$$\sup_{\xi\in\mathbb{R}} \big(\left|\bar{u}'(\xi)\right| + |\bar{u}''(\xi)|\big)\mathrm{e}^{\gamma|\xi|} < +\infty, \tag{3.5}$$

where $\gamma = \min\{D'(\rho_+) - B'(\rho_+); D'(\rho_-) - B'(\rho_-)\}$.

The large deviation asymptotics for the empirical density under the Glauber+Kawasaki dynamics has been analyzed in [12,29,35]. We next recall the associated rate function. Given L positive, let $C(L) := \{\rho \in L^\infty(\mathbb{T}^d_L) \colon 0 \le \rho \le 1\}$, where $\mathbb{T}^d_L$ is the d-dimensional torus of side $L > 0$, equipped with the (metrizable) weak* topology. We define $I_{T,L} \colon C([0,T]; C(L)) \to [0,\infty]$ by

$$I_{T,L}(\phi) := \sup_{H\in C^{1,2}([0,T]\times\mathbb{T}^d_L)} J^H_{T,L}(\phi), \tag{3.6}$$

where

$$\begin{aligned} J^H_{T,L}(\phi) := &\int \mathrm{d}r\, \big[\phi(T,\cdot)H(T,\cdot) - \phi(0,\cdot)H(0,\cdot)\big] \\ &- \int_0^T \mathrm{d}t \int \mathrm{d}r \left[\phi\Big(\partial_t H + \frac{1}{2}\Delta H\Big) + \frac{1}{2}\phi(1-\phi)|\nabla H|^2\right] \\ &- \int_0^T \mathrm{d}t \int \mathrm{d}r \left[B(\phi)(\mathrm{e}^H - 1) + D(\phi)(\mathrm{e}^{-H} - 1)\right]. \end{aligned} \tag{3.7}$$

Under suitable assumptions on the initial conditions, in [12,29,35] it is proven that the empirical magnetization sampled according to the Glauber dynamics, regarded as a random variable taking values in the Skorokhod space $D([0,T]; \mathcal{M}(\mathbb{T}^d_L))$, satisfies a large deviation principle with speed N^d and rate function $\mathcal{I}_{T,L}$ given by $\mathcal{I}_{T,L}(\nu) = I_{T,L}(\phi)$ if $\nu_t = \phi_t\,\mathrm{d}r$ for some $\phi \in C([0,T]; C(L))$ and $+\infty$ otherwise.

By [29, Lemma 2.1], or rather its generalization in dimension $d \ge 1$, if $\phi \in C^{2,3}([0,T] \times \mathbb{T}^d_L; (0,1))$, then the supremum in (3.6) is achieved for

$H = H(\phi) \in C^{1,2}([0,T] \times \mathbb{T}_L^d)$, the unique classical solution to the nonlinear Poisson equation,

$$\partial_t \phi + \nabla \cdot [\phi(1-\phi)\nabla H] = \frac{1}{2}\Delta\phi + B(\phi)\mathrm{e}^{H} - D(\phi)\mathrm{e}^{-H}, \tag{3.8}$$

so that, for such H,

$$\begin{aligned} I_{T,L}(\phi) = J^H_{T,L}(\phi) &= \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}r\, \phi(1-\phi)|\nabla H|^2 \\ &\quad + \int_0^T \mathrm{d}t \int \mathrm{d}r\, B(\phi)\big(1 - \mathrm{e}^{H} + H\mathrm{e}^{H}\big) \\ &\quad + \int_0^T \mathrm{d}t \int \mathrm{d}r\, D(\phi)\big(1 - \mathrm{e}^{-H} - H\mathrm{e}^{-H}\big). \end{aligned} \tag{3.9}$$

Due to the lack of reversibility of the underlying microscopic dynamics, it is not possible to decompose the action function in a form analogous to (2.17).

3.3. Sharp Interface Limit of the Action Functional

To this end, we set $\varepsilon = L^{-1}$, perform a diffusive rescaling of space and time and normalize the resulting action with a factor L^{d-1}. As in the previous section, the space variable in $\mathbb{T}^d$ is denoted by x. We then introduce the rescaled action functional renormalized with a factor L^{d-1}. We thus define the rescaled functional $S_\varepsilon \colon C([0,T]; C(1)) \to [0,\infty]$ by

$$S_\varepsilon(\phi) = \varepsilon^{d-1} I_{\varepsilon^{-2}T,\varepsilon^{-1}}(\phi(\varepsilon^2\cdot, \varepsilon\cdot)), \tag{3.10}$$

whose variational representation is

$$S_\varepsilon(\phi) := \sup_{H \in C^{1,2}([0,T]\times\mathbb{T}^d)} J^H_\varepsilon(\phi), \tag{3.11}$$

where

$$\begin{aligned} J^H_\varepsilon(\phi) &:= \frac{1}{\varepsilon}\int \mathrm{d}x\, \big[\phi(T,\cdot)H(T,\cdot) - \phi(0,\cdot)H(0,\cdot)\big] \\ &\quad -\frac{1}{\varepsilon}\int_0^T \mathrm{d}t \int \mathrm{d}x \left[\phi\Big(\partial_t H + \frac{1}{2}\Delta H\Big) + \frac{1}{2}\phi(1-\phi)|\nabla H|^2\right] \\ &\quad -\frac{1}{\varepsilon}\int_0^T \mathrm{d}t \int \mathrm{d}x \left(B(\phi)\frac{\mathrm{e}^{H}-1}{\varepsilon^2} + D(\phi)\frac{\mathrm{e}^{-H}-1}{\varepsilon^2}\right). \end{aligned} \tag{3.12}$$

Moreover, the representation (3.8), (3.9) gives, for $\phi \in C^{2,3}([0,T]\times\mathbb{T}^d; (0,1))$,

$$\partial_t \phi + \nabla \cdot [\phi(1-\phi)\nabla H] = \frac{1}{2}\Delta\phi + \frac{B(\phi)\mathrm{e}^{H} - D(\phi)\mathrm{e}^{-H}}{\varepsilon^2} \tag{3.13}$$

and

$$\begin{aligned} S_\varepsilon(\phi) &= \frac{1}{2\varepsilon}\int_0^T \mathrm{d}t \int \mathrm{d}x\, \phi(1-\phi)|\nabla H|^2 \\ &\quad + \frac{1}{\varepsilon^3}\int_0^T \mathrm{d}t \int \mathrm{d}x\, B(\phi)\big(1 - \mathrm{e}^{H} + H\mathrm{e}^{H}\big) \end{aligned}$$

$$+\frac{1}{\varepsilon^3}\int_0^T \mathrm{d}t \int \mathrm{d}x\, D(\phi)\big(1-\mathrm{e}^{-H}-H\mathrm{e}^{-H}\big). \tag{3.14}$$

As in the previous section, given a C^1 family of oriented smooth surfaces $\Gamma=\{\Gamma(t)\}_{t\in[0,T]}$, with $\Gamma(t)=\partial\Omega(t)$ for some open $\Omega(t)\subset\mathbb{T}^d$, we denote by $n_t=n_{\Gamma(t)}$ the inward normal of $\Gamma(t)$, by $v_t\colon\Gamma(t)\to\mathbb{R}$ the normal velocity of Γ at time t, by κ_t the mean curvature of $\Gamma(t)$ and by $d(\cdot,\Gamma(t))$ a regularized version of the signed distance from $\Gamma(t)$.

For such families of surfaces, we define the limiting action functional,

$$S_{\mathrm{ac}}(\Gamma)=\frac{1}{4\mu}\int_0^T \mathrm{d}t\int_{\Gamma(t)}\mathrm{d}\sigma\,\left(v_t-\frac{1}{2}\kappa_t\right)^2, \tag{3.15}$$

where the mobility μ is computed according to the following procedure. Recalling the definition (3.4), let $L_{\bar u}$ be the linear operator given by

$$L_{\bar u}\psi=\big[(\bar u(1-\bar u)\psi'\big]'-[B(\bar u)+D(\bar u)]\psi, \tag{3.16}$$

which is obtained by linearizing (3.8) in dimension one at $\phi=\bar u$ around $H=0$. Then,

$$\mu=\frac{2\langle \bar u',(-L_{\bar u})\bar u'\rangle_{L^2}}{\|\bar u'\|_{L^2}^4}. \tag{3.17}$$

For later purpose, we notice that since $B+D$ is strictly positive, $L_{\bar u}$ is bijective on $L^2(\mathbb{R})$. Moreover, the inverse of $L_{\bar u}$ preserves the decays properties of a forcing term, in the sense that if $L_{\bar u}\psi=w$ then, for any $\gamma'>0$,

$$\sup_{\xi\in\mathbb{R}}|w(\xi)|\mathrm{e}^{\gamma'|\xi|}<+\infty \quad\Longrightarrow\quad \sup_{\xi\in\mathbb{R}}\big(|\psi(\xi)| +|\psi'(\xi)|+|\psi''(\xi)|\big)\mathrm{e}^{\gamma'|\xi|}<+\infty. \tag{3.18}$$

Theorem 3.1. *Let $\Gamma=\{\Gamma(t)\}_{t\in[0,T]}$ as before and consider a sequence $\{\phi_\varepsilon\}\subset C([0,T];C(1))$, converging to $\rho_-+(\rho_+-\rho_-)1\!\!1_{\Omega(\cdot)}$, of the form*

$$\phi_\varepsilon(t,x)=\bar u\left(\frac{d(x,\Gamma(t))}{\varepsilon}+\varepsilon Q\left(t,x,\frac{d(x,\Gamma(t))}{\varepsilon}\right)\right)+\varepsilon R_\varepsilon(t,x), \tag{3.19}$$

where $\bar u$ is the instanton, $Q\colon[0,T]\times\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ is a smooth function such that

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}}\left\{\frac{|Q(t,x,\xi)|+|\partial_\xi Q(t,x,\xi)|}{1+|\xi|} +\frac{|\partial_t Q(t,x,\xi)|+|D_x Q(t,x,\xi)|+|D^2_{xx}Q(t,x,\xi)|}{1+|\xi|}\right\}<+\infty, \tag{3.20}$$

and $R_\varepsilon\colon[0,T]\times\mathbb{T}^d\to\mathbb{R}$ is a smooth function.

(a) *If $\|R_\varepsilon\|_\infty+\|\partial_t R_\varepsilon\|_\infty+\|\Delta R_\varepsilon\|_\infty\to 0$ as $\varepsilon\to 0$ then, for any Q,*
$$\liminf_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon)\ge S_{\mathrm{ac}}(\Gamma).$$

(b) *There exist Q^* such that choosing $Q=Q^*$ and $R_\varepsilon=0$ we have,*
$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon)=S_{\mathrm{ac}}(\Gamma).$$

For expository reasons, we prove the statements in reverse order.

Proof of (b). In the sequel, we assume $R_\varepsilon = 0$ and $Q(t,x,\xi) := A(t,x)\bar{Q}(\xi)$, where $A\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}$ and $\bar{Q}\colon \mathbb{R}\to\mathbb{R}$ are smooth functions to be determined later, with $\bar{Q}$ such that

$$\sup_{\xi\in\mathbb{R}}\left\{\frac{|\bar{Q}(\xi)|}{1+|\xi|} + |\bar{Q}'(\xi)| + |\bar{Q}''(\xi)|\right\} < +\infty. \tag{3.21}$$

In order to compute the cost of the sequence (3.19) with these choices, we start by examining the expansions,

$$\begin{aligned}\phi_\varepsilon &= \bar{u}(d_\varepsilon) + \varepsilon\bar{u}'(d_\varepsilon)Q_\varepsilon + \varepsilon R^{(1)}_\varepsilon, \qquad \partial_t\phi_\varepsilon = \bar{u}'(d_\varepsilon)\frac{\partial_t d}{\varepsilon} + R^{(2)}_\varepsilon,\\ \Delta\phi_\varepsilon &= \frac{\bar{u}''(d_\varepsilon)}{\varepsilon^2} + \frac{\bar{u}'(d_\varepsilon)\Delta d + \bar{u}'''(d_\varepsilon)Q_\varepsilon + 2\bar{u}''(d_\varepsilon)Q'_\varepsilon + \bar{u}'(d_\varepsilon)Q''_\varepsilon}{\varepsilon} + R^{(3)}_\varepsilon,\end{aligned} \tag{3.22}$$

where we adopted the notation $d = d(x,\Gamma(t))$, $d_\varepsilon = d/\varepsilon$, $Q_\varepsilon = Q(t,x,d_\varepsilon)$, $Q'_\varepsilon = \partial_\xi Q(t,x,d_\varepsilon)$, and $Q''_\varepsilon = \partial^2_{\xi\xi}Q(t,x,d_\varepsilon)$, and $R^{(i)}_\varepsilon = R^{(i)}_\varepsilon(t,x,d_\varepsilon)$, $i = 1,2,3$, are such that

$$\limsup_{\varepsilon\to 0}\ \sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} \mathrm{e}^{\gamma|\xi|/2}|R^{(i)}_\varepsilon(t,x,\xi)| < \infty,$$

with γ as in (3.5). Equation (3.22) can be easily derived using (3.5) and recalling that $|\nabla d| = 1$ in a neighborhood of $\Gamma(t)$.

Next, assuming for the function H, the unique solution to (3.13), an expansion of the form $H(t,x) = \varepsilon H_1(t,x,d(x,\Gamma(t))/\varepsilon)) + O(\varepsilon^2)$, we deduce a linear equation for H_1. To this end, we write

$$\begin{aligned}\frac{B(\phi_\varepsilon)\mathrm{e}^{H} - D(\phi_\varepsilon)\mathrm{e}^{-H}}{\varepsilon^2} &= \frac{B(\bar{u}(d_\varepsilon)) - D(\bar{u}(d_\varepsilon))}{\varepsilon^2}\\ &\quad + \frac{B'(\bar{u}(d_\varepsilon)) - D'(\bar{u}(d_\varepsilon))}{\varepsilon}\bar{u}'(d_\varepsilon)Q_\varepsilon + \frac{B(\bar{u}(d_\varepsilon)) + D(\bar{u}(d_\varepsilon))}{\varepsilon}H_1 + O(1).\end{aligned} \tag{3.23}$$

Plugging (3.22) and (3.23) in (3.13) and making use of (3.4) and its derivative, we deduce, after some straightforward computations, that, for $(t,x)\in[0,T]\times\mathbb{T}^d$ fixed, $H_1(t,x,\cdot)$ satisfies

$$\begin{aligned}(\bar{u}(1-\bar{u})H_1')' - [B(\bar{u}) + D(\bar{u})]H_1 &= \left(\frac{1}{2}\Delta d - \partial_t d\right)\bar{u}'\\ &\quad + \bar{u}''A\bar{Q}' + \frac{1}{2}\bar{u}'A\bar{Q}'',\end{aligned} \tag{3.24}$$

where $H_1' = \partial_\xi H_1(x,t,\xi)$. Hence, by choosing $A = \partial_t d - \frac{1}{2}\Delta d$ and recalling (3.16), we get $H_1(t,x,\xi) = A(t,x)h(\xi)$ where $h\colon\mathbb{R}\to\mathbb{R}$ solves

$$L_{\bar{u}}h = -\bar{u}' + \bar{u}''\bar{Q}' + \frac{1}{2}\bar{u}'\bar{Q}''. \tag{3.25}$$

For later purpose, we remark that, in view of (3.5), (3.18) and (3.21),

$$\sup_{\xi\in\mathbb{R}} \big(|h(\xi)| + |h'(\xi)| + |h''(\xi)|\big) \mathrm{e}^{\gamma|\xi|} < +\infty. \tag{3.26}$$

With this choice of H_1, the initial assumption on the expansion for H holds. Indeed, in "Appendix A," it is proven that $H(t,x) = \varepsilon H_1(t,x,d(x,\Gamma(t))/\varepsilon)) + \varepsilon^2 \widetilde{H}_\varepsilon(t,x)$ with

$$\limsup_{\varepsilon\to 0} \sup_{(t,x)\in[0,T]\times\mathbb{T}^d} \big(|\widetilde{H}_\varepsilon(t,x)| + \varepsilon|\nabla \widetilde{H}_\varepsilon(t,x)|\big) < \infty. \tag{3.27}$$

From (3.14), the explicit form of H_1 and (3.27), we then have,

$$S_\varepsilon(\phi_\varepsilon) = S_\varepsilon^{(1)} + S_\varepsilon^{(2)} + O(\varepsilon), \tag{3.28}$$

where, after integrating by parts,

$$S_\varepsilon^{(1)} = \frac{1}{2\varepsilon}\int_0^T \mathrm{d}t \int \mathrm{d}x \left[-\nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\,\varepsilon^2\nabla H_1) + \frac{B(\phi_\varepsilon)+D(\phi_\varepsilon)}{2} H_1\right] H_1$$

and

$$\begin{aligned} S_\varepsilon^{(2)} = {} & \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}x\, \phi_\varepsilon(1-\phi_\varepsilon)\,\varepsilon^2 \nabla H_1\cdot\nabla\widetilde{H}_\varepsilon \\ & + \int_0^T \mathrm{d}t \int \mathrm{d}x \left[(B(\phi_\varepsilon)+D(\phi_\varepsilon))H_1\widetilde{H}_\varepsilon + \frac{B(\phi_\varepsilon)-D(\phi_\varepsilon)}{3} H_1^3\right]. \end{aligned}$$

Now, from (3.22), (3.24) and (3.26), we deduce

$$\begin{aligned} & \nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\nabla(\varepsilon H_1)) - \frac{B(\phi_\varepsilon)+D(\phi_\varepsilon)}{\varepsilon^2}(\varepsilon H_1) \\ & \quad = \frac{1}{\varepsilon}(\bar{u}(d_\varepsilon)(1-\bar{u}(d_\varepsilon))H_1')' - \frac{B(\bar{u}(d_\varepsilon))+D(\bar{u}(d_\varepsilon))}{\varepsilon} H_1 + R_\varepsilon^{(4)} \\ & \quad = \frac{1}{\varepsilon}\left[\left(\frac{1}{2}\Delta d - \partial_t d\right) \bar{u}'(d_\varepsilon) + \bar{u}''(d_\varepsilon)A\bar{Q}' + \frac{1}{2}\bar{u}'(d_\varepsilon)A\bar{Q}''\right] + R_\varepsilon^{(4)}, \end{aligned} \tag{3.29}$$

with $R^{(4)} = R_\varepsilon^{(4)}(t,x,d_\varepsilon)$, such that

$$\limsup_{\varepsilon\to 0} \sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} \mathrm{e}^{\gamma|\xi|/2}|R_\varepsilon^{(4)}(t,x,\xi)| < \infty.$$

Recalling that $A(t,x) = \partial_t d(x,\Gamma(t)) - \frac{1}{2}\Delta d(x,\Gamma(t)) = v_t(x) - \frac{1}{2}\kappa_t(x)$ for $x \in \Gamma(t)$ and using (3.5), (3.22), (3.25) and (3.27), by applying the co-area formula as was done in the proof of (2.45), we can compute the limit of $S_\varepsilon^{(1)}$ and $S_\varepsilon^{(2)}$ as $\varepsilon \to 0$. By a few direct calculations (that we omit), we obtain,

$$\lim_{\varepsilon\to 0} S_\varepsilon^{(1)} = C_{\bar{Q}} \int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma \left(v - \frac{1}{2}\kappa_t\right)^2, \qquad \lim_{\varepsilon\to 0} S_\varepsilon^{(2)} = 0, \tag{3.30}$$

where, denoting by $\langle\cdot,\cdot\rangle_{L^2}$ the scalar product in $L^2(\mathbb{R};\mathrm{d}\xi)$,

$$C_{\bar{Q}} = \frac{1}{2}\left\langle \left(\bar{u}' - \bar{u}''\bar{Q}' - \frac{1}{2}\bar{u}'\bar{Q}''\right), (-L_{\bar{u}})^{-1}\left(\bar{u}' - \bar{u}''\bar{Q}' - \frac{1}{2}\bar{u}'\bar{Q}''\right)\right\rangle_{L^2}.$$

We observe that, since $\bar u''\bar Q' + \frac{1}{2}\bar u'\bar Q'' \in L^2(\mathbb{R};\mathrm{d}\xi)$ and $\big\langle \bar u', \bar u''\bar Q' + \frac{1}{2}\bar u'\bar Q''\big\rangle_{L^2} = 0$,

$$C_{\bar Q} \ge C^* := \frac{1}{2} \min_{\psi : \langle \bar u', \psi\rangle_{L^2} = 0} \langle (\bar u' - \psi), (-L_{\bar u})^{-1}(\bar u' - \psi)\rangle_{L^2}. \tag{3.31}$$

The above minimum is achieved at

$$\bar\psi = \bar u' - \frac{\|\bar u'\|^2_{L^2}}{\langle \bar u', (-L_{\bar u})\bar u'\rangle_{L^2}} L_{\bar u}\bar u',$$

so that, recalling (3.17),

$$C^* = \frac{1}{2}\langle (\bar u' - \bar\psi)(-L_{\bar u})^{-1}(\bar u' - \bar\psi)\rangle_{L^2} = \frac{\|\bar u'\|^4_{L^2}}{2\langle \bar u', (-L_{\bar u})\bar u'\rangle_{L^2}} = \frac{1}{4\mu}. \tag{3.32}$$

In view of (3.28), (3.30) and (3.31), to conclude the proof of the statement (b), it remains to show that there is $\bar Q$ for which the minimum is obtained, i.e., there exists a solution $\bar Q$ to the linear equation $\bar u''\bar Q' + \frac{1}{2}\bar u'\bar Q'' = \bar\psi$ satisfying (3.21). This solution can be explicitly computed, precisely,

$$\bar Q(\xi) = 2\int_0^\xi \mathrm{d}\xi' \, \frac{1}{\bar u'(\xi')^2} \int_{-\infty}^{\xi'} \mathrm{d}\xi'' \, \bar u'(\xi'')\bar\psi(\xi''),$$

which satisfies (3.21) in view of (3.5).

Proof of (a). Let ϕ_ε be as in (3.19). By (3.11), (3.12) and integration by parts, we have, for any $H \in C^{1,2}([0,T]\times\mathbb{T}^d)$,

$$\begin{aligned} S_\varepsilon(\phi_\varepsilon) \ge \frac{1}{\varepsilon}\int_0^T \mathrm{d}t \int \mathrm{d}x \, \Big[\Big(\partial_t\phi_\varepsilon - \frac{1}{2}\Delta\phi_\varepsilon\Big)H - \frac{1}{2}\phi_\varepsilon(1-\phi_\varepsilon)|\nabla H|^2\Big] \\ + \frac{1}{\varepsilon}\int_0^T \mathrm{d}t \int \mathrm{d}x \, \Big(B(\phi_\varepsilon)\frac{1-\mathrm{e}^H}{\varepsilon^2} + D(\phi_\varepsilon)\frac{1-\mathrm{e}^{-H}}{\varepsilon^2}\Big). \end{aligned} \tag{3.33}$$

We choose H of the form $H(t,x) = \varepsilon H_1(t,x,d(x,\Gamma(t))/\varepsilon)$, with $H_1 \colon [0,T]\times\mathbb{T}^d\times\mathbb{R} \to \mathbb{R}$ a smooth function to be determined later such that, for some $\gamma' > 0$,

$$\begin{aligned} \sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} \mathrm{e}^{\gamma'|\xi|}\big\{|H_1(t,x,\xi)| + |\nabla_x H_1(t,x,\xi)| \\ + |\partial_\xi H_1(t,x,\xi)|\big\} < \infty. \end{aligned} \tag{3.34}$$

Noticing that the dependence on ϕ_ε of the integrands in the right-hand side of (3.33) is locally Lipschitz and recalling the hypothesis on R_ε, in view of the above assumptions on H_1, it is readily seen that the contribution due to R_ε is $o(1)$ as $\varepsilon \to 0$, and hence it can be neglected.

Therefore, few direct calculations (using (3.22), here applied to $\phi_\varepsilon - \varepsilon R_\varepsilon$, and the co-area formula) give

$$\begin{aligned} \liminf_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) \ge \int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma \int \mathrm{d}\xi \, \Big\{ \Big[\bar u'\Big(\partial_t d - \frac{1}{2}\Delta d\Big) + \bar u'' Q' + \frac{1}{2}\bar u' Q''\Big] H_1 \\ - \frac{1}{2}\bar u(1-\bar u)(H_1')^2 - \frac{B(\bar u) + D(\bar u)}{2} H_1^2 \Big\}, \end{aligned}$$

where the notation $Q' = \partial_\xi Q(x,t,\xi)$, $Q'' = \partial_{\xi\xi} Q(t,x,\xi)$, $H_1' = \partial_\xi H_1(x,t,\xi)$ and $H_1'' = \partial_{\xi\xi} H_1(t,x,\xi)$ has been adopted. The maximum of the expression in the right-hand side is obtained for $H_1 = \mathcal{H}$ with, for any $(t,x) \in [0,T] \times \mathbb{T}^d$ fixed, $\mathcal{H}(t,x,\cdot)$ solution to

$$L_{\bar u} \mathcal{H} = \Big(\frac{1}{2}\Delta d - \partial_t d\Big)\bar u' - \bar u'' Q' - \frac{1}{2}\bar u' Q'' =: F_Q,$$

which satisfies the assumptions (3.34) in view of (3.5), (3.18) and (3.20). Hence,

$$\liminf_{\varepsilon \to 0} S_\varepsilon(\phi_\varepsilon) \geq \frac{1}{2}\int_0^T \mathrm{d}t \int_{\Gamma(t)} \mathrm{d}\sigma \int \mathrm{d}\xi\, F_Q(-L_{\bar u})^{-1} F_Q. \tag{3.35}$$

We next observe that, in view of (3.31), for each $(t,x) \in [0,T] \times \mathbb{T}^d$ fixed,

$$\frac{1}{2}\int \mathrm{d}\xi\, F_Q(-L_{\bar u})^{-1} F_Q \geq \Big(\partial_t d - \frac{1}{2}\Delta d\Big)^2 C^*.$$

As $\partial_t d(x,\Gamma(t)) - \frac{1}{2}\Delta d(x,\Gamma(t)) = v_t(x) - \frac{1}{2}\kappa_t(x)$ for $x \in \Gamma(t)$, the statement (a) follows by (3.32) and (3.35). □

4. Approximating Nucleation Events

In this section, we discuss how the nucleation part of the rate function in (1.5) can be recovered from its absolutely continuous part. The general result should be the following. Given $T > 0$, let $\Gamma = \Gamma(t)$, $t \in [0,T]$, be a path of interfaces satisfying $S(\Gamma) < +\infty$ (with possible nucleation events), then there exists a sequence of smooth paths $\{\Gamma_\delta\}$ such that $\Gamma_\delta \to \Gamma$ and $S_{\mathrm{ac}}(\Gamma_\delta) \to S(\Gamma)$ as $\delta \to 0$. This statement is essentially the content of the density theorem mentioned in the comments just after Theorem 2.1 and appears to be out of reach. In this section, we, however, provide a link between the nucleation cost and the absolutely continuous part in the functional (1.5). More precisely, we consider an example of path Γ with nucleation and show how it can be approximated by a sequence with zero nucleation cost. For convenience, we discuss the two-dimensional isotropic case.

The basic idea is that in dimension $d = 2$ points [i.e., $(d-2)$-dimensional interfaces] can be nucleated with no cost, and we can then let them evolve for a short time (vanishing as $\delta \to 0$) in such a way that at the final time the resulting interface approximates the one we want to nucleate. Moreover, as we are going to argue, it is possible to arrange the evolution so that the corresponding cost indeed approximates the cost of a nucleation event.

Let us consider a path Γ of the form,

$$\Gamma(t) = \begin{cases} \emptyset & \text{if } t \in [0,\bar t), \\ \Gamma^0(t) & \text{if } t \in [\bar t, T], \end{cases} \tag{4.1}$$

where $\bar t \in (0,T)$ is the nucleation time and Γ^0 is a smooth path of smooth one-dimensional interfaces with initial value $\bar\Gamma := \Gamma^0(\bar t)$. We assume that $\Gamma^0(t) =$

$\partial\Omega^0(t)$, for some open set $\Omega^0(t)$, when $t \in (\bar{t}, T]$, while $\bar{\Gamma} = \lim_{t \downarrow \bar{t}} \overline{\Omega^0}(t)$. The corresponding nucleation cost is

$$S_{\mathrm{nucl}}(\Gamma) = 2\tau \mathrm{Per}(\bar{\Gamma}),$$

where we recall that τ is the surface tension and $\mathrm{Per}(\bar{\Gamma})$ denotes here the length of $\bar{\Gamma}$, while the factor 2 is due to the fact that $\bar{\Gamma}$ has be thought as an interface with double multiplicity.

In view of the assumed smoothness of $\bar{\Gamma}$, by localization, it suffices to consider the case in which it is a segment, say of length ℓ. In order to define the corresponding approximating path Γ_δ, we first construct a path $\Sigma_\delta(s)$, $s \in [0, \sigma_\delta]$, with $\sigma_\delta \to 0$, satisfying $\Sigma_\delta(0) \to \bar{\Gamma}$, $\mathrm{Per}\big(\Sigma_\delta(0)\big) \to 2\,\mathrm{Per}(\bar{\Gamma}) = 2\ell$, and $\Sigma_\delta(\sigma_\delta) = \emptyset$. To this end, chop the segment $\bar{\Gamma}$ into N_δ sub-segments (with N_δ diverging as $\delta \to 0$) and then fat each subsegment to an ellipse with major axis of length ℓ/N_δ and minor axis of length $m_\delta \ll \ell/N_\delta$. Denoting by $\bar{\Sigma}_\delta$ the resulting interface, then $\bar{\Sigma}_\delta \to \bar{\Gamma}$ and $\mathrm{Per}\big(\bar{\Sigma}_\delta\big) \to 2\ell$. The path $\Sigma_\delta(s)$, $s \geq 0$ is now defined as the evolution by mean curvature with initial datum $\bar{\Sigma}_\delta$ and transport coefficient θ. Here, we understand that each ellipse evolves by mean curvature separately. By comparing the evolution of each ellipse with that of a circle of initial diameter equal to the major axis, we deduce that $\Sigma_\delta(\sigma_\delta) = \emptyset$ for some $\sigma_\delta \leq (\ell/N_\delta)^2/(8\theta)$.

We now set

$$\Gamma_\delta(t) := \begin{cases} \emptyset & t \in [0, \bar{t} - \sigma_\delta), \\ \Sigma_\delta(\bar{t} - t) & t \in [\bar{t} - \sigma_\delta, \bar{t}), \\ \Gamma_\delta^0(t) & t \in [\bar{t}, T], \end{cases}$$

where Γ_δ^0 is a suitable approximation of the path Γ^0 in (4.1), satisfying $\Gamma_\delta^0(\bar{t}) = \Sigma_\delta(0)$, organized so that $S_{\mathrm{ac},[\bar{t},T]}(\Gamma_\delta^0) \to S_{\mathrm{ac},[\bar{t},T]}(\Gamma^0)$, whose details are omitted. To conclude, we next show that

$$S_{\mathrm{ac},[0,\bar{t}]}(\Gamma_\delta) \to 2\tau\ell.$$

Even if this is essentially a Freidlin–Wentzell argument for evaluating the quasi-potential in the reversible case, we provide the details of the computation. Denoting by v_δ and κ_δ the normal velocity and mean curvature of Γ_δ, we write

$$\frac{1}{4\mu}(v_\delta - \theta\kappa_\delta)^2 = \frac{1}{4\mu}(v_\delta + \theta\kappa_\delta)^2 - \frac{\theta}{\mu}\kappa_\delta v_\delta.$$

By construction of the path that has been obtained by time reversal of motion by mean curvature, the first term on the right-hand side above vanishes. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Per}(\Gamma_\delta(t)) = -\int_{\Gamma_\delta(t)} v_\delta \kappa_\delta,$$

we conclude by using the Einstein relation (1.3).

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Appendix A

Lemma A.1. *For $\widehat{H}$ as in (2.52), there exists a unique solution $\bar{Q}$ to the equation $L(\bar{m}'\bar{Q}) = \widehat{H}$ which satisfies $\bar{Q}(0) = 0$ and Eq. (2.47).*

Proof. Recall the properties of L, given by (2.39) and acting on $L^2(\mathbb{R};\nu)$, described in Sect. 2.2. Letting $E := \{\phi \in L^2(\mathbb{R};\nu)\colon \int \nu(\mathrm{d}\xi)\ \bar{m}'(\xi)\phi(\xi) = 0\}$, the bounded operator $L\colon E \to E$ is symmetric, coercive and therefore a bijection. As $\widehat{H} \in E$, we deduce that there exists a unique solution $\phi^* \in E$ to the equation $L\phi = \widehat{H}$. This implies that the family of functions of the form $\psi_\lambda := \phi^* + \lambda\bar{m}'$, $\lambda \in \mathbb{R}$, coincides with the set of all the solutions to $L\psi = \widehat{H}$ in $L^2(\mathbb{R};\nu)$. Moreover, from the explicit form (2.39) of L and the smoothness of $\widetilde{J}$, $\bar{m}$, c and $\widehat{H}$, the functions ψ_λ turn out to be smooth as well. In particular, since $\bar{m}' > 0$, the values of λ are uniquely determined by the condition $\psi_\lambda(0) = 0$.

So far, we have proven that there exists a unique solution $\bar{\psi} \in L^2(\mathbb{R};\nu)$ to the equation $L\psi = \widehat{H}$ which has the form $\bar{\psi} = \bar{m}'\bar{Q}$ with $\bar{Q}$ a smooth function with $\bar{Q}(0) = 0$. We are left with the proof of Eq. (2.47), which is equivalent, in view of (2.23), to prove that

$$\sup_{\xi\in\mathbb{R}} \frac{\mathrm{e}^{\alpha|\xi|}\big|\bar{\psi}(\xi)\big| + \mathrm{e}^{\alpha|\xi|}\big|\bar{\psi}'(\xi)\big|}{1+|\xi|} < +\infty, \tag{A.1}$$

with α as in (2.24).

Recalling (2.39), (2.50), (2.51) and (2.52), we have that

$$\bar{\psi} = p\widetilde{J} * \bar{\psi} + g, \qquad \bar{\psi}' = p\widetilde{J} * \bar{\psi}' + g', \tag{A.2}$$

where, see (2.4),

$$p := \beta(1 - m_\beta^2) = \frac{1}{1 + f''_\beta(m_\beta)} < 1 \tag{A.3}$$

and

$$g := (m_\beta^2 - \bar{m}^2)\beta\widetilde{J} * \bar{\psi} - (1 - \bar{m}^2)\beta f + \frac{\theta\sqrt{1-\bar{m}^2}}{2\bar{a}}\bar{m}'. \tag{A.4}$$

Since $\bar{\psi} \in L^2(\mathbb{R};\nu)$, both $\widetilde{J} * \bar{\psi}$ and $\widetilde{J}' * \bar{\psi}$ are bounded functions, so that, in view of (2.23) and (2.50),

$$\sup_{\xi\in\mathbb{R}} \mathrm{e}^{\alpha|\xi|}\big(\big|g(\xi)\big| + \big|g'(\xi)\big|\big) < +\infty. \tag{A.5}$$

This immediately implies, by (A.2), that also $\bar{\psi}$ and $\bar{\psi}'$ are bounded functions. To obtain the decay properties (A.1), we now adapt to the present context part of the analysis developed in [18] to study the spatial structure of the traveling fronts. Actually, we only show that

$$\sup_{\xi>0} \frac{\mathrm{e}^{\alpha\xi}\left|\bar{\psi}(\xi)\right|}{1+\xi} < +\infty, \tag{A.6}$$

since the proof of the other bound in (A.1) is similar.

Let

$$K(\xi,\xi') := \mathrm{e}^{\alpha(\xi-\xi')} p\widetilde{J}(\xi-\xi'), \qquad K_{\mathrm{o}}(\xi,\xi') := K(\xi,\xi')1\!\!1_{\xi'>0}. \tag{A.7}$$

By (2.24), $K(\xi,\xi')$ is a probability kernel, i.e., $\int \mathrm{d}\xi'\, K(\xi,\xi') = 1$. By (A.2), for any $\xi > 0$,

$$\begin{aligned}\bar{\psi}(\xi) = &\int_0^{+\infty} \mathrm{d}\xi'\, \mathrm{e}^{-\alpha(\xi-\xi')} K_{\mathrm{o}}(\xi,\xi')\bar{\psi}(\xi') \\ &+ \int_{-1}^{0} \mathrm{d}\xi'\, \mathrm{e}^{-\alpha(\xi-\xi')} K(\xi,\xi')\bar{\psi}(\xi') + g(\xi),\end{aligned}$$

which implies, by iteration,

$$\begin{aligned}\bar{\psi}(\xi) = &\sum_{j=0}^{n-1} \int_0^{+\infty} \mathrm{d}\xi'\, \mathrm{e}^{-\alpha(\xi-\xi')} K_{\mathrm{o}}^j(\xi,\xi') \int_{-1}^{0} \mathrm{d}\xi''\, \mathrm{e}^{-\alpha(\xi'-\xi'')} K(\xi',\xi'')\bar{\psi}(\xi'') \\ &+ \int_0^{+\infty} \mathrm{d}\xi'\, \mathrm{e}^{-\alpha(\xi-\xi')} K_{\mathrm{o}}^n(\xi,\xi')\bar{\psi}(\xi') \\ &+ \sum_{j=0}^{n-1} \int_0^{+\infty} \mathrm{d}\xi'\, \mathrm{e}^{-\alpha(\xi-\xi')} K_{\mathrm{o}}^j(\xi,\xi') g(\xi'), \quad \forall\, \xi \geq 0 \;\; \forall\, n \in \mathbb{N}.\end{aligned} \tag{A.8}$$

Above, the iterated kernel $K_{\mathrm{o}}^j(\xi,\xi')$ is recursively defined by $K_{\mathrm{o}}^0(\xi,\xi') = \delta(\xi-\xi')$, $K_{\mathrm{o}}^1(\xi,\xi') = K_{\mathrm{o}}(\xi,\xi')$, and $K_{\mathrm{o}}^j(\xi,\xi') = \int \mathrm{d}\xi'' K_{\mathrm{o}}(\xi,\xi'') K_{\mathrm{o}}^{j-1}(\xi'',\xi')$ for $j > 1$.

Since $\bar{\psi}$ and g are bounded functions and $\int \mathrm{d}\xi\, \widetilde{J}(\xi) = 1$, the jth terms of the sums in the right-hand side of (A.8) are bounded by a constant multiple of p^j and the term in the middle line by a constant multiple of p^n. As $p < 1$, we thus have, letting $n \to \infty$ in (A.8),

$$\mathrm{e}^{\alpha\xi}\bar{\psi}(\xi) = \int_{-1}^{0} \mathrm{d}\xi'\, \pi(\xi,\xi')\mathrm{e}^{\alpha\xi'}\bar{\psi}(\xi') + G(\xi), \quad \xi > 0, \tag{A.9}$$

where both the series

$$\pi(\xi,\xi') := \sum_{j=0}^{\infty} \int_0^{+\infty} \mathrm{d}\xi''\, K_{\mathrm{o}}^j(\xi,\xi'') K(\xi'',\xi')$$

and

$$G(\xi) := \sum_{j=0}^{\infty} \int_0^{+\infty} \mathrm{d}\xi'\, K_{\mathrm{o}}^j(\xi,\xi')\mathrm{e}^{\alpha\xi'} g(\xi')$$

converge.

The Green function $\pi(\xi,\xi')$, $\xi>0$, $\xi'\in[-1,0]$, can be interpreted as a probability kernel because it is nonnegative and

$$\int_{-1}^{0}\mathrm{d}\xi'\ \pi(\xi,\xi')=1\quad\forall\,\xi>0. \tag{A.10}$$

Moreover, there exists a probability density $\varrho(\xi)$, $\xi\in[-1,0]$, so that, for any function $\varphi\in C([-1,0])$,

$$\lim_{\xi\to+\infty}\int_{-1}^{0}\mathrm{d}\xi'\ \pi(\xi,\xi')\varphi(\xi')=\int_{-1}^{0}\mathrm{d}\xi'\ \varrho(\xi')\varphi(\xi'). \tag{A.11}$$

We omit the proof of (A.10) and (A.11) which are a special case of [18, Eq. (3.69) and Lemma 3.7].

Since $\bar\psi$ is a continuous function, from (A.11) the integral in the right-hand side of (A.9) converges to $\int_{-1}^{0}\mathrm{d}\xi'\ \varrho(\xi')\mathrm{e}^{\alpha\xi'}\bar\psi(\xi')$ as $\xi\to+\infty$. Therefore, to prove (A.6), it remains to show that

$$\sup_{\xi>0}\frac{|G(\xi)|}{1+\xi}<+\infty. \tag{A.12}$$

By (A.5) there is $C>0$ such that

$$|G(\xi)|\le C+C\sum_{n=1}^{\infty}\int_{0}^{+\infty}\mathrm{d}\xi'\ K_{\mathrm{o}}^{n}(\xi,\xi').$$

To estimate the nth term of the sum in the right-hand side, the key observation is taken from the proof of [18, Eq. (3.69)]. We have,

$$\int_{0}^{+\infty}\mathrm{d}\xi'\ K_{\mathrm{o}}^{n}(\xi,\xi')\le\int\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n\ K(\xi,\xi_1)\cdots K(\xi_{n-1},\xi_n)\,1\!\!\mathrm{I}_{\xi_n>0}=:I_n.$$

Since the probability kernel $K(\xi,\xi')$ depends only on the difference $\xi'-\xi$, see (A.7), the multiple integral I_n can be viewed as an expectation with respect to n i.i.d. random variables $Y_j=\xi_j-\xi_{j-1}$, $j=1,\dots,n$, where $\xi_0:=\xi$, each one with the distribution of $\xi'-\xi$ as given by $K(\xi,\xi')\,\mathrm{d}\xi'$. More precisely, as $\{\xi_n>0\}=\{(\xi_n-\xi_{n-1})+(\xi_{n-1}-\xi_{n-2})+\cdots+(\xi_1-\xi)>-\xi\}$ and observing that, in view of (A.7), $\mathbb{E}(Y_1)=\int\mathrm{d}\xi'\ K(\xi,\xi')(\xi'-\xi)=C_1<0$,

$$I_n=\mathbb{P}\left(\sum_{j=1}^{n}Y_j>-\xi\right)=\mathbb{P}\left(\sum_{j=1}^{n}(Y_j-C_1)>n|C_1|-\xi\right).$$

If $n\le 2\xi/|C_1|$, we use the obvious estimate $I_n\le 1$, while for $n>2\xi/|C_1|$, by Chebyshev's inequality,

$$\begin{aligned}I_n&\le\mathbb{P}\left(\sum_{j=1}^{n}(Y_j-C_1)>\frac{n}{2}|C_1|\right)\le\frac{16}{C_1^4n^4}\mathbb{E}\left[\left(\sum_{j=1}^{n}(Y_j-C_1)\right)^4\right]\\&\le\frac{16C_4n+96C_2n^2}{C_1^4n^4},\end{aligned}$$

where $C_2 := \mathbb{E}\big[(Y_1 - C_1)^2\big]$ and $C_4 := \mathbb{E}\big[(Y_1 - C_1)^4\big]$. We conclude that

$$|G(\xi)| \le C + C\sum_{n\ge 1}^{\infty} I_n \le C + \frac{2C}{|C_1|}\xi + \sum_{n>0} \frac{16C_4 n + 96 C_2 n^2}{C_1^4 n^4},$$

from which (A.12) follows. □

Proof of Eq. (3.27). We write $H_\varepsilon = \mathcal{H}_\varepsilon + \mathcal{K}_\varepsilon$ with $\mathcal{H}_\varepsilon$ solving the linear equation,

$$\begin{aligned} -\nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\nabla\mathcal{H}_\varepsilon) + \frac{B(\phi_\varepsilon)+D(\phi_\varepsilon)}{\varepsilon^2}\mathcal{H}_\varepsilon = \partial_t\phi_\varepsilon \\ -\frac{1}{2}\Delta\phi_\varepsilon - \frac{B(\phi_\varepsilon)-D(\phi_\varepsilon)}{\varepsilon^2}, \end{aligned} \tag{A.13}$$

and therefore $\mathcal{K}_\varepsilon$ satisfying

$$\begin{aligned} &\nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\nabla\mathcal{K}_\varepsilon) \\ &= \frac{B(\phi_\varepsilon)(\mathrm{e}^{\mathcal{H}_\varepsilon+\mathcal{K}_\varepsilon} - 1 - \mathcal{H}_\varepsilon) - D(\phi_\varepsilon)(\mathrm{e}^{-\mathcal{H}_\varepsilon-\mathcal{K}_\varepsilon} - 1 + \mathcal{H}_\varepsilon)}{\varepsilon^2}. \end{aligned} \tag{A.14}$$

Since $B(\phi_\varepsilon) + D(\phi_\varepsilon)$ and $\phi_\varepsilon(1-\phi_\varepsilon)$ are strictly positive, the first equation has a unique solution in $L^2(\mathbb{T}^d)$, which is a smooth function of (t,x) by the smoothness of ϕ_ε and elliptic regularity. In addition, in view of the expansions (3.22) and (3.23) (the latter for $H = 0$), the right-hand side in (A.13) is $O(\varepsilon^{-1})$, so that the maximum principle yields $\mathcal{H}_\varepsilon = O(\varepsilon)$.

Combining (3.29) with (A.13), by (3.22) and (3.23) (the latter for $H = 0$),

$$-\nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\nabla(\mathcal{H}_\varepsilon - \varepsilon H_1)) + \frac{B(\phi_\varepsilon)+D(\phi_\varepsilon)}{\varepsilon^2}(\mathcal{H}_\varepsilon - \varepsilon H_1) = O(1), \tag{A.15}$$

hence the maximum principle yields $\mathcal{H}_\varepsilon - \varepsilon H_1 = O(\varepsilon^2)$.

Next, using (A.14), we show that $\mathcal{K}_\varepsilon = O(\varepsilon^2)$, whence $\widetilde{H}_\varepsilon = \varepsilon^{-2}\mathcal{K}_\varepsilon + \varepsilon^{-2}(\mathcal{H}_\varepsilon - \varepsilon H_1) = O(1)$. Indeed, recalling that $B(\phi_\varepsilon)$ and $D(\phi_\varepsilon)$ are strictly positive and noticing that the nonlinear term in (A.14) is an increasing function of $\mathcal{K}_\varepsilon$, by comparison principle it is enough to construct super- and sub-solutions in the form $\mathcal{K}_\varepsilon^\pm = \pm C\varepsilon^2$, where C is a (suitably large) positive constant. This simply follows from the elementary inequality $1 - \mathcal{H} - \mathrm{e}^{C\varepsilon^2 - \mathcal{H}} > 0$, which holds if $|\mathcal{H}| \le C_1\varepsilon$ for given $C_1 > 0$, provided C is large enough and ε is small enough.

We are left with the estimate on the gradient of $\widetilde{H}_\varepsilon$. To this end, we first notice that, as $\mathcal{H}_\varepsilon = O(\varepsilon)$ and $\mathcal{K}_\varepsilon = O(\varepsilon^2)$, Eq. (A.14) can be recast in the form,

$$-\nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\nabla\mathcal{K}_\varepsilon + \frac{B(\phi_\varepsilon)+D(\phi_\varepsilon)}{\varepsilon^2}\mathcal{K}_\varepsilon = O(1). \tag{A.16}$$

By (A.15) and (A.16), we get

$$-\nabla\cdot(\phi_\varepsilon(1-\phi_\varepsilon)\nabla\widetilde{H}_\varepsilon + \frac{B(\phi_\varepsilon)+D(\phi_\varepsilon)}{\varepsilon^2}\widetilde{H}_\varepsilon = O(\varepsilon^{-2}). \tag{A.17}$$

As $H_\varepsilon = O(1)$, the estimate on $\varepsilon|\nabla \widetilde{H}_\varepsilon|$ follows by a standard covering argument with balls of radius ε and applying elliptic regularity. □

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