venerdì 13 novembre 2020 11:02

Esercizi

1) $f(x,y) = \dfrac{x+y}{x^2+1}$ Calcolare la matrice Hessiana nei pti $(0,0)$ e $(1,-1)$

$$f(x,y) = \underbrace{\frac{x}{x^2+1}} + \underbrace{\frac{y}{x^2+1}}$$

$$\frac{\partial f}{\partial x} = \frac{(x^2+1)-(x+y)2x}{(x^2+1)^2} = \frac{-x^2-2xy+1}{(x^2+1)^2} \leftarrow$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+1}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-x^2-2xy+1}{(x^2+1)^2}\right) =$$

$$= \frac{(-2x-2y)(x^2+1)^2-(-x^2-2xy+1)2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

messo in evidenza (x^2+1)

$$= \frac{[(-2x-2y)(x^2+1)-(-x^2-2xy+1)4x]}{(x^2+1)^3}$$

$$= \frac{2x^3+6x^2y-6x-2y}{(x^2+1)^3} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x^2+1}\right) = \frac{\partial}{\partial x}\left((x^2+1)^{-1}\right)$$

$$= -1\,(x^2+1)^{-2}\cdot 2x = \frac{-2x}{(1+x^2)^2} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x^2+1}\right) = 0$$

Calcolare $D^2 f(0,0)$ e in $D^2 f(1,-1)$

$$D^2 f(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D^2 f(1,-1) = \begin{pmatrix} \frac{e-6-6+2}{2^3} & \frac{\quad}{2^2} \\ -\frac{2}{2^2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{8}{8} & -\frac{2}{4} \\ -\frac{2}{4} & 0 \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

Es 4: $f(x,y) = x^2y + 2x - 4y$

Trovare massimi e minimi assoluti in $D = \{(x,y) \text{ t.c. } |x|+|y| \leq 4\}$

1°) Trovare se esistono punti critici all'interno del Dominio D

2°) Trovare i massi e minimi sul bordo di D.

3°) Calcolare f nei p.ti trovati, il valore massimo è il massimo assoluto, il valore minimo è il minimo assoluto.

1°)
$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + 2 = 0 \\ \frac{\partial f}{\partial y} = x^2 - 4 = 0 \longrightarrow x^2 = 4 \end{cases}$$

$$x^2 = 4 \Rightarrow x = 2, \quad x = -2$$

Sostituiamo nella prima equazione

$x=2 \longrightarrow 4y+2=0 \longrightarrow y = -\frac{1}{2}$ $\quad P_1 = (2, -\frac{1}{2})$

$x=-2 \longrightarrow -4y+2=0 \longrightarrow y = \frac{1}{2}$ $\quad P_2 = (-2, \frac{1}{2})$

Controllare se P_1 e P_2 appartengono all'interno di D

Cioè se verificano $|x|+|y| < 4$

Per $P_1 \longrightarrow |2| + |-\frac{1}{2}| = 2 + \frac{1}{2} = \frac{5}{2} < 4$

Per $P_2 \longrightarrow |-2| + |\frac{1}{2}| = 2 + \frac{1}{2} = \frac{5}{2} < 4$

2°) $\partial D = \{(x,y) \text{ t.c } |x|+|y| = 4\}$, $\quad x>0,\ y>0$

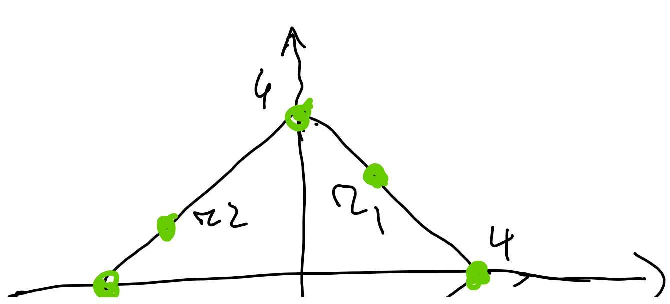
$|x|+|y| = x+y = 4$ $\quad r_1$

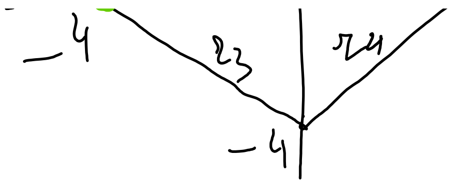
$y = 4-x$

$x<0,\ y>0$

$|x|+|y| = -x+y = 4$

$y = 4+x$





$x<0,\ y<0$

$|x|+|y| = -x-y = 4$ $\quad r_3$

$y = -x-4$

$x>0,\ y<0$

$|x|+|y| = x-y = 4$

$y = x-4$

Restringere f su ognuno dei segmenti r_1, r_2, r_3, r_4 per ognuno ci si riduce a una funzione di 1 variabile e si trova il max e il minimo

$r_1 :\ y = 4-x$, con $\underline{x \in [0,4]}$

$f(x,y) = x^2 y + 2x - 4y$

$f(x, 4-x) = x^2(4-x) + 2x - 4(4-x)$ $\nearrow g(0),\ g(4)$

$= -x^3 + 4x^2 + 6x - 16 = g(x)$

$\sqrt{36} = 6$

$g'(x) = -3x^2 + 8x + 6 = 0$

$$x = \frac{-4 \pm \sqrt{16+18}}{-3} = \frac{-4 \pm \sqrt{34}}{-3}$$

$$x = \frac{4 \pm \sqrt{34}}{3}$$

$$0 < \frac{4+\sqrt{34}}{3} < \frac{4+\sqrt{36}}{3} = \frac{10}{3} < 4$$

NON È INCLUSO $\longrightarrow$ $\dfrac{4-\sqrt{34}}{3} < \dfrac{4-\sqrt{25}}{3} = -\dfrac{1}{3} < 0$

Calcolare

$g(0),\ g(4),\ g\left(\frac{4+\sqrt{34}}{3}\right)$

$f(0,4) = g(0) = -16$

$f(4,0) = g(4) = -x^3 + 4x^2 + 6x - 16 = 24 - 16 = 8$

$$g\left(\frac{4+\sqrt{34}}{3}\right) = -\left(\frac{4+\sqrt{34}}{3}\right)^3 + 4\left(\frac{4+\sqrt{34}}{3}\right)^2 + 6\left(\frac{4+\sqrt{34}}{3}\right) - 16 \simeq 11{,}4$$

In r_2 : $y = 4+x \longrightarrow f(x, 4+x) = x^2(4+x) + 2x - 4(4+x) = g(x)$

$4 < x \le 0$

$$g(x) = x^3 + 4x^2 - 2x - 16$$

$$g(0) = f(0,4) = -16$$

$$g(-4) = 8 - 16 = -8 = f(-4,0)$$

$$g'(x) = 3x^2 + 8x - 2 = 0$$

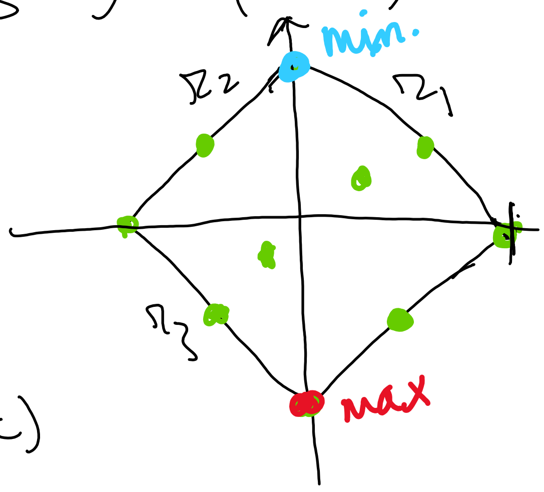
$$x = \frac{-4 \pm \sqrt{16+6}}{3} = \frac{-4 \pm \sqrt{22}}{3}$$

$$x = \frac{-4+\sqrt{22}}{3} > 0 \quad \text{non incluso}$$

$\sqrt{22} < \sqrt{25}$

$$-4 < -3 = \frac{-4-5}{3} < x = \frac{-4-\sqrt{22}}{3} < 0$$

$$-0{,}95 \simeq g\left(\frac{-4-\sqrt{22}}{3}\right) = \left(\frac{-4-\sqrt{22}}{3}\right)^3 + 4\left(\frac{-4-\sqrt{22}}{3}\right)^2 - 2\left(\frac{-4-\sqrt{22}}{3}\right) - 16$$



$r_3 : y = -x - 4 \quad$ con $x \in [-4, 0]$

$$f(x, -x-4) = x^2 \cdot \underbrace{(-x-4)}_{y} + 2x - 4\underbrace{(-x-4)}_{y}$$

$$= -x^3 - 4x^2 + 6x + 16 = g(x)$$

Calcolare $g(-4), g(0)$ e trovare se $g'(x) = 0$

$$g'(x) = -3x^2 - 8x + 6 = 0$$

$$x = \frac{4 \pm \sqrt{16+18}}{-3} = \frac{4 \pm \sqrt{34}}{-3}$$

Determinare quali di questi due punti appartengono a $[-4, 0]$.

$$x = \frac{4-\sqrt{34}}{-3} > 0 \quad \text{non appartiene}$$

$$x = \frac{4+\sqrt{34}}{-3}$$

$$g(-4) = f(-4,0) = -8$$

$$g(0) = f(0,-4) = 16$$

$$\cancel{4}4{,}63 \simeq g\left(\frac{4+\sqrt{34}}{-3}\right) = -\left(\frac{4+\sqrt{34}}{-3}\right)^3 - 4\left(\frac{4+\sqrt{34}}{-3}\right)^2 + 6\left(\frac{4+\sqrt{34}}{-3}\right) + 16$$

In r_4, $\quad x \in [0, 4]$

$y = x-4$, $x \in [\ldots]$

$$f(x, x-4) = x^2(x-4) + 2x - 4(x-4)$$
$$= x^3 - 4x^2 - 2x + 16 = g(x)$$

Calcolare $g(0), g(4)$ e i pt. in $[0,4]$ che verificano $g'(x) = 0$

$$g'(x) = 3x^2 - 8x - 2 = 0$$
$$x = \frac{4 \pm \sqrt{16+6}}{3} = \frac{4 \pm \sqrt{22}}{3}$$

$$x = \frac{4-\sqrt{22}}{3} < 0 \quad \notin [0,4]$$
$$0 < \frac{4+\sqrt{22}}{3} < \frac{4+\sqrt{25}}{3} = \frac{9}{3} = 3 < 4$$

$$g(0) = f(0,-4) = 16$$
$$g(4) = f(4,0) = 8$$
$$0{,}94 = g\left(\frac{4+\sqrt{22}}{3}\right) = \left(\frac{4+\sqrt{22}}{3}\right)^3 - 4\left(\frac{4+\sqrt{22}}{3}\right)^2 - 2\left(\frac{4+\sqrt{22}}{3}\right) + 16$$

Punti Interni critici

$$f\left(2, -\frac{1}{2}\right) = 4 \cdot \left(-\frac{1}{2}\right) + 2 \cdot 2 - 4\left(-\frac{1}{2}\right) = 4$$

$$f\left(-2, \frac{1}{2}\right) = 4 \cdot \left(\frac{1}{2}\right) + 2(-2) - 4\left(\frac{1}{2}\right) = -4$$

$$f(4, 0) = 8$$
$$f(0,4) = -16$$
$$f(-4,0) = -8$$
$$f(0,-4) = 16$$

$$f\left(\frac{4+\sqrt{34}}{3}, 4 - \frac{4+\sqrt{34}}{3}\right) \simeq 1{,}4$$
$$f\left(\frac{-4-\sqrt{22}}{3}, -\frac{4-\sqrt{22}}{3} + 4\right) \simeq 0{,}95$$
$$f\left(\frac{4+\sqrt{34}}{-3}, \ldots\right) \simeq 9{,}63$$
$$f\left(\frac{4+\sqrt{22}}{3}, \frac{4+\sqrt{22}}{3} - 4\right) = 0{,}94$$

$$f(x,y) = yx^2 + 2xy - y^2 - y$$

Trovare i pti critici e determinarne la natura cioè det. se sono pt di max

... o minimo locale o ptr. di sella o nessuno dei tre.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy + 2y = 0 \\ \dfrac{\partial f}{\partial y} = x^2 + 2x - 2y - 1 = 0 \end{cases}$$

$$\boxed{\begin{cases} 2y(x+1) = 0 \\ x^2+2x-2y-1=0 \end{cases}} \longrightarrow y = 0 \quad o \quad x = -1$$

$y=0$:

$x^2+2x-1=0$

$x = -1 \pm \sqrt{1+1}$

$x = -1 + \sqrt{2}$

$x = -1 - \sqrt{2}$

$x=-1$:

$1-2-2y-1=0$

$2y=-2$

$y=-1$

$$P_1 = (-1+\sqrt{2}, 0), \quad P_2 = (-1-\sqrt{2}, 0), \quad P_3 = (-1,-1)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy+2y) = 2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(2xy+2y) = 2x+2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^2+2x-2y-1) = -2$$

$$D^2 f(-1+\sqrt{2}, 0) = \begin{pmatrix} 0 & +2\sqrt{2} \\ +2\sqrt{2} & -2 \end{pmatrix}$$

$$\det D^2 f = -(+2\sqrt{2})^2 = -8 < 0$$

$(-1+\sqrt{2}, 0)$ è ptr di **Sella**

$$D^2 f(-1-\sqrt{2}, 0) = \begin{pmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -2 \end{pmatrix} \rightarrow \det D^2 = -(-2\sqrt{2})^2 = -8 < 0$$

$\downarrow$

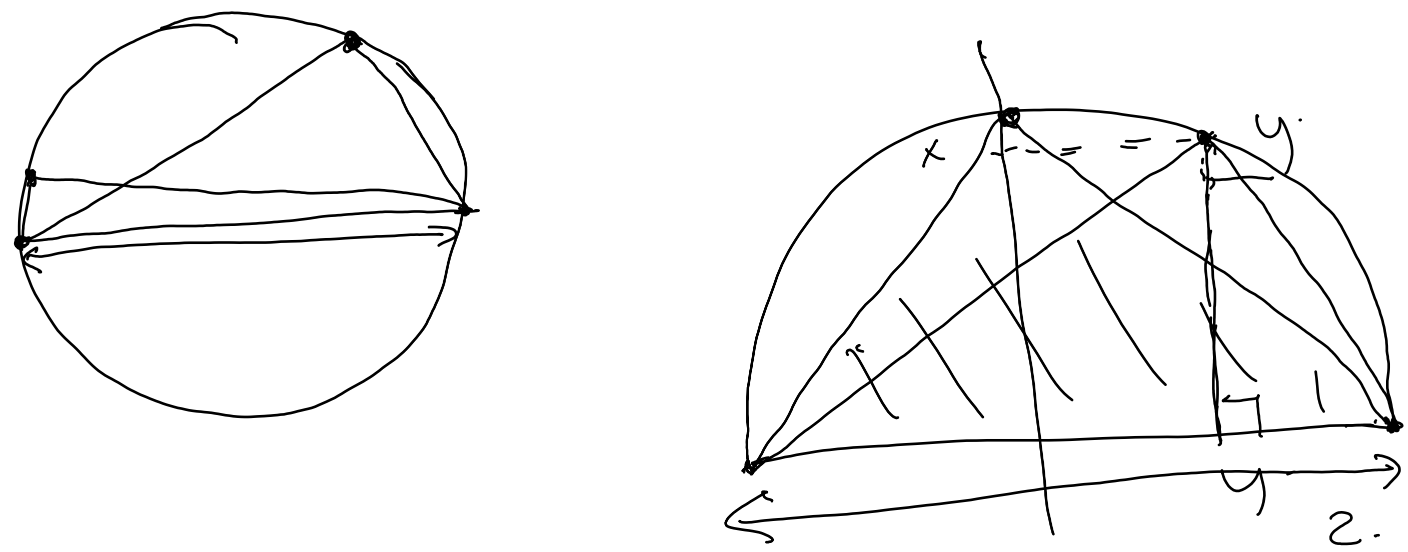
$(-1-\sqrt{2}, 0)$ è ptr di **Sella**

$$D^2 f(-1,-1) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\det D^2 f = -2 \cdot -2 = 4 > 0$$

... ptr di

$\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow (-1,-1)$ è un p.to massimo locale



(x, y)

x^2+y^2

$$A = \frac{2 \cdot y}{2} = y$$