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# Liouville theorems for semilinear equations on the Heisenberg group

by

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ABSTRACT. – In this paper we consider problems of the type

$$\begin{cases} \Delta_H u + h(x)u^p \leq 0, & \text{in } D \subset \mathbb{R}^{2n+1}, \\ u \geq 0 & \text{in } D, \end{cases} \tag{1}$$

where $\Delta_H$ is the Heisenberg Laplacian, $D$ is an unbounded domain and $h$ is a non negative function.

We prove that, under suitable conditions on $h$, $p$ and $D$, the only solution of (1) is $u \equiv 0$.

*Key words:* Liouville property, Heisenberg group.

RÉSUMÉ. – Dans ce travail nous considérons des problèmes du type

$$\begin{cases} \Delta_H u + h(x)u^p \leq 0, & \text{dans } D \subset \mathbb{R}^{2n+1}, \\ u \geq 0 & \text{dans } D, \end{cases} \tag{1}$$

où $\Delta_H$ est le Laplacien de Heisenberg, $D$ est un domaine non borné et $h$ est une fonction positive.

Nous démontrons que sous certaines hypothèses sur $h$, $p$ et $D$, la seule solution de (1) est $u \equiv 0$.

*A.M.S. Classification:* 35 J 60, 35 J 70.

## 1. INTRODUCTION

In this paper we establish some Liouville type theorems for positive functions $u$ satisfying, for example,

$$\begin{cases} \Delta_H u + h(\xi)u^p \leq 0 & \text{in } D, \\ u \geq 0 & \text{in } D, \end{cases} \tag{1.1}$$

where $D$ is an unbounded domain of the Heisenberg group $H^n$. We recall that $H^n$ is the Lie group $(\mathbb{R}^{2n+1}, \circ)$ equipped with the group action

$$\xi_0 \circ \xi = \left( x + x_0,\ y + y_0,\ t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i}) \right), \tag{1.2}$$

for $\xi := (x_1, \ldots, x_n, y_1, \ldots, y_n, t) := (x, y, t) \in \mathbb{R}^{2n+1}$ and $\Delta_H$ is the subelliptic Laplacian on $H^n$ defined by

$$\Delta_H = \sum_{i=1}^{n} X_i^2 + Y_i^2$$

with

$$\begin{cases} X_i = \dfrac{\partial}{\partial x_i} + 2y_i \dfrac{\partial}{\partial t}, \\ Y_i = \dfrac{\partial}{\partial y_i} - 2x_i \dfrac{\partial}{\partial t}. \end{cases}$$

It is easy to check that $\Delta_H$ is a degenerate elliptic operator satisfying the Hormander condition of order one (*see* Section 2).

As an example of our results for the case where $D = H^n$ we prove that, under some conditions on the non negative coefficient $h$ and suitable restriction on the power $p$, any non negative smooth solution $u$ of (1.1) is identically zero. More precisely, denoting by $Q = 2n + 2$ the homogeneous dimension of $H^n$ and by $|\xi|_H$ the intrinsic distance of the point $\xi$ to the origin (*see* [6], [7]), namely

$$|\xi|_H = \left( \sum_{i=1}^{n} (x_i^2 + y_i^2)^2 + t^2 \right)^{\frac{1}{4}}, \tag{1.3}$$

we have:

THEOREM 1.1. – *Let $u$ be a non negative solution of*

$$\Delta_H u(\xi) + a|\xi|_H^{\gamma} u^p(\xi) \leq 0 \ \ in\ H^n, \tag{1.4}$$

*where $a$ is a positive constant and $\gamma > -2$.*

*Then, if* $1 < p \leq \frac{Q+\gamma}{Q-2}$, $u \equiv 0$.

A generalized version of this theorem is proved in section 3 below, where also several variants covering the cases when the equation holds in a half space or some "cone" in $H^n$ are considered (*see* Theorem 3.2, 3.3, 3.4).

Let us point out that a common feature of our results is that we do not impose any condition on the behaviour of $u$ for large $|\xi|_H$, thus allowing $u$ to be, *a priori*, singular at infinity.

Therefore our results can be viewed as the analogues, in the present degenerate elliptic setting, of previous ones due to Gidas-Spruck [10] for the uniformly elliptic case. However, our method of proof is rather inspired by [1], where Liouville type results are established for non negative solutions of

$$\Delta u + a|x|^\gamma u^p \leq 0$$

in a cone of $\mathbb{R}^n$.

We wish to mention that non existence results for non negative solutions of semilinear equations on the Heisenberg group have been obtained previously by Garofalo-Lanconelli in [8]. Note, however, that the theorems in [8], based on Rellich-Pohozaev identities, differ considerably from those in the present paper since they require global integrability conditions on $u$ and on the gradient of $u$. (*see* also [5] for similar results in the uniformly elliptic case).

Finally, we point out that the Liouville theorems presented here are the basic tools for obtaining an *a priori* bound in the sup norm for solutions of the Dirichlet problem

$$\begin{cases} \Delta_H u + f(\xi, u) = 0 & \text{in } \Omega \subset \mathbb{R}^{2n+1}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

under some growth conditions on $f$. This can be done using a blow up technique on the lines of [10], [1], [2] and will be the object of a separate paper [3].

## 2. PRELIMINARY FACTS

In this section we collect for the convenience of the reader some known facts about the Heisenberg group $H^n$ and the operator $\Delta_H$ which will be useful later on. For their proof and more informations we refer for example to [6], [7], [8], [12], [13].

As mentioned in the introduction the Heisenberg group $H^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ ($n \geq 1$), endowed with the group action,

$$\xi_0 \circ \xi = \left( x + x_0,\, y + y_0,\, t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i}) \right),$$

for $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) := (x, y, t)$.

The corresponding Lie Algebra of left-invariant vector fields is generated by $X_i$, $Y_i$ for $i = 1, \ldots, n$, and $T = \frac{\partial}{\partial t}$.

It is easy to check that $X_i$ and $Y_i$ satisfy $[X_i, Y_j] = -4T\delta_{i,j}$, $[X_i, X_j] = [Y_i, Y_j] = 0$ for any $i, j \in \{1, \ldots, n\}$. Therefore, the vector fields $X_i$, $Y_i$ ($i = 1, \ldots, n$) and their first order commutators span the whole Lie Algebra. Hence, the Hormander condition of order one holds true for $\Delta_H$ (*see* [13]); this implies its hypoellipticity (*i.e.* if $\Delta_H u \in C^\infty$ then $u \in C^\infty$ (*see* [13])) and the validity of the maximum principle (*see* [4]).

An intrinsic metric can be defined on $H^n$ by setting

$$d_H(\xi, \eta) = |\eta^{-1} \circ \xi|_H$$

where $|\cdot|_H$ has been defined in (1.3), *see* [6]. Clearly in this metric the open ball of radius $R$ centered at $\xi_o$ is the set:

$$B_H(\xi_o, r) = \{\eta \in H^n \;:\; d_H(\eta, \xi_o) < r\}.$$

It is also important to observe that $\xi \to |\xi|_H$ is homogeneous of degree one with respect to the natural group of dilations (*see* [6], [7]):

$$\delta_\lambda(\xi) = (\lambda x,\, \lambda y,\, \lambda^2 t). \tag{2.1}$$

Since the base $\{X_i, Y_i, T\}$ is obtained by the standard one $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial t}\}$, using the transformation

$$B = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 0 & 0 & 1 \end{pmatrix}$$

whose determinant is identically 1, it follows that the Lebesgue measure is the Haar measure on $H^n$.

This fact, together with the homogeneity property of $|\xi|_H$ described above, implies that

$$|B_H(\xi_o, R)| = |B_H(0,1)| R^Q, \tag{2.2}$$

where $Q = 2n + 2$ is the homogeneous dimension of $H^n$ (*see* [12]) and $|\cdot|$ denotes the Lebesgue measure.

To conclude this section we recall some simple properties of $\Delta_H$. Observe first that

$$\Delta_H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial t^2}.$$

It is easy to check that the operator $\Delta_H$ is homogeneous of degree 2 with respect to the dilation $\delta_\lambda$ defined in (2.1), namely

$$\Delta_H(\delta_\lambda) = \lambda^2 \delta_\lambda(\Delta_H);$$

also, for any fixed $\xi^o$, by the left invariance of the vector fields $X_i$, $Y_i$ with respect to the group action we have:

$$\Delta_H(u(\xi^o \circ \xi)) = (\Delta_H u)(\xi^o \circ \xi) \qquad \forall \xi \in H^n.$$

The next remark concerns the action of $\Delta_H$ on functions $u$ depending only on $\rho := |\xi|_H$. It is easy to show that

$$\Delta_H u(\rho) = \psi \left[\frac{\partial^2 u}{\partial \rho^2} + \frac{Q-1}{\rho}\frac{\partial u}{\partial \rho}\right], \tag{2.3}$$

where the function $\psi$ is defined by

$$\psi(\xi) = \frac{\sum_{i=1}^n (x_i^2 + y_i^2)}{\rho^2} = |\nabla_H \rho|^2 \quad \text{for } \xi \neq 0, \tag{2.4}$$

where with $\nabla_H u$ we denote the vector field $(X_i u, Y_i u)$, for $i = 1, \ldots, n$.

It is useful to observe that

$$\Delta_H = \operatorname{div}(\sigma^T \sigma \nabla) \quad \text{with } \sigma = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.$$

## 3. LIOUVILLE TYPE THEOREMS

In this section we will generalize to the Heisenberg group some Liouville type results which hold for positive solutions of superlinear equations associated to the laplacian, *see* [1], [2], [10].

THEOREM 3.1. – *Let $u$ be a non negative solution of*

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \ \ in\ H^n, \tag{3.1}$$

*where $f$ is a non negative function satisfying*

$$f(\xi, u) \geq h(\xi)u^p \tag{3.2}$$

*for some function $h(\xi) \geq 0$ such that, for $|\xi|_H$ large,*

$$h(\xi) \geq K\psi|\xi|_H^\gamma$$

*for some $K > 0$ and $\gamma > -2$ .*

*If $1 < p \leq \frac{Q+\gamma}{Q-2}$, then $u \equiv 0$.*

Before the proof let us introduce a cut-off function $\phi_R$ which will be used throughout this section. Consider $\phi_R(\rho) := \phi(\frac{\rho}{R})$, where $\rho := |\xi|_H$, $R > 0$, and $\phi$ satisfies:

$$\left| \begin{array}{l} \phi \in C^\infty[0, +\infty), \qquad 0 \leq \phi \leq 1, \\ \phi \equiv 1 \quad \text{on } \left[0, \dfrac{1}{2}\right], \\ \phi \equiv 0 \quad \text{on } [1, +\infty), \\ -\dfrac{C}{R} \leq \dfrac{\partial \phi_R}{\partial \rho} \leq 0, \\ \text{and } \left|\dfrac{\partial^2 \phi_R}{\partial \rho^2}\right| \leq \dfrac{C}{R^2} \quad \text{for some constant } C > 0. \end{array} \right. \tag{3.3}$$

*Proof.* – Set, for $R > 0$,

$$I_R := \int_{H^n} h(\xi)u^p {\phi_R}^q d\xi \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1. \tag{3.4}$$

Observe that $I_R \geq 0$. Moreover, by equation (3.1) and (3.2)

$$I_R \leq \int_{B_H(0,R)} f(\xi, u)\phi_R^q d\xi \leq -\int_{B_H(0,R)} \Delta_H u \phi_R^q d\xi; \tag{3.5}$$

hence an integration by parts yields,

$$\begin{aligned} I_R \leq & -\int_{B_H(0,R)} u\Delta_H(\phi_R^q)d\xi + \int_{\partial B_H(0,R)} u\nabla_H(\phi_R^q)\cdot \nu_H dH_{2n} \\ & - \int_{\partial B_H(0,R)} \phi_R^q \nabla_H u \cdot \nu_H dH_{2n} \leq -\int_{B_H(0,R)} u\Delta_H(\phi_R^q)d\xi \\ & + \int_{\partial B_H(0,R)} uq\phi_R^{q-1}\phi_R' \nabla_H \rho \cdot \nu_H dH_{2n} \leq -\int_{B_H(0,R)} u\Delta_H(\phi_R^q)d\xi, \end{aligned}$$

where $\nu_H(\xi) = \sigma(\xi)\nu(\xi)$ and $\nu$ is the normal to $\partial\Omega$; $dH_{2n}$ denotes the $2n$-dimensional Hausdorff measure. On the other hand, as observed in Section 2 (*see* (2.3)),

$$\Delta_H(\phi_R^q) = \psi\left[\frac{\partial^2}{\partial\rho^2}(\phi_R^q) + \frac{Q-1}{\rho}\frac{\partial}{\partial\rho}(\phi_R^q)\right]. \tag{3.6}$$

Thus we get, using the hypoteses on $\phi_R$ and denoting by $\Sigma_R := B_H(0,R) \setminus B_H(0,\frac{R}{2})$,

$$\begin{aligned} I_R &\leq -\int_{\Sigma_R} u\psi\left[q\phi_R^{q-1}\phi_R'' + \frac{Q-1}{\rho}q\phi_R^{q-1}\phi_R'\right]d\xi \\ &\leq \frac{C}{R^2}\int_{\Sigma_R} u\psi\phi_R^{q-1}d\xi. \end{aligned}$$

Hence, the Hölder inequality yields:

$$I_R \leq \frac{C}{R^2}\left[\int_{\Sigma_R} u^p\rho^\gamma\psi\phi_R^{(q-1)p}d\xi\right]^{\frac{1}{p}}\left[\int_{B_H(0,R)} \psi\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}}. \tag{3.7}$$

Choosing $R > 0$ sufficiently large, in $\Sigma_R$, $h$ satisfies $h \geq \psi K\rho^\gamma$. Therefore,

$$I_R \leq C\left[\int_{\Sigma_R} u^p h\phi_R^q d\xi\right]^{\frac{1}{p}} R^{\left(\frac{-\gamma}{p}+\frac{Q}{q}-2\right)}, \tag{3.8}$$

as $0 \leq \psi \leq 1$. Then,

$$I_R^{1-\frac{1}{p}} \leq CR^{\left(\frac{-\gamma}{p}+\frac{Q}{q}-2\right)}.$$

Hence, if $1 < p < \frac{Q+\gamma}{Q-2}$, letting $R \to +\infty$, we obtain

$$I := \int_{H^n} hu^p d\xi = 0.$$

This implies $u \equiv 0$ for $\rho$ large, since $h$ is strictly positive outside of a set of measure zero and $u$ is *a priori* non negative.

The claim follows now by the maximum principle (*see* [4]). In fact, choose $\overline{R} > 0$ in such a way that, for $\rho \geq \overline{R}$, $h > 0$. Then, $u \equiv 0$ on the complementary of $B_H(0,\overline{R})$, as we proved. Hence, $u$ satisfies:

$$\begin{cases} u \geq 0 & \text{in } B_H(0,\overline{R}+\delta), \\ \Delta_H u \leq 0 & \text{in } B_H(0,\overline{R}+\delta), \\ u \equiv 0 & \text{for } \overline{R} \leq \rho \leq \overline{R}+\delta, \end{cases}$$

for some $\delta > 0$. Therefore, by the maximum principle, since $u$ is not strictly positive, $u$ has to be identically zero.

If $p = \frac{Q+\gamma}{Q-2}$, we obtain, by (3.7), that $I$ is finite and that the right hand side of (3.7) tends to zero when $R$ goes to infinity. This yields $I = 0$ and we can conclude as above.

*Remark* 3.1. – If $h = K > 0$, we get by the previous theorem that, for $1 < p \leq \frac{Q}{Q-2}$, the unique solution of

$$\Delta_H u + K u^p \leq 0 \quad \text{in } H^n \tag{3.9}$$

is $u \equiv 0$.

*Remark* 3.2. – The upper bound of the exponent $p$ is optimal. Indeed, we claim that the function $v(\rho) = C_\varepsilon (1+\rho^2)^{-\frac{\alpha}{2}}$ with $\alpha = Q - 2 - \varepsilon$ and a suitable choice of $C_\varepsilon$ is a positive solution of

$$\Delta_H u(\xi) + \psi(\xi)\rho^\gamma u^p(\xi) \leq 0 \quad \text{in } H^n, \tag{3.10}$$

for $p \geq \frac{Q+\gamma-\varepsilon}{Q-2-\varepsilon}$.

Indeed, let $u(\rho) = (1+\rho^2)^{-\frac{\alpha}{2}}$. Then $u$ satisfies:

$$\begin{aligned} -\Delta_H u &= -\psi \left[ \frac{\partial^2 u}{\partial \rho^2} + \frac{Q-1}{\rho} \frac{\partial u}{\partial \rho} \right] \\ &= \psi \alpha (1+\rho^2)^{-(\frac{\alpha}{2}+2)} [Q(1+\rho^2) - (\alpha+2)\rho^2] \\ &= \psi \alpha (1+\rho^2)^{-(\frac{\alpha}{2}+2)} [\rho^2 (Q - \alpha - 2) + Q] \\ &\geq \psi \alpha (Q-\alpha-2)(1+\rho^2)^{-(\frac{\alpha}{2}+1)}. \end{aligned} \tag{3.11}$$

Hence, if we impose that

$$Q - 2 > \alpha, \qquad p\frac{\alpha}{2} - \frac{\gamma}{2} \geq (\frac{\alpha}{2} + 1), \tag{3.12}$$

we can choose $c = (\alpha(Q-\alpha-2))^{\frac{1}{p-1}}$ and $v = cu$ satisfies:

$$-\Delta_H v \geq \psi(\alpha(Q-\alpha-2))^{\frac{p}{p-1}} (1+\rho^2)^{-p\frac{\alpha}{2}+\frac{\gamma}{2}} \geq \psi \rho^\gamma v^p.$$

Now just choose $\alpha = Q - 2 - \varepsilon$ then (3.12) holds if $p \geq \frac{Q+\gamma-\varepsilon}{Q-2-\varepsilon}$ for any $\varepsilon$ positive.

The idea of the function $v$ was taken from Ramon Soranzo (personal communication to I.B.) who gave a similar counterexample for the Laplacian.

The next result concern the case where the unbounded domain $D$ is an half-space.

THEOREM 3.2. – *Let $D \subset H^n$ be the set*

$$D = \left\{ \xi \in H^n \ : \ \sum_{i=1}^{n} a_i x_i + b_i y_i + d > 0, \right.$$
$$\left. with\ (a,b) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}, d \in \mathbb{R} \right\}.$$

*Let $u$ be a non negative solution of*

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \ \ in\ D, \tag{3.13}$$

*where $f$ is as in Theorem 3.1 with $\gamma > -1$.*
*If $1 < p \leq \frac{Q+\gamma}{Q-1}$, then $u \equiv 0$ in $D$.*

A similar result is valid for half-spaces which do not contain the $t$-direction or for particular cones. However, the upper bound of the exponent $p$ is lower than in the previous case.

The following results hold:

THEOREM 3.3. – *Let $D \subset H^n$ be the set*

$$D = \left\{ \xi \in H^n \ : \ \sum_{i=1}^{n} a_i x_i + b_i y_i + ct + d > 0 \right\},$$
$$for\ a, b \in \mathbb{R}^n,\ \ c \in \mathbb{R} \setminus \{0\},\ \ d \in \mathbb{R},$$

*and let $u$ be a non negative solution of*

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \ \ in\ D, \tag{3.14}$$

*with $f$ as in theorem 3.1 and $\gamma > 0$.*
*Then, if $1 < p \leq \frac{Q+\gamma}{Q}$, $u \equiv 0$ in $D$.*

THEOREM 3.4. – *Let $\Sigma$ be the cone*

$$\Sigma = \left\{ \xi \in H^n \ : \ \sum_{i=1}^{n} (a_i x_i - b_i y_i)(b_i x_i + a_i y_i) > 0 \right\},$$

*and let $u$ be a non negative solution of*

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \quad in\ \Sigma, \tag{3.15}$$

*with $f$ as in theorem 3.1 and $\gamma > 0$.*

*If* $1 < p \leq \frac{Q+\gamma}{Q}$, $u \equiv 0$ *in* $\Sigma$.

The proofs of theorems 3.2, 3.3, 3.4 follow from the next lemma.

LEMMA 3.1. – *Let* $D \subset H^n$ *be an unbounded domain. Assume that* $\eta$ *satisfies:*

$$\begin{cases} \eta > 0 & in\ D, \\ \Delta_H \eta \geq 0 & in\ D, \\ \eta = 0 & on\ \partial D, \end{cases}$$

*and let* $u$ *be a non negative solution of*

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \quad in\ D, \tag{3.16}$$

*with* $f$ *as in Theorem* 3.1. *Then, for*

$$I_R := \int_{D_R} h(\xi) u^p \phi_R^q \eta^q d\xi,$$

*the following estimate holds*

$$I_R \leq I_R^{\frac{1}{p}} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} \eta^q \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \eta \cdot \nabla_H \rho|^q \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \right) \tag{3.17}$$

*for* $R > 0$ *large enough, where* $D_R := B_H(0, R) \cap D$, $\Omega_R := (B_H(0, R) \setminus B_H(0, \frac{R}{2})) \cap D$, *and* $q$ *is the conjugate exponent of* $p$.

*Proof.* – From equation (3.16), assumption (3.2) and the divergence's theorem we get:

$$\begin{aligned} I_R \leq & - \int_{D_R} u \Delta_H (\eta^q \phi_R^q) d\xi + \int_{\partial D_R} u \nabla_H (\eta^q \phi_R^q) \cdot \nu_H dH_{2n} \\ & - \int_{\partial D_R} \eta^q \phi_R^q \nabla_H u \cdot \nu_H dH_{2n}. \end{aligned}$$

Moreover, since $\phi_R = 0$ on $\partial B_H(0, R)$, $\eta = 0$ on $\partial D$, and $q > 1$, the integrals on the boundary of $D_R$ vanish and therefore,

$$I_R \leq - \int_{D_R} u \Delta_H ((\eta \phi_R)^q) d\xi.$$

Thus, using the properties of $\phi_R$ and observing that, by the hypoteses made on $\eta$,

$$\Delta_H(\eta^q) = q(q-1)\eta^{q-2} |\nabla_H \eta|^2 + q\eta^{q-1} \Delta_H \eta > 0 \tag{3.18}$$

it results:

$$I_R \leq -\int_{\Omega_R} u\eta^q \Delta_H(\phi_R^q)d\xi - 2\int_{\Omega_R} u\nabla_H(\eta^q)\cdot\nabla_H(\phi_R^q)d\xi.$$

Using the properties of $\phi_R$, as in the proof of Theorem 3.1 we obtain

$$I_R \leq \frac{C}{R^2}\int_{\Omega_R} u\eta^q\psi\phi_R^{q-1}d\xi + \frac{C}{R}\int_{\Omega_R} u\eta^{q-1}\psi\phi_R^{q-1}\nabla_H\eta\cdot\nabla_H\rho d\xi. \tag{3.19}$$

Thus, the Hölder inequality yields:

$$\begin{aligned} I_R \leq\ & \frac{C}{R^2}\left[\int_{\Omega_R}\psi\rho^\gamma u^p(\eta\phi_R)^{(q-1)p}d\xi\right]^{\frac{1}{p}}\left[\int_{\Omega_R}\eta^q\psi\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}} \\ & + \frac{C}{R}\left[\int_{\Omega_R}\psi\rho^\gamma u^p(\eta\phi_R)^{(q-1)p}d\xi\right]^{\frac{1}{p}}\left[\int_{\Omega_R}|\nabla_H\eta\cdot\nabla_H\rho|^q\psi\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}} \\ \leq\ & I_R^{\frac{1}{p}}\left(\frac{C}{R^2}\left[\int_{\Omega_R}\eta^q\psi\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}}\right. \\ & \left. + \frac{C}{R}\left[\int_{\Omega_R}\psi|\nabla_H\eta\cdot\nabla_H\rho|^q\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}}\right), \end{aligned} \tag{3.20}$$

for $R > 0$ large enough. The statement is proved.

*Proof of Theorem* 3.2. – Consider, without loss of generality, the half space $\{x_1 > 0\}$.

The claim is proved by using the estimate (3.17) applied to $D = \{x_1 > 0\}$ and $\eta = x_1$.

Indeed, by the maximum principle, to show that $u \equiv 0$, it is enough to check that

$$I_R := \int_{\{x_1>0\}} hu^p{\phi_R}^q x_1^q d\xi \to 0 \quad \text{when } R\to\infty, \tag{3.21}$$

where $\phi_R$ is as in (3.3).

If $D_R := B_H(0,R)\cap\{x_1 > 0\}$, then (3.17) becomes:

$$I_R \leq I_R^{\frac{1}{p}}\left(\frac{C}{R^2}\left[\int_{\Omega_R} x_1^q\psi\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}} + \frac{C}{R}\left[\int_{\Omega_R}\psi|\nabla_H\rho|^q\rho^{\frac{-\gamma q}{p}}d\xi\right]^{\frac{1}{q}}\right).$$

Therefore, as $0\leq\psi\leq 1$ and $x_1 \leq CR$ in $\Omega_R$, for $p \leq \frac{Q+\gamma}{Q-1}$ we get:

$$I_R \leq CI_R^{\frac{1}{p}}R^{(\frac{-\gamma}{p}+\frac{Q}{q}-1)}, \tag{3.22}$$

and we can conclude using the same arguments as in Theorem 3.1.

*Proof of Theorem* 3.3. – As in the proof of Theorem 3.2, the claim is proved using the estimate (3.17) of Lemma 3.1 with $\eta = A\cdot x + B\cdot y + ct + d$ and $D_R := B_H(0,R)\cap D$.

Let us consider the integral

$$I_R := \int_D h u^p {\phi_R}^q \eta^q d\xi, \tag{3.23}$$

where $\phi_R$ is as in (3.3). By (3.17), using the fact that

$$\begin{aligned} &\eta \le CR^2 \\ &|\nabla_H \eta| = |(A + 2cy, B - 2cx)| \le CR \end{aligned} \tag{3.24}$$

we obtain:

$$\begin{aligned} I_R &\le I_R^{\frac{1}{p}}\left(\frac{C}{R^2}\left[\int_{\Omega_R} \eta^q \psi \rho^{\frac{-\gamma q}{p}} d\xi\right]^{\frac{1}{q}} + \frac{C}{R}\left[\int_{\Omega_R} \psi|\nabla_H \eta\cdot\nabla_H\rho|^q \rho^{\frac{-\gamma q}{p}} d\xi\right]^{\frac{1}{q}}\right) \\ &\le C I_R^{\frac{1}{p}} R^{\left(\frac{-\gamma}{p}+\frac{Q}{q}\right)}. \end{aligned} \tag{3.25}$$

If $1 < p \le \frac{Q+\gamma}{Q}$ we can conclude as in the previous cases.

*Proof of Theorem* 3.4. – This result follows from the estimate (3.17) by choosing $\eta := \sum_{i=1}^n (a_i x_i - b_i y_i)(b_i x_i + a_i y_i)$ and $D := \Sigma$. Since the function $\eta$ has the same behaviour as the function $\eta$ chosen in the proof of Theorem 3.3, we can conclude in the same way.

*Remark* 3.3. – Let us observe that, instead of inequality (3.17), one can similarly obtain

$$I_R \le I_R^{\frac{1}{p}}\left(\frac{1}{R^2}\left[\int_{\Omega_R} \eta^q \psi h^{-\frac{q}{p}} d\xi\right]^{\frac{1}{q}} + \frac{1}{R}\left[\int_{\Omega_R} \psi h^{-\frac{q}{p}}|\nabla_H \eta\cdot\nabla_H\rho|^q d\xi\right]^{\frac{1}{q}}\right), \tag{3.26}$$

provided $f$ satisfies (3.2) for some $h \ge 0$ such that the right hand side of (3.26) exists.

Consequently, if $h$ verifies:

$$\lim_{R\to+\infty} \frac{1}{R^q}\int_0^R h^{-\frac{q}{p}}(\rho\omega)\rho^{Q-1} d\rho = 0$$

where $\omega = \frac{\xi}{|\xi|_H}$, then the conclusion of Theorem 3.2 holds true. Similar conditions on $h$ and $p$ can be given for Theorems 3.3 and 3.4.

For the sake of completeness, we will also prove a Liouville theorem for bounded solutions of $\Delta_H u = 0$ in the whole space $H^n$.

THEOREM 3.5. – *If $u$ is a bounded function such that $\Delta_H u = 0$ in the whole space $H^n$, then $u$ is a constant.*

The proof is based on the following representation formula for Heisenberg harmonic functions. This formula can be proved easily by using the divergence's theorem, *see* e.g. Gaveau ([9]) for details.

LEMMA 3.2. – *Let $w$ satisfy $\Delta_H w = 0$ in $H^n$. Then, for any $\xi \in H^n$,*

$$w(\xi) = \frac{C_Q}{R^Q} \int_{B_H(\xi,R)} w(\eta)\psi(\eta)d\eta, \tag{3.27}$$

*where $\psi$ is defined in* (2.4), *and* $C_Q = |B_H(\xi,1)|^{-1}$.

*Proof of Theorem* 3.5. – Let us first prove that $\frac{\partial w}{\partial t} \equiv 0$. Observe that, in view of the Hormander condition, the vector field $T = \frac{\partial}{\partial t}$ commutes with $X_i$ and $Y_i$, *i.e.* $T(X_i) = X_i(T)$ and $T(Y_i) = Y_i(T)$. Hence,

$$\Delta_H(Tw) = T(\Delta_H w) = 0.$$

Therefore, applying the previous lemma, we get:

$$\begin{aligned}\frac{\partial w}{\partial t}(\xi) &= \frac{C_Q}{R^Q} \int_{B_H(\xi,R)} \frac{\partial w}{\partial t}(\eta)\psi(\eta)d\eta \\ &= -\frac{C_Q}{R^Q} \int_{B_H(\xi,R)} \frac{\partial \psi}{\partial t}(\eta)w(\eta)d\eta + \frac{C_Q}{R^Q} \int_{\partial B_H(\xi,R)} w\psi\nu_t dH_{2n},\end{aligned}$$

where $\nu_t$ is the $t$-component of the exterior unit normal vector to $B_H(\xi, R)$. Since

$$\begin{aligned}\left|\frac{\partial \psi}{\partial t}\right| &= \frac{|\psi|\,|t|}{\rho^4} \le \frac{1}{\rho^2} \\ |\nu_t| &= \frac{|t|}{2\rho^3} \le \frac{1}{\rho|\nabla\rho|},\end{aligned}$$

from (2.2) we obtain that

$$\left|\frac{\partial w}{\partial t}(\xi)\right| \le \frac{C||w||_{L^\infty}}{R^2}$$

for any $\xi \in H^n$ and for any $R > 0$. Thus, letting $R$ go to infinity, we get $\frac{\partial w}{\partial t}(\xi) = 0$ for any $\xi \in H^n$. Then, $w$ is a bounded solution of

$$\sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} + \frac{\partial^2 w}{\partial y_i^2} + \frac{\partial^2 w}{\partial t^2} = 0 \quad \text{in } \mathbb{R}^{2n+1}.$$

Therefore it has to be constant by the classical Liouville theorem (*see* e.g. [11]).

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