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A negative answer to a one-dimensional symmetry problem in the Heisenberg group

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1 Introduction and main Results

Symmetry properties of solutions to semi-linear elliptic equations have been widely studied in the last decades. In this contest, a long standing conjecture by De Giorgi states that any global solution to the Ginzburg-Landau equation

$$\Delta u + u(1 - u^2) = 0 \text{ in } \mathbb{R}^N \ (N \le 8)$$
(1.1)

satisfying $-1 \le u \le 1$ and $\frac{\partial u}{\partial x_N} > 0$ is constant along hyper-planes. Recently this conjecture was proved to be true by Ghoussoub and Gui for N = 2 ([18]) and by Ambrosio and Cabré for N = 3 ([3]). It is still an open question for N > 3 though Alberti, Ambrosio and Cabré generalized the result for any C^2 non-linearity (when $N \le 3$) [1].

Under the further hypothesis that the solution u satisfies

$$\lim_{x_3 \to \pm \infty} u(x', x_3) = \pm 1 \quad \forall x' \in \mathbb{R}^2$$

the proof that u is constant along hyper-planes given in [3] is somehow simpler. On the other hand, under the hypothesis that this limit is uniform in x', the conjecture was known as Gibbons conjecture and it has been proved for all dimensions independently by Barlow, Bass, Gui in [4], Berestycki, Hamel, Monneau in [5] and Farina in [12].

In recent years symmetry and monotonicity properties of solutions to semilinear equations have been investigated in the more general contest of the Carnot groups, see [7, 8, 9], [2,6] and [15]. The interest in semi-linear equations in these groups has increased as they appear in many theoretical and application fields, such as complex geometry and mathematical models for crystal structures [11].

In [8] Prajapat and the first author studied Gibbons conjecture for the equation

$$\Delta_{\mathbb{H}^n} u + f(u) = 0 \text{ in } \mathbb{H}^n, \tag{1.2}$$

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where $\Delta_{\mathbb{H}^n}$ denotes the Kohn-Laplacian on the Heisenberg group \mathbb{H}^n and f(u) is a non linear term with some general hypothesis (in particular they include the case $f(u) = u(1 - u^2)$). They prove that the conjecture holds true for all directions orthogonal to the center of \mathbb{H}^n .¹ The question of whether the result holds true in the remaining direction was raised in [8].

The aim of this paper is to prove that, with respect to the center direction of \mathbb{H}^n , the stronger De Giorgi conjecture is not true for the equation (1.2). This negative answer will easily follow from next Theorem 1.1, the main result of this note.

In order to clearly state our theorem, we need to recall some known facts about the Heisenberg space \mathbb{H}^n and its intrinsic Laplacian $\Delta_{\mathbb{H}^n}$.

First of all let us say that \mathbb{H}^n is the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$, $n \in N$, endowed with the group action \circ given by

$$\xi_o \circ \xi = (z + z_o, t + t_o + 2\operatorname{Im}(\overline{z} \cdot z_o)).$$
(1.3)

Here and in the rest of the paper we identify \mathbb{C}^n with \mathbb{R}^{2n} and, setting z = x + iy, for the points of \mathbb{H}^n we use the equivalent notations $\xi = (z, t) = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with $z := (z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$. Furthermore, " \cdot " denotes the usual inner product in \mathbb{C}^n .

The Lie Algebra of left invariant vector fields is generated by

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, & \text{for } i = 1, \dots, n, \\ Y_i &= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, & \text{for } i = 1, \dots, n. \end{aligned}$$

The intrinsic Laplacian of \mathbb{H}^n , also called the Kohn Laplacian, is defined as

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^n (X_i^2 + Y_i^2).$$

It is a second order degenerate elliptic operator of Hormander type and hence it is hypoelliptic (see e.g. [13] or [19] for more details about $\Delta_{\mathbb{H}^n}$).

With respect to the group dilation $\delta_{\lambda}\xi = (\lambda z, \lambda^2 t), \Delta_{\mathbb{H}^n}$ is homogeneous of degree two in the following sense

$$\Delta_{\mathbb{H}^n} \circ \delta_\lambda = \lambda^2 \delta_\lambda \circ \Delta_{\mathbb{H}^n}.$$

The Koranyi ball of center ξ_o and radius R is defined by

$$B_{\mathbb{H}^n}(\xi_o, R) := \{\xi \text{ such that } |\xi^{-1} \circ \xi_o|_{\mathbb{H}^n} \le R\}$$

where

$$|\xi|_{\mathbb{H}^n} = \left(|z|^4 + t^2\right)^{\frac{1}{4}}$$

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is a norm with respect to the group dilation and it satisfies

$$|B_{\mathbb{H}^n}(\xi_o, R)| = |B_{\mathbb{H}^n}(0, R)| = CR^Q$$

¹ Very recently, in [6], the results of [8] have been extended to every sub-Laplacian on a Carnot group.

where Q = 2n + 2 is the homogeneous dimension of \mathbb{H}^n .

A fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at the origin is given by:

$$\Gamma(\xi) = \frac{C_Q}{(|\xi|_{\mathbb{H}^n})^{Q-2}}$$

where C_Q is a positive constant.

For our purposes it is convenient to remind that the class of cylindrically symmetric functions is invariant with respect to the action of $\Delta_{\mathbb{H}^n}$. We shall say that a function $(z,t) \rightarrow u(z,t)$ is cylindrically symmetric if there exist a two variables function U such that u(z,t) = U(r,t), r = |z|.

In that case we formally have that

$$\Delta_{\mathbb{H}^n} u(z,t) = \partial_{rr} U + \frac{2n-1}{r} \partial_r U + 4r^2 \partial_{tt} U.$$

The main result of this paper is the following:

Theorem 1.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying the hypotheses listed below:

- (H1) f is odd,
- (H2) f > 0 in]0, 1[, f(0) = f(1) = 0, (H3) $\lim_{s \to 0} \frac{f(s)}{s} = l > 0$.

Then there exists a solution *u* to the equation:

$$\Delta_{\mathbb{H}^n} u + f(u) = 0 \quad in \quad \mathbb{R}^{2n+1} \tag{1.4}$$

satisfying |u| < 1, $\frac{\partial u}{\partial t} > 0$ and

$$\lim_{t \to \pm \infty} u(z, t) = \pm 1.$$

Moreover u is cylindrically symmetric and of class C^{∞} when f is C^{∞} .

For solution u of (1.4) we mean a continuous function u such that:

- 1. For a suitable $\alpha > 0$, $u \in \Lambda^{2+\alpha}_{\text{loc}}(\mathbb{H}^n)$ i.e. $X_j^2 u$ and $Y_j^2 u$, $j = 1, \cdots, n$, exist in the weak sense of distributions and belong to $\Lambda^{\alpha}_{\text{loc}}(\mathbb{H}^n)$
- 2. u satisfies (1.4) pointwise everywhere.

As in [13] we have denoted by $\Lambda_{loc}^{\alpha}(\mathbb{H}^n)$ the class of functions which are locally α -Holder continuous with respect to the intrinsic distance d in \mathbb{H}^n defined by

$$d(\xi,\xi') = |(\xi')^{-1} \circ \xi|_{\mathbb{H}^n}$$

Using the commutators of the Lie Algebra, it is easy to see that $\Lambda^{2+\alpha}_{\text{loc}}(\mathbb{H}^n)$ is continuously embedded in the usual $C^{1+\frac{\alpha}{2}}_{\text{loc}}(\mathbb{R}^{2n+1})$.

From Theorem 1.1 we immediately get the following corollary.

Corollary 1.1 De Giorgi's conjecture in the t-direction is not true in \mathbb{H}^n .

Proof The function $f(s) = s(1 - s^2)$ satisfies all hypotheses of Theorem 1.1, hence there exists a C^{∞} function u such that

$$\begin{cases} \Delta_{\mathbb{H}^n} u + u(1-u^2) = 0 \text{ in } \mathbb{R}^{2n+1}, \\ -1 < u < 1, \ \frac{\partial u}{\partial t} > 0, \\ \lim_{t \to \pm \infty} u(z,t) = \pm 1. \end{cases}$$

Then, if De Giorgi conjecture were true in the t direction there would exist $\alpha \in \mathbb{R}^{2n}$ and $\nu > 0$ such that $u(z,t) = U(\alpha \cdot z + t\nu)$ for some function $U : \mathbb{R} \to \mathbb{R}$. Furthermore U would satisfy

$$(|\alpha|^2 - 4\nu(J\alpha \cdot z) + 4r^2\nu^2) U'' = U(U^2 - 1)$$

where J is the classical symplectic $2n \times 2n$ matrix. This is a contradiction since the right hand side is constant along the hyperplanes $\alpha \cdot z + t\nu = c$ for any $c \in \mathbb{R}$ while the left hand side is not.

It is well known that De Giorgi's conjecture has been sometimes referred to as the ε version of Bernstein Theorem. The reason being that if u is a solution of equation (1.1) then the rescaled energy of the blow-down of $u \Gamma$ -converges to the perimeter functional and in particular the rescaled u, L^1 converges to the characteristic function of a set E (see [20]).

Alberti, Ambrosio and Cabre' in [1] have made rigorous this statement proving that the limit set has minimal local perimeter. Hence by Bernstein's theorem in dimension $N \leq 8$ it is a half-space. This is allegedly the reason why one expects the level sets of u to be minimal.

In the last section we prove that the solution constructed in Theorem 1.1 is a counter-example to the fact that these two properties are related.

Indeed in Proposition 3.3 we prove, using some results of Monti and Serra-Cassano [21] and an energy estimate, that the blow-down of the solution u constructed in Theorem 1.1. converges to the characteristic function of a set F with minimal perimeter (see [14] for the definition of perimeter in \mathbb{H}^n). In fact this is proved for any solution of (1.4) which is \mathbb{H}^n -monotone (see the last section for the definition).

On the other hand, the level sets of u are *not* of minimal perimeter since, as shown in Proposition 3.4, there are regular graphs of minimal perimeter that are defined in the whole space and that have cylindrical symmetry different from the hyper-planes t = c.

Remark 1.1 It would be interesting to know whether the function constructed in Theorem 1.1 has uniform limit with respect to z.

Remark 1.2 It is natural to consider the extension of Theorem 1.1 to the context of Carnot groups. This will be the object of a subsequent study.

Remark 1.3 In [10] Caffarelli, Garofalo e Segala proved among other things the following result. Let u be a classical solution to the semilinear Poisson equation

$$\Delta u = f(u) \text{ in } \mathbb{R}^n$$

and assume F(u(x)) > 0, $\forall x \in \mathbb{R}^n$, when F is an anti-derivative of f. Then if u also satisfies the "Modica-Mortola equation":

$$\frac{1}{2}|\nabla u|^2 = F(u) \text{ in } \mathbb{R}^n,$$

the level sets of u are hyper-planes.

A similar result holds in the Heisenberg setting too. However, the solutions to the corresponding system of equations

$$\Delta_{\mathbb{H}^n} u = f(u), \ \frac{1}{2} |\nabla_{\mathbb{H}^n} u|^2 = F(u)$$
(1.5)

do have one-dimensional character only in directions orthogonal to the center of the group. This result easily follows using the argument in [10] together with a Liouville theorem in \mathbb{H}^n .

Indeed, let u be a solution to the system (1.5) where F is an antiderivative of f such that $F(u(\xi)) > 0 \ \forall \xi \in \mathbb{H}^n$. Following [10], define

$$v(\xi) = H(u(\xi)) := \int_{u(0)}^{u(\xi)} F(s)^{\frac{1}{2}} ds.$$
 (1.6)

An easy computation shows that

$$|\nabla_{\mathbb{H}^n} v| = 1, \Delta_{\mathbb{H}^n} v = 0.$$
(1.7)

Then by a Liouville type theorem given in [17] one obtains that v is a polynome of degree one with respect to δ_{λ} and therefore

$$v(z,t) = \alpha \cdot z + \gamma$$

for suitable $\alpha \in \mathbb{R}^{2n}$ and $\gamma \in \mathbb{R}$. Thus, by (1.6),

$$u(z,t) = H^{-1}(\alpha \cdot z + \gamma)$$

i.e. we have obtained that the level sets of u are hyper planes parallel to the center of the group as claimed.

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2 Proof of Theorem 1.1

For any R > 0 we shall denote by D_R and D_R^+ respectively the cylinders

$$D_R = \{(z,t) \in \mathbb{R}^{2n+1}; |z| < R, |t| < R^2\}$$

and

$$D_R^+ = \{ (z,t) \in \mathbb{R}^{2n+1}; \ |z| < R, \ 0 < t < R^2 \}.$$

Let $\psi(t) = \frac{t}{R^2}$.

We shall split the proof in several steps.

First step. The semilinear Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{H}^n} u = -f(u) \text{ in } D_R^+, \\ u(r,t) = \psi(t), \text{ on } \partial D_R^+. \end{cases}$$
(2.8)

has a solution $u \in \Lambda^{2+\alpha}_{\text{loc}}(D_R^+) \cap \Lambda^{\alpha}(\overline{D_R^+})$ for a suitable $\alpha \in (0,1)$. Furthermore u is cylindrically symmetric, $0 \le u \le 1$ and for any R sufficiently large,

$$u \ge v_o$$

for some function $v_o \ge 0$, $v_o \not\equiv 0$, v_o independent of R.

Let $M \in \mathbb{R}^+$ be larger than the Lipschitz constant of f in [0, 1] and let us define

$$g: \mathbb{R} \to \mathbb{R}, \ g(s) = f(s) + Ms$$

Let \mathcal{T} be the map formally defined by $\mathcal{T}(v) = u$ where u is the only solution to the Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{H}^n} u - Mu = -g(v) \text{ in } D_R^+, \\ u = \psi, \text{ on } \partial D_R^+. \end{cases}$$
(2.9)

The operator \mathcal{T} has the following properties:

(P1) There exists $\alpha \in (0,1)$ such that \mathcal{T} is well defined in $\Lambda^{\alpha}(\overline{D_{R}^{+}})$. Furthermore

$$|u(\xi) - u(\xi')| \le Cd(\xi',\xi)^{\alpha}(1 + \sup|g(v)|)$$
(2.10)

for any $\xi, \xi' \in D_R^+$. We also have that $\mathcal{T}(v) \in A^{2+\alpha}_{\text{loc}}(D_R^+)$ for every $v \in A^{\alpha}(\overline{D_R^+})$.

This statement can be proved by using standard arguments and the results in [13, 19] (see also [16, Theorem 4.1]).

(P2) $\mathcal{T}(v)$ is cylindrically symmetric if v is cylindrically symmetric.

Indeed suppose that $u = \mathcal{T}(v)$. Let S be a unitary rotation in \mathbb{C}^n and define $u_S(z,t) := u(Sz,t)$. Since $\Delta_{\mathbb{H}^n}$ is invariant with respect to S, we have $\Delta_{\mathbb{H}^n} u_S(z,t) = \Delta_{\mathbb{H}^n} u(Sz,t)$, so that u_S is a solution of

$$\begin{cases} \Delta_{\mathbb{H}^n} u_{\mathcal{S}} - M u_{\mathcal{S}} = -g(v(\mathcal{S}z, t)) = -g(v) \text{ in } D_R^+, \\ u_{\mathcal{S}} = \psi, \text{ on } \partial D_R^+. \end{cases}$$

Here we have used the invariance with respect to $\mathcal S$ of v,ψ and D_R^+ .

By the maximum principle we know that the solution of (2.9) is unique, hence $u = u_S$ for any S, i.e. u is a function of $(|z_1|, |z_2|, \ldots, |z_n|, t)$. Then for such functions it is easy to see that $\Delta_{\mathbb{H}^n} u = Gu := \Delta_z u + 4|z|^2 \partial_{tt} u$ and u solves

$$\begin{cases} Gu - Mu = -g(v) \text{ in } D_R^+, \\ u = \psi, \text{ on } \partial D_R^+. \end{cases}$$

Now the operator G is invariant with respect to real rotations around the t axis and it satisfies the maximum principle on D_R^+ . Then, arguing as above, we can prove that

 $u(\mathcal{R}z,t) = u(z,t)$ for every rotation \mathcal{R} in \mathbb{R}^{2n} . This prove that u is cylindrically symmetric.

(P3) \mathcal{T} is monotone. More precisely if $v_1, v_2 \in \Lambda^{\alpha}(\overline{D_R^+})$ and $0 \leq v_1 \leq v_2 \leq 1$, then $\mathcal{T}v_1 \leq \mathcal{T}v_2$.

Let us observe that with our choice of M if $0 \le v_1 \le v_2$ then $g(v_1) \le g(v_2)$. Hence (P3) follows from the maximum principle for $-\Delta_{\mathbb{H}^n} + M$ in D_R^+ .

(P4) If $v \in \Lambda^{\alpha}(\overline{D_R^+})$ and $0 \le v \le 1$ then $0 \le \mathcal{T}(v) \le 1$. Indeed, since g(0) = 0, g(1) = M and $0 \le \psi \le 1$ on ∂D_R^+ , again by the maximum principle we obtain that $\mathcal{T}(1) \le 1$ and $\mathcal{T}(0) \ge 0$. Now we only need to apply property (P3) for $v \in \Lambda^{\alpha}(D_{B}^{+})$ such that $0 \leq v \leq 1$.

We shall now construct a function $v_o \ge 0$ that plays the role of a lower barrier. Let λ_o denote the principal eigenvalue of $-\Delta_{\mathbb{H}^n}$ in D_B^+ and let $\phi_o > 0$ be the corresponding eigenfunction normalized by $\sup \phi_o = 1$.

We choose and fix R_o sufficiently large that

$$\lambda_o \le \frac{l}{2}$$

where l is the limit in condition (H3). Then there exists $\varepsilon \in (0, 1)$ independent of R such that

$$\lambda_o \varepsilon \phi_o \le f(\varepsilon \phi_o).$$

By uniqueness of the normalized eigenfunction ϕ_o , arguing as in the proof of (P2) we can prove that ϕ_o is cylindrically symmetric.

From now on we assume that $R > R_o$. Let us define

$$v_o = \begin{cases} \varepsilon \phi_o & \text{ in } D_{R_o}^+ \\ 0 & \text{ in } D_R^+ \setminus D_{R_o}^+ \end{cases}$$

Standard arguments show that v_0 is locally Holder continuous in \mathbb{R}^{2n+1} , (see e.g. [16, Theorem 4.1], we stress that condition (4.4) in that theorem is satisfied since D_R^+ is convex).

As a consequence $\mathcal{T}(v_o)$ is well defined and since $0 \le v_o \le 1$ using (P4) we get that $0 \leq \mathcal{T}(v_o) \leq 1$. Let us now prove that $v_o \leq u_o := \mathcal{T}(v_o)$. Clearly the inequality holds in $D_R^+ \setminus D_{R_o}^+$, using (P4), hence we just have to prove it in $D_{R_o}^+$. We have

$$\begin{aligned} \Delta_{\mathbb{H}^n} u_o - M u_o &= -g(v_o) = -g(\varepsilon \phi_o) \leq -(M + \lambda_o)(\varepsilon \phi_o) \\ &= -M \varepsilon \phi_o + \Delta_{\mathbb{H}^n} \varepsilon \phi_o = -M v_o + \Delta_{\mathbb{H}^n} v_o, \end{aligned}$$

so that

$$\begin{cases} \Delta_{\mathbb{H}^n}(u_o - v_o) - M(u_o - v_o) \le 0 & \text{in } D^+_{R_o} \\ u_o \ge v_o & \text{on } \partial D^+_{R_o} \end{cases}$$

The maximum principle implies that $u_o \ge v_o$ in $D_{R_o}^+$.

Now we construct the sequence of functions

$$v_k = \mathcal{T}^k(v_o), k \in \mathbb{N}.$$

Clearly using the properties above, all v_k are cylindrically symmetric and

$$1 \ge \mathcal{T}^k(v_o) \ge \mathcal{T}(v_o) \ge v_o \ge 0$$
 for every $k \in \mathbb{N}$.

Let us denote by u the pointwise limit of (v_k) . Then u is cylindrically symmetric, $v_o \leq u \leq 1, u \in \Lambda^{\alpha}(\overline{D_R^+})$ since, by (2.10)

$$|v_k(\xi) - v_k(\xi')| \le Cd(\xi',\xi)^{\alpha}$$

where C > 0 is independent of R. This estimate implies that the v_k uniformly converges to u in $\overline{D_R^+}$, so that $u = \psi$ on ∂D_R^+ .

Furthermore in the weak sense of distributions, u satisfies

$$\Delta_{\mathbb{H}^n} u + f(u) = 0 \quad \text{in} \quad D_R^+. \tag{2.11}$$

From (2.11), the Holder regularity of u and standard bootstrap argument we obtain that $u \in \Lambda^{2+\alpha}_{loc}(D_R^+)$ and it satisfies the equation pointwise. Hence u is the required function.

Remark 2.1 Since u is cylindrically symmetric we have that u(z,t) = U(|z|,t) and U satisfies the semi-linear elliptic equation

$$\partial_{rr}U + \frac{2n-1}{r}\partial_{r}U + 4r^{2}\partial_{tt}U + f(U) = 0$$

in the open subset of \mathbb{R}^2

$$\Omega_R := \{ (r, t) \in \mathbb{R}^2 / 0 < r < R, 0 < t < R^2 \}.$$

Moreover U is locally $\frac{\alpha}{2}$ -Holder continuous, in the usual sense, up to $\partial \Omega_R \setminus \{(0,t)/0 < t < R^2\}$. Then, being U(r,0) = 0 when 0 < r < R, by classical regularity results for elliptic equations, U is of class $C_{\text{loc}}^{2+\frac{\alpha}{2}}$ up to $\Omega_R \cup \{(r,0)/0 < r < R\}$.

Second step. The function constructed in the first step satisfies $\frac{\partial u}{\partial t} > 0$.

In [9] the following definition and theorem are given:

Definition 2.1 Fix $\eta \in \mathbb{H}^n$. A domain $\Omega \subset H$ is said to be η -convex (or convex in the direction η) if for any $\xi_1 \in \Omega$ and any $\xi_2 \in \Omega$ such that $\xi_2 = \alpha \eta \circ \xi_1$ for some $\alpha > 0$, we have $s\eta \circ \xi_1 \in \Omega$ for every $s \in (0, \alpha)$.

Theorem 2.1 Let Ω be an arbitrary bounded domain of \mathbb{H}^n which is η - convex for some $\eta \in H$. Let $u \in \Lambda^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

$$\Delta_{\mathbb{H}^n} u + f(u) = 0 \quad \text{in } \Omega \\ u = \psi \quad \text{on } \partial\Omega \right\}$$
(2.12)

where f is a Lipschitz continuous function. Assume that for any $\xi_1, \xi_2 \in \partial \Omega$, such that $\xi_2 = \alpha \eta \circ \xi_1$ for some $\alpha > 0$, we have for each $s \in (0, \alpha)$

$$\psi(\xi_1) < u(s\eta \circ \xi_1) < \psi(\xi_2) \quad s\eta \circ \xi_1 \in \Omega$$
(2.13)

and

$$\psi(\xi_1) < \psi(s\eta \circ \xi_1) < \psi(\xi_2) \text{ if } s\eta \circ \xi_1 \in \partial\Omega$$
(2.14)

Then *u* satisfies

$$u(s_1\eta\circ\xi) < u(s\eta\circ\xi) \tag{2.15}$$

for any $0 < s_1 < s < \alpha$ and for every $\xi \in \Omega$.

Moreover, u is the unique solution of (2.12) in $\Lambda^2(\Omega) \cap C(\overline{\Omega})$ satisfying (2.13).

Let us choose $\eta = (0, 1)$, clearly D_R^+ is η -convex since:

$$s\eta \circ \xi = (z, t+s).$$

Furthermore $0 = \psi(0) \le u(z,t) \le \psi(1) = 1$ and by construction ψ satisfies (2.14). Hence we are in the hypothesis of Theorem 2.1 and u satisfies

$$u(z, t_1) \le u(z, t_2)$$
 for any $0 \le t_1 \le t_2 \le 1$

in D_R^+ . In particular we get $\frac{\partial u}{\partial t} \ge 0$.

Now since $\frac{\partial}{\partial t}$ commutes with $\Delta_{\mathbb{H}^n}$ and f is Lipschitz continuous then the inequality is strict, just by using the strong Maximum principle.

Third step. We extend to D_R the function u of the previous step by setting

$$v(z,t) = \begin{cases} u(z,t) & \text{for } t \ge 0\\ -u(z,-t) & \text{for } t \le 0. \end{cases}$$

Obviously v is cylindrically symmetric, $-1 \le v \le 1, v \ge v_o$ in $D_R^+, v \in C^{\frac{\alpha}{2}}(D_R)$ and $v = \psi$ on ∂D_R . We want to prove that v satisfies

$$\Delta_{\mathbb{H}^n} v + f(v) = 0 \text{ in } D_R. \tag{2.16}$$

Since f is odd, using the fact that v is odd and cylindrically symmetric it is easy to see that v satisfies (2.16) in $D_R \setminus \{t = 0\}$.

By Remark 2.1 at the end of the first step, we now obtain that $v \in C^{2+\frac{\alpha}{2}}(D_R \setminus$ $\{(0,0)\}\)$ and it solves (2.16) in the same open set. Hence we just have to remove the singularity at the origin. Let us define

$$w(\xi) = -\int_{D_R} \Gamma((\xi')^{-1} \circ \xi) f(v(\xi')) d\xi',$$

where $\Gamma(z,t)$ is the fundamental solution recalled in the Introduction. Since $f(v) \in C^{\frac{\alpha}{2}}(D_R)$ and $C^{\frac{\alpha}{2}}_{loc}(D_R) \subset \Lambda^{\frac{\alpha}{2}}_{loc}(D_R)$, then $w \in \Lambda^{2+\frac{\alpha}{2}}_{loc}(D_R)$ and satisfies

 $\Delta_{\mathbb{H}^n} w = f(v) \text{ in } D_R.$

Hence

$$\Delta_{\mathbb{H}^n}(v+w) = 0 \text{ in } D_R \setminus \{(0,0)\}$$

On the other hand $v + w \in L^{\infty}(D_R)$. Then there exists a C^{∞} -function h, $\Delta_{\mathbb{H}^n}$ harmonic in D_R such that

$$h = v + w$$
 in $D_R \setminus \{(0, 0)\}.$

It follows that v solves (2.16) everywhere in D_R .

This ends the third step. We shall denote $u_R(z,t) = v(z,t)$ the function constructed above.

Fourth step. We let R tend to infinity and obtain a global solution.

Since the functions u_R are equi-bounded and solutions of (2.8) in D_R , then $\Delta_{\mathbb{H}^n} u_R$ are also equi-bounded and by standard arguments, if necessary passing to a subsequence, the u_R 's locally uniformly converge to u, weak solution of

$$\Delta_{\mathbb{H}^n} u + f(u) = 0 \text{ in } \mathbb{R}^{2n+1}.$$
(2.17)

Furthermore

- 1) u is cylindrically symmetric,
- 2) $-1 \le u \le 1$,
- 3) u(z,t) = -u(z,-t),
- 4) for $t \ge 0$, $u(z,t) \ge v_o(z,t)$,
- 5) $t \mapsto u(z,t)$ is monotone increasing.

Since f is locally Lipschitz continuous and $|u| \leq 1$, it follows from (2.17) that $u \in \Lambda_{loc}^{2+\alpha}(\mathbb{H}^n)$ for every $\alpha < 1$. Obviously, the more regular f is, the more regular u is; in particular u is of class C^{∞} when f is C^{∞} .

Moreover, property 5) implies $\frac{\partial u}{\partial t} \ge 0$ so that, since $\frac{\partial}{\partial t}$ commutes with $\Delta_{\mathbb{H}^n}$, by the strong maximum principle either $\frac{\partial u}{\partial t} > 0$ or $\frac{\partial u}{\partial t} \equiv 0$. But by 3) and 4) this second possibility is absurd hence $\frac{\partial u}{\partial t} > 0$.

Last step. We want to prove that

$$\lim_{t \to \pm \infty} u(z, t) = \pm 1.$$

We shall consider only the limit in $+\infty$ since the other case follows similarly. Let us denote $u_o(z) := \lim_{t \to +\infty} u(z,t)$. Since u is bounded and monotone in t the limit is well defined and $0 < u_o(z) \le 1$. We want to prove that $u_o(z) \equiv 1$.

By standard arguments (multiplying equation (2.17) by a sequence of functions $\psi_k(z,t) = \phi(z)\phi_k(t)$ where ϕ has compact support and $\sup \phi_k =]k, k+1[$ and $\int \phi_k dt = 1$ and letting k go to infinity) it easy to see that u_o is a weak solution of

$$\Delta u_o + f(u_o) = 0 \text{ in } \mathbb{R}^{2n}$$

Clearly a bootstrap argument shows that u_o is a classical solution. Moreover $u_o(z) = U_o(r)$ with r = |z| for some function U_o solution of

$$U_o''(r) + \frac{2n-1}{r}U_o'(r) + f(U_o(r)) = 0, \qquad (2.18)$$

$$U_o'(0) = 0 (2.19)$$

The Cauchy problem for (2.18) with initial conditions $U_o(0) = 1$ and $U'_o(0) = 0$ has a unique solution (see e.g. [23]). Thus, since f(1) = 0, if $U_o(0) = 1$ then $U_o \equiv 1$ and we are done. Suppose, by contradiction, that $U_o(0) < 1$.

It is easy to see that $U'_{o} < 0$. Indeed integrating (2.18) one obtains:

$$r^{2n-1}U'_o(r) = -\int_0^r \rho^{2n-1} f(U_o(\rho))d\rho < 0,$$
(2.20)

hence U_o is strictly decreasing and has a finite non–negative limit as $r \to \infty$. More precisely $\lim_{r \to +\infty} U_o(r) = 0$. Indeed otherwise $U_o(r) \to k > 0$ and $f(U_o(r)) \to f(k) > 0$ (by (H2)). This, together with (2.20) implies that $|U'_o(r)| \to \infty$, which is absurd since U_o is bounded. Using hypothesis (H3) on f we obtain that for r large U_o satisfies

$$U_o''(r) + \frac{2n-1}{r}U_o'(r) + K(r)U_o(r) = 0$$

with $K(r) = \frac{f(U_o(r))}{U_o(r)} \rightarrow l > 0.$

Using the substitution $V_o(r) = r^{\frac{2n-1}{2}} U_o(r)$ we obtain that V_o satisfies

$$V''(r) + H(r)V(r) = 0$$

with $H(r) = \frac{2n-1}{2}(1-\frac{N-1}{2})\frac{1}{r^2} + K(r)$. Comparing with

$$U''(r) + \frac{l}{2}U(r) = 0$$

we obtain that V_o i.e. U_o has infinite zeros in a neighborhood of infinity, which is absurd. This conclude the last step and the proof.

3 Minimal surfaces

The energy estimates given in this section are inspired by the work of Alberti, Ambrosio and Cabré [1]. The novelty reside in the fact of defining the right "monotonicity" condition, since the vector fields X_i and Y_i don't commute with one another or with the classical $\frac{\partial}{\partial x_i}$.

We need to introduce other vector fields that will play a crucial role i.e. the right-invariant vector fields:

$$\begin{split} \dot{X}_i &= \frac{\partial}{\partial x_i} - 2y_i \frac{\partial}{\partial t}, \text{ for } i = 1, \dots, n, \\ \tilde{Y}_i &= \frac{\partial}{\partial y_i} + 2x_i \frac{\partial}{\partial t}, \text{ for } i = 1, \dots, n. \end{split}$$

They are obtained through the left action of \circ e.g. for \tilde{X}_1 let $e_1 = (1, 0, \dots, 0)$ then:

$$\tilde{X}_1 u(\xi) = \lim_{h \to 0} \frac{u(he_1 \circ \xi) - u(\xi)}{h}$$

hence they commute with left-invariant vector fields and therefore with $\Delta_{\mathbb{H}^n}$.

Similarly to the Euclidean case (see [1]) we want to prove three propositions in the hypothesis that u is "monotone", concerning the minimality of the Energy functional.

Definition 3.1 We will say that u is \mathbb{H}^n -monotone if either $\tilde{X}_i u > 0$ for one of the indices $1 \le i \le n$ or similarly for $\tilde{Y}_i u > 0$ or $\partial_t u > 0$.

Finally let F be a primitive of -f i.e. let us suppose that u is a solution of

$$\Delta_{\mathbb{H}^n} u - F'(u) = 0 \text{ in } \mathbb{H}^n. \tag{3.21}$$

In all this section we suppose that $F \in C^2$ Then we define

$$E(u) = \int_{\mathbb{H}^n} \frac{1}{2} |\nabla_{\mathbb{H}^n} u|^2 + F(u) dx.$$

Remark. If u is an \mathbb{H}^n -monotone bounded solution of (3.21) than u is a stable solution i.e. the second variation of E at u is semi-positive definite.

Indeed suppose that e.g. u satisfies $\tilde{X}_1 u > 0$. It is enough to remark that since the right-invariant vector fields commute with $\Delta_{\mathbb{H}^n}$ the function $\phi = \tilde{X}_1 u$ is a positive solution of the linearized equation

$$\Delta_{\mathbb{H}^n}\phi = F''(u)\phi.$$

Second we want to see that local minimality holds true for \mathbb{H}^n -monotone bounded solutions. Without loss of generality we shall suppose that $\tilde{X}_1 u > 0$.

We will denote by $\xi_+ = \lim_{s \to +\infty} se_1 \circ \xi$, $\xi_- = \lim_{s \to -\infty} se_1 \circ \xi$ and

$$\bar{u}(\xi_+) = \lim_{s \to +\infty} u(se_1 \circ \xi)$$

and similarly for $\underline{u}(\xi_{-})$.

Proposition 3.1 Let u be a bounded solution of (3.21) such that $\tilde{X}_1 u > 0$. Let Ω be any bounded domain of \mathbb{H}^n then

$$E(u,\Omega) := \int_{\Omega} \frac{1}{2} |\nabla_{\mathbb{H}^n} u|^2 + F(u) dx \le \int_{\Omega} \frac{1}{2} |\nabla_{\mathbb{H}^n} v|^2 + F(v)$$

for any $v \in C^1(\Omega)$ such that u = v on $\partial \Omega$ and satisfying $\underline{u}(\xi_-) \leq v(\xi) \leq \overline{u}(\xi_+)$ in Ω . *Proof* The proof is very similar to the one give in [1] and we give a sketch of it for completeness sake. It will be enough to prove the minimality when v is as in the proposition and satisfies

$$\underline{u}(\xi_{-}) < v(\xi) < \overline{u}(\xi_{+}).$$

Let us call A the set of such functions. Furthermore we define $\tau(\xi, s)$ to be the real number τ such that $u^{\tau}(\xi) := u(\tau e_1 \circ \xi) = s$.

And finally we define

$$\mathcal{F}(v) := \int_{\Omega} \nabla_{\mathbb{H}^n} u^{\tau} \cdot \nabla_{\mathbb{H}^n} v - \frac{1}{2} |\nabla_{\mathbb{H}^n} u^{\tau}|^2 + F(v) dx.$$

for $\tau = \tau(\xi, v(\xi))$.

Claim: \mathcal{F} is constant on A.

In order to do this we introduce the following 2n+1-dimensional vector function $\phi=(\phi^{\xi},\phi^{s})$

$$\phi^{\xi}(\xi, s) = \nabla_{\mathbb{H}^n} u^{\tau}$$
$$\phi^s(\xi, s) = \frac{1}{2} |\nabla_{\mathbb{H}^n} u^{\tau}|^2 - F(s)$$

It is easy to see that since $\nabla_{\mathbb{H}^n}$ commutes with the left action and hence with $\frac{\partial}{\partial \tau}$, we have $\frac{\partial X_1 u^{\tau}}{\partial \tau} = X_1 \frac{\partial u^{\tau}}{\partial \tau}$. Furthermore it is immediate that

$$\tilde{X}_1 u^{\tau} X_1 \tau + X_1 u^{\tau} = 0, \ \tilde{X}_1 u^{\tau} \frac{\partial \tau}{\partial s} = 1.$$

And hence

$$\operatorname{div}_{H,s}\phi := \nabla_{\mathbb{H}^n} \cdot \phi^{\xi} + \partial_s \phi^s = \Delta_{\mathbb{H}^n} u^{\tau} - F'(s) = 0$$

If we call $w_{\sigma} = v + \sigma(w - v)$ with w and v in A, we just need to observe that with w - v = 0 on $\partial \Omega$ integrating by part one gets

$$\partial_{\sigma} \mathcal{F}(w_{\sigma}) = \int_{\Omega} \phi^{\xi} \nabla_{\mathbb{H}^{n}}(w-v) + \partial_{s} \phi^{\xi} \nabla_{\mathbb{H}^{n}} v(w-v) - \int_{\Omega} \partial_{s} \phi^{s}(w-v)$$
$$= -\int_{\Omega} \operatorname{div}_{\mathbb{H}^{n},s} \phi(w-v) = 0.$$

and this concludes the claim.

By construction $\mathcal{F}(u) = E(u, \Omega)$ and $\mathcal{F}(w) \leq E(w, \Omega)$. Hence, for w in A:

$$E(u,\Omega) = \mathcal{F}(u) = \mathcal{F}(w) \le E(w,\Omega).$$

This concludes the proof.

The next Proposition gives a bound of the energy in the Korany ball B_R .

Proposition 3.2 Let u be a bounded and \mathbb{H}^n -monotone solution of (1.2) and suppose that $\bar{u} = M = \max u$ and $\underline{u} = m = \min u$ then

$$\int_{B_R} \frac{1}{2} |\nabla_{\mathbb{H}^n} u|^2 + F(u) - c_u \le C R^{Q-1}.$$

with $c_u = \min\{F(s), s \in [m, M]\}.$

Proof As seen in [1], this is just a consequence of Proposition 3.1. Take $s \in [m, M]$ such that $F(s) = c_u$ and let ϕ_R be a cut-off function equal to one in B_{R-1} and equal to zero outside of B_R such that $|\nabla_{\mathbb{H}^n} \phi_R| \leq 2$.

It is easy to see that $v_R = (1-\phi_R)u + s\phi_R$ satisfies the hypothesis in Proposition 3.1 and hence

$$E(u, B_R) - c_u |B_R| \le E(v_R, B_R) - c_u |B_R| = \int_{B_R \setminus B_{R-1}} \frac{1}{2} |\nabla_{\mathbb{H}^n} v_R|^2 + F(v_R) - c_u dx \le C |B_R \setminus B_{R-1}| = C R^{Q-1}.$$

Before giving the next proposition, let us recall the definition of perimeter in \mathbb{H}^n as given e.g. in [14].

Definition 3.2 $BV_{\mathbb{H}^n}$ is the set of functions $f \in L^1(\Omega)$ such that

$$\|\nabla_{\mathbb{H}^n} f\|(\Omega)$$

:= sup $\left\{ \int_{\Omega} f(\xi) \nabla_{\mathbb{H}^n} \cdot \phi d\xi; \phi = (\phi_1, \cdots, \phi_{2n}) \in C_o^1(\Omega), |\phi(\xi)| \le 1, \forall \xi \in \Omega \right\}$

 $E \subset \mathbb{H}^n$ has local finite perimeter if

$$P(E,\Omega) = \|\partial E\|_{\mathbb{H}^n}(\Omega) := \|\nabla_{\mathbb{H}^n}\chi_E\|(\Omega) < \infty$$

for every open bounded set Ω .

Proposition 3.3 Let u be as in Proposition 3.2 and let R_i be a sequence converging to $+\infty$ and $u_i(\xi) = u(\delta_{R_i}\xi)$.

There exists a subsequence u_{i_k} such that

1. there exists a subset G of \mathbb{H}^n such that

$$\lim_{k \to +\infty} u_{i_k} = \chi_G \text{ in } L^1_{loc}(\mathbb{H}^n)$$

2. *G* has locally finite perimeter and *G* is a local minimizer.

Proof of Proposition 3.3. Let us define

$$E_R(u,\Omega) = \int_{\Omega} \frac{1}{2R} |\nabla_{\mathbb{H}^n} u|^2 + RF(u) dx.$$

From Proposition 3.2

$$E_{R_i}(u_i, B_r) = R_i^{1-Q} E_1(u, B_{R_i r}) \le C r^{Q-1}.$$
(3.22)

Monti and Serra-Cassano in [21] have proved that $E_R(\cdot, \Omega)$ Γ -converges to $C_F P$ (\cdot, Ω) for some constant C_F depending on F.

So the coercivity property of Γ -convergence and (3.22) imply that there is a subsequence u_{i_k} converging in $L^1_{loc}(\mathbb{R}^{2n+1})$ to 1_G for some subset G.

Furthermore the local minimality of u i.e. Proposition 3.1 implies that G is locally minimal i.e. $P(G, \Omega) \leq P(K, \Omega)$ for any K such that $G\Delta K \subset \subset \Omega$.

Proposition 3.4 There exist no regular minimal surfaces in \mathbb{H}^n that are the graph of a regular function depending on |z| i.e. defined by $t = \phi(|z|)$ for $z \in \mathbb{C}^{2n}$, with $\phi \neq Constant$.

Proof Without loss of generality we shall write the proof for n = 1 i.e. $X_1 = X$ and $Y_1 = Y$.

Suppose by contradiction that such a surface exists. It is easy to see that a smooth graph defined by t = f(x, y) that minimizes the perimeter as defined above, satisfies the so called minimal surface equation i.e. if we denote by ν it's normal vector and $\nu_{\mathbb{H}^n} = (\langle \nu, X \rangle, \langle \nu, Y \rangle)$, then $\nu_{\mathbb{H}^n}$ satisfies

$$\operatorname{div}_{\mathbb{H}^n}\left(\frac{\nu_{\mathbb{H}^n}}{|\nu_{\mathbb{H}^n}|}\right) = 0, \qquad (3.23)$$

where $\operatorname{div}_{\mathbb{H}^{n}} \mathbf{v} = Xv_{1} + Yv_{2}$ for $\mathbf{v} = (v_{1}, v_{2})$ (see e.g. [22]).

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Observe that $\nu_{\mathbb{H}^n} = \frac{\phi'(r)}{r}z + 2\bar{z}$ and $|\nu_{\mathbb{H}^n}| = \sqrt{(\phi')^2 + 4r^2}$. Furthermore $\operatorname{div}_{\mathbb{H}^n} z = 2$ while $\operatorname{div}_{\mathbb{H}^n}(\bar{z}) = 0$ and $z \cdot \bar{z} = 0$. Hence if we call

$$\psi(r) = \frac{\phi'(r)}{r\sqrt{(\phi')^2 + 4r^2}}$$

then equation (3.23) becomes

$$\psi'(r) + 2\psi(r) = 0.$$

If ψ is not identically zero then it is given by $\psi(r) = \frac{C}{r^2}$ for some constant C i.e.

$$\frac{\phi'(r)}{r\sqrt{(\phi')^2 + 4r^2}} = \frac{C}{r^2}$$

and this has a solution only for $r \ge |C|$ unless C = 0 and then ϕ is constant. This concludes the proof.

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