One dimensional symmetry in the Heisenberg group.

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Abstract

Let H^n denote the Heisenberg space and let u be a solution of $\Delta_H u + u(1-u^2) = 0$ in H^n satisfying $|u| \leq 1$. Let x_1 be any variable orthogonal to the anisotropic direction t. Assume that for x_1 going to plus or minus infinity u converges uniformly to 1 and -1 respectively. Under these assumptions we prove that u is a function depending only on x_1 and that it is monotone increasing.

This result, which is the analogue for the Heisenberg space of the weak formulation of a conjecture by De Giorgi, is obtained for a wider class of equations; it is a consequence of the invariance of the Heisenberg Laplacian with respect to Heisenberg group. The proof requires a Maximum Principle for unbounded domains which is interesting by itself. We also consider the case when u satisfies the limit condition in the t direction then we conclude that the solution is monotone in t.

1 Introduction

Let u be a classical solution of

$$(1) \qquad \begin{cases} \Delta u + f(u) = 0 \ \text{ in } \ \mathbb{R}^N, \\ |u| \leq 1 \end{cases}$$

here f is a Lipschitz continuous function, non-increasing in $[-1, -1+\delta]$ and in $[1-\delta, 1]$ for some $\delta > 0$, with $f(1) = f(-1) = 0$ and suppose that

$$(2) \qquad \lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$$

where $x = (x_1, x') \in \mathbb{R}^N$.

Under the additional assumption that (2) is uniform in x', [2], [4] and [11] have proved that $\frac{\partial u}{\partial x_1} > 0$ and there exists U such that $u(x_1, x') = U(x_1)$.

Let us recall that this result is related to a conjecture of De Giorgi ([12]) where the question was raised of whether u is constant along hyperplanes without the request that the limit (2) is uniform. The conjecture has been lately solved by Ghoussoub and Gui in dimension $N = 2$ [13] and by Ambrosio and Cabré in dimension $N = 3$ [1].

In this paper we consider the case when the Laplacian is replaced by the Heisenberg Laplacian, precisely

$$\begin{cases} \Delta_H u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^{2n+1}, \\ |u| \leq 1. \end{cases} \tag{3}$$

Here $\mathbb{R}^{2n+1}$ is endowed with the Heisenberg group action $\circ$ and we consider the case when the limit (2) is uniform. Let us recall that Δ_H is a degenerate elliptic operator satisfying Hormander condition and the Heisenberg space $H^n = (\mathbb{R}^{2n+1}, \circ)$ is an anisotropic space, in particular denoting the elements of H^n by $\xi = (x, y, t)$ with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, it is easy to see that Δ_H is homogeneous with respect to the dilation $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$.

Our main result states that under the condition that

$$\lim_{s \to \pm\infty} u(x, y, t) = \pm 1 \quad \text{where} \quad s = v \cdot (x, y) \quad \text{for some unitary vector } v \in \mathbb{R}^{2n}$$

uniformly, then there exists a function $U : \mathbb{R} \to \mathbb{R}$ such that $u(x, y, t) = U(s)$ and $\frac{\partial U}{\partial s} > 0$.

If, on the other hand,

$$\lim_{t \to \pm\infty} u(x, y, t) = \pm 1 \tag{4}$$

uniformly, then we deduce only that $\frac{\partial u}{\partial t} > 0$.

This work has been inspired by [4] of Berestycki, Hamel and Monneau. Their proof is based on two ingredients viz., the maximum principle in unbounded domain contained in cones (see [3]) and the so called "sliding method".

The "sliding method" adapts well to the Heisenberg space since Δ_H is left invariant with respect to the group action $\circ$ (see [10]). On the other hand the maximum principle in domains contained in cones is based on the construction of a comparison function, the existence of which is not known in this setting for general cones. Here we prove it for a large family of cones using some ad hoc argument.

A last remark concerns the case when condition (4) holds. It is not surprising that the situation is different in the t direction. Indeed observe that if the following implication holds true:

$$(4) \Rightarrow u(x,y,t) = U(t) \tag{5}$$

then we would deduce that there are no solutions of (3) satisfying (4), since $U(t)$ cannot be a solution of (3). Still the question of whether (5) is true remains open.

In the next section after a basic introduction to the Heisenberg space is given, we treat the maximum principle in unbounded domains contained in cones and in section 3 we state and prove the symmetry and monotonicity results.

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2 Maximum Principle

Let us recall some known facts about the Heisenberg space H^n.

We will denote by $\xi = (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ the elements of $H^n = (\mathbb{R}^{2n+1}, \circ)$ where the group action $\circ$ is given by

$$\xi_o \circ \xi = (x + x_o, y + y_o, t + t_o + 2\sum_{i=1}^{n}(x_i y_{o_i} - y_i x_{o_i})). \tag{6}$$

The parabolic dilation $\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$ satisfies

$$\delta_\lambda(\xi_o \circ \xi) = \delta_\lambda \xi \circ \delta \xi_o,$$

and

$$|\xi|_H = \left((x^2+y^2)^2 + t^2\right)^{\frac{1}{4}}$$

is a norm with respect to the parabolic dilation.

The Koranyi ball of center ξ_o and radius R is defined by

$$B_H(\xi_o, R) := \{\xi \text{ such that } |\xi^{-1} \circ \xi_o| \le R\}$$

and it satisfies

$$|B_H(\xi_o, R)| = |B_H(0, R)| = CR^Q$$

where $Q = 2n + 2$ is the so called homogeneous dimension.

The Lie Algebra of left invariant vector fields is generated by

$$\begin{array}{l} X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad \text{for} \quad i = 1, \ldots, n, \\ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad \text{for} \quad i = 1, \ldots, n, \\ T = \frac{\partial}{\partial t}. \end{array}$$

Since $[X_i, Y_i] = -4T$, the Heisenberg Laplacian

$$\Delta_H = \sum_{i=1}^{n} X_i^2 + Y_i^2$$

is a second order degenerate elliptic operator of Hormander type and hence it is hypoelliptic (see e.g. [14] for more details about Δ_H).

Clearly the vector fields X_i, Y_i are homogeneous of degree 1 with respect to the norm $|.|_H$ while T is homogeneous of degree 2.

We now want to prove a Maximum Principle result in some unbounded domains of H^n. Precisely

Proposition 2.1 *Let Ω be an open connected subset of H^n such that one of the following conditions holds:*

1. *there exists $\xi_o \in H^n$ and $\lambda \leq 0$ such that $\overline{\xi_o \circ \Omega} \subset \Sigma_\lambda := \{\xi \in H^n : t \geq \lambda(x^2 + y^2)\}$.*
2. *there exists $\xi_o \in H^n$ such that $\overline{\xi_o \circ \Omega}$ lies on one side of an hyperplane parallel to the t axis i.e. there exists $v \in \mathbb{R}^{2n}$ such that $\overline{\xi_o \circ \Omega} \subset \{\xi \in H^n : v \cdot (x, y) > O\}$.*

Suppose that there exists $u \in C(\overline{\Omega})$ bounded above, solution of

$$\left\{ \begin{array}{ll} \Delta_H u + c(\xi) u \geq 0 & \text{in } \Omega, \quad \text{with } c(\xi) \leq 0, \\ u \leq 0 & \text{on } \partial\Omega, \end{array} \right. \tag{7}$$

then $u \leq 0$ in Ω.

When Ω is bounded there is nothing to prove; when Ω is unbounded and it satisfies the first condition the proof is quite standard and similar to the euclidean case proved by Berestycki, Caffarelli and Nirenberg in [3]. We will first give the proof in that case.

Without loss of generality we can suppose in the rest of the section that $\xi_o = 0$ and that $0 \notin \overline{\Omega}$.

Before starting the proof let us introduce some notations.

Let $\rho := |\xi|_H$ and $u : \partial B_H(0,1) \to \mathbb{R}$ be a smooth function. Then

$$X_i(u(\theta)\rho^\alpha) = (\hat{R}_i u + \alpha a_i u)\rho^{\alpha-1},$$

$$Y_i(u(\theta)\rho^\alpha) = (\hat{S}_i u + \alpha b_i u)\rho^{\alpha-1},$$

$$T(u(\theta)\rho^\alpha = (\hat{Z}u + \alpha c u))\rho^{\alpha-1}$$

where $a_i \equiv X_i(\rho)$, $b_i \equiv Y_i(\rho)$, $c = \rho T\rho$ and the vector fields $\hat{R}_i$, $\hat{S}_i$, $\hat{Z}$ are the tangential components of X_i, Y_i and T on the Koranyi unit sphere $S^1_H :=: \partial B_H(0,1)$.

Since Δ_H is homogeneous, a simple computation (see [8] and [14]) shows that

$$\Delta_H(u(\theta)\rho^\alpha) = [\mathcal{L}^\alpha(u(\theta))]\rho^{\alpha-2},$$

where

$$\mathcal{L}^\alpha u = \sum_{i=1}^n \hat{R}_i^2 u + \hat{S}_i^2 u + (2\alpha-1)(a_i\hat{R}_i + b_i\hat{S}_i)u + \alpha(Q-2+\alpha)hu, \tag{8}$$

here $h = \sum_{i=1}^n (a_i^2 + b_i^2) = \frac{x^2+y^2}{\rho^2}$. For simplicity of notations, let us also introduce the following operator

$$D_\alpha = \sum_{i=1}^n \left(\hat{R}_i^2 + \hat{S}_i^2 + (2\alpha-1)(a_i\hat{R}_i + b_i\hat{S}_i)\right). \tag{9}$$

Proof of case 1.

We will first construct an auxiliary function that plays a key role.

Step 1: Let $C_\lambda = \Sigma_\lambda \cap S^1_H$. In lemma 2.1 of [7] it is proved that for any $\lambda_1 \le 0$ there exists a function Ψ depending on $\phi = \frac{t}{\rho^2}$ defined in C_{λ_1} and there exists $\alpha = \alpha(\lambda_1) > 0$ such that

$$\begin{cases} \mathcal{L}^\alpha\Psi = 0 \ \text{ in } \ C_{\lambda_1}, \\ \Psi = 0 \ \text{ on } \ \partial C_{\lambda_1}, \Psi > 0 \ \text{ in } \ C_{\lambda_1}. \end{cases}$$

Let us choose $\lambda_1 < \lambda$ such that $\overline{\Omega \cap S_H^1} \subset\subset C_{\lambda_1}$. Then there exists $\delta > 0$ such that $\Psi \geq \delta > 0$ in $\Omega \cap S_H^1$.

Observe that the function $g = \rho^\alpha \Psi$ satisfies $\Delta_H g = \rho^{\alpha-2} \mathcal{L}^\alpha \Psi$ hence:

$$\begin{cases} \Delta_H g + c(\xi) g = c(\xi) g \leq 0 & \text{in } \Omega, \\ g \geq \delta > 0 & \text{in } \Omega \text{ for some } \delta > 0 \end{cases} \tag{10}$$

Step 2: Since g satisfies (10), the function $\sigma = \frac{u}{g}$ is well defined in Ω. Furthermore it satisfies the following equation

$$\begin{cases} \Delta_H \sigma + \frac{2}{g} \nabla_H \sigma \cdot \nabla_H g + \frac{(\Delta_H g + cg)}{g} \sigma \geq 0 & \text{in } \Omega, \\ \sigma \leq 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

Observe that (10) implies that the zero order coefficient is negative and furthermore, since $\alpha > 0$ and u is bounded above

$$\lim_{\rho \to \infty} \sigma = \lim_{\rho \to \infty} \frac{u}{g} \leq 0. \tag{12}$$

By applying the standard maximum principle we obtain that $\sigma \leq 0$ in Ω i.e. $u \leq 0$ in Ω.

This completes the proof of the Proposition 2.1 for domains Ω satisfying condition 1. $\square$

Before giving the proof for the domains satisfying condition 2, we need to prove a few propositions since for "cones" different from the ones of case 1 the construction of the auxiliary function g is more involved.

Also, without loss of generality, we will suppose that the vector v of case 2 is $v = (1, 0, \ldots, 0)$.

Let us make a few more remarks on the operators $\mathcal{L}^\alpha$ or D_α. Following Kohn and Nirenberg [17], we will say that a point ξ_o of $\partial\Omega'$ is a *characteristic point* for $\mathcal{L}^\alpha$ (or for D_α) if at least one of the vector fields $\hat{R}_i$ or $\hat{S}_i$ is null in ξ_o.

Since $\hat{R}_i$ and $\hat{S}_i$ are respectively the projection on S_H^1 of X_i and Y_i, it is easy to see that all the characteristic points are of the following type $\xi_o = (0, \ldots 1, \ldots, 0)$ where 1 is in one of the first $2n$ positions. Hence, if $\{e_1, \ldots, e_{2n+1}\}$ denotes the standard euclidean basis of $\mathbb{R}^{2n+1}$, then it is easy to see that $\mathcal{L}^\alpha$ is *uniformly elliptic* in $\Omega' \subset S_H^1$ if $e_i \notin \overline{\Omega}'$ for $i = 1, \ldots, 2n$.

On the other hand since $[\hat{R}_i, \hat{S}_i] = -4\hat{Z}$, the operator $\mathcal{L}^\alpha$ is of Hormander type for any $\Omega' \subset S^1_H$.

For $u,\ v \in L^2(\Omega', d\theta)$, let $\langle u, v\rangle = \int_{\Omega'} u(\theta)v(\theta)d\theta$ and $\|u\|^2 = \langle u, u\rangle$. Let us denote by

$$A(u,v) := \left\langle \hat{R}_i u, \hat{R}_i v\right\rangle + \left\langle \hat{S}_i u, \hat{S}_i v\right\rangle . \tag{13}$$

and let B_o be the closure of $C^\infty_o(\Omega')$ with respect to the norm

$$\|u\|_{B_o} = \left(A(u,u) + \|\sqrt{h}u\|^2\right)^{\frac{1}{2}} .$$

Let Ω' be a subdomain of S^1_H that does not have characteristic points on the boundary. Consider the operator $T : L^2(\Omega') \to L^2(\Omega')$ defined by $Tf := u \in B_o$, where u is a solution of

$$\begin{cases} -D_1 u = hf & \text{in } \Omega', \\ u = 0 & \text{on } \partial\Omega'. \end{cases} \tag{14}$$

Proposition 2.2 *T is well defined and it is a compact operator in $L^2(\Omega')$.*

Proof: Observe that we can write the operator $-D_1$ as

$$-D_1 \;=\; -\hat{R}_i^2 - \hat{S}_i^2 - a_i\hat{R}_i - b_i\hat{S}_i$$

A simple computation shows that

$$\sum_{i=1}^{n}(\hat{R}_i a_i + \hat{S}_i b_i) = (Q-1)\sum_{i=1}^{n}(a_i^2 + b_i^2) \tag{15}$$

and for $u,\ v \in C^2_0$

$$\int_{\Omega'} \hat{R}_i uv d\theta = -\int_{\Omega'} u\hat{R}_i v d\theta + (Q-1)\int_{\Omega'} uva_i d\theta \tag{16}$$

(see [8] and [14] for details). Using (15) and (16) it is easy to see that

$$\begin{aligned} \langle -D_1 u, v\rangle \;&=\; \sum_{i=1}^{n}[\left\langle \hat{R}_i u, \hat{R}_i v\right\rangle + \left\langle \hat{S}_i u, \hat{S}_i v\right\rangle + \\ &-\; Q\left(\left\langle a_i\hat{R}_i u, v\right\rangle + \left\langle b_i\hat{S}_i u, v\right\rangle\right)] \end{aligned} \tag{17}$$

Let $a(u,v)$ denote the right hand side of (17).

(16) and (15) imply furthermore that for any $u \in B_o$

$$\sum_{i=1}^{n} \int_{\Omega'} a_i \hat{R}_i u u d\theta + \int_{\Omega'} b_i \hat{S}_i u u d\theta = 0$$

Therefore, using Poincaré inequality for operators satisfying Hormander condition (see [15] and [16]), we have

$$a(u,u) = A(u,u) \geq C\|u\|^2_{B_o}$$

for some constant $C > 0$. Hence $a(u,v)$ is continuous and coercive in B_o and by Lax Milgram theorem for each $f \in L^2(\Omega')$ there exists a unique $u \in B_o$ such that

$$a(u,v) = \langle hf, v \rangle\,, \quad \forall v \in B_o,$$

hence T is well defined.

By a well known result of Kohn

$$a(u,u) \geq C\|u\|^2_{B_o} \geq C\|u\|_{\frac{1}{2}},$$

where $\|.\|_{\frac{1}{2}}$ is the norm of a Sobolev space $H^{\frac{1}{2}}(\Omega')$ of order $\frac{1}{2}$ (see [14]). Hence, by standard embedding theorems, the unit ball of B_o is compact in $L^2(\Omega')$ and therefore T is compact.

□

We now want to use Krein- Rutman theorem under the conditions given in theorem 2.6 of [7], to prove

Proposition 2.3 *T has a positive eigenvalue μ_o and the corresponding eigenfunction ψ is positive in Ω'.*

Proof. Let G denote the cone of positive functions in $L^2(\Omega')$. Clearly, G is closed, convex and $L^2(\Omega') = \overline{G - G}$. Furthermore the L^2 norm is semi-monotone with respect to G.

Theorem 2.6 of [7] claims that if T is compact and there exists e in G and $\gamma > 0$ such that

$$Te - \gamma e \in G$$

then $r(T) := \lim_{k\to\infty} |T^k|^{\frac{1}{k}} := \mu_o > 0$. Hence, from the classical Krein Rutman theorem, μ_o is an eigenvalue of T and the corresponding eigenfunction ψ is positive in Ω'.

Let us construct e and γ as above. Let $\Omega'' \subset \Omega'$ such that there exists Ω_1 without characteristic boundary points, satisfying $\Omega'' \subset \Omega_1 \subset \Omega'$. Let $e \in G$ bounded above such that the support of e is contained in Ω''.

By definition of T and using the maximum principle, the function $v := Te$ satisfies

$$\begin{cases} D_1(v) = -he \leq 0 & \text{in } \ \Omega_1, \\ v \geq 0 & \text{on } \ \partial\Omega_1. \end{cases}$$

By the strong maximum principle we know that $Te = v > 0$ in Ω' hence

$$\inf_D Te = \delta > 0.$$

Let us choose $\gamma := \frac{\delta}{2\|e\|_{L^\infty}}$ then

$$\begin{cases} Te - \gamma e \geq \delta - \gamma e > 0 \ \text{ in } \ \Omega'', \\ Te - \gamma e = Te \geq 0 \ \text{ in } \ \Omega' \setminus \Omega''. \end{cases}$$

e and γ satisfy the required conditions and this completes the proof of the Proposition 2.3.

□

Observe that clearly ψ and μ_o satisfy

$$\begin{cases} D_1\psi + \frac{1}{\mu_o} h\psi = 0 & \text{in } \ \Omega', \\ \psi = 0 & \text{on } \ \partial\Omega'. \end{cases} \tag{18}$$

We will say that $\lambda = \frac{1}{\mu_o}$ is the weighted principal eigenvalue of $-D_1$ in Ω'.

We are now ready to give the

Proof of case 2 of Proposition 2.1: It is enough to construct the auxiliary function, the second step being identical to the one in the proof of case 1 i.e. we want to construct a function g satisfying (10) such that

$$\lim_{|\xi|_H \to \infty} g(\xi) = +\infty \ \text{ in } \ \Omega.$$

Without loss of generality we can suppose that the hyperplane parallel to the t axis is $\{x_1 = 0\}$ and we suppose that $\overline{\Omega} \subset \{\xi; \text{ such that } \ x_1 > 0\} := \Pi$.

Let $\Sigma_o = \Pi \cap S_H^1$. Observe that $\Delta_H x_1 = 0$ implies that the function $u := \frac{x_1}{\rho}$ defined on S_H^1 satisfies

$$\Delta_H x_1 = \Delta_H(\rho u) = \rho^{-1}\mathcal{L}^1(u) = 0.$$

Therefore

$$\begin{aligned}
\mathcal{L}^1(u) &= D_1 u + (Q-1)hu = 0 \;\text{ in }\; \Sigma_o, \\
u &= 0 \;\text{ on }\; \partial\Sigma_o \\
u &> 0 \;\text{ in }\; \Sigma_o,
\end{aligned}$$

i.e. $(Q-1)$ is the principal weighted eigenvalue of $-D_1$ in Σ_o.

Let $\Sigma_\varepsilon \supset \Sigma_o$ close enough to Σ_o that $\lambda_1 = \lambda_1(\varepsilon)$ the principal weighted eigenvalue of $-D_1$ in Σ_ε satisfies

$$Q - 1 - \varepsilon := \lambda_1 < Q - 1$$

for some $\varepsilon > 0$ to be determined. We can choose Σ_ε such that it has no characteristic points on the boundary.

Therefore there exists $\psi_\varepsilon > 0$ in Σ_ε such that

$$\left\{ \begin{array}{ll} D_1\psi_\varepsilon + \lambda_1 h \psi_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ \psi_\varepsilon = 0 & \text{on } \partial\Sigma_\varepsilon. \end{array} \right. \tag{19}$$

The first condition required on ε is that

$$\frac{1}{2}\left(\frac{1}{2} + Q - 2\right) < Q - 1 - \varepsilon. \tag{20}$$

In particular (20) implies that the operator $-\left(D_1 + \frac{1}{2}(\frac{1}{2} + Q - 2)\right)$ has a positive principal weighted eigenvalue in Σ_ε.

It is immediate to see that

$$\mathcal{L}^{\frac{1}{2}} = \sum_{i=1}^{n}(\hat{R}_i^2 + \hat{S}_i^2) + \frac{1}{2}\left(\frac{1}{2} + Q - 2\right)h =$$

$$= D_1 + \frac{1}{2}\left(\frac{1}{2} + Q - 2\right)h - \sum_{i=1}^{n}(a_i\hat{R}_i + b_i\hat{S}_i),$$

this leads to the following

Claim 1. There exists $\varepsilon > 0$ such that there exist a function $\nu > 0$ and a constant $\mu > 0$ such that

$$\begin{cases} \mathcal{L}^{\frac{1}{2}}\nu + \mu\nu \le 0 & \text{in } \Sigma_\varepsilon, \\ \nu = 0 & \text{on } \partial\Sigma_\varepsilon. \end{cases} \tag{21}$$

For some $1 > \beta > 0$, let us compute $\mathcal{L}^{\frac{1}{2}}(\psi_\varepsilon^\beta)$.

Clearly the following equalities hold

$$\begin{aligned} \hat{R}_i(\psi_\varepsilon^\beta) &= \beta\psi_\varepsilon^{\beta-1}\hat{R}_i\psi_\varepsilon, \\ \hat{R}_i^2(\psi_\varepsilon^\beta) &= \beta(\beta-1)\psi_\varepsilon^{\beta-2}(\hat{R}_i\psi_\varepsilon)^2 + \beta\psi_\varepsilon^{\beta-1}\hat{R}_i^2\psi_\varepsilon \end{aligned}$$

and similarly for $\hat{S}_i$. Hence let $\nu = \psi_\varepsilon^\beta$:

$$\mathcal{L}^{\frac{1}{2}}(\nu) = \sum_{i=1}^n (\hat{R}_i^2\psi_\varepsilon^\beta + \hat{S}_i^2\psi_\varepsilon^\beta) + \frac{1}{2}\left(\frac{1}{2} + Q - 2\right)h\nu =$$

$$= \sum_{i=1}^n (\beta\psi_\varepsilon^{\beta-1}(\hat{R}_i^2\psi_\varepsilon + \hat{S}_i^2\psi_\varepsilon) + \beta(\beta-1)\psi_\varepsilon^{\beta-2}((\hat{R}_i\psi_\varepsilon)^2 + (\hat{S}_i\psi_\varepsilon)^2)) + \frac{1}{2}\left(\frac{1}{2} + Q - 2\right)h\nu$$

Using (19) we obtain

$$\mathcal{L}^{\frac{1}{2}}(\nu) = \sum_{i=1}^n \beta(\beta-1)\psi_\varepsilon^{\beta-2}\left((\hat{R}_i\psi_\varepsilon)^2 + (\hat{S}_i\psi_\varepsilon)^2\right) - \beta\psi_\varepsilon^{\beta-1}\left(a_i\hat{R}_i\psi_\varepsilon + b_i\hat{S}_i\psi_\varepsilon\right) +$$

$$+ \left(\frac{1}{2}(\frac{1}{2} + Q - 2) - \beta\lambda_1\right)h\nu.$$

The Young inequality implies:

$$(-\beta\psi_\varepsilon^{\beta-1}a_i\hat{R}_i\psi_\varepsilon) \le \beta(1-\beta)\psi_\varepsilon^{\beta-2}(\hat{R}_i\psi_\varepsilon)^2 + \beta\psi_\varepsilon^\beta\frac{a_i^2}{4(1-\beta)}.$$

Hence:

$$\mathcal{L}^{\frac{1}{2}}(\nu) \le \left(\frac{1}{2}\left(\frac{1}{2} + Q - 2\right) - \beta\lambda_1 + \frac{\beta}{4(1-\beta)}\right)h\nu.$$

Let

$$k(\beta) := -\frac{1}{2}\left(\frac{1}{2} + Q - 2\right) + \beta\lambda_1 - \frac{\beta}{4(1-\beta)}$$

If we prove that, for ε sufficiently small, there exists $\beta_o \in (0,1)$ such that $k(\beta_o) > 0$ then the claim is proved. Indeed by choosing $\beta = \beta_o$ in the definition of ν and $\mu = k(\beta_o)$ we have constructed ν and μ with the required properties.

We define

$$h(\beta) := 4(1-\beta)k(\beta) = -4\lambda_1\beta^2 + 2\beta(2\lambda_1 - 2 + Q) - 2Q + 3,$$

hence it is enough to check that $h(\beta_o) := \max_{\beta\in[0,1]} h(\beta) > 0$ and $\beta_o \in (0,1)$. Observe that $\beta_o = \frac{2\lambda_1-2+Q}{4\lambda_1}$ and

$$h(\beta_o) = \frac{(2\lambda_1 - 2 + Q)^2}{4\lambda_1} + 3 - 2Q = \frac{(Q-2)^2 + 4\varepsilon^2 + 4\varepsilon(1-Q)}{4\lambda_1}.$$

We have used the fact that $\lambda_1 = Q - 1 - \varepsilon$.

It is easy to see that $\beta_o \in (0,1)$ for $\varepsilon < \frac{Q}{2}$ while $h(\beta_o) > 0$ for $\varepsilon > 0$ sufficiently small. This completes the proof of the Claim.

We will choose as auxiliary function $g(\xi) := \rho^{\frac{1}{2}}\nu$. Clearly g satisfies:

$$\begin{cases} \Delta_H g + c(\xi)g = \rho^{-\frac{3}{2}}\mathcal{L}^{\frac{1}{2}}\nu + c(\xi)g \le 0 & \text{in } \Omega, \\ g \ge \delta > 0 & \text{in } \Omega, \\ \lim\limits_{\rho\to\infty} g(\xi) = +\infty & \text{in } \Omega. \end{cases}$$

This completes the proof of Proposition 2.1.

□

3 One dimensional symmetry

An immediate consequence of the Maximum Principle of Proposition 2.1 is the following comparison result:

Corollary 3.1 *Let f be a Lipschitz continuous function, non-increasing on $[-1,-1+\delta]$ and on $[1-\delta,1]$ for some $\delta > 0$. Assume that u_1, u_2 are solutions of*

$$\Delta_H u_i + f(u_i) = 0 \quad \textit{in} \quad \Omega$$

and are such that $|u_i| \le 1$, $i = 1, 2$. Furthermore, assume that

$$u_2 \ge u_1 \quad \textit{on} \quad \partial\Omega$$

and that either

$$u_2 \geq 1 - \delta \quad in \quad \Omega$$

or

$$u_1 \leq 1 + \delta \quad in \quad \Omega.$$

If $\Omega \subset H^n$ is an open connected set satisfying either of the conditions (1) or (2) of Proposition 2.1 then $u_2 \geq u_1$ in Ω.

We shall use Corollary 3.1 to prove the following one dimensional symmetry results:

Theorem 3.1 *Let u be a solution of*

$$\Delta_H u + f(u) = 0 \quad in \quad H^n \tag{22}$$

which satisfies $|u| \leq 1$ together with asymptotic conditions

$$\lim_{x_1 \to \pm\infty} u(x_1, x', y, t) = \pm 1 \tag{23}$$

uniformly in $x' = (x_2, \ldots, n) \in \mathbb{R}^{n-1}$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We assume that f is Lipschitz continuous in $[-1, 1]$, $f(\pm 1) = 0$ and that there exists $\delta > 0$ such that

$$f \quad is \quad nonincreasing \quad on \; [-1, -1+\delta] \quad and \quad on \quad [1-\delta, 1]. \tag{24}$$

Then, $u(x_1, x', y, t) = U(x_1)$ where U is a solution of

$$\left.\begin{array}{rcl} U'' + f(U) & = & 0 \quad in \quad \mathbb{R}, \\ U(\pm\infty) & = & \pm 1, \end{array}\right\} \tag{25}$$

and u is increasing with respect to x_1. The existence of a solution u of (22)-(23) such that $|u| \leq 1$ implies the existence of a solution U of (25). Furthermore, the solution u is unique up to translations of the origin.

Remark: The conclusion of Theorem 3.1 holds if we replace x_1 by any non-anisotropic direction i.e. let $s = a \cdot x + b \cdot y$ for some vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ such that $a^2 + b^2 = 1$, if condition (23) is replaced by

$$\lim_{s \to \pm\infty} u(x, y, t) = \pm 1 \tag{26}$$

uniformly, then there exists U such that $u(x, y, t) = U(a \cdot x + b \cdot y)$.

On the other hand, for the anisotropic direction we have

Proposition 3.1 *Under the hypothesis of Theorem 3.1, replacing condition (23) by*

$$\lim_{t\to\pm\infty} u(x,y,t) = \pm 1 \text{ uniformly in } x = (x_1,\dots,n),\ y = (y_1,\dots,y_n) \tag{27}$$

and further assuming that f is C^1, the function u is monotone along the t-direction, i.e., $\frac{\partial u}{\partial t} > 0$ in H^n.

The equivalent of Theorem 3.1 for the classical Laplacian was obtained by Berestycki, Hamel and Monneau in [4] using the sliding method. Here we shall use the sliding method in H^n with the one parameter family of transformations defined by

$$\begin{aligned} \mathcal{E}_\nu(s)(\xi) &= (sx_0, sy_0, st_0)\circ(x,y,t) \\ &= (x+sx_0, y+sy_0, t+st_0+2s(y_0x-x_0y)) \end{aligned} \tag{28}$$

where $\nu = (x_0, y_0, t_0)$. In [10], we had already used the sliding method to obtain monotonicity results for solutions of semilinear equations in nilpotent, stratified Lie groups.

Proof of Theorem 3.1: The proof of Theorem 3.1 is along the lines of Theorem 1 in [4]. Of course, here we use the group action $\circ$ of H^N and we rely on the fact that the sub-Laplacian is invariant with respect to $\circ$. We begin by proving

Claim 1: For any $\nu = (x_1^0, x^{0\prime}, y^0, t^0) \in \mathbb{R}^{2n+1}$ with $x_1^0 > 0$, we have

$$u_s(\xi) := u(\mathcal{E}_\nu(s)\xi) \geq u(\xi) \text{ for all } \xi \in H^n. \tag{29}$$

Proof: Using the condition (23), for $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$u(x_1, x', y, t) > 1-\delta \quad \text{for } x_1 \geq N \tag{30}$$

$$u(x_1, x', y, t) < -1+\delta \quad \text{for } x_1 \leq -N. \tag{31}$$

Hence for $s > 2N/x_1^0$, we have

$$u_s(\xi) > 1-\delta \quad \text{for } x_1 \geq -N. \tag{32}$$

Furthermore, the function u_s satisfies the equation (22) and

$$u_s(-N, x', y, t) > u(-N, x', y, t).$$

We now apply Corollary 3.1 to the functions u_s and u in the half spaces $\{\xi = (x,y,t) \in H^n : x_1 \geq -N\}$ and $\{\xi = (x,y,t) \in H^n : x_1 \leq -N\}$ to conclude that

$$u_s(x,y,t) \geq u(x,y,t) \text{ for all } (x,y,t) \in H^n.$$

Let $\tau = \inf\{s : u_s(\xi) \geq u(\xi)\ for\ all\ \xi \in H^n\}$. We claim that $\tau = 0$. On the contrary, suppose that $\tau > 0$. We have

$$u_\tau(x, y, t) \geq u(x, y, t) \text{ for all } \ (x, y, t) \in H^n.$$

We consider the following two cases:
Case (i):

$$\inf_{\xi \in [-N,N] \times \mathbb{R}^{2n}} \{u_\tau(\xi) - u(\xi)\} > 0.$$

Since u is bounded and f is Lipschitz continuous, it is easy to see that using e.g. Theorem 2 of chapter XIII of [18] and Corollary IV.7.4 of [19] u is globally Lipschitz continuous.

Hence, there exists $\varepsilon > 0$ small, such that for all s, $\tau - \varepsilon < s < \tau$ we have

$$\inf_{\xi \in [-N,N] \times \mathbb{R}^{2n}} \{u_s(\xi) - u(\xi)\} > 0.$$

Observe that, from (30) we have

$$\begin{aligned} u_s(\xi) &= u(x_1 + sx_1^0, x' + sx^{0'}, y + sy^0, t + st^0 + 2s(y^0 x - x^0 y)) \\ &> 1 - \delta \text{ for all } s > 0 \text{ and for all } x_1 \geq N. \end{aligned} \tag{33}$$

Hence we can again use the comparison principle for u_s and u in the half spaces $\{(x, y, t) \in H^n : x_1 \geq N\}$ and $\{(x, y, t) \in H^n : x_1 \leq -N\}$. Together with (33), we conclude that

$$u_s(\xi) \geq u(\xi) \ \ \forall \xi \in H^n \text{ and for all } \tau - \varepsilon < s < \tau$$

a contradiction to the definition of τ.

Case (ii)

$$\inf_{\xi \in [-N,N] \times \mathbb{R}^{2n}} \{u_\tau(\xi) - u(\xi)\} = 0.$$

Let $\xi_k \in [-N, N] \times \mathbb{R}^{2n}$ such that $u_\tau(\xi_k) - u(\xi_k) \to 0$. Define $v_k(\xi) = u(\xi_k \circ \xi)$ for $\xi \in H^n$.

By regularity estimates and embedding of the non-isotropic Sobolev spaces (see [18] and [19]) we can extract a subsequence of $\{v_k\}$ converging uniformly to a solution v of (22). Moreover, we have $v(0) = v_s(0)$.

Therefore, the function $z(\xi) = v_\tau(\xi) - v(\xi)$ satisfies

$$\left.\begin{aligned} \Delta_H z + c(\xi) z &= 0 \text{ in } H^n, \\ z &\geq 0 \text{ in } H^n, \\ z(0) &= 0, \end{aligned}\right\} \tag{34}$$

where $c(\xi)$ is a bounded function defined by

$$(35) \qquad c(\xi) = \begin{cases} \frac{f(v_\tau(\xi)) - f(v(\xi))}{v_\tau(\xi) - v(\xi)} & \text{if } v_\tau(\xi) \neq v(\xi), \\ 0 & \text{otherwise.} \end{cases}$$

The maximum principle implies that $z \equiv 0$ i.e., $v(\xi) = v((\tau x^0, \tau y^0, \tau t^0) \circ \xi)$ for all $\xi \in H^n$.

However, this is not possible since v also satisfies the asymptotic condition (23). Hence case (ii) does not arise. Therefore, we conclude that $\tau = 0$; which completes the proof of the Claim 1.

From the previous discussion we further conclude that

$$\lim_{s \to 0} \frac{u_s(\xi) - u(\xi)}{s} \geq 0$$

hence for all $\xi \in H^n$ and for every $\nu = (x^0, y^0, t^0) \in \mathbb{R}^{2n+1}$ with $x_1^0 > 0$

$$(36) \qquad (x^0, y^0, t^0 + 2\sum_{i=1}^{n} 2(y_i^0 x_i - x_i^0 y_i) \cdot \nabla u \geq 0$$

By continuity (36) holds for every $\nu \in \mathbb{R}^{2n+1}$ with $x_1^0 = 0$ i.e.

$$(37) \qquad \sum_{i=2}^{n} x_i^0 \frac{\partial u(\xi)}{\partial x_i} + \sum_{i=1}^{n} y_i^0 \frac{\partial u(\xi)}{\partial y_i} + (t^0 + \sum_{i=1}^{n} 2(y_i^0 x_i - x_i^0 y_i)) \frac{\partial u(\xi)}{\partial t} \geq 0$$

It follows from (37) that for every $\nu \in \mathbb{R}^{2n+1}$ with $x_1^0 = 0$ we have

$$(38) \qquad \sum_{i=2}^{n} x_i^0 \frac{\partial u(\xi)}{\partial x_i} + \sum_{i=1}^{n} y_i^0 \frac{\partial u(\xi)}{\partial y_i} + (t^0 + \sum_{i=1}^{n} 2(y_i^0 x_i - x_i^0 y_i)) \frac{\partial u(\xi)}{\partial t} = 0.$$

Varying ν over the standard vectors $e_i = (0, 0, \ldots, 1(i - thplace), \ldots, 0) \in \mathbb{R}^{2n+1}$ with $i = 2, \ldots, 2n+1$, we conclude that all the partial derivatives $\{\frac{\partial u}{\partial x_i}\}_{2 \leq i \leq n}$, $\{\frac{\partial u}{\partial y_i}\}_{1 \leq i \leq n}$ and $\frac{\partial u}{\partial t}$ vanish identically in H^n which implies that u is function of x_1. In particular, the second part of the Theorem 3.1 holds. □

Proof of Proposition 3.1

The proof follows the lines of the proof of Theorem 3.1 choosing $\nu = (0, 0, t_o)$ and applying the Maximum Principle in half spaces $\{t > k\}$.

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