First eigenvalue and Maximum principle for fully nonlinear singular operators.

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Abstract

In this paper, we define the concept of first eigenvalue for a class of fully nonlinear operators modelled on the p-Laplacian operator but not in divergence form. This can be done through the Maximum principle as in [2]. Existence of a first eigenfunction and estimates are given.

Résumé

Dans cet article nous définissons la notion de premi 'ere valeur propre pour des opérateurs complètement non linéaires sur le modèle du p-Laplacien, mais non sous forme de divergence. L'argument utilise le principe du maximum dans l'esprit de [2]. L'existence d'une première fonction propre et d'estimations afférentes sont données.

1 Introduction

Let $Lu = tr(A(x)D^2u)$ where $A(x)$ is a positive definite matrix satisfying $mI \leq A(x) \leq MI$ for some positive constants m and M. If Ω is a bounded smooth domain of $\mathbb{R}^N$, it is well known that there exists $\bar{\lambda}$ such that:

- There exists a positive function ϕ satisfying

$$\begin{cases} L\phi + \bar{\lambda}\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

  Furthermore $\bar{\lambda}$ is the smallest eigenvalue of $-L$ and hence :

- For any $\lambda < \bar{\lambda}$ and for any $f \in L^N(\Omega)$ there exists a unique u such that

$$\begin{cases} Lu + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

  (see [15]).

Let us recall that $L + \lambda$ *satisfies the maximum principle in* Ω if any solution of $Lu + \bar{\lambda}u \leq 0$ in Ω which is positive on the boundary of Ω is positive in Ω.

The first eigenvalue of $-L$ in Ω is characterized by the fact that it is the supremum of the value λ such that $L + \lambda$ satisfies the maximum principle in Ω. (See Protter and Weinberger when L is the Laplacian and Berestycki, Nirenberg and Varadhan [2] for general second order operators in general domains.)

On the other hand if $Lu := \Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ the value

$$\bar{\lambda} = \inf \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

has been called the first eigenvalue for $-\Delta_p$ even though strictly speaking it is not (see e.g. [1, 21]). All the same $\bar{\lambda}$ has the required properties of the eigenvalue:

- There exists a positive function ϕ satisfying

$$\begin{cases} \Delta_p \phi + \bar{\lambda}\phi^{p-1} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

- For any $\lambda < \bar{\lambda}$ and for any $f \in L^p(\Omega)$ there exists a unique u such that

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is important to remark that the higher order term and the zero order term are homogeneous of the same degree and that the definition of $\bar{\lambda}$ is related to the variational nature of the p-Laplacian.

In this paper we introduce a notion of *first eigenvalue* for fully nonlinear operators which are non variational but homogeneous. Following Berestycki, Nirenberg and Varadhan [2] this so called eigenvalue will be defined through the *maximum principle* but it will have the feature of the "eigenvalue" of $-\Delta_p$.

Indeed we consider fully nonlinear elliptic operators $Lu := F(\nabla u, D^2 u)$ which may be singular or degenerate (as the p-Laplacian) that satisfy

(F1) $F(tp, \mu X) = |t|^\alpha \mu F(p, X)$, $\forall t \in \mathbb{R}$, $\mu \in \mathbb{R}^+$, $\alpha > -1$

(F2) $a|p|^\alpha \mathrm{tr} N \leq F(p, M+N) - F(p, M) \leq A|p|^\alpha \mathrm{tr} N$ for $0 < a \leq A$, $\alpha > -1$ and $N \geq 0$.

The class of operators satisfying (F1) and (F2) is large and includes

$$F(\nabla u, D^2 u) = |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u)$$

where $\alpha > -1$ and $\mathcal{M}_{a,A}$ is one of the Pucci operators.

$$F(\nabla u, D^2 u) = \Delta_p u$$

with $\alpha = p + 1$. In [3] many examples of operators satisfying (H2) are given.

Of course the right notion of solution in this context will be that of viscosity solution (see [9]) suitably adapted to our contest. Let us remark that even if u is C^2, when $\alpha < 0$ $F(\nabla u, D^2 u)$ is not define for $\nabla u = 0$.

Before going into details, let us mention that a certain number of interesting papers have appeared that treat viscosity solutions for equations involving the p-Laplacian. In fact in [18, 19] Juutinen, Lindqvist and Manfredi opened the way to this topic. We would like to emphasize that in those papers the point of view is really on the p-Laplacian and the variational structure of the p-Laplacian is used. This is not the case here since the operators we consider are fully nonlinear.

The first key ingredient is the following:

Theorem 1.1 *Suppose that Ω is a bounded open piecewise $\mathcal{C}^1$ domain of $\mathbb{R}^N$. Suppose that for $\lambda \in \mathbb{R}$ there exists a function $v > 0$ such that $F(\nabla v, D^2 v) + \lambda v^{\alpha+1} \leq 0$ in Ω. Then, for $\tau < \lambda$, every viscosity solution of*

$$\begin{cases} F(\nabla \sigma, D^2 \sigma) + \tau |\sigma|^\alpha \sigma \geq 0 \text{ in } \Omega \\ \sigma \leq 0 \text{ on } \partial\Omega \end{cases}$$

satisfies $\sigma \leq 0$ in Ω.

When $\tau < 0$ this result was obtained in [3] for the operators considered here, of course for a large class of elliptic operators see [9] and the references therein. It is well known that to prove maximum principles or comparison principles for viscosity solutions one needs to double the variables and consider the function $\psi(x, y) = u(x) - v(y) + \phi(x, y)$ where ϕ is an appropriate C^2 function (see [9]). On

the other hand here instead of considering a difference of sub and super solutions we consider the ratio of v and *sigma*.

This theorem allows us to define

$$\bar{\lambda} = \sup\{\lambda \in \mathbb{R},\ \ \exists\ \phi > 0\ \text{ in } \Omega,\ F(\nabla\phi, D^2\phi) + \lambda\phi^{\alpha+1} \leq 0 \text{ in the viscosity sense }\}.$$

In other words if we denote by $I_\alpha(u) = |u|^\alpha u$, $\bar{\lambda}$ is the supremum of the value λ such that $F + \lambda I_\alpha$ satisfies the maximum principle in Ω. The main aim of this paper is to convince the reader that it is *correct* to call $\bar{\lambda}$ the first eigenvalue of $-F$ in Ω.

Clearly the set

$$E = \{\lambda \in \mathbb{R},\ \ \exists\ \phi > 0\ \text{ in } \Omega,\ F(\nabla\phi, D^2\phi) + \lambda\phi^{\alpha+1} \leq 0 \text{ in the viscosity sense }\}$$

is an interval; in fact it is an interval which is bounded from above since

Proposition 1.2 *Suppose that R is the radius of the largest ball contained in the bounded set Ω. Then, there exists some constant C which depends only on N and α, such that*

$$\bar{\lambda} \leq \frac{C}{R^{\alpha+2}}.$$

The value $\bar{\lambda}$ has the following features that justify the name of eigenvalue:

Theorem 1.3 *There exists ϕ a continuous positive viscosity solution of*

$$\begin{cases} F(\nabla\phi, D^2\phi) + \bar{\lambda}\phi^{\alpha+1} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore

Theorem 1.4 *For $\lambda < \bar{\lambda}$ if $f < 0$ in Ω and bounded then there exists a unique u nonnegative viscosity solution of*

$$\begin{cases} F(\nabla u, D^2 u) + \lambda u^{\alpha+1} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Let us notice that in order to prove the Theorems 1.3 and 1.4 we need to obtain some estimates which are interesting in their own right:

Theorem 1.5 *Suppose that f is a bounded function in $\bar{\Omega}$ then if u is a bounded nonnegative viscosity solution of $F(\nabla u, D^2u) = f$ in Ω, it is Hölder continuous:*

$$|u(x) - u(y)| \leq M|x - y|^{\gamma}.$$

In [17], among other results, Ishii and Lions prove Hölder and Lipschitz estimates for a class of second order elliptic operators that do not include the operators considered here but the proof of Theorem 1.5 is inspired by [17].We also obtain local Lipschitz regularity using the Hölder regularity of the solution. See also [7] for Hölder's regularity for solution of degenerate elliptic operators.

In the case $\alpha = 0$ Hölder's regularity is proved by Caffarelli Cabré with a different proof that requires Harnack and Alexandrov-Bakelman-Pucci estimates.

After the completion of this work we learned that the eigenvalue problem for Pucci's operators has already been treated; this would correspond to the case $\alpha = 0$ here. An initial work concerning the radial case was completed by Felmer and Quaas [13] then Quaas treated the case of general domains [23]. Later the work was completed by Busca, Esteban and Quaas in [5]. In this interesting paper they denote by μ_1^+ the eigenvalue here denote $\bar{\lambda}$, but they also define

$$\mu_1^- = \sup\{\mu,\ \exists \psi < 0 \text{ in } \Omega : \mathcal{M}^+_{a,A}(\psi) + \mu\psi \geq 0\}.$$

This could be done in our case as well but we have limited ourselves to positive solutions. On the other hand, a priori estimates for $\mathcal{M}^+_{a,A}$ have been given in [6] that allow them to take solution in $W^{2,n}$. These estimates are not known for singular operators. They also consider interesting bifurcation problems.

Let us mention some open problems:

- **Simplicity of the eigenfunction.** The first eigenfunction ϕ is simple for linear second order elliptic operators, for the Pucci's operators and for the p-Laplacian, it would be interesting to know if this is true also in the case treated here i.e. suppose that $\psi > 0$ is another eigenfunction does this imply that there exists $t \in \mathbb{R}^+$ such that $\psi = t\phi$?

- **Fredholm alternative** By the definition of $\bar{\lambda}$ if $f < 0$ in $\bar{\Omega}$ then there are no positive solutions of

$$\begin{cases} F(\nabla u, D^2u) + \bar{\lambda}|u|^{\alpha} + 1 = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

  Is it still true if $f \leq 0$? are there solutions that change sign?

- $\bar{\lambda}$ **is isolated.** Suppose that $\bar{\lambda} + \varepsilon\lambda > \bar{\lambda}$. Is it possible to prove that there exists a solution of (1.1) if ε is sufficiently small?

In the next section we state the precise hypothesis on the fully non linear operator F and we give the notion of "viscosity solution" adapted to the operators considered here. In the third section we prove the maximum principle (Theorem 3.3) and a comparison principle. In the fourth section we give global Hölder and local Lipschitz estimates for the solutions. Finally in the last section we prove different existence results including that of a first eigenfunction. Some properties of the distance function are proved in the appendix.

Acknowledgment: This work was completed while I. Birindelli was visiting the University of Cergy-Pontoise with the support of the Laboratoire d'Analyse, geometrie et Modelisationa and of CNR- Short term program 2004, she would like to thank both institutions.

2 Preliminaries

Let α be some real number, $\alpha > -1$, and let F be a fully nonlinear and singular or degenerate operator

$$F(\nabla u, D^2 u)$$

where $F : \mathbb{R}^N - \{0\} \times S$, S is the set of symmetric matrix in $\mathbb{R}^N$, and we consider the following hypothesis

(H1) $F(tp, \mu X) = |t|^\alpha \mu F(p, X)$, $\forall t \in \mathbb{R}$, $\mu \in \mathbb{R}^+$, $\alpha > -1$ and $F(p, X) \leq F(p, Y)$ for any $p \neq 0$, and $X \leq Y$.

In most of the paper the operator satisfies also the following hypothesis

(H2) $a|p|^\alpha \mathrm{tr} N \leq F(p, M + N) - F(p, M) \leq A|p|^\alpha \mathrm{tr} N$ for $0 < a \leq A$, $\alpha > -1$ and $N \geq 0$.

When only (H1) is required it will be stated explicitly.
Let us recall that (H2) implies

$$|p|^\alpha \mathcal{M}^+_{a,A}(M) \geq F(p, M) \geq |p|^\alpha \mathcal{M}^-_{a,A}(M),$$

where, if e_i are the eigenvalues of M

$$\mathcal{M}^+_{a,A}(M) = a\sum_{e_i<0} e_i + A\sum_{e_i>0} e_i \quad \text{and}$$

$$\mathcal{M}^-_{a,A}(M) = A\sum_{e_i<0} e_i + a\sum_{e_i>0} e_i$$

are the Pucci operators (see e.g. [6]).

Remark 2.1 *Let us observe that if F satisfies (H2), $G(p,X) = -F(p,-X)$ satisfies (H2). With this remark, defining*

$$\bar{\lambda}^- = \sup\{\mu, \exists\ \phi < 0\ , F(\nabla\phi, D^2\phi) + \mu|\phi|^\alpha\phi \geq 0\}$$

and observing that $\bar{\lambda}^- = \bar{\lambda}(G)$ one gets symmetrical results to those enclosed in the sequel for the value $\bar{\lambda}$.

We need first to extend the definitions employed in [3]. Let us recall first the definition of viscosity continuous sub or super solutions for operators that satisfy $(H2)$ and hence may be singular when $\nabla u = 0$.

It is well known that in dealing with viscosity respectively sub and super solutions one works with

$$u^\star(x) = \limsup_{y,|y-x|\leq r} u(y)$$

and

$$u_\star(x) = \liminf_{y,|y-x|\leq r} u(y).$$

It is easy to see that $u_\star \leq u \leq u^\star$ and $u^\star$ is uppersemicontinuous (USC) $u_\star$ is lowersemicontinuous (LSC). See e.g. [9, 16].

Definition 2.2 *Let Ω be an open set in $\mathbb{R}^N$, then v bounded on $\overline{\Omega}$ is called a viscosity super-solution of $F(\nabla v, D^2 v) = g(x,v)$ if for all $x_0 \in \Omega$,*

-Either there exists an open ball $B(x_0,\delta)$, $\delta > 0$ in Ω on which v is constant and equal to c then $g(x,c) \geq 0$

-Or $\forall\varphi \in \mathcal{C}^2(\Omega)$, such that $v_\star - \varphi$ has a local minimum on x_0 and $\nabla\varphi(x_0) \neq 0$, one has

$$F(\nabla\varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, v_\star(x_0)). \tag{2.2}$$

Of course u is a viscosity sub-solution if for all $x_0 \in \Omega$,

-Either there exists a ball $B(x_0,\delta)$, $\delta>0$ on which u is constant and equal to c then $g(x,c)\le 0$,

-Or $\forall\varphi\in\mathcal{C}^2(\Omega)$, such that $u^\star-\varphi$ has a strict local maximum on x_0 and $\nabla\varphi(x_0)\neq 0$, one has

$$F(\nabla\varphi(x_0),D^2\varphi(x_0))\ge g(x_0,u^\star(x_0)). \tag{2.3}$$

See e.g. [8] and [11] for similar definition of viscosity solution for equations with singular operators.

For convenience we recall the definition of semi-jets given e.g. in [9]

$$\begin{aligned} J^{2,+}u(\bar x) &= \{(p,X)\in\mathbb{R}^N\times S,\ u(x)\le u(\bar x)+\langle p,x-\bar x\rangle + \\ &+ \frac{1}{2}\langle X(x-\bar x),x-\bar x)\rangle+o(|x-\bar x|^2)\} \end{aligned}$$

and

$$\begin{aligned} J^{2,-}u(\bar x) &= \{(p,X)\in\mathbb{R}^N\times S,\ u(x)\ge u(\bar x)+\langle p,x-\bar x\rangle + \\ &+ \frac{1}{2}\langle X(x-\bar x),x-\bar x)\rangle+o(|x-\bar x|^2\}. \end{aligned}$$

In the definition of viscosity solutions the test functions can be substituted by the elements of the semi-jets in the sense that if $(p,X)\in J^{2,-}u(\bar x)$ then $\phi(x)=u(\bar x)+\langle p,x-\bar x\rangle+\frac{1}{2}\langle X(x-\bar x),x-\bar x)\rangle$ is a test function for a super solution u at $\bar x$ and similarly if $(q,Y)\in J^{2,+}u(\bar x)$ then $\varphi(x)=u(\bar x)+\langle q,x-\bar x\rangle+\frac{1}{2}\langle Y(x-\bar x),x-\bar x)\rangle$ is a test function for a sub solution u at $\bar x$.

3 Maximum principle and comparison results

As we pointed out in the introduction, we want to generalize the concept of eigenvalue for the Dirichlet problem in a bounded domain Ω associated to the operator $L(u)=F(\nabla u,D^2u)$ satisfying (H2) . It will be defined following the main ideas introduced in [2] for uniformly elliptic operators.

In the introduction we have defined the eigenvalue $\bar\lambda$. In view of the definition given in the previous section the correct definition of the set E becomes

$$E=\{\lambda\in\mathbb{R},\ \ \exists\,\phi,\ \phi_\star>0 \text{ in } \Omega,\ F(\nabla\phi,D^2\phi)+\lambda\phi^{\alpha+1}\le 0 \text{ in the viscosity sense }\}.$$

Throughout the paper we shall denote

$$\bar{\lambda} = \sup E.$$

Remark 3.1 *Of course E is non empty, since 0 obviously belongs to E. Moreover, if $\lambda \in E$, every $\lambda' < \lambda$ is also in E.*

The next proposition proves that $\bar{\lambda} \in \mathbb{R}^+$. In the last section we shall prove that $\bar{\lambda}$ plays the role of the first eigenvalue.

Proposition 3.2 *Suppose that R is the radius of the largest ball contained in Ω and suppose that F satisfies (H2). Then, there exists some constant C which depends only on a, A, N and α, such that*

$$\bar{\lambda} \leq \frac{C}{R^{\alpha+2}}.$$

This proposition is a consequence of the maximum principle stated in the following Theorem and Lemma 3.5 below.

Theorem 3.3 *Suppose that Ω is a bounded open piecewise $\mathcal{C}^1$ domain of $\mathbb{R}^N$. Suppose that $\tau < \bar{\lambda}$ and that F satisfies (H1), then every viscosity solution of*

$$\begin{cases} F(\nabla\sigma, D^2\sigma) + \tau|\sigma|^\alpha\sigma \geq 0 & \text{in } \Omega \\ \sigma \leq 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $\sigma \leq 0$ in Ω.

Remark: In this theorem we don't require F to satisfy (H2), but only (H1).

An immediate consequence of Theorem 3.3 is

Corollary 3.4 *Suppose that F satisfy (H2). If $\lambda < \bar{\lambda}$ and $F(-p,-X) = -F(p,X)$ then every solution of*

$$\begin{cases} F(\nabla\phi, D^2\phi) + \lambda|\phi|^\alpha\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

which is zero on the boundary, is identically zero.

Let $\lambda^+ = \{\lambda \in \mathbb{R},\ \exists\, \phi,\ \phi_\star > 0 \text{ in } \Omega,\ |\nabla\phi|^\alpha \mathcal{M}^+_{a,A}(D^2\phi) + \lambda\phi^{\alpha+1} \leq 0$ in the viscosity sense$\}$, then for any $\lambda < \lambda^+$ the solution of (3.4) is zero.

Proof: In the first case, both ϕ and $-\phi$ are solutions of the equation and this implies that they are both negative. In the general case the hypothesis on F implies that ϕ is a subsolution of

$$|\nabla\phi|^\alpha \mathcal{M}^+_{a,A}(D^2\phi) + \lambda|\phi|^\alpha\phi \geq 0 \text{ in } \Omega, \tag{3.5}$$

and hence $\phi \leq 0$. On the other hand ϕ is a supersolution of

$$|\nabla\phi|^\alpha \mathcal{M}^-_{a,A}(D^2\phi) + \lambda|\phi|^\alpha\phi \leq 0 \text{ in } \Omega,$$

and therefore $-\phi$ is a supersolution (3.5) and $-\phi$ is also negative. This conclude the proof.

Lemma 3.5 *Suppose that $\Omega = B(0,R)$, and let $q = \frac{\alpha+2}{\alpha+1}$ and*

$$\sigma = \frac{1}{2q}(|x|^q - R^q)^2.$$

Let F satisfy (H2). Then there exists some constant C which depends only on a, A, N and α such that

$$\sup_{x\in B(0,R)} \frac{-F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} \leq \frac{C}{R^{\alpha+2}}$$

Proof of Proposition 3.2. Suppose that Theorem 3.3 and Lemma 3.5 hold. Without loss of generality we can suppose that $B(0,R) \subset \Omega$. We shall prove that

$$\bar{\lambda} \leq \sup_{x\in B(0,R)} \frac{-F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} = \tau,$$

by Lemma 3.5 this ends the proof.

Suppose by contradiction that $\tau < \bar{\lambda}$ and let $u = \sigma$ for $|x| \leq R$ and 0 elsewhere. Then one would have

$$F(\nabla u, D^2u) + \tau|u|^\alpha u \geq 0 \text{ in } \Omega.$$

Indeed, for $|x| \leq R$, u is a solution by the definition of τ, for $|x| > R$ the definition of viscosity solution gives the result immediately and for $|x| = R$ all the test functions have zero gradient and so they don't need to be tested. Now since $u = 0$ on $\partial\Omega$, this would imply by Theorem 3.3 that $u \leq 0$ in Ω, a contradiction

with the definition of σ which is positive inside the ball. This ends the proof of Proposition 3.2.

Proof of Lemma 3.5 :

Let $g(r) = \sigma(|x|)$. The computation of $g'(r)$ gives $g'(r) = r^{2q-1} - r^{q-1}R^q$ and

$$g''(r) = (2q-1)r^{2q-2} - (q-1)r^{q-2}R^q.$$

Clearly $g' \leq 0$ while $g'' \leq 0$ for $r \leq \left(\frac{q-1}{2q-1}\right)^{\frac{1}{q}}$ and positive elsewhere. Hence by condition (H2) and using the fact that for radial functions the eigenvalues of the Hessian are $\dfrac{g'}{r}$ with multiplicity N-1 and g'' (see [10]),

$$F(\nabla\sigma, D^2\sigma) \leq |g'|^\alpha \mathcal{M}^+_{a,A} = |g'|^\alpha \left[ag''(r) + a(\frac{N-1}{r})g'(r)\right]$$

or

$$F(\nabla\sigma, D^2\sigma) \leq |g'|^\alpha \mathcal{M}^+_{a,A} = |g'|^\alpha \left[Ag''(r) + a(\frac{N-1}{r})g'(r)\right].$$

In both cases

$$F(\nabla\sigma, D^2\sigma) \leq |g'|^\alpha r^{q-2}(B_1 r^q - B_2 R^q)$$

with either $B_1 = a(N+2q-2)$ and $B_2 = a(N+q-2)$ or

$B_1 = A(2q-1) + a(n-1)$ and $B_2 = A(q-1) + a(N-1)$. Hence one gets:

$$\frac{F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} \leq -\frac{r^{q(\alpha+1)-\alpha-2}(-B_1 r^q + B_2 R^q)}{(R^q - r^q)^{\alpha+2}}.$$

Let

$$\varphi(r) = \frac{(-B_1 r^q + B_2 R^q)}{(R^q - r^q)^{\alpha+2}},$$

since $q = \frac{\alpha+2}{\alpha+1}$ one has that

$$\frac{F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} \leq \sup \varphi(r).$$

It is easy to see that $\sup \varphi(r) = \frac{C}{R^{\alpha+2}}$.

This ends the proof of Lemma 3.5.

Proof of Theorem 3.3. We assume that $\tau < \bar{\lambda}$. Then taking λ such that $\tau < \lambda < \bar{\lambda}$, there exists v, a viscosity sub solution of

$$F(\nabla v, D^2 v) + \lambda v^{1+\alpha} \leq 0 \ \text{ in } \Omega,$$

with $v_\star > 0$ in Ω. Suppose that σ is a viscosity solution of

$$F(\nabla \sigma, D^2 \sigma) + \tau |\sigma|^\alpha \sigma \geq 0 \ \text{ in } \Omega,$$

and $\sigma \leq 0$ on $\partial\Omega$. We need to prove that $\sigma \leq 0$ in Ω. It is sufficient to prove that $\sigma^\star \leq 0$. Using the definition of viscosity solutions one can assume without loss of generality that $\sigma \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ and hence drop the stars.

Let us suppose by contradiction that $\frac{\sigma(x)}{v(x)}$ has a positive supremum inside Ω. For some $q > 2$ let us consider the function

$$\psi_j(x, y) = \frac{\sigma(x)}{v(y)} - \frac{j}{qv(y)}|x - y|^q$$

which is uppersemicontinuous. Then ψ_j also has a positive supremum achieved in some couple of points $(x_j, y_j) \in \Omega^2$. One easily has that $(x_j, y_j) \to (\bar{x}, \bar{x})$, $\bar{x} \in \Omega$ which is a supremum for $\frac{\sigma}{v}$. One can also prove that $j|x_j - y_j|^q \to 0$, and that $\bar{x}$ is a continuity point for σ. For that aim remark that

$$\frac{\sigma(x_j) - \frac{j}{q}|x_j - y_j|^q}{v(y_j)} \geq \frac{\sigma(\bar{x})}{v(\bar{x})}$$

and using the lowersemicontinuity of v on $\bar{x}$ together with $\lim \frac{j}{q}|x_j - y_j|^q = 0$ one gets

$$\liminf \sigma(x_j) \geq \sigma(\bar{x}).$$

Assume for the moment that $x_j \neq y_j$ for j large enough. Take j large enough in order that

$$\sigma(x_j)^{1+\alpha} \geq \frac{3\sigma(\bar{x})^{1+\alpha}}{4}$$

and

$$\frac{j}{q}|x_j - y_j|^q \leq \frac{\sigma(\bar{x})^{1+\alpha}(\lambda - \tau)}{4\lambda}.$$

Using $\psi_j(x, y) \leq \psi_j(x_j, y_j)$, one gets that

$$\sigma(x)v(y_j) - v(y)\left(\sigma(x_j) - \frac{j}{q}|x_j - y_j|^q\right) \leq v(y_j)\frac{j}{q}|x - y|^q. \tag{3.6}$$

We now define

$$\beta_j = \sigma(x_j) - \frac{j}{q}|x_j - y_j|^q$$

and then after some simple calculation (3.6) becomes

$$\begin{aligned} &\left(\sigma(x+x_j) - \sigma(x_j) - j|x_j - y_j|^{q-2}(x_j - y_j.x)\right) v(y_j) + \\ &- \left(v(y+y_j) - v(y_j) - j|x_j - y_j|^{q-2}(x_j - y_j.y)\frac{v(y_j)}{\beta_j}\right)\beta_j \qquad (3.7) \\ \leq\ & v(y_j)\left(\frac{j}{q}|x_j + x - y_j - y|^q - \frac{j}{q}|x_j - y_j|^q - j|x_j - y_j|^{q-2}(x_j - y_j, x - y)\right). \end{aligned}$$

We define the functions

$$U(x) = \left(\sigma(x+x_j) - \sigma(x_j) - j|x_j - y_j|^{q-2}(x_j - y_j.x)\right) v(y_j)$$

and

$$V(y) = -\left(v(y+y_j) - v(y_j) - j|x_j - y_j|^{q-2}(x_j - y_j.y)\frac{v(y_j)}{\beta_j}\right)\beta_j$$

then (3.7) can be written:

$$U(x) + V(y) \leq (x,y)A(x,y)$$

with

$$A = jv(y_j)\begin{pmatrix} D_j & -D_j \\ -D_j & D_j \end{pmatrix}$$

and

$$D_j = 2^{q-3}q|x_j - y_j|^{q-2}\left(I + \frac{(q-2)}{|x_j - y_j|^2}(x_j - y_j)\otimes(x_j - y_j)\right).$$

Then using theorem 3.2' in [9] one gets that there exist X_j and Y_j such that

$$\left(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)}\right) \in J^{2,+}\sigma(x_j)$$

and

$$\left(j|x_j - y_j|^{q-2}(x_j - y_j)\frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}\right) \in J^{2,-}v(y_j)$$

with, for some $\varepsilon > 0$.

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq A + \varepsilon A^2.$$

In particular

$$X_j + Y_j \leq 0.$$

We can conclude using the fact that v and σ are respectively a super and a sub solution and the properties of F. Precisely we have obtained

$$\begin{aligned}
-\tau\sigma(x_j)^{1+\alpha} &\leq F(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)}) \\
&\leq F(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{-Y_j}{v(y_j)}) \\
&\leq \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} F(j|x_j - y_j|^{q-2}(x_j - y_j)\frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \\
&\leq -\lambda\beta_j^{1+\alpha} = -\lambda[\sigma(x_j) - \frac{j}{q}|x_j - y_j|^q]^{1+\alpha}.
\end{aligned}$$

This gives a contradiction, indeed by passing to the limit the inequality becomes

$$-\tau\sigma^{\alpha+1}(\bar{x}) \leq -\lambda\sigma^{\alpha+1}(\bar{x}).$$

It remains to prove that $x_j \neq y_j$ for j large enough. If one assumes that $x_j = y_j$ one has

$$\sigma(x_j) \geq \sigma(x) - \frac{j}{q}|x_j - x|^q$$

and

$$v(x) \geq v(x_j) - \frac{jv(x_j)|x_j - x|^q}{q\sigma(x_j)}.$$

In that case one uses Lemma 2.2 in [3] to get a contradiction. This ends the proof of Theorem 3.3.

Let us recall that in [3] for $\lambda = 0$ we give a comparison principle for continuous viscosity solutions. It is not difficult to see that it can be extended to bounded viscosity solutions. We now prove a further extension adapted to our context.

Theorem 3.6 *Suppose that $\lambda < \bar{\lambda}$, $f \leq 0$, f is upper semicontinuous and g is lower semicontinuous with $f \leq g$ and*

- *either $f \leq -c < 0$ in Ω,*
- *or $g(\bar{x}) > 0$ on every point $\bar{x}$ such that $f(\bar{x}) = 0$.*

Suppose that there exist v nonnegative viscosity sub solution of

$$F(\nabla u, D^2u) + \lambda u^{1+\alpha} = f$$

and σ nonnegative viscosity super solution of

$$F(\nabla \sigma, D^2\sigma) + \lambda \sigma^{1+\alpha} \geq g$$

satisfying $\sigma \leq v$ on $\partial\Omega$.

Then $\sigma \leq v$ in Ω.

As a consequence one has

Corollary 3.7 *Suppose that $\lambda \leq \bar{\lambda}$, there exists at most one nonnegative viscosity solution of*

$$\begin{cases} F(\nabla v, D^2v) + \lambda v^{1+\alpha} = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

for $f < 0$ and continuous.

Proof of Theorem 3.6 First using the strict maximum principle (see [4]) one gets that $F(\nabla v, D^2v) \leq 0$ $v \geq 0$, and since v is not identically zero, $v_\star > 0$ in Ω. Without loss of generality one can assume that σ and v are respectively USC and LSC.

Suppose by contradiction that $\sigma > v$ somewhere in Ω. The supremum of the function $\frac{\sigma}{v}$ on $\partial\Omega$ is less than 1, then its supremum is achieved inside Ω. Let $\bar{x}$ be a point such that

$$1 < \frac{\sigma(\bar{x})}{v(\bar{x})} = \sup_{x\in\overline{\Omega}} \frac{\sigma(x)}{v(x)}.$$

Doing exactly the same construction as in the proof of Theorem 3.3 we similarly get :

$$\begin{aligned} g(x_j) - \lambda\sigma(x_j)^{1+\alpha} &\leq F(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)}) \\ &\leq \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} F(j|x_j - y_j|^{q-2}(x_j - y_j)\frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \\ &\leq -\lambda\beta_j^{1+\alpha} + \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} f(y_j). \end{aligned}$$

Passing to the limit we obtain

$$g(\bar{x}) \leq \left(\frac{\sigma(\bar{x})}{v(\bar{x})}\right)^{\alpha+1} f(\bar{x}). \tag{3.8}$$

Either $f(\bar{x}) = 0$ and $g(\bar{x}) > 0$. but this contradicts (3.8) or $f(\bar{x}) < 0$, and then (3.8) becomes

$$0 < f(\bar{x}) \left[1 - \left(\frac{\sigma(\bar{x})}{v(\bar{x})}\right)^{\alpha+1}\right] \leq f(\bar{x}) - g(\bar{x}) \leq 0,$$

also a contradiction.
This conclude the proof.

4 Hölder and Lipschitz regularity

In all this section we assume that F satisfies (H2) and Ω is a C^2 bounded domain.

Suppose that u is a viscosity solution of

$$\begin{cases} F(\nabla u, D^2 u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Theorem 4.1 *Let f be a bounded function in $\overline{\Omega}$. Let u be a non negative viscosity solution of (4.9) with Ω a $\mathcal{C}^2$ domain. Then for any $\gamma \in (0,1)$, there exists $C > 0$, such that*

$$|u(x) - u(y)| \leq C|x - y|^{\gamma}.$$

Using the result of Theorem 4.1, one obtains the following stronger result :

Theorem 4.2 *Let u be a non negative viscosity solution of equation (4.9), then u is locally Lipschitz continuous when f is bounded.*

Proof of Theorem 4.1: The proof relies on ideas used to prove Hölder and Lipschitz estimates in [17].

First we will prove that u is Hölder near the boundary using the regularity of the boundary and of the distance function near the boundary.

Let $d(x)$ be the function $d(x, \partial\Omega) = \inf\{|x - y|, \text{ for } y \in \partial\Omega\}$.

Claim: $\exists \delta > 0$, *there exist* $M_o > 0$, $1 > \gamma > 0$ *and* $\bar{r}$ *such that* $u(x) \leq M_o d(x)^\gamma$ *for* $d(x) \leq \delta$.

In order to prove the claim we need to show that $g(x) = d(x)^\gamma$ is a super solution of (4.9) in

$$\Omega_\delta = \{x \in \Omega, d(x, \partial\Omega) < \delta\}.$$

It is well known (see [12, 14, 20]), that d is $\mathcal{C}^2$ on Ω_δ for δ small enough since $\partial\Omega$ is $\mathcal{C}^2$. Furthermore the $\mathcal{C}^2$ norm of d is bounded. Then for δ small enough and $d(x) < \delta$,

$$F(\nabla g, D^2 g) \leq \gamma^{1+\alpha} d^{(\gamma(\alpha+1)-\alpha-2)}(\gamma - 1 + cd(x)|D^2 d(x)|_\infty) \leq -\epsilon < 0$$

for some constant c which depends on a and A and some constant $\varepsilon > 0$ which depends on γ, N, α and $\partial\Omega$.

We now define M_o such that

$$M_o \delta^\gamma > \sup_{\partial\Omega_\delta \cap \Omega} u \quad \text{and} \quad M_0^{1+\alpha} > \frac{|f|_\infty}{\epsilon}.$$

By the comparison principle (Theorem 3.6) $u^\star \leq M_o d(x, \partial\Omega)^\gamma$ in Ω_δ and the claim is proved.

We now prove Hölder's regularity inside Ω.

We construct a function Φ as follows: Let M_o and γ be as in the Claim, $M = \sup(M_o, \frac{2 \sup u}{\delta^\gamma})$ and $\Phi(x) = M(|x|^\gamma)$.

We shall consider

$$\Delta_\delta = \{(x, y) \in \Omega^2, \ |x - y| < \delta\}.$$

Claim 2 *For any* $(x, y) \in \Delta_\delta$

$$u^\star(x) - u_\star(y) \leq \Phi(x - y) \tag{4.10}$$

If the Claim 2 holds this completes the proof, indeed taking $x = y$ we would get that $u^\star = u_\star$ and then u is continuous. Therefore going back to (4.10)

$$u(x) - u(y) \leq \frac{2 \sup u}{\delta^\gamma} |x - y|^\gamma,$$

for $(x, y) \in \Delta_\delta$ which is equivalent to the local Hölder continuity.

Let us check first that (4.10) holds on $\partial\Delta_\delta$. On that set,

- either $|x-y|=\delta$ and then $u^\star(x)-u_\star(y)\le M\delta^\gamma$ since $M\delta^\gamma\ge 2\sup u$,

-or $(x,y)\in\partial(\Omega\times\Omega)$. In that case, for $(x,y)\in(\Omega\times\partial\Omega)$ we have just proved that

$$u^\star(x)\le M_o d^\gamma\le M|y-x|^\gamma.$$

Now we consider interior points. Suppose by contradiction that $u^\star(x)-u_\star(y)>\Phi(x-y)$ for some $(x,y)\in\Delta_\delta$. Then there exists $(\bar x,\bar y)$ such that

$$u^\star(\bar x)-u_\star(\bar y)-\Phi(\bar x-\bar y)=\sup(u^\star(x)-u_\star(y)-\Phi(x-y))>0.$$

Clearly $\bar x\ne\bar y$. Then using Ishii's Lemma there exists X and Y such that

$$(\gamma M(\bar x-\bar y)|\bar x-\bar y|^{\gamma-2},X)\in J^{2,+}u^\star(\bar x)$$

$$(\gamma M(\bar x-\bar y)|\bar x-\bar y|^{\gamma-2},-Y)\in J^{2,-}u_\star(\bar y)$$

with

$$\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix}\le\begin{pmatrix} B & -B\\ -B & B\end{pmatrix}$$

and $B=D^2\Phi(\bar x-\bar y)$.

In particular $tr(X+Y)\le 0$. We need a more precise estimate, as in [17]. For that aim let :

$$0\le P:=\frac{(\bar x-\bar y\otimes\bar x-\bar y)}{|\bar x-\bar y|^2}\le I.$$

Remarking that $X+Y\le 4B$, one easily sees that $tr(X+Y)\le tr(P(X+Y))\le 4tr(PB)$. But $tr(PB)=\gamma M(\gamma-1)|\bar x-\bar y|^{\gamma-2}<0$, hence

$$|tr(X+Y)|\ge 4\gamma M(1-\gamma)|\bar x-\bar y|^{\gamma-2}. \tag{4.11}$$

Furthermore by Lemma III.1 of [17] there exists a universal constant C such that

$$|X|,|Y|\le C(|tr(X+Y)|+|B|^{\frac12}|tr(X+Y)|^{\frac12}).$$

Now we can use the fact that u is both a sub and a super solution of (4.9) and applying (H2) condition

$$\begin{aligned} f(\bar x) &\le F(\nabla_x\Phi,X)\\ &\le a|\nabla_x\Phi|^\alpha tr(X+Y)+F(\nabla_y\Phi,tr(-Y))\\ &\le f(\bar y)+|\nabla_x\Phi|^\alpha tr(X+Y).\end{aligned}$$

Which implies, using (4.11),

$$a|\nabla_x \Phi|^\alpha 4\gamma M(1-\gamma)|\bar{x}-\bar{y}|^{\gamma-2} \leq f(\bar{y}) - f(\bar{x}).$$

Recalling that $|\nabla_x \Phi| = \gamma M|\bar{x}-\bar{y}|^{\gamma-1}$ the previous inequality becomes:

$$aM^{\alpha+1}4\gamma^{1+\alpha}(1-\gamma)|\bar{x}-\bar{y}|^{\gamma(\alpha+1)-(\alpha+2)} \leq 2|f|_\infty. \tag{4.12}$$

Using $M \geq \frac{2(\sup u)}{\delta^\gamma}$ and $|\bar{x}-\bar{y}| \leq \delta$ one obtains

$$a(2\sup u)^{1+\alpha}4\gamma^{1+\alpha}(1-\gamma)\delta^{-(\alpha+2)} \leq 2|f|_\infty.$$

This is clearly false for δ small enough and it concludes the proof.

Proof of Theorem 4.2. The proof proceeds similarly to the proof given by Ishii and Lions in [17] but here we shall use the fact that we already know that u is Hölder continuous.

We assume without loss of generality that in hypothesis (H2) $a = A = 1$.

Let μ be an increasing function such that $\mu(0) = 0$, and $\mu(r) \geq r$, let $l(r) = \int_0^r ds \int_0^s \frac{\mu(\sigma)}{\sigma} d\sigma$, let us note that since $\mu \geq 0$ for $r > 0$

$$l(r) \leq rl'(r)$$

Let r_0 be such that $l'(r_0) = \frac{1}{2}$, M such that $Mr_0 \geq 4\sup|u|$. Let also $\delta > 0$ be given, $K = \frac{r_0}{\delta}$, and z be such that $d(z, \partial\Omega) \geq 2\delta$.

We define $\varphi(x,y) = \Phi(x-y) + L|x-z|^k$ where $\Phi(x) = M(K|x| - l(K|x|))$, and

$$\Delta_z = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N, |x-y| < \delta, |x-z| < \delta\}.$$

We shall now choose all the constants above.

Choosing k such that $k > \frac{1}{1-\frac{1}{2\gamma}}$ where γ is such that $\gamma \in]0,1[$ and

$$|u(x) - u(y)| \leq c|x-y|^\gamma,$$

for some constant c which depends on u and γ.

Choosing M and L such that $M \geq \frac{2\sup u}{r_0}$ and $L = c\delta^{k-\gamma}$, using the Hölder continuity of u, one has

$$u(x) - u(y) \leq \varphi(x,y)$$

on $\partial\Delta_z$.

Suppose by contradiction that for some points $\bar{x}, \bar{y}$ one has

$$u(\bar{x}) - u(\bar{y}) > \varphi(\bar{x}, \bar{y}).$$

Clearly $\bar{x} \neq \bar{y}$. Note that

$$L|\bar{x} - z|^k \leq c|\bar{x} - \bar{y}|^\gamma.$$

Proceeding as in the previous proof, there exist X, Y such that

$$\big(MK(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{-1}(1 - l'(K|\bar{x} - \bar{y}|)) + kL|\bar{x} - z|^{k-2}(\bar{x} - z), X\big) \in J^{2,+}u(\bar{x})$$

and

$$\left(MK\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}(1 - l'(K|\bar{x} - \bar{y}|)), -Y\right) \in J^{2,-}u(\bar{y}).$$

The matrices X and Y satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} B + \tilde{L} & -B \\ -B & B \end{pmatrix}$$

with $B = D^2\varphi(\bar{x}, \bar{y})$ and

$$\tilde{L} = kL|\bar{x} - z|^{k-2}\left(I + (k-2)\frac{(\bar{x} - z \otimes \bar{x} - z)}{|\bar{x} - z|^2}\right).$$

Let us note that $X + Y - \tilde{L} \leq 4B$ and then

$$tr(X + Y - \tilde{L}) \leq 4tr(PB)$$

with

$$P = \frac{((\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y}))}{|\bar{x} - \bar{y}|^2}.$$

This allows to have :

$$|tr(X + Y - \tilde{L})| \geq \frac{MK\mu(K|\bar{x} - \bar{y}|))}{|\bar{x} - \bar{y}|} \geq MK^2.$$

Furthermore, as in the previous proof, one has

$$|Y| \leq C(|B|^{\frac{1}{2}}|tr(X + Y - \tilde{L})|^{\frac{1}{2}} + |tr(X + Y - \tilde{L})|).$$

Let us note that

$$|B| \leq C\frac{1}{|\bar{x}-\bar{y}|}$$

and that

$$\nabla_x\varphi(x) = MK(1-l'(K|\bar{x}-\bar{y}|))\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} + kL|\bar{x}-z|^{k-2}(\bar{x}-z).$$

We can now use

$$L|\bar{x}-z|^{k-1} = (L|\bar{x}-z|^k)^{\frac{k-1}{k}}L^{\frac{1}{k}} \leq (c\delta^\gamma)^{\frac{k-1}{k}}(c\delta^{\gamma-k})^{\frac{1}{k}} = O(\delta^{\gamma-1}) = o(K)$$

so that for K large enough

$$2MK \geq |\nabla_x\varphi| \geq \frac{MK}{4}.$$

Then, using the fact that u is both a sub and a super solution, there exist some universal constants c_1, c_2, c_3, such that

$$\begin{aligned}(MK)^\alpha|tr(X+Y-\tilde{L})| &\leq c_1(MK)^{\alpha-1}L|\bar{x}-z|^{k-1}\frac{1}{|\bar{x}-\bar{y}|^{\frac{1}{2}}}(|tr(X+Y-\tilde{L})|^{\frac{1}{2}}) + \\ &+ c_2(MK)^\alpha L|\bar{x}-z|^{k-1}|tr(X+Y-\tilde{L})| + f(\bar{y}) - f(\bar{x}) + \\ &+ c_3|MK|^\alpha L|\bar{x}-z|^{k-2}.\end{aligned}$$

We shall now prove that for K large enough this is absurd by obtaining the following estimates :

$[K1]$ $L|\bar{x}-z|^{k-2} = O(\delta^{\gamma-2}) = o(K^2)$

$[K2]$ $L|\bar{x}-z|^{k-1}\dfrac{|\nabla_x\varphi|^{\alpha-1}|tr(X+Y-\tilde{L})|^{\frac{1}{2}}}{|\bar{x}-\bar{y}|^{\frac{1}{2}}} \leq o(1)|\nabla_x\varphi|^\alpha|tr(X+Y-\tilde{L})|$

or equivalently

$[K3]$ $L\dfrac{|\bar{x}-z|^{k-1}}{|\bar{x}-\bar{y}|^{\frac{1}{2}}} \leq o(1)|\nabla_x\varphi|(K^2)^{\frac{1}{2}} = o(1)K^2.$

We prove $[K1]$:

$$L|\bar{x}-z|^{k-2} \leq (L|\bar{x}-z|^k)^{\frac{k-2}{k}}L^{\frac{2}{k}} \leq c\delta^{\gamma-2} \leq o(K^2).$$

We prove $[K3]$

$$\frac{L|\bar{x}-z|^{k-1}}{|\bar{x}-y|^{\frac{1}{2}}} \leq L^{\frac{1}{k}}\frac{(L|\bar{x}-z|^k)^{\frac{k-1}{k}}}{|\bar{x}-\bar{y}|^{\frac{1}{2}}}$$

$$
\begin{aligned}
&\leq CL^{\frac{1}{k}}(|\bar{x}-\bar{y}|)^{\gamma(1-\frac{1}{k})-\frac{1}{2}} \\
&\leq C(c\delta^{\gamma-k})^{\frac{1}{k}}\delta^{\gamma(1-\frac{1}{k})-\frac{1}{2}} \\
&= O(K^{\frac{3}{2}-\gamma}) \\
&= o(K^{\frac{3}{2}}).
\end{aligned}
$$

We have obtained

$$
\begin{aligned}
CK^{\alpha+2} \quad + \quad & \frac{|\nabla_x\varphi|^\alpha |tr(X+Y-\tilde{L})|}{2} \leq |\nabla_x\varphi|^\alpha |tr(X+Y-\tilde{L})| \\
\leq \quad & 2\sup|f| + o(1)|\nabla_x\varphi|^\alpha |tr(X+Y-\tilde{L})| + o(1)|\nabla_x\varphi|^\alpha + \\
& +o(1)K|tr(X+Y-\tilde{L})|^{\frac{1}{2}}|\nabla_x\varphi|^\alpha \\
\leq \quad & |\nabla_x\varphi|^\alpha |tr(X+Y-\tilde{L})| + o(1)K^2|\nabla_x\varphi|^\alpha
\end{aligned}
$$

which is a contradiction for K large.

We have proved that for all x such that $d(x,\partial\Omega) \geq 2\delta$ and for y such that $|x-y| \leq \delta$

$$
u(x) - u(y) \leq \frac{2\sup M}{r_0}\frac{|x-y|}{\delta}.
$$

The local Lipschitz continuity is proved.

5 Existence results

5.1 The case $\lambda < \bar{\lambda}$

In this subsection we shall prove the esistence of solutions via Perron's method by constructing explicitely a positive super solution.

Theorem 5.1 *Suppose that f is bounded and $f \leq 0$ on $\overline{\Omega}$. Then, for $\lambda < \bar{\lambda}$ there exists u a nonnegative viscosity solution of*

$$
\begin{cases}
F(\nabla u, D^2u) + \lambda u^{1+\alpha} = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Furthermore u is unique.

To prove this theorem, we need the two following propositions

Proposition 5.2 *Suppose that f is bounded, $f \leq 0$ and $\lambda \in \mathbb{R}$. Suppose that there exists $v_1 \geq 0$ and $v_2 \geq 0$ respectively sub solution and super solution of*

$$\begin{cases} F(\nabla v, D^2 v) + \lambda v^{1+\alpha} = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{5.13}$$

with $v_1 \leq v_2$. Then there exists a viscosity solution v of (5.13), such that $v_1 \leq v \leq v_2$. Moreover if $f < 0$ inside Ω the solution is unique.

Proposition 5.3 *For any f bounded and non positive in $\overline{\Omega}$, there exists a unique viscosity solution w of*

$$\begin{cases} F(\nabla w, D^2 w) = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.14}$$

Of course w is nonnegative by the maximum principle and Hölder continuous.

By Proposition 5.2, Proposition 5.3 will be proved if we construct a sub and super solution for (5.14). Since the null function is clearly a sub solution, it is sufficient to construct a viscosity solution u of $F(\nabla u, D^2 u) \leq -1$ which is positive and zero on the boundary, then multiplying by the right constant we get the required super solution of (5.14).

In the next lemma we construct such a super solution:.

Lemma 5.4 *Let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$. Let $d(x) = d(x, \partial\Omega)$ be the distance to the boundary. Then there exist $k \in \mathbf{N}$, $\gamma \in (0,1)$, and $\beta > 0$ such that*

$$u(x) = \beta \left(1 - \frac{1}{(1 + d(x)^\gamma)^k}\right)$$

is a viscosity super solution of

$$F(\nabla u, D^2 u) \leq -1.$$

The proof of this lemma is postponed to the appendix together with some properties of the distance function, while the proof of Proposition 5.2 is at the end of this section.

Proof of Theorem 5.1

For $\lambda < 0$, one can apply directly Proposition 5.2, since 0 is a sub solution for (5.6) and the solution constructed in Proposition 5.3 is a super solution.

We now treat the case $\lambda > 0$.

We define the sequence $u_n = T_f^n(0)$ where $T_f(u)$ is defined as the unique viscosity solution of

$$\begin{cases} F(\nabla T_f(u), D^2 T_f(u)) = f - \lambda u^{1+\alpha} & \text{in } \Omega \\ T_f(u) = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 5.3 implies that $T_f u$ is well defined.

By the comparison principle and the maximum principle for F in [3], u_n is increasing and nonnegative. We want to prove that it is bounded. Suppose not, then $w_n := \frac{u_n}{|u_n|_\infty}$ satisfies

$$F(\nabla w_{n+1}, D^2 w_{n+1}) + \lambda \left(\frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} \right) = \frac{f}{|u_{n+1}|_\infty^{1+\alpha}}.$$

Furthermore

$$F(\nabla w_{n+1}, D^2 w_{n+1}) + \lambda w_{n+1}^{1+\alpha} = \lambda \left(\frac{u_{n+1}^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} - \frac{u_n^{1+\alpha}}{|u_{n+1}|^{1+\alpha}} \right) + \frac{f}{|u_{n+1}|_\infty^{1+\alpha}} \geq \frac{f}{|u_{n+1}|_\infty^{1+\alpha}}.$$

Clearly

$$\left| \lambda \left(\frac{u_{n+1}^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} - \frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} \right) + \frac{f}{|u_{n+1}|_\infty^{1+\alpha}} \right| \leq 2\lambda + \frac{|f|}{|u_1|_\infty^{1+\alpha}}$$

since $0 \leq \frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} \leq 1$.

We can now apply the Hölder estimates in the previous section and this implies that the sequence w_n is relatively compact in $\mathcal{C}(\bar{\Omega})$.

Extracting a subsequence from (w_n) and passing to the limit one gets in particular

$$F(\nabla w, D^2 w) + \lambda w^{1+\alpha} \geq 0.$$

Moreover $w = 0$ on the boundary .

We are in the hypothesis that $\lambda < \bar{\lambda}$ hence we can apply the maximum principle and conclude that $w \leq 0$. We have reached a contradiction since $w \geq 0$ and $|w|_\infty = 1$.

We have obtained that the sequence u_n must be bounded. Since it is increasing and bounded it converges and the convergence is uniform on $\overline{\Omega}$, by the Hölder estimates. Using the properties of uniform limit of viscosity solutions one gets that the limit u is a nonnegative solution of

$$F(\nabla u, D^2 u) + \lambda u^{1+\alpha} = f.$$

Proof of Proposition 5.2 : The proof relies on the Perron's method applied to viscosity solution by Ishii (see [16]).

Let us define

$$v = \sup\{v_1 \leq u \leq v_2, u \text{ is a viscosity sub solution of (5.13)}\}.$$

We want to prove first that $v^\star$ a sub solution. Let u_n be an increasing sequence of sub solutions, $v_1 \leq u_n \leq v_2$, u_n converging to v.

Suppose first that v is equal to a constant C on a ball $B(\bar{x}, r)$. Since $C \geq 0$, it is a sub solution.

We now treat the points where v is not locally constant. Suppose by contradiction that $\bar{x}$ and φ are such that $\nabla\varphi(\bar{x}) \neq 0$ and

$$(v - \varphi)(x) \leq (v^\star - \varphi)(\bar{x}) = 0,$$

and that there exists $r > 0$ with

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \leq f(\bar{x}) - r.$$

Let δ be small enough that for $|\bar{x} - y| \leq \delta$, the following inequalities hold

$$|F(\nabla\varphi, D^2\varphi)(y) - F(\nabla\varphi, D^2\varphi)(\bar{x})| \leq \frac{r}{4},$$

$$|\varphi(y)^{1+\alpha} - \varphi(\bar{x})^{1+\alpha}| \leq \frac{r}{4|\lambda|},$$

$$|f(y) - f(\bar{x})| \leq \frac{r}{4}.$$

One can assume that the supremum of $v^\star - \varphi$ on $\bar{x}$ is strict, so that there exists $\alpha_\delta > 0$ with

$$\sup_{|y-\bar{x}|\geq\delta} (v^\star - \varphi) \leq -\alpha_\delta.$$

Finally take N large enough in order that by the simple convergence of $u_n(\bar{x})$ toward $v(\bar{x})$ one has

$$u_n(\bar{x}) - v^\star(\bar{x}) \geq -\frac{\alpha_\delta}{4}$$

then

$$\sup_{|x-\bar{x}|\leq\delta} (u_n - \varphi)(x) \geq \frac{-\alpha_\delta}{4} \geq -\alpha_\delta \geq \sup_{|x-\bar{x}|\geq\delta} (v^\star - \varphi)(x) \geq \sup_{|x-\bar{x}|\geq\delta} (u_n - \varphi)(x).$$

Furthermore the supremum is achieved inside $B(\bar{x},\delta)$, on some x_n. Then one has

$$\begin{aligned} f(\bar{x}) - r &\geq F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \\ &\geq F(\nabla\varphi, D^2\varphi)(x_n) + \lambda\varphi(x_n)^{1+\alpha} - \frac{r}{2} \\ &\geq f(x_n) - \frac{r}{2} \geq f(\bar{x}) - \frac{3r}{4}, \end{aligned}$$

a contradiction.

We now prove that $v_\star$ is a super solution. If not there would exist $\bar{x} \in \Omega$, $r > 0$ and $\varphi \in \mathcal{C}^2(B(\bar{x}, r)$, with $\nabla\varphi(\bar{x}) \neq 0$, satisfying

$$0 = (v_\star - \varphi)(\bar{x}) \leq (v_\star - \varphi)(x)$$

on $B(\bar{x}, r)$ such that

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} > f(\bar{x}).$$

We prove first that $\varphi(\bar{x}) < v_2(\bar{x})$. If not one would have $\varphi(\bar{x}) = v_\star(\bar{x}) = v_2(\bar{x})$ and then

$$(v_2 - \varphi)(x) \geq (v_\star - \varphi)(x) \geq (v_\star - \varphi)(\bar{x}) = (v_2 - \varphi)(\bar{x}) = 0,$$

hence since v_2 is a super solution and φ is a test function for v_2 on $\bar{x}$,

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \leq f(\bar{x}),$$

a contradiction. Then $\varphi(\bar{x}) < v_2(\bar{x})$. We construct now a sub solution which is greater than v and less than v_2.

Let $\varepsilon > 0$ be such that

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \geq f(\bar{x}) + \varepsilon.$$

Let δ be such that for $|x - \bar{x}| \leq \delta$

$$|F(\nabla\varphi, D^2\varphi)(x) - F(\nabla\varphi, D^2\varphi)(\bar{x})| + |f(x) - f(\bar{x})| + \lambda|\varphi(x)^{1+\alpha} - \varphi(\bar{x})^{1+\alpha}| \leq \frac{\varepsilon}{4}.$$

Then

$$F(\nabla\varphi, D^2\varphi)(x) + \lambda\varphi^{1+\alpha}(x) \geq f(x) + \frac{\varepsilon}{4}.$$

One can assume that

$$(v_\star - \varphi)(x) \geq |x - \bar{x}|^4.$$

We take $r < \delta^4$ and such that $0 < r < \inf_{|x-\bar{x}|\leq\delta}(v_2(x) - \varphi(x))$ and define

$$w = \sup(\varphi(x) + r, v_\star)$$

w is LSC as the supremum of two LSC functions.

One has $w(\bar{x}) = \varphi(\bar{x}) + r$, and $w = v$ for $r < |x - \bar{x}| < \delta$.

w is a sub solution, since when $w = \varphi + r$ one can use $\varphi + r$ as a test function, and since $\varphi(x) > 0$,

$$F(\nabla\varphi, D^2\varphi)(x) + \lambda(\varphi(x) + r)^{1+\alpha} \geq (F(\nabla\varphi, D^2\varphi) + \lambda\varphi^{1+\alpha})(x) \geq f + \frac{\varepsilon}{4}.$$

Elsewhere $w = v$, hence it is a sub solution. Moreover $w \geq v$, $w \neq v$ and $w \leq g$. This contradicts the fact that v is the supremum of the sub solutions. Using Hölder regularity we get that v is Hölder and hence $v^\star = v_\star$.

5.2 The case $\lambda = \bar{\lambda}$.

Theorem 5.5 *Let L be as in the previous section. Then, there exists $\phi > 0$ in Ω such that ϕ is a viscosity solution of*

$$\begin{cases} F(\nabla\phi, D^2\phi) + \bar{\lambda}\phi^{1+\alpha} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover ϕ is γ-Hölder continuous for all $\gamma \in]0, 1[$ and locally Lipschitz.

Proof of Theorem 5.5

Let λ_n be an increasing sequence which converges to $\bar{\lambda}$. Let u_n be a nonnegative viscosity solution of

$$\begin{cases} F(\nabla u_n, D^2 u_n) + \lambda_n u_n^{1+\alpha} = -1 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 5.1 the sequence u_n is well defined. We shall prove that u_n is not bounded. Indeed suppose by contradiction that it is. Then, by Hölder's estimate, one has that a subsequence, still denoted u_n, tends uniformly to a nonnegative continuous function u which is a viscosity solution of

$$F(\nabla u, D^2 u) + \bar{\lambda} u^{1+\alpha} = -1.$$

This contradicts the definition of $\bar{\lambda}$. Indeed $u > 0$ and one can choose ε small enough that

$$F(\nabla u, D^2 u) + (\bar{\lambda} + \varepsilon) u^{1+\alpha} \leq -1 + \varepsilon u^{1+\alpha} \leq 0.$$

We have obtained that the sequence $|u_n|_\infty \to +\infty$. Then defining $w_n = \frac{u_n}{|u_n|_\infty}$ one has

$$F(\nabla w_n, D^2 w_n) + \lambda_n w_n^{1+\alpha} = \frac{f}{|u_n|^{1+\alpha}}$$

and then extracting as previously a subsequence which converges uniformly, one gets that there exists w, $|w|_\infty = 1$ and

$$F(\nabla w, D^2 w) + \bar{\lambda} w^{1+\alpha} = 0.$$

The boundary condition is given by the uniform convergence. Clearly w is Hölder and locally Lipschitz continuous.

6 Appendix: Properties of the distance function.

In all this section Ω is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$. For completeness sake we shall study the regularity of the distance function at points $x \in \Omega$ for which the distance to the boundary is achieved by one single point of the boundary. See e.g. [12, 14, 20] for other interesting results on the distance function.

We shall denote the elements of $\overline{\Omega}$ by $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Without loss of generality we can suppose that $(0,0) \in \partial\Omega$ and that $x_o = (0, d) \in \mathbb{R}^{N-1} \times \mathbb{R}^+$ is at the distance d to $\partial\Omega$ and that $(0,0)$ is the unique point of $\partial\Omega$ at which the distance is achieved.

Always without loss of generality we can suppose that there exist a neighborhood V of $(0,0) \in \mathbb{R}^N$, $r > 0$ and a function $a \in \mathcal{C}^2(B'(0,r))$ (the ball of center 0 and radius r in $\mathbb{R}^{N-1}$) such that

$$\partial\Omega \cap V = \{(x', a(x')),\ x' \in B'(0,r)\}.$$

And we can suppose that the unit interior normal to $\partial\Omega$ at $(0,0)$ is e_N, which implies that $\nabla a(0) = 0$. In the rest of the section we shall consider this setting.

Lemma 6.1 *In the above hypothesis*

$$D^2 a(0) < \frac{1}{d} I,$$

(where I denotes the $N-1$ dimensional matrix identity, in the sense of positivity of symmetric matrices.)

Proposition 6.2 *In the same hypothesis there exists a neighborhood V_0 of $(0,0)$ in $\partial\Omega \times \mathbb{R}^+$ on a neighborhood V_1 of x_o such that $\forall x \in V_1$ there exists one and only one $y \in V_0$ such that*

$$|x-y| = d(x, \partial\Omega).$$

Moreover the map $x \to y(x)$ is $\mathcal{C}^2$ in this neighborhood.

Proof of Lemma 6.1

Since $(0,0)$ is the unique point on the $N-1$ surface $x_N = a(x')$, at which the distance is achieved, we get for $|x'| < r$

$$|(0-x', d-a(x'))|^2 > d^2. \tag{6.15}$$

Using

$$a(x') = \frac{1}{2}\langle D^2 a(0)x', x'\rangle + o(|x|^2),$$

(6.15) implies that

$$|x'|^2 - d\langle D^2 a(0).x', x'\rangle \geq C|x'|^4 + o(|x|^2).$$

This gives the required result.

Proof of Proposition 6.2.

We define a map on a neighborhood of $(0,d)$ as follows

$$\begin{array}{rcl} \Psi : B'(0,r) \times (d-\frac{d}{2}, d+\frac{d}{2}) & \longrightarrow & \Omega \\ (y,t) & \longrightarrow & (y - t\frac{\nabla a(y)}{(1+|\nabla a|^2)^{\frac{1}{2}}}\ ,\ a(y) + \frac{t}{(1+|\nabla a|^2)^{\frac{1}{2}}}). \end{array}$$

We want to prove that this map is invertible around $(0,d)$, since

$$D^2 a(y) < (1/d)I.$$

For that aim we introduce

$$X' = y - t\frac{\nabla a(y)}{(1+|\nabla a|^2)^{\frac{1}{2}}}$$

$$X_N = a(y) + \frac{t}{(1+|\nabla a|^2)^{\frac{1}{2}}}$$

and we prove that the Jacobian is non zero on $(0,0)$. A simple computation gives

$$\frac{\partial X_i}{\partial y_j} = \delta_{ij} - t\frac{a_{,ij}}{(1+|\nabla a|^2)^{\frac{1}{2}}} + \frac{t a_{,k} a_{,i} a_{,kj}}{(1+|\nabla a|^2)^{\frac{3}{2}}}$$

for all $i,j \in [1, N-1]$, and

$$\frac{\partial X_N}{\partial y_j} = a_{,j} + \frac{t(-a_{,k} a_{,kj})}{(1+|\nabla a|^2)^{\frac{3}{2}}}$$

$$\frac{\partial X_i}{\partial t} = -\frac{\nabla a}{(1+|\nabla a|^2)^{\frac{1}{2}}}$$

$$\frac{\partial X_N}{\partial t} = \frac{1}{(1+|\nabla a|^2)^{\frac{1}{2}}}.$$

From this one gets using $\nabla a(0) = 0$ that the Jacobian at $x_0 = (0,d)$ has the value of the determinant of the $N-1$ dimensional matrix $I - d(D^2 a)(0)$. The previous lemma implies that this determinant is strictly positive. Hence in a neighborhood of $(0,d)$ the map $\psi : (y,t) \mapsto (X', X_N)$ is invertible by the local inversion theorem.

Precisely there exists a neighborhood V_1 of $(0,d)$ such that for any $x \in V_1$ there exists a unique $y = y(x) \in B'(0,r)$ and a unique t such that: $d(x) = t = |x - (y(x), a(y(x)))|$. Clearly $y(x)$ is differentiable and

$$\nabla y(x_0) = (I - dD^2 a)^{-1}.$$

This ends the proof .

Corollary 6.3 *The distance d is $\mathcal{C}^2$ around every point x on which the distance is achieved in a unique point of the boundary . Moreover in our setting at x_o the eigenvalues of $D^2 d$ are 0 and the eigenvalues of $(D^2 a)(I - dD^2 a)^{-1}$.*

Proof of Corollary 6.3. We still consider the geometry and the notations of the proof of Proposition 6.2. It is easy to see that for $x = (x', x_N)$

$$y(x) = x' - (x_N - a(y(x))\nabla a(y(x)).$$

This will allow us to compute explicitly $D^2 d$. Indeed, one gets

$$\nabla y(x) = I - (x_N - a(y))D^2 a.\nabla y + (\nabla a \otimes \nabla a).\nabla y.$$

Hence, in particular for $x_o = (0, d)$ with $a(0) = 0$ and $\nabla a(0) = 0$ therefore $\nabla y(x_o) = I - dD^2 a.\nabla y(x_0)$.

Recalling that

$$\nabla d(x) = \frac{1}{d}(x' - y(x), x_n - a(y(x)))$$

then

$$D^2 d(x) = \frac{1}{d}\begin{pmatrix} I - \nabla y & -\nabla y.\nabla a(y) \\ O & 1 \end{pmatrix} - \frac{1}{d^3}(x - (y, a(y)) \otimes (x - (y, a(y)))$$

where $\nabla y.\nabla a = a_{,j} y_{j,i}$. Then if $x_o = (0, d)$, $\nabla a = 0$

$$D^2 d(x_o) = \begin{pmatrix} D^2 a \nabla y & 0 \\ 0 & \frac{1}{d} \end{pmatrix} - \frac{1}{d^3} x_o \otimes x_o$$

where $D^2 a(y)\nabla y$ is the usual product of matrices in $\mathbb{R}^{N-1}$. Clearly $D^2 d(x_o)x_o = 0$ so one of the eigenvalue is 0, while for any $x_1 = (x_1', 0)$ one has

$$D^2 d(x_o)x_1 = \begin{pmatrix} D^2 a \nabla y x_1' \\ 0 \end{pmatrix}.$$

Using the fact that $\nabla y(x_0) = (I - dD^2 a(0))^{-1}$ we get the result, choosing x_1' as an eigenvector of $D^2 a$. This conclude the proof.

For completeness sake let us recall that

Proposition 6.4 *Suppose that $x \in \Omega$ is such that the distance d to $\partial\Omega$ is achieved at least on two points. Then the set $J^{2,-}d(x) = \emptyset$.*

Proof of Proposition 6.4.

Suppose that $x = 0$ and let y_1 and y_2 be two distinct points in $\partial\Omega$ such that $d(0, \partial\Omega) = d = |0 - y_1| = |0 - y_2|$. It is sufficient to prove that $J^{2,-}d^2(0)$ is empty. Suppose that a and A are in $\mathbb{R}^N \times S^N$ such that for all x in a neighborhood of 0

$$d^2 + a.x +^t xAx \leq d(x, \partial\Omega)^2$$

In particular this must be satisfied for all $x = ty_1$ and $|t| < r$ small enough. This implies in particular

$$d^2 + t(a.y_1) + t^2(Ay_1, y_1) \leq \inf_{|t|<r}(|ty_1 - y_1|^2, |ty_1 - y_2|^2)$$

In particular one gets first

$$(a.y_1)t \leq -2d^2 t$$

which implies $a.y_1 = -2d^2$ and secondly one has

$$(a.y_1)t \leq -2(y_1.y_2)t$$

which implies that $(a.y_1) = -2(y_1.y_2) = -2d^2$, a contradiction since $y_1 \neq y_2$ implies that $y_1.y_2 \neq d^2$.

Proposition 6.5 *Let Ω be a bounded open $\mathcal{C}^2$ domain in $\mathbb{R}^N$. Then for all constant $\beta < 0$ there exists a function u which is a viscosity solution of*

$$\begin{cases} F(\nabla u, D^2 u) & \leq & \beta \text{ in } \Omega \\ u & = 0 & \text{on } \partial\Omega \end{cases}$$

$u = 0$ on the boundary.

Proof of proposition 6.5 According to the previous proposition, it is enough to consider a point x where d is achieved on only one point x_o. Hence we can consider that the setting is the one considered in the previous propositions.

Let $K > diam\Omega$. Then $d \leq K$. Let $\gamma \in]0,1[$ and let k be large enough to be chosen later. We construct the following function

$$u(x) = 1 - \frac{1}{(1 + d(x)^\gamma)^k}.$$

Clearly $u = 0$ on the boundary, u is $\mathcal{C}^2$ on the points where d is achieved on a unique point. One has

$$\nabla u = \frac{k\gamma d^{\gamma-1}\nabla d}{(1+d^\gamma)^{k+1}}$$

and

$$D^2 u = \frac{k\gamma d^{\gamma-2}}{(1+d^\gamma)^{k+2}}\left[(\gamma - 1 - (k+2-\gamma)d^\gamma)\nabla d \otimes \nabla d + d(1+d^\gamma)D^2 d\right].$$

We need to evaluate the eigenvalues of D^2u. Using the fact that $D^2 d\nabla d = 0$, we obtain that

$$D^2 u.\nabla d = \frac{k\gamma d^{\gamma-2}}{(1+d^\gamma)^{k+2}}(\gamma - 1 - (k+2-\gamma)d^\gamma)\nabla d,$$

hence $\frac{k\gamma d^{\gamma-2}}{(1+d^\gamma)^{k+2}}(\gamma - 1 - (k+2-\gamma)d^\gamma)$ is a negative eigenvalue of D^2u. While for $x_i = (x_i', 0)$ with x_i' being an eigenvector of $D^2 d$ with corresponding eigenvalue λ_i

$$D^2 u.x_1 = \frac{k\gamma d^{\gamma-1}}{(1+d^\gamma)^{k+1}}\frac{\lambda_i}{1 - d\lambda_i}x_1.$$

Choosing λ_1 to be the greatest eigenvalue of $D^2a(0)$, we obtain that

$$\mathcal{M}^+_{a,A}D^2u \leq \frac{k\gamma d^{\gamma-2}}{(1+d^\gamma)^{k+2}}\left[a(\gamma-1-d^\gamma(k+2-\gamma))+A(N-1)\frac{\lambda_1}{1-d\lambda_1}d(1+d^\gamma)\right]$$

and using that $d(1+d^\gamma) \leq (1+K^\gamma)K^{1-\gamma}d^\gamma$

$$\mathcal{M}^+_{a,A}D^2u \leq \frac{k\gamma d^{\gamma-2}}{(1+d^\gamma)^{k+2}}\left[a(\gamma-1)-d^\gamma\left(a(k+2-\gamma)-A(N-1)\frac{\lambda_1}{1-d\lambda_1}(1+K^\gamma)K^{1-\gamma}\right)\right].$$

We choose k such that

$$a(k+\gamma-2) \geq 2\left(A(N-1)\left|\frac{\lambda_1}{1-d\lambda_1}\right|(1+K^\gamma)K^{1-\gamma}\right).$$

Recalling that $|\nabla d| = 1$, that $\gamma \in (0,1)$ and $\gamma(\alpha+1)-(\alpha+2) < 0$ we have obtained that

$$\begin{aligned} F(\nabla u, D^2u) &\leq |\nabla u|^\alpha \mathcal{M}^+_{a,A}D^2u \\ &\leq \frac{(k\gamma)^{\alpha+1}d^{\gamma(\alpha+1)-(\alpha+2)}}{(1+d^\gamma)^{\alpha(k+1)+k+2}}\frac{1}{2}(\gamma-1-d^\gamma(k+2-\gamma)) \leq \beta < 0. \end{aligned}$$

This conclude the proof.

Lemma 6.6 *Let u be a positive bounded function inside Ω, $u = 0$ on $\partial\Omega$. Then, for all $z \in \partial\Omega$ such that u is not locally constant around z, there exists $C > 0$ and $\bar{x} \in \Omega$, such that $(2C(x-\bar{x}), -CI) \in J^{2,-}u(\bar{x})$*

Proof
Suppose that $C > \frac{2\sup u}{(d(z,\partial\Omega))^2}$ and consider

$$\inf_{x\in\bar{\Omega}}\{u(x)+C|z-x|^2\}$$

Let $\bar{x}$ be a point on which the infimum is achieved. If $\bar{x} \in \partial\Omega$, this contradicts the definition of C. Moreover since u is not locally constant around z the infimum cannot be achieved on z. Then one has for all x

$$u(x) \geq u(\bar{x}) + 2C(z-\bar{x}, x-\bar{x}) - C|x-\bar{x}|^2$$

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