Connected Permutations, Hypermaps and Weighted Dyck Words

Why?

- Graph embeddings
- Nice bijection by Patrice Ossona de Mendez and Pierre Rosenstiehl.
- Deduce enumerative results.
- Extensions?

Cycles (or orbits)

A *permutation* α is a sequence of n distinct integers $a_1, a_2, \ldots, a_n$ all such that $1 \leq a_i \leq n$. It is often useful to consider α as a one to one map from $\{1, 2, \ldots, n\}$ on to itself, denoting a_i by $\alpha(i)$.

A *cycle* is a sequence $b_1, b_2, \ldots, b_p$ of distinct integers such that $b_{i+1} = \alpha(b_i)$ for $1 \leq i < p$ and $b_1 = \alpha(b_p)$

Example : A permutation

$$\alpha = 7, 3, 4, 2, 1, 6, 5$$

and its cycles :

$(1, 7, 5)$ $(2, 3, 4)$ (6)

The set of all permutations (i. e. the symmetric group) is denoted $\mathcal{S}_n$.

Left-to-right maxima

Let $\alpha = a_1, a_2, \ldots, a_n$ be a permutation, a_i is a left-to-right maxima if $a_j < a_i$ for all $1 \leq j < i$.

Remarks :

- for any α, a_1 is a left-to-right maxima
- if $a_k = n$ then it is a left-to -right maxima
- the number of left-to-right maxima of a permutation α is equal to 1 if and only if $a_1 = n$.

Example

4 7 2 1 3 8 5 9 6

Example

4 7 2 1 3 8 5 9 6

Example

4 7 2 1 3 8 5 9 6

Example

4 7 2 1 3 8 5 9 6

Example

4 7 2 1 3 8 5 9 6

Bijection (Foata Transform)

The following algorithm describes a bijection from the set of permutations having k cycles to the set of permutations having k left-to-right maxima.

- Write the permutation α as a product of cycles $\Gamma_1, \Gamma_2, \ldots \Gamma_k$ in which the first element of each cycle Γ_i is the maximum among the elements of Γ_i
- Reorder the Γ_i such that the first elements of the cycles appear in increasing order.
- Delete the parenthesis around the cycles.

Example

4 7 2 1 3 8 5 9 6

= (1, 4) (2, 7, 5, 3) (6, 8 9)

Example

4 7 2 1 3 8 5 9 6

= (1, 4) (2, 7, 5, 3) (6, 8, 9)

= (1, 4) (2, 7 , 5, 3) (6, 8, 9)

= (4, 1) (7, 5, 3, 2) (9, 6, 8)

Example

theta =

4 7 2 1 3 8 5 9 6

= (4, 1) (7, 5, 3, 2) (9, 6, 8)

F (theta) =

4, 1, 7, 5, 3, 2, 9, 6, 8

Enumeration

The number of permutations $s_{n,k}$ of $\mathcal{S}_n$ having k cycles is equal to the coefficient of x^k in the polynomial :

$$A_n(x) \quad = x(x+1)(x+2)\cdots(x+n-1)$$

Proof: In order to build the permutations of $\mathcal{S}_n$ with k cycles, we can start with permutations from $\mathcal{S}_{n-1}$ having $k-1$ cycles and add one cycle containing only n, or with the permutations from $\mathcal{S}_{n-1}$ having k cycles and add n inside one of its cycles. This second construction gives $n-1$ permutations for each permutation of $\mathcal{S}_{n-1}$, hence :

$$s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}$$

Multiplying each equality by x^k and summing up we get :

$$A_n(x) = xA_{n-1}(x) = (n-1)A_{n-1}(x)$$

giving the result.

Stirling numbers

1				
1	1			
2	3	1		
6	11	6	1	
24	50	35	10	1

Connected Permutations

- Seems to be considered for the first time by André Lentin (Thesis, 1969) then by Louis Comtet (note aux Comptes Rendus Acad Sci Paris (1972))

Définition A permutation $\alpha = a_1, a_2, \ldots, a_n$ is connected if it does not contain a left factor (of length p, $0 < p < n$) which is a a permutation of $1, 2, \ldots p$.

Exemple For $n = 3$, there are 3 connected permutations : $2, 3, 1 \quad 3, 2, 1$ and $3, 1, 2$, there are also 3 non connected permutations : $1, 2, 3 \quad 1, 3, 2$ and $2, 1, 3$.

The permutations $2, 4, 1, 3$ and $3, 1, 4, 2$ are connected.

- **The numbers of connected permutations** $1, 1, 3, 13, 71, 461, 3447, \ldots.$

First formula

- Any non connected permutation is the concatenation of a connected permutation on $1, 2, \ldots p$ and a permutation on $p+1, \ldots n$ where : $1 \leq p < n$.
- hence :

$$n! - c_n = \sum_{p=1}^{n-1} c_p \, (n-p)!$$

- Allows us to compute the first terms

Generating functions

$$Fact(x) - C(x) = C(x) Fact(x) \quad \text{where} \quad Fact(x) = \sum_{n \geq 1} n! x^n$$

Giving :

$$C(x) = \frac{Fact(x)}{Fact(x) + 1} = 1 - \frac{1}{1 + Fact(x)}$$

Another formula (Lentin)

A permutation β of $\mathcal{S}_{n-1}$ is k-connectable if there are exactly k positions in β where inserting n gives a connected permutation.

Remarks

- if the insertion of n in position j gives a connected permutation then this is also the case for any insertion in position $i < j$
- Any permutationβ, is p-connectable for some $p \geq 1$
- A permutation is 1-connectable if and only if the first element is 1
- A permutation on $1, 2, \ldots, n-1$ is $(n-1)$ connectable if and only if it is connected.

Another formula (2)

Proposition For $1 \leq k < n$ the number $u_{n,p}$ of p-connectable permutations on $1 \ldots n-1$ is equal to :

$$c_p(n-p-1)!$$

Proof: If a permutation is the concatenation of a connected permutation of length p and of a permutation of length $n-1-p$ then it is p-connectable.

Corollary:

$$c_n = \sum_{p=1}^{n-1} p c_p (n-1-p)!$$

Moreover, the number of connected permutations on $1, 2, \ldots, n$ such that 1 is in position p is given by :

$$\sum_{k=1}^{p-1} c_{n-k}(k-1)!$$

Foata transform

Proposition α is connected if and only if its Foata transform is connected

Consequence The number of connected permutations with k cycles is equal to the number of connected permutations with k left-to-right maxima

Why these permutations?

- Basis for the Hopf Algebra of permutations introduced by Malvenuto and Reutenauer (see also Aguilar Sottile, 2004)
- Counting some configurations in statistical physics
- Maps and Hypermaps

Number of connected permutations with k left-to-right maxima

$$c_{n,k} = \sum_{i=1}^{n-1} \sum_{p=1}^{k} i c_{i,p} s_{n-i-1,k-p}$$

where $s_{m,j}$ is the number of permutations of $\mathcal{S}_m$ with j left-to-right maxima.

$$C_n(x) = \sum_{k=1}^{n-1} k A_{n-1-k}(x) C_k(x)$$

Connected Stirling numbers

1					
2	1				
6	6	1			
24	34	12	1		
120	210	110	20	1	
720	1452	974	270	30	1

Maps

- A (non-oriented) graph $G = (V, E)$ consists of a set V of vertices and a set E of edges, each edge is a subset of V of cardinality 2.
- Each edge gives two arcs, one attached to each vertex contained in it
- An embedding of G in an orientable surface determines a circular order of the arcs incident to each vertex
- This gives a permutation σ on the arcs which cycles consists of the circular order on each vertex
- The edge set defines a fixed point free involution α on the set of arcs, each edge determining a cycle of α consisting of the two arcs associated with it.
- The graph is connecetd if and only if the subgroup generated by α, σ is connected.

Hypermaps

- Pair of permutations σ, α on $B = \{1, 2, \ldots, n\}$ such that the group they generate is transitive on B
- This means that the graph with vertex set B and with the set of edges consists of $\{b, \alpha(b)\}, \ \{b, \sigma(b)\}$ is connected .
- The cycles of σ are the vertices of the hypermap, and those of α the edges.

From connected permutations to Hypermaps

We show the following result : Ossona de Mendez, Rosenstiehl

Theorem There exists a bijection between the set of connected permutations on $1, 2, \ldots, n, n+1$ and the set of (rooted) hypermaps on : $1, 2, \ldots, n$.

A hypermap σ, α is associated to a connected permutation $\theta = a_0, a_1, a_2, \ldots, a_n$ by the following algorithm :

- Détermine the left-to-right maxima of θ, that is the indices such that $i_1, i_2, \ldots i_k$ satisfying : $j < i_p \Rightarrow a_j < a_{i_p}$

  $i_1 = 1 \quad a_{i_k} = n$

- The cycles decomposition of σ is then :

$$(1, 2, \ldots i_2 - 1)(i_2, i_1 + 1, \ldots i_3 - 1) \ldots (i_k \ldots, n)$$

- The permutation α is obtained from θ deleting $n+1$ from its cycle (note that this cycle is of length not less than 2).

Example

4 7 2 1 3 8 5 9 6

Example

4 7 2 1 3 8 5 9 6

sigma1 =

(1)(2, 3, 4, 5)(6,7)(8,9)

Example

4 7 2 1 3 8 5 9 6

sigma1 =

(1)(2, 3, 4, 5)(6,7)(8,9)

theta =

(1, 4) (2, 7, 5, 3) (6, 8, 9)

Example

4 7 2 1 3 8 5 9 6

sigma =

(1)(2, 3, 4, 5)(6,7)(8)

theta =

(1, 4) (2, 7, 5, 3) (6, 8, 9)

Example

4 7 2 1 3 8 5 9 6

sigma =

(1)(2, 3, 4, 5)(6,7)(8)

alpha =

(1, 4) (2, 7, 5, 3)(6,8)

Characterization of the hypermaps obtained

Any hypermap (σ, α) obtained from a connected permutation by the algorithm described above satisfies the following conditions :

- The cycles of the permutation σ consist of consecutive integers in increasing order.
- The seft of right-to-left minima of α^{-1} contains the smallest element of each cycle of σ except possibly the smallest of the last one.

Rooted Hypermaps

Theorem For any hypermap (σ, α) there is an isomorphism ϕ such that $\phi(n) = n$, and such that the hypermap (σ', α') given by :

$$\alpha' = \phi\alpha\phi^{-1} \quad \sigma' = \phi\sigma\phi^{-1}$$

satisfies the conditions above.

Enumeration by number of vertices

- The number of rooted hypermaps with n arcs and p vertices is equal to the number of connected permutations of $\mathcal{S}_{n+1}$ with p cycles, or the number of such permutations with p left-to-right maxima.

Enumeration by number of vertices and edges

Theorem

- The number of rooted hypermaps with n arcs, p vertices and q edges is equal to the number of connected permutations of $\mathcal{S}_{n+1}$ with p cycles, and q left-to-right maxima.

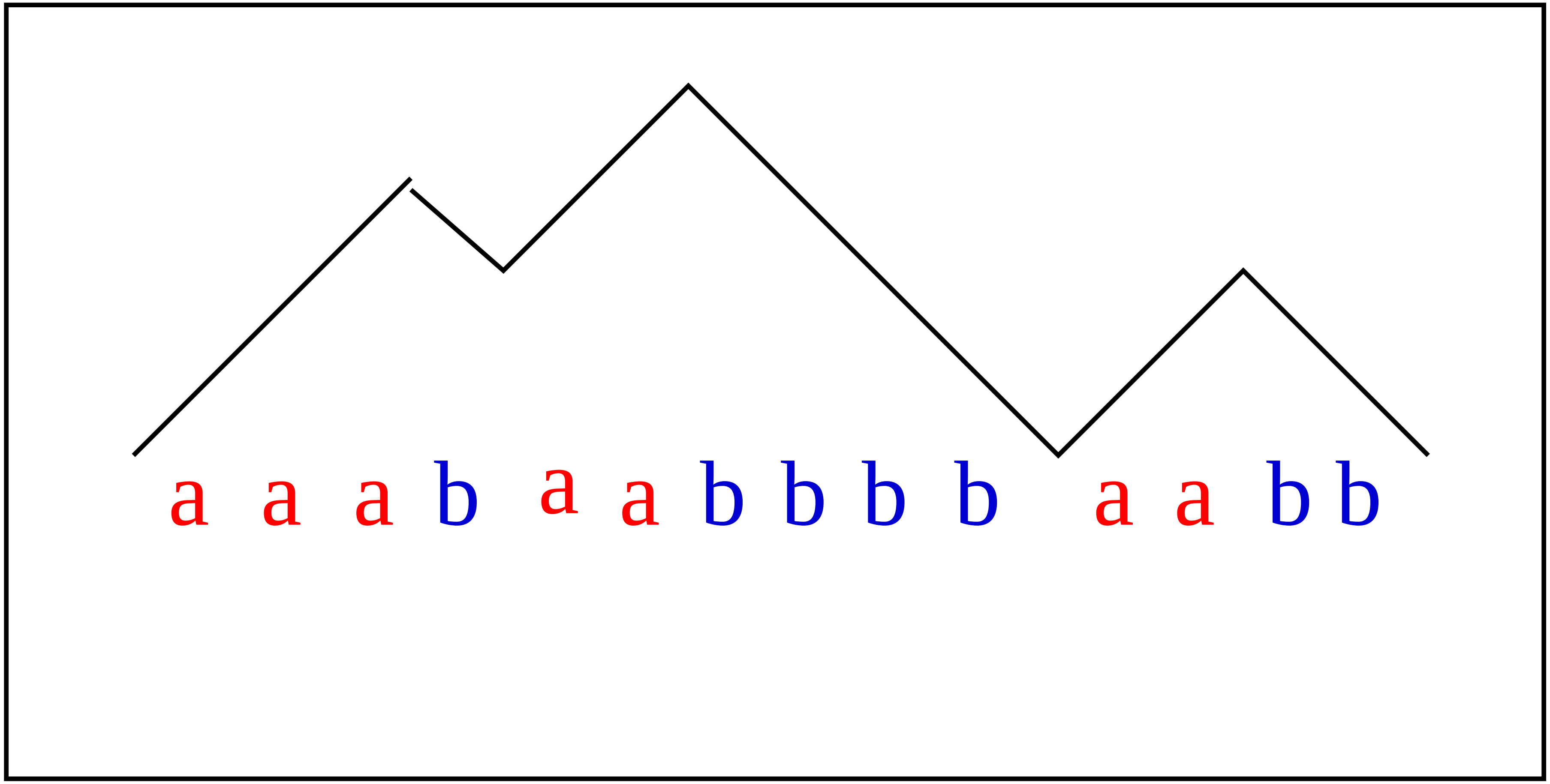
Weighted Dyck words (or paths)

- A Dyck word is a sequence w of letters a and b, having as many a's as b's, and such that for any left factor the number of a's is not less that the number of b's.
- We will write

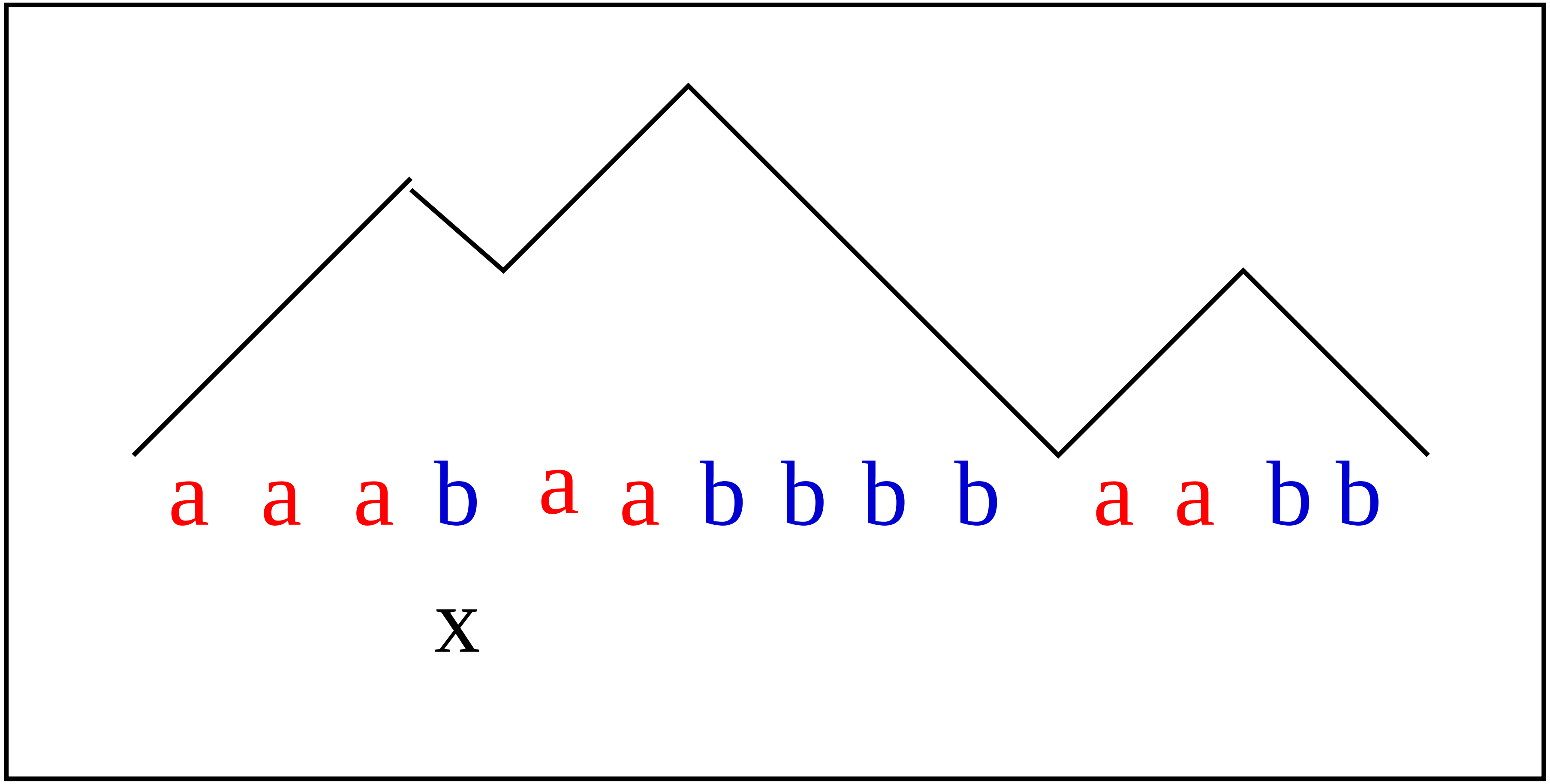
$$|w|_a = |w|_b \quad w = w'w" \Rightarrow |w'|_a \geq |w'|_b$$

- We associate a polynomial $\lambda(w)$ in two variables x, y to each Dyck word by associating to each letter b appearing in w a polynomial of degree 1 λ_i and then taking the product of these λ_i
- For each decomposition $w = w'_i b w"_i$, $\lambda_i = x$ if w' ends with an a and $\lambda_i = y + h_i$ when w' ends with an b, where $h_i = |w'_i b|_a - |w'_i b|_b$

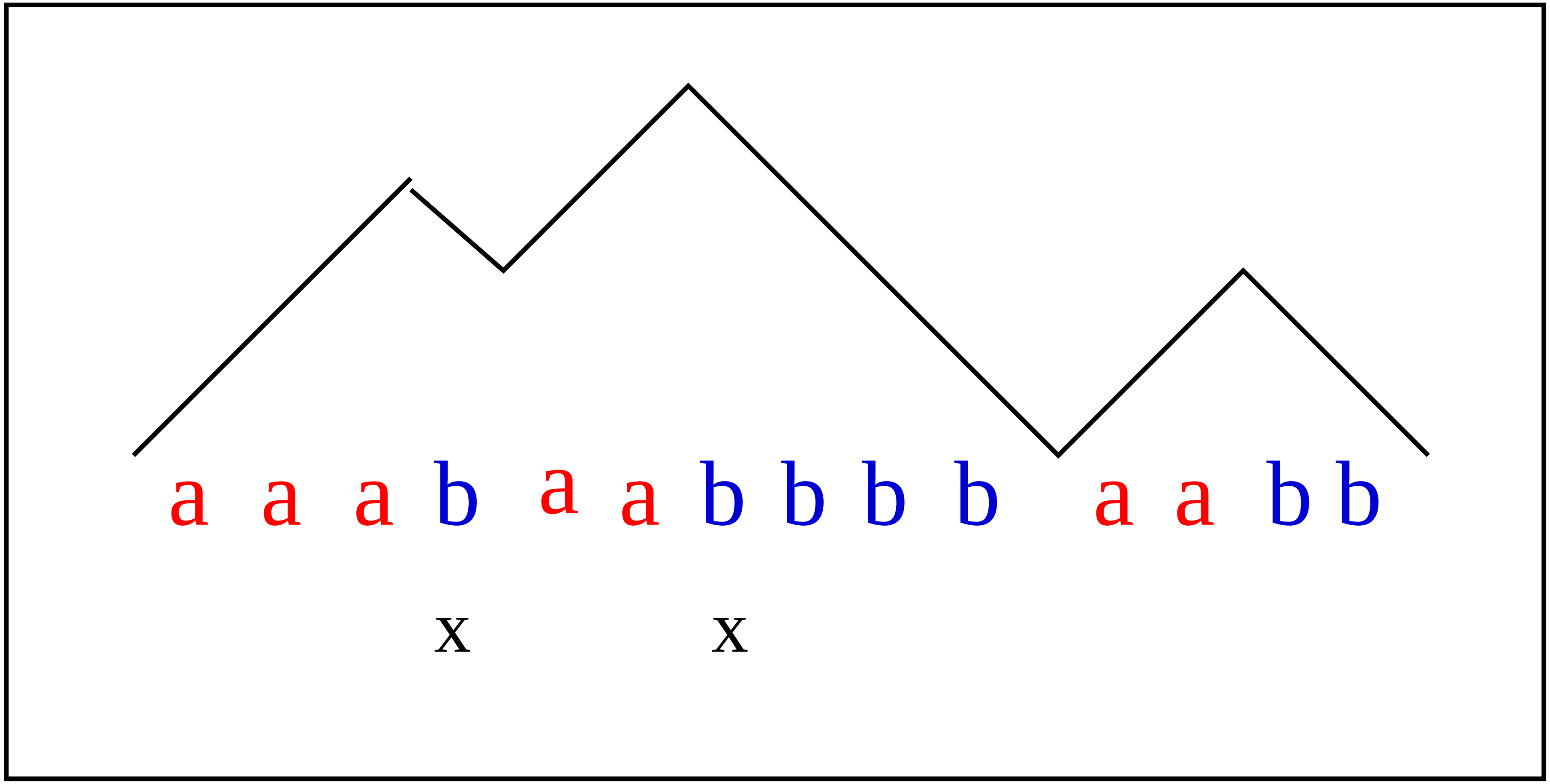
Example

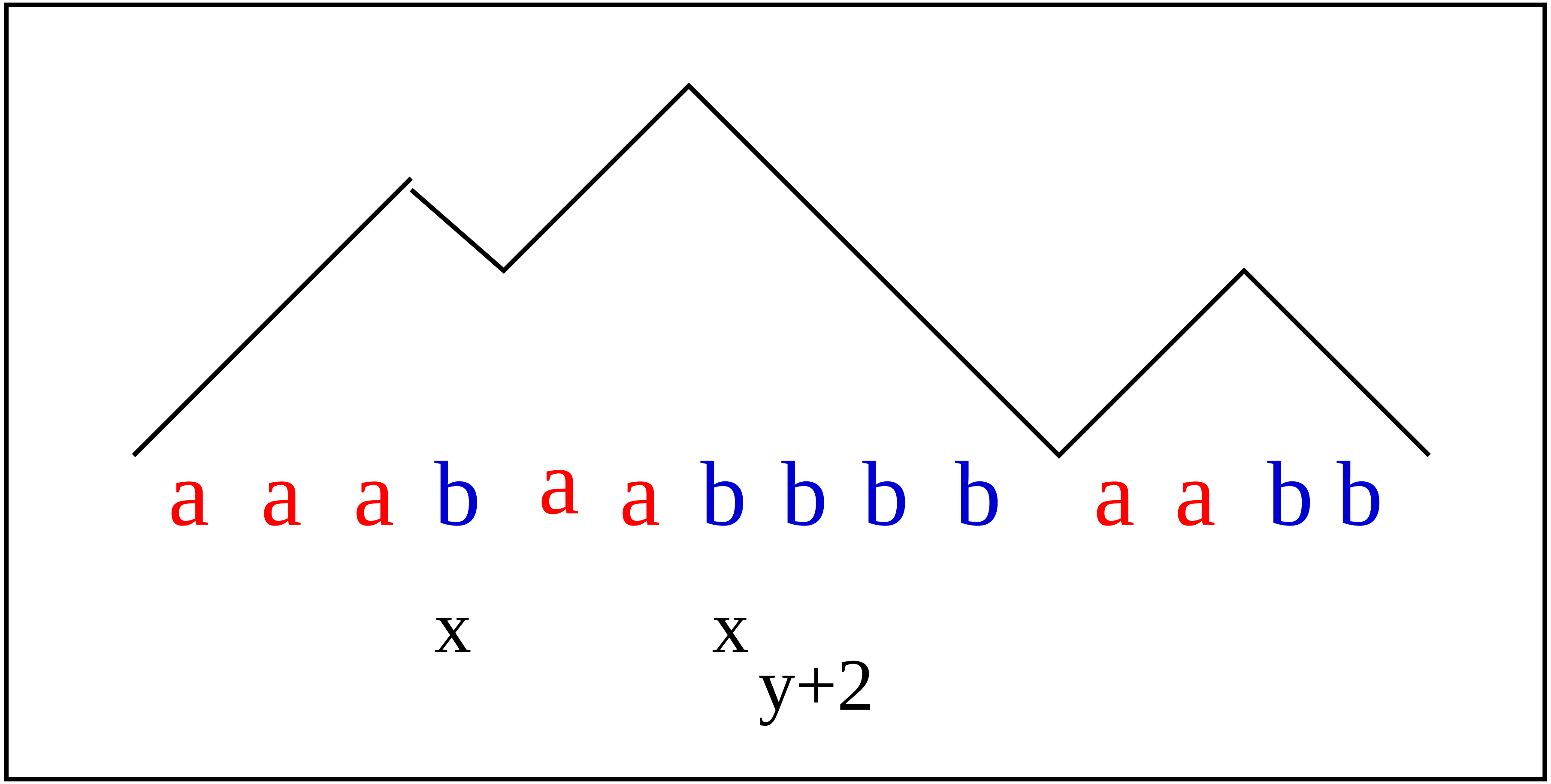


Example

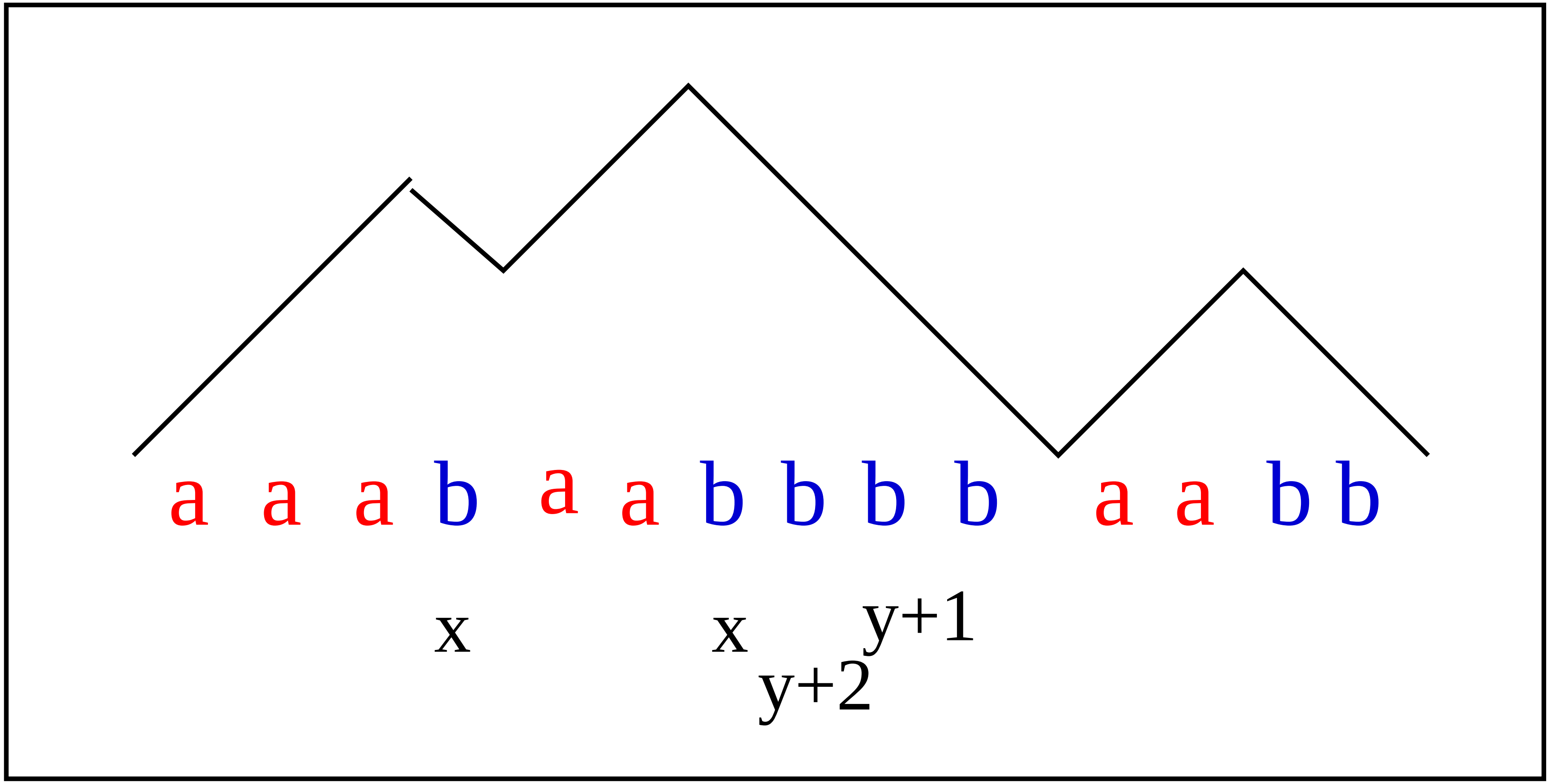


Example




a a a b a a b b b b a a b b
x
x
y+2

Example



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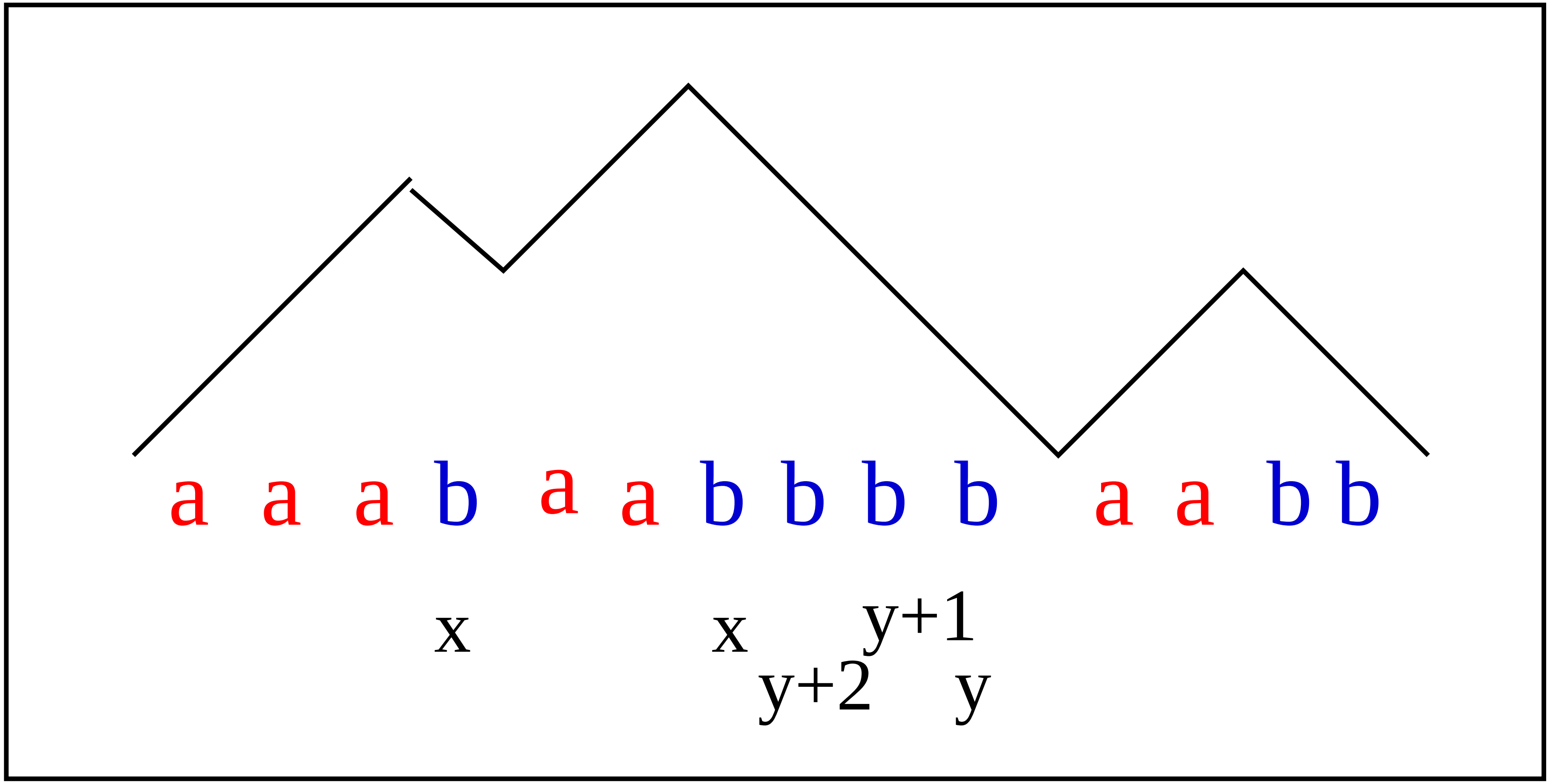

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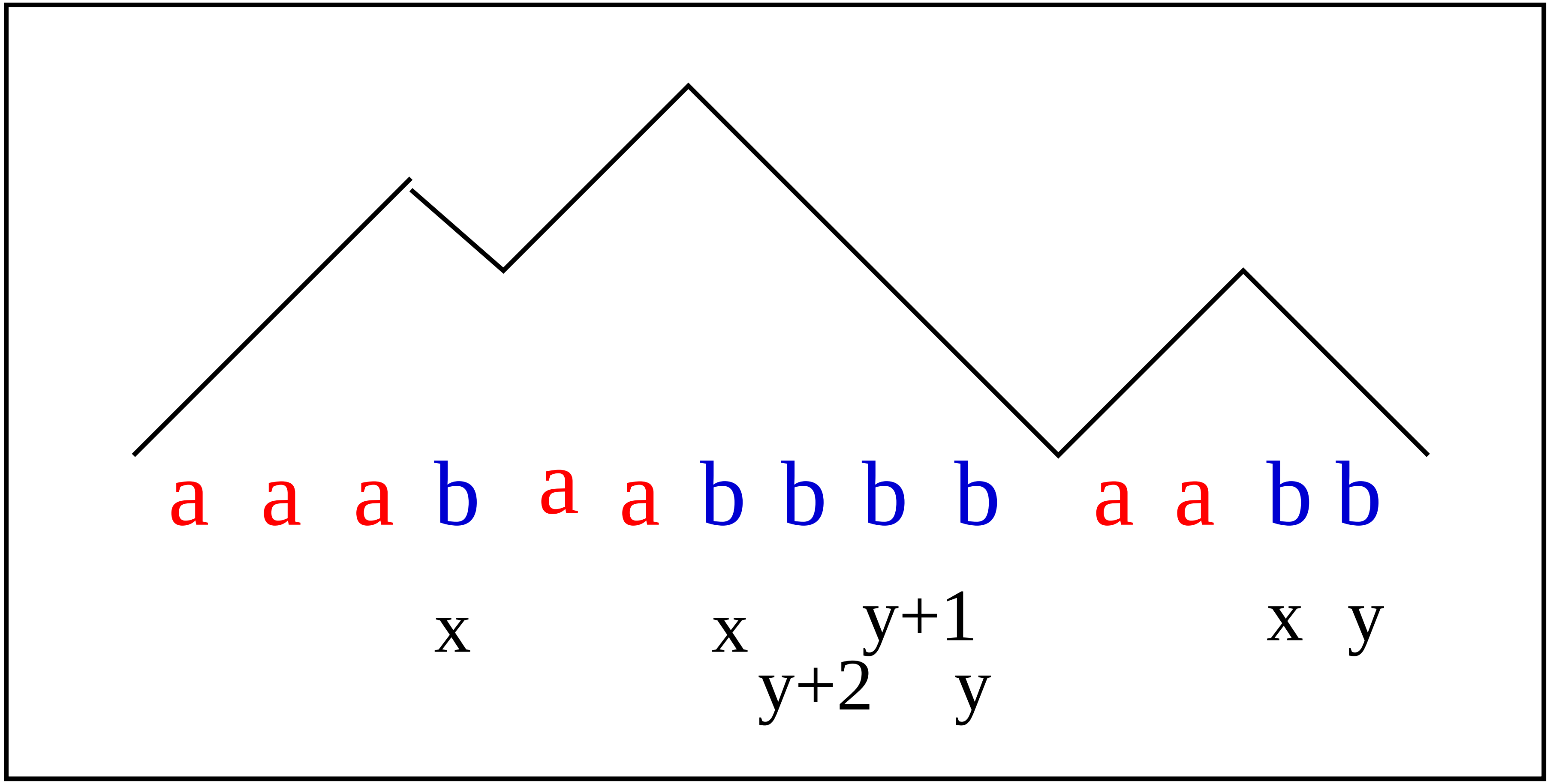

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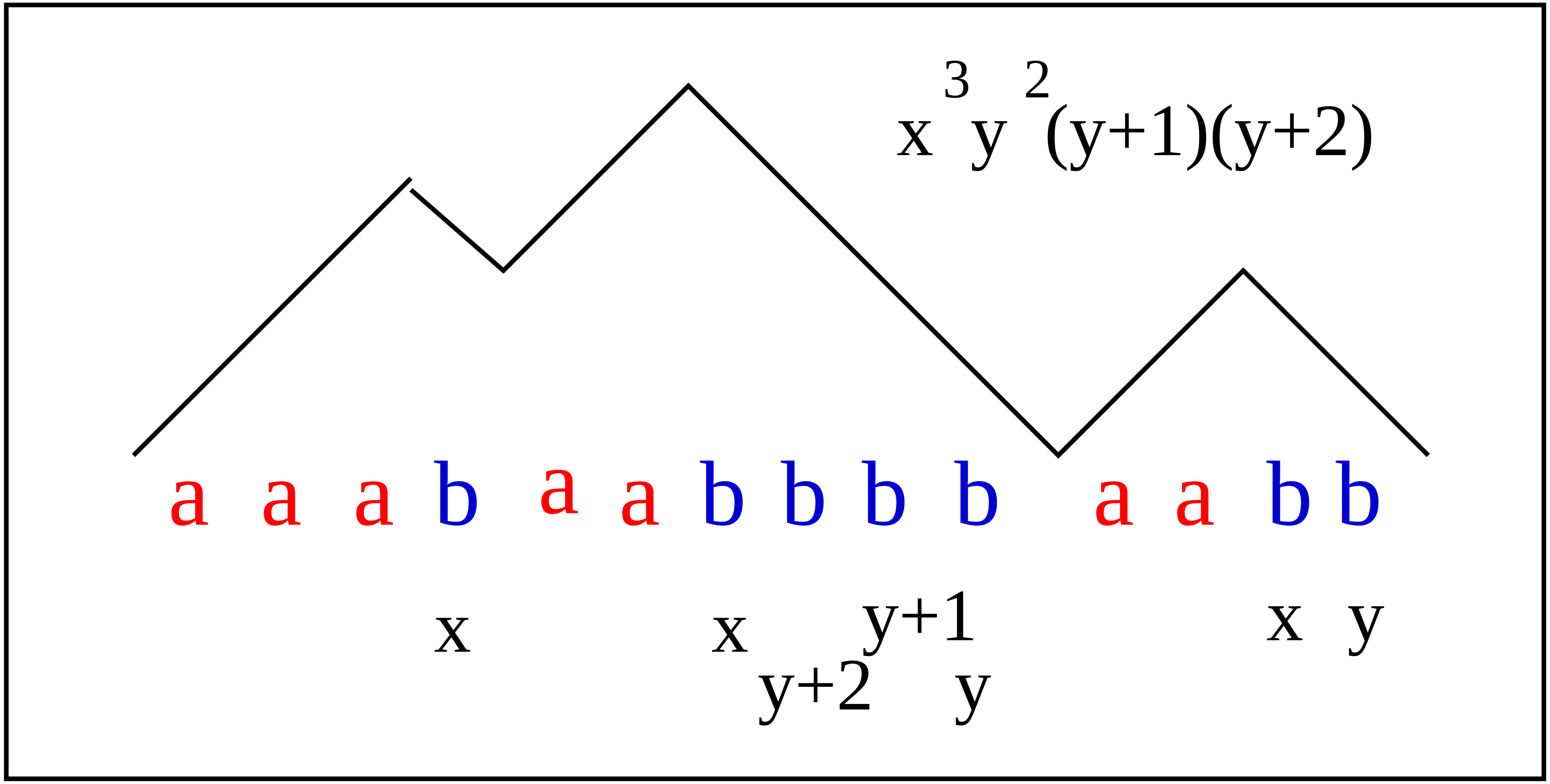
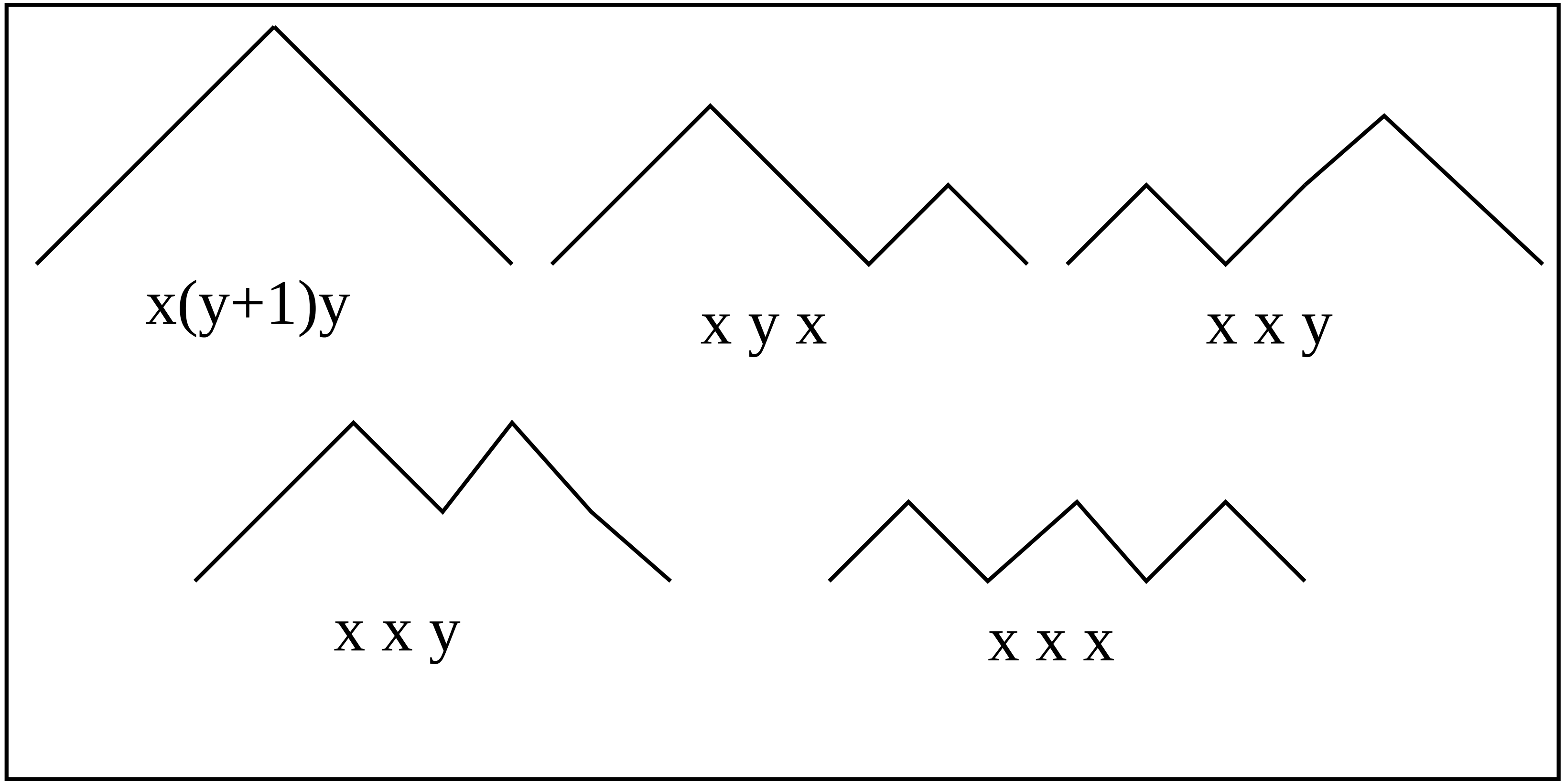


The polynomial $L_n(x, y)$

This polynomial is the sum of the $\lambda(w)$ for all Dyck words w of length $2n$.



Stirling numbers again

We have :

$$L_3(x,y) = x^3 + 3x^2y + xy^2 + xy$$

$$L_3(x,y) = x[(x+1+y)^2 + (x+1)y - 2(x+1+y) + 1]$$

For all n :

$$L_n(x,1) = x(x+1)(x+2)\cdots(x+n-1)$$

$$L_3(x,1) = x(x^2+3x+2)$$

Proof :
Bijection between permutations and labelled Dyck words

From permutations to labelled Dyck words

To any permutation θ of $\mathcal{S}_n$ we associate a labelled Dyck word by the following algorithm :

- Consider
-

Restriction to primitive Dyck Words

This polynomial $L'_n(x,y)$ is the sum of the $\lambda(w)$ for all primitive Dyck words w of length $2n$.

This gives for instance :

$$L'_3(x,y) = x^2y + xy^2 + xy$$

Number of hypermaps with p vertices and q edges.

Theorem

For all n we have. The number of hypermaps with n arcs, p vertices and q edges is given by the coefficient of $x^p y^q$ in $L'_n(x, y)$

Corollary :

The polynomial $L'_n(x, y)$ is symmetric in x, y

Le genre?

- Il est difficile de voir le genre de l'hypercarte sur la permutation connexe associée
- Une des raisons est que l'algorithme d'obtention de θ procède par parcours en largeur, alors que le genre est reflété par le parcours en profondeur (voir algorithme de Tarjan et mon algo de codage).
- On pourrait probablement caractériser le genre en faisant intervenir un algorithme de parcours en profondeur, mais on perdrait très probablement la caractérisation du nombre de sommets et du nombre d'arêtes