Geometric issues in PDE problems related to the infinity Laplace operator

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Abstract. We review some recent results related to the homogeneous Dirichlet problem for the infinity Laplace equation with constant source in a bounded domain. We characterize the geometry of domains for which an overdetermined problem admits a viscosity solutions. An essential tool is a regularity result for viscosity solutions in convex domains, obtained by the convex envelope method introduced by Alvarez, Lasry and Lions.

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1 Introduction

Our primary interest in PDE problems for the infinity Laplacian operator raised from the following overdetermined problem

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

whose study was firstly proposed in [6].

Let us recall that the infinity Laplacian is the strongly non-linear and highly degenerated differential operator defined for smooth functions u by

$$\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u\,.$$

It was firstly discovered by Aronsson in the sixties in connection with the so-called “absolutely minimizing Lipschitz extensions" and later in the nineties a fundamental advance concerning existence and uniqueness of solutions came by Jensen. In the last decade, also due to their connection with tug-of-war games, boundary value problems involving the infinity Laplace operator have received a great impulse thanks to the contribution of several authors; without any attempt of completeness, let us quote the papers [2, 3, 4, 12, 13, 20, 21, 22, 24], where the reader may find further related references.

On the other hand, starting from the fundamental paper by Serrin [23], overdetermined problems of the type (1.1) have been studied for many operators (the basic examples being the Laplace and p–Laplace operator, see for instance [23, 16, 11, 5, 15]), not including the infinite Laplacian operator. In all these cases it is known that, if the overdetermined problem (1.1) admits a solution, then Ω is a ball.

An intriguing discovery is that this is not the case for the infinity Laplacian, unless more regularity (and topological) assumptions are required on the domain Ω.

Motivated by the aim of characterizing the shape of domains where problem (1.1) admits a solution, we were led to study a number of geometrical and regularity matters, going from the concavity properties of the unique solution to the Dirichlet problem given by the first two equations in (1.1), to the study of sets with positive reach and empty interior in $\mathbb{R}^n$.

In this paper we review our achievements on these topics to this day. Our choice is in favor of an intuitive presentation: though the results are rigorously stated, they are introduced in an informal way, enlightening the main ideas and avoiding all technicalities. In this spirit, we invoke more than once heuristic arguments, and we limit ourselves to sketch the proofs, referring for all details to the original papers.

The outline of the paper is as follows.

In Section 2 we recall some basic facts concerning existence, uniqueness and regularity for the homogeneous Dirichlet problem with constant source term.

In Section 3 we deal with a simplified version of problem (1.1) where solutions are searched in the family of functions having prescribed level lines, and precisely the same level lines as the distance function from $\partial\Omega$. Studying the problem in this setting leads to introduce a class of domains, that we call "stadium-like", for which the cut locus agrees with the set of maximal distance from the boundary.

In Section 4 we present the geometric results we obtained for stadium-like domains, which rely on a new classification of closed sets with positive reach and empty interior. These results are essentially 2-dimensional.

In Section 5 we deal with problem (1.1) in its general and quite challenging formulation.

To pursue our attempt of showing that the field is extremely rich, and many relevant questions remain unsolved, we conclude the paper with a short section of open problems.

2 On the Dirichlet problem

In this section we briefly discuss the Dirichlet problem for the infinity Laplace equation with constant source term:

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We begin with a basic example in order to get a feeling with the problem and motivate the use of viscosity solutions.

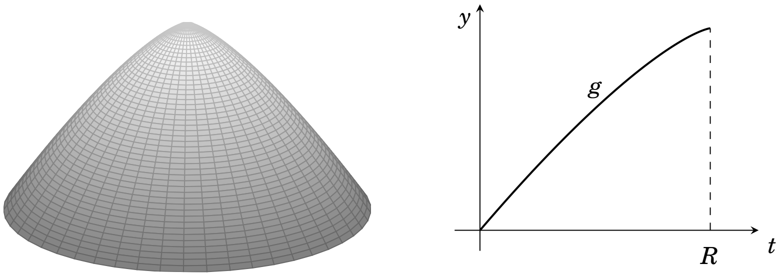

Figure 1. Radial solution of the Dirichlet problem (2.1)

Example 2.1. Let $\Omega = B_R(0)$ be the ball of radius R centered at the origin. Let us look for a radial solution to problem (2.1) of the form $u(x) = g(R-|x|)$, where $g\colon [0, R] \to \mathbb{R}$ is a continuous function, of class C^2 in the interval $(0, R)$. The Dirichlet boundary condition gives $g(0) = 0$. On the other hand, if we want u to be differentiable at $x = 0$ (which is *a posteriori* justified by Theorem 2.2 stated hereafter) we have to require that $g'(R) = 0$. Hence, we have to solve the following one-dimensional boundary value problem for the function g:

$$-\Delta_\infty u(x) = -g''(R - |x|)\,[g'(R - |x|)]^2 = 1, \qquad g(0) = 0, \quad g'(R) = 0.$$

We easily get

$$g(t) = c_0[R^{4/3} - (R - t)^{4/3}], \qquad t \in [0, R] \quad (c_0 = 3^{4/3}/4)$$

(see Figure 1). The function $u(x) = g(R - |x|)$ is of class $C^{1,1/3}(B_R) \cap C^2(B_R)$. This shows that there are no radial solutions of class $C^2(B_R)$.

We shall turn back to the lackness of classical (*i.e.* C^2) solutions for the Dirichlet problem (2.1) in arbitrary domains in Section 5.

By the moment, we limit ourselves to consider the above example as a heuristic explanation why solutions to problem (2.1) cannot be expected to be classical. Moreover, we observe that also the notion of weak solutions is ruled out, because the equation is fully nonlinear and cannot be written in divergence form. In fact, the right notion of solution to problem (2.1) is the one of viscosity solution. We shortly recall it below, for the benefit of the reader, referring to [8] for more details.

A *viscosity subsolution* to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C(\Omega)$ which, for every $x_0 \in \Omega$, satisfies

$$-\Delta_\infty \varphi(x_0) - 1 \leq 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } x_0, \tag{2.2}$$

or equivalently

$$-\langle Xp, p\rangle - 1 \leq 0 \quad \forall (p, X) \in J^{2,+}_\Omega u(x_0)\,. \tag{2.3}$$

Here the second order super-jet $J^{2,+}_{\Omega} u(x_0)$ of a function $u \in C(\overline{\Omega})$ at a point $x_0 \in \Omega$ denotes the set of pairs $(p, A) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\text{sym}}$ such that

$$u(y) \leq \; u(x_0) + \langle p,\, y - x_0 \rangle + \frac{1}{2} \, \langle A(y - x_0),\, y - x_0 \rangle + o(|y - x_0|^2)$$

as $y \to x_0$, $y \in \Omega$.

Similarly, a *viscosity super-solution* to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C(\Omega)$ which, for every $x_0 \in \Omega$, satisfies

$$-\Delta_\infty \varphi(x_0) - 1 \geq 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local minimum at } x_0, \tag{2.4}$$

or equivalently

$$-\langle Xp, p\rangle - 1 \geq 0 \quad \forall (p, X) \in J^{2,-}_{\Omega} u(x_0) \tag{2.5}$$

(the second order sub-jet $J^{2,-}_{\Omega} u(x_0)$ is defined analogously to the super-jet with the inequality reversed).

Finally, a *viscosity solution* to problem (2.1) is a function $u \in C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and u is a viscosity solution to $-\Delta_\infty u = 1$ in Ω, meaning it is both a viscosity sub-solution and a viscosity super-solution on Ω, according to the above definition.

We are now in a position to recall the basic known facts concerning existence, uniqueness, and regularity for viscosity solutions to problem (2.1).

Theorem 2.2 (Basic properties of viscosity solutions to (2.1)). *The Dirichlet problem* (2.1) *admits a unique viscosity solution u. Moreover, u is differentiable at every point of Ω.*

Both existence and uniqueness of viscosity solution have been obtained by Lu and Wang in [21], by adapting the nowadays standard approach for viscosity solutions of nondegenerate second order fully nonlinear equations. In particular, existence is obtained by Perron's method, while uniqueness is a consequence of the following comparison principle.

Theorem 2.3 (Comparison principle). *Let $u, v \in C(\overline{\Omega})$ be respectively viscosity sub- and super-solutions of $-\Delta_\infty u = 1$ in Ω. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

The fact that the unique solution u to (2.1) is differentiable everywhere has been recently proved by Lindgren [20], by adapting the method of Evans and Smart [13] for infinity harmonic functions.

3 On the overdetermined problem: the simple (web) case

In this section we consider a simplified version of the overdetermined problem (1.1) and we introduce a class of domains where such a simplified version turns out to admit a solution.

To follow an intuitive approach, let us present a heuristic argument. Assume for a moment that u is a smooth solution to (1.1), and consider the *gradient flow* associated with u, *i.e.* the flow generated by the ordinary differential equation

$$\dot{x}(t) = \nabla u(x(t)).$$

Solutions of this differential equation will be called *characteristics*. If $x(t)$, $t \in [0, T)$, is a characteristic, and if $\varphi(t) := u(x(t))$ denotes the restriction of u along this solution, we have

$$\begin{aligned} \dot{\varphi}(t) &= |\nabla u(x(t))|^2, \\ \ddot{\varphi}(t) &= 2 \left\langle D^2 u(x) \nabla u(x), \, \nabla u(x) \right\rangle = 2\Delta_\infty u(x) = -2 \end{aligned}$$

i.e., $\varphi(t) = \varphi(0) + \dot{\varphi}(0)\, t - t^2$. Moreover, if $x(0) = y \in \partial\Omega$, from the conditions $u(y) = 0$ and $|\nabla u(y)| = c$ we can determine explicitly φ as

$$\varphi(t) = \sqrt{c}\, t - t^2 \, . \tag{3.1}$$

On the other hand, from this information we cannot reconstruct the expression of the solution u, because in general we do not know the geometry of characteristics, which clearly depends on the solution itself!

However, there is a special case when this geometry is explicitly known, namely when the function u belongs to the following class:

Definition 3.1 (Web-functions). We say that u is a *web function* if it only depends on the distance d from the boundary of $\partial\Omega$, that is it can be written for some function w as $u(x) = w(d(x))$.

As we are going to realize immediately, when dealing with problem (1.1) within the class of web-functions, there are two subsets of $\overline{\Omega}$ related with the geometry of d which turn out to play a crucial role. We introduce them below:

Definition 3.2 (Cut locus and high ridge). The *cut locus* $\overline{\Sigma}(\Omega)$ of Ω is the closure in $\overline{\Omega}$ of the set $\Sigma(\Omega)$ of points of non differentiability of d. The *high ridge* $\mathrm{M}(\Omega)$ of Ω is the set where d achieves its maximum over $\overline{\Omega}$ (called the inradius ρ_Ω of the set Ω).

Figure 2 shows the cut locus and the high ridge when Ω is a rectangle.

Observe now that, for a generic domain Ω, if u is a web-function, ∇u is parallel to ∇d, and hence the characteristics of u are line segments normal to the boundary. More precisely, a characteristic is a line segment which starts at a point of the boundary, is normal to the boundary itself, and reaches a point of the cut locus (for instance, some characteristics of a web function on a rectangle are the dotted line segments in Figure 2).

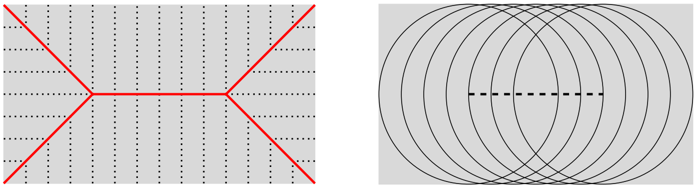

Figure 2. Cut locus (solid), characteristics (dotted), high ridge (dashed)

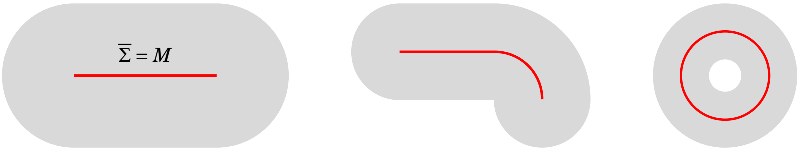

Figure 3. Stadium–like domains

Moreover, if u is written as $w(d)$, we have $|\nabla u(y)| = w'(0)$ for every $y \in \partial\Omega$, so that the condition $|\nabla u| = c$ on $\partial\Omega$ is automatically satisfied, with $c = w'(0)$. Thus, asking that the unique viscosity solution to problem (2.1) is a web function we immediately get a solution to the overdetermined problem (1.1).

By arguing as in Example 2.1, namely solving a one-dimensional boundary value problem for the function w, we obtain

$$w(t) = c_0(R^{4/3} - (R-t)^{4/3}) \qquad c_0 := \frac{3^{4/3}}{4}, \quad R := \frac{c^3}{3}.$$

If we now impose that u is differentiable, we find that all characteristics must have the same length R, and that this length R must coincide with the inradius ρ_Ω.

In other words, the requirement that all characteristics must have the same length is equivalent to ask a precise geometric condition on Ω, which is the coincidence between cut locus and high ridge. Accordingly, we set the following

Definition 3.3 (Stadium–like domains). A set $\Omega \subset \mathbb{R}^n$ is said to be a *stadium–like domain* if $\mathrm{M}(\Omega) = \overline{\Sigma}(\Omega)$.

Clearly, the rectangle is not a stadium-like domain. Some examples of stadium-like domains are represented in Figure 3.

The heuristic arguments presented above can be made rigorous and yield the following result. It has been proved in [6] in the regular case (for C^1 solutions and C^2 domains) and in [10] in the general case (with no regularity assumption on u and Ω).

Theorem 3.4 (Web–viscosity solutions). *The unique viscosity solution to problem* (2.1) *is a web function if and only if* Ω *is a stadium–like domain. In this case, the web–*

viscosity solution is given by

$$u(x) = \psi_\Omega(x) := g(d(x)) = c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d(x))^{4/3}\right] . \tag{3.2}$$

4 On stadium-like domains

In view of Theorem 3.4, a natural question is whether and how is it possible to characterize the geometry of stadium-like domains. A complete classification of them has been given in [9] in dimension $n = 2$; a similar statement in higher dimensions has been proved until now only under the convexity assumption. To prepare our results we have to recall the fundamental notion of set of positive reach introduced by Federer in [14].

Definition 4.1 (Set of positive reach). Let $S \subset \mathbb{R}^n$ be a nonempty closed set, and let d_S denote the distance function from S. We say that S is a set of *positive reach* if there exists $r_S > 0$ (called radius of proximal smoothness) such that every point of the tubular neighbourhood

$$\{x \in \mathbb{R}^n :\ 0 < d_S(x) < r_S\} \tag{4.1}$$

has a unique projection on S.

Federer himself proved that S has positive reach if and only if S is *proximally* C^1, which means that the distance function d_S is of class C^1 in a tubular neighbourhood of the form (4.1). (If this is the case, it can be proved that d_S is of class $C^{1,1}$ in such tubular neighbourhood.)

In [9, Theorem 2], we have obtained the following complete characterization of planar sets with positive reach and empty interior:

Theorem 4.2 (Characterization of planar proximally C^1 sets with empty interior). *Let $S \subset \mathbb{R}^2$ be closed, proximally C^1, with empty interior, and connected. Then S is either a singleton, or a* 1*-dimensional manifold of class $C^{1,1}$.*

Sketch of the proof. The proof is of marked geometric stamp, and here we limit ourselves to give a rough idea of it. It consists basically in performing a careful analysis of the so-called "contact set". Namely, we fix a point $p \in S$ and a positive r smaller that the radius of proximal smoothness, and study the contact set of p into S_r, which is defined as the set where the circumference of radius r centered at p meets the boundary of the tubular neighbourhood $\{d_S(x) < r\}$. The main issue in the proof amounts to show that $C_r(p)$ consists either of two antipodal points, or of a semicircumference. Once one has this geometric characterization of the contact set, it is rather easy to deduce that S is locally the graph of a Lipschitz function g. Finally, the fact that it is of class $C^{1,1}$ comes from the fact that g is both semiconcave and semiconvex. □

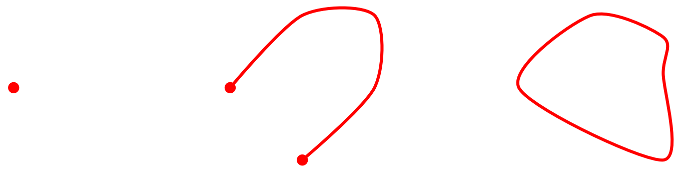

Figure 4. Planar proximally C^1 sets with empty interior

We explicitly note that a 1-dimensional connected manifold can be with boundary (two points) or without boundary (a closed curve), see Figure 4.

It is interesting to observe that, as soon as we require d_S to be of class C^2 in a tubular neighbourhood of S, then the second case in Figure 4 (manifold with boundary) cannot happen. More precisely, let us set the following

Definition 4.3 (Proximally C^k sets). We say that a nonempty closed subset S of $\mathbb{R}^n$ is *proximally* C^k if there exists $r_S > 0$ such that d_S is of class C^k in a tubular neighbourhood of S of the form (4.1).

Then we have (see [9, Theorem 3]):

Theorem 4.4 (characterization of proximally C^2 sets with empty interior). *Let $S \subset \mathbb{R}^2$ be closed, proximally C^2, with empty interior, and connected. Then S is either a singleton, or a 1-dimensional manifold of class C^2 without boundary.*

A direct consequence of Theorems 4.2 and 4.4 is the following characterization of stadium–like domains. To understand it, one has to think of S as playing the role of the set $\mathrm{M}(\Omega) = \overline{\Sigma}(\Omega)$, which is a nonempty closed set with empty interior (notice in fact that the high ridge $\mathrm{M}(\Omega)$ cannot have interior points, since otherwise there would be points where $\nabla d = 0$). Accordingly, the set Ω has to be thought as a tubular neighbourhood of S.

Theorem 4.5 (Characterization of planar domains with $M = \overline{\Sigma}$). *Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected set with $M(\Omega) = \overline{\Sigma}(\Omega)$. Then Ω is either a disk or a parallel neighbourhood of a 1-dimensional $C^{1,1}$ manifold.*

If in addition Ω is C^2, then Ω is either a disk or a parallel neighborhood of a 1-dimensional C^2 manifold with no boundary.

If Ω is also simply connected, then Ω is a disk.

The three possibilities are shown in Figure 5.

In [9, Theorem 12] we also proved a partial extension for convex sets in higher dimension.

Theorem 4.6 (Extension to higher dimensions). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set. If $M(\Omega) = \overline{\Sigma}(\Omega)$ and Ω is C^2, then Ω is a ball.*

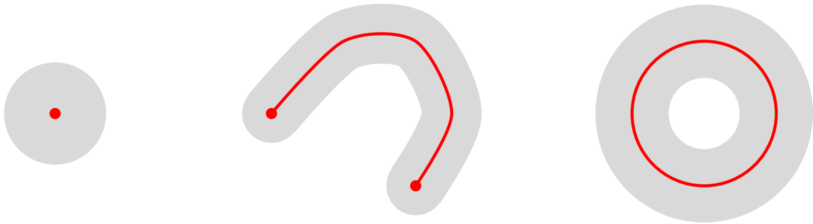

Figure 5. Stadium–like domains

Now our Theorem 3.4 can be rephrased in the following much more “visual” way:

Theorem 4.7 (Web–viscosity solutions). *The unique viscosity solution to problem* (2.1) *is a web function if and only the shape of* Ω *can be characterized as in Theorem 4.5 (in dimension* $n = 2$*) and 4.6 (in any dimension provided* Ω *is assumed to be convex).*

5 On the overdetermined problem: the general (non–web) case

Up to now we have characterized the geometry of sets for which the overdetermined problem (1.1) admits a solution in the class of web functions (we stress once more that, in this class of functions, the overdetermined problem (1.1) is equivalent to the Dirichlet problem (2.1), since the condition $|\nabla u|$ constant on $\partial\Omega$ is automatically satisfied).

In this section we are going to consider what happens in the general case, *i.e.* without the restriction to web functions. Recalling the heuristic argument given at the beginning of Section 3, we see that we have to face with a number of additional difficulties. In particular, the following two main problems emerge.

- Since u is unknown and, *a priori*, its level lines do not have any specific form, the geometry of the trajectories of the gradient flow is unknown.
- Even worse, we do not know if the gradient flow is well–posed. Namely, in general we only know that ∇u is locally bounded, and it is never locally Lipschitz, as we shall see in Theorem 5.4 that u never belongs to $C^{1,1}(\Omega)$. This means that we cannot use the standard Cauchy–Lipschitz theory for ordinary differential equations for the gradient flow $\dot{x} = \nabla u(x)$. Moreover, even if we were able to prove an intermediate regularity result between local boundedness and local Lipschitzianity for ∇u (*e.g.*, that it is locally in BV or in some Sobolev space), we could not even apply the Ambrosio–Di Perna–Lions theory of regular Lagrange flows, because we do not have a lower bound for the measure div∇u.

Our approach is motivated by the above remarks, and in particular it stems from the will of recovering the well-posedness of the gradient flow. In this respect it is well known that, in order to have at least forward well–posedness, it is enough u to be *locally semiconcave*. By definition, this means that there exists a constant $C \geq 0$ such that

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2 \qquad \forall [x-h, x+h] \subset \Omega\,,$$

where $[x-h, x+h]$ denotes the segment in $\mathbb{R}^n$ joining the two points $x-h$ and $x+h$.

In fact, the forward uniqueness of solutions follows from the property

$$\langle \nabla u(y) - \nabla u(x),\, y - x\rangle \leq C|y-x|^2\,,$$

which is the analogous, for differentiable semiconcave functions, of the monotonicity of the gradient of a (differentiable) concave functions. Now, if $x(t)$ and $y(t)$ are two solutions of the gradient flow defined in a common interval $[0, \tau)$, setting $w(t) := |y(t) - x(t)|^2/2$ we get

$$\dot{w}(t) = \langle \nabla u(y(t)) - \nabla u(x(t)),\, y(t) - x(t)\rangle \leq 2\, C\, w(t)\,.$$

Hence, if $w(t_0) = 0$ for some $t_0 \in [0, \tau)$, *i.e.*, if $x(t_0) = y(t_0)$, then by Gronwall's inequality we get that $w(t) = 0$ for every $t \in [t_0, \tau)$ *i.e.*, $x(t) = y(t)$ for every $t \in [t_0, \tau)$.

For a review on semiconcave functions we refer to [7].

In this perspective, our first step will be to set up a regularity result for u, proving that u is locally semiconcave. Unfortunately, we are not able to obtain such a result in full generality, but we have to restrict to convex domains without corners. More precisely, we are going to assume that

$$\Omega \text{ is convex and satisfies an interior sphere condition.} \qquad (H\Omega)$$

Theorem 5.1 (Power-concavity and semiconcavity of solutions). *Assume* $(H\Omega)$ *and let u be the viscosity solution to the Dirichlet problem* (2.1)*. Then $u^{3/4}$ is concave in Ω. In particular, u is locally semiconcave in Ω.*

Sketch of the proof. We have to prove that the function $w = -u^{3/4}$ is convex in Ω. (We remark that w is well defined since $u > 0$ in Ω.)

We first observe that w is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} -\Delta_\infty w - \frac{1}{w}\left[\frac{1}{3}|\nabla w|^4 + \left(\frac{3}{4}\right)^3\right] = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

At first sight the equation satisfied by w looks more complicate than the original one for u. On the other hand, thanks to the structure of such equation (we refer in particular to the factor $1/w$ in front of the brackets), we are enabled to adapt the convex envelop method developed by Alvarez, Lasry and Lions (see [1]). It consists essentially in the following steps.

(i) Prove that the convex envelope w_{**} of w is a viscosity super-solution to (5.1). This is the most challenging task where the structure of the equation intervenes.

(ii) By Step (i) and the comparison principle (that for equation (5.1) has been proved in [21, Thm. 3]), it follows that $w_{**} \geq w$ in Ω.

(iii) By definition of convex envelope, it is immediate that $w_{**} \leq w$ in Ω.

By combining Steps (ii) and (iii), we conclude that w coincides with its convex envelope, so that that $w = -u^{3/4}$ is a convex functions. From this power-concavity property of u, it is straightforward to conclude that u is locally semiconcave in Ω. □

Since u is locally semiconcave and differentiable everywhere, we get at once the following regularity property (see [7, Prop. 3.3.4]).

Corollary 5.2 (C^1-regularity of solutions). *Assume* $(H\Omega)$ *and let* u *be the viscosity solution to the Dirichlet problem* (2.1). *Then* u *is continuously differentiable in* Ω.

Let us now turn back to the overdetermined boundary value problem (1.1), in the light of the regularity results obtained so far for the solution u to problem (2.1) in Ω. In order not to face with boundary regularity matters for u at the boundary of Ω (for which however some results are available in the literature, see [25, 17, 18]), in the following we will assume that u is C^1 *up to* the boundary, namely that

$$\exists\, \delta > 0 \ : \ u \text{ is of class } C^1 \text{ on } \{x \in \overline{\Omega} \ : \ d(x) < \delta\}\,, \tag{Hu}$$

As a consequence of Corollary 5.2 and assumption (Hu), for every initial point $x_0 \in \overline{\Omega}$ the Cauchy problem

$$\begin{cases} \dot{x} = \nabla u(x), \\ x(0) = x_0 \end{cases}$$

turns out to admit a unique forward solution $\mathbf{X}(\cdot, x_0)$, defined on some maximal interval $[0, T(x_0))$. Moreover, we can prove that $t \mapsto \mathbf{X}(t, x_0)$ reaches is finite time a maximum point of u and then stops there.

Characteristics are now back at our disposal! So, let us resume the heuristic approach started in Section 3, consisting in studying the solution along such curves. Assume for a moment that the solution u of the Dirichlet problem (2.1) is smooth enough (let's say C^2), and consider the P–function

$$P(x) := \frac{1}{4}\,|\nabla u(x)|^4 + u(x).$$

If $x(\cdot) = \mathbf{X}(\cdot, y)$ is a characteristic, then

$$\frac{d}{dt} P(x(t)) = |\nabla u(x)|^2 \left\langle D^2 u(x) \nabla u(x),\ \nabla u(x) \right\rangle + |\nabla u(x)|^2 = 0\,,$$

so that the P–function is constant along the gradient flow.

If, in addition, we require the overdetermined condition $|\nabla u| = c$ on $\partial\Omega$ to hold, we have that $P(y) = c^4/4$ at every point $y \in \partial\Omega$. From this information, it follows that the P–function is constant along the set spanned by the gradient flow, *i.e.* on the whole Ω. In turn, the constancy of P over Ω allows to characterize the expression of u and the shape of Ω exactly in the same way as done in Section 3 in the web setting. Indeed, the following result holds.

Theorem 5.3 (P–function). *Under the assumptions* ($H\Omega$)-(Hu)*, let u be the unique solution to problem* (2.1). *If $P(x) = \lambda$ for a.e. $x \in \Omega$, then u is the web-function defined in* (3.2) *and Ω is a stadium-like domain (for which the conclusions of Theorem 4.7 hold).*

Sketch of the proof. The function ψ_Ω in (3.2) is the unique viscosity solution of the Hamilton–Jacobi equation

$$H(u, \nabla u) := \frac{1}{4}|\nabla u|^4 + u - \lambda = 0.$$

On the other hand, $u \in C^1(\Omega)$ is a classical solution of the same equation (since P is continuous and so $P = \lambda$ in Ω). Therefore, $u = \psi_\Omega$. In particular, since u is a web–function, the conclusions of Theorem 4.7 hold. □

Unfortunately, in general u is not regular enough to prove that P is constant a.e. in Ω. Actually, the heuristic argument leading to the constancy of P can be made rigorous only provided u is at least of class $C^{1,1}$, and this kind of regularity never occurs. More precisely, the optimal expected regularity is $C^{1,1/3}$ according to the result below, which is obtained essentially by dealing with ODE's along the gradient flow of u, and in particular exploiting the expression of u along characteristics given by equation (3.1).

Theorem 5.4 (Regularity threshold). *If the unique solution u to problem* (2.1) *is of class $C^{1,1}(A \setminus K)$, where $K := \text{argmax}_{\overline{\Omega}}(u)$ and A is a neighborhood of K, then for any $\alpha > 1/3$ it cannot occur that u is of class $C^{1,\alpha}(A)$.*

Nevertheless, not everything is lost... Still by exploiting characteristics, we can argue to get, in place of the constancy of the P–function, some useful upper and lower bounds for it.

Theorem 5.5 (P–function inequalities). *Under the assumptions* ($H\Omega$)–(Hu)*, let u be the unique solution to problem* (2.1)*. Then*

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} \leq P(x) \leq \max_{\overline{\Omega}} u \qquad \forall x \in \overline{\Omega}\,.$$

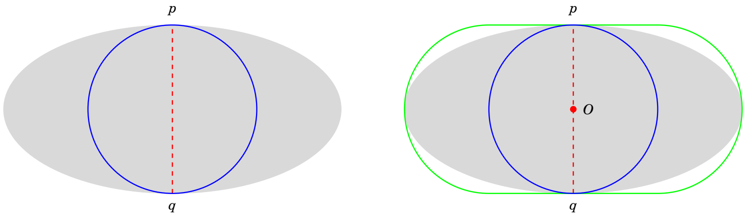

Figure 6. Domains considered in Theorem 5.6

Sketch of the proof. Observe that, if $y \in \partial\Omega$ then $P(\mathbf{X}(0,y)) = P(y) = |\nabla u(y)|^4/4$; on the other hand, for t large enough, $\mathbf{X}(t,y)$ is a maximum point of u, so that $P(\mathbf{X}(t,y)) = \max u$. Then to prove the statement it is enough to show that P is non-decreasing along the gradient flow. To this end, in order to get a bit more of regularity, we consider the supremal convolutions

$$u^\varepsilon(x) = \sup_y \left\{ u(y) - \frac{|x-y|^2}{2\varepsilon} \right\} .$$

By the local semiconcavity of u, these convolutions are of class $C^{1,1}$. Moreover, thanks to the so-called "magical properties" of their superjets, they turn out to be subsolutions of the PDE. Hence the corresponding approximated P–functions

$$P_\varepsilon := \frac{|\nabla u^\varepsilon|^4}{4} + u^\varepsilon$$

are non decreasing along the gradient flow of u^ε. Finally by passing to the limit as $\varepsilon \to 0^+$ we get the desired monotonicity property for P. □

The bounds for the P-function obtained in Theorem 5.5 do not give us enough information to deduce a complete characterization of domains where the overdetermined problem (1.1) admits a solution. However, they are quite helpful to get at least a partial target. Namely we can prove the following result, showing that the same conclusions of Theorem 4.7 continue to hold without asking the solution to be a web function, provided some *a priori* geometric restrictions on Ω are imposed.

Theorem 5.6 (Serrin-type theorem for Δ_∞). *Assume* $(H\Omega)$-(Hu). *Further assume that there exists an inner ball B of radius ρ_Ω which meets $\partial\Omega$ at two diametral points (see Figure 6 left). If there exists a solution u to the overdetermined problem* (1.1)*, then u is the web-function defined in* (3.2)*, and Ω is a stadium-like domain (for which the conclusions of Theorem 4.7 hold).*

Sketch of the proof. Let $p, q \in \partial\Omega$ be the two diametral points belonging to $\partial B \cap \partial\Omega$, and let D be a stadium–like domain D that contains Ω and is tangent to Ω at p and q (see Figure 6 right). Let u_B and u_D denote, respectively, the solutions of the Dirichlet problem (2.1) in B and D. By comparison, we have

$$u_B \leq u \leq u_D \qquad \text{in } \overline{B}\,.$$

In particular, this implies that $u = u_B = u_D$ on the segment $[p, q]$ and that $\nabla u = \nabla u_B = \nabla u_D$ at p and q, so that $|\nabla u_B| = |\nabla u_D| = c$ at these two points. In turn, this gives $\max u_D = c^4/4$ and hence, by Theorem 5.5, we get

$$\frac{c^4}{4} = \min_{\partial\Omega} \frac{|\nabla u|^4}{4} \leq P(x) \leq \max_{\overline{\Omega}} u \leq \frac{c^4}{4}\,.$$

Now the conclusion follows from Theorem 5.3. □

6 Open problems

We list below some open questions related to the results reviewed above, which are in our opinion interesting challenges for further research.

- Provide a complete characterization of stadium-like domains in higher dimensions (*i.e.*, remove the convexity assumption in Theorem 4.6).
- Provide a general version of Serrin theorem for Δ_∞ (*i.e.*, remove the geometric restrictions on Ω in Theorem 5.6).
- Prove that the solution to the Dirichlet problem (2.1) is actually of class $C^{1,1/3}(\Omega)$ (*i.e.*, show that the regularity threshold of Theorem 5.4 is achieved).
- To some extent surprisingly, the geometric condition $\overline{\Sigma}(\Omega) = M(\Omega)$ appears independently in the paper [26], where it is shown that on stadium-like domains the infinity Laplacian admits a unique ground state. (An infinity ground state is, roughly speaking, the limit as $p \to +\infty$ of a sequence of solutions to the Euler-Lagrange equation for the nonlinear Rayleigh quotient associated with the p-Laplacian). As recently shown in [19], the uniqueness of an infinity ground state is false in general, and the geometric characterization of domains where it is true is a completely open problem. It would be interesting to understand whether ∞-ground states are unique in all convex domains or just in stadium-like ones.

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