REGULARITY AND NONUNIQUENESS RESULTS FOR PARABOLIC PROBLEMS ARISING IN SOME PHYSICAL MODELS, HAVING NATURAL GROWTH IN THE GRADIENT

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ABSTRACT. In this paper we study the problem

$$\begin{cases} u_t - \Delta u &= \beta(u)|\nabla u|^2 + f(x,t) & \text{in } Q \equiv \Omega \times (0,+\infty) \\ u(x,t) &= 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded regular domain, β is a positive nondecreasing function and f, u_0 are positive functions satisfying some hypotheses of summability. Among others contribution the main one is to prove a *wild* non-uniqueness result.

1. Introduction

In this paper we will consider the following viscous Hamilton-Jacobi equation

$$(1) \qquad \begin{cases} u_t - \Delta u &= \beta(u)|\nabla u|^2 + f(x,t) & \text{in } Q \equiv \Omega \times (0,+\infty) \\ u(x,t) &= 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded regular domain, β is a positive nondecreasing function and f, u_0 are positive functions satisfying some hypotheses that we will specify later. In the case where $\beta \equiv 1$, this parabolic equation appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation (see [28]). A modification of the problem above is studied by Berestycki, Kamin, and Sivashinsky as a model in flame propagation (see [8]). For constant β, existence results for problem (1) in the whole $\mathbb{R}^N$, with a regular data u_0 and $f \equiv 0$ is well known, we refer to [26], where the Cauchy problem for the equation

$$(2) \qquad u_t - \Delta u = |\nabla u|^q, \ \ q \geq 1$$

is studied. We refer also to the paper [6] where problem (2) is studied in the case $q < 2$, some quantitative properties of the solutions are obtained in that case.

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It is not difficult to obtain an existence result for problem (1) in the case where the data are bounded: it suffices to use a change of unknown of the form $v(x,t) = \Psi(u(x,t))$, also known as Cole-Hopf transformation, to transform the equation into a semilinear problem (or a linear one if the function β is constant), which can then be solved by super/sub-solution methods. In the case where the operator is more general (or in the case where the data are unbounded) this change of variable cannot be done, but it can be replaced with the use of exponential-type test function, whose role is again to get rid of the quadratic term in (1) (see [14], [20]). The case where the Laplace operator is replaced by a nonlinear operator like the p-Laplacian has been studied in [27], [34], [25], [19], [21] and references therein.

In this paper we shall consider the problem of regularity, uniqueness and non uniqueness of solutions to problem (1). For the sake of simplicity, the first part of the article is devoted to the case where the function $\beta(u)$ which appears in the equation is constant. In this case we will prove that all weak solutions of problem (1) satisfy an exponential integrability (see Theorem 3.2). More precisely, we will show that

$$e^{\delta u} - 1 \in L^2(0,T; W_0^{1,2}(\Omega)) \cap \mathcal{C}^0([0,\infty); L^2(\Omega)) \qquad \text{for all } \delta < 1/2, \text{ for all } T > 0, \tag{3}$$

$$\int_\Omega e^{u(x,t)}\, dx < \infty \qquad \text{for all } t \geq 0. \tag{4}$$

The result (3) resembles the corresponding one for elliptic equations with quadratic gradient term, proven by the authors in [2], and has in common with it the fact that the elliptic part of the equation is never used for the regularity, more precisely that the main estimate come from the quadratic term on the right-hand side. Moreover, as in the elliptic case, no regularity on the datum f is assumed (only $f \in L^1_{\text{loc}}(\overline{Q})$ is required). However the proof of the parabolic result is more complicated, since one has to estimate the term with the time derivative of u.

Then we proceed in performing a precise analysis in what happens in the Cole-Hopf change of variable, particularly if one does not assume that the transformed function $v = \Psi(u)$ belongs to the "energy space", that is, $L^2(0,T; W_0^{1,2}(\Omega)) \cap C^0([0,\infty); L^2(\Omega))$, for all $T > 0$. We will show a striking nonuniqueness result[1], and a direct correspondence between solutions of problem (1) and solutions of semilinear problems with measure data, that is, we consider the following linear problem

$$\begin{cases} v_t - \Delta v &= f(x,t)(v+1) + \mu_s & \text{in } \mathcal{D}'(Q) \\ v(x,t) &= 0 & \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) &= v_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where μ_s is a singular positive Radon measure. Here "singular" means that it is concentrated on a set with zero capacity, where by "capacity" we mean the parabolic capacity introduced by Pierre in [38] and studied by Droniou, Porretta and Prignet in [23].

More precisely, under appropriate integrability assumptions on the data f and v_0, we show (Theorem 4.3) that problem (5) admits exactly one solution, and that if we apply the change of variable $u = \Psi^{-1}(v) = \log(1+v)$, then u is a solution of problem (1), with $\beta \equiv 1$. We could summarize this nonuniqueness result by saying that *there exists a one to one correspondence*

[1] However, we remark that in the Kardar-Parisi-Zhang model, problem (1) with $\beta \equiv 1$ appears by approximating $\sqrt{1+|\nabla u|^2} \approx 1 + \frac{1}{2}|\nabla u|^2$. That is, in that model only small, regular solutions are considered.

between solutions to problem (1) *and singular measures concentrated in zero parabolic capacity sets in the cylinder* $Q = \Omega \times [0, \infty)$. Therefore problem (1) admits infinitely many solutions, whose singularities can be prescribed.

The idea behind the result is very simple: if one makes formally the change of variable, then $u = \log(1+v)$ solves the equation

$$u_t - \Delta u = |\nabla u|^2 + f + \frac{\mu_s}{1+u}\,,$$

but if μ_s is a singular measure (for instance, if $\mu_s = \mu_s(x) = \delta_{x_0}(x)$ in space dimension $N \geq 2$), then v is infinite on the set where μ_s is concentrated, therefore the last term in equation (1) vanishes. Of course this is just a formal calculation, but the result will be justified rigorously.

An inverse result can also be proved (see Theorem 4.6): every solution u of problem (1) with $\beta \equiv 1$ corresponds, via the change of variable $v = \Psi(u) = e^u - 1$, to the solution v of an equation of the form (5), for a singular measure μ_s which is determined by u. Among these infinitely many functions there is only one, which we call the "regular" one, which corresponds to $\mu_s = 0$. This function is such that $v = \Psi(u) = e^u - 1 \in L^2(0,T; W_0^{1,2}(\Omega))$, and is unique in the larger class of functions such that $e^{u/2} - 1 \in L^2(0,T; W_0^{1,2}(\Omega))$. All the other solutions only satisfy $e^{\delta u} - 1 \in L^2(0,T; W_0^{1,2}(\Omega))$ for every $\delta < 1/2$.

It is interesting to point out that we also get infinitely many solutions by singular perturbation of the initial data in the transformed problem. More precisely if v is the renormalized solution to problem

$$\begin{cases} v_t - \Delta v &= 0 \quad \text{in } \mathcal{D}'(Q) \\ v(x,t) &= 0 \quad \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) &= \nu_s \quad \text{in } \Omega, \end{cases}$$

where ν_s is a singular positive measure with respect to the classical Lebesgue measure, then $u = \log(v+1)$ solves problem (11) with $f \equiv 0$ and $u_0(x) \equiv 0$. We refer to subsection 4.4 for more details.

The elliptic case was recently studied by the authors in [2], where a similar connection between the stationary solutions of problem (1) and solutions of linear and semilinear problems with measure data is proven. Therefore, the main result of this paper is to prove that the same phenomena occur when one deals with the parabolic problem.

Another interesting result is contained in subsection 4.2, where we prove that the regularity assumptions on f to ensure the existence of positive solutions of (1) are optimal: an explicit example is given when considering $f(x,t) \equiv \frac{\lambda}{|x|^2}$, with large λ.
The case of general β is considered in Section 5 where we assume that β is a non decreasing function such that $\lim\limits_{s\to\infty} \beta(s) = +\infty$. Under this condition we prove the *exponential* regularity of a general solution in subsection (5.1). The existence of a regular solution can be obtained in the same way as in the case $\beta \equiv 1$ with some change of variable which leads to a semilinear parabolic problem with a slightly superlinear term. Existence of infinitely many positive solution in connection with a singular measure is proven in Subsection 5.2 where the inverse problem is also considered. It is worth to point out that the nonuniqueness result opens a large quantity

of questions about the global dynamic of the problem. In Remark 5.8 we give some comments on the uniqueness in the case where $\beta \in \mathcal{C}^0([0,\infty)) \cap L^1(0,\infty)$ or $\beta \in \mathcal{C}^0([0,\infty)) \cap L^1(0,\infty) \cap L^\infty(0,\infty)$. The elliptic case was considered by Korkut, Pašić and Žubrinić in [29].

2. Notations, definitions and useful results

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$. We will denote by Q the cylinder $\Omega \times (0,\infty)$; moreover, for $0 < t_1 < t_2$, we will denote by Q_{t_1}, Q_{t_1,t_2} the cylinders $\Omega \times (0,t_1)$, $\Omega \times (t_1,t_2)$, respectively.

In this paper, we will consider problem (1), where $u_0(x)$ and $f(x,t)$ are positive functions defined in Ω, Q, respectively, such that $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$, for every $T > 0$.

The symbols $L^q(\Omega)$, $L^r(0,T;L^q(\Omega))$, and so forth, denote the usual Lebesgue spaces, see for instance [24]. We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space, of measurable functions having weak derivative in $L^q(\Omega)$ and zero trace on $\partial\Omega$. If $T > 0$, the spaces $L^r(0,T;L^q(\Omega))$ and $L^r(0,T;W_0^{1,q}(\Omega))$ have obvious meanings, see again [24].

Moreover, we will denote by $W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$. Here q' is Hölder's conjugate exponent of $q > 1$, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$. Finally, if $1 \leq q < N$, we will denote by $q^* = Nq/(N-q)$ its Sobolev conjugate exponent.

For the sake of brevity, instead of writing "$u(x,t) \in L^r(0,\tau;W_0^{1,q}(\Omega))$ for every $\tau > 0$", we shall write $u(x,t) \in L^r_{\mathrm{loc}}([0,\infty);W_0^{1,q}(\Omega))$. Similarly, we shall write $u \in L^q_{\mathrm{loc}}(\overline{Q})$ instead of $u \in L^q(Q_\tau)$ for every $\tau > 0$.

Definition 2.1. *We say that $u(x,t)$ is a distributional solution to problem* (1) *if $u \in \mathcal{C}([0,\infty);L^1(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty);W_0^{1,2}(\Omega))$, $\beta(u)|\nabla u|^2 \in L^1_{\mathrm{loc}}(\overline{Q})$, and if for all $\phi(x,t) \in \mathcal{C}_0^\infty(Q)$ one has*

$$-\iint_Q u\,\phi_t\,dx\,dt + \iint_Q \nabla u \cdot \nabla\phi\,dx\,dt = \iint_Q \beta(u)\,|\nabla u|^2\,\phi\,dx\,dt + \iint_Q f\phi\,dx\,dt$$

and

$$u(\cdot,0) = u_0(\cdot) \qquad \text{in } L^1(\Omega).$$

Remark 2.2. Note that the previous definition implies that, for every bounded, Lipschitz continuous function $h(s)$ such that $h(0) = 0$, and for every $\tau > 0$, one has

$$\begin{aligned}\int_\Omega H(u(x,\tau))\,dx - \int_\Omega H(u_0(x))\,dx + \iint_{Q_\tau} |\nabla u|^2\,h'(u)\,dx\,dt \\ = \iint_{Q_\tau} \beta(u)\,|\nabla u|^2\,h(u)\,dx\,dt + \iint_{Q_\tau} f\,h(u)\,dx\,dt\,,\end{aligned}$$

where $H(s) = \int_0^s h(\sigma)\,d\sigma$.

Similarly, if $h(s)$ is Lipschitz continuous and bounded, if $\phi(x,t) \in L^2(0,\tau;W_0^{1,2}(\Omega)) \cap L^\infty(Q_\tau)$ and $\phi_t \in L^2(0,\tau;W^{-1,2}(\Omega))$, then one has

$$\begin{aligned}\int_\Omega H(u(x,\tau))\,\phi(x,\tau)\,dx - \int_\Omega H(u_0(x))\,\phi(x,0)\,dx \\ - \iint_{Q_\tau} \phi_t\, H(u)\,dx\,dt + \iint_{Q_\tau} |\nabla u|^2\, h'(u)\,\phi\,dx\,dt + \iint_{Q_\tau} \nabla u\cdot\nabla\phi\, h(u)\,dx\,dt \\ = \iint_{Q_\tau} \beta(u)\,|\nabla u|^2\, h(u)\,\phi\,dx\,dt + \iint_{Q_\tau} f\,h(u)\,\phi\,dx\,dt\,.\end{aligned}$$

We will consider, for $k > 0$, the usual truncation at level k, i.e.

$$T_k s = \begin{cases} s \text{ if } |s| \le k \\ \\ k\frac{s}{|s|} \text{ if } |s| > k. \end{cases}$$

In order to present some of the results, we need to keep in mind a few regularity and convergence results about parabolic equations with L^1 or measure data (see for instance [13], [12]).

Assume that $v_{0,n}(x)$ and $f_n(x,t)$ are two sequences of nonnegative, bounded functions which have uniformly bounded norms in $L^1(\Omega)$ and $L^1(Q_T)$ (for every $T > 0$), respectively. Then, if one considers the solutions v_n of problems

$$\begin{cases} (v_n)_t - \Delta v_n &= f_n(x,t) & \text{in } Q \\ v_n(x,t) &= 0 & \text{on } \partial\Omega\times(0,\infty), \\ v_n(x,0) &= v_{0,n}(x) & \text{in } \Omega\,, \end{cases}$$

the following estimates hold:

$$\|v_n\|_{L^{r_1}(0,\tau;W^{1,q_1}(\Omega))} \le C(\tau)\,, \quad \text{for every } (r_1,q_1) \text{ such that } 1 \le q_1 < \frac{N}{N-1},\ 1 \le r_1 \le 2 \text{ and } \frac{N}{q_1} + \frac{2}{r_1} > N+1\,; \tag{6}$$

$$\|v_n\|_{\mathcal{C}^0(0,\tau;L^1(\Omega))} \le C(\tau)\,; \tag{7}$$

$$\|T_k v_n\|_{L^2(0,\tau;W_0^{1,2}(\Omega))} \le C(\tau)k\,, \quad \text{for every } k > 0; \tag{8}$$

$$\iint_{Q_\tau} \frac{|\nabla v_n|^2}{(v_n+1)^\alpha} \le C(\tau,\alpha)\,, \quad \text{for every } \alpha > 1. \tag{9}$$

Moreover, if f_n converges to some μ in the weak sense of measures in Q_τ, for every $\tau > 0$, and $v_{0,n}$ converges to v_0 in $L^1(\Omega)$, then for every $\tau > 0$

$$v_n \to v \quad \text{in } L^{r_1}(0,\tau;W_0^{1,q_1}(\Omega)), \text{ for every } (r_1,q_1) \text{ as in (6)}\,, \tag{10}$$

where v is the unique solution of

$$\begin{cases} (v)_t - \Delta v &= \mu & \text{in } Q \\ v(x,t) &= 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) &= v_0(x) & \text{in } \Omega\,, \end{cases}$$

in the sense that

$$\iint_Q (-v\,\phi_t + \nabla v \cdot \nabla \phi)\,dx\,dt - \int_\Omega v_0(x)\,\phi(x,0)\,dx = \langle \mu\,,\, \phi \rangle$$

for every $\phi(x,t) \in \mathcal{C}^1(\overline{Q})$ with compact support in $\Omega \times [0,\infty)$. Moreover, if $\mu = \mu(x,t)$ is a function in $L^1_{\text{loc}}(\overline{Q})$, then

$$v \in \mathcal{C}^0([0,\infty); L^1(\Omega))\,.$$

Finally, if $f_n \to \mu$ strongly in $L^1(Q_T)$, for $T > 0$, then

$$T_k v_n \to T_k v \quad \text{strongly in } L^2(0,T; W_0^{1,2}(\Omega)), \text{ for every } k.$$

The same convergence holds if $f_n \rightharpoonup \mu$ in the weak-$*$ convergence of measures, if μ is concentrated on a set of null parabolic capacity, see Section 4 below. See a detailed proof in [37] where a more general framework is also considered.

3. Regularity of general solutions in the case $\beta \equiv 1$

In this section we deal with the problem

$$\begin{cases} u_t - \Delta u &= |\nabla u|^2 + f(x,t) & \text{in } Q \\ u(x,t) &= 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x) & \text{in } \Omega\,, \end{cases} \tag{11}$$

where u_0 and f are positive functions such that $u_0 \in L^1(\Omega)$ and $f \in L^1_{\text{loc}}(\overline{Q})$. Our first result on the regularity is the following.

Proposition 3.1. *Assume that $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$ is a solution of problem (11), where $f \in L^1_{\text{loc}}(\overline{Q})$ is such that $f(x,t) \geq 0$ a.e. in Q. Then*

$$\int\limits_\Omega e^{u(x,\tau)} d(x)\,dx < \infty \qquad \text{for every } \tau > 0, \tag{12}$$

where $d(x) = dist(x, \partial\Omega)$.

Proof. Let $\varepsilon > 0$, we consider $v_\varepsilon = H_\varepsilon(u)$, where $H_\varepsilon(s) = e^{\frac{s}{1+\varepsilon s}} - 1$, then $v_\varepsilon \in L^\infty(Q) \cap L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$. We claim that v_ε satisfies the inequality

$$(v_\varepsilon)_t - \Delta v_\varepsilon \geq 0$$

in the sense of distributions. Indeed, we consider positive and smooth approximations in L^1, ϕ_n, f_n and $u_{0,n}$ of $|\nabla u|^2$, f and u_0, respectively, and we consider the approximate problems,

$$\begin{cases} (u_n)_t - \Delta u_n &= \phi_n + f_n \quad \text{in } Q \\ u_n(x,t) &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u_n(x,0) &= u_{0,n}(x) \quad \text{in } \Omega\,, \end{cases}$$

and we consider $v_{n,\varepsilon} = H_\varepsilon(u_n)$. Then it is clear that for every positive $\xi(x,t) \in \mathcal{C}_0^\infty(Q)$

$$(13) \quad -\iint_Q v_{n,\varepsilon}\,\xi_t\,dx\,dt + \iint_Q \nabla v_{n,\varepsilon}\cdot\nabla\xi\,dx\,dt \\ = \iint_Q (\phi_n + f_n)\,H'_\varepsilon(u_n)\,\xi\,dx\,dt - \iint_Q |\nabla u_n|^2\,H''_\varepsilon(u_n)\,\xi\,dx\,dt\,.$$

We now wish to pass to the limit in n for fixed ε. By the theory for parabolic equations with data in L^1, the sequence $\{u_n\}$ satisfies the properties stated in the previous Section, and in particular, using convergence (10), we can pass to the limit in n in every term of (13). As far as the last integral is concerned, one has

$$\iint_Q |\nabla u_n|^2\,H''_\varepsilon(u_n)\,\xi\,dx\,dt = \iint_Q |\nabla T_k u_n|^2\,H''_\varepsilon(u_n)\,\xi\,dx\,dt + \iint_{\{u_n>k\}} |\nabla u_n|^2\,H''_\varepsilon(u_n)\,\xi\,dx\,dt\,.$$

The first integral of the r.h.s. passes to the limit by convergence (2), while the second one is small if k is large, uniformly in n, since

$$|H''_\varepsilon(s)| \le \frac{c(\varepsilon)}{(1+\varepsilon s)^3} \qquad \text{for all positive } s\,,$$

and thus, using estimate (9),

$$\iint_{\{u_n>k\}} |\nabla u_n|^2\,|H''_\varepsilon(u_n)|\,\xi\,dx\,dt \le \frac{c}{(1+\varepsilon k)} \iint_{Q\cap \operatorname{supp}\xi} \frac{|\nabla u_n|^2}{(1+\varepsilon u_n)^2}\,dx\,dt \le \frac{c}{(1+\varepsilon k)}\,.$$

Therefore one has

$$-\iint_Q v_\varepsilon\,\xi_t + \iint_Q \nabla v_\varepsilon\cdot\nabla\xi\,dx\,dt = \iint_Q (H'_\varepsilon(u) - H''_\varepsilon(u))\,|\nabla u|^2\,\xi\,dx\,dt + \iint_Q f\,H'_\varepsilon(u)\,\xi\,dx\,dt \ge 0\,,$$

since $H'_\varepsilon(u) - H''_\varepsilon(u) \ge 0$. Moreover $u \in \mathcal{C}([0,\infty); L^1(\Omega))$, therefore $v_\varepsilon \in \mathcal{C}([0,\infty); L^p(\Omega))$ for every $p < \infty$.

Since $u \in L^1(\Omega)$, in particular $e^{u(x,t)} < \infty$ a.e. in Q. For $t_0 > 0$, let w be the solution of problem

$$(14) \quad \begin{cases} w_t - \Delta w &= 0 \quad \text{in } \Omega \times (t_0,\infty) \\ w(x,t) &= 0 \quad \text{on } \partial\Omega \times (t_0,\infty), \\ w(x,t_0) &= v_\varepsilon(x,t_0). \end{cases}$$

Using a result by Martel (see [31] Lemma 2), we obtain that

$$c_1(t)||v_\varepsilon(\cdot,t_0)d(\cdot)||_{L^1}d(x) \le w(x,t) \le c_2(t)||v_\varepsilon(\cdot,t_0)d(\cdot)||_{L^1}d(x) \text{ for all } t > t_0,$$

for some positive functions $c_1(t)$, $c_2(t)$. Since v_ε is a supersolution to problem (14), we conclude that $w \le v_\varepsilon$ in $\Omega \times (t_0, \infty)$. Therefore

$$c_1(t)||v_\varepsilon(\cdot, t_0)d(\cdot)||_{L^1} d(x) \le v_\varepsilon(x,t) \le e^{u(x,t)} < \infty \quad \text{for a.e. } (x,t) \in \Omega \times (t_0, \infty)\,.$$

We fix $(x,t) \in \Omega \times (t_0, \infty)$, such that $u(x,t) < \infty$. Then using Fatou's lemma we obtain

$$\int_\Omega e^{u(x,t_0)}\, d(x)\, dx < \infty.$$

Using the fact that $t_0 > 0$ is arbitrary, we conclude that (12) holds. ■

As a consequence we obtain the following result.

Theorem 3.2. *Under the same hypotheses as in the previous propositions, for all $\tau > 0$ we have*

$$\iint_{Q_\tau} |\nabla u|^2\, e^{\delta u}\, dx\, dt < \infty, \qquad \textit{for all } \delta < 1\,, \tag{15}$$

$$\iint_{Q_\tau} f\, e^u\, dx\, dt < \infty\,, \tag{16}$$

$$\iint_{Q_\tau} e^{\frac{u}{1+\varepsilon u}}\, |\nabla u|^2 \Big(1 - \frac{1}{(1+\varepsilon u)^2}\Big)\, dx\, dt \le C(\tau) \quad \textit{uniformly in } \varepsilon\,, \tag{17}$$

$$\int_\Omega e^{u_0(x)}\, dx < \infty\,. \tag{18}$$

and finally

$$e^u \in L^\infty(0,\tau; L^1(\Omega))\,, \tag{19}$$

Proof. Let us consider an open set $\tilde\Omega \supset\supset \Omega$. For $\tau > 0$, consider the solution $\phi(x,t)$ of problem

$$\begin{cases} -\phi_t - \Delta\phi &= 0 & \text{in } \tilde\Omega \times (0, \tau+1) \\ \phi(x,t) &= 0 & \text{on } \partial\tilde\Omega \times (0,\tau+1), \\ \phi(x,\tau+1) &= \tilde d(x)\,, \end{cases} \tag{20}$$

where

$$\tilde d(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \tilde\Omega \setminus \Omega. \end{cases}$$

Then it is well known that

$$\phi(x,t) \ge c(\tau) > 0\,, \qquad \text{for a.e. } (x,t) \in \Omega \times (0,\tau)\,. \tag{21}$$

Let us define

$$k_{\delta,\varepsilon}(s) = e^{\frac{\delta s}{1+\varepsilon s}}, \quad \Psi_{\delta,\varepsilon}(s) = \int_0^s k_{\delta,\varepsilon}(\sigma) d\sigma \le \frac{1}{\delta} e^{\delta s}\,.$$

We use $\phi(x,t)\,(k_{\delta,\varepsilon}(u(x,t))-1)$ as test function in (11), and we integrate in $Q_{\tau+1}$, obtaining

$$
\begin{aligned}
(22)\quad &\int_\Omega \Psi_{\delta,\varepsilon}(u(x,\tau+1))\,d(x)\,dx - \int_\Omega u(x,\tau+1)\,d(x)\,dx \\
&- \int_\Omega \Psi_{\delta,\varepsilon}(u(x,0))\,\phi(x,0)\,dx + \int_\Omega u(x,0)\,\phi(x,0)\,dx + \iint_{Q_{\tau+1}} k'_{\delta,\varepsilon}(u)\,|\nabla u|^2\,\phi\,dx\,dt \\
= &\iint_{Q_{\tau+1}} k_{\delta,\varepsilon}(u)\,|\nabla u|^2\,\phi\,dx\,dt - \iint_{Q_{\tau+1}} |\nabla u|^2\,\phi\,dx\,dt + \iint_{Q_{\tau+1}} f\,k_{\delta,\varepsilon}(u)\,\phi\,dx\,dt - \iint_{Q_{\tau+1}} f\,\phi\,dx\,dt\,.
\end{aligned}
$$

The first integral in (22) is bounded by (12), therefore, using (21), it follows that

$$
\begin{aligned}
&\iint_{Q_\tau} e^{\frac{\delta\,u}{1+\varepsilon\,u}}\,\Big(1-\frac{\delta}{(1+\varepsilon\,u)^2}\Big)\,|\nabla u|^2\,dx\,dt + \iint_{Q_\tau} e^{\frac{\delta\,u}{1+\varepsilon\,u}}\,f\,dx\,dt + \int_\Omega \Psi_{\delta,\varepsilon}(u_0(x))\,dx \\
&= \iint_{Q_\tau} \big(k_{\delta,\varepsilon}(u)-k'_{\delta,\varepsilon}(u)\big)\,|\nabla u|^2\,dx\,dt + \iint_{Q_\tau} f\,k_{\delta,\varepsilon}(u)\,dx\,dt + \int_\Omega \Psi_{\delta,\varepsilon}(u_0(x))\,dx \le c(\tau)\,.
\end{aligned}
$$

Then, taking $\delta<1$ and passing to the limit as $\varepsilon\to 0$, we obtain (15). Similarly, taking $\delta=1$, we obtain (16), (17) and (18). Finally, let $\omega(x,t)$ be the solution of

$$
\begin{cases}
-\omega_t-\Delta\omega = 0 & \text{in } Q_\tau \\
\omega(x,t) = 0 & \text{on } \partial\Omega\times(0,\tau), \\
\omega(x,\tau) \equiv 1\,.
\end{cases}
$$

Then $0\le\omega(x,t)\le 1$ for every $(x,t)\in Q_\tau$. Multiplying equation (11) by $k_{1,\varepsilon}(u)\,\omega$ gives, with calculations similar to the previous ones,

$$
\begin{aligned}
(23)\quad &\int_\Omega \Psi_{1,\varepsilon}(u(x,\tau))\,dx \\
&\le \iint_{Q_\tau} \big(k_{1,\varepsilon}(u)-k'_{1,\varepsilon}(u)\big)\,|\nabla u|^2\,dx\,dt + \iint_{Q_\tau} f\,k_{1,\varepsilon}(u)\,dx\,dt + \int_\Omega \Psi_{1,\varepsilon}(u_0(x))\,dx\,.
\end{aligned}
$$

Since the right-hand side of (23) is bounded by the previous estimates, (19) follows easily.
■

Remark 3.3. If we consider the following approximating problem

$$
\begin{cases}
(u_n)_t-\Delta u_n = \dfrac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2} + T_n(f(x,t)) & \text{in } Q \\
u_n(x,t) = 0 & \text{on } \partial\Omega\times(0,\infty), \\
u_n(x,0) = T_n(u_0(x)),
\end{cases}
$$

then we can prove using the previous regularity results that $u_n\uparrow u$ and $u_n\to u$ strongly in $L^2(0,\tau;W_0^{1,2}(\Omega))$ for all $\tau>0$.

4. Existence and nonuniqueness.

4.1. Existence of solutions with higher integrability.
Assume that f is a positive function such that

$$f(x,t) \in L^r_{\rm loc}([0,\infty); L^q(\Omega))\,, \qquad \text{with } q,\, r > 1\,, \quad \frac{N}{q} + \frac{2}{r} < 2\,.$$

We perform the change of variable $v = e^u - 1$; then problem (1) becomes

$$(24) \qquad \begin{cases} v_t - \Delta v &=& f(x,t)(v+1) & \text{in } Q \\ v(x,t) &=& 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) &=& v_0(x) = e^{u_0} - 1\,. \end{cases}$$

If we assume that $v_0(x) = e^{u_0} - 1 \in L^2(\Omega)$, then existence of a solution $v \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap L^2_{\rm loc}([0,\infty); W^{1,2}_0(\Omega))$ can be proved using the same argument as in [30]. Using the linearity of the problem the result can be easily adapted to the case where v_0 only belongs to $L^1(\Omega)$, obtaining $v \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^2_{\rm loc}([0,\infty); W^{1,2}_0(\Omega))$. Actually v and ∇v are Hölder continuous (see the classical theory, again in [30]). We set $u = \log(v+1)$, then $u \in L^2(0,T; W^{1,2}_0(\Omega))$ and u satisfies problem (11). The inverse is also true in the sense that if u is a solution to problem (11) with $e^{u_0(x)} - 1 \in L^2(\Omega)$ and $e^u - 1 \in L^2((0,T), W^{1,2}_0(\Omega))$, then if we set $v = e^u - 1$ we obtain that v solves problem (24).

4.2. Optimality of the hypotheses on f: nonexistence result.
To see that the condition on f is optimal in some sense we will assume that $0 \in \Omega$ and that $f(x,t) = f(x) = \dfrac{\lambda}{|x|^2}$. Note that $f(x) \in L^q(\Omega)$ for every $q < N/2$, therefore we are in a limit case of (4.1). Hence we have the following nonexistence result (not even for small times).

Theorem 4.1. *Assume that $N \geq 3$, and that $\lambda > \Lambda_N = (\frac{N-2}{2})^2$, the optimal Hardy constant defined by*

$$\Lambda_N \equiv \inf_{\{\phi \in W^{1,2}_0(\Omega);\, \phi \neq 0\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2 dx}{\displaystyle\int_\Omega \phi^2\, |x|^{-2}\, dx}.$$

Then, for any initial datum $u_0 \geq 0$ and for any $T > 0$, problem

$$(25) \qquad \begin{cases} u_t - \Delta u &=& |\nabla u|^2 + \dfrac{\lambda}{|x|^2} & \textit{in } Q_T \\ u(x,t) &=& 0 & \textit{on } \partial\Omega \times (0,T), \\ u(x,0) &=& u_0(x) & \textit{in } \Omega\,, \end{cases}$$

has no solution.

Proof. The proof uses the same arguments as in [15] and [1] (see also [36]); for the sake of completeness we include here the proof. We argue by contradiction. Assume that u is a solution to problem (25) with $f(x,t) = \dfrac{\lambda}{|x|^2}$, $\lambda > \Lambda_N$. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$, by taking ϕ^2 as a test function in (1) we obtain that

$$\begin{aligned}\int\limits_\Omega u(x,t_2)\,\phi^2\,dx - \int\limits_\Omega u(x,t_1)\,\phi^2 dx \quad &+ \quad 2\iint_{Q_{t_1,t_2}} \phi\,\nabla\phi\cdot\nabla u\,dx\,dt \\ = \iint_{Q_{t_1,t_2}} \phi^2\,|\nabla u|^2\,dx\,dt \quad &+ \quad \lambda\iint_{Q_{t_1,t_2}} \frac{\phi^2}{|x|^2}\,dx\,dt\,,\end{aligned}$$

where we have set $Q_{t_1,t_2} = \Omega\times(t_1,t_2)$. Hence

$$\begin{aligned}-\int\limits_\Omega u(x,t_2)\,\phi^2\,dx &\le 2\iint_{Q_{t_1,t_2}} \phi\,\nabla\phi\cdot\nabla u\,dx\,dt - \iint_{Q_{t_1,t_2}} \phi^2\,|\nabla u|^2\,dx\,dt - \lambda\iint_{Q_{t_1,t_2}} \frac{\phi^2}{|x|^2}\,dx\,dt \\ &= -\iint_{Q_{t_1,t_2}} |\nabla\phi - \phi\,\nabla u|^2\,dx\,dt + \iint_{Q_{t_1,t_2}} |\nabla\phi|^2\,dx\,dt - \lambda\iint_{Q_{t_1,t_2}} \frac{\phi^2}{|x|^2}\,dx\,dt \\ &\le (t_2-t_1)\Big[\int_\Omega |\nabla\phi|^2\,dx - \lambda\int_\Omega \frac{\phi^2}{|x|^2}\,dx\Big]\,.\end{aligned}$$

By the regularity result of Theorem 3.2, we know that $u(\cdot,t)\in L^a(\Omega)$ for all $t\in(0,T)$ and for all $a<\infty$; therefore we obtain that

$$\int_\Omega |\nabla\phi|^2\,dx - \lambda\int_\Omega \frac{\phi^2}{|x|^2}\,dx \ge -\frac{1}{t_2-t_1}\Big(\int_\Omega u^{\frac{N}{2}}(x,t_2)\,dx\Big)^{\frac{2}{N}}\,\Big(\int_\Omega |\phi|^{2^*}\,dx\Big)^{\frac{2}{2^*}}.$$

By density, this implies that

$$I(\Omega) \equiv \inf_{\phi\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2\,dx - \lambda\int_\Omega \frac{\phi^2}{|x|^2}\,dx}{\Big(\displaystyle\int_\Omega |\phi|^{2^*}\,dx\Big)^{\frac{2}{2^*}}} \ge -\frac{1}{t_2-t_1}\Big(\int_\Omega u^{\frac{N}{2}}(x,t_2)\,dx\Big)^{\frac{2}{N}} > -\infty\,.$$

On the other hand taking the sequence $\phi_n(x) = T_n(|x|^{-\frac{N-2}{2}})\eta(x)$, where $\eta(x)$ is a cut-off function with compact support in Ω which is 1 in a neighborhood of the origin, since $\lambda > \left(\frac{N-2}{2}\right)^2$, one can check that $I(\Omega) = -\infty$. Hence we reach a contradiction. ■

Corollary 4.2.
1) If $f(x,t) \ge \dfrac{C(t)}{|x|^{2+\varepsilon}}$ in a neighborhood of the origin, where $C(t)$ is a positive function such that $C(t)\ge a>0$ in $(t_1,t_2)\subset(0,T)$, then problem (25) *has no solution.*

2) Since the argument used in the proof is local, then under the same hypothesis on f we can prove that problem (25) *has no local positive solution.*

Proof. It suffices to observe that in this case, for any $\lambda > \Lambda_N$, one has $f(x,t) \ge \dfrac{\lambda}{|x|^2}$ in a small ball centered at the origin. ■

4.3. **Nonuniqueness: Existence of weaker solutions.**
In this subsection we will show a strong connection between solutions of problem (11) and solutions of a linear problem with measure datum. This will give, as a consequence, a surprising non-uniqueness result for problem (11).
The theory of elliptic and parabolic equations in divergence form with measure data has been strongly developed in the last forty years, starting from the pioneering paper [42] by Guido Stampacchia (see also [13], [9], [12], [22], [10], [7] and references therein). Various definitions of solution have been introduced in order to obtain uniqueness results. Uniqueness is still an open problem for general nonlinear operators. However in the case of problem (27) below, the situation is easier, as far as uniqueness is concerned, because we are considering the heat operator.

The first result we will prove, therefore, is an existence and uniqueness theorem for problem (24) with an additional term which is a finite Radon measure:

Theorem 4.3. *Let f be a function in $L^r_{\rm loc}([0,\infty);L^q(\Omega))$ with*

$$r,q>1\,,\qquad \frac{N}{2q}+\frac{1}{r}<1. \tag{26}$$

Let μ be a Radon measure on Q, which is finite on Q_T for every $T>0$. Then problem

$$\begin{cases} v_t-\Delta v &= f(x,t)\,v+\mu & \text{in } Q\\ \qquad v &= 0 & \text{on } \partial\Omega\times(0,\infty),\\ v(x,0) &= \phi(x)\in L^1(\Omega), \end{cases} \tag{27}$$

has a unique distributional solution such that

$$\begin{cases} i) \quad v\in L^{r_1}_{\rm loc}([0,\infty);W^{1,q_1}_0(\Omega)) \text{ for every } r_1,q_1\geq 1 \text{ such that } \dfrac{N}{q_1}+\dfrac{2}{r_1}>N+1\,;\\ ii) \quad v\in L^\infty_{\rm loc}([0,\infty);L^1(\Omega))\,, \text{ for every } k>0;\\ iii) \quad T_kv\in L^2_{\rm loc}([0,\infty);W^{1,2}_0(\Omega))\,, \text{ for every } k>0;\\ iv) \quad f\,v\in L^1_{\rm loc}(\overline{Q})\,. \end{cases} \tag{28}$$

Proof. Notice that if v satisfies (28) $i)$ and $ii)$, then, using the Gagliardo-Nirenberg inequality, $v\in L^\rho_{\rm loc}([0,\infty);L^\sigma(\Omega))$, for all ρ and σ satisfying

$$\rho,\sigma\geq 1\,,\qquad \frac{N}{\sigma}+\frac{2}{\rho}>N\,. \tag{29}$$

Consider $g_n\in L^\infty(Q)$, such that $\{g_n\}$ is bounded in $L^1(Q_T)$ for every $T>0$ and moreover, as $n\to\infty$,

$$g_n\rightharpoonup\mu \quad \text{weakly in the measures sense in } Q_T, \text{ for every } T>0.$$

Consider $\phi_n\in L^\infty(\Omega)$, $\phi_n\to\phi$ in $L^1(\Omega)$. We solve

$$\begin{cases} (v_n)_t-\Delta v_n &= f\,v_n+g_n & \text{in } Q\\ \qquad v_n &= 0 & \text{on } \partial\Omega\times(0,\infty),\\ v_n(x,0) &= \phi_n(x). \end{cases}$$

Claim.- For every $T > 0$ there exists a constant $C(T) > 0$ such that

$$||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))} \leq C(T)\,.$$

(r', q' Hölder conjugates of r, q in (26)). If the claim holds then $f\, v_n$ is uniformly bounded in $L^1(Q_T)$ for every $T > 0$ and we can conclude in a standard way (see for instance [13] and [12]). Hence it is sufficient to prove the claim above. We argue by contradiction; assume that, up to a subsequence,

$$||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))} \to \infty.$$

Normalizing the sequence, i.e., putting $w_n = \dfrac{v_n}{||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))}}$, then $||w_n||_{L^{r'}(0,T;L^{q'}(\Omega))} = 1$ and for each $n \in \mathbb{N}$, w_n satisfies problem

$$\begin{cases} (w_n)_t - \Delta w_n &=& f(x,t)\, w_n + \dfrac{g_n}{||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))}} & \text{in } Q_T \\ w_n &=& 0 & \text{on } \partial\Omega \times (0,T), \\ w_n(x,0) &=& \dfrac{\phi_n(x)}{||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))}}. & \end{cases}$$

The right-hand side in equation (4.3) is uniformly bounded in $L^1(Q_T)$, hence by using the results (6)–(9) in Section 2 we find that $\{w_n\}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and in $L^{r_1}(0,T;W_0^{1,q_1}(\Omega))$, for all (r_1, q_1) as in (28) *i)*. Therefore by Sobolev's embedding, $\{w_n\}$ is bounded in $L^\rho(0,T;L^\sigma(\Omega))$, for all (ρ, σ) as in (29). Hence there exists w such that $w_n \rightharpoonup w$ weakly in $L^{r_1}(0,T;W_0^{1,q_1}(\Omega))$ for all (r_1, q_1) as in (28) *i)*. Moreover, w verifies

$$\begin{cases} w_t - \Delta w &=& f(x,t)\, w & \text{in } Q_T \\ w &\in& L^\infty(0,T;L^1(\Omega)) \cap L^{r_1}(0,T;W_0^{1,q_1}(\Omega)) & \text{for all } (r_1,q_1) \text{ as in (28)}, \\ w(x,0) &=& 0, & \end{cases} \tag{30}$$

because $\dfrac{g_n}{||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))}} \to 0$ in $L^1(Q_T)$ and $\dfrac{\phi_n(x)}{||v_n||_{L^{r'}(0,T;L^{q'}(\Omega))}} \to 0$ in $L^1(\Omega)$, as $n \to \infty$. We will show that

1) $w_n \to w$ strongly in $L^{r'}(0,T;L^{q'}(\Omega))$, therefore $||w||_{L^{r'}(0,T;L^{q'}(\Omega))} = 1$.
2) Problem (30) admits only the trivial solution.

Notice that 1) and 2) give a contradiction, and then we have proved the claim.
Proof of 1). By using the compact embedding $W_0^{1,q_1}(\Omega) \hookrightarrow L^s(\Omega)$ if $s < q_1^*$, the continuous embedding $L^s(\Omega) \subset W^{-1,q_1'}(\Omega) + L^1(\Omega)$ and the fact that

$$||w_n||_{L^{r_1}(0,T;W_0^{1,q_1}(\Omega))} \leq C \text{ and } ||(w_n)_t||_{L^{r_1'}(0,T;W^{-1,q_1'}(\Omega))+L^1(Q_T)} \leq C,$$

using Aubin's compactness results (see for instance [41]), we conclude that $\{w_n\}$ is relatively compact in $L^{r_1}(0,T;L^s(\Omega))$ for all $s < q_1^*$. Therefore, $\{w_n\}$ is relatively compact in $L^\rho(0,T;L^\sigma(\Omega))$ for all (ρ,σ) as in (29). Therefore we only have to show that one can take $(\rho,\sigma) = (r', q')$ in (29). Indeed, the condition

$$\frac{N}{q'} + \frac{2}{r'} > N$$

is equivalent to the assumption (26). This completes the proof of 1).
Proof of 2). Since uniqueness is trivial in the space $L^2(0,T;W_0^{1,2}(\Omega))$, we only have to show that every solution w of (30) belongs to this space. This is done by a bootstrap method. Indeed, using Hölder's inequality and the regularity of f we find that $f\,w^{m_1} \in L^1(Q_T)$, for every m_1 such that

$$\frac{N}{m_1 q'} + \frac{2}{m_1 r'} > N,$$

and since $\dfrac{1}{q'} + \dfrac{2}{Nr'} > 1$, we can chose $1 < m_1 < \dfrac{1}{q'} + \dfrac{2}{Nr'}$. Therefore, using $\dfrac{w^{m_1-1}}{1+\varepsilon w^{m_1-1}}$ as a test function in (30) and passing to the limit as $\varepsilon \to 0$, we obtain, for every $\tau \in (0,T)$,

$$\begin{aligned}\frac{1}{m_1}\int_\Omega w^{m_1}(x,\tau)\,dx + (m_1-1)\iint_{Q_\tau} w^{m_1-2}\,|\nabla w|^2\,dx\,dt &= \iint_{Q_\tau} f\,w^{m_1}\,dx\,dt \\ &\le \iint_{Q_\tau} f\,w^{m_1}\,dx\,dt = C(T),\ \forall \tau \in [0,T]\,,\end{aligned}$$

Hence, setting

$$v = w^{\frac{m_1}{2}}$$

the last estimate implies

$$v \in L^2((0,T);W_0^{1,2}(\Omega)) \cap L^\infty((0,T);L^2(\Omega))\,,$$

which by Gagliardo-Nirenberg inequality gives

$$v \in L^\delta((0,T);L^\gamma(\Omega)) \text{ with } 2 \le \gamma \le 2^*,\ \delta \le 2 \text{ and } \frac{2}{\delta} + \frac{N}{\gamma} = \frac{N}{2}\,.$$

Hence it follows that $w \in L^\beta((0,T);L^\alpha(\Omega))$ where

$$\alpha = \frac{m_1\gamma}{2},\ \beta = \frac{\delta m_1}{2},\ m_1 \le \alpha \le \frac{2^*}{2}m_1,\ \ m_1 \le \beta \text{ and } \frac{m_1}{\beta} + \frac{Nm_1}{2\alpha} = \frac{N}{2}. \tag{31}$$

This implies that

$$\iint_{Q_T} f w^{m_2}\,dx\,dt < \infty\,, \qquad \text{where } m_2 = m_1\,(\frac{1}{q'} + \frac{2}{Nr'})\,.$$

Iterating the process, if we consider the sequence defined by

$$m_{k+1} = \rho\, m_k\,, \quad \text{with } \rho = \frac{1}{q'} + \frac{2}{Nr'} > 1\,,$$

then

$$\iint_{Q_T} f\,w^{m_k}\,dx\,dt < \infty\,.$$

Thus

$$\iint_{Q_T} w^{m_k-2}\,|\nabla w|^2\,dx\,dt < C(k) \text{ and } \sup_{\tau\in(0,T)}\int_\Omega w^{m_k}(x,t)dx < C(k).$$

As $m_k \to \infty$ and since $T_k(w) \in L^2(0,T;W_0^{1,2}(\Omega))$, then for $k>1$, it follows that

$$\iint_{Q_T} |\nabla w|^2 \,dx\,dt \le \iint_{Q_T} |\nabla T_k(w)|^2 \,dx\,dt + \iint_{Q_T} w^{m_k-2}\, |\nabla w|^2 \,dx\,dt < C(k).$$

Thus $w \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and then the uniqueness result follows. ■

The previous problem (27) with measure datum appears in a natural way when we perform the change of unknown function as before. Theorems 4.6 and 4.8 below will show that there exists a one-to-one correspondence between the solutions of problem (11) and (27), where μ is an arbitrary "singular" measure. To clarify the meaning of "singular" measure we have to use a notion of *parabolic capacity* introduced by Pierre in [38] and by Droniou, Porretta and Prignet in [23].

For $T>0$, we define the Hilbert space $\mathbf{W}$ by setting

$$\mathbf{W} = \mathbf{W}_T = \{u \in L^2(0,T;W_0^{1,2}(\Omega)),\ u_t \in L^2(0,T;W^{-1,2}(\Omega))\},$$

equipped with the norm defined by

$$||u||^2_{\mathbf{W}_T} = \iint_{Q_T} |\nabla u|^2 \,dx\,dt + \int_0^T ||u_t||^2_{W^{-1,2}} dt\,.$$

Definition 4.4. *If $U \subset Q_T$ is an open set, we define*

$$\mathrm{cap}_{1,2}(U) = \inf\{\|u\|_{\mathbf{W}} : u \in \mathbf{W},\ u \ge \chi_U \text{ almost everywhere in } \quad Q_T\}$$

(we will use the convention that $\inf \emptyset = +\infty$*), then for any borelian subset $B \subset Q_T$ the definition is extended by setting:*

$$\mathrm{cap}_{1,2}(B) = \inf\left\{\mathrm{cap}_{1,2}(U),\ U \textit{ open subset of } Q_T,\ B \subset U\right\}.$$

We refer to [23] for the main properties of this capacity. We observe that, if $B \subset Q_T \subset Q_{\tilde{T}}$, then the capacity of B is the same in Q_T and in $Q_{\tilde{T}}$, therefore we will not specify the value of T when speaking of a Borel set compactly contained in $\overline{Q}$.

We recall that, given a Radon measure μ on Q and a Borel set $E \subset Q$, then μ is said to be concentrated on E if $\mu(B) = \mu(B \cap E)$ for every Borel set B.

Definition 4.5. *Let the space dimension N be at least 2. Let μ be a positive Radon measure in Q. We will say that μ is singular if it is concentrated on a subset $E \subset Q$ such that*

$$\mathrm{cap}_{1,2}(E \cap Q_\tau) = 0, \textit{ for every } \tau > 0.$$

As examples of singular measures, one can consider:

i) a space-time Dirac delta $\mu = \delta_{(x_0,t_0)}$ defined by $\langle \mu, \varphi\rangle = \varphi(x_0,t_0)$ for every $\varphi(x,t) \in \mathcal{C}_c(Q)$;

ii) a Dirac delta in space $\mu = \mu(x) = \delta_{x_0}$ defined by $\langle \mu, \varphi\rangle = \int_0^\infty \varphi(x_0,t)\,dt$;

iii) more generally, a measure μ concentrated on the set $E \times (0,+\infty)$, where $E \subset \Omega$ has zero "elliptic" 2-capacity;

iv) a measure μ concentrated on a set of the form $E \times \{t_0\}$, where $E \subset \Omega$ has zero Lebesgue measure.

The main result of this paper is the following multiplicity result.

Theorem 4.6. *Let μ_s be a positive, singular Radon measure such that $\mu_s\big|_{Q_T}$ is bounded for every $T > 0$. Assume that $f(x,t)$ is a positive function such that $f \in L^r_{\mathrm{loc}}([0,\infty); L^q(\Omega))$, where r and q satisfy the Aronson-Serrin hypothesis* (26)*, and that the initial datum u_0 satisfies $v_0 = e^{u_0} - 1 \in L^1(\Omega)$. Consider v, the unique solution of problem*

$$(32)\qquad \begin{cases} v_t - \Delta v &= f(x,t)\,(v+1) + \mu_s \ \text{ in } \mathcal{D}'(Q) \\ v &\in L^\infty_{\mathrm{loc}}([0,\infty); L^1(\Omega)) \cap L^\rho_{\mathrm{loc}}([0,\infty); W_0^{1,\sigma}(\Omega)) \\ & \qquad\qquad \text{where } \sigma, \rho > 1 \text{ verify } \dfrac{N}{\sigma} + \dfrac{2}{\rho} > N+1 \\ v(x,0) &= v_0(x)\,, \qquad f\,v \in L^1_{\mathrm{loc}}(\overline{Q}). \end{cases}$$

We set $u = \log(v+1)$, then $u \in L^2_{\mathrm{loc}}([0,\infty); W_0^{1,2}(\Omega)) \cap \mathcal{C}([0,\infty); L^1(\Omega))$ and is a weak solution of

$$(33)\qquad \begin{cases} u_t - \Delta u &= |\nabla u|^2 + f(x,t) \ \text{ in } \mathcal{D}'(Q) \\ u(x,0) &= u_0(x) \equiv \log(v_0(x)+1)\,. \end{cases}$$

Proof. Let $h_n(x,t) \in L^\infty(Q)$ be a sequence of bounded nonnegative functions such that $\|h_n\|_{L^1(Q_T)} \le C(T)$ for every $T > 0$, and

$$h_n \rightharpoonup \mu_s \ \text{ weakly in the measures sense in } Q_T, \text{ for every } T > 0.$$

Consider now the unique solution v_n to problem

$$(34)\qquad \begin{cases} (v_n)_t - \Delta v_n &= T_n(f\,(v+1)) + h_n \qquad \text{in } Q \\ v_n &\in L^2_{\mathrm{loc}}([0,\infty); W_0^{1,2}(\Omega)) \\ v_n(x,0) &= T_n(v_0(x))\,. \end{cases}$$

Notice that $(v_n)_t \in L^2_{\mathrm{loc}}(\overline{Q})$ (see for instance [24]), and that, for every $T > 0$, $v_n \to v$ in $L^\rho(0,T; W_0^{1,\sigma}(\Omega))$ for all ρ and σ as in (32). We set $u_n = \log(v_n+1)$, then by a direct computation one can check that

$$(u_n)_t - \Delta u_n = |\nabla u_n|^2 + \frac{T_n(f\,(v+1))}{v_n+1} + \frac{h_n}{v_n+1} \text{ in } \mathcal{D}'(Q).$$

Notice that by using the definition of v_n we conclude easily that, for every $T > 0$,

$$(35)\qquad \frac{T_n(f(v+1))}{v_n+1} \to f(x,t) \text{ in } L^1(Q_T) \text{ and } u_n \to u \text{ in } L^1(Q_T).$$

We claim that

$$(36)\qquad \frac{h_n}{v_n+1} \to 0 \text{ in } \mathcal{D}'(Q).$$

To prove the claim let $\phi(x,t)$ be a function in $\mathcal{C}_0^\infty(Q)$; we want to prove that

$$\lim_{n\to\infty} \iint_{Q_T} \phi \frac{h_n}{v_n+1}\,dx = 0\,.$$

We assume that $\operatorname{supp}\phi \subset Q_T$, and we use the assumption on μ_s: let $A \subset Q_T$ be such that $\mathrm{cap}_{1,2}(A) = 0$ and $\mu_s \llcorner Q_T$ is concentrated on A. Then for all $\varepsilon > 0$ there exists an open set

$U_\varepsilon \subset Q_T$ such that $A \subset U_\varepsilon$ and $\mathrm{cap}_{1,2}(U_\varepsilon) \leq \varepsilon/2$. Then, we can find a function $\psi_\varepsilon \in \mathbf{W}_T$ such that $\psi_\varepsilon \geq \chi_{U_\varepsilon}$ and $||\psi_\varepsilon||_{\mathbf{W}_T} \leq \varepsilon$. Let us define the real function

$$m(s) = \frac{2|s|}{|s|+1}\,.$$

Then one has

$$m(\psi_\varepsilon) \leq 2, \qquad m(\psi_\varepsilon) \geq \chi_{U_\varepsilon} \text{ and}$$

$$\iint_{Q_T} |\nabla m(\psi_\varepsilon)|^2\,dx\,dt = \iint_{Q_T} |m'(\psi_\varepsilon)|^2 |\nabla \psi_\varepsilon|^2\,dx\,dt \leq 4\,\varepsilon^2.$$

Using a Picone-type inequality (see [3]), we obtain that

$$4\,\varepsilon^2 \geq \int_\Omega |\nabla m(\psi_\varepsilon)|^2\,dx \geq \int_\Omega \frac{-\Delta(v_n+1)}{v_n+1}\,m^2(\psi_\varepsilon)\,dx \geq \int_\Omega \frac{h_n}{v_n+1}\,m^2(\psi_\varepsilon)\,dx - \int_\Omega \frac{(v_n)_t}{v_n+1}\,m^2(\psi_\varepsilon)\,dx.$$

By integration in t, we get

$$\begin{aligned} \iint_{U_\varepsilon} \frac{h_n}{v_n+1}\,dx\,dt &\leq 4\,\varepsilon^2\,T + \int_\Omega \log(v_n(x,T)+1)\,m^2(\psi_\varepsilon(x,T))\,dx \\ &+ 2\iint_{Q_T} \log(v_n+1)\,m(\psi_\varepsilon)\,m'(\psi_\varepsilon)\,(\psi_\varepsilon)_t\,dx\,dt \\ &= 4\,\varepsilon^2\,T + I_1 + I_2\,. \end{aligned} \tag{37}$$

We begin by estimating the integral I_1. Since $|m(s)| \leq 2$, then using Hölder's inequality we obtain that

$$I_1 \leq C\Big(\int_\Omega \log^2(v_n(x,T)+1)\,dx\Big)^{\frac12}\Big(\int_\Omega m^4(\psi_\varepsilon(x,T))\,dx\Big)^{\frac12} \leq C\,\Big(\int_\Omega m^2(\psi_\varepsilon(x,T))\,dx\Big)^{\frac12}$$

where in the last estimate we have used the inequality $\log(s+1) \leq s^{\frac12} + c$ and the bound

$$\max_{t\in[0,T]} \int_\Omega v_n(x,t)\,dx \leq C(T)\,.$$

Since $m(s) \leq 2\,|s|$, it follows that

$$I_1 \leq C\,\Big(\int_\Omega |\psi_\varepsilon(x,T)|^2\,dx\Big)^{\frac12} \leq \max_{t\in[0,T]}\Big(\int_\Omega \psi_\varepsilon^2(x,t)\,dx\Big)^{\frac12} \leq C\,||\psi_\varepsilon||_{\mathbf{W}_T} \leq C\,\varepsilon, \tag{38}$$

by the fact that $\mathbf{W}_T \subset \mathcal{C}([0,T];L^2(\Omega))$ with a continuous inclusion.

We now estimate I_2. Using $\dfrac{m^2(\psi_\varepsilon)}{v_n+1}$ as a test function in (34), by a direct computation we obtain

$$\begin{aligned}
&\int\limits_\Omega \log(v_n(x,T)+1)\, m^2(\psi_\varepsilon(x,T))\,dx - \int\limits_\Omega \log(T_n(v_0)+1)\, m^2(\psi_\varepsilon(x,0))\,dx\\
&\quad - 2\iint_{Q_T} \log(v_n+1)\, m(\psi_\varepsilon)\, m'(\psi_\varepsilon)\,(\psi_\varepsilon)_t\,dx\,dt + 2\iint_{Q_T} m(\psi_\varepsilon)\, m'(\psi_\varepsilon)\,\nabla\psi_\varepsilon\,\frac{\nabla v_n}{v_n+1}\,dx\,dt\\
&\qquad - \iint_{Q_T} m^2(\psi_\varepsilon)\,\frac{|\nabla v_n|^2}{(v_n+1)^2}\,dx\,dt = \iint_{Q_T} \frac{m^2(\psi_\varepsilon)}{v_n+1}\Big(T_n(f(v+1)) + h_n(x,t)\Big)\,dx\,dt \ge 0\,.
\end{aligned}$$

Thus, recalling (38) and (9) which holds for v_n, we get

$$\begin{aligned}
&2\iint_{Q_T} \log(v_n+1)\, m(\psi_\varepsilon)\, m'(\psi_\varepsilon)\,(\psi_\varepsilon)_t\,dx\,dt\\
&\le I_1 + 2\iint_{Q_T} m(\psi_\varepsilon)\, m'(\psi_\varepsilon)\,\nabla\psi_\varepsilon\frac{\nabla v_n}{v_n+1}\,dx\,dt \le C\,\varepsilon + 8\iint_{Q_T} |\nabla\psi_\varepsilon|\frac{|\nabla v_n|}{v_n+1}\,dx\,dt\\
&\qquad \le C\,\varepsilon + 8\Big(\iint_{Q_T}|\nabla\psi_\varepsilon|^2\,dx\,dt\Big)^{\frac12}\Big(\iint_{Q_T}\frac{|\nabla v_n|^2}{(v_n+1)^2}\,dx\,dt\Big)^{\frac12} \le C\,\varepsilon\,.
\end{aligned} \tag{39}$$

Hence by (37) we conclude that

$$\iint_{U_\varepsilon} \frac{h_n}{v_n+1}\,dx\,dt \le C(\varepsilon+\varepsilon^2)\,. \tag{40}$$

Now, by (39),

$$\begin{aligned}
&\Big|\iint_{Q_T} \phi\frac{h_n}{v_n+1}\,dx\,dt\Big|\\
&\le ||\phi||_\infty \iint_{U_\varepsilon}\frac{h_n}{v_n+1}\,dx\,dt + \iint_{Q_T\setminus U_\varepsilon}|\phi|\,h_n\,dx\,dt \le C||\phi||_\infty\,(\varepsilon+\varepsilon^2) + \iint_{Q_T\setminus U_\varepsilon}|\phi|\,h_n\,dx\,dt\,.
\end{aligned}$$

Since $h_n \to \mu_s$ in $\mathcal{M}_0(Q_T)$ and μ_s is concentrated on $A\subset U_\varepsilon$, we conclude that

$$\iint_{\Omega\setminus U_\varepsilon} |\phi|\,h_n\,dx\,dt \to 0 \qquad \text{as } n\to\infty.$$

Since ε is arbitrary we get the desired result, hence the claim (36) follows.
Let $\phi\in\mathcal{C}_0^\infty(Q_T)$, then we have

$$\begin{aligned}
&\iint_{Q_T} ((u_n)_t - \Delta u_n)\,\phi\,dx\,dt\\
&= \iint_{Q_T} \frac{T_n(f(v+1))}{v_n+1}\phi\,dx\,dt + \iint_{Q_T}|\nabla u_n|^2\phi\,dx\,dt + \iint_{Q_T}\frac{h_n\phi}{v_n+1}\,dx\,dt.
\end{aligned}$$

Hence using (35) and (36) we just have to prove that

$$|\nabla u_n|^2 \to |\nabla u|^2 \text{ in } L^1(Q_T)$$

which means that

$$\frac{|\nabla v_n|^2}{(v_n+1)^2} \to \frac{|\nabla v|^2}{(v+1)^2} \text{ in } L^1(Q_T).$$

Since the sequence $\Big\{\dfrac{|\nabla v_n|^2}{(v_n+1)^2}\Big\}$ converges a.e. in Q_T to $\dfrac{|\nabla v|^2}{(v+1)^2}$, then by Vitali's theorem we only have to prove that it is equi-integrable. Let $E \subset Q_T$ be a measurable set. Then, for every $\delta \in (0,1)$ and $k > 0$,

$$\begin{aligned}\iint_E \frac{|\nabla v_n|^2}{(v_n+1)^2}\,dx\,dt &= \iint_{E\cap\{v_n\le k\}} \frac{|\nabla v_n|^2}{(v_n+1)^2}\,dx\,dt + \iint_{E\cap\{v_n>k\}} \frac{|\nabla v_n|^2}{(v_n+1)^2}\,dx\,dt \\ &\le \iint_E |\nabla T_k(v_n)|^2\,dx\,dt + \frac{1}{(k+1)^{1-\delta}} \iint_{Q_T} \frac{|\nabla v_n|^2}{(v_n+1)^{1+\delta}}\,dx\,dt\,.\end{aligned}$$

By (9), the last integral is uniformly bounded with respect to n, therefore the corresponding term can be made small by choosing k large enough. Moreover, since μ_s is singular and $T_n(f(v+1)) \to f(v+1)$ in $L^1(Q_T)$, one has (see Petitta [37]) $T_k(v_n) \to T_k(v)$ strongly on $L^2(0,T;W_0^{1,2}(\Omega))$ for any $k > 0$, therefore the integral $\int_E |\nabla T_k(v_n)|^2\,dx\,dt$ is uniformly small if meas (E) is small enough. The equi-integrability of $|\nabla u_n|^2$ follows immediately, and the proof is completed. Hence we conclude that

$$u_t - \Delta u = |\nabla u|^2 + f(x,t) \text{ in } \mathcal{D}'(Q).$$

Since $|\nabla u|^2 + f \in L^1(\Omega\times(0,T))$, then using classical result about the regularity and uniqueness of entropy solution we obtain that $u \in \mathcal{C}([0,\infty);L^1(\Omega))$ and the result follows. ∎

Remark 4.7.

(1) An interesting point is the following. If we consider $x_0 \in \Omega$ and $0 < t_0 < T$ and the problem

$$v_t - \Delta u = \delta_{x_0,t_0}, \quad v(x,t) = 0 \text{ on } \partial\Omega\times(0,T), \quad v(x,0) = 0,$$

then it is easy to check that $t \to ||v(t)||_1$, has a jump in $t = t_0$. However, defining $u = \log(1+v)$, u belongs to $\mathcal{C}([0,T];L^1(\Omega))$. The mechanism of this behavior is as follows: 1) u solves the equation $u_t - \Delta u = |\nabla u|^2$ in the sense of distributions; 2) the regularity theory for L^1 data provides the continuity.

(2) In general we can prove that if v is a solution to problem

$$v_t - \Delta v = \mu \text{ in } Q_T, \quad v(x,0) = v_0(x) \in L^1(\Omega),$$

where μ is a positive Radon measure, then $\sup\limits_{t\in[0,T]} \int\limits_\Omega v(x,t)\,dx \le C(\mu(Q_T),T)$. Indeed, consider ω, the solution to problem (3), it is clear that $\omega \le 1$, hence ω is globally defined and therefore using ω as a test function in (2), it follows that

$$\int\limits_\Omega v(x,\tau)\,dx \le \int\limits_\Omega v_0(x)\,\omega(x,0)\,dx + c(T)\mu(Q_T).$$

Hence the result follows by taking the maximum for $\tau \in [0,T]$.

Theorem 4.8. *Let $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega))$ be a solution to problem (11), where $f(x,t)$ is a positive function such that $f \in L^r_{\rm loc}([0,\infty); L^q(\Omega))$, where r and q satisfy the Aronson-Serrin hypothesis (26). Consider $v = e^u - 1$, then $v \in L^1_{\rm loc}(\overline{Q})$, and there exists a bounded positive measure μ in Q_T for every $T > 0$, such that*

(1) v is a distributional solution of

$$v_t - \Delta v = f(x,t)\,(v+1) + \mu \qquad \text{in } Q\,. \tag{41}$$

(2) μ is concentrated on the set $A \equiv \{(x,t):\ u(x,t) = \infty\}$ and $\mathrm{cap}_{1,2}(A \cap Q_T) = 0$ for all $T > 0$, that is μ is a singular measure with respect to $\mathrm{cap}_{1,2}$-capacity.

Moreover μ can be characterized as a weak limit in the space of bounded Radon measures, as follows:

$$\mu = \lim_{\varepsilon\to 0} |\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) \quad \text{in } Q_T, \text{ for every } T > 0. \tag{42}$$

Proof. We set $v = e^u - 1$, then by the regularity results of Theorem 3.2, we obtain that $v \in L^1_{\rm loc}(\overline{Q})$ and

$$\iint_{Q_\tau} f(x,t)\,(v+1)\,dx\,dt + \iint_{Q_\tau} |\nabla u|^2\, e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) dx\,dt \le C(\tau)\,. \tag{43}$$

Therefore, there exists a positive Radon measure μ in Q such that for all $\tau > 0$

$$|\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) \rightharpoonup \mu \quad \text{in the weak measure sense in } Q_\tau.$$

Notice that μ is concentrated in the set $A \equiv \{(x,t) \in Q : u(x,t) = \infty\}$. This follows from the fact that

$$\iint_{Q_\tau \cap \{u \le k\}} |\nabla u|^2\, e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) dx\,dt \to 0 \text{ as } \varepsilon \to 0.$$

We now define

$$v_\varepsilon(x,t) = \int_0^{u(x,t)} e^{\frac{s}{1+\varepsilon s}}\,ds \in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega)).$$

By making an approximation as in the first part of the proof of Proposition 3.1, it is easy to check that v_ε solves

$$(v_\varepsilon)_t - \Delta v_\varepsilon = e^{\frac{u}{1+\varepsilon u}}\,|\nabla u|^2 (1 - \frac{1}{(1+\varepsilon u)^2}) + f(x,t) e^{\frac{u}{1+\varepsilon u}} \tag{44}$$

in the sense of distributions.

By (43) and the monotone convergence theorem we get easily that the last term converges in $L^1(Q_\tau)$ for all $\tau > 0$, while the remaining one converges to μ. Since $v_\varepsilon \to v$ in $L^1(Q_\tau)$ for all $\tau > 0$, we obtain that v solves the equation (41) in the sense of distributions, therefore μ is uniquely determined.

Finally to prove that $\mathrm{cap}_{1,2}(A \cap Q_T) = 0$ and then μ is a singular measure in the sense of Definition 4.5 we use a remark by A. Porretta, [39], that we detail below.

Consider $A_T = A\cap Q_T$, it is clear that $u\in \mathcal{C}([0,T];L^1(\Omega))\cap L^2([0,T];W_0^{1,2}(\Omega))$ solves problem

$$\left\{\begin{array}{rcll} u_t-\Delta u &=& g(x,t)\equiv |\nabla u|^2+f(x,t) & \text{in } Q_T\\ u(x,t) &=& 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0) &=& u_0(x) & \text{in } \Omega, \end{array}\right.$$

Let $\tau\le T$, using $T_k(u)$ as a test function in the above problem it follows that

$$\int_\Omega \Theta_k(u(x,\tau))\,dx\ +\ \iint_{Q_\tau}|\nabla T_k(u)|^2\,dx\,dt\ \ =\ \ \iint_{Q_\tau} g(x,t)T_k(u)\,dx\,dt\ +\ \int_\Omega \Theta_k(u_0(x))\,dx$$

where

$$\Theta_k(s)=\int_0^s T_k(\sigma)d\sigma=\left\{\begin{array}{lll} \frac12 s^2 & \text{if} & |s|\le k,\\ ks-\frac12 k^2 & \text{if} & |s|\ge k.\end{array}\right.$$

Thus

$$\int_\Omega \Theta_k(u(x,\tau))\,dx+\iint_{Q_\tau}|\nabla T_k(u)|^2\,dx\,dt\le k(||g||_{L^1(Q_T)}+||u_0||_{L^1(\Omega)}).$$

Since $\Theta_k(s)\ge \frac12 T_k^2(s)$, we conclude that

$$||T_k(u)||^2_{L^\infty((0,T);L^2(\Omega))}+||T_k(u)||^2_{L^2((0,T);W_0^{1,2}(\Omega))}\le C(T)k.$$

Consider $w_k=\dfrac{T_k(u)}{k}$, it is clear that $w_k\in\mathbf{X}\equiv L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);W_0^{1,2}(\Omega))$ and $||w_k||_X^2\le \dfrac{C(T)}{k}$. Hence $||w_k||^2_{\mathbf{X}}\to 0$ as $k\to\infty$. Using an approximation argument and by Kato type inequality, see for instance [35], there results that

$$(w_k)_t-\Delta w_k\ge 0.$$

Therefore by using Proposition 3 in [37], we obtain $z_k\in\mathbf{W}$ such that $z_k\ge w_k$ and $||z_k||_{\mathbf{W}}\le ||w_k||_{\mathbf{X}}$. It is clear that $z_k\ge 1$ on A_T. Hence

$$\text{cap}_{1,2}(A_T)\le ||z_k||_{\mathbb{W}}\le ||w_k||_X\le (\frac{C(T)}{k})^{\frac12}.$$

Letting $k\to\infty$ it follows that $\text{cap}_{1,2}(A_T)=0$ and then the result follows. ∎

Corollary 4.9. *There exist a unique solution to problem* (11) *in the class*

$$\mathcal{X}=\{u\in L^1_{\text{loc}}(Q)\,:\,e^{\frac u2}-1\in L^2_{\text{loc}}([0,\infty);W_0^{1,2}(\Omega))\}\,.$$

PROOF. It is sufficient to observe that, setting $v=e^u-1$, then by Theorem 4.8, v solves (41). Using (42) we get $\mu=0$. We claim that

$$\int\limits_\Omega v(x,\tau)\phi dx\to\int\limits_\Omega (e^{u_0(x)}-1)\phi dx \text{ as } \tau\to 0 \text{ for all } \phi\in\mathcal{C}^1(\overline{\Omega}),\ \phi|_{\partial\Omega}=0.$$

From the regularity result of Theorem (3.2) we know that $e^u \in L^\infty(0,\tau;L^1(\Omega))$. Let $\phi \in \mathcal{C}^1(\overline{\Omega})$ be such that $\phi|_{\partial\Omega} = 0$, since $e^{\frac{u}{2}} - 1 \in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega))$, then using Theorem (3.2), and by an approximation argument, we can use $e^u\,\phi$ as a test function in (11). Hence it follows that

$$\int_\Omega e^{u(x,\tau)}\phi\,dx + \int_0^\tau \int_\Omega e^u \nabla u\,\nabla\phi\,dx\,dt = \int_0^\tau \int_\Omega e^u\,f\,\phi\,dx\,dt + \int_\Omega e^{u_0(x)}\phi\,dx.$$

Since $f\,e^u \in L^1(Q_{\tau_1})$ where $\tau_1 > 0$, then

$$\lim_{\tau\to 0} \int_0^\tau \int_\Omega e^u\,f\,\phi dx\,dt = 0.$$

Moreover we have

$$\int_0^\tau \int_\Omega e^u|\nabla u||\nabla\phi| dx\,dt \le \frac{1}{2}\int_0^\tau \int_\Omega e^u|\nabla u|^2\,dx\,dt + \int_0^\tau \int_\Omega e^u|\phi|^2\,dx\,dt \to 0 \text{ as } \tau \to 0.$$

Putting together the previous estimates we conclude that

$$\int_\Omega v(x,\tau)\phi dx = \int_\Omega (e^{u(x,\tau)} - 1)\,\phi\,dx \to \int_\Omega (e^{u_0(x)} - 1)\phi\,dx \text{ as } \tau \to 0$$

and then the claim follows. Hence $v \in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega))$ solves

$$v_t - \Delta v = f(x,t)\,(v+1) \qquad \text{in } Q\,.$$

with

$$\int_\Omega v(x,\tau)\phi\,dx \to \int_\Omega (e^{u_0(x)} - 1)\phi\,dx \text{ as } \tau \to 0.$$

The linear classical theory gives the uniqueness. ■

Remark 4.10. A direct computations show that if u is a solution to problem (11), then $u_t, |\nabla u|^2 \in \mathbf{W}'_T$, the dual of $\mathbf{W}_T$ defined in (4.3), for every $T > 0$.

In the same way we have $\dfrac{v_t}{v+1}, \dfrac{|\nabla v|^2}{(1+v)^2} \in \mathbf{W}'_T$ where v is the solution to problem (32). We refer to [23] for a complete characterization of $\mathbf{W}'_T$.

4.4. **Nonuniqueness induced by singular perturbations of the initial data.**

We prove in this subsection nonuniqueness for problem (11) by perturbing the initial data in the associated linear problem with a suitable singular measure. For sake of simplicity, we limit ourselves to the case where $f(x,t) \equiv 0$. In what follows, we will denote by $|E|$ the usual Lebesgue measure $\mathbb{R}^N$. The main result in this direction is the following.

Theorem 4.11. *Let ν_s be a bounded positive singular measure in Ω, concentrated on a subset $E \subset\subset \Omega$ such that $|E| = 0$. Let v be the unique solution of problem*

$$\left\{\begin{array}{rcl} v_t - \Delta v &=& 0 \text{ in } \mathcal{D}'(Q) \\ v(x,t) &=& 0 \text{ on } \partial\Omega \times (0,\infty) \\ v(x,0) &=& \nu_s\,. \end{array}\right. \tag{45}$$

We set $u = \log(v+1)$*, then* $u \in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega))$ *and verifies*

$$\begin{cases} u_t - \Delta u &= |\nabla u|^2 \text{ in } \mathcal{D}'(Q) \\ u(x,0) &= 0. \end{cases} \tag{46}$$

PROOF. Let $h_n \in L^\infty(\Omega)$ be a sequence of nonnegative functions such that $||h_n||_{L^1(\Omega)} \leq C$ and $h_n \rightharpoonup \nu_s$ weakly in the measure sense, namely

$$\lim_{n\to\infty} \int\limits_\Omega h_n(x)\phi(x)dx \to \langle \nu_s, \phi\rangle \text{ for all } \phi \in \mathcal{C}_c(\Omega).$$

Consider now v_n the unique solution to problem

$$\begin{cases} (v_n)_t - \Delta v_n &= 0 \text{ in } Q \\ v_n &\in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega)) \\ v_n(x,0) &= h_n(x)\,. \end{cases} \tag{47}$$

Notice that $v_n \to v$ strongly in $L^r(0,T; W_0^{1,q}(\Omega))$ for all r and q satisfying $\dfrac{N}{q} + \dfrac{2}{r} > N+1$ and

$$\int\limits_\Omega v_n(x,t)\phi(x)dx \to \int\limits_\Omega h_n(x)\phi(x)dx \text{ as } t \to 0, \text{ for all } \phi \in \mathcal{C}(\overline{\Omega}).$$

As in the proof of Theorem 4.6, we can prove that $|\nabla u_n|^2 \to |\nabla u|^2$ strongly in $L^1(Q_T)$ for all $T > 0$, the only difference being that in this case the strong convergence of the truncates is proved in [11].
Moreover to finish we have just to show that $\log(1 + v_n(.,t)) \to 0$ strongly in $L^1(\Omega)$ as $t \to 0$ and $n \to \infty$.
To prove this last affirmation, take $H(v_n)$, where $H(s) = 1 - \dfrac{1}{(1+s)^\alpha}$, $0 < \alpha << 1$, as a test function in (47), then

$$\int\limits_\Omega \overline{H}(v_n(x,\tau))\,dx + \alpha \iint_{Q_\tau} \frac{|\nabla v_n|^2}{(1+v_n)^{1+\alpha}}\,dx\,dt = \int\limits_\Omega \overline{H}(h_n(x))\,dx$$

where $\overline{H}(s) = \int_0^s H(\sigma)d\sigma = s - \frac{1}{1-\alpha}((1+s)^{1-\alpha} - 1)$. Hence $\int_\Omega v_n(x,t)\,dx \leq C$ where C is positive constant independent of n and t. As a consequence we obtain that $\log(1 + v_n(.,t))$ is bounded in $L^p(\Omega)$ for all $p < \infty$ uniformly in n and t.
By the strong convergence of $T_k v_n$, then for small $\varepsilon > 0$ we get the existence of $n(\varepsilon)$ and $\tau(\varepsilon) > 0$ such that for $n \geq n(\varepsilon)$ and $t \leq \tau(\varepsilon)$, we have

$$\iint_{Q_t} \frac{|\nabla v_n|^2}{(1+v_n)^2}\,dx\,ds \leq \varepsilon. \tag{48}$$

Since ν_s is concentrated on a set $E \subset\subset \Omega$ with $|E| = 0$, then for $\varepsilon \in (0,1)$ [[**I have added** $\varepsilon < 1$]] there exists an open set U_ε such that $E \subset U_\varepsilon \subset \Omega$ and $|U_\varepsilon| \leq \varepsilon/2$.
Without loss of generality we can assume that $\operatorname{supp} h_n \subset U_\varepsilon$ for $n \geq n(\varepsilon)$.
Let $\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \phi_\varepsilon \leq 1$, $\phi_\varepsilon = 1$ in U_ε, $\operatorname{supp}\phi_\varepsilon \subset O_\varepsilon$ and $|O_\varepsilon| \leq 2\varepsilon$.

Consider w_ε, the solution to problem

$$\begin{cases} w_{\varepsilon t} - \Delta w_\varepsilon &= 0 \text{ in } Q, \\ w_\varepsilon(x,t) &= 0 \text{ on } \partial\Omega \times (0,\infty), \\ w_\varepsilon(x,0) &= \phi_\varepsilon(x). \end{cases}$$

It is clear that $0 \le w_\varepsilon \le 1$,

$$w_\varepsilon \to 0 \text{ strongly in } L^2(0,\infty); W_0^{1,2}(\Omega)),$$
$$w_\varepsilon \to 0 \text{ strongly in } \mathcal{C}([0,\infty); L^2(\Omega)),$$

(and $\dfrac{dw_\varepsilon}{dt} \to 0$ strongly in $L^2(0,\infty); W^{-1,2}(\Omega))$).

For $t \le \tau \equiv \tau(\varepsilon)$, we set $\widetilde{w}_\varepsilon(x,t) = w(x,\tau - t)$, using $\dfrac{\widetilde{w}_\varepsilon}{1+v_n}$ as a test function in (47), it follows that

$$\int\limits_\Omega \log(1+v_n(x,\tau))\,\widetilde{w}_\varepsilon(x,\tau)\,dx - \iint_{Q_\tau} \frac{|\nabla v_n|^2}{(1+v_n)^2}\widetilde{w}_\varepsilon\,dx\,ds = \int\limits_\Omega \log(1+h_n)\widetilde{w}_\varepsilon(x,0)\,dx.$$

Using (48) and the properties of $\widetilde{w}_\varepsilon$, we get

$$\int_{U_\varepsilon} \log(1+v_n(x,\tau))\,dx \le \varepsilon + \int\limits_\Omega \log(1+h_n)\,\widetilde{w}_\varepsilon(x,0)\,dx \le \varepsilon + \int\limits_\Omega \log(1+h_n)\,dx$$

It is clear that we can obtain the same estimate for any $t \le \tau(\varepsilon)$. Since $\operatorname{supp} h_n \subset U_\varepsilon$, then

$$\int\limits_\Omega \log(1+h_n)\,dx = \int_{U_\varepsilon} \log(1+h_n)\,dx \le C\Big(\varepsilon + \int_{U_\varepsilon} h_n^{1/2}\,dx\Big) \le C(\varepsilon + \varepsilon^{1/2}) \le C\,\varepsilon^{1/2},$$

Hence we conclude that

$$\int_{U_\varepsilon} \log(1+v_n(x,t))\,dx \le C\,\varepsilon^{1/2} \text{ for } n \ge n(\varepsilon) \text{ and } t \le \tau(\varepsilon). \tag{49}$$

We now deal with the complement integral $\int_{\Omega\setminus U_\varepsilon} \log(1+v_n(x,t))\,dx$.

Let $\psi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $0 \le \psi_\varepsilon \le 1$, $\psi_\varepsilon = 0$ in N where N is an open set such that $E \subset\subset N \subset\subset U_\varepsilon$ and $\psi_\varepsilon \equiv 1$ in $\Omega\setminus U_\varepsilon$. [[$\psi_e$, **not** ψ]]

As above, let z_ε, the solution to problem

$$\begin{cases} (z_\varepsilon)_t - \Delta z_\varepsilon &= 0 \text{ in } Q, \\ z_\varepsilon(x,t) &= 0 \text{ on } \partial\Omega \times (0,\infty), \\ z_\varepsilon(x,0) &= \psi_\varepsilon(x). \end{cases}$$

It is not difficult to see that $0 \le z_\varepsilon \le 1$. For $t \le \tau \equiv \tau(\varepsilon)$, we consider $\widetilde{z}_\varepsilon(x,t) = z(x,\tau - t)$, using $\dfrac{\widetilde{z}_\varepsilon}{1+v_n}$ as a test function in (47), and proceeding as above, we get the existence of $\tau(\varepsilon)$ and $n(\varepsilon)$ such that for $n \ge n(\varepsilon)$ and $t \le \tau(\varepsilon)$, we have

$$\int_\Omega \log(1+v_n(x,t))\,dx \le C\,\varepsilon^{1/2}$$

and then we get the desired result.
Hence, as a conclusion we obtain that u solves (46). ■

Remarks 4.12. The previous theorem can also be shown to be true under the presence of an additional initial data $v_0 \in L^1(\Omega)$ and a term $f(x,t)$ in the right-hand side. Therefore, putting together this and the result of Theorem 4.6, the following general multiplicity result can be proved.
Assume that μ_s is a positive Radon measure in Q, singular with respect to the parabolic capacity $\mathrm{cap}_{1,2}$, and $\nu_s \in M(\Omega)$ is a positive Radon measure in Ω, singular with respect to the classical Lebesgue measure, and let v be the unique positive solution to problem

$$\begin{cases} v_t - \Delta v &= f(x,t)\,(v+1) + \mu_s \text{ in } \mathcal{D}'(Q) \\ \\ v(x,0) &= v_0(x) + \nu_s\,, \end{cases}$$

where where $f \in L^r_{\mathrm{loc}}([0,\infty); L^q(\Omega))$, with r and q satisfy the Aronson-Serrin hypothesis (26), and $v_0 \in L^1(\Omega)$. If we set $u = \log(1+v)$, then u solves

$$\begin{cases} u_t - \Delta u &= |\nabla u|^2 + f(x,t) \text{ in } \mathcal{D}'(Q), \\ \\ u(x,0) &= \log(1 + v_0(x)). \end{cases}$$

5. The case of increasing β

We will now consider problem (1), where f is a nonnegative function in $L^\infty_{\mathrm{loc}}(\overline{Q})$ and

$$\beta \,:\, [0,\infty) \longrightarrow [0,\infty)$$

is a continuous nondecreasing function, not identically zero. We set

$$\gamma(t) = \int_0^t \beta(s)ds, \qquad \Psi(t) = \int_0^t e^{\gamma(s)}ds, \tag{50}$$

and we define

$$v(x,t) = \Psi(u(x,t)).$$

Then problem (1) becomes

$$\begin{cases} v_t - \Delta v = f(x,t)\,g(v) & \text{in } Q \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = \Psi(u_0) & \text{in } \Omega, \end{cases} \tag{51}$$

where

$$g(t) = e^{\gamma(\Psi^{-1}(t))} = 1 + \int_0^t \beta(\Psi^{-1}(s))\,ds\,. \tag{52}$$

The main properties of the continuously differentiable function $g : [0,\infty) \longrightarrow [0,\infty)$ are:

$$
\left\{
\begin{array}{ll}
(1) & g(0) = 1, \text{ and } g \text{ is increasing and convex} \\[2ex]
(2) & \displaystyle\lim_{s\to 0} \frac{g(s)-1}{s} = g'(0) = \beta(0) \\[2ex]
(3) & \displaystyle\lim_{s\to\infty} \frac{g(s)}{s} = \lim_{s\to\infty} \beta(s) \in (0,\infty] \\[2ex]
(4) & \displaystyle\int_0^\infty \frac{ds}{g(s)} = +\infty; \text{ indeed} \\[2ex]
 & \displaystyle\int_0^\infty \frac{ds}{g(s)} = \int_0^\infty \frac{ds}{e^{\gamma(\psi^{-1}(s))}} \int_0^\infty \frac{e^{\gamma(t)}}{e^{\gamma(t)}} dt = \infty
\end{array}
\right. \tag{53}
$$

Proposition 5.1. *Assume that g verifies the assumptions above and that f is a bounded function. Let v_0 be a bounded positive function, then there exists a unique positive solution $v \in L^\infty_{\mathrm{loc}}(\overline{Q})$ to problem*

$$
\left\{
\begin{array}{rcll}
v_t - \Delta v &=& f(x,t)\, g(v) & \text{in } Q \\
v &=& 0 & \text{on } \partial\Omega \times (0,\infty) \\
v(x,0) &=& v_0(x) & \text{in } \Omega.
\end{array}
\right. \tag{54}
$$

Therefore problem (1) has at least one positive solution u such that $\Psi(u) \in L^\infty_{\mathrm{loc}}(\overline{Q}) \cap L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_0(\Omega))$ and $u(x,0) = \Psi^{-1}(v_0)$.

Proof. The proof is trivial, using a sub/super-solution argument, considering a super-solution of the form $w = w(t)$. By (4) in (53) all solutions of (54) with bounded data are bounded in Q_T. Since g is locally Lipschitz, the uniqueness follows directly by using Gronwall's inequality. ■

In order to obtain a global solution for unbounded initial data and a measure source term, we will assume the following structural hypotheses on g, which is satisfied by all elementary functions $\beta(u)$:

$$
g(s) \le c(1 + sA(\log^* s)), \text{ for every } s > 0, \tag{55}
$$

where $\log^* s = \max\{\log s, 1\}$, and $A(t) : [0,+\infty) \to [0,+\infty)$ is a continuous, increasing function such that

(1) A satisfies the so-called Δ_2 condition, that is,

$$
A(2t) \le kA(t) \quad \text{for all } t \ge t_0 \tag{56}
$$

for some positive constants k and t_0;

(2) A is at most slightly superlinear, in the sense that

$$
\int^{+\infty} \frac{ds}{A(s)} = +\infty. \tag{57}
$$

The following existence result is proved in [18].

Proposition 5.2. *Assume that g verifies (55), (56) and the (57) condition. If $v_0 \in L^1(\Omega)$, and μ is a positive measure in Q which is bounded in Q_T for every positive T, then there exists a function*

$$v \in L^\infty_{\rm loc}([0,\infty); L^1(\Omega)) \cap L^q_{\rm loc}([0,\infty); W^{1,q}_0(\Omega)) \cap L^\sigma_{\rm loc}(\Omega \times [0,\infty))$$

for every $q < 1 + \frac{1}{N+1}$ and for every $\sigma < 1 + \frac{2}{N}$, such that

a) *For every $\delta < \frac{1}{2}$, $|v|^\delta \in L^2_{\rm loc}([0,\infty); H^1_0(\Omega))$;*

b) *For all $k > 0$, $T_k v \in L^2_{\rm loc}([0,\infty); H^1_0(\Omega))$,*

which is a weak solution to

$$\begin{cases} v_t - \Delta v = f(x,t)\, g(v) + \mu & \text{in } Q \\ v = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{58}$$

Moreover, if $\mu = 0$ and $v_0 \in L^2(\Omega)$, then

$$v \in \mathcal{C}^0([0,\infty); L^2(\Omega)) \cap L^2_{\rm loc}([0,\infty); W^{1,2}_0(\Omega)).$$

Finally, if g satisfies

$$|g(s_1) - g(s_2)| \le C\,(1 + |s_1|^b + |s_2|^b)\, |s_1 - s_2|\,, \qquad 0 < b < \frac{2}{N}\,, \tag{59}$$

for every $s_1, s_2 \in \mathbb{R}$, then the weak solution of (58) is unique.

Remark 5.3. The assumptions (55), (56), (57) and (59) are satisfied in all the model cases (for instance in the case where $\beta(s)$ is a power, an exponential, or a finite iteration of exponentials, however we do not know whether they hold for every choice of β.

5.1. **Regularity and existence of weaker solutions.** Assume that $f \in L^1_{\rm loc}(\overline{Q})$ is a nonnegative function. Let us consider a distributional solution u of problem (1) in the sense of definition 2.1. We start with the following regularity result.

Proposition 5.4. *Assume that $u(x,t)$ is a distributional solution of problem (1), where $f \in L^1_{loc}(\overline{Q})$ is such that $f(x,t) \ge 0$ a.e. in Q. Then*

$$\int_\Omega \Psi(u(x,t))\, d(x)\, dx < \infty, \qquad \text{a.e for every } t > 0, \tag{60}$$

where Ψ is defined as in (50).

Proof. It suffices to consider the function

$$v_\varepsilon = H_\varepsilon(s) = \int_0^{\frac{s}{1+\varepsilon s}} e^{\gamma(\sigma)}\, d\sigma\,,$$

and to follow the lines of Proposition 3.1, using the inequalities

$$\beta(s)\, H'_\varepsilon(s) - H''_\varepsilon(s) \ge 0\,, \qquad |H''(s)| \le \frac{c(\varepsilon)}{(1+\varepsilon s)^3}\,.$$

■

As a consequence and using the same type of computation as in the proof of Theorem 3.2 we get the following main regularity result.

Theorem 5.5. *Under the same hypotheses as in the previous Propositions, for all $\tau > 0$ we have*

$$\iint_{Q_\tau} \beta(u)\,|\nabla u|^2\, e^{\delta\gamma(u)}\,dx\,dt < \infty, \qquad \text{for all } \delta < 1\,, \tag{61}$$

$$\iint_{Q_\tau} f\, e^{\gamma(u)}\,dx\,dt < \infty\,, \tag{62}$$

$$\iint_{Q_\tau} \beta(u)\, e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\,|\nabla u|^2 \Big(1 - \frac{1}{(1+\varepsilon\gamma(u))^2}\Big)\,dx\,dt \le C(\tau) \quad \text{uniformly in } \varepsilon\,, \tag{63}$$

$$\int_\Omega \Psi(u_0(x))\,dx < \infty\,, \tag{64}$$

and finally

$$\Psi(u(x,t)) \in L^\infty_{\rm loc}([0,\infty)\,;\,L^1(\Omega))\,. \tag{65}$$

Proof. It suffices to follow the lines of the proof of Theorem 3.2: first one takes $\phi(x,t)\,\big(k_{\delta,\varepsilon}(u(x,t)) - 1\big)$ as test function in (1), where $\phi(x,t)$ is the solution of problem (20), and

$$k_{\delta,\varepsilon}(s) = e^{\frac{\delta\,\gamma(s)}{1+\varepsilon\,\gamma(s)}}\,, \quad \delta \le 1\,.$$

Using the inequality (60) and passing to the limit as $\varepsilon \to 0$, one obtains (61)– (64). Then one multiplies by $k_{1,\varepsilon}(u(x,t))\,\omega(x,t)$, with $\omega(x,t)$ satisfying (3), to obtain (65). ∎

5.2. **Existence and multiplicity result.** The main result of this subsection is the following.

Theorem 5.6. *Let μ_s be a bounded, positive, singular measure on Q such that $\mu_s(Q_T)$ is bounded for every $T > 0$. Let v be a solution to problem*

$$\begin{cases} v_t - \Delta v &= f(x,t)\,g(v) + \mu_s \ \text{in } \mathcal{D}'(Q) \\ v &\in L^\infty_{\rm loc}([0,\infty);L^1(\Omega)) \cap L^r_{\rm loc}([0,\infty);W^{1,q}_0(\Omega)) \\ f(x,t)\,g(v) &\in L^1_{\rm loc}(\overline{Q}) \\ v(x,0) &= v_0(x) \in L^1(\Omega)\,, \end{cases} \tag{66}$$

for all (r,q) such that

$$q, r \ge 1\,, \qquad \frac{N}{q} + \frac{2}{r} > N+1\,.$$

If we define $u = \Psi^{-1}(v)$, where Ψ is given by (50)*, then u solves*

$$(67)\qquad \begin{cases} u_t - \Delta u &= \beta(u)|\nabla u|^2 + f(x,t) \ \textit{in}\ \mathcal{D}'(Q) \\ u &\in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega)) \\ \beta(u)|\nabla u|^2 &\in L^1_{\rm loc}(\overline{Q}) \\ u(x,0) &= u_0(x) := \Psi^{-1}(u_0(x)) \,. \end{cases}$$

Proof. We begin by proving that $\beta(u)|\nabla u|^2 \in L^1_{\rm loc}(Q)$. Let $\{h_n\}$ be a sequence of a bounded positive function such that $h_n \to \mu_s$ in $\mathcal{M}_0(Q_T)$ for every $T > 0$. Let v_n be the unique solution to problem

$$(68)\qquad \begin{cases} (v_n)_t - \Delta v_n &= T_n(f\, g(v)) + h_n(x,t) \text{ in } Q \\ v_n &\in L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega)) \\ v_n(x,0) &= T_n(v_0(x)) \,. \end{cases}$$

Notice that $v_n \to v$ in $L^r(0,T; W_0^{1,q}(\Omega))$ and $||T_k v_n||_{L^2(0,T;W_0^{1,2}(\Omega))} \le A_k$ for all $T > 0$ and $k > 0$. Fix $T > 0$. By taking $\dfrac{g(v_n)-1}{g(v_n)}$ as a test function in (68), we obtain that

$$\int_\Omega h(v_n(x,T))\,dx - \int_\Omega h(v_0(x))\,dx + \iint_{Q_T} \frac{g'(v_n)|\nabla v_n|^2}{(g(v_n))^2}\,dx\,dt \le C(T)\,,$$

where $h(s) = \displaystyle\int_0^s \frac{g(\sigma)-1}{g(\sigma)}\,d\sigma \le s$. Hence using Fatou's lemma we get

$$\iint_{Q_T} \beta(u)\,|\nabla u|^2\,dx\,dt = \iint_{Q_T} \frac{g'(v)\,|\nabla v|^2}{(g(v))^2}\,dx\,dt \le C(T)\,.$$

Notice that by taking $w_n \equiv 1 - \dfrac{1}{(g(v_n))^\delta}$ as test function in (68), we obtain that, for every $\delta > 0$,

$$\iint_{Q_T} \frac{g'(v_n)\,|\nabla v_n|^2}{(g(v_n))^{1+\delta}}\,dx\,dt \le C(T,\delta) \text{ and then } \iint_{Q_T} \frac{g'(v)\,|\nabla v|^2}{(g(v))^{1+\delta}}\,dx\,dt \le C(T,\delta)\,.$$

Since $g'(s) = \beta(\Psi^{-1}(s))$, the hypothesis on β implies $g'(s) \ge C_1 > 0$ for s large enough; recalling that $T_k v_n$ is bounded in $L^2(0,T; W_0^{1,2}(\Omega))$ for every fixed k, we conclude that, for every $\delta > 0$,

$$\iint_{Q_T} \frac{|\nabla v_n|^2}{(g(v_n))^{1+\delta}}\,dx\,dt \le C(T,\delta) \text{ and then } \iint_{Q_T} \frac{|\nabla v|^2}{(g(v))^{1+\delta}}\,dx\,dt \le C(T,\delta)\,.$$

We set $u_n = \Psi^{-1}(v_n)$, then by a direct computation one can check that

$$(u_n)_t - \Delta u_n = \beta(u_n)|\nabla u_n|^2 + \frac{T_n(f\,g(v))}{g(v_n)} + \frac{h_n}{g(v_n)} \text{ in } \mathcal{D}'(Q).$$

Notice that $\dfrac{T_n(f\,g(v))}{g(v_n)} \to f$ in $L^1(Q_T)$ and $u_n \to u$ in $L^1(Q_T)$. (The last estimate follows by the fact that $|\nabla u_n| = \dfrac{|\nabla v_n|}{g(v_n)}$ is bounded in $L^2(Q_T)$, hence $u_n \to u$ in $L^2(Q_T)$).
We claim that

$$\frac{h_n}{g(v_n)} \to 0 \quad \text{in } \mathcal{D}'(Q_T)\,.$$

We prove the claim, assume that A is a set of zero capacity such that $\mu_s\big|_{Q_T}$ is concentrated on A, and find an open set $U_\varepsilon \supset A$ and a function $\psi_\varepsilon \in \mathbf{W_T}$ such that $\psi_\varepsilon \ge \chi_{U_\varepsilon}$ and $||\psi_\varepsilon||_{\mathbf{W}} \le \varepsilon$. Let $m(s)$ be the function defined in (4.3). Integrating in time and using a Picone-type inequality (see [3]), we obtain

$$\begin{aligned} 4\,\varepsilon^2 &\ge \int_\Omega |\nabla m(\psi_\varepsilon)|^2\,dx \ge \int_\Omega \frac{-\Delta(v_n+1)}{v_n+1}\,m^2(\psi_\varepsilon)\,dx \\ &\ge \int_\Omega \frac{h_n}{v_n+1}\,m^2(\psi_\varepsilon)\,dx - \int_\Omega \frac{(v_n)_t}{v_n+1}\,m^2(\psi_\varepsilon)\,dx \\ &\ge c\int_\Omega \frac{h_n}{g(v_n)}\,m^2(\psi_\varepsilon)dx - \int_\Omega \frac{(v_n)_t}{v_n+1}\,m^2(\psi_\varepsilon)\,dx\,. \end{aligned}$$

Hence, after integrating in time,

$$\begin{aligned} c\iint_{U\varepsilon} \frac{h_n}{g(v_n)}\,dx\,dt \le 4\varepsilon^2 &+ \int_\Omega \log(1+v_n(x,T))\,m^2(\psi_\varepsilon(x,T))\,dx \\ &+ 2\iint_{Q_T} \log(1+v_n)\,m'(\psi_\varepsilon)\,m(\psi_\varepsilon)\,\psi_\varepsilon'\,dx\,dt\,. \end{aligned} \tag{69}$$

The last two integrals in (69) can be estimated exactly as in the proof of Theorem (4.6), and the claim (5.2) follows easily. Finally, by a direct adaptation of the argument used in proof of Theorem 4.6 we can prove that

$$\beta(u_n)\,|\nabla u_n|^2 = \frac{g'(v_n)\,|\nabla v_n|^2}{(g(v_n))^2} \to \beta(u)\,|\nabla u|^2 = \frac{g'(v)\,|\nabla v|^2}{(g(v))^2} \text{ strongly in } L^1(Q_T).$$

Hence we conclude that u is a solution to problem (67), moreover $u \in \mathcal{C}([0,T];L^1(\Omega))$, which shows that the initial datum is $\Psi^{-1}(v_0)$. ■

Let consider now the inverse problem, namely we have the next result.

Theorem 5.7. *Let $u \in \mathcal{C}([0,\infty);L^1(\Omega)) \cap L^2_{\text{loc}}([0,\infty);W_0^{1,2}(\Omega))$ be a solution to problem (1), with $\beta(u)|\nabla u|^2 \in L^1_{\text{loc}}(\overline{Q})$ and $f \in L^\infty_{\text{loc}}(\overline{Q})$, is a positive function. Let $v = \Psi(u)$, then $v \in L^1_{\text{loc}}(Q)$ and there exists a bounded positive Radon measure μ_s, singular with respect to $\text{cap}_{1,2}-$capacity, such that v solves*

$$v_t - \Delta v = f(x,t)\,g(v) + \mu_s \qquad \textit{in } \mathcal{D}'(Q)\,.$$

Moreover μ_s can be characterized as a weak limit in the space of bounded Radon measures, as follows:

$$\mu_s = \lim_{\varepsilon\to 0} e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\,\beta(u)\,|\nabla u|^2\left(1-\frac{1}{(1+\varepsilon\gamma(u))^2}\right)\ \textit{in } Q_\tau \textit{ for every } \tau>0.$$

Proof. We set $v=\Psi(u)$, then by the regularity result of Theorem 5.4 we obtain that $v\in L^1_{\text{loc}}(Q)$. Using the estimate (63), we can extract a sequence of vanishing values of ε such that

$$e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\,\beta(u)\,|\nabla u|^2\left(1-\frac{1}{(1+\varepsilon\gamma(u))^2}\right)dx\,dt\to\mu_s\,,$$

where μ_s is a positive Radon measure defined in Q_T for all $T>0$. Since

$$\iint_{u\le k} e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\,\beta(u)\,|\nabla u|^2\left(1-\frac{1}{(1+\varepsilon\gamma(u))^2}\right)dx\,dt\to 0 \text{ as } \varepsilon\to 0,$$

then μ_s is supported in the set $A\equiv\{x\in\Omega : u(x)=\infty\}$. The rest of the proof consists in taking

$$v_\varepsilon(x,t)=\int_0^{u(x,t)} e^{\frac{\gamma(s)}{1+\varepsilon\gamma(s)}}\,ds\in L^2(0,T;W_0^{1,2}(\Omega)),$$

and in following the same arguments as in the proof of Theorem 4.8. ■

Remark 5.8. Notice that if $\beta\in L^1[0,\infty)$, then necessarily the measure μ_s defined in (5.7) is equivalent to 0. This result follows using the fact that $\gamma(s)\le\int_0^\infty\beta(\sigma)d\sigma$ and that

$$\lim_{\varepsilon\to 0}\iint_{Q_T}\beta(u)|\nabla u|^2 e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\left(1-\frac{1}{(1+\varepsilon\gamma(u))^2}\right)\phi\,dx\,dt=0 \text{ for all } \phi\in\mathcal{C}_0^\infty(Q_T).$$

Moreover if $\beta\in L^1[0,\infty)\cap L^\infty[0,\infty)$, then g is a Lipschitz function, hence problem (5.7) with $\mu_s=0$ has a unique positive local solution, thus problem (1) has a unique local solution. In the elliptic case, the uniqueness result under this condition on β was obtained in [29].

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