STRUCTURE THEORY OF FINITE CONFORMAL ALGEBRAS

ALESSANDRO D'ANDREA AND VICTOR G. KAC

To Victor Guillemin on his 60th birthday

CONTENTS

1. INTRODUCTION

After the seminal paper [BPZ] of Belavin, Polyakov and Zamolodchikov, conformal field theory has become by now a large field with many remarkable ramifications to other fields of mathematics and physics. A rigorous mathematical definition of the "chiral part" of a conformal field theory, called a vertex (= chiral) algebra, was proposed by Borcherds [Bo] more than ten years ago and continued in [DL], [FHL], [FLM], [K], [L] and in numerous other works. However, until now a classification of vertex algebras, similar, for example, to the classification of finite-dimensional Lie algebras, seems to be far away.

In the present paper we give a solution to the special case of this problem when the chiral algebra is generated by a finite number of quantum fields, closed under the operator product expansion (in the sense that only derivatives of the generating fields may occur). In this situation the adequate tool is the notion of a *conformal algebra* [K] which, to some extent, is related to a chiral algebra in the same way a Lie algebra is related to its universal enveloping algebra.

At the same time, the theory of conformal algebras sheds a new light on the problem of classification of infinite-dimensional Lie algebras. About thirty years ago one of the authors posed (and partially solved) the problem of classification of simple $\mathbb{Z}$-graded Lie algebras of finite Gelfand-Kirillov dimension [K1]. This problem was completely solved by Mathieu [M1]-[M3] in a remarkable tour de force. The point of view of the present paper is that the condition of locality (which is the most basic axiom of quantum field theory) along with a finiteness condition, are more natural conditions, which are also much easier to handle.

In this paper we develop a structure theory of finite rank conformal algebras. Applications of this theory are two-fold. On the one hand, the conformal algebra structure is an axiomatic description [K] of the operator product expansion (OPE) of chiral fields in a conformal field theory [BPZ]. Hence the theory of finite conformal algebras provides a classification of finite systems of fields closed under the OPE. On the other hand, the category of finite conformal algebras is (more or less) equivalent to the category of infinite-dimensional Lie algebras spanned by Fourier coefficients of a finite number of pairwise local fields (or rather formal distributions)

that are closed under the OPE [K],[K3]. Hence the theory of finite conformal algebras provides a classification of these finite "formal distribution Lie algebras".

The main idea is to develop a structure theory of finite conformal algebras which is parallel to that of finite-dimensional Lie algebras over through the interplay of formal distribution Lie algebras with their "annihilation algebras". As the latter are filtered Lie algebras in the sense of E. Cartan, we can use the well developed techniques of Cartan's theory.

Using the powerful Cartan-Guillemin theorem [G] (see also [B]) and a conformal analogue of the Killing form, we obtain a classification of finite simple and semi-simple conformal algebras (Theorems 5.1 and 7.1). We also prove conformal analogues of the Lie theorem (Theorem 8.4) and of the Cartan-Jacobson theorem (Theorem 8.6) using a result on filtered Lie algebras from [CK].

Some of the conformal analogues of the finite-dimensional Lie algebra theory fail, however. For example, it is not true that a semi-simple conformal algebra is a direct sum of simple ones (cf. Theorem 7.1). The conformal analogues of the Levi theorem and the Weyl complete reducibility theorem fail as well (see [KW], [BKV], [CKW]). This reflects the fact that the cohomology of simple conformal algebras [BKV] with non-trivial coefficients is highly non-trivial.

Another new important feature of conformal algebra theory is the fact that the conformal analogue of $\mathfrak{gl}_N$, which we call the general conformal algebra and denote by gc_N, is infinite. It is actually the conformal algebra associated to the Lie algebra of regular $N \times N$-matrix valued differential operators on $\mathbb{C}^\times$ (Example 4.3); its central extension, denoted by $W_{1+\infty}^N$, plays an important role in physics. In this paper we develop the relevant conformal linear algebra. This is undoubtedly just the beginning of a "conformal commutative algebra" and a "conformal algebraic geometry", which we are planning to develop in future publications.

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over the field of complex numbers $\mathbb{C}$. We use $\mathbb{Z}_+$ to denote the set of non-negative integers.

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2. BASIC DEFINITIONS

We start by defining the basic objects we want to consider. Let V be a (possibly infinite-dimensional) vector space. A *formal distribution* with values in V is a power series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

where $a_n \in V$. The vector space of these series is denoted by $V[[z, z^{-1}]]$. Such series are called formal distributions since we have a standard V-valued pairing with $\mathbb{C}[z, z^{-1}]$ which is given by

$$\langle a, p \rangle = res_z a(z) p(z)$$

where $res_z a(z) = a_0$ is the coefficient of z^{-1} and p is a Laurent polynomial in z. Of course we have $a_n = \langle a, z^n \rangle$. Vectors a_n are also called *Fourier coefficients* of $a(z)$ and completely determine it. A formal distribution $a(z, w)$ in two variables is similarly defined as a series of the form

$$\sum_{m,n \in \mathbb{Z}} a_{m,n} z^{-m-1} w^{-n-1},$$

and the space of these series is denoted by $V[[z, z^{-1}, w, w^{-1}]]$. Suppose now we have vector spaces U, V, W. Any linear map $\alpha : U \otimes V \to W$ induces a map

$$\tilde{\alpha} : U[[z, z^{-1}]] \otimes V[[w, w^{-1}]] \to W[[z, z^{-1}, w, w^{-1}]].$$

If $u \in U[[z, z^{-1}]]$, $v \in V[[w, w^{-1}]]$ are formal distributions, the pair (u, v) is called a *local pair* whenever for some $N \in \mathbb{Z}_+$:

$$(z - w)^N \tilde{\alpha}(u(z) \otimes v(w)) = 0.$$

Let $\mathfrak{g}$ be a Lie algebra, and let a, b be $\mathfrak{g}$-valued formal distributions. If we choose our bilinear map α to be the Lie bracket $[\,,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, then (a, b) is a local pair if for some $N \in \mathbb{Z}_+$:

$$(z - w)^N [a(z), b(w)] = 0,$$

where we denoted $\widetilde{[\,,]}(a(z) \otimes b(w))$ by $[a(z), b(w)]$. It is clear that in this case (b, a) is also a local pair: we will say a and b are mutually local. When considering locality of formal distributions with values in a Lie algebra, we will always assume it is with respect to $[\,,]$.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{F}$ a family of mutually local formal distributions with values in $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathcal{F})$ is a *formal distribution Lie algebra* if $\mathfrak{g}$ is spanned by Fourier coefficients of formal distributions from $\mathcal{F}$.

Remark 2.1. Every Lie algebra can be trivially made into a formal distribution Lie algebra by choosing $\mathcal{F} = \{g(z) | g \in \mathfrak{g}\}$ where $g(z) = \sum_{n \in \mathbb{Z}} g\, z^{-n-1}$, but we will be mostly concerned with Lie algebras that are spanned by Fourier coefficients of a *finite* number of formal distributions. Also, this formal distribution Lie algebra is not maximal (see Definition 2.4 below).

The Dirac *delta distribution* is the $\mathbb{C}$–valued formal distribution

$$(2.1) \qquad \delta(z, w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}.$$

It will often be denoted by $\delta(z - w)$. This notation is consistent with taking derivatives since $\partial_z \delta(z - w) = -\partial_w \delta(z - w)$. The delta distribution enjoys the property

$$(z - w)\delta(z - w) = 0.$$

In general we shall have

$$(z - w)^m \partial_w^n \delta(z - w) = 0,$$

if $m > n$.

The converse is also true in some sense. If $a(z, w)$ is a distribution such that $(z-w)^N a(z, w) = 0$, then there exist formal distributions $a^n(w)$ such that [K]:

$$(2.2) \qquad a(z, w) = \sum_{n=0}^{N-1} a^n(w) \partial_w^{(n)} \delta(z - w).$$

Here and further we use notation $x^{(n)} = x^n / n!$. Furthermore, the a^n are uniquely determined:

$$(2.3) \qquad a^n(w) = res_z (z - w)^n a(z, w).$$

In this way we know that if (a, b) is a local pair, we can then find formal distributions denoted by $(a_{(n)}b)(w)$ such that

$$(2.4) \qquad \tilde{\alpha}(a(z) \otimes b(w)) = \sum_{n=0}^{N-1} (a_{(n)}b)(w) \partial_w^{(n)} \delta(z - w).$$

The formal distribution $a_{(n)}b$ is $\mathbb{C}$-bilinear in a and b and it is called the *n–th product* of a and b. It is not generally $\mathbb{C}[\partial]$-bilinear in a and b, as from (2.3) we have:

$$(2.5) \qquad (\partial a)_{(n)}b = -na_{(n-1)}b, \quad a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b.$$

In particular, ∂ is a derivation of all n-th products $a_{(n)}b$.

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra. A space $\mathcal{F}$ of $\mathfrak{g}$–valued formal distributions in z is called a *conformal family* if it is closed under derivative ∂_z and all the bilinear products $_{(n)}$ just defined.

One knows [K] that if $(\mathfrak{g}, \mathcal{F})$ is a formal distribution Lie algebra we can always include $\mathcal{F}$ in the minimal conformal family $\overline{\mathcal{F}}$. Note that $\overline{\mathcal{F}}$ can be viewed as a $\mathbb{C}[\partial]$–module, where the action of ∂ is given by $(\partial a)(z) = \partial_z a(z)$. The formal distribution Lie algebra is called *finite* if $\overline{\mathcal{F}}$ is a finitely generated $\mathbb{C}[\partial]$-module.

Proposition 2.1. [K] *The Lie algebra axioms for $\mathfrak{g}$ and the very definition of $a_{(n)}b$ translate into the following properties:*

(C1) $a_{(n)}b = 0$ *for* $n >> 0$,

(C2) $(\partial a)_{(n)}b = -na_{(n-1)}b$, $\qquad a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b$,

(C3) $a_{(n)}b = -\sum_{j=0}^{\infty}(-1)^{n+j}\partial^{(j)}(b_{(n+j)}a)$,

(C4) $a_{(m)}(b_{(n)}c) - b_{(n)}(a_{(m)}c) = \sum_{j=0}^{m}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}c$.

Note that either of formulas (C2) follows from the other one and (C3).

Definition 2.3. A *(Lie) conformal algebra* is a $\mathbb{C}[\partial]$–module R, endowed with a family of $\mathbb{C}$–bilinear products $_{(n)}, n \in \mathbb{Z}_+$, satisfying axioms (C1)-(C4). A conformal algebra R is called *finite* if R is a finitely generated $\mathbb{C}[\partial]$-module. The *rank* of a conformal algebra R, denoted by rkR, is its rank as a $\mathbb{C}[\partial]$–module (recall that this is the dimension over $\mathbb{C}(\partial)$, the field of fractions of $\mathbb{C}[\partial]$, of $\mathbb{C}(\partial) \otimes_{\mathbb{C}[\partial]} R$).

Example 2.1 (Virasoro algebra and Virasoro conformal algebra)**.** The centerless *Virasoro algebra* $\mathfrak{Vect}\,\mathbb{C}^{\times}$ is the Lie algebra of algebraic vector fields on $\mathbb{C}^{\times}$. It is spanned by vector fields $t^n\partial_t, n \in \mathbb{Z}$. The $\mathfrak{Vect}\,\mathbb{C}^{\times}$–valued formal distribution

$$L(z) = -\sum_{n\in\mathbb{Z}}(t^n\partial_t)z^{-n-1}$$

satisfies

(2.6) $$[L(z), L(w)] = \partial_w L(w)\delta(z - w) + 2L(w)\delta'_w(z - w)$$

and is therefore local to itself. The $\mathbb{C}[\partial]$–module generated by $L(z)$ is closed under all the products $_{(n)}$ and thus forms a conformal family. The corresponding conformal algebra *Vir* is the free $\mathbb{C}[\partial]$–module on the generator L with products

(2.7) $$L_{(0)}L = \partial L, \qquad L_{(1)}L = 2L, \qquad L_{(j)}L = 0 \qquad \text{for } j > 1.$$

They uniquely extend to a conformal algebra structure by (C2).

Example 2.2 (Current algebras and current conformal algebras)**.** The centerless *current algebra* associated to the Lie algebra $\mathfrak{g}$ is the space $\mathfrak{g}[t, t^{-1}] = \mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$ endowed with the Lie bracket

$$[gt^m, ht^n] = [g, h]t^{m+n}, \qquad g, h \in \mathfrak{g}, \quad m, n \in \mathbb{Z}.$$

The $\mathfrak{g}[t, t^{-1}]$–valued formal distributions

$$g(z) = \sum_{n\in\mathbb{Z}}(gt^n)z^{-n-1}$$

defined for every $g \in \mathfrak{g}$ satisfy the following commutation relations:

(2.8) $$[g(z), h(w)] = [g, h](w)\delta(z - w)$$

and are therefore local to each other. Their $\mathbb{C}[\partial]$-linear span is closed under all products, and is therefore a conformal family. The conformal algebra describing this structure is given by the $\mathbb{C}[\partial]$–module *Cur* $\mathfrak{g} = \mathbb{C}[\partial] \otimes \mathfrak{g}$ with the products

(2.9) $$g_{(0)}h = [g, h], \qquad g_{(i)}h = 0 \qquad \text{for } i > 0, \quad g, h \in \mathfrak{g},$$

where we identified $\mathfrak{g}$ with the subspace of *Cur* $\mathfrak{g}$ spanned by elements $1 \otimes g, g \in \mathfrak{g}$ as we will often do. The above products extend by (C2) to a unique conformal algebra structure. The conformal algebra *Cur* $\mathfrak{g}$ is called the *current conformal algebra* associated to $\mathfrak{g}$.

Remark 2.2. When R is a free $\mathbb{C}[\partial]$–module, and the products $_{(n)}$ are defined on a $\mathbb{C}[\partial]$–basis for R in such a way that (C1),(C3) and (C4) hold, there is a unique extension of these products to a conformal algebra structure for R, obtained via (C2), and it is easy to show that (C1) and (C4) also hold for this extension. We shall often describe conformal algebras structures on free $\mathbb{C}[\partial]$-modules by giving the products on a fixed $\mathbb{C}[\partial]$–basis.

The notions of subalgebras, ideals, quotients and homomorphisms of conformal algebras are obvious. Note that, due to (C3), any left or right ideal is actually a two-sided ideal.

As we have seen, any formal distribution Lie algebra $(\mathfrak{g}, \mathcal{F})$ gives rise to a conformal algebra $R = \overline{\mathcal{F}}$. Conversely, to any given conformal algebra R we may canonically associate a formal distribution Lie algebra $\mathfrak{Lie}\,R$ with a conformal family isomorphic to R as follows ([K], [K3]). Let

$$(2.10) \qquad \mathfrak{Lie}\,R = R[t, t^{-1}]/(\partial + \partial_t)R[t, t^{-1}],$$

let a_n denote the image of at^n in $\mathfrak{Lie}\,R$, and define a bracket by

$$(2.11) \qquad [a_m, b_n] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}b)_{m+n-j}$$

for $a, b \in R, m, n \in \mathbb{Z}$.

This bracket is a well-defined Lie bracket and the family $\left\{ a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \big| a \in R \right\}$ spans $\mathfrak{Lie}\,R$ and is a conformal algebra isomorphic to R via $a(z) \mapsto a$. Furthermore, since ∂ and ∂_t commute, the derivation ∂_t of $R[t, t^{-1}]$ induces a derivation of $\mathfrak{Lie}\,R$.

Definition 2.4. The pair $(\mathfrak{Lie}\,R, R)$ is called the *maximal* formal distribution algebra associated to the conformal algebra R.

Remark 2.3. [K], [K3] The correspondence which associates to a conformal algebra R the collection of quotients of $(\mathfrak{Lie}\,R, R)$ by ideals with a trivial intersection with R is bijective. In particular, the axioms of a conformal algebra R encode all the algebraic properties of the conformal family for $\mathfrak{Lie}\,R$. Axiom (C3) is equivalent to skew-simmetry and axiom (C4) to the Jacobi identity for the Lie algebra $\mathfrak{Lie}\,R$.

Remark 2.4. Since we know that the Jacobi identity on any triple of elements from a Lie algebra, together with skew-simmetry, give the Jacobi identity on all permutations of the triple, once (C1)-(C3) are established, it is enough to check (C4) on a triple of elements from a conformal algebra, in order for it to hold on all permutations of that triple. Therefore, if R is a free $\mathbb{C}[\partial]$-module on generators $\{a^i, i = 1...n\}$ and products $a^i_{(n)}a^j$ are defined for all $i \leq j$ and $n \in \mathbb{Z}_+$ in such a way that (C3) holds for $a = b = a^i$ and (C4) holds for $a = a^i, b = a^j, c = a^k$ with $i \leq j \leq k$, we can extend them by (C2) and (C3) to a unique conformal algebra structure for R.

3. CONFORMAL LINEAR ALGEBRA

From now on, $\mathcal{A}$ will denote the ring $\mathbb{C}[\partial]$ of polynomials in the indeterminate ∂.

Definition 3.1. A *conformal linear map* between $\mathcal{A}$-modules V and W is a map $\phi : V \to \mathcal{A}[\lambda] \otimes_{\mathcal{A}} W$ that is $\mathbb{C}-$linear and such that

$$(3.1) \qquad \phi\,\partial v = (\partial + \lambda)(\phi\,v).$$

We will often abuse the notation by writing $\phi : V \to W$ any time it is clear from the context that ϕ is conformal linear. We will also write ϕ_λ instead of ϕ to emphasize the dependence of ϕ on λ. The set of all conformal linear maps from V to W is denoted by *Chom*(V, W) and is made into an $\mathcal{A}$–module via

$$(3.2) \qquad (\partial\phi)_\lambda v = -\lambda\phi_\lambda v.$$

We shall write *Cend* V for *Chom*(V, V).

Example 3.1. Let F be a vector space and let $V = \mathcal{A} \otimes F$ be the corresponding free module over $\mathcal{A}$. A conformal linear map $\phi : V \to V$ is uniquely determined by its values on $1 \otimes F$. In fact

$$\phi_\lambda(p(\partial)v) = p(\partial + \lambda)(\phi_\lambda v), \; v \in F$$

determines the value of ϕ_λ on any element of V from its value on $1 \otimes F$, and every *End F*-valued polynomial $a(\partial, \lambda)$ gives rise to such a conformal linear map via:

$$\phi_\lambda v = a(\partial, \lambda)v.$$

We can therefore identify *Chom*(V, V) with the space of all *End F*-valued polynomials in ∂ and λ. Let $\{m_j\}$ be a basis of *End F*; then the polynomials $m_j, m_j\partial, m_j\partial^2, ...$ are easily shown to be an $\mathcal{A}$–basis of $(End\ F)[\partial, \lambda]$. Hence *Chom*$(V, V)$ is an infinite rank $\mathcal{A}$–module an $\mathcal{A}$–basis of which is given by conformal linear maps $T_j^0, T_j^1, T_j^2, ...$ such that:

$$(T_j^i)_\lambda v = \partial^i(m_j v), \ \ v \in F.$$

Let $a(z), u(w)$ be a local pair of formal distributions; we set

$$a_\lambda u = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_{(n)} u.$$

Then we get by (C2):

(3.3) $$(\partial a)_\lambda u = -\lambda a_\lambda u, \qquad a_\lambda \partial u = (\partial + \lambda)(a_\lambda u).$$

Example 3.2. Let A be an associative algebra, V an A-module and $\alpha : A \otimes V \to V$ the action of A on V. If $\mathcal{V}$ is an $\mathcal{A}$-module of formal distributions with values in V, and $T(z)$ is an A-valued formal distribution local to all distributions from $\mathcal{V}$ with respect to the action induced by α, then

$$T_\lambda v = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} T_{(n)} v$$

defines a conformal linear map from $\mathcal{V}$ to itself. Furthermore, if $\mathcal{T}$ is the set of all A-valued formal distributions that are local to all members of $\mathcal{V}$, then the map $\pi : \mathcal{T} \to Cend\ \mathcal{V}$, mapping T to the conformal linear map it defines, commutes with the action of ∂ on both spaces, i.e. it is $\mathcal{A}$–linear.

Let us now define, for a, b in a conformal algebra R, the λ-*bracket*

(3.4) $$[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_{(n)} b.$$

Then axioms (C1)-(C4) are equivalently rephrased as follows:

(C1) $[a_\lambda b] \in \mathcal{A}[\lambda] \otimes_\mathcal{A} R$,
(C2) $[\partial a_\lambda b] = -\lambda[a_\lambda b], \qquad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$,
(C3) $[a_\lambda b] = -[b_{-\partial-\lambda} a]$,
(C4) $[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]$,

for all $a, b, c \in R$. The right-hand side of axiom (C3) – resp. (C4) – means an expression similar to (3.4), i.e. we replace λ by $-\partial - \lambda$ (resp. by $\lambda + \mu$). Axiom (C4) is an equality in $\mathcal{A}[\lambda, \mu] \otimes_\mathcal{A} R$.

Example 3.3. Every $a \in R$ defines a conformal linear map $ad\,a : R \to R$ by

$$(ad\,a)_\lambda b = [a_\lambda b], \ \ b \in R.$$

In fact (C2) and (C3) imply:

$$[a_\lambda (\partial b)] = -[(\partial b)_{-\partial-\lambda} a] = (-\partial - \lambda)[b_{-\partial-\lambda} a] = (\partial + \lambda)[a_\lambda b]$$

Definition 3.2. Let R be a conformal algebra. Then a conformal linear map $d : R \to R$ is a *conformal derivation* of R if

(3.5) $$d_\lambda[a_\mu b] = [(d_\lambda a)_{\lambda+\mu} b] + [a_\mu (d_\lambda b)].$$

By (C4) $ad\,a$ is a conformal derivation of R for every $a \in R$. All derivations of this kind are called *inner*.

A remarkable instance of a non-inner conformal derivation is the following.

Example 3.4 (Semi-direct sum of *Vir* and *Cur* $\mathfrak{g}$). Let *Cur* $\mathfrak{g}$ be the current conformal algebra associated to the finite-dimensional Lie algebra $\mathfrak{g}$. Define a conformal linear map $d^L : Cur\,\mathfrak{g} \to Cur\,\mathfrak{g}$ by $d^L_\lambda g = (\partial + \lambda)g$ for every $g \in \mathfrak{g} \subset Cur\,\mathfrak{g}$. This is a conformal derivation since

$$
\begin{aligned}
[(d^L_\lambda g)_{\,\lambda+\mu}\, h] + [g_{\,\lambda}\,(d^L_\mu h)] &= -\mu[g,h] + [g_{\,\lambda}\,(\partial+\mu)h] \\
&= -\mu[g,h] + (\partial + \lambda + \mu)[g,h] \\
&= (\partial + \lambda)[g,h] \\
&= d^L_\lambda[g_{\,\mu}\,h]
\end{aligned}
$$

This derivation satisfies

$$
[d^L{}_{\,\lambda}\, d^L] = (\partial + 2\lambda)d^L
$$

and allows us to define a semi-direct sum of *Vir* and *Cur* $\mathfrak{g}$, called the *standard* semi-direct sum. The $\mathcal{A}$–module *Vir* $\oplus$ *Cur* $\mathfrak{g}$ can in fact be given a conformal algebra structure by

$$
[L_{\,\lambda}\, L] = (\partial + 2\lambda)L, \quad [g_{\,\lambda}\, h] = [g,h], \quad [L_{\,\lambda}\, g] = d^L_\lambda g,
$$

L being the standard generator of *Vir*, $g, h \in \mathfrak{g}$. This λ–bracket is translated in terms of the $_{(n)}$ products as follows:

$$
\begin{aligned}
L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(0)}g = \partial g, \\
L_{(1)}g = g, \quad g_{(1)}L = g, \quad g_{(0)}h = [g,h],
\end{aligned}
$$

where only the non-zero products of L and elements from $\mathfrak{g} \subset Cur\,\mathfrak{g}$ have been given.

Example 3.5. The λ–bracket on *Cend* V given by

$$
(3.6) \qquad\qquad [\phi_{\,\lambda}\, \psi]_\mu v = \phi_\lambda(\psi_{\mu-\lambda}v) - \psi_{\mu-\lambda}(\phi_\lambda v)
$$

defines a conformal algebra structure on *Cend* V since it satisfies properties (C1)-(C4). This is called the *general conformal algebra* on V and is denoted by $gc\,V$. The set *Der* R of all conformal derivations of a conformal algebra R is a subalgebra of $gc\,R$.

If $V = \mathcal{A}^N$ is a free $\mathcal{A}$-module of rank N, then $gc\,V$ is also denoted by gc_N. We have already seen that the action of a conformal map on V is determined by its action on an $\mathcal{A}$–basis of V via the compatibility relation

$$
\phi_\lambda(p(\partial)v) = p(\partial + \lambda)\phi_\lambda v,
$$

so that the conformal linear maps T^m_A acting as $(T^m_A)_\lambda v = \partial^m A v$, $A \in End\,V, v \in V$, have the following λ–brackets:

$$
(3.7) \qquad [T^m_A{}_{\,\lambda}\, T^n_B] = \sum_{i \in \mathbb{Z}_+} \binom{n}{i}\lambda^i T^{m+n-i}_{AB} - \binom{m}{i}(-\partial - \lambda)^i T^{m+n-i}_{BA},
$$

as one can easily check by applying both sides to any $v \in V$ and using Equation (3.6).

Let now V be an $\mathcal{A}$-module,

$$
v(\lambda) = \sum_{i=0}^{n} \lambda^i v_i \in \mathcal{A}[\lambda] \otimes_\mathcal{A} V,
$$

where $v_i \in V$. Define $\langle v \rangle$ to be the $\mathcal{A}$–submodule of V spanned by all coefficients v_i of v. The following lemma will be very useful.

Lemma 3.1. *If $p(\lambda) = \sum_{i=0}^{m} p_i \lambda^i \in \mathcal{A}[\lambda]$ is a polynomial whose leading coefficient does not depend on ∂ (i.e. is a complex number different from zero), then $\langle p\,v \rangle = \langle v \rangle$.*

Proof. We show, by induction on h, that all v_{n-h} lie in $\langle p\,v \rangle$. Let

$$
p(\lambda)v(\lambda) = \sum_{j=0}^{m+n} \lambda^j (pv)_j.
$$

Then $(pv)_{m+n} = p_m v_n$ and hence $v_n \in \langle p\,v \rangle$. Suppose now $v_{n-i} \in \langle p\,v \rangle$ for all $i \leq h$; we want to show that v_{n-h-1} also lies in $\langle p\,v \rangle$. We have that

$$(pv)_{m+n-h-1} = p_m v_{n-h-1} + p_{m-1} v_{n-h} + ... + p_{m-h-1} v_n \in \langle p\,v \rangle.$$

But by inductive hypothesis, all terms but the first one in the right hand side already lie there. Hence $p_m v_{n-h-1}$, and v_{n-h-1}, also lie in $\langle p\,v \rangle$. The other inclusion $\langle p\,v \rangle \subset \langle v \rangle$ is trivial. $\qquad\square$

The *torsion* of a conformal algebra R, denoted *Tor R* is the torsion of the $\mathcal{A}$–module R, namely the submodule of all elements r for which there exists a non-zero $p \in \mathcal{A}$ such that $p(\partial)r = 0$. The following proposition shows that the torsion does not play any significant role in the conformal algebra structure.

Proposition 3.1. *The torsion Tor R always lies in the* center $Z(R) = \{r \in R | [r_\lambda s] = 0 \text{ for all } s \in R\}$ *of R.*

Proof. Let $t \in$ *Tor R*. Then there is a non-zero $p \in \mathcal{A}$ killing t and we have:

$$0 = [0_\lambda r] = [p(\partial)t_\lambda r] = p(-\lambda)[t_\lambda r]$$

for all $r \in R$. This proves $\langle p(-\lambda)[t_\lambda r] \rangle = 0$, hence by Lemma 3.1 $\langle [t_\lambda r] \rangle = 0$, which shows $[t_\lambda r] = 0$ for every $r \in R$. $\qquad\square$

Remark 3.1. The above proof actually shows that if ϕ is a conformal linear map such that $p(\partial)\phi = 0$ for a non-zero $p \in \mathcal{A}$, then $\phi = 0$.

In view of Proposition 3.1 the λ–bracket of a conformal algebra R defines an adjoint map $ad : R \to Der\,R \subset gc\,R$ whose kernel is the center of R and which is a homomorphism of conformal algebras. Thus all finite rank centerless conformal algebras can be embedded as finite rank conformal subalgebras of gc_N.

Proposition 3.2. *A conformal linear map $\phi : V \to W$ always maps Tor V to zero.*

Proof. Say $v \in V$ is torsion. Then there is a non-zero $p \in \mathcal{A}$ such that $p(\partial)v = 0$. Then

$$0 = \phi_\lambda(p(\partial)v) = p(\partial + \lambda)(\phi_\lambda v).$$

Use now Lemma 3.1 to conclude that $\phi_\lambda v = 0$. $\qquad\square$

We now classify all conformal algebras that are free of rank one as $\mathcal{A}$–modules. This is a joint result with M. Wakimoto.

Proposition 3.3. *Let $R = \mathcal{A}x$ be a conformal algebra that is free of rank one as an $\mathcal{A}$–module. Then either R is commutative, i.e. the λ-bracket is 0, or it is isomorphic to Vir.*

Proof. Axioms (C4) and (C3) give the following relations:

$$(3.8) \qquad\qquad [x_\lambda [x_\mu x]] - [x_\mu [x_\lambda x]] = [[x_\lambda x]_{\lambda+\mu} x],$$

$$(3.9) \qquad\qquad [x_\lambda x] = -[x_{-\partial-\lambda} x].$$

Let us set $[x_\lambda x] = a(\partial, \lambda)x$ for some polynomial a. Then (3.8) is equivalent to

$$(3.10) \qquad a(\partial, \lambda)a(\partial + \lambda, \mu) - a(\partial, \mu)a(\partial + \mu, \lambda) = a(\partial, \lambda + \mu)a(-\lambda - \mu, \lambda).$$

Let

$$a(\partial, \lambda) = \sum_{i=0}^{n} a_i(\lambda)\partial^i$$

with $a_n \neq 0$. Then, assuming $n > 1$, if we equate terms of degree $2n - 1$ in ∂, we get

$$n(\lambda - \mu)a_n(\lambda)a_n(\mu) = 0$$

obtaining a contradiction. So $a(\partial, \lambda) = \alpha(\lambda)\partial + \beta(\lambda)$ is linear in ∂. If we put $\lambda = \mu$ in (3.10), we get

$$a(\partial, 2\lambda)a(-2\lambda, \lambda) = 0,$$

which means $\beta(\lambda) = 2\lambda\alpha(\lambda)$, therefore $a(\partial, \lambda) = \alpha(\lambda)(\partial + 2\lambda)$. Plugging this into (3.9), we get that α is a constant. Up to changing x by a complex multiple we can make it 0 or 1. In the first case we have the trivial product, in the second the Virasoro algebra. $\qquad\square$

The analogy between ordinary linear algebra and conformal linear algebra is very suggestive of the importance of other remarkable constructions. We have already seen that the family of all conformal maps between two $\mathcal{A}$–modules can itself be given an $\mathcal{A}$–module structure. The notion of associative algebra (and of Lie algebra) are modelled on the algebra of linear maps between vector spaces. It is natural to try to extend this construction to the conformal case.

Say we have conformal linear maps $\phi : V \to W$ and $\psi : U \to V$ where U, V, W are $\mathcal{A}$–modules. We would like to define the notion of the composition of ϕ and ψ resulting in a new conformal linear map from U to W. Conformal linearity implies

$$\phi_\lambda \partial v = (\partial + \lambda)(\phi_\lambda v), \qquad \psi_\mu \partial u = (\partial + \mu)(\psi_\mu u),$$

so that the obvious composition would satisfy

$$\phi_\lambda(\psi_\mu \partial u) = (\partial + \lambda + \mu)(\phi_\lambda(\psi_\mu u)).$$

Our composition must therefore be a conformal map indexed by $\lambda + \mu$. The natural choice is:

$$(3.11) \qquad (\phi_\lambda \psi)_{\lambda+\mu} u = \phi_\lambda(\psi_\mu u)$$

for all $u \in U$. This is the only possible choice to be compatible with replacing ϕ by $\partial\phi$ or ψ by $\partial\psi$. This composition satisfies (3.3), and defines on *Cend V* a structure of an associative conformal algebra as defined below:

Definition 3.3. An $\mathcal{A}$–module A is an *associative conformal algebra* if it is endowed with a λ–product

$$A \otimes A \ni a \otimes b \mapsto a_\lambda b \in \mathcal{A}[\lambda] \otimes_{\mathcal{A}} A$$

such that for all $a, b, c \in A$ the following two axioms hold:

(A1) $(\partial a)_\lambda b = -\lambda a_\lambda b, \qquad a_\lambda \partial b = (\partial + \lambda)a_\lambda b,$
(A2) $a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c.$

The next step is taking the commutator in A in order to define the analogue of the notion of a Lie algebra. The properties we want to retain of the commutator in an associative algebra are the bilinearity in the arguments, that we know translates as (C2), and skew-simmetry (C3). There is a unique reasonable choice:

$$(3.12) \qquad [x_\lambda y] = x_\lambda y - y_{-\partial-\lambda} x.$$

It follows from Remark 3.2 below (and it is not difficult to show directly using Remark 3.3) that this λ-bracket defines a (Lie) conformal algebra structure on A. It is also easy to see that the λ-bracket defined by (3.6) coincides with the one defined by (3.12).

Remark 3.2. A formal distribution associative algebra $(A, \mathcal{F})$ gives rise to a conformal associative algebra $\overline{\mathcal{F}}$ in the same way as in the Lie case. Similarly one constructs the maximal formal distribution associative algebra $(Alg\, R, R)$ attached to an associative conformal algebra R. The correspondence between R and $Alg\, R$ is the same as described by Remark 2.3 in the Lie case. Furthermore, passing from a formal distribution associative algebra $(A, \mathcal{F})$ to the Lie algebra $(A, \mathcal{F})$ (obtained from A by imposing the bracket $[a, b] = ab - ba$) corresponds to passing from the λ-product $a_\lambda b$ on $\mathcal{F}$ to the λ-bracket (3.12).

Remark 3.3. The following properties always hold in an associative conformal algebra:

$$a_\lambda(b_{-\partial-\mu}c) = (a_\lambda b)_{-\partial-\mu}c,$$

$$a_{-\partial-\lambda}(b_\mu c) = (a_{-\partial-\mu}b)_{-\partial+\mu-\lambda}c,$$

$$a_{-\partial-\lambda}(b_{-\partial-\mu}c) = (a_{-\partial+\mu-\lambda}b)_{-\partial-\mu}c.$$

4. THE ANNIHILATION ALGEBRA

We have seen in Section 2 how one can associate to a formal distribution Lie algebra a conformal algebra. We have also inverted this construction, i.e. to a given conformal algebra R we have associated its maximal formal distribution Lie algebra $(\mathfrak{Lie}\, R, R)$. Let now R be a finite conformal algebra. Then we may find a free complementary $\mathcal{A}$-submodule R_f to *Tor* R:

$$R = R_f \oplus \text{\textit{Tor}}\, R,$$

where $R_f = \mathcal{A} \otimes \mathfrak{g}$, and $\mathfrak{g} \subset R$ is a finite-dimensional subspace. Then we have the identification [K]

$$(4.1) \qquad \mathfrak{Lie}\, R = \mathfrak{g}[t, t^{-1}] \oplus (\text{\textit{Tor}}\, R)\, t^{-1}$$

with the following brackets (cf. (2.11)):

$$(4.2) \qquad [a_m, b_n] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}b)_{m+n-j},$$

$$(4.3) \qquad [\mathfrak{Lie}\, R, (\text{\textit{Tor}}\, R)\, t^{-1}] = 0,$$

where $a, b \in \mathfrak{g}, m, n \in \mathbb{Z}, a_n$ stands for at^n, and we use the rule

$$(4.4) \qquad (\partial a)_m = -ma_{m-1}, a \in R, m \in \mathbb{Z}.$$

Recall also that $\mathfrak{Lie}\, R$ admits a derivation ∂ defined by

$$(4.5) \qquad \partial(a_n) = -na_{n-1}, a \in R, n \in \mathbb{Z}.$$

Thus, $\mathfrak{Lie}\, R$ may be viewed as a generalization of a current Lie algebra. We have the following corollaries of the above remarks and Remark 2.3.

Proposition 4.1. *Let $(\mathfrak{g}, \mathcal{F})$ be a formal distribution Lie algebra with a conformal family $\mathcal{F}$, which is a free $\mathcal{A}$-module on a basis B. Suppose that all elements $a_n, a \in B, n \in \mathbb{Z}$, are linearly independent. Then $(\mathfrak{g}, \mathcal{F})$ is isomorphic to the maximal formal distribution algebra associated to the conformal algebra $\mathcal{F}$. A similar statement is true for formal distribution associative algebras.*

Proposition 4.2. *Let R be a finite conformal algebra and let $\{a^\alpha\}$ be a finite set of generators for R (as an $\mathcal{A}$-module). Let $\mathfrak{L}_m$ be the linear span of $\{a_i^\alpha, i \geq m\}$; then we have a "quasi-filtration" of $\mathfrak{L}_0$:*

$$(4.6) \qquad \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset ...$$

by subspaces $\mathfrak{L}_i$ of finite codimension satisfying

$$[\mathfrak{L}_i, \mathfrak{L}_j] \subset \mathfrak{L}_{i+j-s} \text{ for some } s \in \mathbb{Z}_+,$$

and

$$[\partial, \mathfrak{L}_i] = \mathfrak{L}_{i-1},$$

for all $i, j \in \mathbb{Z}$ – we let $\mathfrak{L}_j = \mathfrak{L}_0$ if $j < 0$.

Proof. We have:

$$[a^\alpha {}_\lambda a^\beta] = \sum_\gamma p_\gamma^{\alpha\beta}(\partial, \lambda)a^\gamma, \text{ some } p_\gamma^{\alpha\beta}.$$

Let s be the highest degree of the polynomials $p_\gamma^{\alpha\beta}$. The statement clearly follows from (2.11). $\square$

Remark 4.1. One can use the family $\{\mathfrak{L}_m\}$ to define a topology on $(\mathfrak{Lie}\, R)_-$ by setting $\{\mathfrak{L}_m\}$ to be a basis of neighbourhoods of zero. The quasi-filtration $\{\mathfrak{L}_m\}$ depends on the choice of the set of generators for R, but the topology it induces on $\mathfrak{L}_0$ is independent of that choice. The Lie bracket on $\mathfrak{L}_0$ and the action of ∂ are clearly continuous.

Definition 4.1. The *annihilation algebra* associated to a conformal algebra R is the subalgebra
$$(\mathfrak{Lie}\, R)_- = \mathfrak{L}_0 = \{a_n | a \in R, n \geq 0\}$$
of the Lie algebra $\mathfrak{Lie}\, R$, topologized as in Remark 4.1. Obviously, $(\mathfrak{Lie}\, R)_-$ is ∂-invariant. The semi-direct sum $(\mathfrak{Lie}\, R)^- = \mathbb{C}\partial \ltimes (\mathfrak{Lie}\, R)_-$ is called the *extended annihilation algebra*.

Equations (4.1)-(4.5) show that $(\mathfrak{Lie}\, R)_-$ is nothing but $\mathfrak{g}[t]$ with the Lie bracket (4.2). We can now describe (extended) annihilation algebras in all important examples.

Example 4.1. Consider the Virasoro conformal algebra *Vir*, and let L be its standard generator. Then by Proposition 4.1, $\mathfrak{Lie}\, Vir$ is isomorphic to $\mathfrak{Vect}\,\mathbb{C}^\times$ under the identification $L_j = -t^j \partial_t$. The associated annihilation algebra is the Lie subalgebra spanned by $t^j \partial_t$ with $j \geq 0$. In other words, $(\mathfrak{Lie}\, Vir)_- \simeq \mathfrak{Vect}\,\mathbb{C}$. Its topology is the one induced from the standard filtration $\{\mathfrak{L}_n = t^n \mathbb{C}[t]\partial_t\}$ of $\mathfrak{Vect}\,\mathbb{C}$. Since ∂ acts on it as $-ad\, \partial_t$, we see that $(\mathfrak{Lie}\, Vir)^-$ is isomorphic to the direct sum of $\mathfrak{Vect}\,\mathbb{C}$ and the one-dimensional Lie algebra $\mathbb{C}(\partial + \partial_t)$.

Example 4.2. The Lie algebra $\mathfrak{Lie}\, Cur\, \mathfrak{g}$, where $Cur\, \mathfrak{g}$ is the current conformal algebra associated to a complex Lie algebra $\mathfrak{g}$, is isomorphic to $\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes_\mathbb{C} \mathbb{C}[t, t^{-1}]$ where we identified g_n with gt^n. The corresponding annihilation algebra is its positive part $\mathfrak{g}[t]$, its topology being the one induced by the standard filtration $\{\mathfrak{L}_n = t^n \mathfrak{g}[t]\}$, and ∂ acts on this as $-\partial_t$.

Example 4.3. Let *Diff* $\mathbb{C}^\times$ be the associative algebra of algebraic differential operators on $\mathbb{C}^\times$. Its elements are of the form
$$\sum_{i=0}^{k} p_i(t)\partial_t^i,$$
where $p_i \in \mathbb{C}[t, t^{-1}]$. A basis for *Diff* $\mathbb{C}^\times$ is given by $\{t^m \partial_t^n | m \in \mathbb{Z}, n \in \mathbb{Z}_+\}$, and the product is given by the usual composition of operators. Denote by $(\textit{Diff}\,\mathbb{C}^\times)_N$ the algebra $End\,\mathbb{C}^N \otimes$ *Diff* $\mathbb{C}^\times$ of $End\,\mathbb{C}^N$-valued differential operators on $\mathbb{C}^\times$. It acts naturally on the vector space $\mathbb{C}^N \otimes \mathbb{C}[t, t^{-1}]$. The $(\textit{Diff}\,\mathbb{C}^\times)_N$–valued formal distributions
$$J_A^k(z) = \sum_{n \in \mathbb{Z}} At^n(-\partial_t)^k z^{-n-1}$$
act on the family
$$v(z) = \sum_{n \in \mathbb{Z}} (vt^n)z^{-n-1} = v\delta(z - t), v \in \mathbb{C}^N,$$
of $\mathbb{C}^N$-valued formal distributions. Since $J_A^k(z) = A\delta(z - t)(-\partial_t)^k$, J_A^k and v are local under composition, explicitly:

$$
\begin{aligned}
J_A^k(z)\, v(w) &= A\, \delta(z - t)(-\partial_t)^k v\, \delta(w - t) \\
&= Av\, \partial_w^k(\delta(z - t)\delta(w - t)) \\
&= Av \sum_{i=0}^{k} \binom{k}{i} \partial_w^{k-i}\delta(w - t)\partial_w^i\delta(z - w) \\
&= \sum_{i=0}^{k} \binom{k}{i} \partial^{k-i} Av(w)\partial_w^i \delta(z - w).
\end{aligned}
$$

Hence J_A^k induces a conformal linear map of $\mathcal{A} \otimes \mathbb{C}^N$ to itself, given in terms of λ–product as
$$(J_A^k)_\lambda v = \sum_{i=0}^{k} \binom{k}{i} \lambda^i \partial^{k-i} Av = (\partial + \lambda)^k Av.$$

Since the J_A^k's act locally on an $\mathcal{A}$–module, they are local to each other with respect to the λ–product on $Cend\, \mathcal{A}^N$. If $\{m_i\}$ is a basis of $End\,\mathbb{C}^N$, then the formal distributions $J_{m_i}^k$ form a basis of $Cend\, \mathcal{A}^N$ over $\mathcal{A}$ (this is the same as to say that polynomials $(\partial + \lambda)^k$ form an $\mathcal{A}$-basis for

$\mathbb{C}[\partial, \lambda]$) and Fourier coefficients of all the $J_{m_i}^k$ are linearly independent, hence by Proposition 4.1 the associative algebra *Alg Cend* $\mathcal{A}^N$ (see Remark 3.2) is isomorphic to $(Diff \, \mathbb{C}^\times)_N$. Simplicity of $(Diff \, \mathbb{C}^\times)_N$ then implies simplicity of *Cend* $\mathcal{A}^N$ as an associative conformal algebra.

The same is true for gc_N: the Lie algebra $\mathfrak{Lie} \, gc_N$ is isomorphic to the Lie algebra $(Diff \, \mathbb{C}^\times)_N$ of *End* $\mathbb{C}^N$-valued algebraic differential operators on $\mathbb{C}^\times$. The annihilation algebra of *Cend* $\mathcal{A}^N$ (resp. gc_N) is obviously isomorphic to the polynomial part of $(Diff \, \mathbb{C}^\times)_N$ with the usual product (resp. Lie bracket).

Remark 4.2. The J_A^k's constitute an alternative basis for gc_N to the one considered in Example 3.1.

Now we discuss some properties of annihilation algebras that will turn out useful in the sequel. Introduce the following generating series of elements of $(\mathfrak{Lie} \, R)_-$: $a_\lambda = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_n$. Then we have an equivalent form of (2.11) and (4.4):

$$[a_\lambda, b_\mu] = [a_{\,\lambda} \, b]_{\lambda+\mu}, \tag{4.7}$$

$$(\partial a)_\lambda = -\lambda a_\lambda. \tag{4.8}$$

Replacing μ by $\mu - \lambda$, we get $[a_{\,\lambda} \, b]_\mu = [a_\lambda, b_{\mu-\lambda}]$ or, equivalently

$$(a_{(m)} b)_n = \sum_{j=0}^m (-1)^{m+j} \binom{m}{j} [a_j, b_{m+n-j}] \tag{4.9}$$

Lemma 4.1. $a_\lambda = 0$ *if and only if* $a \in Tor \, R$.

Proof. By (4.1) and (4.8) if $a \notin Tor \, R$, then $a_n \neq 0$ for n sufficiently large, and if $a \in Tor \, R$ then $a_n = 0$ for all non-negative n. $\square$

Lemma 4.2. *A torsionless conformal algebra R is commutative if and only if its annihilation algebra $(\mathfrak{Lie} \, R)_-$ is commutative.*

Proof. If the λ-bracket is 0, $(\mathfrak{Lie} \, R)_-$ is trivially commutative. Suppose now $(\mathfrak{Lie} \, R)_-$ is a commutative Lie algebra. From (4.7), we get $[a_{\,\lambda} \, b]_{\lambda+\mu} = 0$ which shows $[a_{\,\lambda} \, b] \in Tor \, R$ because of Lemma 4.1. $\square$

An ideal $\mathfrak{j} \subset (\mathfrak{Lie} \, R)_-$ is said to be *regular* if there is some ideal $J \subset R$ whose Fourier coefficients span $\mathfrak{j}$; then $\mathfrak{j}$ is stable under the action of ∂.

Lemma 4.3. *If R is a torsionless conformal algebra, then every ∂-stable ideal $\mathfrak{J}$ of $(\mathfrak{Lie} \, R)_-$ that is not contained in the center of $(\mathfrak{Lie} \, R)_-$ contains a non-zero regular subideal.*

Proof. Suppose there is an $a \in R$ such that $a_m \in \mathfrak{J}$ for all $m \in \mathbb{Z}_+$. Then (4.9) shows that $(a_{(i)} b)_j \in \mathfrak{J}$ for all $b \in R, i, j \in \mathbb{Z}_+$. Hence the ideal of $(\mathfrak{Lie} \, R)_-$ generated by $\{a_m, m \in \mathbb{Z}_+\}$, which is clearly contained in $\mathfrak{J}$, is $(\mathfrak{Lie} \, J)_-$ where J is the ideal of R generated by a.

Therefore, we are only left with proving that there is such an a. Choose an element $x_n \in \mathfrak{J}$ not lying in the center of $(\mathfrak{Lie} \, R)_-$. There exists a maximal i such that all the $x_j, j < i$ lie in the center of $\mathfrak{J}$. It is clear by (4.9) that $x_{(j)} y = 0$ for all $j < i, y \in R$, so that by (2.11) $[x_i, y_m] = (x_{(i)} y)_m$, for all m. So $a = x_{(i)} y$ is the element we are looking for. $\square$

Remark 4.3. The proof of Lemma 4.3 actually shows that the ideal $[(\mathfrak{Lie} \, R)_-, [(\mathfrak{Lie} \, R)_-, \mathfrak{J}]]$ is always regular. It also follows from (2.11) and (4.9), as well as the proof of Lemma 4.3, that when $\mathfrak{a}$ and $\mathfrak{b}$ are regular ideals, the ideal $[\mathfrak{a}, \mathfrak{b}]$ is also regular. In particular, if $\mathfrak{a}$ (resp. $\mathfrak{b}$) is spanned by all Fourier coefficients of A (resp. B) for ideals A, B of a conformal algebra R, then $[\mathfrak{a}, \mathfrak{b}]$ is regular, and is induced by $\langle [A_\lambda B] \rangle$ (see Lemma 3.1).

Definition 4.2. The *completed* (resp. *completed extended*) annihilation algebra of a conformal algebra R, denoted $\widehat{(\mathfrak{Lie} \, R)}_-$ (resp. $\widehat{(\mathfrak{Lie} \, R)}^-$), is the completion of $(\mathfrak{Lie} \, R)_-$ (resp. $(\mathfrak{Lie} \, R)^-$) with respect to its topology.

For example, the completed annihilation algebra for *Cur* $\mathfrak{g}$ is $\mathfrak{g}[[t]]$, while that for *Vir* is $\mathbb{C}[[t]]\partial_t$. In both cases the topology is induced by filtering with respect to the power series degree in t. There is a *conformal module* action of $(\mathfrak{Lie}\, R)_-$ on R given by $r_n.s = r_{(n)}s$, which we will call the *adjoint* conformal representation of R (see Section 8). Since for any $r \in R$, $\mathfrak{L}_k.r = 0$ for $k >> 0$, this action extends to the whole $\widehat{(\mathfrak{Lie}\, R)}_-$.

Lemma 4.4. *Let $x \in \widehat{(\mathfrak{Lie}\, R)}_-$, $r \in R$. Then we have:*

$$(4.10) \qquad [x, r_n] = \sum_{i=0}^{\infty} ((-\partial)^{(i)} x\, .r)_{n+i},$$

and

$$(4.11) \qquad (x.r)_n = \sum_{i=0}^{\infty} [\partial^{(i)} x, r_{n+i}].$$

Proof. It is enough to show (4.10) for $x = a_m$, $a \in R, m \in \mathbb{Z}_+$. Density of $(\mathfrak{Lie}\, R)_-$ along with convergence of both sides in (4.10) will give the general statement. Indeed, $[a_m, r_n]$ and $\sum_{i \in \mathbb{Z}_+} ((-\partial)^{(i)} a_m\, .r)_{n+i}$ both equal

$$\sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}r)_{m+n-j},$$

since $a_k.r = a_{(k)}r$. The second equality then follows by substituting (4.10) in the right-hand side of (4.11). $\qquad\square$

Proposition 4.3. *Let R be a torsionless conformal algebra, and let $\mathfrak{j}$ be a ∂-stable ideal of $\widehat{(\mathfrak{Lie}\, R)}_-$ such that $\mathfrak{j} \cap (\mathfrak{Lie}\, R)_- = 0$. Then $\mathfrak{j}$ is central in $\widehat{(\mathfrak{Lie}\, R)}_-$.*

Proof. Let $x \in \mathfrak{j}, r \in R$. Then because of (4.11), $(x.r)_n$ lies in $\mathfrak{j}$. But $x.r$ belongs to R, hence $(x.r)_n$ also lies in $(\mathfrak{Lie}\, R)_-$. This shows that $(x.r)_n \in \mathfrak{j} \cap (\mathfrak{Lie}\, R)_- = 0$, and since R is torsionless, $x.r$ must equal 0. Thus $x.r = 0$ for all $x \in \mathfrak{j}, r \in R$, which means, by Lemma 4.4, that $x \in \mathfrak{j}$ is always central in $\widehat{(\mathfrak{Lie}\, R)}_-$. $\qquad\square$

5. Simple conformal algebras

We now want to undertake the task of studying simple conformal algebras through the properties of the corresponding annihilation algebras; throughout this section, annihilation algebras will always be considered along with their topology.

Definition 5.1. A conformal algebra R is *simple* if it is non-commutative and has no non-trivial ideals.

It is clear that *Vir* is simple, and that *Cur* $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ is a simple Lie algebra.

Remark 5.1. In consideration of Proposition 3.1, any simple conformal algebra R must be torsionless.

The final goal of this section is the proof of the following

Theorem 5.1. *A finite simple conformal algebra is isomorphic either to Vir or to the current conformal algebra Cur $\mathfrak{g}$ associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$.*

We shall divide the proof in several steps. The most important result we are going to exploit is the Cartan-Guillemin theorem [G] (see also [B]):

Theorem 5.2. *Let $\mathfrak{L}$ be a linearly compact Lie algebra having an open subalgebra containing no ideals of $\mathfrak{L}$. Then:*

(a) *Every non-commutative minimal closed ideal of $\mathfrak{L}$ is of the form $\mathfrak{S} = S \widehat{\otimes} \mathbb{C}[[t_1, t_2, \ldots, t_r]]$, for some $r \in \mathbb{Z}_+$ and some simple Lie algebra S (the simple Lie algebra S is either finite-dimensional or one of the complete Cartan type Lie algebras W_N, S_N, H_N, K_N; see e.g. [G1] or [K1]).*

(b) *There exists a descending family of closed ideals of $\mathfrak{L}$*

(5.1) $$\mathfrak{L} = \mathfrak{J}_0 \supset \mathfrak{J}_1 \supset \cdots \supset \mathfrak{J}_n \supset \mathfrak{J}_{n+1} = 0$$

such that each quotient $\mathfrak{J}_k/\mathfrak{J}_{k+1}, 0 \leq k \leq n$, is either commutative or isomorphic to $\mathfrak{S}$ described in (a), in which case there are no closed ideals strictly between $\mathfrak{J}_k$ and $\mathfrak{J}_{k+1}$.

Recall that a *linearly compact* vector space is a topological vector space which is the topological product of discrete finite dimensional vector spaces. The linearly compact Lie algebra we will apply Theorem 5.2 to is $(\widehat{\mathfrak{Lie}\, R})^-$, which is the topological product of $\mathbb{C}\partial$ and $\mathfrak{L}_i/\mathfrak{L}_{i+1}, i > 0$.

Lemma 5.1. *If R is a finite conformal algebra, then $(\widehat{\mathfrak{Lie}\, R})^-$ has a subalgebra of finite codimension containing no ideals of $(\widehat{\mathfrak{Lie}\, R})^-$.*

Proof. By (4.1) we can assume R to be torsionless. Use Proposition 4.2 with an $\mathcal{A}$-basis of R. The closure of $\mathfrak{L}_1$ in $(\widehat{\mathfrak{Lie}\, R})^-$ does not contain any ideal of $(\widehat{\mathfrak{Lie}\, R})^-$, and is clearly of finite codimension. $\qquad\square$

We can therefore apply Theorem 5.2 to $(\widehat{\mathfrak{Lie}\, R})^-$. It is also easy to find out what the non-commutative quotients in (5.1) for $(\widehat{\mathfrak{Lie}\, R})^-$ can be.

Lemma 5.2. *$(\widehat{\mathfrak{Lie}\, R})^-$ is a Lie algebra of growth ≤ 1 (see [GK], [K1] for the definition of growth=Gelfand-Kirillov dimension of a Lie algebra and its properties).*

Proof. Let $\{a^\alpha, \alpha = 1 \ldots n\}$ be a set of generators of the $\mathcal{A}$–module R. Then the elements ∂, a_m^α with $\alpha = 1, \ldots, N, m \in \mathbb{Z}_+$ span the Lie algebra $(\mathfrak{Lie}\, R)^-$. Consider a finite subset X of this set of elements, so that for $a_m^\alpha \in X$ we have $m \leq M$, where M is a positive integer. Then, due to (2.11), commutators of length $\leq n$ of elements from X produce linear combinations of the a_m^α with $m \leq Mn$ and the total number of these elements is at most MNm, which is a linear function in n, hence the growth of $(\mathfrak{Lie}\, R)^-$ is at most one. $\qquad\square$

The growth of $(\mathfrak{Lie}\, R)^-$ is certainly not less than the growth of the graded algebra associated to $(\widehat{\mathfrak{Lie}\, R})^-$ with its standard filtration, and the same applies to all subquotients. Therefore the only possible non-commutative quotients in (5.1) have an associated graded algebra of growth at most one. Since W_1 is the only complete Cartan type Lie algebra that contains a dense subalgebra of growth one, and all finite dimensional Lie algebras have growth zero, we obtain:

Lemma 5.3. *The only non-commutative quotients showing up in (5.1) for the Lie algebra $(\widehat{\mathfrak{Lie}\, R})^-$ are isomorphic to $\widehat{\mathfrak{Vect}\, \mathbb{C}}$, $\mathfrak{g}$ or $\mathfrak{g}[[t]]$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra.*

Lemma 5.4. *If R is a finite conformal algebra with no non-zero commutative ideals, then $\mathfrak{J}_n$ in (5.1) can be chosen to be non-commutative.*

Proof. Suppose $\mathfrak{J}_n$ to be commutative; then it must be central, otherwise Lemma 4.3 would give a commutative ideal of R.

If $\mathfrak{J}_n$ is central, then $\mathfrak{J}_{n-1}$ is a central extension of $\mathfrak{J} = \mathfrak{J}_{n-1}/\mathfrak{J}_n$ by $\mathfrak{J}_n$ that we may assume without loss of generality to be non-trivial: indeed, if it were trivial, then either $\mathfrak{J}_{n-1}$ would be commutative, or $\mathfrak{J}$ would be a non-commutative minimal ideal. So $\mathfrak{J}_{n-1}$ is non-commutative, and $\mathfrak{J}_{n-1} \cap (\mathfrak{Lie}\, R)_-$, which is nonzero by Proposition 4.3, contains a non-zero regular ideal $\mathfrak{r}$ (Lemma 4.3).

If $\mathfrak{J}$ is commutative, then $\mathfrak{J}_{n-1}$ is solvable, and $[\mathfrak{J}_{n-1}, \mathfrak{J}_{n-1}]$ lies inside $\mathfrak{J}_n$. Then either $\mathfrak{r}$ is a commutative regular ideal, or it is non-commutative, in which case $[\mathfrak{r}, \mathfrak{r}] \neq 0$ is still regular (Remark 4.3) and commutative, since it is central in $(\mathfrak{Lie}\, R)_-$. But from Lemma 4.2 every

commutative regular ideal corresponds to a commutative ideal of the conformal algebra R, thus giving a contradiction.

If $\mathfrak{J}$ is non-commutative, then by Lemma 5.3 it is isomorphic to one of the following: $\widehat{\mathfrak{Vect}\,\mathbb{C}}$, $\mathfrak{g}$ or $\mathfrak{g}[[t]]$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. But those Lie algebras have no non-trivial (continuous) central extensions. So $[\mathfrak{J}_{n-1}, \mathfrak{J}_{n-1}]$ is a closed ideal isomorphic to $\mathfrak{J}$, hence minimal. $\qquad\square$

Let us now classify all surjective continuous derivations of $\mathfrak{g}[[t]]$ and $\widehat{\mathfrak{Vect}\,\mathbb{C}}$.

Lemma 5.5. *If $\mathfrak{S} = \mathfrak{g}[[t]]$, then $\partial = -\partial_t$ up to a continuous automorphism of $\mathfrak{L}$.*

Proof. All continuous derivations ∂ of $\mathfrak{S}$ are of the form $\partial = a(t)\partial_t + ad\, g(t)$, $a(t) \in \mathbb{C}[[t]]$, $g(t) \in \mathfrak{g}[[t]]$ (cf. [K2], Ex. 7.4).

Surjectivity forces $a(0) \neq 0$: since $\partial x(t) = a(t)x'(t) + [g(t), x(t)]$, computing everything at 0, and assuming $a(0) = 0$, we get $(\partial x(t))|_{t=0} = [g(0), x(0)]$. Since ∂ is surjective, all possible elements of $\mathfrak{g}$ must show up in the form $[g(0), x(0)]$ for some $x(0)$. But this is clearly impossible if $\mathfrak{g}$ is finite-dimensional. If $a(0) \neq 0$, the differential equation

$$c(t)^{-1}c'(t) = a(t)^{-1}ad\, g(t) \in ad\, \mathfrak{g}[[t]]$$

always has a solution $c(t) \in (Ad\, G)[[t]]$, where G is the Lie group with Lie algebra $\mathfrak{g}$. But this is exactly the condition for $c(t)\partial c(t)^{-1}$ to be equal to $a(t)\partial_t$. Therefore ∂ can be conjugated to it; the automorphism mapping $\mathfrak{g}$ identically, and sending $t \mapsto -\int a(t)^{-1}dt$ clearly conjugates $a(t)\partial_t$ to $-\partial_t$. $\qquad\square$

Lemma 5.6. *If $\mathfrak{S} = \widehat{\mathfrak{Vect}\,\mathbb{C}}$, then $\partial = -ad\, \partial_t$ up to a continuous automorphism.*

Proof. It is a well-known fact that all continuous derivations of $\mathfrak{S}$ are inner: $\partial = ad\, a(t)\partial_t$, $a(t) \in \mathbb{C}[[t]]$. Also ∂ is surjective if and only if $a(0) \neq 0$. If so, ∂ is conjugated to $-\partial_t$ by the automorphism sending $t \mapsto -\int a(t)^{-1}dt$. $\qquad\square$

Corollary 5.1. *If R is a simple finite conformal algebra, then $(\mathfrak{Lie}\, R)_-$ is isomorphic to either $\mathfrak{Vect}\,\mathbb{C}$ or $\mathfrak{g}[t]$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra.*

Proof. R has no commutative ideals so, by Lemma 5.4, $(\widehat{\mathfrak{Lie}\, R})_-$ has a minimal non-commutative closed ideal $\mathfrak{J}$. $\mathfrak{J}$ cannot be central hence $\mathfrak{j} = \mathfrak{J} \cap (\mathfrak{Lie}\, R)_- \neq 0$, by Proposition 4.3. $\mathfrak{j}$ must be dense in $\mathfrak{J}$ because of minimality of $\mathfrak{J}$ and it must be all of $(\mathfrak{Lie}\, R)_-$ since otherwise Lemma 4.3 would give a strict ideal of R, which is simple. Since $(\mathfrak{Lie}\, R)_-$ is clearly infinite-dimensional, the case of a simple finite-dimensional Lie algebra is ruled out.

Now, $(\mathfrak{Lie}\, R)_-$ is exactly the subspace of all elements on which ∂ acts nilpotently, and since we know we can conjugate the action of ∂ to $-\partial_t$ (Lemmas 5.5, 5.6) it is clear that $(\mathfrak{Lie}\, R)_-$ will be isomorphic to either $\mathfrak{Vect}\,\mathbb{C}$ or $\mathfrak{g}[t]$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra. $\qquad\square$

Our classification will be proved as soon as we show that the extended annihilation algebra completely describes the conformal algebra structure.

Proposition 5.1. *Let R and S be torsionless finite conformal algebras and $\phi : (\mathfrak{Lie}\, R)_- \to (\mathfrak{Lie}\, S)_-$ be a homomorphism of topological Lie algebras compatible with the action of ∂. Then there is a unique homomorphism of conformal algebras $\tilde{\phi} : R \to S$ inducing ϕ.*

Proof. We are going to show that as soon as ϕ is a continuous linear map which is compatible with the action of ∂, we can find a unique $\tilde{\phi}$ inducing it. This will prove the statement since (4.9) shows how to recover the conformal algebra structure from the Lie algebra structure of the annihilation algebra. Uniqueness of $\tilde{\phi}$ will then show that $\tilde{\phi}$ is a homomorphism of $\mathcal{A}$-modules.

Since S has no torsion, by Lemma 4.1 for every choice of $r \in R$ there is at most one $s \in S$ such that $s_i = \phi(r_i)$ for every i. Let us fix a basis $\{s^i\}$ of S and consider an element $r \in R$.

Because of the compatibility of ϕ with the action of ∂, there always exist numbers c_j^i such that

$$(5.2) \qquad \phi(r_n) = \sum_{j=0}^{n} \sum_{i} (-1)^{n-j} \frac{n!}{j!} c_{n-j}^i s_j^i,$$

for every $n \in \mathbb{Z}_+$. Then set

$$(5.3) \qquad \tilde{\phi}(r) = \sum_{j \in \mathbb{Z}_+} c_j^i \partial^j s^i.$$

The formal element $\tilde{\phi}(r)$ is such that $(\tilde{\phi}(r))_n = \phi(r_n)$ for all $n \in \mathbb{Z}_+$. Because of continuity of ϕ, for every $r \in R$, $\phi(r_i)$ lies in $\mathfrak{L}_1$ for $i >> 0$ (see (4.6)). But this forces $\tilde{\phi}(r)$ to be polynomial in ∂ and therefore to belong to S. Then, due to uniqueness of $\tilde{\phi}(r)$ the map $r \mapsto \tilde{\phi}(r)$ is $\mathcal{A}$-linear. Moreover $\tilde{\phi}$ clearly induces ϕ. $\square$

We are now able to classify all finite simple conformal algebra.

Proof of Theorem 5.1. We already know that *Vir* and *Cur* $\mathfrak{g}$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra, are simple conformal algebras. We want to show they are the only ones.

Let R be a simple conformal algebra. Then Corollary 5.1 shows $(\mathfrak{Lie}\, R)_-$ is isomorphic to either $\mathfrak{Vect}\, \mathbb{C}$ or $\mathfrak{g}[t]$ with the topology induced by the standard filtrations and that ∂ can be assumed to act as $-\partial_t$. So we get ∂-compatible continuous isomorphisms between $(\mathfrak{Lie}\, R)_-$ and either $(\mathfrak{Lie}\, Vir)_-$ or $(\mathfrak{Lie}\, Cur\, \mathfrak{g})_-$, resulting by Proposition 5.1 in isomorphisms of R with either *Vir* or *Cur* $\mathfrak{g}$. This concludes the proof. $\square$

The correspondence between conformal algebras and formal distribution Lie algebras (Remark 2.3) implies the following corollary of Theorem 5.1.

Corollary 5.2. *A finite simple (i.e. without non-trivial regular ideals) formal distribution Lie algebra is isomorphic either to the Virasoro Lie algebra or to a quotient of a current Lie algebra $\mathfrak{g}[t, t^{-1}]$ with $\mathfrak{g}$ a finite-dimensional simple Lie algebra.*

6. The Killing form

The next step towards a complete structure theory of conformal algebras is the classification of semi-simple algebras. In the theory of finite-dimensional Lie algebras we can rely on the inestimable use of the Killing form. It is evident we need an analogous notion in the conformal algebra case, but the very notion of trace of a conformal linear map is ill-defined. Life would be much easier if we were handling the usual $\mathcal{A}$–linear maps.

Definition 6.1. Let M be a finite rank $\mathcal{A}$–module and $T : M \to M$ an $\mathcal{A}$–linear map. Let $\mathcal{K} = \mathbb{C}(\partial)$ be the field of fractions of $\mathcal{A}$ and let $M_{\mathcal{K}} = \mathcal{K} \otimes_{\mathcal{A}} M$. Then T extends by linearity to a $\mathcal{K}$-linear map $M_{\mathcal{K}} \to M_{\mathcal{K}}$ and the *trace* of T on M is defined as $Tr_M T = Tr_{M_{\mathcal{K}}} T_{\mathcal{K}}$.

Remark 6.1. This trace inherits all standard properties of a trace on a vector space. In particular $Tr_M AB = Tr_M BA$. Furthermore, if M is a free module, then $Tr_M T$ is equal to the sum of elements on the diagonal of any matrix representation of T. Also $Tr_M T = Tr_{M/Tor\, M} T$.

In spite of the fact that conformal linear maps are not $\mathcal{A}$–linear, one obviously has:

Proposition 6.1. *Let ϕ^i be conformal linear maps, λ_i indeterminates. Then the composition $\phi^1_{\lambda_1} \phi^2_{\lambda_2} \ldots \phi^n_{\lambda_n}$ is $\mathcal{A}$–linear whenever $\sum_i \lambda_i = 0$.*

In this way, the trace of the composition of conformal linear maps is well-defined as soon as the λ's add up to 0. The analogue of the commutativity property of trace in this case is:

Proposition 6.2. *Let M be an $\mathcal{A}$–module of finite rank, and for $p(\partial) \in \mathcal{A}$ let $s_c p(\partial) = p(\partial + c)$. If $\phi^i \in Cend\, M$, then one has*

$$(6.1) \qquad Tr_M \phi^1_{\lambda_1} \phi^2_{\lambda_2} \ldots \phi^n_{\lambda_n} = s_{\lambda_1} Tr_M \phi^2_{\lambda_2} \ldots \phi^n_{\lambda_n} \phi^1_{\lambda_1}.$$

Proof. We may assume that M is a free $\mathcal{A}$-module (see Remark 6.1). Choose an $\mathcal{A}$–basis $\{m^i\}$ of M. Then there is a unique conformal linear map $I : M \to M$ such that $I_\lambda m^i = m^i$ for all i. The composition $I_\lambda I_{-\lambda}$ is always the identity.

Let T be an $\mathcal{A}$–linear map on M. Then $I_\lambda T I_{-\lambda}$ is $\mathcal{A}$–linear and we have:

$$(6.2) \qquad Tr_M I_\lambda T I_{-\lambda} = s_\lambda Tr_M T,$$

as can be easily shown by adding up elements on the diagonal of the matrix representation of the composition. Proposition follows now by noticing that Remark 6.1 implies:

$$Tr_M \phi_{\lambda_1}^1 \phi_{\lambda_2}^2 \ldots \phi_{\lambda_n}^n = Tr_M (\phi_{\lambda_1}^1 I_{-\lambda_1}) I_{\lambda_1} \phi_{\lambda_2}^2 \ldots \phi_{\lambda_n}^n = Tr_M I_{\lambda_1} \phi_{\lambda_2}^2 \ldots \phi_{\lambda_n}^n \phi_{\lambda_1}^1 I_{-\lambda_1}$$

$$\square$$

We are now able to define our "Killing form".

Definition 6.2. Let R be a finite conformal algebra. The *Killing form* of R is the $\mathbb{C}$-bilinear pairing

$$(x|y)_\lambda = Tr_R (ad\, x)_\lambda (ad\, y)_{-\lambda}.$$

Due to (6.1), the Killing form satisfies

$$(6.3) \qquad (x|y)_\lambda = s_\lambda (y|x)_{-\lambda}.$$

It is also clear by definition that

$$(6.4) \qquad (\partial x|y)_\lambda = -(x|\partial y)_\lambda = -\lambda(x|y)_\lambda.$$

Example 6.1. Let $R = Vir = \mathcal{A}L$. Then the Killing form is

$$(L|L)_\lambda = (\partial - \lambda)(\partial + 2\lambda).$$

Example 6.2. If $R = Cur\, \mathfrak{g}$, $g, h \in \mathfrak{g} \subset Cur\, \mathfrak{g}$ then

$$(g|h)_\lambda = \kappa(g|h),$$

where κ is the Killing form of the Lie algebra $\mathfrak{g}$.

Remark 6.2. Notice that the Killing form of a simple conformal algebra is always non-degenerate.

For a subalgebra I of R set $I^\perp = \{x \in R|\ (x|I)_\lambda = 0\}$. Due to (6.3) we get $I^\perp = \{x \in R|\ (I|x)_\lambda = 0\}$, and due to (6.4), $I^\perp$ is an $\mathcal{A}$-submodule. The following example shows that $I^\perp$ does not need to be an ideal of R even if I is.

Example 6.3. Let R be the semi-direct sum of *Vir* and *Cur* $\mathfrak{g}$, where $\mathfrak{g}$ is semi-simple, given by $L_\lambda r = d_\lambda^L r$ as in Example 3.4. Then

$$(g|h)_\lambda = \kappa(g|h), \quad (L|L)_\lambda = (\partial - \lambda)(\partial + 2\lambda), \quad (L|g)_\lambda = 0$$

for every $g, h \in \mathfrak{g}$. Therefore, *Cur* $\mathfrak{g}$ is an ideal of R, *Cur* $\mathfrak{g}^\perp = Vir$ but *Vir* is not an ideal.

What is instead true is that $I^\perp$ is always a subalgebra.

Proposition 6.3. *Let R be a conformal algebra and $I \subset R$ an ideal. Then $I^\perp$ is a subalgebra of R.*

Proof. Let $a \in I, b, c \in I^\perp$. Then, since I is closed under $\lambda-$bracket with every element of R, we have:

$$([a\,_\lambda\, b]|c)_{\lambda+\mu} = ([a\,_\lambda\, c]|b)_\mu = (c|[a\,_\lambda\, b])_{-\lambda-\mu} = 0.$$

Vanishing of the first term implies

$$(6.5) \qquad Tr_R\, ad\, a_\lambda\, ad\, b_\mu ad\, c_{-\lambda-\mu} = Tr_R\, ad\, b_\mu\, ad\, a_\lambda ad\, c_{-\lambda-\mu},$$

while that of the other two shows

$$(6.6) \qquad \begin{aligned} Tr_R\, ad\, a_\lambda\, ad\, c_{-\lambda-\mu}\, ad\, b_\mu = \\ = Tr_R\, ad\, c_{-\lambda-\mu}\, ad\, a_\lambda\, ad\, b_\mu = Tr_R\, ad\, c_{-\lambda-\mu}\, ad\, b_\mu\, ad\, a_\lambda. \end{aligned}$$

We want to show that $(a|[b_\mu c])_\lambda = 0$, that is:

$$(6.7) \qquad Tr_R \, ad \, a_\lambda \, ad \, b_\mu \, ad \, c_{-\lambda-\mu} = Tr \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu.$$

But we know from Proposition 6.2 and (6.6) that

$$Tr_R \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu =$$
$$= s_\lambda Tr_R \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu \, ad \, a_\lambda = s_\lambda Tr_R \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu.$$

This shows that the polynomial $Tr_R \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu$ is independent of ∂. The rest of the proof follows easily. Using Proposition 6.2, independence of $Tr_R \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu$ with respect to ∂, and (6.5) we obtain:

$$Tr_R \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} \, ad \, b_\mu =$$
$$= Tr_R \, ad \, b_\mu \, ad \, a_\lambda \, ad \, c_{-\lambda-\mu} = Tr_R \, ad \, a_\lambda \, ad \, b_\mu \, ad \, c_{-\lambda-\mu},$$

establishing (6.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.3. All properties of the Killing form extend to the trace form in any finite R-module.

The subalgebra $I^\perp$ does not even need to be a complementary submodule to I. Nevertheless one has:

Proposition 6.4. *Let $I \subset R$ be an ideal isomorphic to either Vir or Cur $\mathfrak{g}$, for a simple finite-dimensional Lie algebra $\mathfrak{g}$. Then $R = I^\perp \oplus I$ as $\mathcal{A}$–modules.*

In order to prove this statement, we need a classification of all conformal derivations (see Definition 3.2) of *Vir* and *Cur* $\mathfrak{g}$.

Lemma 6.1. *Every conformal derivation d of the conformal algebra Vir is inner.*

Proof. Let us assume that $d_\lambda L = a(\partial, \lambda)L$ where $a \in \mathbb{C}[\partial, \lambda]$. We write $a = \sum_{i=1}^n a_i(\lambda)\partial^i$. If d is a conformal derivation, it needs to satisfy $d_\lambda[L_\mu L] = [L_\mu (d_\lambda L)] + [(d_\lambda L)_{\lambda+\mu} L]$. This is equivalent to saying that

$$(\partial + \lambda + 2\mu)a(\partial, \lambda)L = a(\partial + \mu, \lambda)(\partial + 2\mu)L + a(-\lambda - \mu, \lambda)(\partial + 2\lambda + 2\mu)L.$$

Assuming all a_i's are non-zero and equating terms of degree n in ∂ in both sides, we get, if $n > 1$, $(\lambda - n\mu)a_n(\lambda) = 0$. This shows that $a_n(\lambda) = 0$, a contradiction. Therefore the degree of $a(\partial, \lambda)$ in ∂ can be at most one. In this case, substituting into the derivation requirement, one gets $a_0(\lambda) = 2\lambda a_1(\lambda)$, hence $d_\lambda L = a_1(\lambda)(\partial + 2\lambda)L$, which is an inner derivation induced by the element $a_1(-\partial)L$. $\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.2. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Then every conformal derivation d of Cur $\mathfrak{g}$ is of the form $p(\partial)d^L + d$ where d is inner and d^L is as in Example 3.4.*

Proof. Set $d_\lambda g = \sum_{i \in \mathbb{Z}_+} \partial^i d^i_\lambda(g)$, for every $g \in \mathfrak{g}$, where d^i_λ are $\mathbb{C}$-linear maps of $\mathfrak{g}$ to $\mathbb{C}[\lambda] \otimes \mathfrak{g}$. Then (3.5) tells us that

$$(6.8) \qquad \sum_i \partial^i d^i_\lambda([g, h]) = \sum_i ((-\lambda - \mu)^i[d^i_\lambda g, h] + (\partial + \mu)^i[g, d^i_\lambda h]).$$

Setting $\mu = 0$, switching the roles of g and h and adding up, one gets:

$$\sum_i \partial^i([g, d^i_\lambda h] + [h, d^i_\lambda g]) = \sum_i (-\lambda)^i([g, d^i_\lambda h] + [h, d^i_\lambda g]).$$

The right hand side does not depend on ∂, hence all coefficients of non-zero powers of ∂ must be 0. This means that $[d^i_\lambda g, h] + [d^i_\lambda h, g] = 0$ for every $i > 0$ and every $g, h \in \mathfrak{g}$. If $\mathfrak{g}$ is simple, such a linear map can only be a multiple of the identity, as next lemma will show, hence d must be of the form:

$$d_\lambda g = d^0_\lambda(g) + p(\partial, \lambda)g$$

for every $g \in \mathfrak{g}$. But d is a conformal derivation, so if $n > 1$ and $p(\partial, \lambda) = \sum_{i=0}^{n} p_i(\lambda)\partial^i$, equating $n-1$ degree terms in (3.5), we get $n\mu p_n(\mu) = 0$. This proves that p is of degree at most one in ∂.

Let $d'_\lambda g = p_1(\lambda)(\partial + \lambda)g$, i.e. $d' = p_1(-\partial)d^L$. This is a conformal derivation of $\mathfrak{g}$ and the action of $d - d'$ on $\mathfrak{g}$ is independent of ∂. Hence, by (6.8), $d_\lambda - d'$ is an ordinary derivation from $\mathfrak{g}$ to $\mathbb{C}[\lambda] \otimes \mathfrak{g}$ and is therefore an inner conformal derivation, proving the lemma. $\square$

Lemma 6.3. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and $T : \mathfrak{g} \to \mathfrak{g}$ a linear map. If $[Tx, y] = [x, Ty]$ for all $x, y \in \mathfrak{g}$ then T is a multiple of the identity map.*

Proof. Let $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ be a root space decomposition with respect to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$. Since $[Tx, x] = 0$, if $x \in \mathfrak{h}$ is a regular element, then Tx lies in the centralizer of x, hence in $\mathfrak{h}$. This shows that $T\mathfrak{h} \subset \mathfrak{h}$. This means that T preserves the root decomposition of $\mathfrak{g}$ modulo elements in $\mathfrak{h}$. In fact if $x \in \mathfrak{g}_\alpha$, then $[Th, x] = [h, Tx]$, $h \in \mathfrak{h}$, and since the left hand side is a multiple of x, the right hand side must be too, showing that Tx is a multiple of x plus some element from $\mathfrak{h}$: $Tx = c_\alpha x + h(x)$.

Let now $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$. From $[Tx, y] = [x, Ty]$ we obtain

$$c_\alpha[x, y] + [h(x), y] = c_\beta[x, y] + [x, h(y)].$$

When α and β are unequal roots, this means

$$(c_\alpha - c_\beta)[x, y] = 0, \quad [h(x), y] = 0, \quad [x, h(y)] = 0.$$

Since $\mathfrak{g}$ is simple, its Dynkin diagram is connected, and this allows us to show that all $c_\alpha = c_\beta = c$ are the same for all roots α, β. But $[h(x), y] = 0$ for all root vectors y gives $h(x) = 0$. Hence T equals $c\,Id$ on all $\mathfrak{g}_\alpha, \alpha \neq 0$. It is left to show that T equals $c\,Id$ also on $\mathfrak{h}$. But $[Th, x] = [h, Tx]$ gives $[Th - ch, x] = 0$ as soon as x is a root vector. This tells us that $Th = ch$ also for $h \in \mathfrak{h}$. $\square$

Proof of Proposition 6.4. Let $a \in R, j \in J$. Then in taking $(a|j)_\lambda$ it is enough to consider the trace over J since this is an ideal. Element a acts on J via a conformal derivation. But any conformal derivation of *Vir* is inner, and any non-inner conformal derivation of *Cur* $\mathfrak{g}$ is as in Lemma 6.2. Any element acting as d^L on *Cur* $\mathfrak{g}$ has a zero Killing product with it.

In any case $j \mapsto (a|j)_\lambda$ is equal to $j \mapsto (i|j)_\lambda$ for some $i \in J$, and all $j \in J$. This, and the fact that the Killing form is non-degenerate on simple algebras (which we need in order to show uniqueness in the choice of i), show that $J + J^\perp$ is a direct sum decomposition. $\square$

Remark 6.4. One can prove the proposition when the ideal J is isomorphic to the Virasoro conformal algebra without using the Killing form. Indeed, let C be the centralizer of this ideal. It will be an ideal, and since all derivation of J are inner and its center is trivial, every element x in our conformal algebra admits a unique decomposition $x = x_C + x_J$ where x_C lies in the C and x_J lies in J.

Thus, $C + J$ is a direct sum decomposition, and C is shown to be a complementary ideal, and not only a subalgebra. This argument is applicable any time J is centerless and has no outer derivations.

7. SEMI-SIMPLE CONFORMAL ALGEBRAS

We have all the tools we need in order to attack the problem of classification of semi-simple conformal algebras, now. Let us start by giving the basic definitions.

Definition 7.1. Let R be a conformal algebra, I and J its ideals. The bracket $[I \cdot J]$ of these is the subspace of R that is spanned by all products $i_{(n)}j$ with $i \in I, j \in J, n \in \mathbb{Z}_+$. Note that this is an $\mathcal{A}$-module due to (C2) and an ideal due to (C4). In other words $[I \cdot J] = \langle [I_\lambda J] \rangle$ (see Lemma 3.1).

The *derived conformal algebra* of R is $R' = [R \cdot R]$. We set $R^{(1)} = R'$, $R^{(n+1)} = (R^{(n)})'$, $n \geq 1$. Then R is a *solvable conformal algebra* if $R^{(n)} = 0$ for some $n \geq 1$. An ideal $I \subset R$ is

solvable if it is solvable as a conformal algebra. If I is an ideal of R, we define $I^1 = I$ and $I^{n+1} = [R \cdot I^n], n \geq 1$. R is a *nilpotent conformal algebra* if $R^n = 0$ for some $n \geq 1$.

A conformal algebra R is *semi-simple* if it has no non-zero solvable ideals.

Recall that by (2.11) and (4.9) (see also Remark 4.3) we have for any two ideals I and J of a conformal algebra R:

$$(7.1) \qquad\qquad (\mathfrak{Lie}\,[I \cdot J])_- = [(\mathfrak{Lie}\,I)_-, (\mathfrak{Lie}\,J)_-].$$

Lemma 7.1. *A conformal algebra R is solvable (resp. nilpotent) if and only if its annihilation algebra $(\mathfrak{Lie}\,R)_-$ is.*

Proof. We proceed as in Lemma 4.2. R is solvable (resp. nilpotent) if and only if $R/Tor\,R$ is, so we will assume R to be torsionless. Due to (7.1), we have: $(\mathfrak{Lie}\,R^{(n)})_- = (\mathfrak{Lie}\,R)_-^{(n)}$, $(\mathfrak{Lie}\,R^n)_- = (\mathfrak{Lie}\,R)^n$. The result follows form the fact that $J \subset R$, $(\mathfrak{Lie}\,J)_- = 0$ implies $J = 0$, when R is torsionless (see Lemma 4.1). $\qquad\square$

Remark 7.1. Here and further, we have denoted the subspace $\mathfrak{j}$ of $(\mathfrak{Lie}\,R)_-$ spanned by all Fourier coefficients of members from an ideal J of a conformal algebra R by $(\mathfrak{Lie}\,J)_-$. Even though the inclusion of an ideal J in a conformal algebra R is clearly injective, the map it induces at the level of annihilation algebras might have a non-trivial kernel. However this kernel $\mathfrak{k}$ must be central in $(\mathfrak{Lie}\,J)_-$. In fact, if $\mathfrak{k}$ were not central, Lemma 4.3 would locate a non-zero regular subideal of $\mathfrak{k}$, which amounts to finding a non-zero ideal K in J that is mapped to 0 by the inclusion $J \to R$, thus obtaining a contradiction. Since one can usually reconstruct the algebraic properties of the ideal J from $\mathfrak{j}$, this abuse of notation should cause no confusion.

Lemma 7.2. *If R is a finite semi-simple conformal algebra, then any minimal ideal of R is simple.*

Proof. Let I be a minimal ideal of R. Then $\widehat{(\mathfrak{Lie}\,I)}_-$ must contain a minimal closed ideal $\mathfrak{I}$ of $\widehat{(\mathfrak{Lie}\,R)}^-$ due to finiteness of descending chains of ideals [G], and we can assume it to be non-commutative by arguing as in Lemma 5.4. Since $\mathfrak{I}$ is not central, it has a non-zero intersection with $(\mathfrak{Lie}\,R)_-$, which must contain a non-zero regular ideal $\mathfrak{i}$ by Lemma 4.3. Minimality of I then shows that $\mathfrak{i}$ equals $(\mathfrak{Lie}\,I)_-$. As in Corollary 5.1, $\mathfrak{i}$ is the space of all elements in the non-commutative closed ideal $\mathfrak{I}$ on which ∂ acts nilpotently, and by Lemmas 5.5 and 5.6 it is clear that $\mathfrak{i}$ must be isomorphic to either $\mathfrak{Vect}\,\mathbb{C}$ or $\mathfrak{g}[t]$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, and that ∂ can be assumed to act on $\mathfrak{i}$ as $-\partial_t$.

This allows us to give a continuous embedding ϕ of $\mathfrak{Vect}\,\mathbb{C}$ (resp. $\mathfrak{g}[t]$) in $\mathfrak{Lie}\,R$ which is compatible with the action of ∂. Proposition 5.1 then shows that, since I is clearly torsionless, there is an embedding $\tilde{\phi}$ of *Vir* (resp. *Cur* $\mathfrak{g}$) in R which is an isomorphism onto I. Therefore I is simple as a conformal algebra. $\qquad\square$

The main theorem we are going to prove is the following.

Theorem 7.1. *Any finite semi-simple conformal algebra can be uniquely decomposed in a finite direct sum of conformal algebras each of which is isomorphic to one of the following:*
  (a) Vir
  (b) Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra
  (c) The semi-direct sum of Vir and Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a semi-simple finite-dimensional Lie algebra (Example 3.4).

Proof. Let R be a semi-simple conformal algebra. We prove the theorem by induction on rkR. If R has an ideal isomorphic to *Vir* then by Remark 6.4 it has a complementary ideal that centralizes it. Therefore it splits up in a direct sum decomposition. Hence, we may assume that R does not contain any ideal isomorphic to *Vir*.

Consider all minimal ideals $I_1, I_2, ..., I_n$ of R. They are simple by Lemma 7.2. We assumed that no ideal of R is isomorphic to *Vir*, hence $I = I_1 + I_2 + ... + I_n$ is an ideal of R isomorphic

to *Cur* $\mathfrak{g}$, where $\mathfrak{g}$ is a semi-simple finite-dimensional Lie algebra. The centralizer C of I is an ideal of R intersecting I trivially. Since C must contain a minimal subideal, $C = 0$.

Consider the orthocomplement $I^\perp$ of I with respect to the Killing form. Proposition 6.4 also applies (by induction) to direct sums of simple algebras, so $R = I^\perp \oplus I$ as $\mathcal{A}$-modules. If $I^\perp$ had a commutative ideal A, it may act on every I_i (hence on all of I) only by inner derivations (by Lemma 6.2), hence since $C = 0$, we get $A = 0$. Thus, $I^\perp$ is semi-simple and we can apply the inductive assumption. Note that $I^\perp$ contains no currents (i.e. elements a such that $a_\lambda a = 0$), otherwise $C \neq 0$. So $I^\perp$ is a direct sum of (its) ideals isomorphic to *Vir*.

The action of all such ideals on I_i is either zero or the standard one (Example 3.4) since they lie inside $I^\perp$ and no two *Vir*'s can act in a non-zero way on the same I_i because of [CK], Proposition 3.1, or Proposition 8.1 below. $\qquad\square$

As in the case of simple conformal algebras, Theorem 7.1 implies

Corollary 7.1. *A finite semi-simple formal distribution Lie algebra can be uniquely decomposed in a finite direct sum of formal distribution Lie algebras each of which is isomorphic either to a finite simple formal distribution Lie algebra (classified by Corollary 5.2), or to the semi-direct sum of the Virasoro Lie algebra with a current Lie algebra $\mathfrak{g}[t, t^{-1}]$, where $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra.*

8. REPRESENTATION THEORY

Definition 8.1. Let R be a conformal algebra. Then an $\mathcal{A}$–module V is a *representation* of R, or an *R–module*, if for every $r \in R$ a conformal linear map $r : V \to V$ is defined such that ($r, s \in R, v \in V$):

$$r_\lambda(s_\mu v) - s_\mu(r_\lambda v) = [r \,_\lambda s]_{\lambda+\mu} v, \quad (\partial r)_\lambda v = -\lambda r_\lambda v.$$

As before, we will often write $r_\lambda = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} r_n$, $r_n \in End\, V$. An R-module V is called *finite* if it is a finitely generated $\mathcal{A}$-module. Note also that a representation of R in an $\mathcal{A}$-module V is the same as a homomorphism (of conformal algebras) $R \to gc\, V$.

Remark 8.1. Let R be a finite conformal algebra with a faithful representation $V = \mathcal{A}v$ which is free of rank one, i.e. we have an injective homomorphism of conformal algebras $R \to gc_1$. By (3.7) an element $\sum_{n \in \mathbb{Z}_+} p_n(\partial) T^n$ generates an infinite rank subalgebra of gc_1 as soon as $p_i \neq 0$ for some $i > 1$. Therefore, the image of R in gc_1 must lie in the subalgebra $\mathcal{A}T^0 + \mathcal{A}T^1$. This implies that the only finite conformal algebras that have a faithful representation which is free of rank one as an $\mathcal{A}$-module are subalgebras of the standard semi-direct sum (see Example 3.4) of *Vir* and a commutative current algebra *Cur* $\mathbb{C}$.

It is immediate to see that a representation of a conformal algebra R in V is the same as a representation of the Lie algebra $(\mathfrak{Lie}\, R)^-$ in V satisfying the local nilpotency condition

$$(8.1) \qquad\qquad r_n v = 0 \text{ for } n \text{ sufficiently large}, r \in R, v \in V.$$

A $(\mathfrak{Lie}\, R)^-$-module satisfying this condition is called *conformal*.

This turns out to be a more convenient language for the study of representations of conformal algebras, using which Cheng and Kac [CK] classified all irreducible representations of *Vir*, *Cur* $\mathfrak{g}$ and their non-trivial semi-direct sum.

We shall often write $R \cdot V$ for $\sum_{j \in \mathbb{Z}_+} R_j V$. A representation V of the conformal algebra R is called *trivial* if $R \cdot V = 0$. Recall that a finite-dimensional Lie algebra $\mathfrak{g}$ is called *reductive* is it is a direct sum of a semi-simple Lie algebra $\mathfrak{g}'$ and a commutative Lie algebra $\mathfrak{a}$.

Theorem 8.1. *Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{a}$ be a reductive Lie algebra. Then any non-trivial finite rank irreducible representation of Cur $\mathfrak{g}$ is of the form $V(U, \phi) = \mathcal{A} \otimes U$, where U is an irreducible finite-dimensional $\mathfrak{g}$-module, $\mathfrak{a} \ni a \mapsto \phi_a \in \mathbb{C}[\lambda]$ is a linear map such that $a = \phi_a(0)$ on U and either U is non-trivial or $\phi \neq 0$, and the action of Cur $\mathfrak{g}$ on $V(U, \phi)$ is the unique one extending*

$$g_\lambda u = g.u, \quad a_\lambda u = \phi_a(\lambda) u$$

where $g \in \mathfrak{g}' \subset Cur\,\mathfrak{g}, a \in \mathfrak{a} \subset Cur\,\mathfrak{g}, u \in U$ and $g.u$ denotes the action of $g \in \mathfrak{g}'$ on the $\mathfrak{g}$-module U.

Theorem 8.2. *Any non-trivial representation V of Vir that is free of rank one as an $\mathcal{A}$-module is of the form*

$$L_\lambda v = (c + \partial + \Delta\lambda)v,$$

where $\Delta, c \in \mathbb{C}$ and v is a free generator of V. This representation is irreducible if and only if Δ is non-zero, and all irreducible representations of Vir are of this kind.

Theorem 8.3. *Any non-trivial irreducible representation of the semi-direct sum of Vir and Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a non-zero reductive Lie algebra, is of the form $V(U) = \mathcal{A} \otimes U$, where U is an irreducible finite-dimensional $\mathfrak{g}$-module, and the action is the unique one extending*

$$g_\lambda u = g.u, \quad L_\lambda u = (c + \partial + \Delta\lambda)u,$$

where $c, \Delta \in \mathbb{C}, g \in \mathfrak{g} \subset Cur\,\mathfrak{g}, u \in U$, and U is a non-trivial $\mathfrak{g}$-module if $\Delta = 0$.

Proposition 8.1. *Let R be a direct sum of conformal algebras isomorphic to one of the following:*
   (i) Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a non-zero reductive Lie algebra;
   (ii) the standard semi-direct sum of Vir and Cur $\mathfrak{g}$, where $\mathfrak{g}$ is reductive or 0.
   Suppose that R has a finite faithful irreducible representation. Then R is either of type (i) or of type (ii) where $\mathfrak{g}$ has at most one-dimensional center. Finite irreducible representations of R are described by Theorems 8.1-8.3.

The proof of these results is the same as in [CK]. It relies on the following key lemma, that we will occasionally use later.

Lemma 8.1. [CK] *Let $\mathfrak{L}$ be a Lie algebra, with a distinguished element ∂ and a descending sequence of subspaces*

$$\mathfrak{L} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \dots$$

such that $[\partial, \mathfrak{L}_n] = \mathfrak{L}_{n-1}$ for all $n > 0$. Let V be a $\mathfrak{L}$-module and let $V_n = \{v \in V | \mathfrak{L}_n v = 0\}$; suppose that $V_n \neq 0$ for n sufficiently large, and set N to be minimal such that $V_N \neq 0$. Then, provided that $N \geq 1$, $\mathcal{A}V_N = \mathcal{A} \otimes V_N$. In particular, V_N is a finite-dimensional vector space (over $\mathbb{C}$) if $N \geq 1$ and V is a finitely generated $\mathcal{A}$-module.

Clearly, if $\mathfrak{L} = (\mathfrak{Lie}\,R)^-$ and V is a module over R (i.e. a conformal module over $\mathfrak{L}$), the conditions of Lemma 8.1 are satisfied (cf. Proposition 4.2).

Remark 8.2. In fact [CK] contains only classification of finite irreducible representations of *Vir*. The classification of all rank one *Vir*-modules is obtained by using Remark 8.1 or by the following simple argument: if $V = \mathcal{A}v$ is a free of rank one non-trivial representation of *Vir*, then it contains a minimal rank one submodule W (see Corollary 8.1 below), which is clearly irreducible. Let $w = p(\partial)v$ be a generator for W; we can assume p to be a monic polynomial. Then by classification of irreducibles of *Vir*, we have:

$$p(\partial + \lambda)L_\lambda v = L_\lambda w = (c + \partial + \Delta\lambda)w = (c + \partial + \Delta\lambda)p(\partial)v.$$

If $L_\lambda v = q(\partial, \lambda)v$, then clearly $p(\partial + \lambda) = (c + \partial + \Delta\lambda)$ and $q(\partial, \lambda) = p(\partial)$, showing $\Delta = 1$ and $L_\lambda v = (c + \partial)v$.

What we want to investigate now is how basic results from Lie algebra representation theory extend to the conformal algebra case. First, let us establish a few facts:

Lemma 8.2. *If R is a conformal algebra and V is an R-module, then R acts trivially on Tor V.*

Proof. R acts via conformal linear maps. Use Proposition 3.2. $\qquad\qquad\square$

Corollary 8.1. *Let V_0 be the intersection of all submodules of the R-module V having the same rank as V. Then $R \cdot V \subset V_0$.*

Lemma 8.3. *If R is a conformal algebra and V its representation, then Tor R acts trivially on V.*

Proof. *Tor* R acts via elements of *Tor* $(gc\, V)$, which is is zero (Remark 3.1). $\square$

Definition 8.2. Let $R \ni r \mapsto \phi_r(\lambda) \in \mathbb{C}[\lambda]$ be a $\mathbb{C}$-linear map. A ϕ-*weight space* of a module V over a conformal algebra R is the R-invariant subspace

$$V_\phi = \{v \in V \,|\, a_\lambda v = \phi_a(\lambda)v, a \in R\}.$$

A non-zero vector v from V_ϕ is called a *weight vector* with *weight* ϕ.

Remark 8.3. If $V_\phi \neq 0$ (in which case ϕ is called a *weight* of V), then $\phi_{\partial a}(\lambda) = -\lambda\phi_a(\lambda)$.

Theorem 8.4. *(Conformal version of Lie's Theorem) Let R be a finite rank solvable conformal algebra. Then any non-trivial irreducible finite rank conformal representation V of R is free of rank one: $V = \mathcal{A}v$, where v is a weight vector.*

In order to prove this statement, we need a simple lemma about finite solvable conformal algebras:

Lemma 8.4. *If R is a finite solvable conformal algebra, then the rank of R' is strictly lower than that of R.*

This is an easy corollary of the following proposition.

Proposition 8.2. *If R is a finite conformal algebra such that $rkR' = rkR$ then $R^{(n)} = R'$ for all $n \geq 1$.*

Proof. If $rkR' = rkR$, then $R_{(n)}R \subset P(\partial)R$ for some non-zero polynomial $P \in \mathcal{A}$ and all n. Hence the second derived algebra $R^{(2)} = [R' \cdot R']$ contains $\langle P(-\partial)P(\partial + \lambda)[R_\lambda R]\rangle$ which is equal to $\langle[R_\lambda R]\rangle$ by Lemma 3.1, hence to R'. $\square$

Proof of Theorem 8.4. Since *Tor* R always acts trivially (Lemma 8.3), we can assume R to be free as an $\mathcal{A}$–module. We prove the statement by induction on the rank of R. We have

$$R \supset R^{(1)} \supset R^{(2)} \supset ... \supset R^{(n+1)} = 0$$

with $R^{(n)}$ non-zero and commutative. Then R acts on $R^{(n)}$ via the adjoint representation, $R^{(n)}$ acting trivially. So we get a representation of $R/R^{(n)}$ in $R^{(n)}$, hence by inductive assumption (cf. Lemma 8.4) we have a weight vector $b \in R^{(n)}$ with weight independent of ∂ (if R itself is commutative, any vector is a weight vector of weight zero for this action). This means $[a_\lambda b] = f_a(\lambda)b$ for all $a \in R$. Denote by $\mathfrak{b}_i$ the subspace of $(\mathfrak{Lie}\, R)_-$ spanned by elements $b_m, m \geq i$. It is clear from (2.11) that $\mathfrak{b}_i$ is normalized by $(\mathfrak{Lie}\, R)_-$. But $\mathfrak{b}_i$ kills a vector $u \in V$ for i sufficiently large, by local nilpotency assumption (8.1). Let V_j be the set of all vectors killed by $\mathfrak{b}_j$ and let U be V_N for N minimal such that $V_N \neq (0)$. Because of Lemma 8.1, when $N > 0$, U is a vector space of finite dimension, and it is invariant under the action of $(\mathfrak{Lie}\, R)_-$, since $\mathfrak{b}_N$ is normalized by $(\mathfrak{Lie}\, R)_-$. But now we can consider the image of $(\mathfrak{Lie}\, R)_-$ inside $\mathfrak{gl}\, U$. This will be a solvable Lie algebra, hence we have an eigenvector v by the classical Lie's theorem (see e.g. [S]), hence $V = \mathcal{A}v$ and theorem is proved.

In the other case $N = 0$, b kills some non-zero vector v: $b_\lambda v = 0$. Since $\mathcal{A}b$ forms an ideal, the set of vectors killed by b is a non-zero submodule of V, hence V itself. Hence $R/\mathcal{A}b$ acts on V. But $R/\mathcal{A}b$ is of lower rank than R, it is still solvable, and V is its irreducible representation. Then theorem follows by induction. $\square$

Corollary 8.2. *The derived algebra of a solvable conformal algebra R always acts trivially on an irreducible representation of R.*

Lemma 8.5. *If $V = \mathcal{A}v$ is a free of rank one representation of a solvable conformal algebra R, then V is either trivial or irreducible.*

Proof. It follows form Remark 8.1. We also give here an alternative proof. If V is non-trivial, there is a minimal submodule W of V of rank one (Corollary 8.1). If $W \neq V$, then W is spanned by a vector $w = p(\partial)v$, with p non-constant. The module W is irreducible, hence it is a weight module. We have:

$$p(\partial + \lambda)q(\partial, \lambda)v = p(\partial + \lambda)s_\lambda v = s_\lambda w = \phi_s(\lambda)w = \phi_s(\lambda)p(\partial)v,$$

for some polynomial $q(\partial, \lambda)$. This gives $\phi_s(\lambda)p(\partial) = p(\partial+\lambda)q(\partial, \lambda)$. Since the polynomial p is non-constant, it must have a complex zero c. Substituting $c - \lambda$ for ∂, we get $\phi_s(\lambda)p(c-\lambda) = 0$, hence $\phi_s(\lambda) = 0$. But this tells us that the action of S on W is trivial, giving a contradiction. Hence $W = V$, and V is irreducible. $\qquad\square$

Corollary 8.3. *Let R be a finite solvable conformal algebra and V a finite conformal module for R. Then it is always possible to find a family of submodules*

$$Tor\ V = V_0 \subset V_1 \subset ... \subset V_N = V$$

such that V_{i+1}/V_i is free of rank one for $i \geq 0$. The action of R on V_{i+1}/V_i is either trivial or as in Theorem 8.4. In particular, R' is a nilpotent conformal algebra.

If V is a free $\mathcal{A}$-module, then we can choose an $\mathcal{A}$-basis of V in which the action of R is expressed via upper triangular matrices.

Corollary 8.3 implies:

Theorem 8.5. *If R is a finite solvable conformal algebra, then there is a sequence of ideals*

$$Tor\ R = R_0 \subset R_1 \subset ... \subset R_N = R$$

such that R_{i+1}/R_i is free of rank one as an $\mathcal{A}$-module. In particular, if R is free as an $\mathcal{A}$-module, it can be obtained by a sequence of extensions by commutative conformal algebras that are free as $\mathcal{A}$-modules.

We finish by proving the useful analogue of a classical result of Lie representation theory (see e.g. [S]) sometimes referred to as the Cartan-Jacobson theorem.

Theorem 8.6. *(Conformal version of Cartan-Jacobson Theorem) Let R be a finite conformal algebra which has a finite irreducible faithful representation V. Then R is isomorphic to one of the following conformal algebras:*

(i) Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a non-zero reductive Lie algebra whose center is at most one-dimensional;
(ii) the standard semi-direct sum of Vir and Cur $\mathfrak{g}$, where $\mathfrak{g}$ is as in (i) or zero.

This result allows us to easily prove the conformal version of Engel's theorem for Lie algebras.

Corollary 8.4. *Let $R \subset gcV$ be a finite conformal algebra, with V of finite rank. If a_n is nilpotent on V for any $a \in R$ and $n \in \mathbb{Z}_+$ and $V \neq 0$, then there exists a non-zero $v \in V$ such that $a_\lambda v = 0$ for all $a \in R$.*

Proof. By contradiction: suppose there is no vector killed by the whole of R. First of all, V must be torsionless, due to Proposition 3.2.

Let us take a maximal proper submodule W (there always exists one). If $W = 0$ then V is irreducible. But Theorems 8.1-8.3 and 8.6 classify all faithful irreducible representations of conformal algebras, and no one of them is acted on by all nilpotent maps. This means that all of R acts as 0, If $W \neq 0$ and $rkW < rkV$ then the statement holds for W by induction on rkV.

Let us take the intersection W_0 of all submodules of V with the same rank as V. By Corollary 8.1, $R \cdot V \subset W_0$.

If $rkW_0 = rkV$, then W_0 has no submodules of the same rank as V, so we can use the above proof for W_0. If $0 < rkW_0 < rkV$ then we have found a submodule of lower rank than V, and we can use induction. If $W_0 = 0$ then $R \cdot V = 0$. $\qquad\square$

Corollary 8.5. *(Conformal version of Engel's Theorem) If R is a finite rank conformal algebra and all elements of R are ad -nilpotent then R is a nilpotent conformal algebra.*

Proof. By induction on rkR. Since $ad\,R \subset gcR$ satisfies hypotheses of Corollary 8.4, there is $x \neq 0$ in R such that $R \cdot x = 0$, i.e. the center Z of R is non-trivial. But R/Z also consists of ad-nilpotent elements and its rank is less than that of R, since gcR is free, and the rank of $Z \subset gcR$ is therefore of rank at least one.

Inductive hypothesis shows that R/Z is nilpotent. But it is clear that R is nilpotent if and only if R/Z is. $\square$

We will divide the proof of Theorem 8.6 in several lemmas.

Lemma 8.6. *If R is a finite conformal algebra having a faithful irreducible finite representation V then the radical of R is a commutative ideal.*

Proof. Let $J = RadR$. Since J is solvable, there is a $v \in V$ that is a common eigenvector for J; furthermore, we know that J acts on v with weights that are independent of ∂ (Theorem 8.4). Hence $J' = [J \cdot J]$ is an ideal that kills v. But the set of all vectors killed by J' is a non-zero submodule, so it is the whole V. By faithfulness of V, $J' = 0$. $\square$

Definition 8.3. Let J be a conformal algebra and let V be a J-module. Let $J \ni a \mapsto \phi_a(\lambda) \in \mathbb{C}[\lambda]$ be a linear map. For a non-negative integer n denote by V_ϕ^n the subspace of all elements of V that are killed by n factors of the form $a_\lambda - \phi_a(\lambda)$, where $a \in J, \lambda \in \mathbb{C}$. The *generalized weight space* of V corresponding to ϕ is

$$V_\phi = \bigcup_{n=1}^{\infty} V_\phi^n.$$

(Note that V_ϕ^1 is the ordinary weight space).

Lemma 8.7. *Let V be an R-module, and let J be an ideal of R and $V_\phi = \bigcup_{n \geq 1} V_\phi^n$ be a generalized weight space for J. Then for each $n \in \mathbb{Z}_+$ we have:*
  (a) $\partial V_\phi^n \subset V_\phi^{n+1}$.
  (b) Each V_ϕ^n is J-invariant.
  (c) $R \cdot V_\phi^n \subset V_\phi^{n+1}$.
  (d) $V_\phi^{n+1} = \{v \in V \,|\, (a_\lambda - \phi_a(\lambda))v \in V_\phi^n \text{ for all } a \in J, \lambda \in \mathbb{C}\}$.
  (e) V_ϕ is an R-submodule of V.

Proof. (d) is clear. (a) is proved by induction, observing that $(a_\lambda - \phi_a(\lambda))(\partial v) = \partial((a_\lambda - \phi_a(\lambda))v) + \lambda a_\lambda v$; (d) takes care of the basis of induction, since if $v \in V_\phi^1$, then $(a_\lambda - \phi_a(\lambda))(\partial v) = \lambda a_\lambda v \in V_\phi^1$. (b) is immediate since $a_\lambda v = (a_\lambda - \phi_a(\lambda))v + \phi_a(\lambda)v$.

Proof of (c) is as follows: If $v \in V_\phi^n$ then for all $r \in R, a \in J$,

$$(8.2) \qquad r_\lambda a_\mu v - a_\mu r_\lambda v = [r_{\,\lambda}\, a]_{\lambda+\mu} v.$$

From this we get $r_\lambda(a_\mu - \phi_a(\mu))v - (a_\mu - \phi_a(\mu))r_\lambda v = -[r_{\,\lambda}\, a]_{\lambda+\mu}v$ hence, since $[r_{\,\lambda}\, a] \in J$ and $(a_\mu - \phi_a(\mu))v \in V_\phi^{n-1}$, $(a_\mu - \phi_a(\mu))r_\lambda v \in V_\phi^n$ by inductive hypothesis and (a). (e) is implied by (a) and (c). $\square$

It will be useful below to set an appropriate basis in order to compare matrix representations of the action of two elements that need to be equal.

Lemma 8.8. *There exists an $\mathcal{A}$-basis $\{v_1^1, v_2^1, ..., v_{k_1}^1, v_1^2, ..., v_{k_2}^2, v_1^3, ...\}$ of V_ϕ such that*

$$(8.3) \qquad V_\phi^n = \{\sum_{i,j} p_j^i(\partial)v_j^i| \deg p_j^i(\partial) \leq n - i\}.$$

Proof. For every $i \in \mathbb{Z}_+$ let $\{v_j^i\}_{j \in J_i}$ be a $\mathbb{C}$-basis of V_ϕ^i modulo $V_\phi^i \cap \mathcal{A}V_\phi^{i-1}$. We want to show that all $\{v_j^i\}$ are linearly independent over $\mathcal{A}$. In particular, if V_ϕ is finite, then $\{v_j^i\}$ is a finite set of vectors.

If we have a non-trivial relation between the v_j^i's, we can choose

$$(8.4) \qquad \sum_{\substack{i=1\ldots N-1 \\ j}} p_j^i(\partial)v_j^i + \sum_j q_j(\partial)v_j^N = 0$$

with $q_j \neq 0$ for some j, and of minimal maximal degree d of the q_j's. Then $d = 0$. In fact, if $d > 0$, then applying $a_\lambda - \phi(\lambda)$ to both sides of (8.4), one gets:

$$(8.5) \qquad \sum_{\substack{i=1\ldots N-1 \\ j}} \tilde{p}_j^i(\partial, \lambda)v_j^i + \phi_a(\lambda) \sum_j (q_j(\partial + \lambda) - q_j(\partial))v_j^N = 0$$

for some polynomials $\tilde{p}_j^i$. If the degree of ϕ_a is m, the coefficient of λ^{m+d} in (8.5) gives a linear relation as in (8.4) where all q_j's are constant, and not all of them are 0.

Once we know $d = 0$, (8.4) shows that $v = \sum_j q_j v_j^N$ is a non-trivial $\mathbb{C}$-linear combination of the v_j^N which lies in $\mathcal{A}V_\phi^{N-1}$ by construction. But v represents a non-zero class in $V_\phi^N/(V_\phi^N \cap \mathcal{A}V_\phi^{N-1})$, giving a contradiction. Therefore, the v_j^i's are linearly independent, and they clearly span V_ϕ.

We are left with showing that $V_\phi^n = \{\sum_{i,j} p_j^i(\partial)v_j^i | \deg p_j^i \leq n - i\}$. This is obviously true for $n = 1$, and we want to establish it for all n using induction on n. Every element inside V_ϕ^{n+1} is clearly an $\mathcal{A}$-linear combination of the $v_j^i, i \leq n + 1$. Let

$$\sum_{\substack{i=1\ldots N-1 \\ j}} p_j^i(\partial)v_j^i + \sum_j q_j(\partial)v_j^N$$

be an element of V_ϕ^{n+1}. We show by induction on N that polynomials p_j^i and q_j are as desired. Applying $a_\lambda - \phi(\lambda), \lambda \in \mathbb{C}$, one gets

$$\sum_{\substack{i=1\ldots N-1 \\ j}} \tilde{p}_j^i(\partial, \lambda)v_j^i + \phi_a(\lambda) \sum_j (q_j(\partial + \lambda) - q_j(\partial))v_j^N$$

for some choice of polynomials $\tilde{p}_j^i$, and this must lie in V_ϕ^n. But inductive assumption tells us that the degree in ∂ of $q_j(\partial + \lambda) - q_j(\partial)$ must be $\leq n - N$ for all $\lambda \in \mathbb{C}$, hence $\deg q_j \leq n + 1 - N$.

If so, the $\mathcal{A}$-linear combination $\sum_j q_j(\partial)v_j^N$ certainly belongs to V_ϕ^{n+1}, hence we are left with proving the statement for

$$\sum_{\substack{i=1\ldots N-1 \\ j}} p_j^i(\partial)v_j^i,$$

which is true by inductive assumption. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We will refer to any basis of V_ϕ obtained in this way as to a *standard basis*. Matrix representations of conformal linear maps are analogous to those of linear maps between vector spaces. If $A(\partial, \lambda)$ is the matrix representing the action of a_λ and $B(\partial, \mu)$ is that representing the action of b_μ, then the matrix representing $a_\lambda b_\mu$ is clearly given by the product matrix $A(\partial, \lambda)B(\partial + \lambda, \mu)$.

We are going to consider generalized weight spaces for the action of $J = RadR$ on V. They come useful, since they are fixed by the action of R. Since J is solvable, we know by the conformal version of Lie's theorem (Theorem 8.4) that at least one of the weight-spaces is non-trivial. Let ϕ be the corresponding weight. Note that $\phi \neq 0$, since otherwise V_ϕ^1 is R-invariant, hence $V = \mathcal{A}V_\phi^1$, J acts trivially on V, and Theorem 8.6 follows from Proposition 8.1.

From now on, R will be a finite conformal algebra, J its radical, V a faithful irreducible finite R-module such that $V = V_\phi$, the generalized weight space with respect to the action of J for the weight $\phi \neq 0$, for which $V_\phi^1 \neq 0$.

Lemma 8.9. *The rank of J is at most one.*

Proof. Suppose that $rkJ > 1$. Then, by Remark 8.3, $\phi_{\partial a} = -\lambda\phi_a$, hence we can always find a non-zero $a \in J$ for which $\phi_a = 0$. Now, all elements in J whose weight is 0 form an ideal J_0 of R. In fact, since the action of $r \in R$ increases the filtration by one (Lemma 8.7c), and $a_\lambda - \phi_a(\lambda), \lambda \in \mathbb{C}$, obviously decreases it by one, their matrix representation in any fixed standard basis is given by the following block matrices for the filtration $\{V_\phi^n\}$:

$$r_\lambda = \begin{pmatrix} * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ R_1 & * & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & R_2 & * & & \cdot & \cdot \\ \vdots & \ddots & \ddots & & \ddots \\ 0 & \cdots & 0 & R_{n-1} & * \end{pmatrix}$$

$$a_\mu = \begin{pmatrix} \phi_a(\mu) & J_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \phi_a(\mu) & J_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & & 0 & \ddots & \ddots \\ 0 & & & \ddots & \phi_a(\mu) & J_{n-1} \\ 0 & & \cdot & \cdot & \cdot & \cdot & 0 & \phi_a(\mu) \end{pmatrix}$$

where $R_i = R_i(\lambda)$ and $J_i = J_i(\mu)$ are independent of ∂ (cf. Lemma 8.8); then taking trace of the matrices representing both sides of

$$r_\lambda a_\mu - a_\mu r_\lambda = [r_\lambda a]_{\lambda+\mu}$$

we easily get that if ϕ_a is zero, then $\phi_{[r_\lambda a]}$ is also zero. The same reasoning as before shows that J_0 acts trivially on V, a contraiction with faithfulness of the R-module V. $\square$

Once we know J is commutative of rank one, we can consider the adjoint representation of R/J on J. This is a rank one representation of a semi-simple conformal algebra. We know from Theorem 8.1 that the current part of R/J must act as 0, and that of all the Virasoro algebras, only one can act non-trivially. Let $S \subset R$ be the centralizer of J in R.

Lemma 8.10. *S stabilizes the filtration $\{V_\phi^n\}$.*

Proof. By definition of $S, [S \cdot J] = 0$. Then using this in (8.2) gives the proof. $\square$

This tells us that in the standard basis the matrices representing the action of elements from S are block upper triangular for the filtration $\{V_\phi^n\}$, with diagonal blocks independent of ∂. In particular, if S/J contains *Vir*'s, diagonal blocks in their matrix representation are trivial (by Theorem 8.2). Hence their action is trivial on V, so S/J is a current conformal algebra by Theorem 7.1.

Lemma 8.11. *All central extensions of a finite simple conformal algebra R by a free $\mathcal{A}$-module of rank one are trivial.*

Proof. Let R be a central extension of a finite simple current conformal algebra *Cur* $\mathfrak{g}$ by a rank one center $\mathcal{A}$. The λ-bracket in such a conformal algebra is given by

(8.6) $$[g_\lambda h] = [g, h] + \alpha_\lambda(g, h),$$

where $g, h \in \mathfrak{g} \subset$ *Cur* $\mathfrak{g}$ and $\alpha_\lambda(g, h) \in \mathcal{A}[\lambda]$. Then axiom (C4) gives

(8.7) $$\alpha_\lambda(a, [b, c]) - \alpha_\mu(b, [a, c]) = \alpha_{\lambda+\mu}([a, b], c),$$

and setting $\alpha_\lambda(g, h) = \sum_i \lambda^i \alpha_i(g, h)$:

$$\sum_i \lambda^i \alpha_i(a, [b, c]) - \sum_j \mu^j \alpha_j(b, [a, c]) = \sum_k (\lambda + \mu)^k \alpha_k([a, b], c),$$

for every $a, b, c \in \mathfrak{g}$.

This immediately shows $\alpha_i = 0$ for $i \geq 2$, since $\mathfrak{g}' = \mathfrak{g}$. Moreover:

$$\alpha_0(a, [b, c]) - \alpha_0(b, [a, c]) = \alpha_0([a, b], c), \tag{8.8}$$

$$\alpha_1(a, [b, c]) = \alpha_1([a, b], c). \tag{8.9}$$

Axiom (C3) is equivalent to

$$\alpha_0(a, b) + \alpha_0(b, a) = \partial \alpha_1(a, b). \tag{8.10}$$

It follows from (8.9) and (8.10) that α_1 is a symmetric invariant $\mathcal{A}$-valued bilinear form on $\mathfrak{g}$, hence it is a multiple of the Killing form $(\cdot|\cdot)$: $\alpha_1(g, h) = p(\partial)(g|h)$.

Set $\beta(g, h) = \alpha_0(g, h) - \frac{1}{2}\partial p(\partial)(g|h)$. Then (8.10) implies skew-symmetry of β and (8.8) translates as

$$\beta([a, b], c) - \beta(a, [b, c]) + \beta(b, [a, c]) = \frac{1}{2}\partial p(\partial)([a, b]|c).$$

Plugging in $a = h \in \mathfrak{h}, b = e_\alpha, c = e_{-\alpha}$ for some root α, we get:

$$\beta(h, [e_\alpha, e_{-\alpha}]) = \frac{1}{2}\partial p(\partial)\alpha(h)(e_\alpha|e_{-\alpha}). \tag{8.11}$$

If h_α is the bracket of root vectors $e_\alpha, e_{-\alpha}$ such that $(e_\alpha|e_{-\alpha}) = 1$, we get $\beta(h_\alpha, h_\alpha) = \frac{1}{2}\partial p(\partial)\alpha(h_\alpha)$. But the left-hand side of (8.11) must be 0 due to skew-symmetry of β. Hence $p = 0$. This shows that α_1 is 0 on the whole $\mathfrak{g}$, hence α_0 is a Lie algebra 2-cocycle for $\mathfrak{g}$, so it is trivial, since any central extension of the simple Lie algebra $\mathfrak{g}$ is trivial.

In the case of a central extension of *Vir*, axiom (C4) for the λ-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L + p(\partial, \lambda)$$

gives the following condition on p:

$$(\partial + \lambda + 2\mu)p(\partial, \lambda) - (\partial + 2\lambda + \mu)p(\partial, \mu) = (\lambda - \mu)p(\partial, \lambda + \mu) \tag{8.12}$$

Setting $\mu = 0$ in (8.12) we get $\partial p(\partial, \lambda) = (\partial + 2\lambda)p(\partial, 0)$, whence $p(\partial, \lambda) = q(\partial)(\partial + 2\lambda)$, for $q(\partial) = \frac{p(\partial)}{\partial}$. Then $L + q(\partial)$ is a standard generator of a Virasoro conformal algebra. $\square$

Lemma 8.12. *If R is a finite semi-simple conformal algebra, then all central extensions of R by a free $\mathcal{A}$-module of rank one are trivial.*

Proof. As an $\mathcal{A}$-module, a semi-simple conformal algebra R is a finite direct sum $\bigoplus_i S_i$ of subalgebras S_i (Theorem 7.1). We can make a 2-cocycle $\alpha_\lambda(\cdot, \cdot)$ on R trivial on $S_i \times S_i$ for all i. We will call $\alpha_\lambda|_{S_i \times S_j}$ a cross-extensions of S_i and S_j. Our goal is to prove that cross-extensions of a 2-cocycle on R are zero if its restrictions on all S_i are zero.

Without loss of generality, we can assume R to be the (semi-direct) sum of *two* simple algebras.

If $R = Cur\,\mathfrak{g}$ is a semi-simple current conformal algebra, one shows as in Lemma 8.11 that α_1 is an invariant bilinear form on $\mathfrak{g}$, hence $\alpha_1(g, h) = 0$ if g and h belong to distinct simple ideals. So α_1 is identically zero on $\mathfrak{g}$, hence α_0 is a Lie algebra 2-cocycle on $\mathfrak{g}$, hence it is trivial (since $\mathfrak{g}$ is a semi-simple Lie algebra). This takes care of cross extensions of a simple conformal algebra by another simple conformal algebra.

Next, we need to figure out what $\alpha_\lambda(L, g)$ can be if L is the standard generator of *Vir*, and $g \in Cur\,\mathfrak{g}$, with $\mathfrak{g}$ a finite-dimensional simple Lie algebra. In a semi-direct sum, L will act trivially or in the standard way on g. Then setting $a = L, b = g, c = h, g, h \in \mathfrak{g}$, in (8.7) gives $\alpha_\lambda(L, [g, h]) = 0$ as soon as α_λ is zero on $\mathfrak{g} \times \mathfrak{g}$. If we make α_λ zero in advance, this shows that $\alpha_\lambda(L, g) = 0$ if g lies inside the derived Lie algebra $\mathfrak{g}'$ which equals $\mathfrak{g}$ when $\mathfrak{g}$ is simple.

In order to show that also cross-extensions of *Vir* by *Vir* are zero, it is enough to substitute $a = L^1, b = c = L^2$ in (8.7), where L^1 and L^2 are standard generators for the two Virasoro conformal algebras. $\square$

Remark 8.4. (a) The universal central extension of the conformal algebra *Vir* has one-dimensional (over $\mathbb{C}$) center and the corresponding cocycle is a multiple of $\alpha_\lambda(L, L) = \lambda^3$ (of course the associated formal distribution Lie algebra is the usual Virasoro algebra).

(b) The center of the universal central extension of *Cur* $\mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra is canonically identified with the space B of all invariant bilinear forms on $\mathfrak{g}$ and the corresponding cocycle is $\alpha_\lambda^f(a, b) = f(a, b), f \in B, a, b \in \mathfrak{g}$ (of course, for simple $\mathfrak{g}$, the associated formal distribution Lie algebra is the usual affine Kac-Moody algebra).

(c) For an arbitrary semi-simple conformal algebra the universal central extension is the obvious combination of (a) and (b).

Corollary 8.6. $S = S/J \oplus J$ *(direct sum of conformal algebras).*

If $R = S$, i.e. J is centralized by the whole R, we are finished, since S is the current conformal algebra associated to a reductive Lie algebra with one-dimensional center. If $R \neq S$, we need to do some extra work.

We are in the following situation: R is an extension of R/J by J which is trivial on S/J, and $R/S = $ *Vir*. From now on, denote by a a generator of J, by ϕ the polynomial ϕ_a, and by L a representative in R of the standard generator for the quotient conformal algebra R/S. It is clear that we can choose L to act on S/J in the standard way, and with no contribution in J (since S/J is isomorphic to S'). Next, we prove:

Lemma 8.13. $[L_\lambda a] = (\partial + \lambda)a$.

Proof. By now we only know that $[L_\lambda a] = p(\partial, \lambda)a$ for some polynomial $p \in \mathbb{C}[\partial, \lambda]$. Writing down matrix representations of the action of L and a in the standard basis

$$
L_\lambda = \begin{pmatrix} M_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_1(\lambda) & M_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & L_2(\lambda) & \ddots & & & & \\ \vdots & & \ddots & \ddots & M_{n-1} & \\ 0 & \cdots & 0 & L_{n-1}(\lambda) & M_n \end{pmatrix}
$$

$$
a_\mu = \begin{pmatrix} \phi(\mu) & C_1(\mu) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \phi(\mu) & C_2(\mu) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & 0 & \ddots & \ddots & & & \\ 0 & & & \ddots & \phi(\mu) & C_{n-1}(\mu) \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \phi(\mu) \end{pmatrix},
$$

where $L_i(\lambda)$'s and $C_i(\mu)$'s are independent of ∂ and $M_i(\partial, \lambda)$'s are linear in ∂ (cf. Lemma 8.8), and taking trace of matrices representing both sides of:

$$(8.13) \qquad\qquad [L_\lambda, a_\mu] = p(-\lambda - \mu, \lambda)a_{\lambda+\mu}$$

we get

$$(8.14) \qquad \phi(\mu)(\sum_i Tr(M_i(\partial, \lambda) - M_i(\partial + \mu, \lambda))) = p(-\lambda - \mu, \lambda)\phi(\lambda + \mu) \cdot rk(V).$$

Set $M(\partial, \lambda) = Tr \sum_i M_i(\partial, \lambda)$ and write $M(\partial, \lambda) = M_0(\lambda) + N(\lambda)\partial$. Then (8.14) can be rewritten as

$$(8.15) \qquad\qquad -\mu\phi(\mu)N(\lambda) = p(-\lambda - \mu, \lambda)\phi(\lambda + \mu)rk(V).$$

Setting $\mu = 0$ in (8.15) we obtain $p(-\lambda, \lambda)\phi(\lambda) = 0$ and since ϕ is not identically zero, $p(-\lambda, \lambda) = 0$ for all $\lambda \in \mathbb{C}$. This means that $p(\partial, \lambda) = (\partial + \lambda)q(\partial, \lambda)$ for some other polynomial q. But the adjoint action of L on a defines a rank one representation of the Virasoro conformal algebra, and we know all such (Theorem 8.2). The only possible values of q are zero and one. If q is zero, then L centralizes J and $R = S$. If $q = 1$ then $[L_\lambda a] = (\partial + \lambda)a$. $\qquad\square$

Remark 8.5. Substituting $p(\partial, \lambda) = \partial + \lambda$ in (8.15) it is immediate to show that ϕ must be a constant polynomial and $N(\lambda) = Id$. Up to rescaling a by a complex multiple, ϕ can therefore be made equal to 1.

The only λ–bracket we now need to take care of is $[L_\lambda L]$. In order to show it is nothing but $(\partial + 2\lambda)L$, or rather, that it can be made into this form up to shifting the representative L by an element in J, we prove the following lemma.

Lemma 8.14. $L_\lambda - \partial$ stabilizes V_ϕ^1.

Proof. Using Lemma 8.13 and axioms of an R-module, we get:

$$(8.16) \qquad [L_\lambda, a_\mu] = -\mu a_{\lambda+\mu}$$

Applying both sides of this equation to $v \in V_\phi^1$, we get

$$(8.17) \qquad L_\lambda v - s_\mu L_\lambda v = -\mu v,$$

where s_μ is as in Proposition 6.2. Using the notation of Lemma 8.13, the left top block in the matrix representation of $-\mu a_{\lambda+\mu}$, obtained from (8.16), must be equal to $-\mu - C_1(\mu)L_1(\lambda)$. But by (8.17) this equals to $-\mu$, so $C_1(\mu)L_1(\lambda) = 0$.

The way they have been obtained, it is clear that the columns of $C_1(\mu)$ are linearly independent over $\mathbb{C}$, otherwise we would get a linear combination of the v_j^2 which lies in V_ϕ^1. Therefore $L_1(\lambda)$ itself must be zero. This shows that $\mathcal{A}V_\phi^1$ is stabilized by the action of L and that $L_\lambda - \partial$ acts independently of ∂ on a basis. $\square$

Combining Lemma 8.7b, Lemma 8.10 and Lemma 8.14, we see that $\mathcal{A}V_\phi^1$ is an R-submodule of V, hence $V = \mathcal{A}V_\phi^1$. So there is a basis of V of eigenvectors for a, and the action of a is given in this basis by the identity matrix.

Lemma 8.15. L can be chosen such that $[L_\lambda L] = (\partial + 2\lambda)L$.

Proof. The matrix representation of the action of L_λ in the eigenvector basis of V is $\partial + C(\lambda)$, where C is some matrix independent of ∂ (Lemma 8.14). Up to adding to L an appropriate element j from J, we can make the trace of $C(\lambda)$ equal to zero.

Let $\tilde{L} = L - j$. Then the action of $[\tilde{L}_\lambda, \tilde{L}_\mu] - (\lambda - \mu)\tilde{L}_{\lambda+\mu}$ is given by the matrix $\lambda C(\lambda) - \mu C(\mu) - (\lambda-\mu)C(\lambda+\mu)$, whose trace is clearly 0. This shows that if $[\tilde{L}_\lambda \tilde{L}] = (\partial+2\lambda)\tilde{L}+j', j' \in J$, then the sum of elements on the diagonal of the matrix representation of j' is zero. Hence $j' = 0$, and $\tilde{L}$ is Virasoro-like. $\square$

End of proof of Theorem 8.6. By Lemma 8.9, rkJ is at most one. If $J = 0$, R is semi-simple. In view of Proposition 8.1, a finite semi-simple conformal algebra having a finite faithful irreducible representation is as in (i) or in (ii).

If $J \neq 0$, and R centralizes J, then $R = R/J \oplus J$ is isomorphic to *Cur* $\mathfrak{g}$, where $\mathfrak{g}$ is a reductive Lie algebra with one-dimensional center.

If $J \neq 0$, and R does not centralize J, then R is the standard semi-direct sum of *Vir* with $S = S/J \oplus J$ which is the current conformal algebra associated to a reductive Lie algebra $\mathfrak{g}$ with one-dimensional center. $\square$

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DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE MA 02139
E-mail address: dandrea@mat.uniroma1.it