ALGEBRAIC ASPECTS OF OPERATOR PRODUCT EXPANSION IN A CONFORMAL FIELD THEORY: AN INTRODUCTION TO PSEUDOALGEBRAS

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ABSTRACT. The concept of Lie pseudoalgebra over a cocommutative Hopf algebra, introduced in [BDK], is a generalization of that of a Lie algebra. The basic motivating example is provided by the algebra underlying Operator Product Expansion of fields in a vertex algebra, i.e. in the chiral part of a Conformal Field Theory. I give a short introduction to the algebraic structures of vertex algebras and pseudoalgebras, introduce the basic tools used in the theory, and list the main results of the theory of finite Lie pseudoalgebras. A few applications are given.

CONTENTS

Algebraic properties of chiral quantum fields in a Conformal Field Theory in dimension two have been recently axiomatized in the notion of *vertex algebra*. This concept has undergone mathematical investigation because of its relevance in Representation Theory – see for instance Borcherds' work [B] on the Moonshine conjecture.

However vertex algebra are hard to handle, mostly because they are huge objects. The language in which they are often described is that of conformal algebras, or Operator Product Expansion. In this talk I give a short introduction to conformal algebra and pseudoalgebras, their higher dimensional generalization. I begin by defining vertex algebras, and show how this notion motivates that of conformal algebras. I will then briefly sketch the recent results that have provided a structure theory of conformal algebras and their representations, and show how these results generalize to a broader algebraic structure encompassing both Lie algebras and conformal algebras.

The standard introductory reference on vertex algebras is [K]. Results on conformal algebras mainly refer to [DK] for classification, [CK] for representation, and [BKV] for cohomology. Pseudoalgebras are introduced in [BD] and studied in [BDK].

1. VERTEX ALGEBRAS

What is a commutative algebra? We would typically answer by saying it is a vector space endowed with an associative and commutative bilinear product. However, my ultimate goal is that of defining vertex algebras: I will therefore give a less usual definition which has a definite *vertex flavour*. As far as I am concerned, a commutative algebra – "over $\mathbb{C}$" will be understood throughout – is a vector space A endowed with a linear map

$$Y : A \to \operatorname{End} A$$

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satisfying

$$[Y(a), Y(b)] = 0 \text{ for all } a, b \in A. \tag{1.1}$$

Moreover A contains an element 1 with the property that

$$Y(1) = \mathrm{id}_A \qquad Y(a)1 = a. \tag{1.2}$$

How do we understand this as the usual definition of a commutative algebra? We can define a product on A by $a \cdot b = Y(a)b$; then 1 is the multiplicative identity, due to (1.2), whereas associativity and commutativity follow from (1.1). Indeed

$$ab = Y(a)Y(b)1 = Y(b)Y(a)1 = ba$$

and

$$a(bc) = a(cb) = Y(a)Y(c)b = Y(c)Y(a)b = c(ab) = (ab)c.$$

In other words we have defined a commutative algebra via the collection of left multiplications by its elements $\{Y(a), a \in A\}$.

In a vertex algebra setting A is the Fock space, and its elements are *physical states* whereas their images under Y are the (quantum) *fields* acting on A. Thus, Y establishes a *state-field correspondence*, which is indeed injective by (1.2). An interesting property of multiplication operators $Y(a)$ is that every endomorphism of A commuting with all $Y(a)$ also is multiplication by some element. Let $\phi \in \mathrm{End}_A$, $[\phi, Y(a)] = 0$ for all $a \in A$. Then we have $\phi(a) = \phi(Y(a)1) = Y(a)(\phi(1)) = Y(a)Y(\phi(1))1 = Y(\phi(1))Y(a)1 = Y(\phi(1))(a)$. In other words ϕ coincides with $Y(\phi(1))$.

Our point of view is that a vertex algebra is a commutative algebra where we have replaced products by functions of an indeterminate z. However, to make the whole thing interesting, we must allow these functions to be singular. Indeed, in a vertex algebra we endow the vector space V with a correspondence

$$Y : V \to (\mathrm{End}\, V)[[z, z^{-1}]]$$

with the property that $Y(a, z)$ acts as a *field* on V. A formal distribution $\phi(z) \in (\mathrm{End}\, V)[[z, z^{-1}]]$ is a field if $\phi(z)v$ lies inside $V((z))$ - i.e. if it has finitely many negative powers of z - for every $v \in V$. Fields of the form $Y(a, z)$ are called vertex operators. Commutativity is replaced by a *locality* axiom. For all $a, b \in V$ we have:

$$(z - w)^N [Y(a, z), Y(b, w)] = 0 \tag{1.3}$$

for N sufficiently large. As $Y(a, z)$ and $Y(b, w)$ are formal distributions, this does not force their commutator to be zero. We will see that the commutator of $Y(a, z)$ and $Y(b, w)$ can be expressed as a linear combination of Dirac δ distributions. The unit element in a vertex algebra is called *vacuum*, and is denoted by $\mathbf{1}$. It satisfies:

$$Y(\mathbf{1}, z) = \mathrm{id}_V \qquad Y(a, z)\mathbf{1} = a \mod zV[[z]]. \tag{1.4}$$

This is the core of the definition of a vertex algebra, but before we are done, a last remark is needed. We would like a field satisfying locality with respect to all of our $Y(a, z)$ to be a vertex operator as well, in analogy with the case of commutative algebras. Indeed this is not guaranteed by the definition of a vertex algebra we have just sketched. In fact, $\frac{d}{dz}Y(a, z)$ is clearly local with respect to all vertex operators, but it might fail to be a vertex operator, unless we explicitly impose it to be. We denote by Ta the element of V such that:

$$Y(Ta, z) = \frac{d}{dz}Y(a, z).$$

Definition 1.1. A vertex algebra is a vector space V endowed with

- a linear map $Y : V \to (\mathrm{End}\, V)[[z, z^{-1}]]$ — the *state-field correspondence*,
- an element $\mathbf{1} \in V$ — the *vacuum*,
- a linear endomorphism $T : V \to V$ — the *translation operator*

such that:
(i) all $Y(a, z)$ are pairwise local fields
(ii) $Y(\mathbf{1}, z) = \mathrm{id}_V, \qquad Y(a, z)\mathbf{1} = a \mod zV[[z]], \qquad$ for all $a \in V$
(iii) $Y(Ta, z) = [T, Y(a, z)] = \frac{d}{dz} Y(a, z), \qquad$ for all $a \in V$.

The difference between a commutative algebra and a vertex algebra is encoded in the locality axiom. Indeed, $a(z, w)$ is killed by multiplication by $(z - w)^N$ if and only if $a(z, w)$ can be expressed as [K]:

$$a(z, w) = \sum_{j=0}^{N-1} a^j(w)\delta^{(j)}(z - w), \tag{1.5}$$

for some choice of formal distributions $a^j(w)$, where

$$\delta(z - w) = \frac{1}{w} \sum_{k \in \mathcal{Z}} \left(\frac{z}{w}\right)^k$$

is the Dirac delta distribution and $\delta^{(j)}(z - w)$ is its j-th derivative with respect to w.

Once this is established, it is easy to check that if $(z - w)^N [Y(a, z), Y(b, w)]$ equals 0 then expressing $a(z, w) = [Y(a, z), Y(b, w)]$ as in (1.5), all coefficients $a^j(w)$ are indeed fields, and are also local with respect to all vertex operators. In other words they are vertex operators themselves. One may eventually prove that if

$$Y(a, z) = \sum_j z^{-j-1} a_j \tag{1.6}$$

then:

$$[Y(a, z), Y(b, w)] = \sum_j Y(a_j(b), w) \frac{\delta^{(j)}(z - w)}{j!}. \tag{1.7}$$

The above expression[1] (or rather, an equivalent formula used by physicists) is traditionally called Operator Product Expansion (OPE) of the fields $Y(a, z)$ and $Y(b, w)$. It encodes the commutation structure of the "commutative algebra" V. Indeed interesting vertex algebras are typically huge objects. They commonly arise in the fashion of graduate vector spaces of exponential growth, and their theory encompasses that of affine Kac-Moody algebras, unimodular lattices, and so forth. A classification of simple vertex algebras is clearly hopeless.

The algebraic structure underlying OPE (which is introduced in [K] and studied in [DK] under the name of "conformal algebra") is instead much tamer. First of all, there are small interesting instances of such structures. Secondly, a broad class of vertex algebras can be defined in terms of conformal algebras by means of a universal envelope construction. Thirdly, every vertex algebra is also a conformal algebra, and we hope that the study of a vertex algebra structure can be made easier and more insightful by first studying the underlying conformal algebra.

Lastly, the conformal algebra underlying a vertex algebra measures, in a way, how far the vertex algebra is from being a commutative algebra. Indeed, when all OPE of fields are trivial, the structure of a vertex algebra collapses to that of a commutative algebra with a derivation. In this talk, I will expose the structure theory of conformal algebras and pseudoalgebras, their higher dimensional generalization.

2. Conformal algebras

It can be proved that every family of pairwise local fields acting on some vector space V can be embedded inside a vertex algebra. In other words the vertex algebra structure captures all algebraic properties of families of local fields containing the identity field and closed under normally ordered product and coefficients of the OPE.

[1]the $j!$ denominator is just a convenient normalization.

If we retain the OPE and discard the normally ordered product, our formal distributions do not need to be fields anymore. In fact, the field property was only used in order to define the normally ordered product (by avoiding divergence problems). Indeed, the action itself of the formal distributions on the vector space V becomes unnecessary: all that is needed is just a family of formal distributions satisfying locality.

If we axiomatize the structure of OPE, then we obtain something called *conformal algebra*, which can be characterized by properties that are remindful of the axioms for a Lie algebra. A conformal algebra is meant to be a family F of local formal distributions with coefficients in some Lie algebra $\mathfrak{g}$ (even though our final definition will not mention $\mathfrak{g}$). F will be closed under $\mathbb{C}[\partial]$ linear combination, where

$$(\partial a)(z) = \frac{d}{dz} a(z)$$

whenever $a(z) \in F$. Moreover, we require F to contain all formal distributions showing up as coefficients of the OPE of any two elements of F – in other words F must be OPE closed. Indeed if $a_{(j)}b$ denotes the j-th coefficient in the expression

$$[a(z), b(w)] = \sum_j (a_{(j)}b)(w) \cdot \frac{\delta^{(j)}(z-w)}{j!} \tag{2.1}$$

for the commutator of two local formal distributions $a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]]$, then setting:

$$[a\,{}_\lambda b] = \sum_j \frac{\lambda^j}{j!} a_{(j)}b, \tag{2.2}$$

makes $[a\,{}_\lambda b]$ a polynomial expression in λ with coefficients in F, which satisfies the following properties:

(C1) $[\partial a\,{}_\lambda b] = -\lambda[a\,{}_\lambda b] \qquad [a\,{}_\lambda \partial b] = (\partial + \lambda)[a\,{}_\lambda b]$,
(C2) $[a\,{}_\lambda b] = -[b\,{}_{-\partial-\lambda} a]$,
(C3) $[a\,{}_\lambda [b\,{}_\mu c]] = [b\,{}_\mu [a\,{}_\lambda c]] + [[a\,{}_\lambda b]\,{}_{\lambda+\mu} c]$.

Definition 2.1. A conformal algebra is a $\mathbb{C}[\partial]$-module L, endowed with a λ-bracket $[{}_\lambda] : L \otimes L \to L[\lambda]$ satisfying the properties (C1-3).

Apart from axiom (C1), which is just $\mathbb{C}$-linearity of some kind, the other two axioms are very similar to skew-symmetry and Jacobi identity in a Lie algebra. We will see later that a conformal algebra is in fact a Lie algebra in a different multilinear sense than usual.

Notice also that the Lie algebra $\mathfrak{g}$ is never mentioned in the above definition. Its structure is encoded inside the conformal algebra. The Lie algebra $\mathfrak{g}$ can be recovered as a certain quotient of a universal Lie algebra attached to the conformal algebra F, called the *Lie algebra of Fourier coefficients* of F. In fact, denote by $\mathcal{A}(L)$ the $\mathbb{C}$-linear span of symbols a_i where $a \in L, i \in \mathbb{Z}$, and take its quotient by relations $(\lambda a + \mu b)_n = \lambda a_n + \mu b_n$ and $(\partial a)_n = -n a_{n-1}$. Then $\mathcal{A}(L)$ becomes a Lie algebra with the bracket:

$$[a_m, b_n] = \sum_j \binom{m}{j} (a_{(j)}b)_{m+n-j}. \tag{2.3}$$

This Lie bracket is engineered in such a way that the $\mathcal{A}(L)$-valued formal distributions defined as:

$$a(z) = \sum_i a_i z^{-i-1}$$

satisfy precisely the OPE encoded in the conformal algebra L. Moreover the $\mathbb{C}[\partial]$-module structure on L induces a derivation $d(a_n) = (\partial a)_n = -n a_{n-1}$ of $\mathcal{A}(L)$, which is therefore a differential Lie algebra. There is a canonical projection $\mathcal{A}(L) \to \mathfrak{g}$, so that $\mathcal{A}(L)$ is the largest Lie algebra having a set of linear generators which can be assembled into formal distribution having the desired OPE.

Conformal (super)algebra structures on finitely generated $\mathbb{C}[\partial]$-modules have been extensively considered by physicists. The most common examples are the following.

Example 2.2 (Current conformal algebras)**.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. We can define a conformal algebra on $L(\mathfrak{g}) = \mathbb{C} \otimes \mathfrak{g}$ by setting

$$[1 \otimes g \,{}_\lambda 1 \otimes h] = 1 \otimes [g, h]$$

for all $g, h \in \mathfrak{g}$ and extending by axiom (C1). The conformal algebra $L(\mathfrak{g})$ has no nontrivial ideals if and only if $\mathfrak{g}$ is simple. The Lie algebra $\mathcal{A}(L(\mathfrak{g}))$ is linearly generated by elements $g_i, g \in \mathfrak{g}, i \in \mathbb{Z}$, the Lie bracket being

$$[g_m, h_n] = [g, h]_{m+n}.$$

The Lie algebra $\mathcal{A}(L)$ is therefore isomorphic to the affinization $\mathfrak{g}[z, z^{-1}]$ of $\mathfrak{g}$.

Example 2.3 (Virasoro conformal algebra)**.** Let $V = \mathbb{C}[\partial]x$ be a free module of rank one. Then

$$[x_\lambda x] = (\partial + 2\lambda)x$$

defines a conformal algebra structure. A linear basis for the Lie algebra $\mathcal{A}(V)$ is given by elements x_i satisfying the Lie bracket:

$$[x_m, x_n] = (m - n)x_{m+n-1}.$$

This is isomorphic to the *centerless Virasoro Lie algebra* $\mathbb{C}[z, z^{-1}]d/dz$ of regular vector fields on $\mathbb{C}^*$ via the map $x_m \mapsto -z^m d/dz$. The conformal algebra V is simple.

It can be shown that with any differential commutative associative algebra D one can associate a differential Lie algebra $\mathcal{A}_D(L)$. The Lie algebra of Fourier coefficients of L is then obtained for $D = \mathbb{C}[z, z^{-1}]$. Indeed, any conformal algebra establishes a functor from the category of differential commutative algebras to the category of differential Lie algebras.[2]

The interplay between the $\mathbb{C}[\partial]$-module structure and the λ-bracket on one side, and the Lie algebra $\mathcal{A}(L)$ on the other, gives a powerful tool for the study of conformal algebras. In [DK] a classification of all simple conformal algebras on finitely generated $\mathbb{C}[\partial]$-module is established.

Theorem 2.4. *A complete list of finite simple conformal algebras is as follows:*
1) the Virasoro conformal algebra V*,*
2) all current conformal algebras $L(\mathfrak{g})$*, where* $\mathfrak{g}$ *if a simple finite dimensional Lie algebra.*

The classification is obtained by studying the *annihilation Lie algebra* $\mathcal{A}_+(L)$ of L, i.e. the subalgebra of $\mathcal{A}(L)$ spanned by elements a_i with non-negative i. One can build up a filtration on $\mathcal{A}_+(L)$, and consider the completion of $\mathcal{A}_+(L)$ with respect to it. The result is a linearly compact topological Lie algebra, which can be studied by means of Cartan's classical classification of infinite Lie algebras of vector fields (see [DK], [G1], [G2] for the statement on linearly compact infinite dimensional Lie algebras satisfying a descending chain condition by means of which we employ Cartan's result). A correspondence between ideals of L and ideals of the corresponding annihilation algebra, together with a reconstruction functor providing a conformal algebra model to all interesting linearly compact Lie algebras, proves the classification result.

The point I want to stress here is that the Lie algebra bracket (2.3) has a definite Hopf algebra flavour. If we write $a_{(t^i)}b$ instead of $a_{(i)}b$, and $t^n a$ instead of a_n, Equation (2.3) becomes:

$$[t^m a, t^n b] = \sum_j \binom{m}{j} (t^n \cdot t^{m-j}) a_{(t^j)} b, \tag{2.4}$$

which is a close relative of the comultiplication structure

$$\Delta(t^m) = \sum_j \binom{m}{j} t^{m-j} \otimes t^j$$

[2]This is remindful of the definition of group schemes, if we understand differential Lie algebras as formal differential Lie groups.

on the bialgebra $\mathbb{C}[\partial]$. This hints to a generalization of conformal algebras which will be introduced in the sequel. As a last remark on conformal algebras, I notice that associative or commutative conformal algebras could be defined similarly. Lie conformal algebras are our main example of such structures just because of their connection to vertex algebras.

3. Conformal algebras and Lie* algebras

The usual definition of a vertex algebras is far from being coordinate free. The setting is that of formal distributions that are expressed in terms of Laurent power series in z and z^{-1}. Moreover the whole theory has a kind of arithmetic flavour, as fields are required to be closed under derivation but not under multiplication by regular functions. Above all, the theory is forced to be set over $\mathbb{C}^*$, due to its nature. A more geometric definition of vertex algebras has been undertaken by Beilinson and Drinfeld in [BD]. Their definition uses some algebraic geometry, together with $\mathcal{D}$-modules over algebraic varieties, and is rather hard to read. The setting is nicer, but the structure loses much of its algebraic flavour.

They also construct OPE algebras over vertex algebras. A toy model of such OPE algebras is also introduced, which is more algebraic, and closely resembles the structure of a conformal algebra. Let $\mathfrak{d}$ be a finite-dimensional Lie algebra, $H = U(\mathfrak{d})$ its universal enveloping algebra. A Lie* algebra structure on an H-module L is an $H \otimes H$ linear map:

$$[\,,] : L \boxtimes L \to (H \otimes H) \otimes_H L$$

satisfying skew-symmetry and Jacobi identity. By $L \boxtimes L$, I mean the tensor product $L \otimes L$ seen as a (left) $H \otimes H$-module. The target space is the tensor product of $H \otimes H$ with L via the left H-module structure of L and the right diagonal H module structure of $H \otimes H$: $(h \otimes k) \otimes_H \alpha x = (h \otimes k)\Delta(\alpha) \otimes_H x$. $(H \otimes H) \otimes_H L$ is then naturally an $H \otimes H$-module, and we ask $[\,,]$ to be $H \otimes H$-linear. Skew-symmetry axiom is easy to explain whereas the Jacobi identity is slightly more involved. Skew-symmetry of the Lie bracket is replaced by

$$[a, b] = -\sigma[b, a],$$

where the right hand side is obtained by applying the flip $\sigma : H \otimes H \to H \otimes H$, $\sigma(h \otimes k) = k \otimes h$ on the first $\otimes_H$ factor.

Before explaining Jacobi identity let us see how conformal algebras become an instance of such a structure. Let $\mathfrak{d} = \mathbb{C}\partial$ be an abelian Lie algebra. Then $U(\mathfrak{d})$ is nothing else but $\mathbb{C}[\partial]$. We now rewrite OPE in a different fashion. For instance, the OPE of a Virasoro formal distribution has been written as

$$[L(z), L(w)] = \partial L(w)\delta(z - w) + 2L(w)\delta'_w(z - w). \tag{3.1}$$

Notice we take derivative of both coefficients and of the $\delta(z - w)$ term. Indeed, it is possible to reexpress OPE using only derivatives of the whole product. In fact, as

$$\frac{\partial}{\partial z}(a(w) \cdot \delta(z - w)) = a(w) \cdot \delta'_z(z - w) = -a(w) \cdot \delta'_w(z - w)$$

and

$$\frac{\partial}{\partial w}(a(w) \cdot \delta(z - w)) = a'(w) \cdot \delta(z - w) + a(w) \cdot \delta'_w(z - w),$$

we can write:

$$(\partial a)(w) \cdot \delta(z - w) = a'(w) \cdot \delta(z - w) = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right)(a(w) \cdot \delta(z - w)). \tag{3.2}$$

With this understanding, the OPE (3.1) becomes:

$$\begin{aligned}[L(z), L(w)] &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right)(L(w) \cdot \delta(z - w)) - 2\frac{\partial}{\partial z}(L(w) \cdot \delta(z - w)) \\ &= \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)(L(w) \cdot \delta(z - w)).\end{aligned}$$

If we agree on writing $\partial \otimes 1$ for $\partial/\partial z$ and $1 \otimes \partial$ for $\partial/\partial w$, then setting $[L * L] = (1 \otimes \partial - \partial \otimes 1) \otimes_H L$ gives the Lie*-algebra incarnation of the Virasoro conformal algebra. The general correspondence between a λ-bracket and a Lie*-bracket is the following. If

$$[x\,{}_\lambda y] = \sum_i P_i(\partial, \lambda) z_i, \tag{3.3}$$

then:

$$[x * y] = \sum_i P_i(\partial \otimes 1 + 1 \otimes \partial, -\partial \otimes 1) \otimes_H z_i. \tag{3.4}$$

Notice that $(h \otimes k) \otimes_H \alpha x = (h \otimes k)\Delta(\alpha) \otimes_H x$ in this context is nothing but (3.2). Our aim is now to reformulate all of the constructions made in the case of conformal algebras in the new Lie* setting, and obtain a classification of the latter structures as well.

4. Pseudo-linear algebra and pseudoalgebras

As with conformal algebras, Lie* algebras are just the Lie manifestation of a deeper phenomenon. We will set up the basic linear algebra underlying such algebraic structure, and will then define Lie, associative, commutative algebras in the new environment.

Definition 4.1. ([BD]) A *pseudotensor category* is a class of objects $\mathcal{M}$ together with vector spaces $\mathrm{Lin}(\{L_i\}_{i\in I}, M)$ whose elements are called *polylinear maps*. There is an action of the symmetric groups S_I permuting them. Moreover polylinear maps can be composed as follows.

For any surjection of finite sets $\pi\colon J \twoheadrightarrow I$ and a collection $\{N_j\}_{j\in J}$, we have the compositions rules of polylinear maps given by maps

$$\mathrm{Lin}(\{L_i\}_{i\in I}, M) \otimes \bigotimes_{i\in I} \mathrm{Lin}(\{N_j\}_{j\in J_i}, L_i) \to \mathrm{Lin}(\{N_j\}_{j\in J}, M), \tag{4.1}$$

$$\phi \times \{\psi_i\}_{i\in I} \mapsto \phi \circ (\otimes_{i\in I}\psi_i) \equiv \phi(\{\psi_i\}_{i\in I}), \tag{4.2}$$

where $J_i = \pi^{-1}(i)$ for $i \in I$.

The composition maps have the following properties:

- **Associativity:** If $K \twoheadrightarrow J$, $\{P_k\}_{k\in K}$ is a family of objects and $\chi_j \in \mathrm{Lin}(\{P_k\}_{k\in K_j}, N_j)$, then $\phi\big(\{\psi_i(\{\chi_j\}_{j\in J_i})\}_{i\in I}\big) = \big(\phi(\{\psi_i\}_{i\in I})\big)(\{\chi_j\}_{j\in J}) \in \mathrm{Lin}(\{P_k\}_{k\in K}, M)$.
- **Unit:** For any object M there is an element $\mathrm{id}_M \in \mathrm{Lin}(\{M\}, M)$ such that for any $\phi \in \mathrm{Lin}(\{L_i\}_{i\in I}, M)$ one has $\mathrm{id}_M(\phi) = \phi(\{\mathrm{id}_{L_i}\}_{i\in I}) = \phi$.
- **Equivariance:** The compositions (4.1) are equivariant with respect to the natural action of the symmetric group.

A standard example of a pseudotensor category, is the category of vector spaces endowed with multilinear maps. The example which interest us more is given by the construction of Beilinson and Drinfeld [BD].

Let H be a cocommutative bialgebra with a comultiplication Δ. We introduce a pseudotensor category $\mathcal{M}^*(H)$ whose objects are left H-modules but with another pseudotensor structure:

$$\mathrm{Lin}(\{L_i\}_{i\in I}, M) = \mathrm{Hom}_{H^{\otimes I}}(\boxtimes_{i\in I} L_i, H^{\otimes I} \otimes_H M). \tag{4.3}$$

Here $\boxtimes_{i\in I} L_i$ is the tensor product of the H-modules L_i viewed as an $H^{\otimes n}$-module. For a surjection $\pi\colon J \twoheadrightarrow I$, the composition of polylinear maps is defined as follows:

$$\phi\big(\{\psi_i\}_{i\in I}\big) = \Delta^{(\pi)}\big(\phi\big) \circ \big(\boxtimes_{i\in I}\psi_i\big). \tag{4.4}$$

Here $\Delta^{(\pi)}$ is the functor associating to the left $H^{\otimes I}$-module M the $H^{\otimes J}$-module $H^{\otimes J} \otimes_{H^{\otimes I}} M$, where $H^{\otimes I}$ acts on $H^{\otimes J}$ via the iterated comultiplication determined by π.

Explicitly, let $n_j \in N_j$ $(j \in J)$, and write

$$\psi_i\big(\otimes_{j\in J_i} n_j\big) = \sum_r g_i^r \otimes_H l_i^r, \qquad g_i^r \in H^{\otimes J_i},\ l_i^r \in L_i, \tag{4.5}$$

where, as before, $J_i = \pi^{-1}(i)$ for $i \in I$. Let

$$\phi\big(\otimes_{i\in I} l_i^r\big) = \sum_s f^{rs} \otimes_H m^{rs}, \qquad f^{rs} \in H^{\otimes I},\ m^{rs} \in M. \tag{4.6}$$

Then, by definition,

$$\big(\phi(\{\psi_i\}_{i\in I})\big)\big(\otimes_{j\in J} n_j\big) = \sum_{r,s} (\otimes_{i\in I} g_i^r)\Delta^{(\pi)}(f^{rs}) \otimes_H m^{rs}, \tag{4.7}$$

where $\Delta^{(\pi)}\colon H^{\otimes I} \to H^{\otimes J}$ is the iterated comultiplication determined by π. For example, if $\pi\colon \{1,2,3\} \to \{1,2\}$ is given by $\pi(1) = \pi(2) = 1$, $\pi(3) = 2$, then $\Delta^{(\pi)} = \Delta \otimes \mathrm{id}$; if $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, then $\Delta^{(\pi)} = \mathrm{id} \otimes \Delta$.

The symmetric group S_I acts among the spaces $\mathrm{Lin}(\{L_i\}_{i\in I}, M)$ by simultaneously permuting the factors in $\boxtimes_{i\in I} L_i$ and $H^{\otimes I}$. This is the only place where we need the cocommutativity of H; for example, the permutation $\sigma_{12} = (12) \in S_2$ acts on $(H \otimes H) \otimes_H M$ by

$$\sigma_{12}\big((f \otimes g) \otimes_H m\big) = (g \otimes f) \otimes_H m,$$

and this is well defined only when H is cocommutative.

Definition 4.2. An *algebra* in a pseudotensor category $\mathcal{M}$ is an object A in $\mathcal{M}$ endowed with a bilinear map $\mu \in \mathrm{Lin}(\{A, A\}, A)$. We will say that μ is an *associative product* if

$$\mu(\mu(\cdot,\cdot),\cdot) = \mu(\cdot,\mu(\cdot,\cdot)).$$

The product μ is *commutative* if

$$\mu = \sigma_{12}\mu.$$

Similarly, a *Lie algebra* in $\mathcal{M}$ is an object L endowed with a bilinear map $\beta \in \mathrm{Lin}(\{L, L\}, L)$ satisfying *skew-symmetry*

$$\beta = -\sigma_{12}\beta$$

and the *Jacobi identity*

$$\beta(\beta(\cdot,\cdot),\cdot) = \beta(\cdot,\beta(\cdot,\cdot)) + \sigma_{12}\beta(\cdot,\beta(\cdot,\cdot)).$$

I will call a (resp. Lie, associative, commutative) algebra in $\mathcal{M}^*(H)$ a (resp. Lie, associative, commutative) H-pseudoalgebra.

Remark 4.3. We already have many examples of pseudoalgebras, as we obtain the standard notion of algebras over $\mathbb{C}$ when $H = \mathbb{C}$ and their conformal algebraic analogues for $H = \mathbb{C}[\partial]$.

As we mentioned before, (Lie, associative, commutative) conformal algebras actually represent functors from the category of differential algebras to that of (resp. Lie, associative, commutative) differential algebras. This is true for H-pseudoalgebras as well. Let Y be an H-differential commutative associative algebra. This means that Y is an associative algebra together left and right actions of H on it. These actions satisfy $h.(fg) = (h_{(1)}f)(h_{(2)}g)$, and similarly for the right action, where we use Sweedler's notation for the coproduct $\Delta(h) = h_{(1)} \otimes h_{(2)}$ on H.

Whenever L is a pseudoalgebra, then we can set up on $\mathcal{A}_Y A = Y \otimes_H A$ an H-differential algebra by

$$(x \otimes_H a)(y \otimes_H b) = \sum_i (xf_i)(yg_i) \otimes_H e_i$$

if $\mu(a, b) = \sum_i (f_i \otimes g_i) \otimes_H e_i$. This product turns out to be associative, commutative or Lie when μ is of the corresponding type in the pseudotensor category sense. In particular $X = H^*$ is naturally a commutative associative H-differential algebra, and we call $\mathcal{A}_X A$ the *annihilation algebra* of A, in analogy with the conformal algebra case. Indeed, when $H = \mathbb{C}[\partial]$, we obtain $X = H^* = \mathbb{C}[[t]]$. Then the isomorphisms between $\mathcal{A}_X A$ and the completion of the annihilation Lie algebra $\mathcal{A}_+ A$ constructed in the previous section is given by $t^i \otimes_H a \mapsto a_i$ and the requirement $(\partial a)_n = -na_{n-1}$ follows from the definition of the right H-action on X.

The natural formal topology on X induces a corresponding topology on $\mathcal{A}_X A$ whenever A is a finitely generated H-module (see [BDK]). $\mathcal{A}_X A$ is a linearly compact vector space with respect to this topology. In case A is a Lie H-pseudoalgebra, we can again use Cartan's result on simple linearly compact Lie algebras. I introduce now the most remarkable instance of simple Lie pseudoalgebra.

Example 4.4 ($W(\mathfrak{d})$)**.** Let $H = U(\mathfrak{d})$, $L = H \otimes \mathfrak{d}$. The H-module L becomes a Lie H-pseudoalgebra – denoted $W(\mathfrak{d})$ – when endowed with the pseudobracket

$$[1 \otimes a * 1 \otimes b] = (1 \otimes 1) \otimes_H (1 \otimes [a, b]) - (1 \otimes a) \otimes_H (1 \otimes b) + (b \otimes 1) \otimes_H (1 \otimes a).$$

Example 4.5 (Current pseudoalgebras)**.** Let $\mathfrak{d}' \subset \mathfrak{d}$ be Lie algebras, $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$. We can induce an H'-pseudoalgebra structure from an H'-module L' to the H-module $L = H \otimes_{H'} L$ via

$$[1 \otimes_{H'} a * 1 \otimes_{H'} b] = \sum_i (f_i \otimes g_i) \otimes_H (1 \otimes_{H'} c_i)$$

if $[a * b] = \sum_i (f_i \otimes g_i) \otimes_H c_i$ in L'. L is called the *current pseudoalgebra* induced from the pseudoalgebra L'. L is then simple whenever L' is simple.

Notice that this is abelian when $H = \mathbb{C}$, and gives the Virasoro conformal algebra when $H = \mathbb{C}[\partial]$. The Lie pseudoalgebra $W(\mathfrak{d})$ has several interesting properties [BDK]. First of all, it is simple, and all H-pseudoalgebras, that are not current pseudoalgebras over a finite dimensional Lie algebra, uniquely occur as subalgebras of $W(\mathfrak{d})$. I call them *pseudoalgebras of vector fields*. They are *primitive simple Lie pseudoalgebras* when they cannot be obtained as current pseudoalgebras.

The correspondence between a pseudoalgebra and its annihilation algebra provides a classification of primitive simple Lie pseudoalgebras corresponding to the four series of Cartan type simple Lie algebras. In particular, $W(\mathfrak{d})$ is the unique pseudoalgebra corresponding to the Lie algebra $W_N, N = \dim \mathfrak{d}$. As far as other types are concerned, there are usually several pseudoalgebras attached to the same Lie algebra. In types H and K, all corresponding primitive pseudoalgebras are indeed free H-modules of rank one. They are characterized by the following statement.

Theorem 4.6. *Let $H = U(\mathfrak{d})$ and let $L = Hx$ be a Lie pseudoalgebra which is free of+ rank one as an H-module. If $[x, x] = \alpha \otimes_H x, \alpha \in H \otimes H$ we have $\alpha = r + s \otimes 1 - 1 \otimes s$ for some $r \in \mathfrak{d} \wedge \mathfrak{d}, s \in \mathfrak{d}$.*

L is a primitive simple Lie H-pseudoalgebra, then either $\dim \mathfrak{d}$ is even, and r is non-degenerate, in which case L is of type H, or $\dim \mathfrak{d} = 2N + 1$ is odd, and r is of rank $2N$, and its support generates $\mathfrak{d}$ modulo s, in which case L is of type K.

Remark 4.7. Not all values of r and s give a Lie pseudoalgebra structure on L. Indeed, they must satisfy the identities:

$$[r, \Delta(s)] = 0,$$

$$[r_{12}, r_{13}] + r_{12}s_3 + \text{ cyclic permutations } = 0.$$

Notice that the latter equation generalizes the classical Yang Baxter Equation and reduce to it when $s = 0$.

Theorem 4.8. *A complete list of finite primitive simple Lie H-pseudoalgebras is as follows:*

1) $W(\mathfrak{d})$

2) The subalgebras $S(\mathfrak{d}, \chi) = \{\sum_i h_i \otimes d_i | \sum h_i(d_i + \chi(d_i)) = 0\}$ of $W(\mathfrak{d})$, where χ is a trace form[3] on $\mathfrak{d}$.

3) The primitive simple Lie pseudoalgebras of type H and K listed in Theorem 4.6.

[3] i.e. a Lie algebra homomorphism $\chi : \mathfrak{d} \to \mathbb{C}$.

5. PSEUDO-LINEAR ALGEBRA AND REPRESENTATION THEORY

When we have a bilinear map between vector spaces, $\phi : U \otimes V \to W$, the specializations $\phi_u(v) = \phi(u, v)$ turn out to be linear maps. In a pseudotensor category, this is no more the case. However, in a pseudoalgebra left multiplications by elements are maps of such a type, so such specializations are worth considering. I call them *pseudolinear maps*:

Definition 5.1. Let M, N be H-modules. Then $\phi : M \to (H \otimes H) \otimes_H N$ is a pseudolinear map if $\phi(hm) = (1 \otimes h)\phi(m)$.

The collection of all pseudolinear maps from M to N can be given an H-module structure via $(h\phi)(m) = (h \otimes 1)\phi(m)$. The set of all pseudolinear map from M to itself is denoted by $\mathrm{Cend}\, M$, and is never finitely generated over H unless $H = \mathbb{C}$ or M is torsion (in which case there is no nonzero pseudolinear map).

Composition as defined in the pseudotensor category $\mathcal{M}$ endows $\mathrm{Cend}\, M$ (M a finitely generated H-module) with an associative pseudoalgebra structure – with respect to ordinary composition – and a Lie pseudoalgebra structure denoted $\mathrm{gc}\, M$ – with respect to a naturally defined commutator. When M is infinitely generated $\mathrm{Cend}\, M$ and $\mathrm{gc}\, M$ are ill behaved.[4] It is natural to define the concept of representation of an associative (resp. Lie) pseudoalgebra A on a module V as a pseudoalgebra homomorphism $A \to \mathrm{Cend}(V)$ (resp. $A \to \mathrm{gc}\, V$). Remarkably enough, representation theory of solvable and nilpotent Lie pseudoalgebras is identical to the Lie theoretic case.

Theorem 5.2 (Lie Theorem)**.** *Let S be a solvable Lie pseudoalgebra acting on a finitely generated H-module V. Then there exists a common* eigenvector *for the action of S on V.*

In other words the action of S on V can be put in an upper triangular form. This also implies that if there are no zero weights, the module V must be free.

Theorem 5.3 ([BDK])**.** *If N is a nilpotent Lie pseudoalgebra acting on a finitely generated H-module V then V can be decomposed in a direct sum of generalized weight submodules for the action of N.*

Notice that the above statements are trivially true for ordinary linear endomorphisms on a finite-dimensional vector space. However this is no longer the case in the pseudoalgebraic setting, and a pseudolinear map $\phi \in \mathrm{Cend}\, V$ can be put in upper triangular case only if it generates a solvable subalgebra $\langle\phi\rangle \subset \mathrm{gc}\, V$. Moreover a decomposition à la Jordan of V into direct sum of generalized eigenspaces only holds for pseudolinear maps generating a nilpotent subalgebra.

Irreducible representations for finite simple conformal algebras have been determined [CK]. Complete reducibility fails, and conformal algebras, and pseudoalgebras in general, have a rich cohomology theory [BKV]. The study of irreducible representations of finite primitive Lie pseudoalgebra is in progress.

6. APPLICATIONS OF CONFORMAL ALGEBRAS AND PSEUDOALGEBRAS.

We introduced conformal algebras because they provided a tool for studying the surface of a vertex algebra structure. In particular the analysis of a vertex algebra can be initially restricted to that of the underlying conformal algebra, which will then give big constraints on what kind of normally ordered product can extend the particular conformal algebra structure. This strategy can be used in particular on vertex algebras defined on finitely generated $\mathbb{C}[\partial]$-modules, as pretty much everything is known of such conformal algebras. In [D] the following statement is proved.

Theorem 6.1. *Let V be a finitely generated $\mathbb{C}[\partial]$-module, endowed with a vertex algebra structure. Then*[5] *if the minimal conformal subalgebra $\langle x\rangle$ of V containing some given element x is*

[4] Indeed, the natural composition and commutator do not make them into pseudoalgebras.

[5] Under the technical condition that V contain no strongly nilpotent element, see below.

solvable, it follows that the adjoint action of x (on the conformal algebra V) is nilpotent. In particular $\langle x\rangle$ is nilpotent.

This result hints at the fact that conformal algebras underlying vertex algebras behave more regularly – at least under some finiteness assumption. Representation theory of affine Kac-Moody algebras and of the Virasoro algebra can be used to prove that no finite vertex algebra can have a simple conformal algebra as a subquotient. Therefore the underlying conformal algebra needs to be solvable. Then the above theorem can be used to give a nilpotence characterization of the conformal algebra underlying a finite vertex algebra.

An element a in a vertex algebra V is called *strongly nilpotent* if there is some integer n such that the product of vertex operators $Y(a_i, z_i)$ is zero as soon as at least n of the a_i equal a. Strongly nilpotent elements form an ideal N of the vertex algebra V, which is obviously nilpotent. The ideal N is the *nilpotent radical* of V, and V is reduced if its nilpotent radical equals zero. Notice that V/N is always reduced. Then one has:

Theorem 6.2. *The conformal algebra underlying V/N is always nilpotent. In other words, V is always the extension of a nilpotent conformal algebra by N, which is nilpotent as a vertex algebra (hence also as a conformal algebra). In particular, a reduced finite vertex algebra always has nilpotent OPE structure.*

The classification of finite Lie pseudoalgebras has a surprising application. When we have $H = \mathbb{C}[x_1, ..., x_n]$, the structure of a Lie pseudoalgebra is equivalent to that of a linear *Poisson bracket of hydrodynamic type*. Such a structure had been studied rather extensively (see [DN1], [DN2] and the other references in [BDK]). The classification of simple and semisimple Lie pseudoalgebras gives as a byproduct a classification of simple and semisimple linear Poisson brackets of hydrodynamic type. This result is described in [BDK] along with a description of central extensions of such structures, which turn out to be physically relevant.

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