FORMAL DE RHAM THEORY: IRREDUCIBLE REPRESENTATIONS OF FINITE SIMPLE LIE PSEUDOALGEBRAS

ALESSANDRO D'ANDREA

ABSTRACT. In this communication, I recall the main results [BDK1] in the classification of finite Lie pseudoalgebras, which generalize several previously known algebraic structures, and announce some new results [BDK2] concerning their representation theory.

In questa comunicazione, elenco i principali risultati [BDK1] di classificazione delle pseudoalgebre di Lie finite, che generalizzano diverse strutture algebriche precedentemente note, e annuncio alcuni nuovi risultati [BDK2] che riguardano le loro rappresentazioni.

CONTENTS

1. INTRODUCTION

The aim of the present communication is that of describing a certain algebraic structure, called Lie pseudoalgebra, and its representation theory.

The concept of Lie pseudoalgebra is a natural generalization of many algebraic structures, such as conformal algebras [K, DK], Lie* algebras [BeDr] and Poisson algebras of hydrodynamic type [DuN1, DuN2]. Lie pseudoalgebras structures on finitely generated modules over a cocommutative Hopf algebra were studied in [BDK1]. Here I give an account of the results therein, and of some more recent ones [BDK2] in the representation theory of simple Lie pseudoalgebras obtained jointly with B. Bakalov and V. Kac.

The plan of the exposition is as follows: I will briefly recall a classical theorem [Ca, Gu1, Gu2] by E. Cartan on the classification of infinite dimensional Lie algebras of vector fields. I will then give the definition of Lie pseudoalgebra, along with a few examples, and explain the relation between Lie pseudoalgebras and Lie algebras of Cartan type. I will then move to describing the classification of irreducible representations of both Lie pseudoalgebras and Lie algebras of

The author was supported in part by a prize fellowship by Clay Mathematics Institute.

Cartan type [Rud1, Rud2, Ko], via the study of *singular vectors* in some peculiar representations called *tensor modules*.

In the final part of the communication, I will explain how representation theory of simple Lie pseudoalgebras can be understood in terms of some naturally constructed complexes of modules, that are generalizations of the de Rham complex in the special case of symplectic and contact manifolds.

2. Cartan's Classification of Linearly Compact Simple Lie Algebras

A classical theorem of E. Cartan, whose proof was later completed by V. Guillemin, classifies infinite dimensional linearly compact simple Lie algebras over an algebraically closed field of zero characteristic. The condition of linear compactness is a topological translation of natural properties of Lie algebras of vectors fields.

Let M be a manifold, and $\mathcal{L}$ denote the algebra of (regular) vector fields on M. Then $\mathcal{L}$ has the following properties:

$\mathcal{L}$ is an infinite-dimensional Lie algebra
$\mathcal{L}$ possesses a filtration by order of zero at a point $P \in M$.
The Lie bracket on $\mathcal{L}$ is continuous with respect to the topology induced by this filtration.

In order to translate such properties in an algebraically suitable language, we may take the completion of $\mathcal{L}$ with respect to the filtration. This is more or less equivalent to the process of taking vector fields on a formal neighbourhood of P. A *linearly compact* vector space is nothing but a vector space which is complete with respect to a filtration by subspaces of finite codimension.

The above-mentioned theorem claims that the characterization just given captures all algebraic properties of Lie algebras of vector fields. Its exact statement is the following:

Theorem 2.1 (Cartan, Guillemin)**.** *Every linearly compact infinite dimensional simple Lie algebra is isomorphic to one of the following:*

— *The Lie algebra W_n of all vector fields*

$$A = \sum_{i=1}^{n} a_i(x_1, ..., x_n) \cdot \frac{\partial}{\partial x_i};$$

— *The Lie subalgebra S_n of all elements in W_n preserving a volume form;*
— *The Lie subalgebra H_{2n} of all elements in W_{2n} preserving a symplectic form;*
— *The Lie subalgebra K_{2n+1} of all elements in W_{2n+1} preserving a contact form (up to homotheties).*

The Lie algebras W_n, S_n, H_{2n} and K_{2n+1} are called *Lie algebras of Cartan type*.

Let $\mathcal{L}$ be the Lie algebra W_n and $\mathcal{L}_k$ denote the subalgebra of vector fields that have a zero of order at least $k+1$. Then the filtration

$$\mathcal{L} = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset ...$$

is such that $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$, and

$$\mathcal{L}_0/\mathcal{L}_1 \simeq \mathfrak{gl}_n.$$

The ideal $\mathcal{L}_1/\mathcal{L}_k$ lies in the radical of the quotient Lie algebra $\mathcal{L}_0/\mathcal{L}_k$, hence acts trivially on all irreducible $\mathcal{L}_0/\mathcal{L}_k$-modules.

The other simple Lie algebras of Cartan type have a similar structure. The quotient Lie algebra $\mathcal{L}_0/\mathcal{L}_1$ is isomorphic to $\mathfrak{sl}_n$ (resp. $\mathfrak{sp}_{2n}$, $\mathfrak{csp}_{2n}$) when $\mathcal{L} = S_n$ (resp. H_{2n}, K_{2n+1}).

3. Lie Pseudoalgebras

Definition 3.1. An H-Lie pseudoalgebra is a module L over a cocommutative Hopf algebra H endowed with an $H\otimes H$-linear Lie bracket

$$[\,,]: L\otimes L \to (H\otimes H)\otimes_H L$$

which is *skew-symmetric* and satisfies the *Jacobi identity*.

In the above definition, $H\otimes H$ is understood as a right H-module via the right multiplication by $\Delta(h) = h_{(1)}\otimes h_{(2)}$, skew-symmetry means

$$[a,b] = -\sigma[b,a]$$

where $\sigma : H\otimes H \to H\otimes H$ is the flip $\sigma(h\otimes k) = k\otimes h$. The Jacobi identity is more complicated to describe, as we do not have an understanding of double Lie brackets $[a,[b,c]], [[a,b],c] \in (H\otimes H\otimes H)\otimes_H L$ yet. If

$$[a,b] = \sum_i (f_i\otimes g_i)\otimes_H e_i,$$

and

$$[e_i,c] = \sum_j (f_{ij}\otimes g_{ij})\otimes_H e_{ij},$$

then we set

$$[[a,b],c] = \sum_{ij} (f_i f_{ij\,(1)}\otimes g_i f_{ij\,(2)}\otimes g_{ij})\otimes_H e_{ij}. \tag{3.1}$$

Similarly, if

$$[c,e_i] = \sum_j (h_{ij}\otimes k_{ij})\otimes d_{ij},$$

then we set

$$[c,[a,b]] = \sum_{ij} (h_{ij}\otimes f_i k_{ij\,(1)}\otimes g_i k_{ij\,(2)})\otimes_H d_{ij}. \tag{3.2}$$

Jacobi identity reads then as

$$[[x,y],z] = [x,[y,z]] - \sigma_{12}[y,[x,z]],$$

where σ_{12} flips the first two tensor factors in $H\otimes H\otimes H$. The name of *pseudoalgebra* is justified by the fact that this is a Lie algebra structure in a suitable pseudotensor category [BeDr, So].

If H is a Hopf algebra, elements $p\in H$ such that $\Delta(p) = p\otimes 1 + 1\otimes p$ are called *primitive* or *Lie-like*, and span a Lie algebra $\mathfrak{p}(H)$. Elements $g\in H$ such that $\Delta(g) = g\otimes g$ are called *group-like*, and constitute a group $G(H)$. The following fact allows one to restrict the study of H-Lie pseudoalgebra to the case when H is a universal enveloping algebra.

Theorem 3.2 (Kostant)**.** *A cocommutative Hopf algebra over an algebraically closed field* $\mathbf{k}$ *of zero characteristic is isomorphic to the smash product of the universal enveloping algebra* $U(\mathfrak{p}(H))$ *with the group algebra* $\mathbf{k}[G(H)]$.

Indeed, a H-Lie pseudoalgebra is nothing but a $U(\mathfrak{p}(H))$-Lie pseudoalgebra along with a $G(H)$-action.

3.1. **Examples.**

3.1.1. $H=\mathbb{C}$. In this case $H\otimes H\simeq H$ and $\Delta : H\to H\otimes H\simeq H$ is the identity mapping. Then the axioms for a Lie pseudoalgebra reduce to those of an ordinary Lie algebra.

3.1.2. $H = \mathbb{C}[\partial]$. In this case the axioms for an H-Lie pseudoalgebras are equivalent to those for a conformal algebra as in [DK]. Indeed, if

$$[a, b] = \sum_i P_i(\partial \otimes 1, 1 \otimes \partial) \otimes_h e_i,$$

then

$$[a_\lambda b] = \sum_i P_i(-\lambda, \partial + \lambda) e_i$$

satisfies the axioms for a λ-bracket if and only if $[\,,]$ is a Lie pseudoalgebra bracket.

3.1.3. $H = \mathbb{C}[\partial_1, ..., \partial_n]$. The notion of H-Lie pseudoalgebra is equivalent to that of a (linear) Poisson algebra of hydrodynamic type as in [DuN1, DuN2]. The equivalence is shown as above with conformal algebras, see [BDK1].

3.1.4. $H = U(\mathfrak{d})$. In this case H-Lie pseudoalgebras were introduced in [BeDr], where they were called Lie*-algebras.

4. Pseudoalgebras and their Annihilation Algebras.

Let us choose $H = U(\mathfrak{d})$ as the base Hopf algebra, where $\mathfrak{d}$ is some finite dimensional Lie algebra. If $\dim \mathfrak{d} = N$ then the topological algebra H^* is isomorphic to formal power series $\mathbb{C}[[t_1, ..., t_N]]$, endowed with the formal topology.

The natural action of H on H^* is differential, i.e. elements of $\mathfrak{d} \subset H$ act as derivations on H^*, hence as (formal) vector fields on the space H^* of formal functions.

The Lie bracket on $\mathfrak{d}$ is a translation of the Lie bracket between the corresponding vector fields on a manifold. There are minor sign convention problems, so if $\mathfrak{d}$ is abelian, and $\phi = \phi(t_1, ..., t_N) \in H^*$, then the right and left H-actions on H^* satisfy:

$$\partial_i(\phi) = (\phi)\partial_i = -\partial\phi/\partial t_i.$$

In this setting, it is useful to think of a Lie pseudoalgebra as a shorthand notation for some special kind of infinite dimensional Lie algebras.

Example 4.1. Let ϕ and ψ be formal series in the N variables $t_1, ..., t_N$. Then the Lie bracket of vector fields satisfies:

$$[\phi \cdot \partial_i, \psi \cdot \partial_j] = -\phi(\psi\partial_i) \cdot \partial_j + (\phi\partial_j)\psi \cdot \partial_j + (\phi\psi) \cdot [\partial_i, \partial_j].$$

We can associate with this Lie bracket, the following Lie pseudoalgebra bracket:

$$[\partial_i, \partial_j]_{\text{pseudo}} = -(1 \otimes \partial_i) \otimes_H \partial_j + (\partial_j \otimes 1) \otimes_H \partial_i + (1 \otimes 1) \otimes_H [\partial_i, \partial_j]$$

on the H-module $H \otimes \mathfrak{d}$. The H-Lie pseudoalgebra thus defined is called $W(\mathfrak{d})$.

Example 4.2. If $\mathfrak{g}$ is a Lie algebra, then the Lie bracket on its loop algebra

$$[\phi \otimes g, \psi \otimes h] = (\phi\psi) \otimes [g, h]$$

corresponds to the Lie pseudoalgebra bracket

$$[g, h]_{\text{pseudo}} = (1 \otimes 1) \otimes_H [g, h]$$

on the H-module $H \otimes \mathfrak{g}$. The H-Lie pseudoalgebra thus defined is called $\mathrm{Cur}\, \mathfrak{g}$.

According to this shorthand convention, the pseudoalgebra axioms for L are those needed to make sure that $H^* \otimes_H R$ is a Lie algebra.

4.1. **Annihilation algebra of a Lie pseudoalgebra.** The shorthand notation hints to a more general fact: any Lie pseudoalgebra L establishes a functor from the category of (commutative) H-differential algebras to that of H-differential Lie algebras

$$X \mapsto X \otimes_H L.$$

The investigation of pseudoalgebras exploits this correspondence between Lie pseudoalgebras and Lie algebras: if $X = H^*$, then the natural filtration of H^* induces a filtration on $H^* \otimes_H L$ that makes it linearly compact.

Algebraic properties of L are connected with those of $H^* \otimes_H L$. The study of simple Lie pseudoalgebras can then be done by means of Cartan's classification theorem.

The linearly compact Lie algebra $H^* \otimes_H L$ is called *Lie algebra of annihilation operators*, or simply *annihilation algebra* associated to L, as it is related to the algebra of (quantum) annihilation operators on the Fock space of quantum states.

Example 4.3. Let $H = U(\mathfrak{d})$. We have seen that the H-module $H \otimes \mathfrak{d}$, endowed with the Lie bracket

$$[1 \otimes a, 1 \otimes b] = (1 \otimes 1) \otimes_H (1 \otimes [a, b]) - (1 \otimes a) \otimes_H (1 \otimes b) + (b \otimes 1) \otimes_H (1 \otimes a),$$

is a Lie pseudoalgebra on H, denoted by $W(\mathfrak{d})$. Its annihilation algebra $H^* \otimes_H W(\mathfrak{d}) \simeq H^* \otimes \mathfrak{d}$ is then isomorphic to the Lie algebra W_N of vector fields, where $N = \dim \mathfrak{d}$, thus recovering the "short-hand" motivation for $W(\mathfrak{d})$.

The pseudoalgebra $W(\mathfrak{d})$ is the new phenomenon making the theory of H-Lie pseudoalgebra dramatically different from the classical theory of Lie algebras. It is a much less commutative object than ordinary finite dimensional Lie algebra.

5. Structure of Lie Pseudoalgebras

5.1. **Primitive pseudoalgebras.** Simple pseudoalgebras can be constructed from special instances called *primitive simple pseudoalgebras*.

Example 5.1. $S(\mathfrak{d}, \chi)$ is the H-submodule of all elements $\sum_i h_i \otimes a_i$ from $W(\mathfrak{d})$ satisfying

$$\operatorname{div}^\chi(\sum_i h_i \otimes a_i) = \sum_i h_i(a_i + \chi(a_i)) = 0,$$

where $\chi : \mathfrak{d} \to \mathbb{C}$ is a trace form, i.e. a Lie algebra homomorphism.

Then $S(\mathfrak{d}, \chi)$ is a subalgebra of $W(\mathfrak{d})$ with respect to its Lie bracket, which makes it into a simple Lie pseudoalgebra. Its annihilation algebra is isomorphic to S_N.

Example 5.2. Lie pseudoalgebras on a free module He of rank one are determined by the only structure constant $\alpha \in H \otimes H$ such that

$$[e, e] = \alpha \otimes_H e.$$

It can be easily showed that $\alpha \in H \otimes H$ is necessarily of the form $e = r + s \otimes 1 - 1 \otimes s$ where $r \in \mathfrak{d} \otimes \mathfrak{d}$ is skew-symmetric. r and s must also satisfy some technical commutation relations.

When r is non degenerate, then $\dim \mathfrak{d} = 2n$ is an even number, and He is denoted by $H(\mathfrak{d}, \chi, \omega)$. Its annihilation algebra is isomorphic to a central extension of the Lie algebra H_{2n}, which can be also viewed as the Poisson algebra P_{2n} determined by the bi-vector field r on the space H^* of formal power series (i.e. regular functions).

When r has a one-dimensional kernel on which s projects non trivially, the $\dim \mathfrak{d} = 2n + 1$ is an odd number, and He is denoted by $K(\mathfrak{d}, \theta)$. Its annihilation algebra is isomorphic to K_{2n+1}. Parameters χ, ω and θ are obtained as functions of r and s.

All pseudoalgebra structures on free H-modules of rank one can be realized as subalgebras of $W(\mathfrak{d})$ via the embedding $e \mapsto -r + 1 \otimes s$.

5.2. **Current pseudoalgebras.** If L is a H-Lie pseudoalgebra, and $\bar{H}$ is a Hopf algebra containing H as a subalgebra, then we may construct a new Lie pseudoalgebra by taking $\bar{L} = \bar{H} \otimes_H L$ with the Lie bracket induced by that of L. This is called the current pseudoalgebra of L, and is denoted by $\mathrm{Cur}_H^{\bar{H}} L$.

Example 5.3. Let $\mathfrak{g}$ be a Lie algebra, i.e. a $\mathbb{C}$-Lie pseudoalgebra. Then for any Hopf algebra H we can construct the current Lie pseudoalgebra $\mathrm{Cur}_{\mathbb{C}}^{H} \mathfrak{g} = H \otimes \mathfrak{g}$. Its bracket is as given in Example 4.2.

Let $\bar{L}$ be a finite[1] simple $\bar{H}$-Lie pseudoalgebra, and let H be the smallest Hopf subalgebra of $\bar{H}$ containing all structure constant of $\bar{L}$. Then $\bar{L}$ is a current pseudoalgebra of a finite simple H-Lie pseudoalgebra L. Such simple pseudoalgebras of minimal H are either finite dimensional simple Lie algebras or one of the primitive H-Lie pseudoalgebras $W(\mathfrak{d})$, $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$, $K(\mathfrak{d}, \theta)$.

5.3. **Properties of the Lie pseudoalgebra $W(\mathfrak{d})$.** The theory of pseudoalgebras is in a sense parallel to that of ordinary Lie algebras. Main differences are exemplified by $W(\mathfrak{d})$. These are its main properties:

- There are no non-zero commuting elements in $W(\mathfrak{d})$, i.e. if $[a, b] = 0$ then $a = 0$ or $b = 0$. This is in contrast with ordinary Lie algebras, where every element commutes with itself.
- The only subalgebras of $W(\mathfrak{d})$ are simple.
- Every primitive simple pseudoalgebra embeds in $W(\mathfrak{d})$.
- This embedding is unique. The only automorphism of $W(\mathfrak{d})$ and of its subalgebra is identity.

6. Representation Theory of Primitive Simple Lie Pseudoalgebras

A representation of an H-Lie pseudoalgebra L is an H-module M endowed with an L-action, i.e. with a map

$$L \otimes M \ni a \otimes m \mapsto a \cdot m \in (H \otimes H) \otimes_H M$$

satisfying

$$[a, b] \cdot m = a \cdot (b \cdot m) - \sigma_{12} b \cdot (a \cdot m),$$

where composition of actions $a \cdot (b \cdot m)$ is understood as in (3.1) and (3.2).

Representations of a finite H-Lie pseudoalgebra L are in bijection with (topologically) discrete representations of the Lie algebra $\tilde{\mathcal{L}} = \mathfrak{d} \ltimes (H^* \otimes L)$. In other words, a pseudoalgebra representation of L is the same as a representation of the corresponding annihilation algebra, along with a $\mathfrak{d}$-module, i.e. an $H = U(\mathfrak{d})$-module, structure, satisfying suitable compatibility conditions.

There are explicit formulas to obtain the action of L on the module M out of the action of the annihilation algebra of L, and vice versa. For instance, if $\{h_i\}$ and $\{x_i\}$ are dual bases of H and H^*, then one can recover $r \cdot m, r \in L, m \in M$ as follows:

$$r \cdot m = \sum_i (h_i \otimes 1) \otimes_H (x_i \otimes_H r).m.$$

Representations of finite solvable Lie pseudoalgebras follow closely the behaviour of representations of finite dimensional solvable Lie algebras: one has analogues of Lie's and Engel's theorem, and every finite representation has a basis making the action upper triangular. The representation theory of finite simple Lie pseudoalgebras is more involved.

[1] A pseudoalgebra is finite if the underlying H-module is finitely generated.

6.1. **Representations of $W(\mathfrak{d})$.** The study of irreducible representations of primitive Lie pseudoalgebras is done by studying the action of the corresponding annihilation algebras. Discrete representations of Lie algebras of Cartan type were studied by A. Rudakov and I. Kostrikhin [Rud1, Rud2, Ko]. However, the language of pseudoalgebras gives a different, and easier, approach to the classification of irreducible modules.

Here I list a few facts on (irreducible) representations of the pseudoalgebra $W(\mathfrak{d})$. Similar statements are true for the other primitive pseudoalgebras. We will focus on distinguished elements, called *singular vectors*, whose stabilizer is maximal.

— Every $W(\mathfrak{d})$-module contains singular vectors.
— An irreducible $W(\mathfrak{d})$-module is H-linearly generated by its singular vectors.
— The $\tilde{\mathcal{L}} = H^* \otimes_H W(\mathfrak{d})$-action on singular vectors is particularly easy to describe.
— The space of singular vectors of a module containing no trivial submodule is a finite dimensional vector space.

Let M be an irreducible $W(\mathfrak{d})$-module, and $S \subset M$ be the finite dimensional vector space of singular vectors. If M is viewed as an $\tilde{\mathcal{L}}$-module, then S is invariant under the action of $\mathcal{L}_1 \subset \mathcal{L} \subset \tilde{\mathcal{L}}$, and is stable under the action of the normalizer $\mathcal{N}$ of $\mathcal{L}_0$ in $\tilde{\mathcal{L}}$. The action of $\mathcal{N}$ is easily described in terms of commuting actions of $\mathfrak{d}$ and $\mathcal{L}_0/\mathcal{L}_1 \simeq \mathfrak{gl}_N$. The representation of $W(\mathfrak{d})$ obtained induced such $\mathfrak{d} \oplus \mathfrak{gl}_N$-action to all of $\tilde{\mathcal{L}}$ is isomorphic — as an H-module — to the free module $H \otimes S$, and possesses a canonical projection onto V. A representation of this kind is called *tensor module*.

Theorem 6.1. *The $W(\mathfrak{d})$-action on each singular vector s lying in an irreducible module M is such that*

$$(1 \otimes \partial) \cdot s = -\sum_i (\partial_i \otimes 1) \otimes_H \rho(\partial_i^* \otimes \partial).s + (1 \otimes 1) \otimes_H ((\phi(\partial) + \rho(\operatorname{ad}\partial)).s - \partial s), \tag{6.1}$$

where $\partial \in \mathfrak{d}$, elements ∂_i constitute a basis of $\mathfrak{d}$, and σ, ρ are commuting actions of $\mathfrak{d}$ and $\mathfrak{gl}(\mathfrak{d}) = \mathfrak{gl}_N$.

Theorem 6.2. *Every finite irreducible $W(\mathfrak{d})$-module is a quotient of a finite tensor module $H \otimes S$, where all elements from $S = 1 \otimes S \subset H \otimes S$ are singular vectors, and the $\mathfrak{d} \oplus \mathfrak{gl}_N$-representation on S describing the action* (6.1) *of $W(\mathfrak{d})$ on singular vectors is irreducible.*

6.2. **Representations of primitive pseudoalgebras of type S, H, K.** Theorem 6.2 applies to all other primitive pseudoalgebras, the only difference being in the expression describing the action on singular vectors. Indeed, the annihilation algebra of pseudoalgebras of type S, H, K are isomorphic[2] to Lie algebras of type S, H, K, and the argument described above extends verbatim to these new cases. While the action of $S(\mathfrak{d}, \chi)$ can be extended to $W(\mathfrak{d})$ — so that the action of $S(\mathfrak{d}, \chi)$ on singular vectors can be recovered from (6.1) — that of $K(\mathfrak{d}, \theta)$ and $H(\mathfrak{d}, \chi, \omega)$ on singular vectors from an irreducible module can be described as follows. If $\psi \in \mathfrak{gl}(\mathfrak{d})$, let $\psi^{\mathfrak{sp}}$ denote its orthogonal projection to $\mathfrak{sp}(\mathfrak{d}, \omega)$. Then

Theorem 6.3. *The $K(\mathfrak{d}, \theta)$-action on each singular vector s lying in an irreducible module M is such that*

$$\begin{aligned}(1 \otimes e) \cdot s = &-\sum_{ij} (\partial_i \partial_j \otimes 1) \otimes_H \rho(x_{ij}).s \\ &+ \sum_i (\partial_i \otimes 1) \otimes_H (\partial^i s - (\phi(\partial^i) + \rho((\operatorname{ad}\partial^i)^{\mathfrak{sp}})).s) \\ &- (1 \otimes 1) \otimes_H (\partial_0 s - (\phi(\partial_0) + \rho(\operatorname{ad}\partial_0)).s) \\ &- (\partial_0 \otimes 1) \otimes_H \rho(E).s\end{aligned} \tag{6.2}$$

[2] Actually, the annihilation algebra of a pseudoalgebra of type H is a central extension of H_{2n}, but this fact plays no major role.

where e is the free H-generator of $H(\mathfrak{d},\chi,\omega)$, elements ∂_i and ∂^i constitute bases of $\mathfrak{d}$ dual with respect to $\omega = \mathrm{d}\theta$, and ϕ, ρ are commuting actions of $\mathfrak{d}$ and of $\mathfrak{csp}_N = \mathrm{span}\langle E, x_{ij}\rangle$.

Theorem 6.4. *The $H(\mathfrak{d},\chi,\omega)$-action on each singular vector s lying in an irreducible module M is such that*

$$
\begin{aligned}
(1\otimes e)\cdot s = &-\sum_{ij}(\bar\partial_i\bar\partial_j\otimes 1)\otimes_H \rho(x_{ij}).s\\
&+\sum_i(\bar\partial_i\otimes 1)\otimes_H(\partial^i s-(\phi(\partial^i)+\rho((\mathrm{ad}\,\partial^i+\chi\otimes\partial^i)^{\mathfrak{sp}})).s)\\
&+(1\otimes 1)\otimes_H(1\otimes\phi(E).s)
\end{aligned}
\tag{6.3}
$$

where e is the free H-generator of $K(\mathfrak{d},\theta)$, $\chi : \mathfrak{d}\to\mathbb{C}$ is a trace form, elements ∂_i and ∂^i constitute bases of $\mathfrak{d}$ dual to each other with respect to ω, $\bar\partial_i = \partial_i+\chi(\partial_i)$, and ϕ,ρ are commuting actions of $\mathfrak{sp}_N = \mathrm{span}\langle x_{ij}\rangle$ and of the central extension $\mathfrak{d}' = \mathrm{span}\langle\partial_i, E\rangle$ of $\mathfrak{d}$ determined by the cocycle ω.

A curious consequence of (6.1) is the following. A classification of singular vectors for the action of $W(\mathfrak{d})$ on a reducible tensor module shows that those not lying in S are vectors of the form

$$\sum_l \partial_l\otimes\rho(e_{kl}).s-\partial_k\otimes s+1\otimes s_0,$$

where $s\in 1, S\subset H\otimes S$, and s_0 is uniquely detemined from knowledge of k and s. Then, substituting this into (6.1) and imposing it to be a singular vector gives:

$$\rho(e_{ab})\rho(e_{cd})+\rho(e_{ad})\rho(e_{cb})=\delta_{bc}\rho(e_{ad})+\delta_{cd}\rho(e_{ab}), \tag{6.4}$$

which is valid for all $\mathfrak{gl}(\mathfrak{d})$-representations ρ for which the corresponding tensor module is reducible. In other words the only irreducible representations of $\mathfrak{gl}(\mathfrak{d})$ on which the quadratic relations (6.4) are valid are those of the form $\bigwedge^i\mathfrak{d}^*\simeq\bigwedge^{N-i}\mathfrak{d}$.

In the same way, (6.3) and (6.2) leads to quadratic relations that are valid for irreducible representations of $\mathfrak{sp}(\mathfrak{d},\omega)$ that occur as highest weight components of $\bigwedge^i\mathfrak{d}^*$. They are

$$\rho(x_{ab})\rho(x_{cd})+\text{ all permutations of } a,b,c,d\ =0, \tag{6.5}$$

where $x_{ij}=\frac{1}{2}(\partial_i^*\otimes\partial^j+\partial_j^*\otimes\partial_i)$ are generators of $\mathfrak{sp}(\mathfrak{d},\omega)$.

6.3. **An irreducibility criterion for tensor modules.** Simple computations show that a tensor module $H\otimes S$ is irreducible for all actions of $\mathfrak{gl}(\mathfrak{d})$ (resp. $\mathfrak{sp}(\mathfrak{d},\omega),\mathfrak{csp}(\mathfrak{d},\omega)$) but finitely many. In the $W(\mathfrak{d})$ case, for instance, representations of $\mathfrak{gl}(\mathfrak{d})$ not giving rise to irreducible representations of $W(\mathfrak{d})$ are all $\bigwedge$ powers of the contragradient representation $\mathfrak{d}^*$. The corresponding $W(\mathfrak{d})$-modules are those[3] appearing in the de Rham complex. Indeed, all modules of differential forms possess an action of the Lie algebra of vector fields, which is nothing but the annihilation algebra of $W(\mathfrak{d})$. This translates the de Rham complex into a complex of $W(\mathfrak{d})$-modules, in which the differential d is a $W(\mathfrak{d})$-homomorphism. In order to classify irreducible quotients of non irreducible tensor modules, one needs to find singular vectors in $H\otimes S$ not lying in U. The differential d serve this purpose, in that $\Omega^i/\mathrm{d}\Omega^{i-1}$ is the only irreducible quotient of the tensor module $\Omega^i=H\otimes\bigwedge^i\mathfrak{d}^*$.

[3]More precisely, they are "twists" of those modules, where twisting a $\mathfrak{d}\oplus\mathfrak{gl}(\mathfrak{d})$-module basically consists of keeping the same $\mathfrak{gl}(\mathfrak{d})$-action, while changing the $\mathfrak{d}$-action.

7. De Rham Complexes and Representation Theory

The setting we have outlined for $W(\mathfrak{d})$-representations extends to other primitive pseudoalgebras. In pretty much the same way as we have an exact sequence

$$\Omega^0(\mathfrak{d}) \to \Omega^1(\mathfrak{d}) \to \dots \to \Omega^n(\mathfrak{d})$$

of reducible tensor modules for $W(\mathfrak{d})$, all reducible tensor modules for the pseudoalgebras $K(\mathfrak{d}, \theta)$ and $H(\mathfrak{d}, \chi, \omega)$ fall in sequences that are generalizations of the de Rham complex.

For instance, one can build up an exact complex of $K(\mathfrak{d}, \theta)$-modules

$$\begin{aligned} \Omega^0(\mathfrak{d}) \to \Omega^1(\mathfrak{d})/I_1 \to \dots \to & \Omega^n(\mathfrak{d})/I_n \to \\ & \to J^{n+1} \to J^{n+2} \to \dots \to J^{2n+1}, \end{aligned} \tag{7.1}$$

where

$$I_k = \{\theta \wedge \eta + \omega \wedge \gamma | \beta \in \Omega^{k-1}(\mathfrak{d}), \gamma \in \Omega^{k-2}(\mathfrak{d})\}$$

and

$$J^k = \{\alpha \in \Omega^k(\mathfrak{d}) | \theta \wedge \alpha = \omega \wedge \alpha = 0\}.$$

The modules showing up in (7.1) are the only reducible tensor modules of $K(\mathfrak{d}, \theta)$. The above complex is completely analogous to that introduced by M. Rumin [Rum] in the context of the geometry of contact manifolds.

One can build up a complex à la Rumin in the case of $H(\mathfrak{d}, \chi, \omega)$ by taking

$$\Omega^0(\mathfrak{d}) \to \Omega^1(\mathfrak{d})/I_1 \to \dots \to \Omega^n(\mathfrak{d})/I_n \simeq J^n \to J^{n+1} \to \dots \to J^{2n},$$

where $I_k = \omega \wedge \Omega^{k-2}(\mathfrak{d})$ and $J^k = \{\alpha \in \Omega^k(\mathfrak{d}) | \omega \wedge \alpha = 0\}$.

One learns a general principle, which also applies in some way to the case of super Lie pseudoalgebras [KRud1, KRud2]: primitive Lie pseudoalgebras describe *"geometric structures"* on formal manifolds. Reducible tensor modules for primitive Lie pseudoalgebras arise in families, or rather complexes, which are often exact. These complexes are related to differential geometry, and are generalization of the de Rham complex, in the case of a particular choice of a (e.g. symplectic or contact) geometry.

References

[BDK1] B. Bakalov, A. D'Andrea, and V. G. Kac, *Theory of finite pseudoalgebras*, Adv. Math. **162** (2001), 1–140.

[BDK2] ———, *Irreducible representations of primitive simple Lie pseudoalgebras*, in preparation.

[BeDr] A. Beilinson and V. Drinfeld, *Chiral algebras*, preprint.

[Ca] E. Cartan, *Les groupes de transformation continus, infinis, simples*, Ann. Sci. ENS, **26** (1909), 93–161.

[DK] A. D'Andrea and V. G. Kac, *Structure theory of finite conformal algebras*, Selecta Math. (N.S.) **4** (1998), no. 3, 377–418.

[DuN1] B. A. Dubrovin and S. P. Novikov, *Poisson brackets of hydrodynamic type* (Russian), Dokl. Akad. Nauk SSSR **279** (1984), no. 2, 294–297. English translation in Soviet Math. Dokl. **30** (1984), no.2, 651-654.

[DuN2] B. A. Dubrovin and S. P. Novikov, *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory* (Russian), Uspekhi Mat. Nauk **44** (1989), no. 6(270), 29–98, 203. English translation in Russian Math. Surveys **44** (1989), no.6, 35–124.

[Gu1] V. Guillemin, *A Jordan-Hölder decomposition for a certain class of infinite dimensional Lie algebras*, J. Diff. Geom. **2** (1968) 313–345.

[Gu2] ———, *Infinite-dimensional primitive Lie algebras*, J. Diff. Geom. **4** (1970), 257–282.

[K] V. G. Kac, *Vertex algebras for beginners*, University Lecture Series, **10**. American Mathematical Society, Providence, RI, 1996. Second edition 1998.

[KRud1] V. G. Kac and A. N. Rudakov, *Representations of the exceptional Lie superalgebra $E(3,6)$. I. Degeneracy conditions*, Transform. Groups **7** (2002), no. 1, 67–86.

[KRud2] V. G. Kac and A. N. Rudakov, *Representations of the exceptional Lie superalgebra $E(3,6)$. II. Four series of degenerate modules*, Comm. Math. Phys. **222** (2001), no. 3, 611–661.

[Ko] I. A. Kostrikin, *Representations of height 1 of infinite-dimensional Lie algebras of the series K_n*, (Russian) Uspekhi Mat. Nauk **34** (1979), no. 1(205), 229–230. English translation in Russ. Math. Surv. **34** (1979), no. 1, 225–226.

[Rud1] A. N. Rudakov, *Irreducible representations of infinite-dimensional Lie algebras of Cartan type*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 835–866. English transl. in Math. USSR Izv. **8** (1974) 836–866.

[Rud2] ———, *Irreducible representations of infinite-dimensional Lie algebras of types S and H*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 496–511. English translation in Math. USSR Izv. **9** (1976) 465–480.

[Rum] M. Rumin, *Formes diffrentielles sur les varits de contact*, (French) J. Diff. Geom. **39** (1994), 281–330.

[So] Y. Soibelman, *Meromorphic tensor categories*, preprint `q-alg/9709030`.

DIPARTIMENTO DI MATEMATICA, ISTITUTO "GUIDO CASTELNUOVO", UNIVERSITÀ DI ROMA "LA SAPIENZA", 00185 ROME, ITALY

E-mail address: `dandrea@mat.uniroma1.it`