IRREDUCIBLE MODULES OVER FINITE SIMPLE LIE PSEUDOALGEBRAS I. PRIMITIVE PSEUDOALGEBRAS OF TYPE W AND S

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1. INTRODUCTION

One of the algebraic structures that has emerged recently in the study of the operator product expansions of chiral fields in conformal field theory is that of a *Lie conformal algebra* [K]. Recall that this is a module L over the algebra of polynomials $\mathbb{C}[\partial]$ in the indeterminate ∂, endowed with a $\mathbb{C}$-linear map

$$L \otimes L \to \mathbb{C}[\lambda] \otimes L\,, \quad a \otimes b \mapsto [a_\lambda b]\,,$$

satisfying axioms similar to those of a Lie algebra (see [DK, K]).

Choosing a set of generators $\{a^i\}_{i\in I}$ of the $\mathbb{C}[\partial]$-module L, we can write:

$$[a^i_\lambda a^j] = \sum_k Q^{ij}_k(\lambda,\partial)a^k\,,$$

where Q^{ij}_k are some polynomials in λ and ∂. The commutators of the corresponding chiral fields $\{a^i(z)\}_{i\in I}$ then are:

$$[a^i(z)\,,\,a^j(w)] = \sum_k Q^{ij}_k(\partial_z,\partial_t)(a^k(t)\delta(z-w))|_{t=w}\,.$$

Letting $P^{ij}_k(x,y) = Q^{ij}_k(-x,x+y)$, we can rewrite this in a more symmetric form

$$[a^i(z),a^j(w)] = \sum_k P^{ij}_k(\partial_z,\partial_w)(a^k(w)\delta(z-w))\,.$$

We thus obtain an $H=\mathbb{C}[\partial]$-bilinear map (i.e., a map of $H\otimes H$-modules):

$$L \otimes L \to (H\otimes H)\otimes_H L\,, \quad a\otimes b \mapsto [a*b] \tag{1.1}$$

(where H acts on $H\otimes H$ via the comultiplication map $\Delta(\partial)=\partial\otimes 1+1\otimes\partial$), defined by

$$[a^i * a^j] = \sum_k P^{ij}_k(\partial\otimes 1, 1\otimes\partial)\otimes_H a^k\,.$$

Hence the notion of a λ-bracket $[a_\lambda b]$ is equivalent to the notion of a $*$-bracket $[a*b]$, as introduced by Beilinson and Drinfeld [BD]. For example, the Virasoro conformal algebra $\mathrm{Vir} = \mathbb{C}[\partial]\ell$ with $[\ell_\lambda\ell] = (\partial+2\lambda)\ell$ corresponds to the Virasoro $*$-bracket

$$[\ell * \ell] = (1\otimes\partial - \partial\otimes 1)\otimes_{\mathbb{C}[\partial]}\ell\,. \tag{1.2}$$

A *Lie pseudoalgebra* is a generalization of the notion of a Lie conformal algebra for which $\mathbb{C}[\partial]$ is replaced by the Hopf algebra $H=U(\mathfrak{d})$, where $\mathfrak{d}$ is a finite-dimensional Lie algebra and $U(\mathfrak{d})$ is its universal enveloping algebra. It is defined as an H-module L endowed with an H-bilinear map (1.1) subject to certain skew-symmetry and Jacobi identity axioms (see [BD, BDK] and Section 2.2 below). The name *pseudoalgebra* is motivated by the fact that this is an algebra in a pseudotensor category, as introduced in [L, BD]. Accordingly, the $*$-bracket is also called a pseudobracket.

In [BDK] we gave a complete classification of finite (i.e., finitely generated as an H-module) simple Lie pseudoalgebras. In order to state the result, we introduce a generalization of the Virasoro pseudoalgebra (1.2) defined for $H=\mathbb{C}[\partial]$, to the case $H=U(\mathfrak{d})$, where $\mathfrak{d}$ is any finite-dimensional Lie algebra. This is the Lie pseudoalgebra $W(\mathfrak{d}) = H\otimes\mathfrak{d}$ with the pseudobracket

$$[(1\otimes a)*(1\otimes b)] = (1\otimes 1)\otimes_H(1\otimes[a,b]) + (b\otimes 1)\otimes_H(1\otimes a) - (1\otimes a)\otimes_H(1\otimes b)\,.$$

It is shown in [BDK] that all subalgebras of the Lie pseudoalgebra $W(\mathfrak{d})$ are simple and, along with current Lie pseudoalgebras $\mathrm{Cur}\,\mathfrak{g} = H\otimes\mathfrak{g}$ with pseudobracket

$$[(1\otimes a)*(1\otimes b)] = (1\otimes 1)\otimes_H[a,b]\,,$$

where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra, they form a complete list of finite simple Lie pseudoalgebras.

The notion of a Lie pseudoalgebra is intimately related to the more classical notion of a differential Lie algebra. Let L be a Lie pseudoalgebra, and let Y be a commutative associative algebra

with compatible left and right actions of the Hopf algebra H. Then we define a Lie algebra $\mathcal{A}_Y L = Y \otimes_H L$ with the obvious left H-module structure and the following Lie bracket:

$$[x \otimes_H a, y \otimes_H b] = \sum_i (xf_i)(yg_i) \otimes_H c_i\,, \quad \text{if} \quad [a * b] = \sum_i (f_i \otimes g_i) \otimes_H c_i\,.$$

The main tool in the study of Lie pseudoalgebras and their representations is the *annihilation algebra* $\mathcal{A}_X L$, where $X = H^*$ is the commutative associative algebra dual to the coalgebra H. In particular a module over a Lie pseudoalgebra L is the same as a "conformal" module over the extended annihilation Lie algebra $\mathfrak{d} \ltimes \mathcal{A}_X L$ (see [BDK] and Section 2.3 below).

The annihilation algebra of the Lie pseudoalgebra $W(\mathfrak{d})$ turns out to be isomorphic to the linearly compact Lie algebra of all formal vectors fields on a Lie group whose Lie algebra is $\mathfrak{d}$. This leads to a formalism of pseudoforms, similar to the usual formalism of differential forms, which allows us to define three series of subalgebras $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$ of $W(\mathfrak{d})$. The annihilation algebras of the simple Lie pseudoalgebras $W(\mathfrak{d})$, $S(\mathfrak{d}, \chi, \omega)$, $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$ are isomorphic to the four series of Lie–Cartan linearly compact Lie algebras W_N, S_N, P_N (which is an extension of H_N by a 1-dimensional center) and K_N, where $N = \dim \mathfrak{d}$.

However, the Lie pseudoalgebras $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$ depend on certain parameters χ, ω and θ due to inequivalent actions of $\mathfrak{d}$ on the annihilation algebra [BDK]. It is shown in [BDK] that these series of subalgebras along with their current generalizations, associated to subalgebras of $\mathfrak{d}$, exhaust all subalgebras of $W(\mathfrak{d})$.

The main goal of the present paper is to give a complete list and an explicit construction of all irreducible finite modules over the Lie pseudoalgebras $W(\mathfrak{d})$ and $S(\mathfrak{d}, \chi)$. Representation theory of the series $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$ will be treated in sequel papers.

The simplest example of a nontrivial $W(\mathfrak{d})$-module is the module $\Omega^0(\mathfrak{d}) = H$ (of rank 1 over H) given by:

$$(f \otimes a) * g = -(f \otimes ga) \otimes_H 1\,, \quad f, g \in H\,,\ a \in \mathfrak{d}\,.$$

The corresponding module over the annihilation Lie algebra is just the representation of the Lie algebra of all formal vector fields in the space of formal power series. As in the latter case, the $W(\mathfrak{d})$-module $\Omega^0(\mathfrak{d})$ is the first member of the pseudo de Rham complex

$$0 \to \Omega^0(\mathfrak{d}) \xrightarrow{\mathrm{d}} \Omega^1(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^N(\mathfrak{d})\,,$$

where $\Omega^n(\mathfrak{d}) = H \otimes \Omega^n$, $\Omega^n = \bigwedge^n \mathfrak{d}^*$, and $N = \dim \mathfrak{d}$ (see Section 5.2).

The $W(\mathfrak{d})$-modules $\Omega^n(\mathfrak{d})$ of pseudodifferential forms are special cases of *tensor modules* $\mathcal{T}(U) = H \otimes U$ over $W(\mathfrak{d})$, associated to any $\mathfrak{gl}\,\mathfrak{d}$-module U, given by:

$$\begin{aligned}(1 \otimes \partial_i) * (1 \otimes u) = (1 \otimes 1) \otimes_H (1 \otimes (\operatorname{ad} \partial_i)u) + \sum_{j=1}^{N} (\partial_j \otimes 1) \otimes_H (1 \otimes e_i^j u) \\ - (1 \otimes \partial_i) \otimes_H (1 \otimes u)\,,\end{aligned} \tag{1.3}$$

where $\{\partial_i\}$ is a basis of $\mathfrak{d}$ and $e_i^j \partial_k = \delta_k^j \partial_i$ (see Section 4.3). Then $\Omega^n(\mathfrak{d}) = \mathcal{T}(\Omega^n)$.

Furthermore, for a finite-dimensional $\mathfrak{d}$-module Π we define the *twisting* of $\mathcal{T}(U)$ by Π by $\mathcal{T}(\Pi, U) = H \otimes (\Pi \otimes U)$ and by adding the term $(1 \otimes 1) \otimes_H (1 \otimes \partial_i u)$ in the right-hand side of (1.3). Then we have the Π-twisted pseudo de Rham complex of $W(\mathfrak{d})$-modules:

$$0 \to \mathcal{T}(\Pi, \Omega^0) \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi, \Omega^1) \xrightarrow{\mathrm{d}_\Pi} \cdots \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi, \Omega^N)$$

(see Section 5.3).

The first main result of the present paper (Theorem 6.6) states that:

(a) The $W(\mathfrak{d})$-module $\mathcal{T}(\Pi, U)$ is irreducible if and only if Π and U are irreducible and U is not isomorphic to one of the $\mathfrak{gl}\,\mathfrak{d}$-modules $\Omega^n = \bigwedge^n \mathfrak{d}^*$ for $n \geq 1$;
(b) The $W(\mathfrak{d})$-submodule $\mathrm{d}_\Pi \mathcal{T}(\Pi, \Omega^n)$ of $\mathcal{T}(\Pi, \Omega^{n+1})$ is irreducible, provided that Π is irreducible, for all $0 \leq n \leq N - 1$;

(c) The irreducible $W(\mathfrak{d})$-modules listed in (a) and (b) exhaust all irreducible finite $W(\mathfrak{d})$-modules.

The corresponding result for $S(\mathfrak{d},\chi)$ is Theorem 7.6. We also describe the structure of submodules of the $W(\mathfrak{d})$- and $S(\mathfrak{d},\chi)$-modules $\mathcal{T}(\Pi,\Omega^n)$ (Lemmas 6.12 and 7.10).

As in the Lie algebra case, the main part of the problem is the computation of singular vectors. However, in the Lie pseudoalgebra framework the calculations are much simpler. In particular, we obtain simpler and more transparent proofs of the results of Rudakov [R1, R2].

In the case when $\mathfrak{d}=\mathbf{k}\partial$ is 1-dimensional, the Lie pseudoalgebra $W(\mathbf{k}\partial)$ is isomorphic to the Virasoro pseudoalgebra Vir with the pseudobracket (1.2). Now Theorem 6.6 states that every irreducible $W(\mathbf{k}\partial)$-module is of the form $\mathcal{T}(\Pi,\mathbf{k})$, where Π is an irreducible $\mathbf{k}\partial$-module and $\mathbf{k}$ is the trivial $\mathfrak{gl}_1$-module. The modules Π are 1-dimensional over $\mathbf{k}$ and are uniquely determined by the eigenvalue $\alpha\in\mathbf{k}$ of ∂. Thus we recover the classification result of [CK].

Note that the category of representations of a Lie pseudoalgebra is not semisimple in general, i.e., complete reducibility of modules does not hold. To study extensions of modules, as well as central extensions and infinitesimal deformations of Lie pseudoalgebras, one defines *cohomology* of Lie pseudoalgebras (see [BKV, BDK]). The cohomology of the Virasoro conformal algebra Vir was computed in [BKV]. The cohomology of $W(\mathfrak{d})$ and its subalgebras will be computed in a future publication.

2. Basic Definitions

In this section, we review some facts and notation from [BDK], which will be used throughout the paper. We will work over an algebraically closed field $\mathbf{k}$ of characteristic 0. Unless otherwise specified, all vector spaces, linear maps and tensor products will be considered over $\mathbf{k}$. We will denote by $\mathbb{Z}_+$ the set of non-negative integers.

2.1. **Preliminaries on Hopf Algebras.** Let H be a Hopf algebra with a coproduct Δ, a counit ε, and an antipode S. We will use the following notation (cf. [Sw]):

$$\Delta(h)=h_{(1)}\otimes h_{(2)},\qquad h\in H, \tag{2.1}$$

$$(\Delta\otimes\mathrm{id})\Delta(h)=(\mathrm{id}\otimes\Delta)\Delta(h)=h_{(1)}\otimes h_{(2)}\otimes h_{(3)}, \tag{2.2}$$

$$(S\otimes\mathrm{id})\Delta(h)=h_{(-1)}\otimes h_{(2)},\quad \text{etc.} \tag{2.3}$$

Note that notation (2.2) uses coassociativity of Δ. The axioms of antipode and counit can be written as follows:

$$h_{(-1)}h_{(2)}=h_{(1)}h_{(-2)}=\varepsilon(h), \tag{2.4}$$

$$\varepsilon(h_{(1)})h_{(2)}=h_{(1)}\varepsilon(h_{(2)})=h, \tag{2.5}$$

while the fact that Δ is a homomorphism of algebras translates as:

$$(fg)_{(1)}\otimes(fg)_{(2)}=f_{(1)}g_{(1)}\otimes f_{(2)}g_{(2)},\qquad f,g\in H. \tag{2.6}$$

Equations (2.4) and (2.5) imply the following useful relations:

$$h_{(-1)}h_{(2)}\otimes h_{(3)}=1\otimes h=h_{(1)}h_{(-2)}\otimes h_{(3)}. \tag{2.7}$$

Let $X=H^*:=\mathrm{Hom}_{\mathbf{k}}(H,\mathbf{k})$ be the dual of H. Recall that H acts on X by the formula ($h,f\in H$, $x,y\in X$):

$$\langle hx,f\rangle=\langle x,S(h)f\rangle, \tag{2.8}$$

so that

$$h(xy)=(h_{(1)}x)(h_{(2)}y). \tag{2.9}$$

Moreover, X is commutative when H is cocommutative. Similarly, one can define a right action of H on X by

$$\langle xh,f\rangle=\langle x,fS(h)\rangle, \tag{2.10}$$

and then we have

$$(xy)h = (xh_{(1)})(yh_{(2)}). \tag{2.11}$$

Associativity of H implies that X is an H-bimodule, i.e.,

$$f(xg) = (fx)g, \qquad f, g \in H,\ x \in X. \tag{2.12}$$

Throughout the paper, $H = U(\mathfrak{d})$ will be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. In this case,

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \quad S(a) = -a, \qquad a \in \mathfrak{d}\,; \tag{2.13}$$

hence, Δ is cocommutative and $S^2 = \mathrm{id}$. Set $N = \dim \mathfrak{d}$ and fix a basis $\{\partial_i\}_{i=1,\dots,N}$ of $\mathfrak{d}$. Then

$$\partial^{(I)} = \partial_1^{i_1} \cdots \partial_N^{i_N}/i_1! \cdots i_N!\,, \qquad I = (i_1, \dots, i_N) \in \mathbb{Z}_+^N\,, \tag{2.14}$$

is a basis of H (similar to the Poincaré–Birkhoff–Witt basis). Moreover, it is easy to see that

$$\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}. \tag{2.15}$$

For a multi-index $I = (i_1, \dots, i_N)$, let $|I| = i_1 + \cdots + i_N$. Recall that the canonical increasing filtration of $H = U(\mathfrak{d})$ is given by

$$\mathrm{F}^p U(\mathfrak{d}) = \mathrm{span}_{\mathbf{k}}\{\partial^{(I)} \mid |I| \le p\}\,, \qquad p = 0, 1, 2, \dots \tag{2.16}$$

and does not depend on the choice of basis of $\mathfrak{d}$. This filtration is compatible with the structure of Hopf algebra (see, e.g., [BDK, Section 2.2] for more details). We have: $\mathrm{F}^{-1} H = \{0\}$, $\mathrm{F}^0 H = \mathbf{k}$, $\mathrm{F}^1 H = \mathbf{k} \oplus \mathfrak{d}$.

We define a filtration of $H \otimes H$ in the usual way:

$$\mathrm{F}^n(H \otimes H) = \sum_{i+j=n} \mathrm{F}^i H \otimes \mathrm{F}^j H\,. \tag{2.17}$$

The following lemma, which is a reformulation of [BDK, Lemma 2.3], plays an important role in the paper. (This lemma holds for any Hopf algebra H.)

Lemma 2.1. (i) *The linear maps*

$$H \otimes H \to H \otimes H\,, \qquad f \otimes g \mapsto (f \otimes 1)\Delta(g)$$

and

$$H \otimes H \to H \otimes H\,, \qquad f \otimes g \mapsto (1 \otimes f)\Delta(g)$$

are isomorphisms of vector spaces. These isomorphisms are compatible with the filtration (2.17)*.*

(ii) *For any H-module V, the linear maps*

$$H \otimes V \to (H \otimes H) \otimes_H V\,, \qquad h \otimes v \mapsto (h \otimes 1) \otimes_H v$$

and

$$H \otimes V \to (H \otimes H) \otimes_H V\,, \qquad h \otimes v \mapsto (1 \otimes h) \otimes_H v$$

are isomorphisms of vector spaces. In addition, we have:

$$(\mathrm{F}^n H \otimes \mathbf{k}) \otimes_H V = \mathrm{F}^n(H \otimes H) \otimes_H V = (\mathbf{k} \otimes \mathrm{F}^n H) \otimes_H V\,.$$

Let us define elements $x_I \in X$ by $\langle x_I, \partial^{(J)} \rangle = \delta_I^J$, where, as usual, $\delta_I^J = 1$ if $I = J$ and $\delta_I^J = 0$ if $I \ne J$. Then (2.15) implies $x_J x_K = x_{J+K}$; hence,

$$x_I = (x^1)^{i_1} \cdots (x^N)^{i_N}\,, \qquad I = (i_1, \dots, i_N) \in \mathbb{Z}_+^N\,, \tag{2.18}$$

where

$$x^i = x_{\varepsilon_i}\,, \qquad \varepsilon_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)\,, \quad i = 1, \dots, N\,. \tag{2.19}$$

Therefore, X can be identified with the ring $\mathcal{O}_N = \mathbf{k}[[t^1, \dots, t^N]]$ of formal power series in N indeterminates. We have a ring isomorphism

$$\varphi\colon X \xrightarrow{\sim} \mathcal{O}_N\,, \quad \varphi(x^i) = t^i\,,\ \varphi(x_I) = t_I\,, \tag{2.20}$$

where t_I is given by a formula similar to (2.18).

Let $\mathrm{F}_p X = (\mathrm{F}^p H)^\perp$ be the set of elements from $X = H^*$ that vanish on $\mathrm{F}^p H$. Then $\{\mathrm{F}_p X\}$ is a decreasing filtration of X such that $\mathrm{F}_{-1} X = X$, $X/\mathrm{F}_0 X \simeq \mathbf{k}$, $\mathrm{F}_0 X/\mathrm{F}_1 X \simeq \mathfrak{d}^*$. Under the isomorphism (2.20), the filtration $\{\mathrm{F}_p X\}$ becomes

$$\mathrm{F}_p \mathcal{O}_N = (t^1, \dots, t^N)^{p+1} \mathcal{O}_N\,, \qquad p = -1, 0, 1, \dots\,. \tag{2.21}$$

This filtration has the following properties:

$$(\mathrm{F}_n X)(\mathrm{F}_p X) \subset \mathrm{F}_{n+p+1} X\,, \quad \mathfrak{d}(\mathrm{F}_p X) \subset \mathrm{F}_{p-1} X\,, \quad (\mathrm{F}_p X)\mathfrak{d} \subset \mathrm{F}_{p-1} X\,. \tag{2.22}$$

We can consider x^i as elements of $\mathfrak{d}^*$; then $\{x^i\}$ is a basis of $\mathfrak{d}^*$ dual to the basis $\{\partial_i\}$ of $\mathfrak{d}$, i.e., $\langle x^i, \partial_j \rangle = \delta^i_j$.

We define a topology of X by considering $\{\mathrm{F}_p X\}$ as a fundamental system of neighborhoods of 0. We will always consider X with this topology, while H and $\mathfrak{d}$ with the discrete topology. Then X is linearly compact (see [BDK, Chapter 6]), and the multiplication of X and the (left and right) actions of $\mathfrak{d}$ on it are continuous (see (2.22)).

Example 2.1. When $\mathfrak{d}$ is commutative, its left and right actions on $\mathcal{O}_N$ coincide and are given by $\partial_i \mapsto -\partial/\partial t^i$ for $i = 1, \dots, N$.

The following lemma is well known (see also [Re, Section 6]).

Lemma 2.2. *Let c^k_{ij} be the structure constants of $\mathfrak{d}$ in the basis $\{\partial_i\}$, so that $[\partial_i, \partial_j] = \sum c^k_{ij} \partial_k$. Then we have the following formulas for the left and right actions of $\mathfrak{d}$ on X:*

$$\partial_i x^j = -\delta^j_i - \sum_{k<i} c^j_{ik} x^k \mod \mathrm{F}_1 X\,,$$
$$x^j \partial_i = -\delta^j_i + \sum_{k>i} c^j_{ik} x^k \mod \mathrm{F}_1 X\,.$$

In particular,

$$\partial_i x^j - x^j \partial_i = -\sum_k c^j_{ik} x^k \mod \mathrm{F}_1 X$$

is the coadjoint action of $\mathfrak{d}$ on $\mathfrak{d}^* \simeq \mathrm{F}_0 X/\mathrm{F}_1 X$.

Proof. We will prove the first equality. The second one is proved in the same way, while the third follows from the other two. If we express $\partial_i x^j$ in the basis $\{x_K\}$ of X, we have

$$\partial_i x^j = \sum_{K \in \mathbb{Z}^N_+} a_K x_K \iff a_K = \langle \partial_i x^j, \partial^{(K)} \rangle\,,$$

where $\partial^{(K)}$ are from (2.14). Since we are interested in $\partial_i x^j \mod \mathrm{F}_1 X$, we need to compute a_K only for $|K| \le 1$, i.e., only for $\partial^{(K)} = 1$ or $\partial^{(K)} = \partial_k$. Using (2.8), we obtain

$$\langle \partial_i x^j, 1 \rangle = -\langle x^j, \partial_i \rangle = -\delta^j_i\,,$$
$$\langle \partial_i x^j, \partial_k \rangle = -\langle x^j, \partial_i \partial_k \rangle = -\langle x^j, \partial_k \partial_i \rangle - \langle x^j, [\partial_i, \partial_k] \rangle\,.$$

If $i \le k$, then $\partial_i \partial_k$ is (up to a constant) an element of the basis (2.14) and $\partial_i \partial_k \ne \partial_j$; hence, $\langle x^j, \partial_i \partial_k \rangle = 0$. If $i > k$, then by the same argument $\langle x^j, \partial_k \partial_i \rangle = 0$, while $\langle x^j, [\partial_i, \partial_k] \rangle = c^j_{ik}$. This completes the proof. □

2.2. **Pseudoalgebras and Their Representations.** In this subsection, we recall the definition of a pseudoalgebra from [BDK, Chapter 3]. Let A be a (left) H-module. A *pseudoproduct* on A is an H-bilinear map

$$A \otimes A \to (H \otimes H) \otimes_H A\,, \quad a \otimes b \mapsto a * b\,, \tag{2.23}$$

where we use the comultiplication $\Delta\colon H \to H \otimes H$ to define $(H \otimes H) \otimes_H A$. A *pseudoalgebra* is a (left) H-module A endowed with a pseudoproduct (2.23). The name is motivated by the fact that this is an algebra in a pseudotensor category, as introduced in [L, BD] (see [BDK, Chapter 3]).

In order to define associativity of a pseudoproduct, we extend it from $A \otimes A \to H^{\otimes 2} \otimes_H A$ to $(H^{\otimes 2} \otimes_H A) \otimes A \to H^{\otimes 3} \otimes_H A$ and to $A \otimes (H^{\otimes 2} \otimes_H A) \to H^{\otimes 3} \otimes_H A$ by letting:

$$(h \otimes_H a) * b = \sum (h \otimes 1)\,(\Delta \otimes \mathrm{id})(g_i) \otimes_H c_i\,, \tag{2.24}$$

$$a * (h \otimes_H b) = \sum (1 \otimes h)\,(\mathrm{id} \otimes \Delta)(g_i) \otimes_H c_i\,, \tag{2.25}$$

where $h \in H^{\otimes 2}$, $a, b \in A$, and

$$a * b = \sum g_i \otimes_H c_i \qquad \text{with} \quad g_i \in H^{\otimes 2},\, c_i \in A. \tag{2.26}$$

Then the associativity property is given by the usual equality (in $H^{\otimes 3} \otimes_H A$):

$$(a * b) * c = a * (b * c)\,. \tag{2.27}$$

The main objects of our study are Lie pseudoalgebras. The corresponding pseudoproduct is conventionally called *pseudobracket* and denoted by $[a * b]$. A *Lie pseudoalgebra* is a (left) H-module equipped with a pseudobracket satisfying the following skew-commutativity and Jacobi identity axioms:

$$[b * a] = -(\sigma \otimes_H \mathrm{id})\,[a * b]\,, \tag{2.28}$$

$$[[a * b] * c] = [a * [b * c]] - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,[b * [a * c]]\,. \tag{2.29}$$

Here, $\sigma\colon H \otimes H \to H \otimes H$ is the permutation of factors, and the compositions $[[a * b] * c]$, $[a * [b * c]]$ are defined using (2.24), (2.25).

Remark 2.1. Let A be an associative pseudoalgebra with a pseudoproduct $a * b$. Define a pseudobracket on A as the commutator

$$[a * b] = a * b - (\sigma \otimes_H \mathrm{id})\,(b * a)\,. \tag{2.30}$$

Then, with this pseudobracket, A is a Lie pseudoalgebra.

Example 2.2. For any $\mathbf{k}$-algebra A, let its associated *current* H-pseudoalgebra be $\mathrm{Cur}\,A = H \otimes A$ with the pseudoproduct

$$(f \otimes a) * (g \otimes b) = (f \otimes g) \otimes_H (1 \otimes ab)\,. \tag{2.31}$$

Then the H-pseudoalgebra $\mathrm{Cur}\,A$ is Lie (or associative) iff the $\mathbf{k}$-algebra A is.

The definitions of modules over Lie (or associative) pseudoalgebras are obvious modifications of the above. A *module* over a Lie pseudoalgebra L is a left H-module V together with an H-bilinear map

$$L \otimes V \to (H \otimes H) \otimes_H V\,, \quad a \otimes v \mapsto a * v \tag{2.32}$$

that satisfies ($a, b \in L$, $v \in V$)

$$[a * b] * v = a * (b * v) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,(b * (a * v))\,. \tag{2.33}$$

An L-module V will be called *finite* if it is finite (i.e., finitely generated) as an H-module. The *trivial* L-module is the set $\{0\}$.

A subspace $W \subset V$ is an *L-submodule* if it is an H-submodule and $L * W \subset (H \otimes H) \otimes_H W$. (Here $L * W$ is the linear span of all elements $a * w$, where $a \in L$ and $w \in W$.) A submodule

$W \subset V$ is called *proper* if $W \neq V$. An L-module V is *irreducible* (or *simple*) if it does not contain any nontrivial proper L-submodules and $L * V \neq \{0\}$.

Let U and V be two L-modules. A map $\beta\colon U \to V$ is a *homomorphism* of L-modules if β is H-linear and it satisfies

$$\big((\mathrm{id} \otimes \mathrm{id}) \otimes_H \beta\big)(a * u) = a * \beta(u)\,, \qquad a \in L\,,\ u \in U\,. \tag{2.34}$$

Remark 2.2. (i) Let V be a module over a Lie pseudoalgebra L and let W be an H-submodule of V. By Lemma 2.1(ii), for each $a \in L$, $v \in V$, we can write

$$a * v = \sum_{I \in \mathbb{Z}_+^N} (\partial^{(I)} \otimes 1) \otimes_H v'_I\,, \qquad v'_I \in V\,,$$

where the elements v'_I are uniquely determined by a and v. Then $W \subset V$ is an L-submodule iff it has the property that all $v'_I \in W$ whenever $v \in W$. This follows again from Lemma 2.1(ii).

(ii) Similarly, for each $a \in L$, $v \in V$, we can write

$$a * v = \sum_{I \in \mathbb{Z}_+^N} (1 \otimes \partial^{(I)}) \otimes_H v''_I\,, \qquad v''_I \in V\,,$$

and W is an L-submodule iff $v''_I \in W$ whenever $v \in W$.

Example 2.3. Let L be a Lie pseudoalgebra, and let V be an L-module, which is finite dimensional (over $\mathbf{k}$). Then the action of L on V is trivial, i.e., $L * V = \{0\}$. Indeed, since $\dim H = \infty$, every element $v \in V$ is torsion, i.e., such that $hv = 0$ for some nonzero $h \in H$. Then the statement follows from [BDK, Corollary 10.1].

2.3. **Annihilation Algebras of Lie Pseudoalgebras.** For a Lie H-pseudoalgebra L, we set $\mathcal{A}(L) = X \otimes_H L$, where as before $X = H^*$. We define a Lie bracket on $\mathcal{L} = \mathcal{A}(L)$ by the formula (cf. [BDK, Eq. (7.2)]):

$$[x \otimes_H a, y \otimes_H b] = \sum (xf_i)(yg_i) \otimes_H c_i\,, \quad \text{if} \quad [a * b] = \sum (f_i \otimes g_i) \otimes_H c_i\,. \tag{2.35}$$

Then $\mathcal{L}$ is a Lie algebra, called the *annihilation algebra* of L (see [BDK, Section 7.1]). We define a left action of H on $\mathcal{L}$ in the obvious way:

$$h(x \otimes_H a) = hx \otimes_H a. \tag{2.36}$$

In the case $H = U(\mathfrak{d})$, the Lie algebra $\mathfrak{d}$ acts on $\mathcal{L}$ by derivations. The semidirect sum $\widetilde{\mathcal{L}} = \mathfrak{d} \ltimes \mathcal{L}$ is called the *extended annihilation algebra*.

Similarly, if V is a module over a Lie pseudoalgebra L, we let $\mathcal{A}(V) = X \otimes_H V$, and define an action of $\mathcal{L} = \mathcal{A}(L)$ on $\mathcal{A}(V)$ by:

$$(x \otimes_H a)(y \otimes_H v) = \sum (xf_i)(yg_i) \otimes_H v_i\,, \quad \text{if} \quad a * v = \sum (f_i \otimes g_i) \otimes_H v_i\,. \tag{2.37}$$

We also define an H-action on $\mathcal{A}(V)$ similarly to (2.36). Then $\mathcal{A}(V)$ is an $\widetilde{\mathcal{L}}$-module [BDK, Proposition 7.1].

When L is a finite H-module, we can define a filtration on $\mathcal{L}$ as follows (see [BDK, Section 7.4] for more details). We fix a finite-dimensional vector subspace L_0 of L such that $L = HL_0$, and set

$$\mathrm{F}_p\,\mathcal{L} = \{x \otimes_H a \in \mathcal{L} \mid x \in \mathrm{F}_p\,X\,,\ a \in L_0\}\,, \qquad p \geq -1\,. \tag{2.38}$$

The subspaces $\mathrm{F}_p\,\mathcal{L}$ constitute a decreasing filtration of $\mathcal{L}$, satisfying

$$[\mathrm{F}_n\,\mathcal{L}, \mathrm{F}_p\,\mathcal{L}] \subset \mathrm{F}_{n+p-\ell}\,\mathcal{L}\,, \quad \mathfrak{d}(\mathrm{F}_p\,\mathcal{L}) \subset \mathrm{F}_{p-1}\,\mathcal{L}\,, \tag{2.39}$$

where ℓ is an integer depending only on the choice of L_0. Notice that the filtration just defined depends on the choice of L_0, but the topology it induces does not [BDK, Lemma 7.2]. We set $\mathcal{L}_p = \mathrm{F}_{p+\ell}\,\mathcal{L}$, so that $[\mathcal{L}_n, \mathcal{L}_p] \subset \mathcal{L}_{n+p}$. In particular, $\mathcal{L}_0$ is a Lie algebra.

We also define a filtration of $\widetilde{\mathcal{L}}$ by letting $\mathrm{F}_{-1}\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}$, $\mathrm{F}_p\widetilde{\mathcal{L}} = \mathrm{F}_p\mathcal{L}$ for $p \geq 0$, and we set $\widetilde{\mathcal{L}}_p = \mathrm{F}_{p+\ell}\widetilde{\mathcal{L}}$. An $\widetilde{\mathcal{L}}$-module V is called *conformal* if every $v \in V$ is killed by some $\mathcal{L}_p$; in other words, if V is a topological $\widetilde{\mathcal{L}}$-module when endowed with the discrete topology.

The next two results from [BDK] will play a crucial role in our study of representations (see [BDK], Propositions 9.1 and 14.2, and Lemma 14.4).

Proposition 2.1. *Any module V over the Lie pseudoalgebra L has a natural structure of a conformal $\widetilde{\mathcal{L}}$-module, given by the action of $\mathfrak{d}$ on V and by*

$$(x \otimes_H a) \cdot v = \sum \langle x, S(f_i g_{i(-1)})\rangle \, g_{i(2)} v_i \,, \quad \text{if} \quad a * v = \sum (f_i \otimes g_i) \otimes_H v_i \tag{2.40}$$

for $a \in L$, $x \in X$, $v \in V$.

Conversely, any conformal $\widetilde{\mathcal{L}}$-module V has a natural structure of an L-module, given by

$$a * v = \sum_{I \in \mathbb{Z}_+^N} \left(S(\partial^{(I)}) \otimes 1\right) \otimes_H \left((x_I \otimes_H a) \cdot v\right). \tag{2.41}$$

Moreover, V is irreducible as an L-module iff it is irreducible as an $\widetilde{\mathcal{L}}$-module.

Lemma 2.3. *Let L be a finite Lie pseudoalgebra and V be a finite L-module. For $p \geq -1-\ell$, let*

$$\ker_p V = \{v \in V \mid \mathcal{L}_p\, v = 0\},$$

so that, for example, $\ker_{-1-\ell} V = \ker V$ and $V = \bigcup \ker_p V$. Then all vector spaces $\ker_p V / \ker V$ are finite dimensional. In particular, if $\ker V = \{0\}$, then every vector $v \in V$ is contained in a finite-dimensional subspace invariant under $\mathcal{L}_0$.

3. Primitive Lie Pseudoalgebras of Type W and S

Here we introduce the main objects of our study: the *primitive* Lie pseudoalgebras $W(\mathfrak{d})$ and $S(\mathfrak{d}, \chi)$ and their annihilation algebras $\mathcal{W}$ and $\mathcal{S}$ (see [BDK, Chapter 8]).

3.1. **Definition of $W(\mathfrak{d})$ and $S(\mathfrak{d}, \chi)$.** We define the Lie pseudoalgebra $W(\mathfrak{d})$ as the free H-module $H \otimes \mathfrak{d}$ with the pseudobracket

$$\begin{aligned}[(f \otimes a) * (g \otimes b)] &= (f \otimes g) \otimes_H (1 \otimes [a, b]) \\ &\quad - (f \otimes ga) \otimes_H (1 \otimes b) + (fb \otimes g) \otimes_H (1 \otimes a)\,.\end{aligned} \tag{3.1}$$

There is a structure of a $W(\mathfrak{d})$-module on H given by:

$$(f \otimes a) * g = -(f \otimes ga) \otimes_H 1\,. \tag{3.2}$$

Let χ be a trace form on $\mathfrak{d}$, i.e., a linear functional from $\mathfrak{d}$ to $\mathbf{k}$ that vanishes on $[\mathfrak{d}, \mathfrak{d}]$. Define an H-linear map $\mathrm{div}^\chi \colon W(\mathfrak{d}) \to H$ by the formula:

$$\mathrm{div}^\chi \Big(\sum h_i \otimes \partial_i\Big) = \sum h_i(\partial_i + \chi(\partial_i))\,. \tag{3.3}$$

Then

$$S(\mathfrak{d}, \chi) := \{s \in W(\mathfrak{d}) \mid \mathrm{div}^\chi s = 0\} \tag{3.4}$$

is a subalgebra of the Lie pseudoalgebra $W(\mathfrak{d})$. It was shown in [BDK, Proposition 8.1] that $S(\mathfrak{d}, \chi)$ is generated over H by the elements

$$s_{ab} := (a + \chi(a)) \otimes b - (b + \chi(b)) \otimes a - 1 \otimes [a, b] \qquad \text{for} \quad a, b \in \mathfrak{d}\,. \tag{3.5}$$

Pseudobrackets of the elements s_{ab} are explicitly calculated in [BDK, Proposition 8.1]. Notice that when $\dim \mathfrak{d} > 2$, $S(\mathfrak{d}, \chi)$ is not free as an H-module, because the elements s_{ab} satisfy the relations [BDK, Eq. (8.23)].

Remark 3.1. If $\dim \mathfrak{d} = 1$, then $S(\mathfrak{d}, \chi) = \{0\}$. If $\dim \mathfrak{d} = 2$, the Lie pseudoalgebra $S(\mathfrak{d}, \chi)$ is free as an H-module and it is isomorphic to a primitive Lie pseudoalgebra of type H (see [BDK], Section 8.6 and Example 8.1).

Irreducible modules over primitive Lie pseudoalgebras of type H will be studied in a sequel paper. From now on, whenever we consider the Lie pseudoalgebra $S(\mathfrak{d}, \chi)$, we will assume that $\dim \mathfrak{d} > 2$.

3.2. **Annihilation Algebra of $W(\mathfrak{d})$.** Let $\mathcal{W} = \mathcal{A}(W(\mathfrak{d}))$ be the annihilation algebra of the Lie pseudoalgebra $W(\mathfrak{d})$. Since $W(\mathfrak{d}) = H \otimes \mathfrak{d}$, we have $\mathcal{W} = X \otimes_H (H \otimes \mathfrak{d}) \equiv X \otimes \mathfrak{d}$, so we can identify $\mathcal{W}$ with $X \otimes \mathfrak{d}$. Then the Lie bracket (2.35) in $\mathcal{W}$ becomes ($x, y \in X$, $a, b \in \mathfrak{d}$):

$$[x \otimes a, y \otimes b] = xy \otimes [a, b] - x(ya) \otimes b + (xb)y \otimes a\,, \tag{3.6}$$

while the left action (2.36) of H on $\mathcal{W}$ is given by: $h(x \otimes a) = hx \otimes a$. The Lie algebra $\mathfrak{d}$ acts on $\mathcal{W}$ by derivations. We denote by $\widetilde{\mathcal{W}}$ the extended annihilation algebra $\mathfrak{d} \ltimes \mathcal{W}$, where

$$[\partial, x \otimes a] = \partial x \otimes a\,, \qquad \partial, a \in \mathfrak{d},\ x \in X\,. \tag{3.7}$$

We choose $L_0 = \mathbf{k} \otimes \mathfrak{d}$ as a subspace of $W(\mathfrak{d})$ such that $W(\mathfrak{d}) = HL_0$, and we obtain the following filtration of $\mathcal{W}$:

$$\mathcal{W}_p = \mathrm{F}_p \mathcal{W} = \mathrm{F}_p X \otimes_H L_0 \equiv \mathrm{F}_p X \otimes \mathfrak{d}\,. \tag{3.8}$$

This is a decreasing filtration of $\mathcal{W}$, satisfying $\mathcal{W}_{-1} = \mathcal{W}$ and (2.39) for $\ell = 0$. Note that $\mathcal{W}/\mathcal{W}_0 \simeq \mathbf{k} \otimes \mathfrak{d} \simeq \mathfrak{d}$ and $\mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{d}^* \otimes \mathfrak{d}$.

Let us fix a basis $\{\partial_i\}_{i=1,\dots,N}$ of $\mathfrak{d}$, and let $x^i \in X$ be given by (2.19). We can view x^i as elements of $\mathfrak{d}^*$; then $\{x^i\}$ is a basis of $\mathfrak{d}^*$ dual to the basis $\{\partial_i\}$ of $\mathfrak{d}$. Let $e_i^j \in \mathfrak{gl}\,\mathfrak{d}$ be given by $e_i^j \partial_k = \delta_k^j\, \partial_i$; in other words, e_i^j corresponds to $\partial_i \otimes x^j$ under the isomorphism $\mathfrak{gl}\,\mathfrak{d} \simeq \mathfrak{d} \otimes \mathfrak{d}^*$.

Lemma 3.1. *In the Lie algebra $\mathcal{W} = X \otimes \mathfrak{d}$, we have the following*:

$$[x^j \otimes \partial_i, 1 \otimes \partial_k] = -\delta_k^j\, 1 \otimes \partial_i \mod \mathcal{W}_0\,,$$
$$[x^j \otimes \partial_i, x^l \otimes \partial_k] = \delta_i^l\, x^j \otimes \partial_k - \delta_k^j\, x^l \otimes \partial_i \mod \mathcal{W}_1\,.$$

Proof. This follows from (3.6) and Lemma 2.2. □

Corollary 3.1. *For $x \in \mathrm{F}_0 X$, $a \in \mathfrak{d}$, the map*

$$x \otimes a \mod \mathcal{W}_1 \mapsto -a \otimes (x \bmod \mathrm{F}_1 X)$$

is a Lie algebra isomorphism from $\mathcal{W}_0/\mathcal{W}_1$ to $\mathfrak{d} \otimes \mathfrak{d}^* \simeq \mathfrak{gl}\,\mathfrak{d}$. Under this isomorphism, the adjoint action of $\mathcal{W}_0/\mathcal{W}_1$ on $\mathcal{W}/\mathcal{W}_0$ coincides with the standard action of $\mathfrak{gl}\,\mathfrak{d}$ on $\mathfrak{d}$.

Proof. The above map takes $x^j \otimes \partial_i \mod \mathcal{W}_1$ to $-e_i^j \in \mathfrak{gl}\,\mathfrak{d}$. □

The action of $W(\mathfrak{d})$ on H induces a corresponding action of the annihilation algebra $\mathcal{W} = \mathcal{A}(W(\mathfrak{d}))$ on $\mathcal{A}(H) \equiv X$ given by (2.37):

$$(x \otimes a)y = -x(ya)\,, \qquad x, y \in X,\ a \in \mathfrak{d}\,. \tag{3.9}$$

Recall from Section 2.1 that we have a ring isomorphism $\varphi\colon X \xrightarrow{\sim} \mathcal{O}_N$, which is compatible with the corresponding filtrations and topologies (see (2.20), (2.21)). Since $\mathfrak{d}$ acts on X by continuous derivations, the Lie algebra $\mathcal{W}$ acts on X by continuous derivations. Hence, (3.9) defines a Lie algebra homomorphism

$$\varphi\colon \mathcal{W} \to W_N \quad \text{such that} \quad \varphi(Ay) = \varphi(A)\varphi(y) \ \text{ for } \ A \in \mathcal{W}\,,\ y \in X\,, \tag{3.10}$$

where W_N is the Lie algebra of continuous derivations of $\mathcal{O}_N$.

There is a natural filtration of W_N given by

$$\mathrm{F}_p W_N = \{D \in W_N \mid D(\mathrm{F}_n \mathcal{O}_N) \subset \mathrm{F}_{n+p} \mathcal{O}_N \ \text{ for all } n\}, \qquad p \geq -1\,. \tag{3.11}$$

Explicitly, by (2.21), we have

$$\mathrm{F}_p W_N = \left\{ \sum_{i=1}^{N} f_i \frac{\partial}{\partial t^i} \;\Big|\; f_i \in \mathrm{F}_p \mathcal{O}_N \right\} . \tag{3.12}$$

The filtration (3.11) has the following important property for $D \in W_N$:

$$[D, \mathrm{F}_p W_N] \subset \mathrm{F}_{p+n} W_N \iff D \in \mathrm{F}_n W_N . \tag{3.13}$$

Proposition 3.1. (i) *We have*:

$$\varphi(x \otimes a) = \varphi(x)\varphi(1 \otimes a) , \qquad x \in X , \ a \in \mathfrak{d} ,$$
$$\varphi(1 \otimes \partial_i) = -\frac{\partial}{\partial t^i} \mod \mathrm{F}_0 W_N , \qquad i = 1, \dots, N .$$

(ii) *The homomorphism* (3.10) *is an isomorphism and* $\varphi(\mathcal{W}_p) = \mathrm{F}_p W_N$ *for all* $p \geq -1$.

Proof. Part (i) follows from (3.10) and Lemma 2.2. Part (ii) follows from (i) and (3.8), (3.12). □

The adjoint action of the Euler vector field

$$E := \sum_{i=1}^{N} t^i \frac{\partial}{\partial t^i} \in \mathrm{F}_0 W_N \tag{3.14}$$

decomposes W_N as a direct product of eigenspaces $W_{N;j}$ ($j \geq -1$), on which the action of E is multiplication by j. One clearly has:

$$\mathrm{F}_p W_N = \prod_{j \geq p} W_{N;j} , \quad \mathrm{F}_p W_N / \mathrm{F}_{p+1} W_N \simeq W_{N;p} . \tag{3.15}$$

Notice that $W_{N;0} = \ker(\operatorname{ad} E)$ is a Lie algebra isomorphic to $\mathfrak{gl}_N$ and each space $W_{N;p}$ is a module over $W_{N;0}$.

Definition 3.1. The preimage $\mathcal{E} = \varphi^{-1}(E) \in \mathcal{W}_0$ of the Euler vector field (3.14) under the isomorphism (3.10) will be called the *Euler element* of $\mathcal{W}$.

By Proposition 3.1 and Corollary 3.1, we have:

$$\mathcal{E} = -\sum_{i=1}^{N} x^i \otimes \partial_i \mod \mathcal{W}_1 , \quad \text{i.e.,} \quad \mathcal{E} \bmod \mathcal{W}_1 = \mathrm{Id} \in \mathfrak{gl}\,\mathfrak{d} \simeq \mathcal{W}_0/\mathcal{W}_1 . \tag{3.16}$$

3.3. **The Normalizer $\mathcal{N}_{\mathcal{W}}$.** In this subsection, we study the normalizer of $\mathcal{W}_p$ ($p \geq 0$) in the extended annihilation algebra $\widetilde{\mathcal{W}}$. These results will be used later in our classification of finite irreducible $W(\mathfrak{d})$-modules.

We denote by ad the adjoint action of $\mathfrak{d}$ on itself (or on $H = U(\mathfrak{d})$), and by coad the coadjoint action of $\mathfrak{d}$ on $X = H^*$. For $\partial \in \mathfrak{d}$, we will also consider $\operatorname{ad}\partial$ as an element of $\mathfrak{gl}\,\mathfrak{d}$. Note that, by (2.8), (2.10), we have

$$(\operatorname{coad}\partial)x = \partial x - x\partial , \qquad \partial \in \mathfrak{d} , \ x \in X . \tag{3.17}$$

Since $\operatorname{ad}\partial$ preserves the filtration (2.16) of H, it follows that $\operatorname{coad}\partial$ preserves the filtration $\{\mathrm{F}_p X\}$ of X.

Lemma 3.2. (i) *For* $\partial, a \in \mathfrak{d}$ *and* $x \in X$*, the following formula holds in* $\widetilde{\mathcal{W}}$:

$$[\partial + 1 \otimes \partial, x \otimes a] = (\operatorname{coad}\partial)x \otimes a + x \otimes [\partial, a] .$$

In particular, the adjoint action of $\partial + 1 \otimes \partial \in \widetilde{\mathcal{W}}$ *on* $\mathcal{W} \subset \widetilde{\mathcal{W}}$ *preserves the filtration* $\{\mathcal{W}_p\}$.

(ii) *The adjoint action of* $\partial + 1 \otimes \partial$ *on* $\mathcal{W}/\mathcal{W}_0$ *coincides with the standard action of* $\operatorname{ad}\partial \in \mathfrak{gl}\,\mathfrak{d}$ *on* $\mathfrak{d} \simeq \mathcal{W}/\mathcal{W}_0$.

Proof. Part (i) follows from (3.6)–(3.8), (3.17), and the above observation that $\operatorname{coad}\partial$ preserves the filtration $\{\mathrm{F}_p X\}$ of X. Part (ii) is obvious from (i). □

It is well known that all derivations of W_N are inner. Since $\mathcal{W} \simeq W_N$ and $\mathfrak{d}$ acts on $\mathcal{W}$ by derivations (see (3.7)), there is an injective Lie algebra homomorphism

$$\gamma\colon \mathfrak{d} \hookrightarrow \mathcal{W} \quad \text{such that} \quad [\partial, A] = [\gamma(\partial), A]\,, \qquad \partial \in \mathfrak{d} \subset \widetilde{\mathcal{W}}\,,\ A \in \mathcal{W} \subset \widetilde{\mathcal{W}}\,. \tag{3.18}$$

Definition 3.2. For $\partial \in \mathfrak{d}$, let $\widetilde{\partial} = \partial - \gamma(\partial) \in \widetilde{\mathcal{W}}$, where γ is given by (3.18). Let $\widetilde{\mathfrak{d}} = (\mathrm{id} - \gamma)(\mathfrak{d}) \subset \widetilde{\mathcal{W}}$ and $\mathcal{N}_{\mathcal{W}} = \widetilde{\mathfrak{d}} + \mathcal{W}_0 \subset \widetilde{\mathcal{W}}$.

Proposition 3.2. (i) *The space $\widetilde{\mathfrak{d}}$ is a subalgebra of $\widetilde{\mathcal{W}}$ centralizing $\mathcal{W}$, i.e., $[\widetilde{\mathfrak{d}}, \mathcal{W}] = \{0\}$. The map $\partial \mapsto \widetilde{\partial}$ is a Lie algebra isomorphism from $\mathfrak{d}$ to $\widetilde{\mathfrak{d}}$.*

(ii) *The space $\mathcal{N}_{\mathcal{W}}$ is a subalgebra of $\widetilde{\mathcal{W}}$, and it decomposes as a direct sum of Lie algebras, $\mathcal{N}_{\mathcal{W}} = \widetilde{\mathfrak{d}} \oplus \mathcal{W}_0$.*

Proof. It follows from (3.18) that $[\widetilde{\partial}, A] = 0$ for all $\partial \in \mathfrak{d}$, $A \in \mathcal{W}$. Then for $\partial, \partial' \in \mathfrak{d}$, we have

$$[\partial, \partial'] = [\widetilde{\partial} + \gamma(\partial), \widetilde{\partial}' + \gamma(\partial')] = [\widetilde{\partial}, \widetilde{\partial}'] + [\gamma(\partial), \gamma(\partial')]\,,$$

which implies $[\widetilde{\partial}, \widetilde{\partial}'] = \widetilde{[\partial, \partial']}$ since γ is a Lie algebra homomorphism. This proves (i). Part (ii) follows from (i) and Definition 3.2. □

Lemma 3.3. *For every $\partial \in \mathfrak{d}$, the element $\partial + 1 \otimes \partial - \widetilde{\partial} \in \widetilde{\mathcal{W}}$ belongs to $\mathcal{W}_0$. Its image in $\mathcal{W}_0/\mathcal{W}_1$ coincides with $\operatorname{ad}\partial \in \mathfrak{gl}\,\mathfrak{d} \simeq \mathcal{W}_0/\mathcal{W}_1$.*

Proof. First note that $\partial + 1 \otimes \partial - \widetilde{\partial} = \gamma(\partial) + 1 \otimes \partial$ belongs to $\mathcal{W}$. By (3.18) and Lemma 3.2(i), the adjoint action of this element on $\mathcal{W}$ preserves the filtration $\{\mathcal{W}_p\}$. Therefore, by (3.13), $\gamma(\partial) + 1 \otimes \partial$ belongs to $\mathcal{W}_0$. By (3.18) and Lemma 3.2(ii), its image in $\mathcal{W}_0/\mathcal{W}_1$ coincides with $\operatorname{ad}\partial$. □

Proposition 3.3. *For every $p \geq 0$, the normalizer of $\mathcal{W}_p$ in the extended annihilation algebra $\widetilde{\mathcal{W}}$ is equal to $\mathcal{N}_{\mathcal{W}}$. In particular, it is independent of p. There is a decomposition as a direct sum of subspaces, $\widetilde{\mathcal{W}} = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{W}}$.*

Proof. First, to show that $\widetilde{\mathcal{W}} = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{W}}$, we have to check that $\widetilde{\mathcal{W}} = \mathfrak{d} \oplus \widetilde{\mathfrak{d}} \oplus \mathcal{W}_0$ is a direct sum of subspaces. This follows from Definition 3.2, Lemma 3.3 and the fact that $\widetilde{\mathcal{W}} = \mathfrak{d} \oplus \mathcal{W}$, $\mathcal{W} = (\mathbf{k} \otimes \mathfrak{d}) \oplus \mathcal{W}_0$ as vector spaces.

Next, it is clear that $\mathcal{N}_{\mathcal{W}}$ normalizes $\mathcal{W}_p$, because $[\widetilde{\mathfrak{d}}, \mathcal{W}_p] = \{0\}$ and $[\mathcal{W}_0, \mathcal{W}_p] \subset \mathcal{W}_p$. Assume that an element $\partial \in \mathfrak{d}$ normalizes $\mathcal{W}_p$. By (3.7), we obtain that in this case $\partial(\mathrm{F}_p X) \subset \mathrm{F}_p X$. However, one can deduce from Lemma 2.2 that $\partial(\mathrm{F}_p X) = \mathrm{F}_{p-1} X$, which is strictly larger than $\mathrm{F}_p X$. This contradiction shows that the normalizer of $\mathcal{W}_p$ is equal to $\mathcal{N}_{\mathcal{W}}$. □

In order to understand the irreducible representations of $\mathcal{N}_{\mathcal{W}}$, we need the following lemma, which appeared (in the more difficult super case) in [CK, Erratum].

Lemma 3.4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{g}_0 \subset \mathfrak{g}$ be either a simple Lie algebra or a 1-dimensional Lie algebra. Let I be a subspace of the radical of $\mathfrak{g}$, stabilized by $\operatorname{ad}\mathfrak{g}_0$ and having the property that $[\mathfrak{g}_0, a] = 0$ for $a \in I$ implies $a = 0$. Then I acts trivially on any irreducible finite-dimensional $\mathfrak{g}$-module V.*

Proof. By Cartan–Jacobson's Theorem (see, e.g., [Se, Theorem VI.5.1]), every $a \in \operatorname{Rad}\mathfrak{g}$ acts by scalar multiplication on V. Let $J = \{a \in I \mid a(V) = 0\}$. Then $[\mathfrak{g}_0, I] \subset J$.

Now, if $\mathfrak{g}_0$ is simple, then J is a $\mathfrak{g}_0$-submodule of I and, by complete reducibility, $I = J \oplus J^{\perp}$ as $\mathfrak{g}_0$-modules for some complement $J^{\perp}$. Hence, $[\mathfrak{g}_0, J^{\perp}] = 0$, so $J^{\perp} = 0$ and $I = J$.

If instead $\mathfrak{g}_0 = \mathbf{k}e$ is 1-dimensional, then $[e, I] \subset J$. If $J \neq I$, then $\operatorname{ad}e\colon I \to J$ is not injective, which is a contradiction. We conclude that $J = I$. □

An $\mathcal{N}_\mathcal{W}$-module V will be called *conformal* if it is conformal as a module over the subalgebra $\mathcal{W}_0 \subset \mathcal{N}_\mathcal{W}$, i.e., if every vector $v \in V$ is killed by some $\mathcal{W}_p$.

Proposition 3.4. *The subalgebra $\mathcal{W}_1 \subset \mathcal{N}_\mathcal{W}$ acts trivially on any irreducible finite-dimensional conformal $\mathcal{N}_\mathcal{W}$-module. Irreducible finite-dimensional conformal $\mathcal{N}_\mathcal{W}$-modules are in one-to-one correspondence with irreducible finite-dimensional modules over the Lie algebra $\mathcal{N}_\mathcal{W}/\mathcal{W}_1 \simeq \mathfrak{d} \oplus \mathfrak{gl}_\mathfrak{d}$.*

Proof. A finite-dimensional vector space V is a conformal $\mathcal{N}_\mathcal{W}$-module iff it is an $\mathcal{N}_\mathcal{W}$-module on which $\mathcal{W}_p$ acts trivially for some $p \geq 0$, i.e., iff it is a module over the finite-dimensional Lie algebra $\mathfrak{g} = \mathcal{N}_\mathcal{W}/\mathcal{W}_p = \widetilde{\mathfrak{d}} \oplus (\mathcal{W}_0/\mathcal{W}_p)$.

We will apply Lemma 3.4 for $I = \mathcal{W}_1/\mathcal{W}_p$ and $\mathfrak{g}_0 = \mathbf{k}\mathcal{E} \mod \mathcal{W}_p \subset \mathcal{W}_0/\mathcal{W}_p$, where $\mathcal{E} \in \mathcal{W}_0$ is the Euler element (see Definition 3.1). Note that $I \subset \operatorname{Rad}\mathfrak{g}$ and $[\mathcal{E}, I] \subset I$, because $\operatorname{ad} E$ is injective on $\operatorname{F}_1 W_N / \operatorname{F}_p W_N = \prod_{j=1}^{p-1} W_{N;j}$ (see (3.15)). We conclude that I acts trivially on any finite-dimensional conformal $\mathcal{N}_\mathcal{W}$-module. Hence, we can take $p = 1$. Then $\mathfrak{g} = \widetilde{\mathfrak{d}} \oplus (\mathcal{W}_0/\mathcal{W}_1) \simeq \mathfrak{d} \oplus \mathfrak{gl}_\mathfrak{d}$, since $\widetilde{\mathfrak{d}} \simeq \mathfrak{d}$ and $\mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{gl}_\mathfrak{d}$. □

3.4. **Annihilation Algebra of $S(\mathfrak{d}, \chi)$.** Assume that $N = \dim \mathfrak{d} > 2$. In this subsection, we study the annihilation algebra $\mathcal{S} = \mathcal{A}(S(\mathfrak{d}, \chi)) := X \otimes_H S(\mathfrak{d}, \chi)$ of the Lie pseudoalgebra $S(\mathfrak{d}, \chi)$ defined in Section 3.1. Our treatment here is more detailed than in [BDK, Section 8.4].

We choose

$$L_0 = \operatorname{span}_\mathbf{k}\{s_{ab} \mid a, b \in \mathfrak{d}\} \subset S(\mathfrak{d}, \chi) \tag{3.19}$$

as a subspace such that $S(\mathfrak{d}, \chi) = HL_0$, where the elements s_{ab} are given by (3.5). We obtain a decreasing filtration of $\mathcal{S}$:

$$\mathcal{S}_p = \operatorname{F}_{p+1}\mathcal{S} = \operatorname{F}_{p+1} X \otimes_H L_0\,, \qquad p \geq -2\,, \tag{3.20}$$

satisfying $\mathcal{S}_{-2} = \mathcal{S}$ and (2.39) for $\ell = 1$. Then $[\mathcal{S}_n, \mathcal{S}_p] \subset \mathcal{S}_{n+p}$ for all n, p.

The canonical injection of the subalgebra $S(\mathfrak{d}, \chi)$ into $W(\mathfrak{d})$ induces a Lie algebra homomorphism $\iota\colon \mathcal{S} \to \mathcal{W}$. Explicitly, we have:

$$\begin{aligned} &\iota(x \otimes_H s) = \sum xh_i \otimes \partial_i \in \mathcal{W} \equiv X \otimes \mathfrak{d} \\ &\qquad \text{for} \quad x \in X\,,\ s = \sum h_i \otimes \partial_i \in S(\mathfrak{d}, \chi) \subset W(\mathfrak{d}) = H \otimes \mathfrak{d}\,. \end{aligned} \tag{3.21}$$

Here, as before, we identify $\mathcal{W} = X \otimes_H W(\mathfrak{d})$ with $X \otimes \mathfrak{d}$.

We define a map $\operatorname{div}^\chi\colon \mathcal{W} \to X$ by the formula (cf. (3.3)):

$$\operatorname{div}^\chi\Big(\sum y_i \otimes \partial_i\Big) = \sum y_i(\partial_i + \chi(\partial_i))\,. \tag{3.22}$$

It is easy to see that

$$\operatorname{div}^\chi[A, B] = A(\operatorname{div}^\chi B) - B(\operatorname{div}^\chi A)\,, \qquad A, B \in \mathcal{W}\,, \tag{3.23}$$

where the action of $\mathcal{W}$ on X is given by (3.9). This implies that

$$\overline{\mathcal{S}} := \{A \in \mathcal{W} \mid \operatorname{div}^\chi A = 0\} \tag{3.24}$$

is a Lie subalgebra of $\mathcal{W}$. It was shown in [BDK, Section 8.4] that $\overline{\mathcal{S}}$ is isomorphic to the Lie algebra of divergence-zero vector fields

$$S_N := \left\{ \sum_{i=1}^N f_i \frac{\partial}{\partial t^i} \in W_N \;\Big|\; \sum_{i=1}^N \frac{\partial f_i}{\partial t^i} = 0 \right\}. \tag{3.25}$$

Lemma 3.5. *If $N = \dim \mathfrak{d} > 2$, the map* (3.21) *is an embedding of Lie algebras $\iota\colon \mathcal{S} \hookrightarrow \overline{\mathcal{S}}$.*

Proof. It follows from (3.3), (3.21) and (3.22) that

$$x(\mathrm{div}^\chi\, s) = \mathrm{div}^\chi\, \iota(x \otimes_H s)\,, \qquad x \in X\,,\ s \in W(\mathfrak{d})\,.$$

Therefore, $\iota(\mathcal{S})$ is contained in $\overline{\mathcal{S}}$. Next, note that for $N > 2$, $\mathcal{S}$ is isomorphic to S_N by [BDK, Theorem 8.2(i)]. It is well known that the Lie algebra S_N is simple; hence, $\mathcal{S}$ is simple. Since ι is a nonzero homomorphism, it must be injective. □

Remark 3.2. When $N = \dim \mathfrak{d} = 2$, the Lie algebra $\mathcal{S}$ is isomorphic to P_2, which is an extension of $S_2 = H_2$ by a 1-dimensional center (cf. Remark 3.1). In this case, the homomorphism (3.21) has a 1-dimensional kernel.

We will prove in Proposition 3.5 below that, in fact, $\iota(\mathcal{S}) = \overline{\mathcal{S}}$. Recall that we have a Lie algebra isomorphism $\varphi\colon \mathcal{W} \xrightarrow{\sim} W_N$, given by (3.10). However, although $\overline{\mathcal{S}} \simeq S_N \subset W_N$, the image $\varphi(\overline{\mathcal{S}}) \subset W_N$ is not equal to S_N in general. Instead, we will show that the images of $\varphi(\overline{\mathcal{S}})$ and S_N coincide in the associated graded algebra of W_N (see Proposition 3.6 below).

Lemma 3.6. *For every $p \geq -1$, we have*

$$\varphi(\overline{\mathcal{S}} \cap \mathrm{F}_p\,\mathcal{W}) \subset (S_N \cap \mathrm{F}_p\, W_N) + \mathrm{F}_{p+1}\, W_N\,.$$

Proof. Take an element $A = \sum y_i \otimes \partial_i \in \mathrm{F}_p\,\mathcal{W}$; then each $y_i \in \mathrm{F}_p\, X$. By Proposition 3.1, we have $\varphi(A) = \sum f_i\, \varphi(1 \otimes \partial_i)$, where $f_i = \varphi(y_i) \in \mathrm{F}_p\,\mathcal{O}_N$. Since $\varphi(1 \otimes \partial_i) = -\partial/\partial t^i \mod \mathrm{F}_0\, W_N$, we have $\varphi(A) = -\sum f_i\, \partial/\partial t^i \mod \mathrm{F}_{p+1}\, W_N$. It follows from (3.22) and Lemma 2.2 that $\varphi(\mathrm{div}^\chi\, A) = -\sum \partial f_i/\partial t^i \mod \mathrm{F}_p\,\mathcal{O}_N$. If $A \in \overline{\mathcal{S}} \cap \mathrm{F}_p\,\mathcal{W}$, then $\sum \partial f_i/\partial t^i = 0 \mod \mathrm{F}_p\,\mathcal{O}_N$. Then there exist elements $\hat{f}_i \in \mathrm{F}_p\,\mathcal{O}_N$ such that $\hat{f}_i = f_i \mod \mathrm{F}_{p+1}\,\mathcal{O}_N$ and $\sum \partial \hat{f}_i/\partial t^i = 0$. This means that $\hat{A} := -\sum \hat{f}_i\, \partial/\partial t^i \in S_N \cap \mathrm{F}_p\, W_N$ and $\varphi(A) = \hat{A} \mod \mathrm{F}_{p+1}\, W_N$. □

Consider the associated graded of $\mathcal{W}$,

$$\mathrm{gr}\,\mathcal{W} := \bigoplus_{p=-1}^{\infty} \mathrm{gr}_p\,\mathcal{W}\,, \qquad \mathrm{gr}_p\,\mathcal{W} := \mathrm{F}_p\,\mathcal{W}/\,\mathrm{F}_{p+1}\,\mathcal{W}\,. \tag{3.26}$$

Note that, by (3.8), we have $\mathrm{gr}_p\,\mathcal{W} = (\mathrm{gr}_p\, X) \otimes \mathfrak{d}$. Similarly, we have $\mathrm{gr}_p\, W_N = \sum_{i=1}^N (\mathrm{gr}_p\,\mathcal{O}_N)\, \partial/\partial t^i$. The maps $\varphi\colon X \to \mathcal{O}_N$ and $\varphi\colon \mathcal{W} \to W_N$ (see (2.20), (3.10)) preserve the corresponding filtrations and induce maps $\mathrm{gr}\,\varphi\colon \mathrm{gr}\, X \to \mathrm{gr}\,\mathcal{O}_N$ and $\mathrm{gr}\,\varphi\colon \mathrm{gr}\,\mathcal{W} \to \mathrm{gr}\, W_N$. Note also that the map $\mathrm{div}^\chi\colon \mathcal{W} \to X$ takes $\mathrm{F}_p\,\mathcal{W}$ to $\mathrm{F}_{p-1}\, X$, and hence induces a map $\mathrm{gr}\,\mathrm{div}^\chi\colon \mathrm{gr}\,\mathcal{W} \to \mathrm{gr}\, X$ of degree -1. The same is true for the map $\mathrm{div}\colon W_N \to \mathcal{O}_N$ given by $\mathrm{div}(\sum f_i\, \partial/\partial t^i) := \sum \partial f_i/\partial t^i$. From the proof of Lemma 3.6 we deduce:

Corollary 3.2. *The above maps satisfy*

$$(\mathrm{gr}\,\varphi)\Big(\sum_{i=1}^N \bar{y}_i \otimes \partial_i\Big) = -\sum_{i=1}^N (\mathrm{gr}\,\varphi)(\bar{y}_i)\, \frac{\partial}{\partial t^i}\,, \qquad \bar{y}_i \in \mathrm{gr}\, X$$

and

$$\mathrm{gr}\,\varphi \circ \mathrm{gr}\,\mathrm{div}^\chi = \mathrm{gr}\,\mathrm{div} \circ \mathrm{gr}\,\varphi\,.$$

The Lie algebra $\mathcal{S}$ has a filtration (3.20), while $\overline{\mathcal{S}} \subset \mathcal{W}$ has one obtained by restricting the filtration (3.8) of $\mathcal{W}$. Using Lemma 3.6, we can prove that ι is compatible with the filtrations.

Proposition 3.5. *Let $\mathcal{S}$ be the annihilation algebra of $S(\mathfrak{d}, \chi)$, and let $\overline{\mathcal{S}} \subset \mathcal{W}$ be defined by* (3.24). *Then for $\dim \mathfrak{d} > 2$, the map* (3.21) *is an isomorphism of Lie algebras $\iota\colon \mathcal{S} \xrightarrow{\sim} \overline{\mathcal{S}}$ such that $\iota(\mathcal{S}_p) = \overline{\mathcal{S}} \cap \mathcal{W}_p$ for all $p \geq -1$.*

Proof. It is clear from definitions that

$$\iota(\mathcal{S}_p) = \mathrm{F}_{p+1} X \otimes_H \mathrm{span}_{\mathbf{k}}\{s_{ab}\} \subset \mathrm{F}_p X \otimes_H (\mathbf{k} \otimes \mathfrak{d}) \equiv \mathrm{F}_p X \otimes \mathfrak{d} = \mathcal{W}_p\,.$$

In addition, $\iota(\mathcal{S}) \subset \overline{\mathcal{S}}$ by Lemma 3.5; hence, $\iota(\mathcal{S}_p) \subset \overline{\mathcal{S}} \cap \mathcal{W}_p$.

Conversely, let $A \in \overline{\mathcal{S}} \cap \mathcal{W}_p$. By Lemma 3.6, we can find $\hat{A} \in S_N \cap \mathrm{F}_p W_N$ such that $\varphi(A) = \hat{A} \mod \mathrm{F}_{p+1} W_N$. Any element of $S_N \cap \mathrm{F}_p W_N$ can be written in the form

$$\hat{A} = \sum_{i,j=1}^{N} \frac{\partial f_{ij}}{\partial t^i}\frac{\partial}{\partial t^j} - \frac{\partial f_{ij}}{\partial t^j}\frac{\partial}{\partial t^i}\,, \qquad f_{ij} \in \mathrm{F}_{p+1}\mathcal{O}_N\,.$$

Now consider the following element of $\mathcal{S}_p$:

$$\tilde{A} := -\sum_{i,j=1}^{N} y_{ij} \otimes_H s_{\partial_i \partial_j}\,, \qquad y_{ij} := \varphi^{-1}(f_{ij}) \in \mathrm{F}_{p+1} X\,.$$

Then we have $\tilde{A} \in \mathcal{S}_p$ and $\iota(\tilde{A}) = A \mod \mathcal{W}_{p+1}$.

Let $A_1 = A - \iota(\tilde{A})$; then $A_1 \in \overline{\mathcal{S}} \cap \mathcal{W}_{p+1}$ and $A - A_1 \in \iota(\mathcal{S}_p)$. By the above argument, we can find an element $\tilde{A}_1 \in \mathcal{S}_{p+1}$ such that $\iota(\tilde{A}_1) = A_1 \mod \mathcal{W}_{p+2}$. Let $A_2 = A_1 - \iota(\tilde{A}_1)$; then $A_2 \in \overline{\mathcal{S}} \cap \mathcal{W}_{p+2}$ and $A_1 - A_2 \in \iota(\mathcal{S}_{p+1})$. Continuing this way, we obtain a sequence of elements $A_n \in \overline{\mathcal{S}} \cap \mathcal{W}_{p+n}$ such that $A_n - A_{n+1} \in \iota(\mathcal{S}_{p+n})$ for all $n \ge 0$, where $A_0 := A$. The sequence A_n converges to 0 in $\mathcal{W}$ and $A - A_n \in \iota(\mathcal{S}_p)$ for all $n \ge 0$; therefore, $A \in \iota(\mathcal{S}_p)$.

This proves that $\iota(\mathcal{S}_p) = \overline{\mathcal{S}} \cap \mathcal{W}_p$. Taking $p = -1$, we get $\iota(\mathcal{S}) \supset \iota(\mathcal{S}_{-1}) = \overline{\mathcal{S}}$, because $\mathcal{W}_{-1} = \mathcal{W} \supset \overline{\mathcal{S}}$. Now Lemma 3.5 implies that ι is an isomorphism. □

Recall that any ring automorphism ψ of $\mathcal{O}_N$ induces a Lie algebra automorphism ψ of $W_N = \mathrm{Der}\,\mathcal{O}_N$ such that $\psi(Ay) = \psi(A)\psi(y)$ for $A \in W_N$, $y \in \mathcal{O}_N$. Any $\psi \in \mathrm{Aut}\,\mathcal{O}_N$ preserves the filtration, because $\mathrm{F}_0\mathcal{O}_N$ is the unique maximal ideal of $\mathcal{O}_N$ and $\mathrm{F}_p\mathcal{O}_N = (\mathrm{F}_0\mathcal{O}_N)^{p+1}$ for $p \ge 0$ (see (2.21)). Then it follows from (3.11) that ψ preserves the filtration $\{\mathrm{F}_p W_N\}$.

Proposition 3.6. *There exists a ring automorphism ψ of $\mathcal{O}_N$ such that the induced Lie algebra automorphism ψ of W_N satisfies $\varphi(\overline{\mathcal{S}}) = \psi(S_N)$ and*

$$(\psi - \mathrm{id})(\mathrm{F}_p W_N) \subset \mathrm{F}_{p+1} W_N\,, \qquad p \ge -1\,. \tag{3.27}$$

Proof. In [BDK, Remark 8.2] the image $\varphi(\overline{\mathcal{S}})$ is described as the Lie algebra of all vector fields annihilating a certain volume form. But any two volume forms are related by a change of variables, i.e., by a ring automorphism of $\mathcal{O}_N$, and the subalgebra S_N corresponds to the standard volume form $\mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^N$. Hence, there exists an automorphism ψ of $\mathcal{O}_N$ such that $\varphi(\overline{\mathcal{S}}) = \psi(S_N)$. Due to Corollary 3.2, we can choose ψ such that

$$\psi(t^i) = t^i \mod \mathrm{F}_1\mathcal{O}_N\,, \qquad i = 1, \dots, N\,,$$

i.e., such that $\mathrm{gr}\,\psi = \mathrm{id}$. Since the latter is equivalent to (3.27), this completes the proof. □

Corollary 3.3. *The Lie algebra isomorphism $\psi^{-1}\varphi\iota\colon \mathcal{S} \xrightarrow{\sim} S_N$ maps $\mathcal{S}_p$ onto $S_N \cap \mathrm{F}_p W_N$ for all $p \ge -1$. In particular, $\mathcal{S}_{-2} = \mathcal{S}_{-1} = \mathcal{S}$.*

Proof. By Proposition 3.5, $\iota(\mathcal{S}_p) = \overline{\mathcal{S}} \cap \mathcal{W}_p$. Then under the isomorphism $\varphi\colon \mathcal{W} \to W_N$, we have $\varphi\iota(\mathcal{S}_p) = \varphi(\overline{\mathcal{S}}) \cap \mathrm{F}_p W_N$. But, by Proposition 3.6, $\varphi(\overline{\mathcal{S}}) = \psi(S_N)$ and $\psi(\mathrm{F}_p W_N) = \mathrm{F}_p W_N$; hence, $\varphi\iota(\mathcal{S}_p) = \psi(S_N \cap \mathrm{F}_p W_N)$. □

Recall that $W_{N;p}$ is the subspace of W_N on which the adjoint action of the Euler vector field (3.14) is multiplication by p. We let $S_{N;p} = S_N \cap W_{N;p}$. Since S_N is preserved by $\mathrm{ad}\,E$, it admits a decomposition similar to (3.15):

$$S_N \cap \mathrm{F}_p W_N = \prod_{j \ge p} S_{N;j}\,, \quad (S_N \cap \mathrm{F}_p W_N)/(S_N \cap \mathrm{F}_{p+1} W_N) \simeq S_{N;p}\,. \tag{3.28}$$

The following facts about the Lie algebra $S_N \subset W_N$ are well known.

Lemma 3.7. (i) *The Lie algebra $S_{N;0}$ is isomorphic to $\mathfrak{sl}_N$.*

(ii) *For every $p \geq -1$, the $S_{N;0}$-module $S_{N;p}$ is isomorphic to the highest component of the $\mathfrak{sl}_N$-module $\mathbf{k}^N \otimes (\mathrm{S}^{p+1}\,\mathbf{k}^N)^*$. In particular, $S_{N;p}$ has no trivial components in its decomposition as a sum of irreducible $\mathfrak{sl}_N$-modules.*

(iii) *The normalizer of S_N in W_N is $S_N \oplus \mathbf{k}E$.*

Definition 3.3. We let $\widehat{E} = \psi(E) \in W_N$ and $\widehat{\mathcal{E}} = \varphi^{-1}(\widehat{E}) \in \mathcal{W}$, where E is the Euler vector field (3.14), φ is from (3.10) and ψ is from Proposition 3.6.

Combining the above results with (3.16), we obtain the following corollary.

Corollary 3.4. (i) *The Lie algebra $\mathcal{S}_0/\mathcal{S}_1$ is isomorphic to $\mathfrak{gl}\,\mathfrak{d}$.*

(ii) *For every $p \geq -1$, the $(\mathcal{S}_0/\mathcal{S}_1)$-module $\mathcal{S}_p/\mathcal{S}_{p+1}$ has no trivial $\mathfrak{gl}\,\mathfrak{d}$-components.*

(iii) *The normalizer of $\overline{\mathcal{S}}$ in $\mathcal{W}$ is $\overline{\mathcal{S}} + \mathbf{k}\widehat{\mathcal{E}}$.*

(iv) *$\widehat{\mathcal{E}}$ belongs to $\mathcal{W}_0$ and its image in $\mathcal{W}_0/\mathcal{W}_1$ coincides with $\mathrm{Id} \in \mathfrak{gl}\,\mathfrak{d} \simeq \mathcal{W}_0/\mathcal{W}_1$.*

3.5. **The Normalizer $\mathcal{N}_\mathcal{S}$.** In this subsection, we study the normalizer of $\mathcal{S}_p$ $(p \geq 0)$ in the extended annihilation algebra $\widetilde{\mathcal{S}} = \mathfrak{d} \ltimes \mathcal{S}$. We will use extensively the results and notation of Sections 3.3 and 3.4, and we will identify $\mathcal{S}$ with the subalgebra $\overline{\mathcal{S}}$ of $\mathcal{W}$ (see Proposition 3.5).

Recall that the filtration $\{\mathcal{S}_p\}$ of $\mathcal{S}$ has the properties: $\mathcal{S}_{-2} = \mathcal{S}_{-1} = \mathcal{S}$ and $[\mathcal{S}_n, \mathcal{S}_p] \subset \mathcal{S}_{n+p}$ for all n, p. In addition, by Corollary 3.4, we have: $\mathcal{W}_0 = \mathcal{S}_0 + \mathbf{k}\widehat{\mathcal{E}} + \mathcal{W}_1$, where the element $\widehat{\mathcal{E}} \in \mathcal{W}_0$ is from Definition 3.3.

Lemma 3.8. *For every $\partial \in \mathfrak{d}$, we have*: $1 \otimes \partial - (\chi(\partial)/N)\,\widehat{\mathcal{E}} \in \mathcal{S} + \mathcal{W}_1$.

Proof. As before, let $\{\partial_i\}_{i=1,\dots,N}$ be a basis of $\mathfrak{d}$, and let $x^i \in X$ be given by (2.19). Denote by c_{ij}^k the structure constants of $\mathfrak{d}$ in the basis $\{\partial_i\}$, and let $\chi_i = \chi(\partial_i)$ for short. Using (3.5), (3.21) and Lemma 2.2, we compute for $i < j$:

$$\begin{aligned}\iota(x^i \otimes_H s_{\partial_i,\partial_j}) &= \chi_i x^i \otimes \partial_j - \chi_j x^i \otimes \partial_i - x^i \otimes [\partial_i, \partial_j] + x^i\partial_i \otimes \partial_j - x^i\partial_j \otimes \partial_i \\ &= \chi_i x^i \otimes \partial_j - \chi_j x^i \otimes \partial_i - \sum_k c_{ij}^k x^i \otimes \partial_k - 1 \otimes \partial_j \\ &\qquad + \sum_{k>i} c_{ik}^i x^k \otimes \partial_j - \sum_{k>j} c_{jk}^i x^k \otimes \partial_i \mod \mathcal{W}_1\,.\end{aligned}$$

From here, we see that the element $\iota(x^i \otimes_H s_{\partial_i,\partial_j}) + 1 \otimes \partial_j$ belongs to $\mathcal{W}_0$. Next, using Corollary 3.1, we find that the image of this element in $\mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{gl}\,\mathfrak{d}$ has trace χ_j. Therefore, by Corollary 3.4 (i), (iv),

$$\iota(x^i \otimes_H s_{\partial_i,\partial_j}) + 1 \otimes \partial_j - (\chi_j/N)\,\widehat{\mathcal{E}} \in \mathcal{S}_0 + \mathcal{W}_1\,,$$

which implies $1 \otimes \partial_j - (\chi_j/N)\,\widehat{\mathcal{E}} \in \mathcal{S} + \mathcal{W}_1$. □

Lemma 3.9. *For every $\partial \in \mathfrak{d}$, we have*: $\gamma(\partial) + 1 \otimes \partial - (\operatorname{tr}\operatorname{ad}(\partial)/N)\,\widehat{\mathcal{E}} \in \mathcal{S}_0 + \mathcal{W}_1$, *where γ is from* (3.18).

Proof. By Lemma 3.3, $\gamma(\partial) + 1 \otimes \partial \in \mathcal{W}_0$ and its image in $\mathcal{W}_0/\mathcal{W}_1$ coincides with $\operatorname{ad}\partial \in \mathfrak{gl}\,\mathfrak{d}$. Now the statement follows from Corollary 3.4 (i), (iv). □

Definition 3.4. For $\partial \in \mathfrak{d}$, let

$$\widehat{\gamma}(\partial) = \gamma(\partial) + \big((\chi - \operatorname{tr}\operatorname{ad})(\partial)/N\big)\widehat{\mathcal{E}} \in \mathcal{W}\,,$$

where γ is given by (3.18). Let $\widehat{\partial} = \partial - \widehat{\gamma}(\partial)$, $\widehat{\mathfrak{d}} = (\mathrm{id} - \widehat{\gamma})(\mathfrak{d}) \subset \widetilde{\mathcal{W}}$, and $\mathcal{N}_\mathcal{S} = \widehat{\mathfrak{d}} + \mathcal{S}_0 \subset \widetilde{\mathcal{W}}$.

Note that

$$\widehat{\partial} = \widetilde{\partial} - \big((\chi - \operatorname{tr}\operatorname{ad})(\partial)/N\big)\widehat{\mathcal{E}}\,, \qquad \partial \in \mathfrak{d}\,, \tag{3.29}$$

where $\widetilde{\partial}$ is from Definition 3.2.

Proposition 3.7. (i) *We have* $\widehat{\gamma}(\mathfrak{d}) \subset \mathcal{S}$ *and* $\widehat{\mathfrak{d}} \subset \widetilde{\mathcal{S}}$.

(ii) *The map* $\partial \mapsto \widehat{\partial}$ *is a Lie algebra isomorphism from* $\mathfrak{d}$ *to* $\widehat{\mathfrak{d}}$.

(iii) *The Lie algebra* $\widehat{\mathfrak{d}}$ *normalizes* $\mathcal{S}_p$ *for all* $p \geq -1$.

(iv) *The Lie algebra* $\widehat{\mathfrak{d}}$ *centralizes* $\mathcal{S}_0/\mathcal{S}_1$.

Proof. (i) Combining Lemmas 3.8 and 3.9, we get $\widehat{\gamma}(\partial) \in \mathcal{S} + \mathcal{W}_1$ for all $\partial \in \mathfrak{d}$. On the other hand, we deduce from (3.18) and Corollary 3.4 that $\widehat{\gamma}(\partial)$ normalizes $\mathcal{S}$. Hence, again by Corollary 3.4, $\widehat{\gamma}(\partial) \in \mathcal{S} + \mathbf{k}\widehat{\mathcal{E}}$. However, the intersection $(\mathcal{S} + \mathcal{W}_1) \cap (\mathcal{S} + \mathbf{k}\widehat{\mathcal{E}})$ is equal to $\mathcal{S}$. This shows that $\widehat{\gamma}(\partial) \in \mathcal{S}$.

(ii) Recall from Section 3.3 that $\partial \mapsto \widetilde{\partial}$ is a Lie algebra isomorphism and $\widetilde{\mathfrak{d}} \subset \widetilde{\mathcal{W}}$ centralizes $\mathcal{W}$. Then part (ii) follows from (3.29) and the fact that $\chi - \operatorname{tr}\operatorname{ad}$ is a trace form on $\mathfrak{d}$.

(iii) and (iv) follow from (3.29), Corollary 3.4 and $[\widetilde{\mathfrak{d}}, \mathcal{W}] = 0$. □

It follows from Proposition 3.7 that $\mathcal{N}_S$ is a Lie subalgebra of $\widetilde{\mathcal{S}}$, isomorphic to the semidirect sum $\widehat{\mathfrak{d}} \ltimes \mathcal{S}_0$.

Proposition 3.8. *For every* $p \geq 0$*, the normalizer of* $\mathcal{S}_p$ *in the extended annihilation algebra* $\widetilde{\mathcal{S}}$ *is equal to* $\mathcal{N}_{\mathcal{S}}$*. In particular, it is independent of* p*. There is a decomposition as a direct sum of subspaces,* $\widetilde{\mathcal{S}} = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{S}}$.

Proof. The proof is similar to that of Proposition 3.3. □

An $\mathcal{N}_{\mathcal{S}}$-module V is called *conformal* if it is conformal as a module over the subalgebra $\mathcal{S}_0 \subset \mathcal{N}_{\mathcal{S}}$, i.e., if every vector $v \in V$ is killed by some $\mathcal{S}_p$.

Proposition 3.9. *The subalgebra* $\mathcal{S}_1 \subset \mathcal{N}_{\mathcal{S}}$ *acts trivially on any irreducible finite-dimensional conformal* $\mathcal{N}_{\mathcal{S}}$*-module. Irreducible finite-dimensional conformal* $\mathcal{N}_{\mathcal{S}}$*-modules are in one-to-one correspondence with irreducible finite-dimensional modules over the Lie algebra* $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1 \simeq \mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d}$.

Proof. As in Proposition 3.4, the $\mathcal{N}_{\mathcal{S}}$-action factors via the finite-dimensional Lie algebra $\mathfrak{g} := \mathcal{N}_{\mathcal{S}}/\mathcal{S}_p$ for some $p \geq 1$. Recall that $[\mathcal{S}_i, \mathcal{S}_j] \subset \mathcal{S}_{i+j}$ for all i, j, so that $I := \mathcal{S}_1/\mathcal{S}_p$ is contained in the radical of $\overline{\mathfrak{g}} = \mathcal{S}_0/\mathcal{S}_p \subset \mathfrak{g}$. Moreover, the quotient $\overline{\mathfrak{g}}/I = \mathcal{S}_0/\mathcal{S}_1$ is isomorphic to the simple Lie algebra $\mathfrak{sl}\,\mathfrak{d}$, so that I coincides with the radical of $\overline{\mathfrak{g}}$ and we can locate a subalgebra $\mathfrak{g}_0$ of $\overline{\mathfrak{g}}$ isomorphic to the semisimple quotient $\overline{\mathfrak{g}}/I \simeq \mathfrak{sl}\,\mathfrak{d}$.

Similarly, the space I is contained in the radical of $\mathfrak{g}$, and the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$ preserves it. Moreover, by Corollary 3.4(ii), I has no trivial $\mathfrak{g}_0$-components. We can now apply Lemma 3.4 to deduce that I acts trivially on any irreducible finite-dimensional conformal $\mathcal{N}_{\mathcal{S}}$-module. Therefore, the $\mathcal{N}_{\mathcal{S}}$-action factors via $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1$. By Proposition 3.7(iv), $\widehat{\mathfrak{d}}$ centralizes $\mathcal{S}_0/\mathcal{S}_1$. Hence, $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1$ is isomorphic to a direct sum of Lie algebras $\widehat{\mathfrak{d}} \oplus (\mathcal{S}_0/\mathcal{S}_1) \simeq \mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d}$. □

4. Pseudo Linear Algebra

In this section, we generalize several linear algebra constructions to the pseudoalgebra context. We introduce an important class of $W(\mathfrak{d})$-modules called tensor modules.

4.1. **Pseudolinear Maps.**

The definition of a module over a pseudoalgebra motivates the following definition of a pseudolinear map.

Definition 4.1 ([BDK])**.** Let V and W be two H-modules. An *H-pseudolinear map* from V to W is a $\mathbf{k}$-linear map $\phi\colon V \to (H \otimes H) \otimes_H W$ such that

$$\phi(hv) = ((1 \otimes h) \otimes_H 1)\,\phi(v), \qquad h \in H, v \in V. \tag{4.1}$$

We denote the space of all such ϕ by $\operatorname{Chom}(V, W)$. We will also use the notation $\phi * v \equiv \phi(v)$ for $\phi \in \operatorname{Chom}(V, W)$, $v \in V$. We define a left action of H on $\operatorname{Chom}(V, W)$ by:

$$(h\phi)(v) = ((h \otimes 1) \otimes_H 1)\,\phi(v). \tag{4.2}$$

When $V = W$, we set $\operatorname{Cend} V = \operatorname{Chom}(V, V)$.

Example 4.1. Let A be an H-pseudoalgebra, and let V be an A-module. Then for every $a \in A$ the map $m_a\colon V \to (H \otimes H) \otimes_H V$ defined by $m_a(v) = a * v$ is an H-pseudolinear map. Moreover, we have $h\, m_a = m_{ha}$ for $h \in H$.

Remark 4.1. Given two homomorphisms of left H-modules $\beta\colon V' \to V$ and $\gamma\colon W \to W'$, we define a homomorphism

$$\mathrm{Chom}(\beta,\gamma)\colon\ \mathrm{Chom}(V,W) \to \mathrm{Chom}(V',W') \tag{4.3}$$

by the formula

$$\phi \mapsto ((\mathrm{id}\otimes\mathrm{id})\otimes_H \gamma)\circ\phi\circ\beta\,. \tag{4.4}$$

Then we can view $\mathrm{Chom}(-,-)$ as a bifunctor from the category of left H-modules to itself, contravariant in the first argument and covariant in the second one.

Recall from [BDK, Chapter 10] that when V is a finite H-module, $\mathrm{Cend}\, V$ has a unique structure of an associative pseudoalgebra such that V is a module over it via $\phi * v = \phi(v)$. Denote by $\mathrm{gc}\, V$ the Lie pseudoalgebra obtained from $\mathrm{Cend}\, V$ by the construction of Remark 2.1. Then V is also a module over $\mathrm{gc}\, V$.

Proposition 4.1 ([BDK])**.** *Let L be a Lie pseudoalgebra, and let V be a finite H-module. Then giving a structure of an L-module on V is equivalent to giving a homomorphism of Lie pseudoalgebras from L to $\mathrm{gc}\, V$.*

Proof. If V is a finite L-module, we define a map $\rho\colon L \to \mathrm{gc}\, V$ by $a \mapsto m_a$, where m_a is from Example 4.1. Then ρ is a homomorphism of Lie pseudoalgebras (cf. [BDK, Proposition 10.1]). Conversely, given a homomorphism $\rho\colon L \to \mathrm{gc}\, V$, we define an action of L on V by $a * v = \rho(a) * v$. □

In the case when V is a free H-module of finite rank, one can give an explicit description of $\mathrm{Cend}\, V$, and hence of $\mathrm{gc}\, V$, as follows (see [BDK, Proposition 10.3]). Let $V = H \otimes V_0$, where H acts trivially on V_0 and $\dim V_0 < \infty$. Then $\mathrm{Cend}\, V$ is isomorphic to $H \otimes H \otimes \mathrm{End}\, V_0$, with H acting by left multiplication on the first factor, and with the following pseudoproduct:

$$(f\otimes a\otimes A) * (g\otimes b\otimes B) = (f\otimes ga_{(1)})\otimes_H (1\otimes ba_{(2)}\otimes AB)\,. \tag{4.5}$$

The action of $\mathrm{Cend}\, V$ on $V = H \otimes V_0$ is given by:

$$(f\otimes a\otimes A) * (h\otimes v) = (f\otimes ha)\otimes_H (1\otimes Av)\,. \tag{4.6}$$

The pseudobracket in $\mathrm{gc}\, V$ is given by:

$$\begin{aligned}[(f\otimes a\otimes A) * (g\otimes b\otimes B)] &= (f\otimes ga_{(1)})\otimes_H (1\otimes ba_{(2)}\otimes AB)\\ &\quad - (fb_{(1)}\otimes g)\otimes_H (1\otimes ab_{(2)}\otimes BA)\,.\end{aligned} \tag{4.7}$$

The action of $\mathrm{gc}\, V$ on V is also given by (4.6).

Remark 4.2. Let L be a Lie pseudoalgebra. Let $V = H \otimes V_0$ be a finite L-module, which is free as an H-module. For all $a \in L$, $v \in V_0$ we can write

$$a * (1\otimes v) = \sum (f_i\otimes g_i)\otimes_H (1\otimes A_i v)\,, \tag{4.8}$$

where $f_i, g_i \in H$, $A_i \in \mathrm{End}\, V_0$. Then the homomorphism $L \to \mathrm{gc}\, V$ is given by $a \mapsto \sum f_i \otimes g_i \otimes A_i$. This follows from (4.6) and the proof of Proposition 4.1.

Example 4.2. (i) The action (3.2) of $W(\mathfrak{d})$ on H gives an embedding of Lie pseudoalgebras $W(\mathfrak{d}) \hookrightarrow \mathrm{gc}\, H = H \otimes H$, $f \otimes a \mapsto -f \otimes a$ ($f \in H$, $a \in \mathfrak{d} \subset H$).

(ii) Consider the semidirect sum $H \rtimes W(\mathfrak{d})$, where H is regarded as a commutative Lie pseudoalgebra and $W(\mathfrak{d})$ acts on H via (3.2). Then we have an embedding $H \rtimes W(\mathfrak{d}) \hookrightarrow \mathrm{gc}\, H$ given by $g + f \otimes a \mapsto g \otimes 1 - f \otimes a$ for $f, g \in H$, $a \in \mathfrak{d} \subset H$.

Remark 4.3. For any Lie algebra $\mathfrak{g}$, we have a semidirect sum $\operatorname{Cur}\mathfrak{g} \rtimes W(\mathfrak{d})$, where $\operatorname{Cur}\mathfrak{g}$ is defined in Example 2.2 and $W(\mathfrak{d})$ acts on $\operatorname{Cur}\mathfrak{g} = H \otimes \mathfrak{g}$ via

$$(f \otimes a) * (g \otimes B) = -(f \otimes ga) \otimes_H (1 \otimes B)\,, \qquad f, g \in H\,,\ a \in \mathfrak{d}\,,\ B \in \mathfrak{g}\,.$$

Let V_0 be a finite-dimensional $\mathfrak{g}$-module, and let ρ be the corresponding homomorphism $\mathfrak{g} \to \mathfrak{gl}\, V_0$. Then we have a homomorphism of Lie pseudoalgebras $\operatorname{Cur}\mathfrak{g} \rtimes W(\mathfrak{d}) \to \operatorname{gc}(H \otimes V_0)$, given by

$$g \otimes B + f \otimes a \mapsto g \otimes 1 \otimes \rho(B) - f \otimes a \otimes \operatorname{Id}\,. \tag{4.9}$$

4.2. **Duals and Twistings of Representations.** Let L be a Lie H-pseudoalgebra, and let Π be any finite-dimensional $\mathfrak{d}$-module. We consider Π as an L-module equipped with the trivial action of L and with the action of $H = U(\mathfrak{d})$ induced from the action of $\mathfrak{d}$. In particular, $\mathbf{k}$ has the trivial action of both L and H.

Lemma 4.1 ([BDK])**.** *Let L be a Lie pseudoalgebra, and let V, W be finite L-modules. Then the formula* ($a \in L$, $v \in V$, $\phi \in \operatorname{Chom}(V, W)$)

$$(a * \phi) * v = a * (\phi * v) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,(\phi * (a * v)) \tag{4.10}$$

provides $\operatorname{Chom}(V, W)$ *with the structure of an L-module.*

Note that if $\beta\colon V' \to V$ and $\gamma\colon W \to W'$ are homomorphisms of L-modules, the map (4.3) is a homomorphism of L-modules.

Definition 4.2. (i) For any finite L-module V, the L-module $D(V) = \operatorname{Chom}(V, \mathbf{k})$ is called the *dual* of V. If $\beta\colon V' \to V$ is a homomorphism of L-modules, we define a homomorphism $D(\beta)\colon D(V) \to D(V')$ as $D(\beta) = \operatorname{Chom}(\beta, \mathrm{id})$ (see Remark 4.1). Then D is a contravariant functor from the category of finite L-modules to itself.

(ii) For any finite L-module V and any finite-dimensional $\mathfrak{d}$-module Π, the L-module $T_\Pi(V) = \operatorname{Chom}(D(V), \Pi)$ is called the *twisting* of V by Π. If $\beta\colon V \to V'$ is a homomorphism of L-modules, we define a homomorphism $T_\Pi(\beta)\colon T_\Pi(V) \to T_\Pi(V')$ as $T_\Pi(\beta) = \operatorname{Chom}(D(\beta), \mathrm{id})$. Then T_Π is a covariant functor from the category of finite L-modules to itself.

Now let V be a free H-module of finite rank, $V = H \otimes V_0$, where H acts by left multiplication on the first factor and $\dim V_0 < \infty$. Then for any H-module W we can identify $\operatorname{Chom}(V, W)$ with $H \otimes (W \otimes V_0^*)$, where H acts on the first factor. Explicitly, by Lemma 2.1(ii), for any fixed $v \in V_0$, we can write

$$\phi(1 \otimes v) = \sum (h_i \otimes 1) \otimes_H w_i\,, \tag{4.11}$$

where $h_i \in H$, $w_i \in W$. Then ϕ corresponds to the $\mathbf{k}$-linear map $V_0 \to H \otimes W$, $v \mapsto \sum h_i \otimes w_i$.

In particular, we have isomorphisms $D(V) \simeq H \otimes V_0^*$ and $T_\Pi(V) \simeq H \otimes (\Pi \otimes V_0)$ as H-modules. Now we will describe the action of L on them.

Proposition 4.2. *Let $V = H \otimes V_0$ be a finite L-module, which is free as an H-module. Let $\{v_i\}$ be a $\mathbf{k}$-basis of V_0, and let $\{\psi_i\}$ be the dual basis of V_0^*, so that $\psi_i(v_j) = \delta_{ij}$. For a fixed $a \in L$, write*

$$a * (1 \otimes v_i) = \sum_j (f_{ij} \otimes g_{ij}) \otimes_H (1 \otimes v_j) \tag{4.12}$$

where $f_{ij}, g_{ij} \in H$. Then the action of L on $D(V) \simeq H \otimes V_0^*$ is given by

$$a * (1 \otimes \psi_k) = -\sum_j \left(f_{jk} g_{jk_{(-1)}} \otimes g_{jk_{(-2)}}\right) \otimes_H \left(1 \otimes \psi_j\right). \tag{4.13}$$

The action of L on $T_\Pi(V) \simeq H \otimes (\Pi \otimes V_0)$ is given by

$$a * (1 \otimes u \otimes v_i) = \sum_j \left(f_{ij} \otimes g_{ij_{(1)}}\right) \otimes_H \left(1 \otimes g_{ij_{(-2)}} u \otimes v_j\right). \tag{4.14}$$

Both (4.13) and (4.14) can be easily derived from the following lemma.

Lemma 4.2. *Under the assumptions of Proposition* 4.2*, the action of* L *on* $\mathrm{Chom}(V,\Pi) \simeq H \otimes (\Pi \otimes V_0^*)$ *is given by*

$$a * (1 \otimes u \otimes \psi_k) = -\sum_j \left(f_{jk} g_{jk_{(-1)}} \otimes g_{jk_{(-2)}}\right) \otimes_H \left(1 \otimes g_{jk_{(3)}} u \otimes \psi_j\right). \tag{4.15}$$

Proof. First, note that by (4.11), we have

$$(1 \otimes u \otimes \psi_k) * (1 \otimes v_i) = (1 \otimes 1) \otimes_H \delta_{ki} u\,. \tag{4.16}$$

We will compute $(a*(1\otimes u\otimes\psi_k))*(1\otimes v_i)$ using (4.10). The first term in the right-hand side of (4.10) vanishes because the action of L on Π is trivial. By (2.25), (4.12) and (4.16), the second term is equal to

$$\begin{aligned} -((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id}) \left((1 \otimes u \otimes \psi_k) * (a * (1 \otimes v_i))\right) \\ = -\sum_j (\sigma \otimes \mathrm{id})\,(1 \otimes f_{ij} \otimes g_{ij}) \otimes_H \delta_{kj} u \\ = -(f_{ik} \otimes 1 \otimes g_{ik}) \otimes_H u \\ = -(f_{ik} g_{ik_{(-1)}} \otimes g_{ik_{(-2)}} \otimes 1) \otimes_H g_{jk_{(3)}} u\,, \end{aligned}$$

where we used (2.7) in the last equality. We will obtain the same result if we apply the right-hand side of (4.15) to $1 \otimes v_i$ and use (2.24) and (4.16). □

Example 4.3. Consider H as a $W(\mathfrak{d})$-module via (3.2). Then $T_\Pi(H) = H \otimes \Pi$ with the following action of $W(\mathfrak{d})$:

$$(1 \otimes a) * (1 \otimes u) = (1 \otimes 1) \otimes_H (1 \otimes au) - (1 \otimes a) \otimes_H (1 \otimes u) \tag{4.17}$$

for $a \in \mathfrak{d}$, $u \in \Pi$.

Remark 4.4. There is an embedding of Lie pseudoalgebras

$$W(\mathfrak{d}) \hookrightarrow \mathrm{Cur}\,\mathfrak{d} \rtimes W(\mathfrak{d})\,, \quad 1 \otimes a \mapsto 1 \otimes a + 1 \otimes a\,, \tag{4.18}$$

where the first summand is in $\mathrm{Cur}\,\mathfrak{d} = H \otimes \mathfrak{d}$, and the second one is in $W(\mathfrak{d}) = H \otimes \mathfrak{d}$. By Remark 4.3, the representation of $\mathfrak{d}$ on Π gives rise to a homomorphism $\mathrm{Cur}\,\mathfrak{d} \rtimes W(\mathfrak{d}) \to \mathrm{gc}(H \otimes \Pi)$. Composing (4.9) with (4.18), we obtain a homomorphism $W(\mathfrak{d}) \to \mathrm{gc}(H \otimes \Pi)$, which corresponds to the $W(\mathfrak{d})$-module $T_\Pi(H)$ from Example 4.3 (see Remark 4.2).

Next, we will describe explicitly the homomorphisms $D(\beta)$ and $T_\Pi(\beta)$ from Definition 4.2.

Proposition 4.3. *Let* $V = H \otimes V_0$ *and* $V' = H \otimes V_0'$ *be finite free* H*-modules. Let* $\{v_i\}$ (*respectively* $\{v_i'\}$) *be a* **k***-basis of* V_0 (*respectively* V_0')*, and let* $\{\psi_i\}$ (*respectively* $\{\psi_i'\}$) *be the dual basis of* V_0^* (*respectively* $(V_0')^*$)*. For a homomorphism of* H*-modules* $\beta\colon V \to V'$*, write*

$$\beta(1 \otimes v_i) = \sum_j h_{ij} \otimes v_j' \tag{4.19}$$

where $h_{ij} \in H$*. Then we have*:

$$D(\beta)(1 \otimes \psi_k') = \sum_j S(h_{jk}) \otimes \psi_j \tag{4.20}$$

and

$$T_\Pi(\beta)(1 \otimes u \otimes v_i) = \sum_j h_{ij_{(1)}} \otimes h_{ij_{(-2)}} u \otimes v_j'\,. \tag{4.21}$$

By linearity, Proposition 4.3 follows from the following special case, which we formulate as a lemma for future reference.

Lemma 4.3. *Let $V = H \otimes V_0$ and $V' = H \otimes V_0'$ be finite free H-modules. For fixed $h \in H$, $B \in \mathrm{Hom}_{\mathbf{k}}(V_0, V_0')$, consider the homomorphism of H-modules $\beta\colon V \to V'$ given by*

$$\beta(1 \otimes v) = h \otimes Bv\,, \qquad v \in V_0\,. \tag{4.22}$$

Then we have:

$$D(\beta)(1 \otimes \psi') = S(h) \otimes (\psi' \circ B)\,, \qquad \psi' \in (V_0')^* = \mathrm{Hom}_{\mathbf{k}}(V_0', \mathbf{k}) \tag{4.23}$$

and

$$T_\Pi(\beta)(1 \otimes u \otimes v) = h_{(1)} \otimes h_{(-2)}u \otimes Bv\,, \qquad u \in \Pi\,,\ v \in V_0\,. \tag{4.24}$$

Proof. The proof is straightforward from definition, and it is left to the reader. □

4.3. **Tensor Modules for $W(\mathfrak{d})$.** The adjoint representation of $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ gives rise to the following homomorphism of Lie pseudoalgebras $W(\mathfrak{d}) \to \mathrm{gc}(H \otimes \mathfrak{d})$ (see Eq. (3.1) and Remark 4.2):

$$1 \otimes a \mapsto 1 \otimes 1 \otimes \mathrm{ad}\, a - 1 \otimes a \otimes \mathrm{Id} + \varepsilon_a\,, \tag{4.25}$$

where the pseudolinear map ε_a is given by

$$\varepsilon_a(g \otimes b) = (b \otimes g) \otimes_H (1 \otimes a)\,, \qquad g \in H\,,\ b \in \mathfrak{d}\,. \tag{4.26}$$

In (4.25) we have identified $\mathrm{gc}(H \otimes \mathfrak{d})$ with $H \otimes H \otimes \mathrm{End}\,\mathfrak{d}$; in this identification

$$\varepsilon_{\partial_i} = \sum_{j=1}^{N} \partial_j \otimes 1 \otimes e_i^j\,, \tag{4.27}$$

where $\{\partial_i\}_{i=1,\dots,N}$ is a basis of $\mathfrak{d}$ and $e_i^j(\partial_k) = \delta_k^j \partial_i$.

Lemma 4.4. *The map*

$$1 \otimes \partial_i \mapsto \Big(1 \otimes \mathrm{ad}\,\partial_i + \sum_{j=1}^{N} \partial_j \otimes e_i^j\Big) + 1 \otimes \partial_i \tag{4.28}$$

is an embedding of Lie pseudoalgebras $W(\mathfrak{d}) \hookrightarrow (\mathrm{Cur}\,\mathfrak{gl}\,\mathfrak{d}) \rtimes W(\mathfrak{d})$.

Proof. The image of $W(\mathfrak{d})$ under the embedding (4.25) is contained in

$$H \otimes \mathbf{k} \otimes \mathrm{End}\,\mathfrak{d} + H \otimes \mathfrak{d} \otimes \mathrm{id} \subset H \otimes H \otimes \mathrm{End}\,\mathfrak{d}\,,$$

which is isomorphic to $(\mathrm{Cur}\,\mathfrak{gl}\,\mathfrak{d}) \rtimes W(\mathfrak{d})$ by Remark 4.3. □

By Remark 4.3, for any finite-dimensional $\mathfrak{gl}\,\mathfrak{d}$-module V_0, we have a homomorphism of Lie pseudoalgebras $(\mathrm{Cur}\,\mathfrak{gl}\,\mathfrak{d}) \rtimes W(\mathfrak{d}) \to \mathrm{gc}\,V$, where $V = H \otimes V_0$. After composing it with the embedding from Lemma 4.4, we obtain a homomorphism $W(\mathfrak{d}) \to \mathrm{gc}\,V$, i.e., a representation of $W(\mathfrak{d})$ on V. Explicitly, the action of $W(\mathfrak{d})$ on V is given by:

$$\begin{aligned}(1 \otimes \partial_i) * (1 \otimes v) = (1 \otimes 1) \otimes_H (1 \otimes (\mathrm{ad}\,\partial_i)v) + \sum_{j=1}^{N} (\partial_j \otimes 1) \otimes_H (1 \otimes e_i^j v)\\ - (1 \otimes \partial_i) \otimes_H (1 \otimes v)\,.\end{aligned} \tag{4.29}$$

Now let Π be any finite-dimensional $\mathfrak{d}$-module. The twisting of V by Π is $T_\Pi(V) = H \otimes (\Pi \otimes V_0)$ with the following action of $W(\mathfrak{d})$ (see Proposition 4.2):

$$\begin{aligned}(1 \otimes \partial_i) * (1 \otimes w) = (1 \otimes 1) \otimes_H (1 \otimes (\mathrm{ad}\,\partial_i)w) + \sum_{j=1}^{N} (\partial_j \otimes 1) \otimes_H (1 \otimes e_i^j w)\\ - (1 \otimes \partial_i) \otimes_H (1 \otimes w) + (1 \otimes 1) \otimes_H (1 \otimes \partial_i w)\end{aligned} \tag{4.30}$$

for $w \in \Pi \otimes V_0$, where $\mathfrak{d}$ acts on the factor Π and $\mathfrak{gl}\,\mathfrak{d}$ acts on V_0.

Definition 4.3. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras, and let U_i be a $\mathfrak{g}_i$-module ($i = 1, 2$). Then we will denote by $U_1 \boxtimes U_2$ the $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$-module $U_1 \otimes U_2$, where $\mathfrak{g}_1$ acts on the first factor and $\mathfrak{g}_2$ acts on the second one.

The above formulas (4.29), (4.30) motivate the introduction of an important class of $W(\mathfrak{d})$-modules.

Definition 4.4. (i) Let W_0 be a finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module. The $W(\mathfrak{d})$-module $H \otimes W_0$, with the action of $W(\mathfrak{d})$ given by (4.30) for $w \in W_0$, is called a *tensor module* and is denoted as $\mathcal{T}(W_0)$.

(ii) Let $W_0 = \Pi \boxtimes V_0$, where Π is a finite-dimensional $\mathfrak{d}$-module and V_0 is a finite-dimensional $\mathfrak{gl}\,\mathfrak{d}$-module. Then the tensor module $\mathcal{T}(W_0)$ will also be denoted as $\mathcal{T}(\Pi, V_0)$.

(iii) Occasionally, we will denote $\mathcal{T}(\Pi, V_0)$ also by $\mathcal{T}(\Pi, V_0, c)$, where V_0 is viewed as a module over $\mathfrak{sl}\,\mathfrak{d} \subset \mathfrak{gl}\,\mathfrak{d}$, and $c \in \mathbf{k}$ denotes the scalar action of $\mathrm{Id} \in \mathfrak{gl}\,\mathfrak{d}$ on V_0.

Remark 4.5. By definition, we have $\mathcal{T}(\Pi, V_0) = T_\Pi(\mathcal{T}(\mathbf{k}, V_0))$.

Remark 4.6. Combining the embeddings (4.18) and (4.28), we get an embedding of Lie pseudoalgebras $W(\mathfrak{d}) \hookrightarrow \mathrm{Cur}(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d}) \rtimes W(\mathfrak{d})$,

$$1 \otimes \partial_i \mapsto \Big(1 \otimes \partial_i + 1 \otimes \operatorname{ad} \partial_i + \sum_{j=1}^{N} \partial_j \otimes e_i^j\Big) + 1 \otimes \partial_i \,. \tag{4.31}$$

Given a $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module W_0, the $W(\mathfrak{d})$-module obtained from it by Remark 4.3 is exactly the tensor module $\mathcal{T}(W_0) = H \otimes W_0$ corresponding to W_0.

5. Tensor Modules of de Rham Type

Throughout this section, $\mathfrak{d}$ will be an N-dimensional Lie algebra. We fix a basis $\{\partial_i\}_{i=1,\dots,N}$ of $\mathfrak{d}$ with structure constants c_{ij}^k: $[\partial_i, \partial_j] = \sum c_{ij}^k \partial_k$. Define elements $e_j^i \in \mathfrak{gl}\,\mathfrak{d}$ by $e_j^i(\partial_k) = \delta_k^i \partial_j$.

5.1. **Forms with Constant Coefficients.** The material in this subsection is completely standard; our purpose is just to fix the notation. For $0 \le n \le N$, let

$$\Omega^n = \textstyle\bigwedge^n \mathfrak{d}^* \,, \quad \Omega = \bigwedge^\bullet \mathfrak{d}^* = \bigoplus_{n=0}^{N} \Omega^n \,. \tag{5.1}$$

Set $\Omega^n = \{0\}$ if $n < 0$ or $n > N$. We will think of the elements of Ω^n as skew-symmetric n-*forms*, i.e., linear maps from $\bigwedge^n \mathfrak{d}$ to $\mathbf{k}$.

Consider the cohomology complex of $\mathfrak{d}$ with trivial coefficients,

$$0 \to \Omega^0 \xrightarrow{\mathrm{d}_0} \Omega^1 \xrightarrow{\mathrm{d}_0} \cdots \xrightarrow{\mathrm{d}_0} \Omega^N, \tag{5.2}$$

where the differential d_0 is given by the formula ($\alpha \in \Omega^n$, $a_i \in \mathfrak{d}$):

$$\begin{aligned} &(\mathrm{d}_0\alpha)(a_1 \wedge \cdots \wedge a_{n+1}) \\ &\quad = \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{n+1}) \end{aligned} \tag{5.3}$$

if $n \ge 1$, and $\mathrm{d}_0\alpha = 0$ for $\alpha \in \Omega^0 = \mathbf{k}$. Here a hat over a_i means that the term a_i is omitted in the wedge product.

For $a \in \mathfrak{d}$, define operators $\iota_a \colon \Omega^n \to \Omega^{n-1}$ by

$$(\iota_a\alpha)(a_1 \wedge \cdots \wedge a_{n-1}) = \alpha(a \wedge a_1 \wedge \cdots \wedge a_{n-1}) \,, \qquad a_i \in \mathfrak{d} \,. \tag{5.4}$$

For $A \in \mathfrak{gl}\,\mathfrak{d}$, denote by $A\cdot$ its action on Ω; explicitly,

$$(A \cdot \alpha)(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} (-1)^i \alpha(Aa_i \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_n) \,. \tag{5.5}$$

Then we have the following Cartan formula for the coadjoint action of $\mathfrak{d}$:

$$(\operatorname{ad} a)\cdot = \mathrm{d}_0\iota_a + \iota_a\mathrm{d}_0\,. \tag{5.6}$$

This, together with $\mathrm{d}_0^2 = 0$, implies that $(\operatorname{ad} a)\cdot$ commutes with d_0.

5.2. **Pseudo de Rham Complex.** Following [BDK, Section 8.3], we define the spaces of *pseudo-forms* $\Omega^n(\mathfrak{d}) = H\otimes\Omega^n$ and $\Omega(\mathfrak{d}) = H\otimes\Omega = \bigoplus_{n=0}^N \Omega^n(\mathfrak{d})$. They are considered as H-modules, where H acts on the first factor by left multiplication. We can identify $\Omega^n(\mathfrak{d})$ with the space of linear maps from $\bigwedge^n \mathfrak{d}$ to H, and $H^{\otimes 2}\otimes_H \Omega^n(\mathfrak{d})$ with $\operatorname{Hom}(\bigwedge^n \mathfrak{d}, H^{\otimes 2})$. Note that $\Omega^n(\mathfrak{d}) = \{0\}$ if $n<0$ or $n>N$.

Let us consider $H = U(\mathfrak{d})$ as a left $\mathfrak{d}$-module with respect to the action $a\cdot h = -ha$, where ha is the product of $a\in\mathfrak{d}\subset H$ and $h\in H$ in H. Consider the cohomology complex of $\mathfrak{d}$ with coefficients in H:

$$0\to\Omega^0(\mathfrak{d})\xrightarrow{\mathrm{d}}\Omega^1(\mathfrak{d})\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\Omega^N(\mathfrak{d})\,. \tag{5.7}$$

Explicitly, the differential d is given by the formula ($\alpha\in\Omega^n(\mathfrak{d})$, $a_i\in\mathfrak{d}$):

$$\begin{aligned}
&(\mathrm{d}\alpha)(a_1\wedge\cdots\wedge a_{n+1})\\
&\qquad=\sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j]\wedge a_1\wedge\cdots\wedge\widehat{a}_i\wedge\cdots\wedge\widehat{a}_j\wedge\cdots\wedge a_{n+1})\\
&\qquad+\sum_i(-1)^i\alpha(a_1\wedge\cdots\wedge\widehat{a}_i\wedge\cdots\wedge a_{n+1})\,a_i\qquad\text{if }\ n\geq 1,\\
&(\mathrm{d}\alpha)(a_1)=-\alpha a_1\qquad\text{if }\ \alpha\in\Omega^0(\mathfrak{d})=H\,,
\end{aligned} \tag{5.8}$$

where a hat over a_i means that the term a_i is omitted. Notice that d is H-linear.

Proposition 5.1 ([BDK])**.** *The n-th cohomology of the complex $(\Omega(\mathfrak{d}),\mathrm{d})$ is trivial for $n\neq N=\dim\mathfrak{d}$ and 1-dimensional for $n=N$. In particular, the sequence* (5.7) *is exact.*

Proof. By Poincaré duality $\mathrm{H}^n(\mathfrak{d},U(\mathfrak{d}))\simeq\mathrm{H}^{N-n}(\mathfrak{d},U(\mathfrak{d})^*)$. But $\mathrm{H}^n(\mathfrak{d},U(\mathfrak{d})^*)\simeq\mathrm{H}_n(\mathfrak{d},U(\mathfrak{d}))^*$ is trivial for $n>0$ and 1-dimensional for $n=0$; see, e.g., [F]. □

Definition 5.1. The sequence (5.7) is called the *pseudo de Rham complex*.

The following lemma provides another formula for the differential (5.8), which will be useful later.

Lemma 5.1. *For every $\alpha\in\Omega^n$, $n\geq 0$, and $i=1,\dots,N$, consider the element $1\otimes\iota_{\partial_i}\alpha\in\Omega^{n-1}(\mathfrak{d})$. Then we have*:

$$\mathrm{d}(1\otimes\iota_{\partial_i}\alpha)=\sum_{k=1}^N\partial_k\otimes e_i^k\alpha-\sum_{\substack{j,k,l=1\\k<l}}^N 1\otimes c_{kl}^j e_i^k e_j^l\alpha-\sum_{\substack{k,l=1\\k<l}}^N 1\otimes c_{kl}^k e_i^l\alpha\,, \tag{5.9}$$

where the action of $\mathfrak{gl}\,\mathfrak{d}$ on Ω^n is given by (5.5)*.*

Proof. For $n=0$, both sides of (5.9) are trivial; so we can assume that $n\geq 1$. Denote the three terms in the right-hand side of (5.9) by β_1, β_2, β_3. Using (5.5), we compute for $1\leq i_1<\cdots<i_n\leq N$:

$$\beta_1(\partial_{i_1}\wedge\cdots\wedge\partial_{i_n})=\sum_{k=1}^N\sum_{r=1}^n\partial_k\otimes(-1)^r\delta_{i_r}^k\,\alpha(\partial_i\wedge\partial_{i_1}\wedge\cdots\wedge\widehat{\partial}_{i_r}\wedge\cdots\wedge\partial_{i_n})$$

and

$$\beta_2(\partial_{i_1}\wedge\cdots\wedge\partial_{i_n}) = \sum_{\substack{j,k,l=1\\k<l}}^{N}\sum_{\substack{r,s=1\\r<s}}^{n} 1\otimes(-1)^{r+s+1}\delta^k_{i_r}\delta^l_{i_s}c^j_{kl}\,\alpha(\partial_i\wedge\partial_j\wedge\partial_{i_1}\wedge\cdots \wedge\widehat{\partial}_{i_r}\wedge\cdots\wedge\widehat{\partial}_{i_s}\wedge\cdots\wedge\partial_{i_n}) + \sum_{\substack{j,k,l=1\\k<l}}^{N}\sum_{s=1}^{n} 1\otimes(-1)^{s+1}\delta^k_j\delta^l_{i_s}c^j_{kl}\,\alpha(\partial_i\wedge\partial_{i_1}\wedge\cdots\wedge\widehat{\partial}_{i_s}\wedge\cdots\wedge\partial_{i_n})\,.$$

Similarly,

$$\beta_3(\partial_{i_1}\wedge\cdots\wedge\partial_{i_n}) = \sum_{\substack{k,l=1\\k<l}}^{N}\sum_{s=1}^{n} 1\otimes(-1)^{s}\delta^l_{i_s}c^k_{kl}\,\alpha(\partial_i\wedge\partial_{i_1}\wedge\cdots\wedge\widehat{\partial}_{i_s}\wedge\cdots\wedge\partial_{i_n})\,.$$

These formulas, together with (5.8), (5.4) and the equation

$$\sum_{\substack{j,k,l=1\\k<l}}^{N}\delta^k_{i_r}\delta^l_{i_s}c^j_{kl}\,\partial_j = [\partial_{i_r},\partial_{i_s}]\,,\qquad r<s\,,\ \ i_r<i_s\,,$$

imply that $\mathrm{d}(1\otimes\iota_{\partial_i}\alpha) = \beta_1-\beta_2-\beta_3$. □

Next, we introduce H-bilinear maps

$$*_\iota\colon W(\mathfrak{d})\otimes\Omega^n(\mathfrak{d})\to H^{\otimes 2}\otimes_H\Omega^{n-1}(\mathfrak{d})\,, \tag{5.10}$$

$$*\colon W(\mathfrak{d})\otimes\Omega^n(\mathfrak{d})\to H^{\otimes 2}\otimes_H\Omega^{n}(\mathfrak{d})\,, \tag{5.11}$$

by the formulas:

$$(f\otimes a)*_\iota(g\otimes\alpha) = (f\otimes g)\otimes_H\iota_a\alpha\,, \tag{5.12}$$

$$w*\gamma = ((\mathrm{id}\otimes\mathrm{id})\otimes_H\mathrm{d})(w*_\iota\gamma) + w*_\iota(\mathrm{d}\gamma)\,, \tag{5.13}$$

for $w=f\otimes a\in W(\mathfrak{d})$, $\gamma=g\otimes\alpha\in\Omega^n(\mathfrak{d})$. Eq. (5.13) is an analog of Cartan's formula (5.6). Explicitly, we have (see [BDK, Eq. (8.7)]):

$$\begin{aligned}(w*\gamma)(a_1\wedge\cdots\wedge a_n) &= -(f\otimes ga)\,\alpha(a_1\wedge\cdots\wedge a_n)\\ &+\sum_{i=1}^{n}(-1)^i(fa_i\otimes g)\,\alpha(a\wedge a_1\wedge\cdots\wedge\widehat{a}_i\wedge\cdots\wedge a_n)\\ &+\sum_{i=1}^{n}(-1)^i(f\otimes g)\,\alpha([a,a_i]\wedge a_1\wedge\cdots\wedge\widehat{a}_i\wedge\cdots\wedge a_n)\in H^{\otimes 2}\end{aligned} \tag{5.14}$$

for $n\geq 1$, and $w*\gamma = -f\otimes ga$ for $\gamma=g\in\Omega^0(\mathfrak{d})=H$. Note that the latter coincides with the action (3.2) of $W(\mathfrak{d})$ on H.

Theorem 5.1. *The maps* (5.11) *provide each* $\Omega^n(\mathfrak{d})$ *with a structure of a tensor* $W(\mathfrak{d})$*-module corresponding to the* $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$*-module* $\mathbf{k}\boxtimes\Omega^n$*, i.e.,* $\Omega^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k},\Omega^n)$*. The differential* $\mathrm{d}\colon\Omega^n(\mathfrak{d})\to\Omega^{n+1}(\mathfrak{d})$ *is a homomorphism of* $W(\mathfrak{d})$*-modules.*

Proof. Comparing (5.14) with (5.5), we obtain for $\alpha\in\Omega^n$:

$$\begin{aligned}(1\otimes\partial_k)*(1\otimes\alpha) &= -(1\otimes\partial_k)\otimes_H(1\otimes\alpha)\\ &+\sum_{j=1}^{N}(\partial_j\otimes 1)\otimes_H(1\otimes e^j_k\alpha)+(1\otimes 1)\otimes_H(1\otimes(\mathrm{ad}\,\partial_k)\alpha)\,.\end{aligned}$$

But this is exactly (4.29); hence, $\Omega^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k}, \Omega^n)$.

To prove that d is a homomorphism, we have to check that it satisfies (2.34). This follows from (5.13) and $\mathrm{d}^2 = 0$. Indeed, replacing γ with $\mathrm{d}\gamma$ in (5.13), we get

$$w * (\mathrm{d}\gamma) = ((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(w *_\iota \mathrm{d}\gamma)\,,$$

while applying $((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})$ to both sides of (5.13) gives

$$((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(w * \gamma) = ((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(w *_\iota \mathrm{d}\gamma)\,.$$

This completes the proof. □

5.3. **Twisting of the Pseudo de Rham Complex.** As before, let Π be a finite-dimensional $\mathfrak{d}$-module, which we consider as an H-module. We will apply the twisting functor T_Π (see Definition 4.2(ii)) to the pseudo de Rham complex (5.7). Note that, by Theorem 5.1 and Remark 4.5, we have $T_\Pi(\Omega^n(\mathfrak{d})) = \mathcal{T}(\Pi, \Omega^n)$. We obtain a complex of $W(\mathfrak{d})$-modules

$$0 \to \mathcal{T}(\Pi, \Omega^0) \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi, \Omega^1) \xrightarrow{\mathrm{d}_\Pi} \cdots \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi, \Omega^N)\,, \qquad \mathrm{d}_\Pi \equiv T_\Pi(\mathrm{d})\,, \tag{5.15}$$

which we call the Π-*twisted pseudo de Rham complex.*

It follows from (5.8) and Proposition 4.3 that the complex (5.15) coincides with the cohomology complex of $\mathfrak{d}$ with coefficients in $H \otimes \Pi$ considered with the action

$$a \cdot (h \otimes u) = -ha + au\,, \qquad a \in \mathfrak{d}\,,\ h \in H\,,\ u \in \Pi\,. \tag{5.16}$$

Lemma 5.2. *The $\mathfrak{d}$-module $H \otimes \Pi$, equipped with the action* (5.16)*, is isomorphic to $H \otimes \Pi$ with $\mathfrak{d}$ acting only on H via*

$$a(h \otimes u) = -ha \otimes u\,, \qquad a \in \mathfrak{d}\,,\ h \in H\,,\ u \in \Pi\,.$$

In other words, $H \otimes \Pi$ is isomorphic to a direct sum of $\dim \Pi$ copies of the $\mathfrak{d}$-module H.

Proof. Consider the linear map

$$F\colon H \otimes \Pi \to H \otimes \Pi\,, \qquad h \otimes u \mapsto h_{(1)} \otimes h_{(-2)}u\,.$$

From (2.7) it is easy to see that F is a linear isomorphism and

$$F^{-1}(h \otimes u) = h_{(1)} \otimes h_{(2)}u$$

(see [BDK, Section 2.3] for a similar argument). Using (2.6) and (2.13), we compute

$$\begin{aligned} F(-ha \otimes u) &= -(ha)_{(1)} \otimes (ha)_{(-2)}u \\ &= -h_{(1)}a \otimes h_{(-2)}u + h_{(1)} \otimes ah_{(-2)}u = a \cdot F(h \otimes u)\,. \end{aligned}$$

This shows that F is an isomorphism of the corresponding $\mathfrak{d}$-modules. □

Now Proposition 5.1 and Lemma 5.2 immediately imply:

Proposition 5.2. *The sequence* (5.15) *is exact. The image of d_Π in $\mathcal{T}(\Pi, \Omega^N)$ has codimension $\dim \Pi$.*

Finally, let us give a formula for the differential d_Π, which is similar to (5.9).

Lemma 5.3. *For every $\alpha \in \Omega^n$, $n \geq 0$, $u \in \Pi$, and $i = 1, \dots, N$, we have*:

$$\begin{aligned} \mathrm{d}_\Pi(1 \otimes u \otimes \iota_{\partial_i}\alpha) &= \sum_{k=1}^N \partial_k \otimes u \otimes e_i^k\alpha - \sum_{k=1}^N 1 \otimes \partial_k u \otimes e_i^k\alpha \\ &\quad - \sum_{\substack{j,k,l=1 \\ k<l}}^N 1 \otimes u \otimes c_{kl}^j e_i^k e_j^l \alpha - \sum_{\substack{k,l=1 \\ k<l}}^N 1 \otimes u \otimes c_{kl}^k e_i^l \alpha\,, \end{aligned} \tag{5.17}$$

where the action of $\mathfrak{gl}\,\mathfrak{d}$ on Ω^n is given by (5.5)*.*

Proof. For a fixed i, extend by H-linearity the map ι_{∂_i} to a map from $\Omega^n(\mathfrak{d})$ to $\Omega^{n-1}(\mathfrak{d})$. Note that, by Proposition 4.3 (or Lemma 4.3), we have

$$(T_\Pi(\iota_{\partial_i}))(1\otimes u\otimes\alpha) = 1\otimes u\otimes\iota_{\partial_i}\alpha\,.$$

Consider the H-linear map $\mathrm{d}\circ\iota_{\partial_i}\colon \Omega^n(\mathfrak{d})\to\Omega^n(\mathfrak{d})$. This map sends $1\otimes\alpha$ to the right-hand side of (5.9). Hence, by Proposition 4.3, $(T_\Pi(\mathrm{d}\circ\iota_{\partial_i}))(1\otimes u\otimes\alpha)$ is given by the right-hand side of (5.17). On the other hand, we have $T_\Pi(\mathrm{d}\circ\iota_{\partial_i}) = T_\Pi(\mathrm{d})\circ T_\Pi(\iota_{\partial_i})$, which gives $\mathrm{d}_\Pi(1\otimes u\otimes\iota_{\partial_i}\alpha)$ when applied to $1\otimes u\otimes\alpha$. □

6. Classification of Irreducible Finite $W(\mathfrak{d})$-Modules

In this section we provide a complete classification of all irreducible finite $W(\mathfrak{d})$-modules. Our main result is Theorem 6.6.

6.1. **Singular Vectors and Tensor Modules.** Recall that the annihilation algebra $\mathcal{W}$ of $W(\mathfrak{d})$ has a decreasing filtration $\{\mathcal{W}_p\}_{p\geq -1}$ given by (3.8). For a $\mathcal{W}$-module V, we denote by $\ker_p V$ the set of all $v\in V$ that are killed by $\mathcal{W}_p$. Then a $\mathcal{W}$-module V is conformal iff $V=\bigcup\ker_p V$. Recall also that the extended annihilation algebra is defined as $\widetilde{\mathcal{W}}=\mathfrak{d}\ltimes\mathcal{W}$, where $\mathfrak{d}$ acts on $\mathcal{W}$ by (3.7). By Proposition 2.1, any $W(\mathfrak{d})$-module has a natural structure of a conformal $\widetilde{\mathcal{W}}$-module and vice versa.

For every $p\geq 0$, the normalizer of $\mathcal{W}_p$ in $\widetilde{\mathcal{W}}$ is equal to $\mathcal{N}_\mathcal{W}$ (see Definition 3.2 and Proposition 3.3). Therefore, each $\ker_p V$ is an $\mathcal{N}_\mathcal{W}$-module. In fact, $\ker_p V$ is a module over the finite-dimensional Lie algebra $\mathcal{N}_\mathcal{W}/\mathcal{W}_p = \widetilde{\mathfrak{d}}\oplus(\mathcal{W}_0/\mathcal{W}_p)$. The Lie algebra $\mathcal{N}_\mathcal{W}/\mathcal{W}_1$ is isomorphic to the direct sum of Lie algebras $\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d}$.

Definition 6.1. For a $W(\mathfrak{d})$-module V, a *singular vector* is an element $v\in V$ such that $\mathcal{W}_1\cdot v=0$. The space of singular vectors in V will be denoted by $\operatorname{sing} V$. We will denote by $\rho_{\mathrm{sing}}\colon \mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d}\to\mathfrak{gl}(\operatorname{sing} V)$ the representation obtained from the $\mathcal{N}_\mathcal{W}$-action on $\operatorname{sing} V\equiv\ker_1 V$ via the isomorphism $\mathcal{N}_\mathcal{W}/\mathcal{W}_1\simeq\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d}$.

Recall that $\ker V\equiv\ker_{-1}V$ is the space of all $v\in V$ such that $\mathcal{W}\cdot v=0$. Then, obviously, $\ker V\subset\operatorname{sing} V$. Note also that $\ker V=\{0\}$ when V is irreducible.

Theorem 6.1. *For any nontrivial finite $W(\mathfrak{d})$-module V, we have $\operatorname{sing} V\neq\{0\}$ and the space $\operatorname{sing} V/\ker V$ is finite dimensional.*

Proof. The second statement is a special case of Lemma 2.3. To show that $\operatorname{sing} V\neq\{0\}$, we can assume without loss of generality that $\ker V=\{0\}$. Since V is a conformal $\mathcal{W}$-module, $\ker_p V\neq\{0\}$ for some $p\geq 1$. By Lemma 2.3, the space $\ker_p V$ is finite dimensional. Let us choose a minimal $\mathcal{N}_\mathcal{W}$-submodule R of $\ker_p V$. Then R is an irreducible $\mathcal{N}_\mathcal{W}$-module; hence, by Proposition 3.4, $\mathcal{W}_1$ acts trivially on R. This means that $R\subset\operatorname{sing} V$. □

Remark 6.1. It follows from (3.8) and Proposition 2.1 that a vector $v\in V$ is singular if and only if

$$(1\otimes\partial)*v\in(\mathrm{F}^1 H\otimes\mathbf{k})\otimes_H V\,,\qquad \partial\in\mathfrak{d}\,, \tag{6.1}$$

where $\mathrm{F}^1 H=\mathbf{k}\oplus\mathfrak{d}$. Similarly, by Lemma 2.1(ii), a vector $v\in V$ is singular if and only if

$$(1\otimes\partial)*v\in(\mathbf{k}\otimes\mathrm{F}^1 H)\otimes_H V\,,\qquad \partial\in\mathfrak{d}\,. \tag{6.2}$$

As before, let $\{\partial_i\}_{i=1,\dots,N}$ be a basis of $\mathfrak{d}$, and let $x^i\in X$ be given by (2.19). We view x^i as elements of $\mathfrak{d}^*$; then $\{x^i\}$ is a basis of $\mathfrak{d}^*$ dual to the basis $\{\partial_i\}$ of $\mathfrak{d}$. Let $e_i^j\in\mathfrak{gl}\,\mathfrak{d}$ be given by $e_i^j\partial_k=\delta_k^j\,\partial_i$; in other words, e_i^j corresponds to $\partial_i\otimes x^j$ under the isomorphism $\mathfrak{gl}\,\mathfrak{d}\simeq\mathfrak{d}\otimes\mathfrak{d}^*$.

Note that, by Definition 3.2 and Corollary 3.1, we have

$$\rho_{\mathrm{sing}}(\partial)v=\widetilde{\partial}\cdot v\,,\quad \rho_{\mathrm{sing}}(e_i^j)v=-(x^j\otimes\partial_i)\cdot v\,,\qquad \partial\in\mathfrak{d}\,,\ v\in\operatorname{sing} V\,. \tag{6.3}$$

Lemma 6.1. *Let V be a $W(\mathfrak{d})$-module. Then for every singular vector $v \in \operatorname{sing} V$, the action of $W(\mathfrak{d})$ on v is given by*

$$(1\otimes\partial_i) * v = \sum_{j=1}^{N} (\partial_j\otimes 1)\otimes_H \rho_{\mathrm{sing}}(e_i^j)v - (1\otimes 1)\otimes_H \partial_i v + (1\otimes 1)\otimes_H \rho_{\mathrm{sing}}(\partial_i + \operatorname{ad}\partial_i)v\,. \tag{6.4}$$

Proof. Since $\mathcal{W}_1 \cdot v = 0$, it follows from Proposition 2.1 that for $\partial \in \mathfrak{d}$

$$(1\otimes\partial) * v = (1\otimes 1)\otimes_H (1\otimes\partial)\cdot v - \sum_j (\partial_j\otimes 1)\otimes_H (x^j\otimes\partial)\cdot v\,,$$

while Lemma 3.3 implies

$$(\partial + 1\otimes\partial - \widetilde{\partial})\cdot v = \rho_{\mathrm{sing}}(\operatorname{ad}\partial)v\,.$$

Combining the above equations with (6.3) proves (6.4). □

Corollary 6.1. *Let V be a $W(\mathfrak{d})$-module and let R be a nontrivial $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-submodule of $\operatorname{sing} V$. Denote by HR the H-submodule of V generated by R. Then HR is a $W(\mathfrak{d})$-submodule of V. In particular, if V is irreducible, then $V = HR$.*

Proof. It follows from (6.4) that $W(\mathfrak{d}) * R \subset (H\otimes H)\otimes_H HR$. Then, by H-bilinearity, $W(\mathfrak{d}) * HR \subset (H\otimes H)\otimes_H HR$, which means that HR is a $W(\mathfrak{d})$-submodule of V. □

Let R be a finite-dimensional $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-module, with an action denoted as ρ_R. Let $V = H\otimes R$ be the free H-module generated by R, where H acts by left multiplication on the first factor. We define a pseudoproduct

$$(1\otimes\partial_i) * (1\otimes u) = \sum_{j=1}^{N} (\partial_j\otimes 1)\otimes_H (1\otimes\rho_R(e_i^j)u) - (1\otimes 1)\otimes_H (\partial_i\otimes u) + (1\otimes 1)\otimes_H (1\otimes\rho_R(\partial_i + \operatorname{ad}\partial_i)u)\,, \qquad u\in R\,, \tag{6.5}$$

and then extend it by H-bilinearity to a map $*\colon W(\mathfrak{d})\otimes V \to (H\otimes H)\otimes_H V$.

Lemma 6.2. *Let R be a finite-dimensional $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-module with an action ρ_R. Then formula* (6.5) *defines a structure of a $W(\mathfrak{d})$-module on $V = H\otimes R$. We have $\mathbf{k}\otimes R \subset \operatorname{sing} V$ and*

$$\rho_{\mathrm{sing}}(A)(1\otimes u) = 1\otimes\rho_R(A)u\,, \qquad A\in\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d}\,,\ u\in R\,. \tag{6.6}$$

Proof. The fact that V is a $W(\mathfrak{d})$-module can be proved by a straightforward computation, using (2.33) and (3.1). Instead, we will show that V is a tensor module (see Definition 4.4). Let us compare (6.5) to (4.30), keeping in mind that, by definition,

$$(1\otimes 1)\otimes_H (\partial_i\otimes u) = (\partial_i\otimes 1)\otimes_H (1\otimes u) + (1\otimes\partial_i)\otimes_H (1\otimes u)\,.$$

We see that $V = H\otimes R$ coincides with the tensor module $\mathcal{T}(R)$, where R is equipped with the following modified action of $\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d}$:

$$\partial u = \big(\rho_R(\partial) + \operatorname{tr}(\operatorname{ad}\partial)\big)u\,, \qquad \partial\in\mathfrak{d}\,,\ u\in R\,, \tag{6.7}$$

$$Au = \big(\rho_R(A) - \operatorname{tr} A\big)u\,, \qquad A\in\mathfrak{gl}\,\mathfrak{d}\,,\ u\in R\,. \tag{6.8}$$

The fact that $\mathbf{k}\otimes R\subset\operatorname{sing} V$ follows from Remark 6.1, and (6.6) follows from comparing (6.4) with (6.5). This completes the proof. □

Definition 6.2. (i) Let R be a finite-dimensional $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-module with an action ρ_R. Then the $W(\mathfrak{d})$-module $H\otimes R$, with the action of $W(\mathfrak{d})$ given by (6.5), will be denoted as $\mathcal{V}(R)$.

(ii) Let $R = \Pi\boxtimes U$, where Π is a finite-dimensional $\mathfrak{d}$-module and U is a finite-dimensional $\mathfrak{gl}\,\mathfrak{d}$-module. Then the module $\mathcal{V}(R)$ will also be denoted as $\mathcal{V}(\Pi, U)$.

Remark 6.2. If R is a finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module, we can define an action of $\mathcal{N}_\mathcal{W}$ on it by letting $\mathcal{W}_1$ act as zero. Then as a $\widetilde{\mathcal{W}}$-module, $\mathcal{V}(R)$ is isomorphic to the induced module $\mathrm{Ind}_{\mathcal{N}_\mathcal{W}}^{\widetilde{\mathcal{W}}} R$. This follows from the fact that $\widetilde{\mathcal{W}} = \mathfrak{d} \oplus \mathcal{N}_\mathcal{W}$ as a vector space (see Proposition 3.3).

For a Lie algebra $\mathfrak{g}$ and a trace form χ on $\mathfrak{g}$, we denote by $\mathbf{k}_\chi$ the 1-dimensional $\mathfrak{g}$-module such that each $a \in \mathfrak{g}$ acts as the scalar $\chi(a)$. Then (6.7) and (6.8) are equivalent to:

$$\mathcal{V}(R) = \mathcal{T}(R \otimes (\mathbf{k}_{\mathrm{tr}\,\mathrm{ad}} \boxtimes \mathbf{k}_{-\mathrm{tr}}))\,, \quad \mathcal{T}(R) = \mathcal{V}(R \otimes (\mathbf{k}_{-\mathrm{tr}\,\mathrm{ad}} \boxtimes \mathbf{k}_{\mathrm{tr}}))\,, \tag{6.9}$$

This can also be written as follows (cf. Definition 4.4(ii)):

$$\mathcal{V}(\Pi, U) = \mathcal{T}(\Pi \otimes \mathbf{k}_{\mathrm{tr}\,\mathrm{ad}}, U \otimes \mathbf{k}_{-\mathrm{tr}})\,, \quad \mathcal{T}(\Pi, U) = \mathcal{V}(\Pi \otimes \mathbf{k}_{-\mathrm{tr}\,\mathrm{ad}}, U \otimes \mathbf{k}_{\mathrm{tr}})\,. \tag{6.10}$$

Theorem 6.2. *Let V be an irreducible finite $W(\mathfrak{d})$-module, and let R be an irreducible $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-submodule of $\mathrm{sing}\,V$. Then V is a homomorphic image of $\mathcal{V}(R)$. In particular, every irreducible finite $W(\mathfrak{d})$-module is a quotient of a tensor module.*

Proof. By Corollary 6.1, we have $V = HR$. Consider the natural projection

$$\pi\colon \mathcal{V}(R) = H \otimes R \to HR = V\,, \qquad h \otimes u \mapsto hu\,.$$

Note that π is H-linear. Comparing (6.4) with (6.5), we see that

$$\big((\mathrm{id} \otimes \mathrm{id}) \otimes_H \pi\big)\big((1 \otimes \partial_i) * (1 \otimes u)\big) = (1 \otimes \partial_i) * u\,, \qquad i = 1, \dots, N\,,\ u \in R\,.$$

By H-bilinearity, this leads to

$$\big((\mathrm{id} \otimes \mathrm{id}) \otimes_H \pi\big)(a * v) = a * \pi(v)\,, \qquad a \in W(\mathfrak{d})\,,\ v \in \mathcal{V}(R)\,,$$

which means that π is a homomorphism of $W(\mathfrak{d})$-modules (cf. (2.34)). □

6.2. **Filtration of Tensor Modules.** Let $\mathcal{V}(R)$ be a tensor $W(\mathfrak{d})$-module, as defined in Definition 6.2(i). Recall the canonical increasing filtration $\{\mathrm{F}^p H\}$ of H given by (2.16). We introduce an increasing filtration of $\mathcal{V}(R) = H \otimes R$ as follows:

$$\mathrm{F}^p\,\mathcal{V}(R) = \mathrm{F}^p\,H \otimes R\,, \qquad p = -1, 0, \dots\,. \tag{6.11}$$

Note that $\mathrm{F}^{-1}\,\mathcal{V}(R) = \{0\}$, $\mathrm{F}^0\,\mathcal{V}(R) = \mathbf{k} \otimes R$.

The associated graded space of $\mathcal{V}(R)$ is

$$\mathrm{gr}\,\mathcal{V}(R) = \bigoplus_{p \geq 0} \mathrm{gr}^p\,\mathcal{V}(R)\,, \qquad \mathrm{gr}^p\,\mathcal{V}(R) = (\mathrm{F}^p\,H \otimes R)/(\mathrm{F}^{p-1}\,H \otimes R)\,. \tag{6.12}$$

We have isomorphisms of vector spaces:

$$\mathrm{gr}^p\,\mathcal{V}(R) \simeq \mathrm{gr}^p\,H \otimes R \simeq \mathrm{S}^p\,\mathfrak{d} \otimes R\,, \tag{6.13}$$

where $\mathrm{S}^p\,\mathfrak{d}$ is the p-th symmetric power of the vector space $\mathfrak{d}$.

Next, we study the action of the extended annihilation algebra $\widetilde{\mathcal{W}}$ on the filtration (6.11).

Lemma 6.3. *For every $p \geq 0$, we have*:
(i) $\mathfrak{d} \cdot \mathrm{F}^p\,\mathcal{V}(R) \subset \mathrm{F}^{p+1}\,\mathcal{V}(R)$,
(ii) $\mathcal{N}_\mathcal{W} \cdot \mathrm{F}^p\,\mathcal{V}(R) \subset \mathrm{F}^p\,\mathcal{V}(R)$,
(iii) $\mathcal{W}_1 \cdot \mathrm{F}^p\,\mathcal{V}(R) \subset \mathrm{F}^{p-1}\,\mathcal{V}(R)$.

Proof. Part (i) is obvious from definitions, since

$$\partial \cdot (h \otimes u) = \partial h \otimes u\,, \qquad \partial \in \mathfrak{d}\,,\ h \in H\,,\ u \in R\,.$$

We will prove parts (ii) and (iii) by induction on p. For $p = 0$, we have $\mathrm{F}^0\,\mathcal{V}(R) = \mathbf{k} \otimes R \subset \mathrm{sing}\,\mathcal{V}(R)$; hence, (ii) and (iii) hold by the definition of a singular vector.

Now assume that (ii) is satisfied for some $p \geq 0$. Then it is enough to show that

$$A \cdot (\partial v) \in \mathrm{F}^{p+1}\,\mathcal{V}(R) \qquad \text{for all} \quad A \in \mathcal{N}_\mathcal{W}\,,\ \partial \in \mathfrak{d}\,,\ v \in \mathrm{F}^p\,\mathcal{V}(R)\,.$$

Note that, since $\widetilde{\mathcal{W}} = \mathfrak{d} + \mathcal{N}_{\mathcal{W}}$ (see Proposition 3.3), statements (i) and (ii) imply $\widetilde{\mathcal{W}} \cdot v \subset \mathrm{F}^{p+1}\,\mathcal{V}(R)$. Then we have

$$A \cdot (\partial v) = \partial \cdot (A \cdot v) + [A, \partial] \cdot v \in \mathfrak{d} \cdot (\mathcal{N}_{\mathcal{W}} \cdot v) + \widetilde{\mathcal{W}} \cdot v \subset \mathrm{F}^{p+1}\,\mathcal{V}(R)\,,$$

by part (i) and the inductive assumption. This proves (ii).

Similarly, assume that (iii) holds for some $p \geq 0$. Then we want to show that

$$B \cdot (\partial v) \in \mathrm{F}^{p}\,\mathcal{V}(R) \qquad \text{for all} \quad B \in \mathcal{W}_1\,,\ \partial \in \mathfrak{d}\,,\ v \in \mathrm{F}^{p}\,\mathcal{V}(R)\,.$$

We have

$$B \cdot (\partial v) = \partial \cdot (B \cdot v) + [B, \partial] \cdot v \in \mathfrak{d} \cdot (\mathcal{W}_1 \cdot v) + \mathcal{N}_{\mathcal{W}} \cdot v \subset \mathrm{F}^{p}\,\mathcal{V}(R)\,,$$

by (i), (ii) and the inductive assumption, because $[B, \partial] \in \mathcal{W}_0 \subset \mathcal{N}_{\mathcal{W}}$. This completes the proof. □

Lemma 6.3(ii) implies that the Lie algebra $\mathcal{N}_{\mathcal{W}}$ acts on the associated graded space $\operatorname{gr}\mathcal{V}(R)$. By Lemma 6.3(iii), the same is true for the Lie algebra $\mathcal{N}_{\mathcal{W}}/\mathcal{W}_1 = \widetilde{\mathfrak{d}} \oplus (\mathcal{W}_0/\mathcal{W}_1) \simeq \mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d}$. This action is described in the next two lemmas.

Lemma 6.4. *For every $\partial \in \mathfrak{d}$, $h \in \mathrm{F}^{p}\,H$, $u \in R$, we have*

$$\widetilde{\partial} \cdot (h \otimes u) = h \otimes \rho_R(\partial) u \mod \mathrm{F}^{p-1}\,\mathcal{V}(R)\,.$$

Proof. The proof is by induction on p and is similar to that of Lemma 6.3(ii). First, for $p = 0$, we have $\mathrm{F}^{0}\,H = \mathbf{k}$ and $1 \otimes u \in \operatorname{sing}\mathcal{V}(R)$. Hence, $\widetilde{\partial} \cdot (1 \otimes u) = 1 \otimes \rho_R(\partial) u$ by (6.3), (6.6).

Now assume the statement holds for $h \in \mathrm{F}^{p}\,H$, and consider $\widetilde{\partial} \cdot (\partial' h \otimes u)$ for $\partial' \in \mathfrak{d}$. Note that, by Proposition 3.2(i), we have: $[\widetilde{\partial}, \partial'] = [\widetilde{\partial}, \widetilde{\partial}' + \gamma(\partial')] = [\widetilde{\partial}, \widetilde{\partial}'] \in \widetilde{\mathfrak{d}}$. From the inductive assumption, we get $[\widetilde{\partial}, \partial'] \cdot (h \otimes u) \in \mathrm{F}^{p}\,\mathcal{V}(R)$. Therefore,

$$\widetilde{\partial} \cdot (\partial' h \otimes u) = \partial' \cdot (\widetilde{\partial} \cdot (h \otimes u)) \mod \mathrm{F}^{p}\,\mathcal{V}(R) = \partial' h \otimes \rho_R(\partial) u \mod \mathrm{F}^{p}\,\mathcal{V}(R)$$

by the inductive assumption. □

Lemma 6.5. *The action of $\mathfrak{gl}\,\mathfrak{d} \simeq \mathcal{W}_0/\mathcal{W}_1$ on the space $\operatorname{gr}^{p}\mathcal{V}(R) \simeq \mathrm{S}^{p}\,\mathfrak{d} \otimes R$ is given by*

$$A \cdot (f \otimes u) = Af \otimes u + f \otimes \rho_R(A) u\,, \qquad A \in \mathfrak{gl}\,\mathfrak{d}\,,\ f \in \mathrm{S}^{p}\,\mathfrak{d}\,,\ u \in R\,,$$

where Af is the standard action of $\mathfrak{gl}\,\mathfrak{d}$ on $\mathrm{S}^{p}\,\mathfrak{d}$.

Proof. The proof uses the same argument as in Lemmas 6.3(ii) and 6.4, and the fact that via the isomorphisms $\mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{gl}\,\mathfrak{d}$ and $\mathcal{W}/\mathcal{W}_0 \simeq \mathfrak{d}$ the adjoint action $[A, \partial]$ becomes the standard action of $\mathfrak{gl}\,\mathfrak{d}$ on $\mathfrak{d}$ (see Corollary 3.1). □

When $R = \Pi \boxtimes U$, the above two lemmas can be summarized as follows.

Corollary 6.2. *We have $\operatorname{gr}^{p}\mathcal{V}(\Pi, U) \simeq \Pi \boxtimes (\mathrm{S}^{p}\,\mathfrak{d} \otimes U)$ as $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-modules.*

6.3. **Submodules of Tensor Modules.** Let $\mathcal{T}(R) = H \otimes R$ be a tensor module (see Definition 4.4). We will assume that R is an irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module. Then $R = \Pi \boxtimes U$, where Π (respectively U) is an irreducible finite-dimensional module over $\mathfrak{d}$ (respectively $\mathfrak{gl}\,\mathfrak{d}$). In this case, $\mathcal{T}(R) = \mathcal{T}(\Pi, U)$.

As usual, we fix a basis $\{\partial_i\}$ of $\mathfrak{d}$. Recall that the action of $1 \otimes \partial_i \in W(\mathfrak{d})$ on an element $1 \otimes u \in \mathbf{k} \otimes R \subset \mathcal{T}(R)$ is given by (4.30). For us, it will be convenient to rewrite (4.30) as follows (making use of (2.13)):

$$\begin{aligned}(1 \otimes \partial_i) * (1 \otimes u) &= (1 \otimes 1) \otimes_H (1 \otimes (\partial_i + \operatorname{ad}\partial_i) u) \\ &\quad + \sum_{j=1}^{N} (1 \otimes 1) \otimes_H (\partial_j \otimes e_i^j u) - \sum_{j=1}^{N} (1 \otimes \partial_j) \otimes_H (1 \otimes (e_i^j + \delta_i^j) u)\,.\end{aligned} \tag{6.14}$$

Introduce the following notation:

$$s(\partial_i, u) = \sum_{j=1}^{N} \partial_j \otimes e_i^j u\,, \qquad u \in R\,,\ i = 1, \dots, N\,. \tag{6.15}$$

By linearity, we define $s(\partial, u)$ for all $\partial \in \mathfrak{d}$. Then $s(\partial, u)$ does not depend on the choice of basis $\{\partial_i\}$ of $\mathfrak{d}$ (cf. (4.26), (4.27)).

Since $\mathcal{T}(R) = H \otimes R$, any element $v \in \mathcal{T}(R)$ can be written uniquely in the form

$$v = \sum_{I \in \mathbb{Z}_+^N} \partial^{(I)} \otimes v_I\,, \qquad v_I \in R\,, \tag{6.16}$$

where $\partial^{(I)} \in H$ are given by (2.14). Note that the above sum is finite, i.e., $v_I \neq 0$ only for finitely many I. From (6.14)–(6.16) and H-bilinearity, we find

$$\begin{aligned}(1 \otimes \partial_i) * v = &\sum_I (1 \otimes \partial^{(I)}) \otimes_H (1 \otimes (\partial_i + \operatorname{ad}\partial_i) v_I) \\ &+ \sum_I (1 \otimes \partial^{(I)}) \otimes_H s(\partial_i, v_I) \\ &- \sum_I \sum_{j=1}^{N} (1 \otimes \partial^{(I)} \partial_j) \otimes_H (1 \otimes (e_i^j + \delta_i^j) v_I)\,.\end{aligned} \tag{6.17}$$

Definition 6.3. The nonzero elements v_I in the expression (6.16) are called *coefficients* of $v \in \mathcal{T}(R)$. For a submodule $M \subset \mathcal{T}(R)$, we denote by $\operatorname{coeff} M$ the subspace of R linearly generated by all coefficients of elements of M.

Recall that $\mathcal{T}(R)$ has a filtration given by $\mathrm{F}^p \mathcal{T}(R) = \mathrm{F}^p H \otimes R$ (cf. (6.11) and (6.9)). We have: $\mathrm{F}^{-1} \mathcal{T}(R) = \{0\}$, $\mathrm{F}^0 \mathcal{T}(R) = \mathbf{k} \otimes R$ and $\mathrm{F}^1 \mathcal{T}(R) = (\mathbf{k} + \mathfrak{d}) \otimes R$.

Lemma 6.6. *For any nontrivial proper $W(\mathfrak{d})$-submodule M of $\mathcal{T}(R)$, we have $M \cap \mathrm{F}^0 \mathcal{T}(R) = \{0\}$.*

Proof. Let M_0 be the set of all $u \in R$ such that $1 \otimes u \in M$. By (6.14) and Remark 2.2(i), we have:

$$(\partial_i + \operatorname{ad}\partial_i) u \in M_0\,, \quad (e_i^j + \delta_i^j) u \in M_0 \quad \text{for all} \quad i, j = 1, \dots, N\,,\ u \in M_0\,.$$

This means that M_0 is a $(\mathfrak{d} \oplus \mathfrak{gl}\mathfrak{d})$-submodule of R. Since R is irreducible, either $M_0 = \{0\}$ or $M_0 = R$. In the latter case, we obtain $M \supset H \otimes M_0 = \mathcal{T}(R)$, which is a contradiction. Therefore, $M \cap (\mathbf{k} \otimes R) = \{0\}$. □

Corollary 6.3. *If $\operatorname{sing} \mathcal{T}(R) = \mathrm{F}^0 \mathcal{T}(R)$, then the tensor $W(\mathfrak{d})$-module $\mathcal{T}(R)$ is irreducible.*

Proof. If $M \subset \mathcal{T}(R)$ is a nontrivial proper submodule, then by Theorem 6.1 it contains a nonzero singular vector. This contradicts Lemma 6.6. □

Corollary 6.3 will play a crucial role in our classification of irreducible finite $W(\mathfrak{d})$-modules.

Lemma 6.7. *For any nontrivial proper $W(\mathfrak{d})$-submodule M of $\mathcal{T}(R)$, we have $\operatorname{coeff} M = R$.*

Proof. Pick a nonzero element $v \in M$ and write in the form (6.16). Then, for fixed I, the coefficient multiplying $1 \otimes \partial^{(I)}$ in the right-hand side of (6.17) equals

$$s(\partial_i, v_I) + 1 \otimes \partial_i v_I + \text{terms in } \mathbf{k} \otimes (\mathfrak{gl}\mathfrak{d} + \mathbf{k})(\operatorname{coeff} M)\,. \tag{6.18}$$

By Remark 2.2(ii), this is an element of M. Hence, for each coefficient v_I of v, we have $e_i^j v_I \in \operatorname{coeff} M$. Then from (6.18) we also get $\partial_i v_I \in \operatorname{coeff} M$. Therefore, $\operatorname{coeff} M$ is a nontrivial $(\mathfrak{d} \oplus \mathfrak{gl}\mathfrak{d})$-submodule of R. But R is irreducible; hence, $\operatorname{coeff} M = R$. □

Lemma 6.8. *Let M be a nontrivial proper $W(\mathfrak{d})$-submodule of $\mathcal{T}(R)$. Then for every $\partial \in \mathfrak{d}$ and $u \in R$, there is a unique element $s_M(\partial, u) \in M$ such that*

$$s_M(\partial, u) = s(\partial, u) \mod \mathrm{F}^0\,\mathcal{T}(R)\,, \tag{6.19}$$

where $s(\partial, u)$ is given by (6.15)*. The element $s_M(\partial, u)$ depends linearly on both ∂ and u.*

Proof. Uniqueness follows from Lemma 6.6. From uniqueness we deduce that $s_M(\partial, u)$ depends linearly on ∂ and u. Then, to prove existence, it is enough to consider the case $\partial = \partial_i$ and $u = v_I$ for some $v \in M$ (because $R = \operatorname{coeff} M$ by Lemma 6.7). In this case $s_M(\partial_i, v_I)$ is exactly the element (6.18). □

Elements $s_M(\partial, u)$ will be used in the next subsection to determine all singular vectors of M.

6.4. **Computation of Singular Vectors.** In this subsection, we continue to use the notation of Section 6.3. Our goal is to find all singular vectors of $\mathcal{T}(R) = \mathcal{T}(\Pi, U)$. Given a nontrivial proper $W(\mathfrak{d})$-submodule M of $\mathcal{T}(R)$, we also find all singular vectors of M. These results will be used in Section 6.5 to classify irreducible finite $W(\mathfrak{d})$-modules.

First, we consider the case when the $\mathfrak{gl}\,\mathfrak{d}$-action on R is trivial.

Proposition 6.1. *For any irreducible finite-dimensional $\mathfrak{d}$-module Π, we have*:
(i) $\operatorname{sing}\mathcal{T}(\Pi, \mathbf{k}) = \mathrm{F}^0\,\mathcal{T}(\Pi, \mathbf{k})$;
(ii) $\mathcal{T}(\Pi, \mathbf{k})$ *is an irreducible $W(\mathfrak{d})$-module.*

Proof. Pick a singular vector $v \in \mathcal{T}(\Pi, \mathbf{k})$, and write it in the form (6.16). Then, by (6.17),

$$(1 \otimes \partial_i) * v = \sum_I (1 \otimes \partial^{(I)}) \otimes_H (1 \otimes \partial_i v_I) - \sum_I (1 \otimes \partial^{(I)}\partial_i) \otimes_H (1 \otimes v_I)\,.$$

Now Remark 6.1 implies that $v_I = 0$ whenever $|I| \geq 1$. This proves (i).

(ii) follows from (i) and Corollary 6.3. □

Lemma 6.9. *For any irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module R, the tensor $W(\mathfrak{d})$-module $\mathcal{T}(R)$ satisfies*

$$\mathrm{F}^0\,\mathcal{T}(R) \subset \operatorname{sing}\mathcal{T}(R) \subset \mathrm{F}^1\,\mathcal{T}(R)\,.$$

Proof. First of all, by Proposition 6.1(i), we can assume that the $\mathfrak{gl}\,\mathfrak{d}$-action on R is nontrivial. Since $\mathrm{F}^0\,\mathcal{T}(R) = \mathbf{k} \otimes R$, the first inclusion follows from (6.14) and Remark 6.1. To prove the second one, pick a nonzero singular vector v and write in the form (6.16). Then $(1 \otimes \partial_i) * v$ is given by formula (6.17). The coefficient multiplying $1 \otimes \partial^{(I)}$ in (6.17) is given by (6.18). By Remark 6.1, this coefficient must vanish whenever $|I| > 1$. Hence, $s(\partial_i, v_I) = 0$ for all i, which implies $(\mathfrak{gl}\,\mathfrak{d})v_I = 0$. Therefore, $v_I = 0$. This proves that $\operatorname{sing}\mathcal{T}(R) \subset \mathrm{F}^1\,\mathcal{T}(R)$. □

Lemma 6.10. *An element*

$$v = \sum_{k=1}^{N} \partial_k \otimes v^k \in \mathfrak{d} \otimes R \subset \mathcal{T}(R) \tag{6.20}$$

is a singular vector iff it satisfies the equations

$$(e_i^j + \delta_i^j)v^k + (e_i^k + \delta_i^k)v^j = 0 \qquad \textit{for all} \quad i, j, k = 1, \dots, N\,. \tag{6.21}$$

In this case, for the action ρ_{sing} of $\mathfrak{gl}\,\mathfrak{d}$ on v, we have (*see* (6.15)):

$$\rho_{\mathrm{sing}}(e_i^k)v = -s(\partial_i, v^k) \mod \mathrm{F}^0\,\mathcal{T}(R)\,. \tag{6.22}$$

Proof. As a special case of (6.17), we have:

$$\begin{aligned}(1 \otimes \partial_i) * v = &\sum_{k=1}^{N} (1 \otimes \partial_k) \otimes_H (1 \otimes (\partial_i + \operatorname{ad}\partial_i)v^k) + \sum_{k=1}^{N} (1 \otimes \partial_k) \otimes_H s(\partial_i, v^k) \\ &- \sum_{k,j=1}^{N} (1 \otimes \partial_k\partial_j) \otimes_H (1 \otimes (e_i^j + \delta_i^j)v^k)\,.\end{aligned}$$

For $k \leq j$, the coefficient multiplying $1 \otimes \partial_k \partial_j$ is up to a sign equal to

$$1 \otimes (e_i^j + \delta_i^j) v^k + 1 \otimes (e_i^k + \delta_i^k) v^j \,. \tag{6.23}$$

By Remark 6.1, v is a singular vector iff this coefficient vanishes for all j, k. This proves (6.21). On the other hand, the coefficient multiplying $1 \otimes \partial_k$ is equal to $s(\partial_i, v^k)$ modulo $\mathrm{F}^0\,\mathcal{T}(R)$. Then (6.22) follows from (6.4). □

Our next result describes all singular vectors in a tensor $W(\mathfrak{d})$-module $\mathcal{T}(R)$.

Theorem 6.3. *For any irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module R, we have*:

$$\operatorname{sing} \mathcal{T}(R) = \mathrm{F}^0\,\mathcal{T}(R) + \{s(\partial, u) \mid \partial \in \mathfrak{d}\,,\ u \in R_0\}\,, \tag{6.24}$$

where $s(\partial, u)$ is defined by (6.15) *and R_0 is the subspace of all $u \in R$ satisfying the equations*

$$(e_i^j + \delta_i^j) e_l^k u + (e_i^k + \delta_i^k) e_l^j u = 0\,, \qquad i, j, k, l = 1, \dots, N\,. \tag{6.25}$$

The subspace R_0 is either $\{0\}$ or the whole R.

Proof. When $R = \Pi\boxtimes\mathbf{k}$, (6.24) follows from Proposition 6.1(i), because in this case all $s(\partial, u) = 0$. Let us assume that the $\mathfrak{gl}\,\mathfrak{d}$-action on R is nontrivial, and denote the space in the right-hand side of (6.24) by S. Notice that $s(\partial, u)$ is a singular vector iff $u \in R_0$, because for $v = s(\partial_l, u)$ we have $v^k = e_l^k u$ and (6.21) becomes (6.25). Hence, Lemmas 6.9 and 6.10 imply $S \subset \operatorname{sing} \mathcal{T}(R)$.

From these lemmas, we also deduce that the action ρ_{sing} of $\mathfrak{gl}\,\mathfrak{d}$ on $\operatorname{sing} \mathcal{T}(R)$ maps $\operatorname{sing} \mathcal{T}(R)$ into S. Consider the finite-dimensional $\mathfrak{gl}\,\mathfrak{d}$-module $\operatorname{sing} \mathcal{T}(R)/\,\mathrm{F}^0\,\mathcal{T}(R)$. We claim that its decomposition as a direct sum of irreducibles does not contain the trivial $\mathfrak{gl}\,\mathfrak{d}$-module. Indeed, let $v \in \operatorname{sing} \mathcal{T}(R)$ be such that $\rho_{\mathrm{sing}}(e_i^k) v \in \mathrm{F}^0\,\mathcal{T}(R)$ for all i, k. We want to show that $v \in \mathrm{F}^0\,\mathcal{T}(R)$. Without loss of generality, we can assume that $v \in \mathfrak{d} \otimes R$. Then, by (6.22), all $s(\partial_i, v^k) = 0$, and from (6.15), $e_i^j v^k = 0$ for all i, j, k. This implies $v^k = 0$ for all k, and $v = 0$. Therefore, the $\mathfrak{gl}\,\mathfrak{d}$-action on $\operatorname{sing} \mathcal{T}(R)/\,\mathrm{F}^0\,\mathcal{T}(R)$ is surjective, and $\operatorname{sing} \mathcal{T}(R) = S$.

Clearly, if $R = \Pi \boxtimes U$, then $R_0 = \Pi \boxtimes U_0$, where U_0 is the subspace of all $u \in U$ satisfying (6.25). We claim that U_0 is a $\mathfrak{gl}\,\mathfrak{d}$-submodule of U. Since U is an irreducible $\mathfrak{gl}\,\mathfrak{d}$-module, this would imply that either $U_0 = \{0\}$ or $U_0 = U$. Now if $u \in R_0$, then $v = s(\partial_l, u)$ is a singular vector for all l. By (6.22), all $s(\partial_i, v^k)$ are singular vectors too. Hence, $v^k = e_l^k u$ belongs to R_0 for all k, l. Therefore, $(\mathfrak{gl}\,\mathfrak{d}) R_0 \subset R_0$, which implies $(\mathfrak{gl}\,\mathfrak{d}) U_0 \subset U_0$. □

Corollary 6.4. *If the $W(\mathfrak{d})$-module $\mathcal{T}(R)$ is not irreducible, then equations* (6.25) *are satisfied for all $u \in R$.*

Proof. This follows from Corollary 6.3 and Theorem 6.3. □

Next, we find all singular vectors in a nontrivial proper $W(\mathfrak{d})$-submodule M of $\mathcal{T}(R)$. Recall the elements $s_M(\partial, u) \in M$, constructed in Lemma 6.8.

Theorem 6.4. *For any nontrivial proper $W(\mathfrak{d})$-submodule M of $\mathcal{T}(R)$, we have*:
(i) $\operatorname{sing} M = M \cap \mathrm{F}^1\,\mathcal{T}(R) = \{s_M(\partial, u) \mid \partial \in \mathfrak{d}\,,\ u \in R\}$;
(ii) $\operatorname{sing} \mathcal{T}(R) = \mathrm{F}^0\,\mathcal{T}(R) \oplus \operatorname{sing} M$ *as $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-modules.*

Proof. (i) Note that $\operatorname{sing} M \subset M \cap \mathrm{F}^1\,\mathcal{T}(R)$ by Lemma 6.9. Conversely, pick $v' \in M \cap \mathrm{F}^1\,\mathcal{T}(R)$, and write $v' = v + 1 \otimes u$ with $v \in \mathfrak{d} \otimes R$, $u \in R$. Since $1 \otimes u \in \operatorname{sing} \mathcal{T}(R)$, the vector v' is singular if and only if v is. In the proof of Lemma 6.10 we saw that the coefficient multiplying $1 \otimes \partial_k \partial_j$ in $(1 \otimes \partial_i) * v$ is up to a sign equal to (6.23). By Remark 6.1, $(1 \otimes \partial_i) * v'$ has the same coefficient. Now Remark 2.2(ii) implies that the elements (6.23) belong to M. Hence, they vanish by Lemma 6.6. Then, by Lemma 6.10, v is a singular vector, and $v' \in \operatorname{sing} M$.

This proves the first equality in (i). The second equality follows from the first one, Lemma 6.8 and Theorem 6.3.

(ii) The sum is direct because of Lemma 6.6. The equality follows from part (i), Lemma 6.8 and Theorem 6.3. □

Corollary 6.5. *Let R be an irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module, and let M, M' be two nontrivial proper $W(\mathfrak{d})$-submodules of $\mathcal{T}(R)$. Then* $\operatorname{sing} M = \operatorname{sing} M'$.

Proof. Consider the canonical projection of $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-modules

$$\pi\colon \operatorname{sing}\mathcal{T}(R) \to \operatorname{sing}\mathcal{T}(R)/\,\mathrm{F}^0\,\mathcal{T}(R) \subset \mathrm{gr}^1\,\mathcal{T}(R)\,.$$

By Theorem 6.4(ii), the restriction of π to $\operatorname{sing} M$ is an isomorphism. On the other hand, combining (6.10) with Corollary 6.2, we obtain isomorphisms of $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-modules

$$\mathrm{F}^0\,\mathcal{T}(R) \simeq \widetilde{\Pi} \boxtimes \widetilde{U}\,, \quad \mathrm{gr}^1\,\mathcal{T}(R) \simeq \widetilde{\Pi} \boxtimes (\mathfrak{d} \otimes \widetilde{U})\,,$$

where

$$R = \Pi \boxtimes U\,, \quad \widetilde{\Pi} = \Pi \otimes \mathbf{k}_{-\operatorname{tr}\operatorname{ad}}\,, \quad \widetilde{U} = U \otimes \mathbf{k}_{\mathrm{tr}}\,.$$

The $\mathfrak{gl}\,\mathfrak{d}$-module $\widetilde{U}$ is irreducible. Say that $\mathrm{Id} \in \mathfrak{gl}\,\mathfrak{d}$ acts as the scalar c on $\widetilde{U}$. Then it acts as $c+1$ on $\mathfrak{d} \otimes \widetilde{U}$. It follows that $\operatorname{sing} M$ is precisely the set of all vectors $v \in \operatorname{sing}\mathcal{T}(R)$ such that $\mathrm{Id}\cdot v = (c+1)v$. The same is true for M' instead of M. □

6.5. **Irreducible Finite $W(\mathfrak{d})$-Modules.** This subsection contains our main results about irreducible finite $W(\mathfrak{d})$-modules. As before, let Π (respectively U) be an irreducible finite-dimensional representation of $\mathfrak{d}$ (respectively $\mathfrak{gl}\,\mathfrak{d}$). First, we determine which tensor $W(\mathfrak{d})$-modules are irreducible.

Theorem 6.5. *The tensor $W(\mathfrak{d})$-module $\mathcal{T}(\Pi, U)$ is irreducible if and only if, as a $\mathfrak{gl}\,\mathfrak{d}$-module, U is not isomorphic to $\bigwedge^n \mathfrak{d}^*$ for any $n \geq 1$.*

Proof. Assume that $\mathcal{T}(\Pi, U)$ is not irreducible. Then, by Corollary 6.4, equations (6.25) are satisfied for all $u \in R$. In the special case $i = j = k = l$ they give

$$(e_i^i + 1)e_i^i u = 0 \qquad \text{for all} \quad u \in R = \Pi \boxtimes U\,. \tag{6.26}$$

We claim that the $\mathfrak{gl}\,\mathfrak{d}$-module U is isomorphic to $\Omega^n := \bigwedge^n \mathfrak{d}^*$ for some n.

To prove this, first note that the matrix $\mathrm{Id} \in \mathfrak{gl}\,\mathfrak{d}$ acts as a scalar on U, and the module U remains irreducible when restricted to $\mathfrak{sl}\,\mathfrak{d}$. Then U has a highest weight vector v, and the $\mathfrak{gl}\,\mathfrak{d}$-module U is uniquely determined by its highest weight, i.e., by the eigenvalues λ_i of e_i^i on v. Furthermore, all $\lambda_i - \lambda_{i+1}$ are non-negative integers (see, e.g., [Se, Chapter VII]). But by (6.26), all $\lambda_i = 0$ or -1; hence the N-tuple $(\lambda_1, \dots, \lambda_N)$ has the form $(0, \dots, 0, -1, \dots, -1)$. The module Ω^n has such a highest weight, where the number of -1's is n.

Therefore, $U \simeq \Omega^n$, and the case $n = 0$ is excluded by Proposition 6.1(ii). Next, recall the Π-twisted pseudo de Rham complex (5.15), and introduce the shorthand notation

$$T^n := \mathcal{T}(\Pi, \Omega^n)\,, \quad I^n := \mathrm{d}_\Pi(T^{n-1}) \subset T^n\,. \tag{6.27}$$

Since the differential d_Π is a homomorphism of $W(\mathfrak{d})$-modules, the image I^n is a submodule of T^n for each $n = 1, \dots, N$. It is easy to see from Proposition 5.2 and Lemma 5.3 that I^n is nontrivial and proper. Therefore, the tensor modules T^n are not irreducible for $n \geq 1$. □

Corollary 6.6. *Let R be an irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module. Then the $W(\mathfrak{d})$-module $\mathcal{T}(R)$ is irreducible if and only if* $\operatorname{sing}\mathcal{T}(R) = \mathrm{F}^0\,\mathcal{T}(R)$.

Proof. In one direction, the statement is exactly Corollary 6.3. The opposite direction follows from Theorem 6.3 and the proof of Theorem 6.5. □

Our next goal is to study the submodules I^n of T^n (see (6.27)).

Lemma 6.11. *For $1 \leq n \leq N$, the $W(\mathfrak{d})$-submodule $I^n \subset T^n$ has the following properties*:
(i) *I^n is nontrivial and proper*;
(ii) $\operatorname{sing} I^n = \mathrm{d}_\Pi(\mathbf{k} \otimes \Pi \otimes \Omega^{n-1})$;
(iii) *I^n is generated by* $\operatorname{sing} I^n$ *as an H-module*;
(iv) *Any nontrivial proper submodule M of T^n contains I^n*;
(v) *I^n is an irreducible $W(\mathfrak{d})$-module.*

Proof. (i) is easy to see from Proposition 5.2 and Lemma 5.3.

(ii) Formula (5.17) and Lemma 6.8 imply

$$s_{I^n}(\partial_i, u\otimes\alpha) = \mathrm{d}_\Pi(1\otimes u\otimes\iota_{\partial_i}\alpha)\,, \qquad u\in\Pi\,,\ \alpha\in\Omega^n\,,\ i=1,\dots,N\,.$$

Then (ii) follows from Theorem 6.4(i) and the fact that Ω^{n-1} is linearly spanned by all $\iota_{\partial_i}\alpha$.

(iii) is obvious from (ii) and the H-linearity of d_Π.

(iv) By Corollary 6.5, we have $\operatorname{sing} M = \operatorname{sing} I^n$. Then $M \supset H(\operatorname{sing} M) = H(\operatorname{sing} I^n)$, which is equal to I^n by part (iii).

(v) is obvious from (iv). □

Note that, from the exactness of the complex (5.15), we have $I^1 \simeq T^0 = \mathcal{T}(\Pi,\Omega^0) = \mathcal{T}(\Pi,\mathbf{k})$.

Lemma 6.12. *For $1\le n\le N-1$, I^n is the unique nontrivial proper $W(\mathfrak{d})$-submodule of T^n.*

Proof. If M is a nontrivial proper submodule of T^n, it contains I^n. The image $\mathrm{d}_\Pi M$ is a submodule of I^{n+1}; hence, $\mathrm{d}_\Pi M$ is either $\{0\}$ or the whole I^{n+1}. But the kernel of d_Π is equal to I^n, by the exactness of the complex (5.15). We obtain that either $M=I^n$ or $M=T^n$. □

Now we can classify all irreducible finite $W(\mathfrak{d})$-modules.

Theorem 6.6. *Any irreducible finite $W(\mathfrak{d})$-module is isomorphic to one of the following*:

(i) *Tensor modules $\mathcal{T}(\Pi,U)$, where Π is an irreducible finite-dimensional $\mathfrak{d}$-module, and U is an irreducible finite-dimensional $\mathfrak{gl}\,\mathfrak{d}$-module not isomorphic to $\bigwedge^n\mathfrak{d}^*$ for any $n\ge1$;*

(ii) *Images $\mathrm{d}_\Pi\mathcal{T}(\Pi,\Omega^n)$, where Π is an irreducible finite-dimensional $\mathfrak{d}$-module, and $1\le n\le\dim\mathfrak{d}-1$ (see* (5.15)).

Proof. Let V be an irreducible finite $W(\mathfrak{d})$-module. Then, by Theorem 6.2 and (6.9), V is a quotient of some tensor module $\mathcal{T}(R)=\mathcal{T}(\Pi,U)$.

If $U\not\simeq\bigwedge^n\mathfrak{d}^*(=\Omega^n)$ as a $\mathfrak{gl}\,\mathfrak{d}$-module for any $n\ge1$, then $\mathcal{T}(R)$ is irreducible by Theorem 6.5. In this case, $V\simeq\mathcal{T}(R)$.

Assume that $U\simeq\Omega^n$ for some $n\ge1$; then $\mathcal{T}(R)\simeq\mathcal{T}(\Pi,\Omega^n)=T^n$ (see (6.27)). Now if $n\le N-1$, $N=\dim\mathfrak{d}$, Lemma 6.12 implies that $V\simeq T^n/I^n$. By the exactness of (5.15), we get $V\simeq I^{n+1}=\mathrm{d}_\Pi\mathcal{T}(\Pi,\Omega^n)$.

Finally, it remains to consider the case when V is a quotient of T^N. Then $V\simeq T^N/M$, where $M\supset I^N$ due to Lemma 6.11(iv). Now Proposition 5.2 implies that V is finite dimensional; hence, $W(\mathfrak{d})$ acts trivially on it by Example 2.3. So in this case V cannot be irreducible. □

Theorem 6.7. *The irreducible finite $W(\mathfrak{d})$-modules listed in Theorem* 6.6 *satisfy*:

(i) $\operatorname{sing}\mathcal{T}(\Pi,U)\simeq(\Pi\otimes\mathbf{k}_{-\operatorname{tr}\operatorname{ad}})\boxtimes(U\otimes\mathbf{k}_{\mathrm{tr}})$ *as $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-modules*;

(ii) $\operatorname{sing}(\mathrm{d}_\Pi\mathcal{T}(\Pi,\Omega^n))\simeq(\Pi\otimes\mathbf{k}_{-\operatorname{tr}\operatorname{ad}})\boxtimes(\Omega^n\otimes\mathbf{k}_{\mathrm{tr}})$ *as $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-modules.*

In particular, no two of them are isomorphic to each other.

Proof. First, note that if $\beta\colon V\to V'$ is a homomorphism of $W(\mathfrak{d})$-modules, then its restriction to $\operatorname{sing}V$ is a homomorphism of $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-modules $\operatorname{sing}V\to\operatorname{sing}V'$. In particular, if V and V' are isomorphic, then $\operatorname{sing}V\simeq\operatorname{sing}V'$.

(i) If $\mathcal{T}(R)=\mathcal{T}(\Pi,U)$ is irreducible, then by Corollary 6.6, $\operatorname{sing}\mathcal{T}(R)=\mathrm{F}^0\mathcal{T}(R)=\mathbf{k}\otimes R$. Now (i) follows from (6.6) and (6.10).

(ii) By Lemma 6.11(ii), we have: $\operatorname{sing}(\mathrm{d}_\Pi\mathcal{T}(\Pi,\Omega^n))=\mathrm{d}_\Pi(\mathrm{F}^0\mathcal{T}(\Pi,\Omega^n))$. But $\mathrm{F}^0\mathcal{T}(\Pi,\Omega^n)\simeq(\Pi\otimes\mathbf{k}_{-\operatorname{tr}\operatorname{ad}})\boxtimes(\Omega^n\otimes\mathbf{k}_{\mathrm{tr}})$ is an irreducible $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-module. Therefore, d_Π is an isomorphism from $\mathrm{F}^0\mathcal{T}(\Pi,\Omega^n)$ onto $\operatorname{sing}(\mathrm{d}_\Pi\mathcal{T}(\Pi,\Omega^n))$. □

Remark 6.3. Let R and R' be two non-isomorphic irreducible finite-dimensional $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-modules. Using Theorem 6.4(ii) and the same argument as in the proof of Theorem 6.7, one can show that the only nonzero homomorphisms of $W(\mathfrak{d})$-modules $\mathcal{T}(R)\to\mathcal{T}(R')$ are, up to a constant factor, the differentials d_Π.

7. Classification of Irreducible Finite $S(\mathfrak{d},\chi)$-Modules

In this section we adapt the classification results of Section 6 to the case of the Lie pseudoalgebra $S(\mathfrak{d},\chi)$. Our main result is Theorem 7.6.

7.1. **Singular Vectors.** Recall that the annihilation algebra $\mathcal{S}$ of $S(\mathfrak{d},\chi)$ possesses a decreasing filtration $\{\mathcal{S}_p\}_{p\geq -1}$ by subspaces of finite codimension, as given by (3.20). This filtration is compatible with that of $\mathcal{W}$ given by (3.8), (3.11), in the sense made clear by Lemma 3.6 and Proposition 3.5. In particular, we know that $\mathcal{S}\subset\mathcal{W}\simeq W_N$ is a graded Lie algebra isomorphic to S_N, the grading being given by the eigenspace decomposition with respect to the adjoint action of the element $\widehat{\mathcal{E}}\in\mathcal{W}$ described in Definition 3.3. We denote the i-eigenspace by $\mathfrak{s}_i$; hence, $\mathcal{S}_p=\prod_{i\geq p}\mathfrak{s}_i$. Note that we can do the same with the extended annihilation algebra $\widetilde{\mathcal{S}}$, as $\widehat{\mathcal{E}}$ commutes with $\widetilde{\mathfrak{d}}$. We denote the corresponding eigenspaces by $\widetilde{\mathfrak{s}}_i$; then $\widetilde{\mathfrak{d}}\subset\widetilde{\mathfrak{s}}_0$ and $\widetilde{\mathfrak{s}}_i=\mathfrak{s}_i$ for $i\neq 0$.

In analogy with the case of $W(\mathfrak{d})$, for an $\mathcal{S}$-module V, we denote by $\ker_p V$ the space of all elements $v\in V$ that are killed by $\mathcal{S}_p$. We denote by $\ker V$ the space $\ker_{-1}V$ killed by $\mathcal{S}=\mathcal{S}_{-1}$. Then the module V is *conformal* iff $V=\bigcup_p\ker_p V$. Any $S(\mathfrak{d},\chi)$-module has a natural structure of a conformal module over the extended annihilation algebra $\widetilde{\mathcal{S}}=\mathfrak{d}\ltimes\mathcal{S}$ (see Proposition 2.1). The normalizer of $\mathcal{S}_p$ in $\widetilde{\mathcal{S}}$ was computed in Section 3.5. It is independent of $p\geq 0$, and is denoted by $\mathcal{N}_{\mathcal{S}}$. Note that $\mathcal{N}_{\mathcal{S}}=\prod_{i\geq 0}\widetilde{\mathfrak{s}}_i$ and $\widetilde{\mathcal{S}}=\widetilde{\mathfrak{s}}_{-1}\oplus\mathcal{N}_{\mathcal{S}}$ as a vector space.

Each $\ker_p V$ is a module over the finite-dimensional quotient $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_p$; moreover, the Lie algebra $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1=\widehat{\mathfrak{d}}\oplus(\mathcal{S}_0/\mathcal{S}_1)$ is isomorphic to the direct sum $\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d}$.

Definition 7.1. A *singular vector* in an $S(\mathfrak{d},\chi)$-module V is an element $v\in V$ such that $\mathcal{S}_1\cdot v=0$. The space of singular vectors $\ker_1 V$ is also denoted by $\operatorname{sing} V$.

Theorem 7.1. *For any nontrivial finite $S(\mathfrak{d},\chi)$-module V, we have $\operatorname{sing} V\neq\{0\}$, and the space $\operatorname{sing} V/\ker V$ is finite dimensional.*

Proof. The proof is the same as that of Theorem 6.1, making use of Proposition 3.9 instead of Proposition 3.4. □

Recall that $S(\mathfrak{d},\chi)$ is generated over H by the elements s_{ab} defined in (3.5). It will be convenient to introduce the notation

$$\bar\partial=\partial+\chi(\partial)\,,\qquad \partial\in\mathfrak{d}\,, \tag{7.1}$$

and

$$s_{ij}\equiv s_{\partial_i\partial_j}=\bar\partial_i\otimes\partial_j-\bar\partial_j\otimes\partial_i-1\otimes[\partial_i,\partial_j]\,, \tag{7.2}$$

where, as before, $\{\partial_i\}$ is a fixed basis of $\mathfrak{d}$.

Remark 7.1. By (3.20) and Proposition 2.1, a vector $v\in V$ is singular if and only if

$$s_{ij}*v\in(\mathrm{F}^2\,H\otimes\mathbf{k})\otimes_H V\,,\qquad i,j=1,\dots,N\,,$$

or, equivalently,

$$s_{ij}*v\in(\mathbf{k}\otimes\mathrm{F}^2\,H)\otimes_H V\,,\qquad i,j=1,\dots,N\,.$$

7.2. **Tensor Modules for $S(\mathfrak{d},\chi)$.** Let R be a finite-dimensional $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-module, with an action denoted by $\widehat{\rho}_R$. Then the isomorphism $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1\simeq\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d}$ can be employed to make R an $\mathcal{N}_{\mathcal{S}}$-module with a trivial action of $\mathcal{S}_1$. For example, the action of the subalgebra $\widehat{\mathfrak{d}}\subset\mathcal{N}_{\mathcal{S}}$ is given by:

$$\widehat{\partial}\cdot u=\widehat{\rho}_R(\partial)u\,,\qquad \partial\in\mathfrak{d}\,,\ u\in R\,. \tag{7.3}$$

Consider the induced $\widetilde{\mathcal{S}}$-module $V=\operatorname{Ind}_{\mathcal{N}_{\mathcal{S}}}^{\widetilde{\mathcal{S}}}R$. Since as a vector space $\widetilde{\mathcal{S}}=\mathfrak{d}\oplus\mathcal{N}_{\mathcal{S}}$ (see Proposition 3.8), as an H-module V is isomorphic to the free module $H\otimes R$.

Definition 7.2. The $S(\mathfrak{d},\chi)$-module $\mathrm{Ind}_{\mathcal{N}_\mathcal{S}}^{\widetilde{\mathcal{S}}} R$ constructed above will be denoted by $\mathcal{V}_\chi(R)$, and will be called a *tensor module* for the Lie pseudoalgebra $S(\mathfrak{d},\chi)$. If R is an irreducible $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-module isomorphic to $\Pi\boxtimes U$, then we will also write $\mathcal{V}_\chi(R)=\mathcal{V}_\chi(\Pi,U)$.

The name tensor module is justified by the fact that, as we will show in Theorem 7.3 below, the $S(\mathfrak{d},\chi)$-module $\mathcal{V}_\chi(R)$ is the restriction to $S(\mathfrak{d},\chi)\subset W(\mathfrak{d})$ of a tensor module for $W(\mathfrak{d})$ (see also Remark 6.2).

Theorem 7.2. *Let V be an $S(\mathfrak{d},\chi)$-module, and let R be a $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-submodule of $\operatorname{sing}V$. Then HR is an $S(\mathfrak{d},\chi)$-submodule of V, and there is a natural surjective homomorphism $\mathcal{V}_\chi(R)\to HR$. In particular, every irreducible finite $S(\mathfrak{d},\chi)$-module is a quotient of a tensor module.*

Proof. Since $\widetilde{\mathcal{S}}=\mathfrak{d}\oplus\mathcal{N}_\mathcal{S}$ as a vector space, and $R\subset\operatorname{sing}V$, it follows that HR is preserved by the action of $\widetilde{\mathcal{S}}$. Then by Proposition 2.1, HR is an $S(\mathfrak{d},\chi)$-submodule of V. The existence of a natural surjective homomorphism $\mathcal{V}_\chi(R)\to HR$ follows from the definition of $\mathcal{V}_\chi(R)$. Finally, if V is irreducible and finite, then by Theorem 7.1, $\operatorname{sing}V\neq\{0\}$, and we have $H(\operatorname{sing}V)=V$. □

Lemma 7.1. *The unique injection $\iota\colon S(\mathfrak{d},\chi)\to W(\mathfrak{d})$ induces an injective Lie algebra homomorphism $\iota_*\colon\mathcal{N}_\mathcal{S}/\mathcal{S}_1\to\mathcal{N}_\mathcal{W}/\mathcal{W}_1$. The homomorphism ι_* satisfies $\iota_*(\widehat{\mathfrak{d}})\subset\widetilde{\mathfrak{d}}\oplus\mathbf{k}\,\mathrm{Id}$. More precisely (see* (3.29)*),*

$$\iota_*(\widehat{\partial})=\widetilde{\partial}+\frac{1}{N}(\operatorname{tr}\operatorname{ad}-\chi)(\partial)\,\mathrm{Id}\,,\qquad\partial\in\mathfrak{d}\,. \tag{7.4}$$

Furthermore, ι_* embeds $\mathcal{S}_0/\mathcal{S}_1\simeq\mathfrak{sl}\,\mathfrak{d}$ as the Lie subalgebra $\mathfrak{sl}\,\mathfrak{d}\subset\mathfrak{gl}\,\mathfrak{d}\simeq\mathcal{W}_0/\mathcal{W}_1$.

Proof. By Proposition 3.5, the induced Lie algebra homomorphism $\iota\colon\mathcal{S}\to\mathcal{W}$ is injective and satisfies $\iota(\mathcal{S}_1)\subset\mathcal{W}_1$. Hence, ι_* is injective. The rest of the lemma follows from (3.29) and Corollary 3.4. □

Lemma 7.1 shows that $\mathcal{N}_\mathcal{W}/\mathcal{W}_1=\iota_*(\mathcal{N}_\mathcal{S}/\mathcal{S}_1)\oplus\mathbf{k}\,\mathrm{Id}$. Hence, every $\mathcal{N}_\mathcal{S}/\mathcal{S}_1$-module can be extended to an $\mathcal{N}_\mathcal{W}/\mathcal{W}_1$-module by imposing the element Id to act as multiplication by a scalar $c\in\mathbf{k}$. These are the only possible extensions if the action of $\mathcal{N}_\mathcal{S}/\mathcal{S}_1$ is irreducible.

Theorem 7.3. *Every tensor module for $S(\mathfrak{d},\chi)$ can be obtained as the restriction of a tensor module for $W(\mathfrak{d})$. More precisely, for every $c\in\mathbf{k}$, $\mathcal{V}_\chi(R)=\mathcal{V}_\chi(\Pi,U)$ is isomorphic to the restriction of $\mathcal{V}(\Pi\otimes\mathbf{k}_{c(\chi-\operatorname{tr}\operatorname{ad})/N},U,c)$.*

Proof. Note that, as an H-module, $V=\mathcal{V}(\Pi\otimes\mathbf{k}_{c(\chi-\operatorname{tr}\operatorname{ad})/N},U,c)$ can be identified with $H\otimes R$. Moreover, since $R\subset\operatorname{sing}V$, we have $\mathcal{W}_1\cdot R=\{0\}$. Then R becomes an $\mathcal{N}_\mathcal{S}/\mathcal{S}_1$-module via the embedding ι_* from Lemma 7.1.

We identify each of the Lie algebras $\widetilde{\mathfrak{d}}$ and $\widehat{\mathfrak{d}}$ with $\mathfrak{d}$. It follows from (7.4) that if $\widetilde{\mathfrak{d}}$ acts on $\Pi\otimes\mathbf{k}_{c(\chi-\operatorname{tr}\operatorname{ad})/N}$ and Id acts as c, then $\widehat{\mathfrak{d}}$ acts on Π. Similarly, $\mathfrak{sl}\,\mathfrak{d}$ acts as U, so the action of $\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d}\simeq\mathcal{N}_\mathcal{S}/\mathcal{S}_1$ on R is isomorphic to $\Pi\boxtimes U$.

Then, by the definition of $\mathcal{V}_\chi(R)$, there is a natural surjective homomorphism of $S(\mathfrak{d},\chi)$-modules

$$\pi\colon\mathcal{V}_\chi(R)=\mathcal{V}_\chi(\Pi,U)\to HR=V\,.$$

The homomorphism π takes an element $u\in R\subset\mathcal{V}_\chi(R)$ to the element $u\in R\subset V$. But V is free as an H-module; hence, π is injective and $\mathcal{V}_\chi(R)\simeq V$. □

We will denote the restriction of $\mathcal{T}(\Pi,U)$ to $S(\mathfrak{d},\chi)$ by $\mathcal{T}_\chi(\Pi,U)$, and similarly for $\mathcal{T}(\Pi,U,c)$ (cf. Definition 4.4). Note that by (6.10), we have

$$\mathcal{T}(\Pi\otimes\mathbf{k}_{\chi+c(\chi-\operatorname{tr}\operatorname{ad})/N},U,c)=\mathcal{V}(\Pi\otimes\mathbf{k}_{(N+c)(\chi-\operatorname{tr}\operatorname{ad})/N},U,N+c)\,. \tag{7.5}$$

Then Theorem 7.3 implies

$$\mathcal{V}_\chi(\Pi,U)\simeq\mathcal{T}_\chi(\Pi\otimes\mathbf{k}_{\chi+c(\chi-\operatorname{tr}\operatorname{ad})/N},U,c)\,,\qquad c\in\mathbf{k}\,. \tag{7.6}$$

Observe that $\chi = \operatorname{tr}\operatorname{ad}$ is the only case for which the restriction of $\mathcal{T}(\Pi, U, c)$ to $S(\mathfrak{d}, \chi)$ is independent of c.

Example 7.1. Note that $\mathcal{T}(\Pi, \Omega^n) = \mathcal{T}(\Pi, \Omega^n, -n)$, because the element $\mathrm{Id} \in \mathfrak{gl}\,\mathfrak{d}$ acts on $\Omega^n = \bigwedge^n \mathfrak{d}^*$ as $-n$. Then it follows from (7.6) that

$$\mathcal{T}_\chi(\Pi, \Omega^n) \simeq \mathcal{V}_\chi(\Pi \otimes \mathbf{k}_{-\chi+n(\chi-\operatorname{tr}\operatorname{ad})/N}, \Omega^n)\,. \tag{7.7}$$

In particular, we have

$$\mathcal{V}_\chi(\Pi, \mathbf{k}) \simeq \mathcal{T}_\chi(\Pi \otimes \mathbf{k}_\chi, \Omega^0) \simeq \mathcal{T}_\chi(\Pi \otimes \mathbf{k}_{\operatorname{tr}\operatorname{ad}}, \Omega^N)\,. \tag{7.8}$$

One can use (7.6) for $c = 0$ to write an explicit expression for the action of $S(\mathfrak{d}, \chi)$ on its tensor module $\mathcal{V}_\chi(\Pi, U)$. First, we note that if we identify $\mathcal{T}(\Pi \otimes \mathbf{k}_\chi, U, 0)$ with $H \otimes R$, then $\partial \in \mathfrak{d}$ acts on R as $\bar\partial$ (see (7.1)). Then we use (7.2) and (6.14) to compute $s_{ij} * (1 \otimes u)$ for $u \in R$. The full expression is too cumbersome to write here. Because of (7.2), it is a sum of three terms. The third term is just a direct application of (6.14) for $[\partial_i, \partial_j]$. The second term is obtained from the first one by switching the roles of i and j. Finally, using H-bilinearity, (6.14), and (6.15), we find that the first term is equal to:

$$\begin{aligned}(\bar\partial_i \otimes \partial_j) * (1 \otimes u) &= (\bar\partial_i \otimes 1) \otimes_H (1 \otimes (\bar\partial_j + \operatorname{ad}\partial_j)u) \\ &+ (\bar\partial_i \otimes 1) \otimes_H s(\partial_j, u) - \sum_{k=1}^N (\bar\partial_i \otimes \partial_k) \otimes_H (1 \otimes (e_j^k + \delta_j^k)u)\,.\end{aligned} \tag{7.9}$$

Recall that $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$ for $\partial \in \mathfrak{d}$. This implies $\Delta(\bar\partial) = \bar\partial \otimes 1 + 1 \otimes \partial$ and

$$\begin{aligned}(\bar\partial \otimes g) \otimes_H (h \otimes u) = (1 \otimes g) \otimes_H (\bar\partial h \otimes u) - (1 \otimes g\partial) \otimes_H (h \otimes u)\,,\\ \partial \in \mathfrak{d}\,,\ g, h \in H\,,\ u \in R\,.\end{aligned} \tag{7.10}$$

Applying (7.10), we rewrite (7.9) as follows:

$$\begin{aligned}(\bar\partial_i \otimes \partial_j) * (1 \otimes u) &= (1 \otimes 1) \otimes_H (\bar\partial_i \otimes (\bar\partial_j + \operatorname{ad}\partial_j)u) \\ &- (1 \otimes \partial_i) \otimes_H (1 \otimes (\bar\partial_j + \operatorname{ad}\partial_j)u) + (1 \otimes 1) \otimes_H \bar\partial_i s(\partial_j, u) \\ &- (1 \otimes \partial_i) \otimes_H s(\partial_j, u) - \sum_{k=1}^N (1 \otimes \partial_k) \otimes_H (\bar\partial_i \otimes (e_j^k + \delta_j^k)u) \\ &+ \sum_{k=1}^N (1 \otimes \partial_k\partial_i) \otimes_H (1 \otimes (e_j^k + \delta_j^k)u)\,.\end{aligned} \tag{7.11}$$

When the action of $\mathfrak{gl}\,\mathfrak{d}$ on R is trivial, things can be rearranged in a more elegant form as follows:

$$\begin{aligned}s_{ij} * (1 \otimes u) &= (1 \otimes 1) \otimes_H (\partial_i \otimes \partial_j u - \partial_j \otimes \partial_i u - 1 \otimes [\partial_i, \partial_j]u) \\ &+ (1 \otimes \partial_i) \otimes_H (\partial_j \otimes u - 1 \otimes \partial_j u) \\ &- (1 \otimes \partial_j) \otimes_H (\partial_i \otimes u - 1 \otimes \partial_i u)\,.\end{aligned} \tag{7.12}$$

Even though the expression (7.11) is not very inspiring, it will turn out to be useful. We state as a lemma the properties that we are going to need later. Before that let us introduce the notation (see (6.15)):

$$a_{ij}(u) = \partial_i s(\partial_j, u) - \partial_j s(\partial_i, u) = \sum_{k=1}^N \left(\partial_i\partial_k \otimes e_j^k u - \partial_j\partial_k \otimes e_i^k u\right)\,. \tag{7.13}$$

Note that $a_{ii}(u) = 0$.

Lemma 7.2. *Consider the tensor $S(\mathfrak{d},\chi)$-module $\mathcal{V}_\chi(R)$. Then the action of the elements $s_{ij}\in S(\mathfrak{d},\chi)$, defined in* (7.2)*, on an element $1\otimes u\in\mathbf{k}\otimes R\subset\mathcal{V}_\chi(R)$ has the form*

$$\begin{aligned} s_{ij}*(1\otimes u) &= (1\otimes 1)\otimes_H A_{ij}(u)+\sum_{k=1}^N (1\otimes\partial_k)\otimes_H A_{ij}^k(u)\\ &\quad+\sum_{\substack{k,l=1\\ k\le l}}^N (1\otimes\partial_k\partial_l)\otimes_H A_{ij}^{kl}(u)\,, \end{aligned} \tag{7.14}$$

where

$$\begin{aligned} &A_{ij}^{kl}(u)\in\mathbf{k}\otimes(\mathbf{k}+\mathfrak{gl}\,\mathfrak{d})u\,,\\ &A_{ij}^{k}(u)\in\mathfrak{d}\otimes(\mathbf{k}+\mathfrak{gl}\,\mathfrak{d})u+\mathbf{k}\otimes(\mathbf{k}+\widehat{\mathfrak{d}}+\mathfrak{gl}\,\mathfrak{d})u\,, \end{aligned}$$

and

$$\begin{aligned} A_{ij}(u)\in a_{ij}(u)&+\partial_i\otimes\widehat{\partial}_j u-\partial_j\otimes\widehat{\partial}_i u\\ &+\mathfrak{d}\otimes(\mathbf{k}+\mathfrak{gl}\,\mathfrak{d})u+\mathbf{k}\otimes(\mathbf{k}+\widehat{\mathfrak{d}}+\mathfrak{gl}\,\mathfrak{d})u\,. \end{aligned}$$

7.3. **Filtration of Tensor Modules.** In analogy with tensor modules for $W(\mathfrak{d})$, one can define an increasing filtration $\mathrm{F}^p\,\mathcal{V}_\chi(R)$ of $\mathcal{V}_\chi(R)=H\otimes R$ by

$$\mathrm{F}^p\,\mathcal{V}_\chi(R)=\mathrm{F}^p\,H\otimes R\,,\qquad p\ge -1\,. \tag{7.15}$$

Note that $\mathrm{F}^{-1}\,\mathcal{V}_\chi(R)=\{0\}$ and $\mathrm{F}^0\,\mathcal{V}_\chi(R)=\mathbf{k}\otimes R$. The associated graded space of $\mathcal{V}_\chi(R)$ is

$$\mathrm{gr}\,\mathcal{V}_\chi(R)=\bigoplus_{p\ge -1}\mathrm{gr}^p\,\mathcal{V}_\chi(R)\,,\qquad \mathrm{gr}^p\,\mathcal{V}_\chi(R)=\mathrm{F}^p\,\mathcal{V}_\chi(R)/\,\mathrm{F}^{p-1}\,\mathcal{V}_\chi(R)\,. \tag{7.16}$$

The proof of the following lemma is completely similar to that of Lemma 6.3, so we omit it.

Lemma 7.3. *The action of $\widetilde{\mathcal{S}}$ on $\mathcal{V}_\chi(R)$ satisfies*:
(i) $\mathfrak{d}\cdot\mathrm{F}^p\,\mathcal{V}_\chi(R)\subset\mathrm{F}^{p+1}\,\mathcal{V}_\chi(R)$,
(ii) $\mathcal{N}_\mathcal{S}\cdot\mathrm{F}^p\,\mathcal{V}_\chi(R)\subset\mathrm{F}^p\,\mathcal{V}_\chi(R)$,
(iii) $\mathcal{S}_1\cdot\mathrm{F}^p\,\mathcal{V}_\chi(R)\subset\mathrm{F}^{p-1}\,\mathcal{V}_\chi(R)$.

Lemma 7.3 implies that each $\mathrm{gr}^p\,\mathcal{V}_\chi(R)$ is a module over the Lie algebra $\mathcal{N}_\mathcal{S}/\mathcal{S}_1\simeq\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d}$. This module is described in the next lemma.

Lemma 7.4. *We have*

$$\mathrm{gr}^p\,\mathcal{V}_\chi(\Pi,U)\simeq(\Pi\otimes\mathbf{k}_{p(\mathrm{tr\,ad}-\chi)/N})\boxtimes(\mathrm{S}^p\,\mathfrak{d}\otimes U)$$

as $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-modules.

Proof. Let us extend the $\mathfrak{gl}\,\mathfrak{d}$-action on U to an action of $\mathfrak{gl}\,\mathfrak{d}$ by letting Id act as 0. Then, by Theorem 7.3, $\mathcal{V}_\chi(\Pi,U)$ is the restriction to $S(\mathfrak{d},\chi)$ of the tensor $W(\mathfrak{d})$-module $\mathcal{V}(\Pi,U,0)$. Moreover, the filtration (7.15) coincides with the one defined in Section 6.2. The structure of a $(\mathfrak{d}\oplus\mathfrak{gl}\,\mathfrak{d})$-module on $\mathrm{gr}^p\,\mathcal{V}(\Pi,U,0)$ is described in Corollary 6.2. Note that this describes the action of $\widetilde{\mathfrak{d}}$. Using (7.4), we find that $\widehat{\mathfrak{d}}$ acts as $\Pi\otimes\mathbf{k}_{p(\mathrm{tr\,ad}-\chi)/N}$, because Id acts as p on $\mathrm{S}^p\,\mathfrak{d}\otimes U$. □

The grading of $\widetilde{\mathcal{S}}$ can be used to endow $\mathcal{V}_\chi(R)$ with a graded module structure as follows. Recall that $\mathcal{V}_\chi(R)=\mathrm{Ind}_{\mathcal{N}_\mathcal{S}}^{\widetilde{\mathcal{S}}}R$ and $\widetilde{\mathcal{S}}=\mathfrak{s}_{-1}\oplus\mathcal{N}_\mathcal{S}$ as a vector space. Therefore, as a vector space, $\mathcal{V}_\chi(R)=U(\mathfrak{s}_{-1})\otimes R$. However, the Lie algebra $\mathfrak{s}_{-1}$ is commutative, because the degree -1 part in S_N is commutative and because the isomorphism $\mathcal{S}\simeq S_N$ is compatible with the grading (see Corollary 3.3). Then $U(\mathfrak{s}_{-1})$ is the symmetric algebra generated by $\mathfrak{s}_{-1}$, and we grade $\mathcal{V}_\chi(R)$ by letting $\mathfrak{s}_{-1}$ have degree -1 and R have degree 0.

By definition, the above grading of $\mathcal{V}_\chi(R)$ is compatible with the grading of $\widetilde{\mathcal{S}}$. It is also compatible with the filtration (7.15).

7.4. **Submodules of Tensor Modules.** In what follows, $\mathcal{V}_\chi(R)$ will be a tensor module for $S(\mathfrak{d},\chi)$. We will assume that $R = \Pi \boxtimes U$, where Π (respectively U) is an irreducible finite-dimensional representation of $\mathfrak{d}$ (respectively $\mathfrak{sl}\,\mathfrak{d}$).

Recall that every element $v \in \mathcal{V}_\chi(R)$ can be expressed uniquely as a finite sum

$$v = \sum_{I\in\mathbb{Z}_+^N} \partial^{(I)} \otimes v_I\,, \qquad v_I \in R\,. \tag{7.17}$$

As in Section 6.3, nonzero elements v_I are called *coefficients* of v, and we denote by $\operatorname{coeff} M$ the subspace of R linearly spanned by coefficients of elements $v \in M$.

Lemma 7.5. *For any nontrivial proper $S(\mathfrak{d},\chi)$-submodule $M \subset \mathcal{V}_\chi(R)$, we have $M \cap \mathrm{F}^0\,\mathcal{V}_\chi(R) = \{0\}$.*

Proof. The action of $\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d}$ preserves both $\mathbf{k}\otimes R$ (by the definition of $\mathcal{V}_\chi(R)$) and M (because it is an $S(\mathfrak{d},\chi)$-submodule). Thus, their intersection $M \cap \mathrm{F}^0\,\mathcal{V}_\chi(R)$ is a $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-submodule of R. Irreducibility of R implies that this intersection is trivial. □

Lemma 7.6. *For any nontrivial proper $S(\mathfrak{d},\chi)$-submodule $M \subset \mathcal{V}_\chi(R)$, we have* $\operatorname{coeff} M = R$.

Proof. Take an element $v \in M$ and write it in the form (7.17). For $i \neq j$, we compute $s_{ij} * v$ using H-bilinearity and Lemma 7.2. Denote by m the coefficient of $1 \otimes \partial^{(I)}$ in the expression for $s_{ij} * v$; then, by Remark 2.2(ii), $m \in M$. By Lemma 7.2, we have

$$m = a_{ij}(v_I) \mod \mathrm{F}^1\,H \otimes R\,.$$

Note that $e_i^k v_I$ for $k \neq i$, $e_j^k v_I$ for $k \neq j$, and $(e_j^j - e_i^i)v_I$ are coefficients of $a_{ij}(v_I)$ (see (7.13)). Hence, they are coefficients of m, and we conclude that $(\mathfrak{sl}\,\mathfrak{d})v_I \subset \operatorname{coeff} M$.

In order to show that $\widehat{\mathfrak{d}}\cdot v_I \subset \operatorname{coeff} M$, we look at the degree one part of the above element m. Again by Lemma 7.2, it is equal to

$$\partial_i \otimes \widehat{\partial}_j v_I - \partial_j \otimes \widehat{\partial}_i v_I + \mathfrak{d} \otimes (\mathbf{k} + \mathfrak{sl}\,\mathfrak{d})(\operatorname{coeff} M)\,.$$

Hence, the action of $\widehat{\mathfrak{d}}$ preserves $\operatorname{coeff} M$. Then $\operatorname{coeff} M$ is a $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-submodule of R, which is irreducible. This shows that $\operatorname{coeff} M = R$, as it cannot be $\{0\}$. □

Lemma 7.7. *Let M be a nontrivial proper $S(\mathfrak{d},\chi)$-submodule of $\mathcal{V}(R)$. Then for every $i \neq j$ and $u \in R$, there exists an element $m \in M \cap \mathrm{F}^2\,\mathcal{V}_\chi(R)$ such that*

$$m = a_{ij}(u) \mod \mathrm{F}^1\,\mathcal{V}_\chi(R)\,,$$

where $a_{ij}(u)$ is given by (7.13).

Proof. As $\operatorname{coeff} M = R$, it is enough to prove the statement when u is a coefficient v_I of some element $v \in M$. Then m is the element considered in the proof of Lemma 7.6. □

Our next result describes which tensor $S(\mathfrak{d},\chi)$-modules are irreducible.

Theorem 7.4. *Let Π (respectively U) be an irreducible finite-dimensional module over $\mathfrak{d}$ (respectively $\mathfrak{sl}\,\mathfrak{d}$). Then the $S(\mathfrak{d},\chi)$-module $\mathcal{V}_\chi(\Pi,U)$ is irreducible if and only if U is not isomorphic to $\Omega^n = \bigwedge^n \mathfrak{d}^*$ for any $n \geq 0$.*

Proof. Recall from Theorem 6.5 that the tensor $W(\mathfrak{d})$-module $\mathcal{T}(\Pi,\Omega^n)$ is not irreducible for $n \geq 1$. Then its restriction $\mathcal{T}_\chi(\Pi,\Omega^n)$ is not irreducible either. It follows from (7.8) and Proposition 5.2 that $\mathcal{V}_\chi(\Pi,\Omega^0)$ is not irreducible as well.

Conversely, assume that $\mathcal{V}_\chi(\Pi,U)$ is not irreducible, and let M be a nontrivial proper submodule. Apply s_{ij} on an element $m \in M$ satisfying the conditions of Lemma 7.7 (for the same i,j).

Then for $i<j$ the coefficient multiplying $1\otimes\partial_i^2\partial_j^2$ in the expression for $s_{ij}*m$ is equal to (see Lemma 7.2):

$$1\otimes\big((e_i^i-e_j^j)^2-(e_i^je_j^i+e_j^ie_i^j)\big)u\,.$$

By Remark 2.2(ii), this is an element of M. Lemma 7.5 implies that this element vanishes for all $u\in R=\Pi\boxtimes U$.

Note that $h=e_i^i-e_j^j$, $e=e_i^j$, and $f=e_j^i$ are standard generators of a subalgebra of $\mathfrak{sl}\,\mathfrak{d}$ isomorphic to $\mathfrak{sl}_2$. We know that $h^2-(ef+fe)$ acts trivially on R. The element h^2 is a linear combination of $h^2-(ef+fe)$ and of the Casimir element; hence, it acts on any irreducible $\mathfrak{sl}_2$-submodule $W\subset U$ as a scalar. This means that $\dim W=0$ or 1; hence, for $i<j$ the eigenvalues of $e_i^i-e_j^j$ on weight vectors can only be 0 or 1.

Recall that every irreducible $\mathfrak{sl}\,\mathfrak{d}$-module U has a highest weight vector, and U is uniquely determined by its highest weight (see, e.g., [Se, Chapter VII]). Let us denote by λ_{ij} the eigenvalue of $e_i^i-e_j^j$ on the highest weight vector of U. Then $\lambda_{ij}+\lambda_{jk}=\lambda_{ik}$ but all $\lambda_{ij}=0$ or 1. This implies that $\lambda_{i,i+1}=\delta_{i,n}$ for some n; in other words, U is the n-th fundamental representation, which is isomorphic to Ω^{N-n}. □

Corollary 7.1. *Let V be a finite $S(\mathfrak{d},\chi)$-module, and let $R\subset\operatorname{sing}V$ be an irreducible $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-submodule. Assume that $R\simeq\Pi\boxtimes U$ with $U\not\simeq\Omega^n$ for any n. Then as an H-module, $HR\simeq H\otimes R$.*

Proof. By the definition of $\mathcal{V}_\chi(R)$, there is a natural surjective homomorphism of $S(\mathfrak{d},\chi)$-modules $\mathcal{V}_\chi(R)\to HR\subset V$. However, by Theorem 7.4, the tensor module $\mathcal{V}_\chi(R)$ is irreducible. Therefore, $HR\simeq\mathcal{V}_\chi(R)=H\otimes R$ is free as an H-module. □

7.5. **Computation of Singular Vectors.** In this section, we will compute singular vectors for all tensor $S(\mathfrak{d},\chi)$-modules of the form $\mathcal{V}_\chi(\Pi,\Omega^n)$, where Π is an irreducible finite-dimensional representation of $\mathfrak{d}$ and $\Omega^n=\bigwedge^n\mathfrak{d}^*$. This will be used in Section 7.6 for the classification of all irreducible quotients of tensor modules.

Proposition 7.1. *For $V=\mathcal{V}_\chi(\Pi,\Omega^n)$, we have $\mathrm{F}^0V\subset\operatorname{sing}V\subset\mathrm{F}^2V$. Furthermore, if $V=\mathcal{V}_\chi(\Pi,\mathbf{k})$, then $\operatorname{sing}V\subset\mathrm{F}^1V$.*

Proof. Let us consider first the case $n=0$, i.e., $V=\mathcal{V}_\chi(\Pi,\mathbf{k})$. Let $v\in V$ be a singular vector written in the form (7.17). Then using (7.12) and H-bilinearity, we get:

$$\begin{aligned}s_{ij}*v=&\sum_I(1\otimes\partial^{(I)})\otimes_H(\partial_i\otimes\partial_jv_I-\partial_j\otimes\partial_iv_I-1\otimes[\partial_i,\partial_j]v_I)\\&+\sum_I(1\otimes\partial^{(I)}\partial_i)\otimes_H(\partial_j\otimes v_I-1\otimes\partial_jv_I)\\&-\sum_I(1\otimes\partial^{(I)}\partial_j)\otimes_H(\partial_i\otimes v_I-1\otimes\partial_iv_I)\,.\end{aligned}$$

Assume that $v\notin\mathrm{F}^1V$, and choose I so that $|I|$ is maximal among those for which $v_I\neq0$. Then, by Remark 7.1, the element multiplying $1\otimes\partial^{(I)}\partial_i$ in the above expression must vanish. Hence, $v_I=0$, which is a contradiction.

Now let us assume that $n\neq0,N$, i.e., $\Omega^n\not\simeq\mathbf{k}$. We proceed as above: consider a singular vector $v=\sum\partial^{(I)}\otimes v_I$ and use (7.14) to compute $s_{ij}*v$. Then the coefficient of $1\otimes\partial^{(I)}$ is equal to $a_{ij}(v_I)$ mod F^1V. If $|I|>2$ and $v_I\neq0$, this contradicts Remark 7.1. □

Recall that the tensor $S(\mathfrak{d},\chi)$-module $V=\mathcal{V}_\chi(R)$ has a filtration $\{\mathrm{F}^pV\}$, given by (7.15). If $v\in V$ is a nonzero singular vector, we can find a unique $p\geq0$ such that $v\in\mathrm{F}^pV\setminus\mathrm{F}^{p-1}V$. Note that, by Lemma 7.3, both F^pV and gr^pV are $(\mathfrak{d}\oplus\mathfrak{sl}\,\mathfrak{d})$-modules, and the natural projection $\pi^p\colon\mathrm{F}^pV\to\mathrm{gr}^pV$ is a homomorphism. Therefore, the restriction

$$\pi^p\colon\operatorname{sing}V\cap\mathrm{F}^pV\to\mathrm{gr}^pV \tag{7.18}$$

is a homomorphism of $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-modules. Since $\pi^p(v) \neq 0$, we obtain that the $\mathfrak{sl}\,\mathfrak{d}$-modules $\operatorname{sing} V$ and $\operatorname{gr}^p V$ contain an isomorphic irreducible summand. We will utilize these remarks, together with the next lemma, to study singular vectors.

Lemma 7.8. *Let $V = \mathcal{V}_\chi(\Pi, U)$, and let U' be an irreducible $\mathfrak{sl}\,\mathfrak{d}$-submodule of $\operatorname{gr}^p V$. Assume that $U' \not\simeq \Omega^m$ for any m, and that $\dim U' > \dim U$. Then the submodule $\pi^p(\operatorname{sing} V \cap \mathrm{F}^p V) \subset \operatorname{gr}^p V$ does not intersect U'.*

Proof. Let

$$R = \Pi \boxtimes U\,, \quad R' = (\Pi \otimes \mathbf{k}_{p(\operatorname{tr}\operatorname{ad} - \chi)/N}) \boxtimes U'\,.$$

By Lemma 7.4, R' is an irreducible $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-submodule of $\operatorname{gr}^p V$. If we assume that $\pi^p(\operatorname{sing} V \cap \mathrm{F}^p V)$ intersects U', then $\operatorname{sing} V$ contains a $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-submodule isomorphic to R'. Now Corollary 7.1 implies that $HR' \subset V$ is free as an H-module. But

$$\dim R' = (\dim \Pi)(\dim U') > (\dim \Pi)(\dim U) = \dim R\,.$$

Therefore, the H-submodule $HR' \subset V$ has a larger rank than $V = H \otimes R$, which is impossible. □

Lemma 7.8 is a powerful tool for studying singular vectors, when combined with the explicit knowledge of the $\mathfrak{sl}\,\mathfrak{d}$-modules $\operatorname{gr}^p V$. It follows from Lemma 7.4 that $\operatorname{gr}^p V$ is a completely reducible $\mathfrak{sl}\,\mathfrak{d}$-module, all of whose irreducible $\mathfrak{sl}\,\mathfrak{d}$-components are contained in $\mathrm{S}^p\,\mathfrak{d} \otimes U$. In addition, since $\operatorname{sing} V \subset \mathrm{F}^2 V$ by Proposition 7.1, we can restrict our attention to the cases $p = 1$ or 2. Our next result shows a typical application of these ideas.

Proposition 7.2. *If $V = \mathcal{V}_\chi(\Pi, \Omega^n)$, $n \neq 1$, then $\operatorname{sing} V \subset \mathrm{F}^1 V$.*

Proof. Recall that $\operatorname{sing} V \subset \mathrm{F}^2 V$ by Proposition 7.1. We want to show that $\pi^2(\operatorname{sing} V) = \{0\}$ (see (7.18)). We know from Lemma 7.4 that

$$\operatorname{gr}^2 V \simeq (\Pi \otimes \mathbf{k}_{2(\operatorname{tr}\operatorname{ad} - \chi)/N}) \boxtimes (\mathrm{S}^2\,\mathfrak{d} \otimes \Omega^n)\,.$$

Thus, any irreducible $\mathfrak{sl}\,\mathfrak{d}$-submodule $U' \subset \operatorname{gr}^2 V$ is contained in $\mathrm{S}^2\,\mathfrak{d} \otimes \Omega^n$. One can check (see Lemma 7.9(iii) below) that all such U' satisfy $\dim U' > \dim \Omega^n$ and $U' \not\simeq \Omega^m$ for any m. Hence, we can apply Lemma 7.8 to conclude that $\pi^2(\operatorname{sing} V) \cap U' = \{0\}$. Therefore, $\operatorname{sing} V \subset \mathrm{F}^1 V$. □

Note that the above proof does not hold in the case $n = 1$, as one of the irreducible $\mathfrak{sl}\,\mathfrak{d}$-summands in $\mathrm{S}^2\,\mathfrak{d} \otimes \Omega^1$ is isomorphic to $\mathfrak{d} \simeq \Omega^{N-1}$. To get a complete description of all singular vectors, we need a detailed study of the $\mathfrak{sl}\,\mathfrak{d}$-modules $\mathrm{S}^2\,\mathfrak{d} \otimes \Omega^1$ and $\mathfrak{d} \otimes \Omega^n$.

Lemma 7.9. (i) *For $1 \leq n \leq N-1$, we have a direct sum of $\mathfrak{sl}\,\mathfrak{d}$-modules: $\mathfrak{d} \otimes \Omega^n = \Omega^{n-1} \oplus U'$, where U' is irreducible, $\dim U' > \dim \Omega^n$ and $U' \not\simeq \Omega^m$ for any m.*

(ii) *We have a direct sum of $\mathfrak{sl}\,\mathfrak{d}$-modules: $\mathrm{S}^2\,\mathfrak{d} \otimes \Omega^1 = \mathfrak{d} \oplus U''$, where U'' is irreducible, $\dim U'' > \dim \Omega^1 = N$ and $U'' \not\simeq \Omega^m$ for any m.*

(iii) *For $2 \leq n \leq N-1$, every irreducible $\mathfrak{sl}\,\mathfrak{d}$-submodule $U' \subset \mathrm{S}^2\,\mathfrak{d} \otimes \Omega^n$ satisfies $\dim U' > \dim \Omega^n$ and $U' \not\simeq \Omega^m$ for any m.*

Proof. We will use Table 5 from the Reference Chapter of [OV]. Following [OV], we will denote by $R(\Lambda)$ the irreducible representation of $\mathfrak{sl}\,\mathfrak{d} \simeq \mathfrak{sl}_N$ with highest weight Λ. We will denote by π_n the n-th fundamental weight of $\mathfrak{sl}_N$, and we will set $\pi_0 = \pi_N = 0$. Note that $R := R(\pi_1) = \mathfrak{d}$ is the vector representation of $\mathfrak{sl}\,\mathfrak{d}$, and $R(0) = \mathbf{k}$ is the trivial one. Then we have:

$$\Omega^n = \bigwedge^n \mathfrak{d}^* \simeq \left(\bigwedge^n \mathfrak{d}\right)^* \simeq \bigwedge^{N-n} \mathfrak{d} = \bigwedge^{N-n} R \simeq R(\pi_{N-n})$$

and

$$\mathrm{S}^2\,\mathfrak{d} = \mathrm{S}^2\,R \simeq R(2\pi_1)\,.$$

Using [OV], we find:

$$R(\pi_1)\otimes R(\pi_p)\simeq R(\pi_1+\pi_p)\oplus R(\pi_{p+1})\,,$$
$$R(2\pi_1)\otimes R(\pi_p)\simeq R(2\pi_1+\pi_p)\oplus R(\pi_1+\pi_{p+1})\,,$$

and

$$\dim R(\pi_p)=\binom{N}{p}\,,$$
$$\dim R(\pi_1+\pi_p)=\frac{p}{p+1}(N+1)\binom{N}{p}\,,$$
$$\dim R(2\pi_1+\pi_p)=\frac{p}{p+2}\binom{N+2}{2}\binom{N}{p}\,.$$

From here, it is easy to finish the proof. □

Let us introduce some notation. For a $\mathfrak{d}$-module Π, we set

$$\Pi_n=\Pi\otimes\mathbf{k}_{-\chi+n(\chi-\operatorname{tr}\operatorname{ad})/N}\,,\qquad \Pi'=\Pi\otimes\mathbf{k}_{\operatorname{tr}\operatorname{ad}-\chi}\,. \tag{7.19}$$

Then we can restate (7.7) as

$$\mathcal{T}_\chi(\Pi,\Omega^n)\simeq\mathcal{V}_\chi(\Pi_n,\Omega^n)\,, \tag{7.20}$$

while by (7.8) we have an isomorphism of $S(\mathfrak{d},\chi)$-modules

$$\psi\colon\mathcal{T}_\chi(\Pi',\Omega^N)\xrightarrow{\sim}\mathcal{T}_\chi(\Pi,\Omega^0)\,. \tag{7.21}$$

Also, recall the Π-twisted pseudo de Rham complex of $W(\mathfrak{d})$-modules (5.15). When we restrict these modules to $S(\mathfrak{d},\chi)$, we obtain a complex of $S(\mathfrak{d},\chi)$-modules

$$0\to\mathcal{T}_\chi(\Pi,\Omega^0)\xrightarrow{\mathrm{d}_\Pi}\mathcal{T}_\chi(\Pi,\Omega^1)\xrightarrow{\mathrm{d}_\Pi}\cdots\xrightarrow{\mathrm{d}_\Pi}\mathcal{T}_\chi(\Pi,\Omega^N)\,. \tag{7.22}$$

Note that the isomorphism ψ is compatible with the filtrations (i.e., it maps each F^p to F^p), while d_Π has degree 1 (i.e., it maps each F^p to F^{p+1}).

Theorem 7.5. *Let Π be an irreducible finite-dimensional $\mathfrak{d}$-module. Then we have the following equalities and isomorphisms of $(\mathfrak{d}\oplus\mathfrak{sl}(\mathfrak{d})$-modules*:

(i)
$$\begin{aligned}\operatorname{sing}\mathcal{T}_\chi(\Pi,\Omega^n)&=\mathrm{F}^0\,\mathcal{T}_\chi(\Pi,\Omega^n)+\mathrm{d}_\Pi\,\mathrm{F}^0\,\mathcal{T}_\chi(\Pi,\Omega^{n-1})\\&\simeq(\Pi_n\boxtimes\Omega^n)\oplus(\Pi_{n-1}\boxtimes\Omega^{n-1})\,,\qquad 2\le n\le N\,,\end{aligned}$$

(ii)
$$\begin{aligned}\operatorname{sing}\mathcal{T}_\chi(\Pi,\Omega^1)&=\mathrm{F}^0\,\mathcal{T}(\Pi,\Omega^1)+\mathrm{d}_\Pi\,\mathrm{F}^0\,\mathcal{T}_\chi(\Pi,\Omega^0)+\mathrm{d}_\Pi\psi\mathrm{d}_{\Pi'}\,\mathrm{F}^0\,\mathcal{T}_\chi(\Pi',\Omega^{N-1})\\&\simeq(\Pi_1\boxtimes\Omega^1)\oplus(\Pi_0\boxtimes\Omega^0)\oplus(\Pi_{-1}\boxtimes\Omega^{N-1})\,,\end{aligned}$$

where we use the notation from (7.19)–(7.21).

Proof. Let $V=\mathcal{T}_\chi(\Pi,\Omega^n)$. Then by (7.20), $V\simeq\mathcal{V}_\chi(\Pi_n,\Omega^n)$, and by Lemma 7.4, we have an isomorphisms of $(\mathfrak{d}\oplus\mathfrak{sl}(\mathfrak{d})$-modules

$$\operatorname{gr}^p V\simeq\Pi_{n-p}\boxtimes(\mathrm{S}^p\,\mathfrak{d}\otimes\Omega^n)\,.$$

In particular, $\mathrm{F}^0\,V=\operatorname{gr}^0 V\simeq\Pi_n\boxtimes\Omega^n$. Note that the latter is an irreducible $(\mathfrak{d}\oplus\mathfrak{sl}(\mathfrak{d})$-module. This implies the isomorphisms in (i) and (ii) above, because d_Π and ψ are homomorphisms and because $(\Pi')_{N-1}\simeq\Pi_{-1}$ (see (7.19)).

Recall that $\mathrm{F}^0\,V\subset\operatorname{sing}V\subset\mathrm{F}^2\,V$, and $\operatorname{sing}V\subset\mathrm{F}^1\,V$ for $n\neq1$ (see Propositions 7.1 and 7.2). Since d_Π and ψ are homomorphisms of $S(\mathfrak{d},\chi)$-modules, they map singular vectors to singular vectors. Then it is clear that the right-hand sides of (i) and (ii) are contained in $\operatorname{sing}V$.

Next, we describe the image of $\operatorname{sing}V\cap\mathrm{F}^1\,V$ in $\operatorname{gr}^1 V$ under the natural projection (7.18). On one hand, we have

$$\pi^1(\operatorname{sing}V\cap\mathrm{F}^1\,V)\supset\pi^1(\mathrm{d}_\Pi\,\mathrm{F}^0\,\mathcal{T}_\chi(\Pi,\Omega^{n-1}))\simeq\Pi_{n-1}\boxtimes\Omega^{n-1}\,.$$

On the other hand, every irreducible $\mathfrak{sl}\,\mathfrak{d}$-submodule of $\operatorname{gr}^1 V$ is contained in $\mathfrak{d} \otimes \Omega^n$. By Lemma 7.9(i), we have a direct sum of $\mathfrak{sl}\,\mathfrak{d}$-modules: $\mathfrak{d}\otimes\Omega^n = \Omega^{n-1}\oplus U'$, where U' is irreducible, $\dim U' > \dim \Omega^n$ and $U' \not\simeq \Omega^m$ for any m. Now, by Lemma 7.8, the image $\pi^1(\operatorname{sing} V \cap \mathrm{F}^1 V)$ does not intersect $\Pi_{n-1} \boxtimes U'$. Therefore, the above inclusion is an equality. In particular, we get statement (i).

To finish the proof of (ii), we note that

$$\pi^2(\operatorname{sing} V) \supset \pi^2(\mathrm{d}_\Pi \psi \mathrm{d}_{\Pi'} \mathrm{F}^0\, \mathcal{T}_\chi(\Pi', \Omega^{N-1})) \simeq \Pi_{-1} \boxtimes \Omega^{N-1}\,.$$

By the same argument as above, this is an equality, because of Lemma 7.9(ii). □

Remark 7.2. It follows from Theorem 7.5 and the isomorphism (7.21) that

$$\begin{aligned}\operatorname{sing} \mathcal{T}_\chi(\Pi, \Omega^0) &= \mathrm{F}^0\, \mathcal{T}_\chi(\Pi, \Omega^0) + \psi \mathrm{d}_{\Pi'} \mathrm{F}^0\, \mathcal{T}_\chi(\Pi', \Omega^{N-1}) \\ &\simeq (\Pi_0 \boxtimes \Omega^0) \oplus (\Pi_{-1} \boxtimes \Omega^{N-1})\,.\end{aligned}$$

7.6. **Irreducible Finite $S(\mathfrak{d}, \chi)$-Modules.** We can now complete the classification of irreducible finite $S(\mathfrak{d}, \chi)$-modules. Our fist result describes all submodules of the tensor $S(\mathfrak{d}, \chi)$-module $\mathcal{T}_\chi(\Pi, \Omega^n)$.

Lemma 7.10. *Let Π be an irreducible finite-dimensional $\mathfrak{d}$-module, let $T^n = \mathcal{T}_\chi(\Pi, \Omega^n)$, and let $M \subset T^n$ be a nontrivial proper $S(\mathfrak{d}, \chi)$-submodule. Then*:
(i) $\operatorname{sing} M = \mathrm{d}_\Pi \mathrm{F}^0\, T^{n-1}$, *if* $2 \le n \le N$;
(ii) $M \supset \mathrm{d}_\Pi T^{N-1}$, *if* $n = N$;
(iii) $M = \mathrm{d}_\Pi T^{n-1}$, *if* $2 \le n \le N-1$;
(iv) $\mathrm{d}_\Pi T^{n-1}$ *is irreducible for* $2 \le n \le N$.

Proof. Let $2 \le n \le N$, and let $M \subset T^n$ be a nontrivial proper $S(\mathfrak{d}, \chi)$-submodule. Then $\operatorname{sing} M \subset \operatorname{sing} T^n$ is a $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-submodule, and $M \cap \mathrm{F}^0\, T^n = \{0\}$ by Lemma 7.5. Now Theorem 7.5(i) and an argument similar to the one used in the proof of Corollary 6.5 imply part (i). Then

$$M \supset H(\operatorname{sing} M) = \mathrm{d}_\Pi(H(\mathrm{F}^0\, T^{n-1})) = \mathrm{d}_\Pi T^{n-1}\,.$$

The rest of the proof of (iii) is the same as that of Lemma 6.12, while (iv) follows from (ii) and (iii). □

Remark 7.3. Recall that the $W(\mathfrak{d})$-module $\mathcal{T}(\Pi, \Omega^1)$ has a unique nontrivial proper $W(\mathfrak{d})$-submodule, namely $\mathrm{d}_\Pi \mathcal{T}(\Pi, \Omega^0)$ (see Lemma 6.12). However, the restriction $\mathcal{T}_\chi(\Pi, \Omega^1)$ of $\mathcal{T}(\Pi, \Omega^1)$ to $S(\mathfrak{d}, \chi)$ has two nontrivial proper $S(\mathfrak{d}, \chi)$-submodules:

$$\mathrm{d}_\Pi \psi \mathrm{d}_{\Pi'} \mathcal{T}_\chi(\Pi', \Omega^{N-1}) \subset \mathrm{d}_\Pi \mathcal{T}_\chi(\Pi, \Omega^0)$$

(cf. Theorem 7.5(ii)). Because of (7.21) and the exactness of (7.22), these two $S(\mathfrak{d}, \chi)$-modules are isomorphic to the following ones:

$$\mathrm{d}_{\Pi'} \mathcal{T}_\chi(\Pi', \Omega^{N-1}) \subset \mathcal{T}_\chi(\Pi', \Omega^N)\,.$$

Now we can state the main result of this section.

Theorem 7.6. *Any irreducible finite $S(\mathfrak{d}, \chi)$-module is isomorphic to one of the following*:
(i) *Tensor modules $\mathcal{T}_\chi(\Pi, U, 0)$, where Π is an irreducible finite-dimensional $\mathfrak{d}$-module, and U is an irreducible finite-dimensional $\mathfrak{sl}\,\mathfrak{d}$-module not isomorphic to $\bigwedge^n \mathfrak{d}^*$ for any $n \ge 0$*;
(ii) *Images $\mathrm{d}_\Pi \mathcal{T}_\chi(\Pi, \Omega^n)$, where Π is an irreducible finite-dimensional $\mathfrak{d}$-module, and $1 \le n \le \dim \mathfrak{d} - 1$ (see* (7.22)).

Proof. The proof is similar to that of Theorem 6.6. Let V be an irreducible finite $S(\mathfrak{d}, \chi)$-module. Then, by Theorem 7.2 and (7.6), $V \simeq T/M$, where $T = \mathcal{T}_\chi(\Pi, U)$ is a tensor module and $M \subset T$ is an $S(\mathfrak{d}, \chi)$-submodule.

If $U \not\simeq \bigwedge^n \mathfrak{d}^* = \Omega^n$ as an $\mathfrak{sl}\,\mathfrak{d}$-module for any $n \ge 0$, then T is irreducible by Theorem 7.4 and (7.6). In this case, $V \simeq \mathcal{T}_\chi(\Pi, U)$.

Assume that $U \simeq \Omega^n$ for some $n \geq 0$; then $T \simeq \mathcal{T}_\chi(\Pi, \Omega^n) = T^n$ is not irreducible. Because of (7.8), we can assume without loss of generality that $1 \leq n \leq N = \dim \mathfrak{d}$. Now if $2 \leq n \leq N-1$, Lemma 7.10(iii) implies that $M = \mathrm{d}_\Pi T^{n-1}$. By the exactness of (7.22), we get $V \simeq T^n/\mathrm{d}_\Pi T^{n-1} \simeq \mathrm{d}_\Pi T^n$.

Next, consider the case when V is a quotient of T^N. Then, by Lemma 7.10(ii), we have $M \supset \mathrm{d}_\Pi T^{N-1}$. Now Proposition 5.2 implies that V is finite dimensional; hence, $S(\mathfrak{d}, \chi)$ acts trivially on it by Example 2.3, and V is not irreducible.

Finally, it remains to consider the case when V is a quotient of T^1. Note that $\mathrm{d}_\Pi M$ is a proper $S(\mathfrak{d}, \chi)$-submodule of T^2; hence, by Lemma 7.10(iii), it must be either trivial or equal to $\mathrm{d}_\Pi T^1$. First, if $\mathrm{d}_\Pi M = \{0\}$, then $M \subset \mathrm{d}_\Pi T^0$ and we have a surjective homomorphism $T^1/M \to T^1/\mathrm{d}_\Pi T^0$. But $T^1/M \simeq V$ is irreducible; therefore, $V \simeq T^1/\mathrm{d}_\Pi T^0 \simeq \mathrm{d}_\Pi T^1$. Second, if $\mathrm{d}_\Pi M = \mathrm{d}_\Pi T^1$, then $M + \mathrm{d}_\Pi T^0 = T^1$ and we have isomorphisms $V \simeq T^1/M \simeq (\mathrm{d}_\Pi T^0)/(\mathrm{d}_\Pi T^0 \cap M)$. Since the map $\mathrm{d}_\Pi \colon T^0 \to T^1$ is injective, we get that $V \simeq T^0/K$ for some $S(\mathfrak{d}, \chi)$-submodule K of T^0. This case was already considered above, because of (7.8). □

Finally, for each irreducible finite $S(\mathfrak{d}, \chi)$-module V, we will describe the space $\operatorname{sing} V$ of singular vectors of V.

Lemma 7.11. *Let R be an irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-module. Then $V = \mathcal{V}_\chi(R)$ is an irreducible $S(\mathfrak{d}, \chi)$-module if and only if $\operatorname{sing} V = \mathrm{F}^0 V$.*

Proof. It is clear from Lemma 7.5 that V is irreducible when $\operatorname{sing} V = \mathrm{F}^0 V$. Conversely, assume that V is irreducible. Consider the grading of $V = U(\mathfrak{s}_{-1}) \otimes R$ constructed at the end of Section 7.3. All homogeneous components of a singular vector are still singular, so we have to show that the only homogeneous singular vectors in V are of degree zero. If $v \in \operatorname{sing} V$ is a singular vector of negative degree, then the $\widetilde{\mathcal{S}}$-submodule generated by v is contained in the negatively graded part of V, which contradicts the irreducibility of V. Therefore, $\operatorname{sing} V = R = \mathrm{F}^0 V$. □

Theorem 7.7. *The irreducible finite $S(\mathfrak{d}, \chi)$-modules listed in Theorem* 7.6 *satisfy* (*see* (7.19)):
(i) $\operatorname{sing} \mathcal{T}_\chi(\Pi, U, 0) \simeq \Pi_0 \boxtimes U$ *as $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-modules*;
(ii) $\operatorname{sing}(\mathrm{d}_\Pi \mathcal{T}_\chi(\Pi, \Omega^n)) \simeq \Pi_n \boxtimes \Omega^n$ *as $(\mathfrak{d} \oplus \mathfrak{sl}\,\mathfrak{d})$-modules.*
In particular, no two of them are isomorphic to each other.

Proof. The proof is similar to that of Theorem 6.7, and it uses (7.20), Theorem 7.5(i), and Lemmas 7.10(i) and 7.11. □

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