# COMMUTATIVITY AND ASSOCIATIVITY OF VERTEX ALGEBRAS

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ABSTRACT. I define products on subspaces of a vertex algebra that behave in a better way than residual products of elements. In particular, they satisfy commutativity and associativity properties, and make it possible to easily prove many vertex algebra generalizations of standard facts in commutative algebra theory.

CONTENTS



## 1. INTRODUCTION

My aim in these notes is to present a vertex algebra identity – similar to the Leibniz rule in a Poisson algebra, and generalizing the method introduced by Wick [7] for computing the OPE of normally ordered products of quantum fields – which has immediate and deep algebraic consequences. It allows one (Propositions 3.1 and 4.2) to produce a wealth of vertex ideals out of ideals of both the vertex algebra $V$ and the Lie conformal algebra $V^{Lie}$ underlying it. Among its consequences, one can produce an unexpected characterization (Theorem 3.1) of ideals of $V^{Lie}$ in terms of those of $V$. Using Proposition 3.1, I easily deduce (Proposition 4.1) the associativity property of a suitably defined product between $\mathbb{C}[T]$-submodules in a vertex algebra. This result is not new, as it has already been observed in a more general form in [1], but the present proof seems somewhat more natural and straightforward. Using the above-mentioned associativity, together with standard skew-symmetry in a vertex algebra, I show vertex analogues of well-known facts in commutative algebra theory: existence of a nilradical, the fact that the annihilator of any given set of elements is an ideal, a no zero-divisor statement for simple vertex algebra structures. I finally list a few applications towards the study of vertex algebra structures over finitely generated $\mathbb{C}[T]$-modules, that are proved in [4]. My overall point of view is that the Poisson-Wick formula (3) is strong enough to make it possible to deduce many interesting algebraic properties of vertex algebras without much effort.

## 2. Basic definitions

2.1. **Vertex algebras.** Let $V$ be a complex vector space. A *field* on $V$ is a formal distribution $\phi \in (\mathrm{End}V)[[z, z^{-1}]]$ with the property that $\phi(v) \in V((z))$ for every $v \in V$. In other words, if

$$\phi(z) = \sum_{i \in \mathbb{Z}} \phi_i \, z^{-i-1}$$

then $\phi_n(v) = 0$, for $n \gg 0$.

**Definition 2.1** ([6]). A vertex algebra is a (complex) vector space $V$ endowed with a linear *state-field correspondence* $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$, a *vacuum element* $\mathbf{1} \in V$ and a linear endomorphism $T \in \mathrm{End}V$ satisfying the following properties:

- **Field axiom**: $Y(v, z)$ is a field for all $v \in V$.
- **Locality**: For every $a, b \in V$ there exists $N$ such that:
$$(z-w)^N [Y(a,z), Y(b,w)] = 0.$$
- **Vacuum axiom**: $Y(\mathbf{1}, z) = \mathrm{id}_V, \qquad Y(a,z)\mathbf{1} \equiv a \mod zV[[z]]$, for all $a \in V$.
- **Translation invariance**: $[T, Y(a,z)] = Y(Ta, z) = dY(a,z)/dz$, for all $a \in V$.

Every vertex algebra carries a natural $\mathbb{C}[T]$-module structure. Coefficients $Y(a,z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}$ of quantum fields in a vertex algebra span a Lie algebra under the commutator Lie bracket, and more explicitly satisfy [6]

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}, \tag{1}$$

for all $a, b \in V, m, n \in \mathbb{Z}$. In particular, applying both sides of (1) to $c \in V$ gives

$$a_{(m)}(b_{(n)}c) = b_{(n)}(a_{(m)}c) + \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c. \tag{2}$$

If $A$ and $B$ are subsets of $V$, then we may define $A \cdot B$ as the $\mathbb{C}$-linear span of all products $a_{(j)}b$, where $a \in A, b \in B, j \in \mathbb{Z}$; clearly $A \cdot B$ does not change if we replace $A, B$ by the linear subspaces they generate. Translation invariance implies

$$[T, a_{(n)}] = (Ta)_{(n)} = -na_{(n-1)},$$

hence $A \cdot B = (\mathbb{C}[T]A) \cdot B$. It follows that if $B$ is a $\mathbb{C}[T]$-submodule of $V$, then the same holds for $A \cdot B$, as by translation invariance $T$ is a derivation of all $j$-products; in particular, $A \cdot V$ is always a $\mathbb{C}[T]$-submodule of $V$. Note that in general $(\mathbb{C}[T]A) \cdot (\mathbb{C}[T]B) = \mathbb{C}[T](A \cdot B)$. The $\mathbb{C}[T]$-submodules generated by $A \cdot B$ and $B \cdot A$ always coincide by skew-commutativity, and $A \subset A \cdot V$ by the vacuum axiom. An *ideal* of $V$ is a $\mathbb{C}[T]$-submodule $I \subset V$ such that $V \cdot I \subset I$. A vertex algebra is *simple* if its only ideals are trivial.

2.2. **Lie conformal algebras.**

**Definition 2.2** ([5]). A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module $R$ with a $\mathbb{C}$-bilinear product $(a, b) \mapsto [a\,{}_\lambda\, b] \in R[\lambda]$ satisfying the following axioms:

**(C1)** $[a\,{}_\lambda\, b] \in R[\lambda]$,
**(C2)** $[\partial a\,{}_\lambda\, b] = -\lambda[a\,{}_\lambda\, b], \;\; [a\,{}_\lambda\, \partial b] = (\partial + \lambda)[a\,{}_\lambda\, b]$,
**(C3)** $[a\,{}_\lambda\, b] = -[b\,{}_{-\partial-\lambda}\, a]$,
**(C4)** $[a\,{}_\lambda\, [b\,{}_\mu\, c]] - [b\,{}_\mu\, [a\,{}_\lambda\, c]] = [[a\,{}_\lambda\, b]\,{}_{\lambda+\mu}\, c]$,

for every $a, b, c \in R$.

Any vertex algebra $V$ can be given a $\mathbb{C}[\partial]$-module structure by setting $\partial = T$. Then defining

$$[a\,{}_\lambda\, b] = \sum_{n \in \mathbb{N}} \frac{\lambda^n}{n!} a_{(n)} b$$

endows $V$ with a Lie conformal algebra structure, denoted by $V^{Lie}$. Indeed (C1) follows from the field axiom, (C2) from translation invariance, (C3) from skew-commutativity, and (C4) from (1). If $A$ and $B$ are subsets of a Lie conformal algebra $R$, then we may define $[A, B]$ to be the $\mathbb{C}$-linear span of all $\lambda$-coefficients in the products $[a\,{}_\lambda b]$, where $a \in A, b \in B$; as above, $[A, B]$ does not change if we replace $A$ and $B$ by the linear subspaces they generate. It follows from axiom (C2) that if $B$ is a $\mathbb{C}[\partial]$-submodule of $R$, then $[A, B]$ is also a $\mathbb{C}[\partial]$-submodule, and that $[\mathbb{C}[\partial]A, \mathbb{C}[\partial]B] = \mathbb{C}[\partial][A, B]$. Notice that by axiom (C3) $[A, B]$ and $[B, A]$ generate over $\mathbb{C}[\partial]$ the same submodule of $V$, hence they are equal as soon as they are both $\mathbb{C}[\partial]$-submodules. An *ideal* of a Lie conformal algebra $R$ is a $\mathbb{C}[\partial]$-submodule $I \subset R$ such that $[R, I] \subset I$. If $I, J$ are ideals of $R$, then $[I, J]$ is an ideal too. An ideal $I$ is said to be *central* if $[R, I] = 0$, i.e., if it is contained in the *centre* $Z(R) = \{r \in R | [r\,{}_\lambda s] = 0 \text{ for all } s \in R\}$ of $R$; $R$ is *abelian* if it coincides with its centre. A Lie conformal algebra $R$ is *simple* if it is not abelian, and has no nontrivial ideals; it is *strongly simple* if $[a, R] = R$ for every nonzero $a \in R$. A strongly simple (nontrivial) Lie conformal algebra is clearly simple. Notice that, when $A, B$ are subsets of a vertex algebra $V$, then $[A, B]$ is the subspace of $V$ linearly generated by elements $a_{(j)}b$, for all choices of $a \in A, b \in B, j \in \mathbb{N}$. As a consequence, $[A, B] \subset A \cdot B$ holds for all choices of $A, B \subset V$. Ideals of a vertex algebra structure and ideals of the underlying Lie conformal algebra are not equivalent notions. Indeed, ideals of the vertex algebra are also ideals of the Lie conformal algebra, but the converse is generally false, as it can be seen by noticing that $\mathbb{C}\mathbf{1}$ is always a central ideal of the Lie conformal algebra structure, but it never is a proper ideal of the vertex algebra. The following example will be mentioned later on.

**Example 2.1** (Virasoro conformal algebra)**.** Let $R = \mathbb{C}[\partial]L \oplus \mathbb{C}\mathbf{1}$, where $\partial\mathbf{1} = 0$. Then the $\lambda$-brackets

$$[L_\lambda L] = (\partial + 2\lambda)L + \lambda^3\mathbf{1}, \qquad [L_\lambda \mathbf{1}] = 0, \qquad [\mathbf{1}_\lambda \mathbf{1}] = 0,$$

uniquely extend to a Lie conformal algebra structure, called *Virasoro conformal algebra*. The quotient *Vir* $= R/\mathbb{C}\mathbf{1}$ is called *centreless Virasoro conformal algebra* and is the unique finite strongly simple Lie conformal algebra up to isomorphism [5]. $R$ is its only nontrivial irreducible central extension [4].

## 3. The Poisson-Wick formula and ideals of $V^{Lie}$

3.1. **The Poisson-Wick formula.** After multiplying both sides in (2) by $\lambda^m z^{-n-1}/m!$ and adding up over all $m \in \mathbb{N}, n \in \mathbb{Z}$, one obtains the following *Poisson-Wick formula:*

$$[a\,{}_\lambda\, Y(b, z)\, c] = e^{\lambda z} Y([a\,{}_\lambda\, b], z)\; c + Y(b, z)\, [a\,{}_\lambda\, c]. \tag{3}$$

Observe the similarity of the above equation with Leibniz identity in a Poisson algebra. Indeed, the most evident difference lies in the term $e^{\lambda z}$ in front of the first summand on the right-hand side. The unusual presence of such a multiplying factor is indeed useful and allows us to prove the following mixed associativity between the bracket product and the dot product of subsets.

**Lemma 3.1.** *Let $A, B, C$ be subsets of a vertex algebra $V$. Then $[A, B] \cdot C \subset [A, B \cdot C]$.*

*Proof.* Follows easily by applying [3, Lemma 4.1] to (3). □

**Proposition 3.1.** *Let $U, I$ be subspaces of a vertex algebra $V$. Then $[U, I]$ is an ideal of $V$ provided that $I$ is an ideal of $V$. In particular, $[U, V]$ is always an ideal of $V$.*

*Proof.* Since $I$ is an ideal, we have $I \cdot V = I$. Then Lemma 3.1 gives $[U, I] \cdot V \subset [U, I \cdot V] = [U, I]$. □

**Corollary 3.1.** *If $I$ is an ideal of $V^{Lie}$, then $[I, V]$ is an ideal of $V$ contained in $I$.*

In principle, there are many more ideals of $V^{\mathit{Lie}}$ than there are of $V$. The following theorem shows that ideals of $V^{\mathit{Lie}}$ are always *"commutatively commensurable"* with those of $V$.

**Theorem 3.1.** *A $\mathbb{C}[T]$-submodule $I \subset V$ is an ideal of $V^{\mathit{Lie}}$ if and only if it contains an ideal $J$ of $V$ such that $I/J$ lies in the centre of $V/J$.*

*Proof.* If $I$ is an ideal of $V^{\mathit{Lie}}$, then $I/[I,V]$ is central in $V/[I,V]$. Conversely, if $I/J$ is central in $V/J$, then $[I/J, V/J] = 0$, whence $[I,V] \subset J \subset I$. $\square$

3.2. **Lie structure of simple vertex algebras.** Theorem 3.1 has interesting consequences when applied to a simple vertex algebra.

**Proposition 3.2.** *A proper $\mathbb{C}[T]$-submodule of a simple vertex algebra $V$ is an ideal of $V^{\mathit{Lie}}$ if and only if it lies in the centre of $V$.*

*Proof.* The only proper ideal of $V$ is $0$. Next use Theorem 3.1. $\square$

The most striking consequence of Proposition 3.1 is the following characterization of the Lie conformal algebra underlying a simple vertex algebra structure.

**Theorem 3.2.** *Let $V$ be a simple vertex algebra. Then either $V$ is commutative, or $V^{\mathit{Lie}}$ is an irreducible central extension of a strongly simple Lie conformal algebra.*

*Proof.* Assume $V$ is not commutative. Then $0 \neq [V,V]$ is an ideal of $V$, hence it equals $V$. In particular, $V^{\mathit{Lie}}$ cannot be solvable. If $a \in V$, then $[a,V]$ is an ideal of $V$. Then either $a \in Z(V)$ or $[a,V] = V$. This shows that the Lie conformal algebra $R = V^{\mathit{Lie}}/Z(V)$ satisfies $[a,R] = R$ for all $a \neq 0$. Thus $R$ is strongly simple, and since $V = [V,V]$ we conclude that $V$ is an irreducible central extension of $R$ by $Z(V)$. $\square$

The above result allows one to construct many new examples of simple Lie conformal algebras by taking the centreless quotient of $V^{\mathit{Lie}}$ whenever $V$ is a simple Lie algebra. When $V$ is a vertex operator algebra one obtains instances of simple Lie conformal algebra structures on graded vector spaces of superpolynomial growth.

## 4. Associativity in vertex algebras

4.1. **Associativity and commutativity of products of subspaces.**

**Lemma 4.1.** *Let $A, B, C$ be subsets of a vertex algebra $V$. Then $A \cdot (B \cdot C) = B \cdot (A \cdot C)$.*

*Proof.* Observe that the second summand in (2) only contains products $a_{(j)}b$ with non-negative $j$. Then choosing $a \in A, b \in B, c \in C$ gives $A \cdot (B \cdot C) \subset B \cdot (A \cdot C) + [A,B] \cdot C$. However

$$\begin{aligned}[A,B] \cdot C = (\mathbb{C}[\partial][A,B]) \cdot C &= (\mathbb{C}[\partial][B,A]) \cdot C \\ &= [B,A] \cdot C \subset [B, A \cdot C] \subset B \cdot (A \cdot C),\end{aligned}$$

hence $A \cdot (B \cdot C) \subset B \cdot (A \cdot C)$. Reversing the roles of $A$ and $B$ proves equality. $\square$

**Proposition 4.1.** *Let $A, B, C$ be $\mathbb{C}[\partial]$-submodules of a vertex algebra $V$. Then $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.*

*Proof.* Repeated application of skew-commutativity and Lemma 4.1 gives

$$(A \cdot B) \cdot C = C \cdot (A \cdot B) = A \cdot (C \cdot B) = A \cdot (B \cdot C).$$

$\square$

**Corollary 4.1.** *The product $U_1 \cdot U_2 \cdot \ldots \cdot U_n$ of $\mathbb{C}[\partial]$-modules in a vertex algebra depends neither on the order of the factors nor on the parenthesization of the product.*

*Proof.* Follows easily from Proposition 4.1 and skew-commutativity. $\square$

**Proposition 4.2.** *Let $U$ be a subset and $I$ an ideal of the vertex algebra $V$. Then $U \cdot I$ is an ideal of $V$.*

*Proof.* As $I$ is a $\mathbb{C}[\partial]$-submodule of $V$, $U \cdot I$ is a $\mathbb{C}[\partial]$-submodule of $V$. Proposition 4.1 gives $(U \cdot I) \cdot V = U \cdot (I \cdot V) = U \cdot I$ showing that $U \cdot I$ is an ideal. □

### 4.2. **The nilradical of a vertex algebra.**

**Corollary 4.2.** *Let $V$ be a vertex algebra, $a \in V$. Then $a \cdot V$ is the ideal of $V$ generated by $a$. Similarly, if $U \subset V$ is any subset, then $U \cdot V$ is the ideal of $V$ generated by $U$.*

*Proof.* Let $(a)$ denote the minimal ideal of $V$ containing $a$. By Proposition 4.2, $a \cdot V$ is an ideal of $V$, which must contain $a$ as $a_{(-1)}\mathbf{1} = a$, hence $(a) \subset a \cdot V$, the opposite inclusion being obvious. The remaining claim is similarly proved. □

**Proposition 4.3.** *Let $a \in V$ be such that $Y(a,z)a = 0$. Then $a$ generates an abelian ideal of $V$.*

*Proof.* First of all, notice that $(\mathbb{C}[\partial]a) \cdot (\mathbb{C}[\partial]a) = \mathbb{C}[\partial](a \cdot a) = 0$. Now, $(a) = a \cdot V$ is a $\mathbb{C}[\partial]$-submodule of $V$, so it equals $(\mathbb{C}[\partial]a) \cdot V$. Using associativity and commutativity we obtain

$$(a) \cdot (a) = ((\mathbb{C}[\partial]a) \cdot V) \cdot ((\mathbb{C}[\partial]a) \cdot V) = (\mathbb{C}[\partial]a) \cdot (\mathbb{C}[\partial]a) \cdot V \cdot V = 0.$$

□

An element $a \in V$ is *nilpotent* if

$$Y(a, z_1)Y(a, z_2) \ldots Y(a, z_k)a = 0,$$

for sufficiently large values of $k$.

If $N \subset V$ is any subset, define $N^1 = N$, $N^{k+1} = N \cdot N^k$ for $k > 1$. Then an ideal $N$ of $V$ is a *nil-ideal* if $N^k = 0$ for sufficiently large values of $k$. Abelian ideals are always nil-ideals, as in this case $N^2 = N \cdot N = 0$.

**Lemma 4.2.** *Let $V$ be a vertex algebra, $a \in V$. Then the following are equivalent:*

(1) *$a$ is a nilpotent element;*
(2) *$a$ generates a nil-ideal;*
(3) *$a$ is contained in a nil-ideal.*

*Proof.* (2) ⇒ (3) ⇒ (1) are obvious, so it suffices to show (1) ⇒ (2). If $a$ is nilpotent, then $a^k = 0$ for some $k$, hence $(\mathbb{C}[\partial]a)^k = \mathbb{C}[\partial](\mathbb{C}a)^k = 0$. Then $I = (a) = (\mathbb{C}[\partial]a) \cdot V$ satisfies $I^k = (\mathbb{C}[\partial]a)^k \cdot V^k = 0$. □

**Theorem 4.1.** *The set* $\mathrm{Nil}\, V$ *of all nilpotent elements in a vertex algebra $V$ is an ideal of $V$.*

*Proof.* Let $a \in \mathrm{Nil}\, V$. Then $Ta \in (a)$ shows $\mathrm{Nil}\, V$ is $T$-stable. Moreover, $a \cdot V \subset (a)$ shows $(\mathrm{Nil}\, V) \cdot V \subset \mathrm{Nil}\, V$. We are left with showing that $\mathrm{Nil}\, V$ is a subspace of $V$. Indeed, if $a, b \in \mathrm{Nil}\, V$, then $a + b$ lies in $(a) + (b)$. Using associativity and commutativity, it is easy to show that, for all choices of ideals $I, J \subset V$, one has

$$(I + J)^n \subset \sum_{i+j=n} I^i \cdot J^j.$$

Now, if $(a)^h = (b)^k = 0$ and $n \geq h + k - 1$, we obtain $((a) + (b))^n = 0$, showing that any $\mathbb{C}$-linear combination of $a$ and $b$ lies in $\mathrm{Nil}\, V$. □

$\mathrm{Nil}\, V$ is the *nil-radical* of the vertex algebra $V$. It is a nil-ideal as soon as $V$ is Noetherian. The quotient vertex algebra $V/\mathrm{Nil}\, V$ contains no nonzero nilpotent elements.

4.3. **Other consequences of associativity.** The annihilator of an element $u \in V$ is the subspace

$$\mathrm{Ann}(u) = \{v \in V | Y(u,z)v = 0\}$$

of all elements killed by all coefficients of $u$. Clearly, it is also a $\mathbb{C}[\partial]$-submodule, as $v \in \mathrm{Ann}(u)$ implies

$$Y(u,z)(Tv) = T(Y(u,z)v) + \frac{d}{dz}(Y(u,z)v) = 0.$$

**Theorem 4.2.** $\mathrm{Ann}(u)$ *is an ideal of* $V$ *for all* $u \in V$.

*Proof.* We know that $u \cdot \mathrm{Ann}(u) = 0$. Then $u \cdot (\mathrm{Ann}(u) \cdot V) = (u \cdot \mathrm{Ann}(u)) \cdot V = 0$, hence $\mathrm{Ann}(u) \cdot V \subset \mathrm{Ann}(u)$. □

The following is an analogue of the integral domain property of a field.

**Corollary 4.3.** *Let* $V$ *be a simple vertex algebra. Then* $Y(a,z)b = 0$ *implies* $a = 0$ *or* $b = 0$.

*Proof.* Let $b \neq 0$. Then $\mathrm{Ann}(a)$ contains $b$, hence it is a non-zero ideal of $V$. By simplicity, $\mathrm{Ann}(a) = V$. Then $Y(a,z)\mathbf{1} = 0$, whence $a = 0$. □

## 5. Applications

An interesting application of Theorem 3.2 is the following characterization of simple vertex algebra structures over finitely generated $\mathbb{C}[\partial]$-modules.

**Theorem 5.1.** *Every finite simple vertex algebra is commutative.*

*Proof.* Let $V$ be a finite simple vertex algebra. Then $V^{Lie}$ is an irreducible central extension of a strongly simple Lie conformal algebra, hence it is isomorphic to the Virasoro Lie conformal algebra (see Example 2.1). An easy computation then shows that it cannot support a compatible vertex algebra structure, see [4]. □

Recall [2] that the structure of a commutative vertex algebra degenerates to that of a differential commutative associative unital algebra. Indeed the product $a \circ b = a_{(-1)}b$ is both commutative and associative, $\mathbf{1}$ is its identity and $T$ a derivation. It can then be shown that $Y(a,z)b = (e^{zT}a) \circ b$ for all $a, b$. The following are examples of finite simple vertex algebras.

**Example 5.1.** The trivial vertex algebra $V = \mathbb{C}\mathbf{1}$, $T = 0$ is a finite simple vertex algebra.

**Example 5.2.** The commutative algebra $A = \mathbb{C}[x,y]/(y^2 - 4x^3 - 1)$ possesses a unique derivation $\partial$ such that $\partial x = y, \partial y = 6x^2$. Then $A$ is a finitely generated $\mathbb{C}[\partial]$-module, and has no nontrivial $\partial$-stable ideals. Therefore, it is a (differentiably) simple differential commutative unital algebra.

Using commutativity of finite simple vertex algebras, it is easy to show that every finite vertex algebra $V$ has a solvable underlying Lie conformal algebra $V^{Lie}$. A more detailed investigation [4] shows that $(V/\mathrm{Nil}\, V)^{Lie}$ is always nilpotent; in other words, a finite vertex algebra must either possess nontrivial nilpotent elements, or have a nilpotent underlying Lie conformal algebra.

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