REPRESENTATIONS OF LIE PSEUDOALGEBRAS WITH COEFFICIENTS

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A Corrado per il suo sessantesimo compleanno

CONTENTS

1. INTRODUCTION

The recent notion of Lie pseudoalgebra [BDK1] over a cocommutative Hopf algebra is a multivariable generalization of the concept of (Lie) conformal algebra [DK], introduced by Kac [K] in connection with vertex algebras.

Lie pseudoalgebras have proved useful in the study of representations of linearly compact Lie algebras especially because of the possibility of associating with a given Lie pseudoalgebra L over the cocommutative Hopf algebra H the Lie algebra $\mathcal{L} = H^* \otimes_H L$ of its annihilation operators, or *annihilation algebra* for short. When L is finitely generated as an H-module, and H is Noetherian, then $\mathcal{L}$ is a linearly compact Lie algebra. Moreover there is a one-to-one correspondence between pseudoalgebra representations of L and discrete continuous representations of $\mathcal{L}$ satisfying a technical condition: the representation space must possess an H-module structure satisfying a suitable compatibility with the natural H-module structure of $\mathcal{L}$.

By a theorem [C, G1, G2] of Cartan and Guillemin, infinite dimensional simple linearly compact Lie algebras are isomorphic either to the Lie algebra W_n of all (formal) vector fields in n indeterminates, or to one the subalgebras S_n, H_n, K_n, whose elements preserve a volume form, a symplectic form or a contact structure respectively. The Lie algebras W_n, S_n, K_n are all obtained as annihilation algebras of certain "primitive" finite simple Lie pseudoalgebras. Moreover, the relevant irreducible representations all possess a compatible H-module structure. The above correspondence can then be exploited to obtain a complete classification of irreducible modules – see, e.g., [BDK2, BDK3].

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The role of the Lie algebra H_n is somewhat special. There exist primitive simple Lie pseudoalgebras $H(\mathfrak{d},\chi,\omega)$ of type H, yet their annihilation algebra is a non-trivial irreducible central extension of H_n – namely the Lie algebra structure defined on $\mathbf{k}[[t_1,\dots,t_n]]$ by the Poisson bracket of (formal) functions, that we denote by P_n. Moreover, discrete representations of P_n corresponding to finite irreducible pseudoalgebra representations of $H(\mathfrak{d},\chi,\omega)$ satisfy the above-mentioned technical condition if and only if they factor via the centerless quotient H_n. As a consequence, the representation theory of the Lie pseudoalgebra $H(\mathfrak{d},\chi,\omega)$ reflects that of H_n, and not that of P_n.

This is a strange and unexpected twist, in that the actual annihilation algebra P_n disappears from the description. Our aim in this paper is that of introducing a generalization of the concept of (Lie) pseudoalgebra representation which makes it possible to also treat irreducible representations of P_n that do not factor via H_n. This is done by representing a Lie pseudoalgebra over H not on an H-module, but rather on a D-module M, where D is a suitable comodule algebra over H: we call such a construction *representation with coefficients in* D.

All main constructions using the concept of pseudoalgebra representation carry over to the case of representations with coefficients. We take care of adjusting the old language to the new definition, and show that all discrete complex irreducible representations of P_n may be lifted to representations with coefficients of the primitive simple Lie pseudoalgebra $H(\mathfrak{d},\chi,\omega)$, at least in the case of an abelian Lie algebra $\mathfrak{d}$ with a trivial trace form χ. We would like to thank the referees for their insightful comments and a patient and careful inspection of the paper.

2. Pseudoalgebras and Rings of Coefficients

We review the definitions and results about Lie pseudoalgebras which will be needed later. In most of the paper, H will be the universal enveloping algebra $U(\mathfrak{d})$ of a finite-dimensional Lie algebra $\mathfrak{d}$, endowed with the standard cocommutative Hopf algebra structure. All vector spaces, linear maps, tensor products, etc. are considered over an algebraically closed field $\mathbf{k}$ of characteristic 0.

2.1. **Hopf algebra notations.** Let H be a cocommutative Hopf algebra with a coproduct Δ, a counit ε, and an antipode S. By using the following notation:

$$\Delta(h)=h_{(1)}\otimes h_{(2)}=h_{(2)}\otimes h_{(1)}\,,$$
$$(\Delta\otimes\mathrm{id})\Delta(h)=(\mathrm{id}\otimes\Delta)\Delta(h)=h_{(1)}\otimes h_{(2)}\otimes h_{(3)}\,,\qquad h\in H,$$

the axioms of antipode and counit can be written as

$$S(h_{(1)})h_{(2)}=h_{(1)}S(h_{(2)})=\varepsilon(h),\qquad \varepsilon(h_{(1)})h_{(2)}=h_{(1)}\varepsilon(h_{(2)})=h,$$

so that

$$S(h_{(1)})h_{(2)}\otimes h_{(3)}=1\otimes h=h_{(1)}S(h_{(2)})\otimes h_{(3)},$$
$$h_{(1)}\otimes S(h_{(2)})h_{(3)}=h\otimes 1=h_{(1)}\otimes h_{(2)}S(h_{(3)}),$$

while the fact that Δ is a homomorphism of algebras translates as:

$$(hk)_{(1)}\otimes(hk)_{(2)}=h_{(1)}k_{(1)}\otimes h_{(2)}k_{(2)},\qquad h,k\in H.$$

Setting $\Delta^1=\Delta,\Delta^{i+1}=(\Delta^i\otimes\mathrm{id}_H)\circ\Delta$ gives us the iterated coproduct maps $\Delta^m: H\to H^{\otimes(m+1)}$,

$$\Delta^m(h)=h_{(1)}\otimes h_{(2)}\otimes\ldots\otimes h_{(m+1)},$$

which define on $H^{\otimes(m+1)}$ both a left and a right H-module structure.

The dual $X=H^*:=\mathrm{Hom}_{\mathbf{k}}(H,\mathbf{k})$ becomes a commutative associative algebra under the product defined as

$$\langle xy,h\rangle=\langle x,h_{(1)}\rangle\langle y,h_{(2)}\rangle\,,\qquad h\in H,x,y\in X\,.$$

It admits left and right actions of H, given by

$$\langle hx, k\rangle = \langle x, S(h)k\rangle\,,$$
$$\langle xh, k\rangle = \langle x, kS(h)\rangle\,, \qquad h,k\in H,\ x\in X\,,$$

satisfying

$$h(xy) = (h_{(1)}x)(h_{(2)}y)\,,$$
$$(xy)h = (xh_{(1)})(yh_{(2)})\,,$$
$$h(xk) = (hx)k, \qquad h,k\in H,\ x,y\in X.$$

Let now $H = U(\mathfrak{d})$ be the universal enveloping algebra of the finite dimensional Lie algebra $\mathfrak{d}$. We choose a basis $\partial_1, \dots \partial_n$ of $\mathfrak{d}$, and set

$$\partial^{(I)} = \frac{\partial_1^{i_1}\dots\partial_n^{i_n}}{i_1!\dots i_n!} \in H, \tag{2.1}$$

where $I = (i_1, \dots, i_n) \in \mathbb{N}^n$. By Poincaré-Birkhoff-Witt theorem, elements $\partial^{(I)}, I \in \mathbb{N}^n$ constitute a basis of H. The standard coproduct on H given by $\Delta(\partial) = \partial\otimes 1 + 1\otimes\partial$ then satisfies

$$\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)}\otimes\partial^{(K)}.$$

Elements $t^I \in X = H^*$ such that $\langle t^I, \partial^{(J)}\rangle = \delta_{I,J}$ are then linearly independent, and satisfy $t^I\cdot t^J = t^{I+J}$. We will write $t_i = t^{\varepsilon_i}$, where $\varepsilon_i = (0,\dots,0,\underset{i}{1},0,\dots,0)$, so that $I = (i_1,\dots,i_n)$ implies $t^I = t_1^{i_1}\dots t_n^{i_n}$.

$H = U(\mathfrak{d})$ may be provided with an increasing family of subspaces $\{\mathrm{F}^i H\}_{i\in\mathbb{Z}}$ linearly generated by PBW monomials of degree $\le i$ — here we agree that $\mathrm{F}^i H = (0)$ whenever $i < 0$. The dual space $X = H^*$ is correspondingly filtered by a decreasing family of subspaces $\mathrm{F}_i X = (\mathrm{F}^i H)^\perp$ which define a linearly compact topology on X. The two filtrations are compatible in the sense that the action of H on X satisfies $(\mathrm{F}^i H)\cdot(\mathrm{F}_m X) \subset \mathrm{F}_{m-i} X$ for all i, m.

Any element $a \in X$ is uniquely determined by its values $a_I = \langle a, \partial^{(I)}\rangle$. It makes sense to write $a = \sum_{I\in\mathbb{N}^n} a_I t^I$ as the right-hand side becomes a finite sum when computed on any element of H. This gives an identification of X with $\mathcal{O}_n = \mathbf{k}[[t_1,\dots,t_n]]$ as topological commutative algebras, once we endow $\mathcal{O}_n$ with the formal topology.

Our most typical situation will be when the Lie algebra $\mathfrak{d}$ is abelian. In this case, the Hopf algebra $H = U(\mathfrak{d})$ is isomorphic to the symmetric algebra $\mathrm{S}(\mathfrak{d})$. The left and right actions of H on X then coincide, and are given by $\partial_i.f = -\partial f/\partial t_i$. In this case, both H and X are graded vector spaces and the action of H on X is homogeneous of degree zero.

2.2. **Rings of coefficients.**

Definition 2.1. Let H be a Hopf algebra over $\mathbf{k}$ with coproduct Δ_H. A *comodule algebra* over H is an associative $\mathbf{k}$-algebra D endowed with a homomorphism of $\mathbf{k}$-algebras $\Delta_D : D \to H\otimes D$ satisfying the comodule property

$$(\Delta_H\otimes \mathrm{id}_D)\circ\Delta_D = (\mathrm{id}_H\otimes\Delta_D)\circ\Delta_D,$$

and such that $\varepsilon(d_{(1)})d_{(2)} = d$ for all $d\in D$, where as usual we use the notation $\Delta_D(d) = d_{(1)}\otimes d_{(2)}$. In other words, it is an associative $\mathbf{k}$-algebra D which is a left H-comodule such that the comodule map is a $\mathbf{k}$-algebra homomorphism.

Remark 2.1. Notice that $\varepsilon(d_{(1)})$ makes sense in the above equation, as $d_{(1)}\in H$.

Example 2.1. Let H' be a Hopf subalgebra of H. Then the restriction to H' of the comultiplication $\Delta : H \to H\otimes H$ defines a comodule algebra structure on H'. Indeed, $\Delta|_{H'}$ is a homomorphism of associative algebra by restriction, and it maps H' into $H'\otimes H' \subset H\otimes H'$. In particular, both H and the base field $\mathbf{k}$ have a structure of comodule algebras over H.

Example 2.2. Let V be a vector space of even dimension, $\omega \in \bigwedge^2 V^*$. Set:

$$D(V,\omega) = \mathrm{T}(V)/\langle uv - vu - \omega(u,v),\ u,v \in V\rangle,$$

where $\mathrm{T}(V)$ denotes the tensor algebra over the vector space V.

Then $H = D(V,0) = \mathrm{S}(V)$ has the usual Hopf algebra structure, and $D = D(V,\omega)$ is an associative algebra. Observe that V linearly embeds in both H and $D(V,\omega)$. The latter has a structure of comodule algebra over H defined by the $\mathbf{k}$-algebra homomorphism $\Delta_D : D(V,\omega) \to H \otimes D(V,\omega)$ extending $\Delta_D(\bar v) = v \otimes 1 + 1 \otimes \bar v$, where $\bar v \in V \subset D(V,\omega)$ and v denotes its image in $H = D(V,0)$.

Henceforth, when D is a comodule algebra over the cocommutative Hopf algebra H, we will say that D is a *ring of coefficients* over H.

2.3. **Left and right straightening.** It was showed in [BDK1] that if H is a Hopf algebra, then $H \otimes H = (H \otimes \mathbf{k})\Delta(H) = (\mathbf{k} \otimes H)\Delta(H)$. Only part of this statement holds in the general case of rings of coefficients.

Lemma 2.1. *If D is a ring of coefficients over H, then $H \otimes D = (H \otimes \mathbf{k})\Delta_D(D)$*

Proof. It is immediate to check that $h \otimes d = (hS(d_{(1)}) \otimes 1)\Delta_D(d_{(2)})$. □

Remark 2.2. When $D = H$, one also has $H \otimes H = (\mathbf{k} \otimes H)\Delta(H)$, as $h \otimes k = (1 \otimes kS(h_{(2)}))\Delta(h_{(1)})$. Notice that in general, $H \otimes D \neq (\mathbf{k} \otimes D)\Delta_D(D)$. For instance, if $D = H'$ is a proper Hopf subalgebra of H, then $\Delta_D(D) \subset D \otimes D$, so that $(\mathbf{k} \otimes D)\Delta_D(D) \subset D \otimes D$, which is strictly contained in $H \otimes D$, and equality cannot hold.

The following lemma will be important later on.

Lemma 2.2. *Let $H = D(V,0)$, $D = D(V,\omega)$. Then $H \otimes D = (\mathbf{k} \otimes D)\Delta_D(D)$.*

Proof. The statement becomes obvious after describing the comodule algebra map $\Delta_D : D \to H \otimes D$ as follows: let $\widetilde V$ be central the extension of V, viewed as an abelian Lie algebra, by a one-dimensional ideal $\mathbf{k}c$, as defined by the 2-cocycle ω, and endow $\widetilde H = U(\widetilde V)$ with the standard Hopf algebra structure satisfying $\widetilde\Delta(x) = x \otimes 1 + 1 \otimes x,\ x \in \widetilde V$.

Notice now that both H and D are quotients of $\widetilde H$, namely $H = \widetilde H/c\widetilde H$ and $D = \widetilde H/(c-1)\widetilde H$. Moreover $\widetilde\Delta(c-1) = c \otimes 1 + 1 \otimes (c-1)$, so that

$$\widetilde\Delta((c-1)\widetilde H) \subset (c\widetilde H) \otimes \widetilde H + \widetilde H \otimes ((c-1)\widetilde H).$$

It is then easy to check that the well-defined map

$$D = \widetilde H/(c-1)\widetilde H \longrightarrow \widetilde H/c\widetilde H \otimes \widetilde H/(c-1)\widetilde H = H \otimes D$$

induced by $\widetilde\Delta$ coincides with Δ_D. Projecting $\widetilde H \otimes \widetilde H = (\mathbf{k} \otimes \widetilde H)\widetilde\Delta(\widetilde H)$ to $H \otimes D$ one obtains $H \otimes D = (\mathbf{k} \otimes D)\Delta_D(D)$. □

Remark 2.3. Let D be a ring of coefficients over H, and $\Delta_D : D \to H \otimes D$ be the corresponding comodule map. Then $H \otimes D$ has a right D-module structure induced by Δ_D, so that one may define the tensor product $(H \otimes D) \otimes_D M$ — which has a natural left $H \otimes D$-module structure given by multiplication — whenever M is a left D-module. The right H-module structure on $H \otimes H$ is a particular instance of this construction.

We may also define $\Delta_D^m : D \to H^{\otimes m} \otimes D$ by repeated application of Δ_D and Δ_H. By the comodule property and coassociativity of Δ_H, the composition Δ_D^m does not depend on which factors we apply each occurrence of Δ_H or Δ_D. We may then use Δ_D^m to endow $H^{\otimes m} \otimes D$ with left and right D-module structures.

The importance of the above lemmas sits in the following statement.

Corollary 2.1. *Let D be a ring of coefficients over H, M be a left D-module. Then every element of $(H \otimes D) \otimes_D M$ can be* left-straightened *to the form*

$$\sum (h_i \otimes 1) \otimes_D m_i,$$

for a suitable choice of $h_i \in H, m_i \in M$.

Similarly, if $H \otimes D = (\mathbf{k} \otimes D)\Delta_D(D)$, then every element of $(H \otimes D) \otimes_D M$ may be right-straightened *to the form*

$$\sum (1 \otimes d_i) \otimes_D m_i,$$

for a suitable choice of $d_i \in D, m_i \in M$.

Proof. The first statement follows by Lemma 2.1 and $((h \otimes 1)\Delta_D(d)) \otimes_D m = (h \otimes 1) \otimes_D dm, h \in H, d \in D, m \in M$. Similarly, if $H \otimes D = (\mathbf{k} \otimes D)\Delta_D(D)$, then one may use $((1 \otimes d)\Delta_D(d')) \otimes_D m = (1 \otimes d) \otimes_D d'm$, where $d, d' \in D, m \in M$. □

2.4. **Associative and Lie pseudoalgebras.** In this section, we recall some standard facts on (associative and Lie) pseudoalgebras from [BDK1]. A *pseudoalgebra over H* or *H-pseudo-algebra* is a left H-module A endowed with a *pseudoproduct*, i.e., an $H \otimes H$-linear map (see Remark 2.3)

$$A \otimes A \to (H \otimes H) \otimes_H A\,, \quad a \otimes b \mapsto a * b\,.$$

Explicitly, for any $h, k \in H,\ a, b \in A$, if $a * b = \sum_i (h^i \otimes k^i) \otimes_H c_i$ then

$$(ha) * (kb) = \sum_i (hh^i \otimes kk^i) \otimes_H c_i.$$

If A and A' are H-pseudoalgebras, then an H-linear map $\phi : A \to A'$ is a *pseudoalgebra homomorphism* if

$$\phi(a) * \phi(b) = ((\mathrm{id}_H \otimes \mathrm{id}_H) \otimes_H \phi)\,(a * b),$$

for all $a, b \in A$. It is possible to extend (see [Ko]) any pseudoproduct on A to a well-defined $H^{\otimes(m+n)}$-linear map

$$(H^{\otimes m} \otimes_H A) \otimes (H^{\otimes n} \otimes_H A) \to H^{\otimes(m+n)} \otimes_H A$$

by letting

$$(F \otimes_H a) * (G \otimes_H b) = \sum_i (F \otimes G) \cdot ((\Delta^{m-1} \otimes \Delta^{n-1}) \otimes_H \mathrm{id}_A)((h^i \otimes k^i) \otimes_H c_i)\,,$$

where $F \in H^{\otimes m}, G \in H^{\otimes n}$, $a, b \in A$, and

$$a * b = \sum_i (h^i \otimes k^i) \otimes_H c_i, \qquad \text{with} \quad h^i, k^i \in H,\, c_i \in A.$$

This makes it possible to compute and compare "multilinear" iterated products such as $a * (b * c), (a * b) * c \in H^{\otimes 3} \otimes_H A$.

Definition 2.2. An *associative H-pseudoalgebra* is an H-pseudoalgebra A whose pseudoproduct $a \otimes b \mapsto a * b$ satisfies the following associativity axiom:

$$(a * b) * c = a * (b * c)\,,$$

for all choices of $a, b, c \in A$.

A *Lie H-pseudoalgebra* is an H-pseudoalgebra L endowed with a pseudoproduct $a \otimes b \mapsto [a * b]$, called *Lie pseudobracket*, satisfying the following skew-commutativity and Jacobi identity axioms:

$$[b * a] = -(\sigma \otimes_H \mathrm{id}_L)\,[a * b]\,,$$

$$[[a * b] * c] = [a * [b * c]] - ((\sigma \otimes \mathrm{id}_H) \otimes_H \mathrm{id}_L)\,[b * [a * c]]\,,$$

where $a, b, c \in L$, and $\sigma : H \otimes H \to H \otimes H$ denotes the flip $\sigma(h \otimes k) = k \otimes h$.

Every associative pseudoalgebra A can be turned into a Lie pseudoalgebra A^- by setting

$$[a * b] = a * b - (\sigma \otimes_H \mathrm{id}_A)(b * a).$$

If A is a pseudoalgebra over H, and $U, V \subset A$ are H-submodules of A, denote by $U \cdot V$ the smallest among all H-submodules $W \subset A$ such that $u * v \in (H \otimes H) \otimes_H W$ for all $u \in U, v \in V$. Then $S \subset A$ is a *subalgebra* if $S \cdot S \subset S$, and $I \subset A$ is an *ideal* if $I \cdot A, A \cdot I \subset I$. A pseudoalgebra A is *simple* if its only ideals are 0 and A. We say that a pseudoalgebra over H is *finite* if it is finitely generated as an H-module.

Example 2.3. Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{d}$. Then $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ is given a structure of Lie pseudoalgebra by setting

$$\begin{aligned}[(h \otimes a) * (k \otimes b)] = (h \otimes k) \otimes_H (1 \otimes [a, b]) \\ - (h \otimes ka) \otimes_H (1 \otimes b) + (hb \otimes k) \otimes_H (1 \otimes a)\,,\end{aligned}$$

where $h, k \in H, a, b \in \mathfrak{d}$.

Example 2.4. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbf{k}$. Then $H \otimes \mathfrak{g}$ has the structure of a Lie pseudoalgebra with the Lie pseudobracket:

$$[(h \otimes a) * (k \otimes b)] = (h \otimes k) \otimes_H (1 \otimes [a, b]), \qquad h, k \in H,\ a, b \in \mathfrak{g}.$$

$H \otimes \mathfrak{g}$ is called *current H-pseudoalgebra* of $\mathfrak{g}$ and it is denoted by $\mathrm{Cur}_{\mathbf{k}}^H \mathfrak{g}$.

Example 2.5. The construction described in Example 2.4 is an instance of *extension of scalars*, or base change, for pseudoalgebras: let $\phi : H' \to H$ be a homomorphism of Hopf algebras, A be a (Lie, associative) pseudoalgebra over H'. We may use ϕ to endow H with a right H'-module structure and give the left H-module $H \otimes_{H'} A$ a (Lie, associative) pseudoalgebra (over H) structure by setting

$$(h \otimes_{H'} a) * (k \otimes_{H'} b) = \sum_i (h\phi(f^i) \otimes k\phi(g^i)) \otimes_H (1 \otimes_{H'} c_i),$$

if $a * b = \sum_i (f^i \otimes g^i) \otimes_H c_i$, where $a, b, c_i \in A$, $f^i, g^i \in H'$. This is clearly $H \otimes H$-linear, and it can be easily showed to provide a well-defined pseudoproduct.

When $H' \subset H$ and ϕ is the inclusion homomorphism, the above construction reduces to the *current H-pseudoalgebra* $\mathrm{Cur}_{H'}^H A$.

Remark 2.4. All nonzero subalgebras of $W(\mathfrak{d})$ are simple. Indeed, subalgebras of $W(\mathfrak{d})$ and current H-pseudoalgebras obtained from simple finite-dimensional Lie algebras over $\mathbf{k}$ provide a complete list of finite simple Lie pseudoalgebras over $H = U(\mathfrak{d})$.

3. Primitive Lie Pseudoalgebras of Type H

In the rest of the paper, $\mathfrak{d}$ will be an abelian Lie algebra of even dimension $n = 2N$, and $\omega \in \bigwedge^2 \mathfrak{d}^*$ will be a symplectic form. As before, $H = U(\mathfrak{d})$.

3.1. Definition of $H(\mathfrak{d}, 0, \omega)$.

We recall from [BDK1] the main facts on primitive pseudoalgebras of type H that we are going to need. The pseudoproduct

$$[e * e] = (r + s \otimes 1 - 1 \otimes s) \otimes_H e$$

endows He with a Lie pseudoalgebra structure as soon as $r \in \bigwedge^2 \mathfrak{d}$ and $s \in \mathfrak{d}$ satisfy the following equations:

$$[r, \Delta(s)] = 0\,, \qquad ([r_{12}, r_{13}] + r_{12}s_3) + \text{cyclic permutations} = 0\,, \tag{3.1}$$

where we use the standard notation $r_{12} = r \otimes 1$, $s_3 = 1 \otimes 1 \otimes s$, etc. In particular, since $\mathfrak{d}$ is abelian, setting $s = 0$ satisfies (3.1) for every choice of r. When r is of maximal rank, it induces an isomorphism $\mathfrak{d}^* \simeq \mathfrak{d}$. Then the symplectic 2-form $\omega \in \bigwedge^2 \mathfrak{d}^*$ corresponding to its inverse is a 2-cocycle of $\mathfrak{d}$.

In this case He is a simple Lie pseudoalgebra over H, which is denoted by $H(\mathfrak{d},0,\omega)$. Note that if ∂_i, $i=1,\dots,2N$, is a basis of $\mathfrak{d}$, $r=\sum_{ij} r^{ij}\partial_i\otimes\partial_j$, and $\omega_{ij}=\omega(\partial_i,\partial_j)$, then the matrices (r^{ij}) and (ω_{ij}) are inverse to each other. We will use the notation:

$$\partial^i=\sum_{j=1}^{2N} r^{ij}\partial_j,$$

so that

$$\partial_i=\sum_{j=1}^{2N}\omega_{ij}\partial^j,\qquad r=\sum_{i=1}^{2N}\partial_i\otimes\partial^i=-\sum_{i=1}^{2N}\partial^i\otimes\partial_i.$$

Moreover

$$\omega(\partial^i,\partial_j)=\delta^i_j=-\omega(\partial_i,\partial^j),\qquad \omega(\partial^i,\partial^j)=-r^{ij}.$$

There exists a unique nontrivial (injective) Lie pseudoalgebra homomorphism

$$\iota\colon H(\mathfrak{d},0,\omega)\to W(\mathfrak{d})\,,\qquad e\mapsto -r\,, \tag{3.2}$$

so that we may identify $H(\mathfrak{d},0,\omega)$ with a subalgebra of $W(\mathfrak{d})$.

3.2. **Annihilation Algebra of $W(\mathfrak{d})$ and of $H(\mathfrak{d},0,\omega)$.** For a Lie pseudoalgebra L, we set $\mathcal{L}=\mathcal{A}(L)=X\otimes_H L$, and define a bracket on $\mathcal{L}$ by the formula (cf. [BDK1, Eq. (7.2)]):

$$[x\otimes_H a,y\otimes_H b]=\sum(xh^i)(yk^i)\otimes_H c_i\,,\quad\text{if}\quad [a*b]=\sum(h^i\otimes k^i)\otimes_H c_i\,. \tag{3.3}$$

Then $\mathcal{L}$ is a Lie algebra, called the *annihilation algebra* of L. If $\phi:L\to L'$ is a Lie pseudoalgebra homomorphism, then $x\otimes_H a\mapsto x\otimes_H\phi(a), x\in X, a\in L$ is a Lie algebra homomorphism $\mathcal{A}(\phi):\mathcal{A}(L)\to\mathcal{A}(L')$, thus making $\mathcal{A}$ into a functor. There is a left action of H on $\mathcal{L}$ given by:

$$h(x\otimes_H a)=hx\otimes_H a. \tag{3.4}$$

As $\Delta(\partial)=\partial\otimes 1+1\otimes\partial$ for all $\partial\in\mathfrak{d}$, then elements from $\mathfrak{d}\subset H$ act on $\mathcal{L}$ by derivations. Let $\mathcal{W}=\mathcal{A}(W(\mathfrak{d}))$ be the annihilation algebra of the Lie pseudoalgebra $W(\mathfrak{d})$. Since $W(\mathfrak{d})=H\otimes\mathfrak{d}$, we have $\mathcal{W}=X\otimes_H(H\otimes\mathfrak{d})\simeq X\otimes\mathfrak{d}$ and we can identify $\mathcal{W}$ with $X\otimes\mathfrak{d}$. Then the Lie bracket (3.3) reads as follows ($x,y\in X$, $a,b\in\mathfrak{d}$):

$$[x\otimes a,y\otimes b]=-x(ya)\otimes b+(xb)y\otimes a\,.$$

Choosing a basis $\partial_1,\dots,\partial_{2N}$ identifies $\mathcal{W}$ with

$$W_{2N}=\operatorname{Der}\mathcal{O}_{2N}=\mathbf{k}[[t_1,\dots,t_{2N}]]\langle\partial/\partial t_1,\dots,\partial/\partial t_{2N}\rangle,$$

endowed with the standard Lie bracket. Under this identification, $x\otimes\partial_i\mapsto x\,\partial/\partial t_i$.

As before, let $r\in\bigwedge^2\mathfrak{d}$ be a skew-symmetric non-degenerate tensor. We may then choose a basis $\partial_1,\dots,\partial_{2N}$ of $\mathfrak{d}$ so that

$$r=\sum_{i=1}^{N}(\partial_i\otimes\partial_{N+i}-\partial_{N+i}\otimes\partial_i)\,. \tag{3.5}$$

Let $\mathcal{P}=\mathcal{A}(H(\mathfrak{d},0,\omega))=X\otimes_H(He)=X\otimes_H e$ be the annihilation algebra of $H(\mathfrak{d},0,\omega)$. According to (3.3), the Lie bracket on $\mathcal{P}$ is given by

$$[\phi\otimes_H e,\psi\otimes_H e]=\sum_{i=1}^{N}\left(\frac{\partial\phi}{\partial t_i}\frac{\partial\psi}{\partial t_{N+i}}-\frac{\partial\phi}{\partial t_{N+i}}\frac{\partial\psi}{\partial t_i}\right)\otimes_H e=\{\phi,\psi\}\otimes_H e, \tag{3.6}$$

where $\{\phi,\psi\}$ denotes the standard Poisson bracket on $\mathcal{O}_{2N}$. In other words, the map $\mathcal{O}_{2N}\ni\phi\to\phi\otimes_H e$ provides a Lie algebra isomorphism between the Poisson type linearly compact Lie algebra P_{2N} with the annihilation algebra $\mathcal{P}$ of $H(\mathfrak{d},0,\omega)=He$. We may define a filtration $\mathcal{P}=\mathcal{P}_{-2}\supset\mathcal{P}_{-1}\supset\dots$ on $\mathcal{P}=X\otimes_H He$ by

$$\mathcal{P}_p=\mathrm{F}_p\,\mathcal{P}=\mathrm{F}_{p+1}\,X\otimes_H e,$$

which satisfies $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$. In particular, $\mathcal{P}_0$ is a subalgebra of $\mathcal{P}$ and normalizes all $\mathcal{P}_i$. Since we have chosen $\mathfrak{d}$ to be abelian, this filtration is indeed induced by a grading on $\mathcal{P}$, and the Lie bracket is homogeneous of degree 0.

Recall that the canonical injection ι of the subalgebra $H(\mathfrak{d}, 0, \omega)$ in $W(\mathfrak{d})$ induces a Lie algebra homomorphism $\iota_* = \mathcal{A}(\iota) : \mathcal{P} \to \mathcal{W}$, which is however not injective, contrary to what happens with primitive Lie pseudoalgebras of all other types. Indeed:

Lemma 3.1. *ι_* has a one-dimensional kernel, linearly generated by $1 \otimes_H e$, which coincides with the centre of $\mathcal{P}$. In particular, $\mathcal{P}$ is a central extension of $\iota_*\mathcal{P} = \mathcal{H} \subset \mathcal{W}$ by a one-dimensional ideal.*

Proof. Using (3.2), we obtain $\iota_*(x \otimes_H e) = -\sum_i (x\partial^i) \otimes \partial_i$. Then $\iota_*(x \otimes_H e) = 0$ if and only if $x\partial^i = 0$ for all i, which only happens when x lies in $\mathbf{k} \subset X$.

The fact that $1 \otimes_H e$ is central in $\mathcal{P}$ easily follows from (3.6). In order to show that $1 \otimes_H e$ linearly spans the center of $\mathcal{P}$, notice that if $\phi \otimes_H e$ is central, then we argue by (3.6) that

$$0 = [t_i \otimes_H e, \phi \otimes_H e] = \frac{\partial \phi}{\partial t_{N+i}}, \qquad 0 = [t_{N+i} \otimes_H e, \phi \otimes_H e] = -\frac{\partial \phi}{\partial t_i},$$

for all $i = 1, \dots, N$, so that $\phi \in \mathbf{k}$. □

Remark 3.1. Notice that if the basis $\partial_1, \dots, \partial_{2N}$ is chosen so that (3.5) holds, the isomorphism $\mathcal{W} \simeq W_{2N}$ identifies $\mathcal{H}$ with the subalgebra $H_{2N} \subset W_{2N}$ of all formal vector fields preserving the standard symplectic form $\sum_{i=1}^N dt_i \wedge dt_{N+i}$.

3.3. **A central extension of $\mathfrak{d}$.** Recall that the linearly compact Lie algebra $\mathcal{P}$ is endowed by (3.4) with a left H-module structure given by $h.(x \otimes_H e) = hx \otimes_H e$. As the Lie algebra $\mathfrak{d}$ is abelian, the Hopf algebra H is commutative, and the left and right action of H on $X = H^*$ coincide. Let us introduce the notation $\widehat{\partial^i} = -t_i \otimes_H e$, and extend $\partial^i \mapsto \widehat{\partial^i}$ to a linear map $\mathfrak{d} \ni \partial \mapsto \widehat{\partial} \in \mathcal{P}$. Then

$$[-t_i \otimes_H e\,,\, x \otimes_H e] = -\sum_{j=1}^{2N} (t_i\partial_j)(x\partial^j) = (x\partial^i) \otimes_H e = (\partial^i x) \otimes_H e,$$

for all $i = 1, \dots, 2N$, so that

$$[\widehat{\partial}, x \otimes_H e] = \partial(x \otimes_H e),$$

for every choice of $\partial \in \mathfrak{d}, x \in X$. In particular,

$$[\widehat{\partial^i}, \widehat{\partial^j}] = r^{ij} \otimes_H e = -\omega(\partial^i, \partial^j) \otimes_H e,$$

hence $[\widehat{\partial}, \widehat{\partial'}] = -\omega(\partial, \partial') \otimes_H e$. In conclusion, we have:

Proposition 3.1. *Let $\widehat{\mathfrak{d}}$ be the Lie subalgebra of $\mathcal{P}$ generated by elements $\widehat{\partial}$ along with $c = -1 \otimes_H e$. Then c is central in $\widehat{\mathfrak{d}}$ and $[\widehat{\partial}, \widehat{\partial'}] = \omega(\partial, \partial')c$. In other words, $\widehat{\mathfrak{d}}$ is a central extension of $\mathfrak{d}$ of 2-cocycle ω.*

Let $\pi : \widehat{\mathfrak{d}} \to \mathfrak{d}$ be the canonical projection $\pi(\widehat{\partial}) = \partial$, and denote by $i : \widehat{\mathfrak{d}} \to \mathcal{P}$ the inclusion of $\widehat{\mathfrak{d}}$ as a subalgebra of $\mathcal{P}$.

Proposition 3.2. *Let $\delta \in \widehat{\mathfrak{d}}, \phi \in \mathcal{P}$. Then*

$$[i(\delta), \phi] = \pi(\delta).\phi.$$

In other words,

$$[\widehat{\partial}, \phi] = \partial.\phi,$$

for all $\partial \in \mathfrak{d}, \phi \in \mathcal{P}$. In particular, if M is a representation of $\mathcal{P}$, then

$$\widehat{\partial}(\phi.m) = (\partial.\phi).m + \phi.(\widehat{\partial}.m), \tag{3.7}$$

for all $\partial \in \mathfrak{d}, \phi \in \mathcal{P}, m \in M$.

4. Representations with coefficients

4.1. **Definition of representation with coefficients.** Let A be a pseudoalgebra over H, D a ring of coefficients over H, M a left D-module. By Remark 2.3 we may construct $(H \otimes D) \otimes_D M$, which has a natural left $H \otimes D$-module.

A *pseudoaction of A on M with coefficients in D* is then an $H \otimes D$-linear map $A \otimes M \ni a \otimes m \longrightarrow a * m \in (H \otimes D) \otimes_D M$. Explicitly, if $a * m = \sum_i (h^i \otimes d^i) \otimes_D m_i$ then

$$(ha) * (dm) = (h \otimes d) \cdot (a * m) = \sum_i (hh^i \otimes dd^i) \otimes_D m_i.$$

We may, as in Section 2.4, extend any pseudoaction of A on M with coefficients in D to $H^{\otimes m} \otimes (H^{(n-1)} \otimes D) = H^{\otimes(m+n-1)} \otimes D$-linear maps

$$(H^{\otimes m} \otimes_H A) \otimes ((H^{\otimes(n-1)} \otimes D) \otimes_D M) \to (H^{\otimes(m+n-1)} \otimes D) \otimes_D M \tag{4.1}$$

by setting

$$(F \otimes_H a) * (G \otimes_D m) = (F \otimes G) \cdot ((\Delta_H^{m-1} \otimes \Delta_D^{n-1}) \otimes_D \mathrm{id}_M)\ (a * m), \tag{4.2}$$

where $F \in H^{\otimes m}$, $G \in H^{\otimes(n-1)} \otimes D$, $a \in A$, $m \in M$. Here we are using the right D-module structure on $H^i \otimes D$ mentioned in Remark 2.3.

Definition 4.1. Let A be an associative pseudoalgebra over H, D a ring of coefficients over H. A D-module M is a *representation of A with coefficients in D* if it is endowed with a pseudoaction of A on M with coefficients in D

$$A \otimes M \to (H \otimes D) \otimes_D M\,, \quad a \otimes m \mapsto a * m$$

that satisfies ($a, b \in A$, $m \in M$)

$$(a * b) * m = a * (b * m),$$

where $(a * b) * m, a * (b * m) \in (H^{\otimes 2} \otimes D) \otimes_D M$ are understood by means of (4.1), (4.2).

Definition 4.2. Let L be a Lie pseudoalgebra over H, D a ring of coefficients over H. A D-module M is a *representation of L with coefficients in D* if it is endowed with a pseudoaction of L on M with coefficients in D

$$L \otimes M \to (H \otimes D) \otimes_D M\,, \quad a \otimes m \mapsto a * m$$

that satisfies ($a, b \in L$, $m \in M$)

$$[a * b] * m = a * (b * m) - ((\sigma \otimes \mathrm{id}_D) \otimes_D \mathrm{id}_M)(b * (a * m)),$$

where once again, all terms are computed by using (4.1), (4.2).

Example 4.1. When $D = H$, we recover the usual notion of representation or module of a Lie pseudoalgebra over H (see [BDK1, BDK2]).

Example 4.2. Let L be a Lie pseudoalgebra over H, M be a representation of L (with coefficients in H). If $H' \subset H$ is a Hopf subalgebra of H, then M is an H'-module by restriction of scalars. If $N \subset M$ is an H'-submodule such that $L * N \subset (H \otimes H') \otimes_H N$ then N is a representation of L with coefficients in H'.

Let M, N be D-modules. A D-conformal linear map from M to N is a $\mathbf{k}$-linear homomorphism $\phi : M \to (H \otimes D) \otimes_D N$ such that $\phi(dm) = (1 \otimes d) \cdot \phi(m)$. The space $\mathrm{Chom}^D(M, N)$ of all D-conformal linear maps from M to N is made into a left H-module via $(h\phi)(m) = (h \otimes 1) \cdot \phi(m)$. Then

$$\mathrm{Chom}^D(M, N) \otimes M \ni \phi \otimes m \mapsto \phi(m) \in (H \otimes D) \otimes_D N$$

is clearly an $H \otimes D$-linear map. When $\phi \in \mathrm{Chom}^D(M, N)$ and $m \in M$ we will denote $\phi(m)$ by $\phi * m$.

Proposition 4.1. *Let M be a (finitely generated) D-module. Then there exists a unique associative H-pseudoalgebra structure on $\mathrm{Cend}^D M = \mathrm{Chom}^D(M, M)$ such that the equality in $(H^{\otimes 2} \otimes D) \otimes_D M$*

$$(\phi * \psi) * m = \phi * (\psi * m),$$

holds for all $\phi, \psi \in \mathrm{Cend}^D M, m \in M$. Here both sides are understood according to (4.1) *and* (4.2).

Proof. The proof is analogous to [BDK1, Lemma 10.1]. □

The Lie H-pseudoalgebra structure $(\mathrm{Cend}^D M)^-$ obtained by taking the corresponding commutator

$$[\phi * \psi] = \phi * \psi - (\sigma \otimes_H \mathrm{id})\,(\psi * \phi),$$

is denoted by $\mathrm{gc}^D M$. Notice that M is a representation with coefficients in D of both $\mathrm{Cend}^D M$ and $\mathrm{gc}^D M$.

Remark 4.1. The assumption that M be a finitely generated D-module is necessary in order to set up the pseudoalgebra structure on $\mathrm{Cend}^D M$ (see [BDK1]).

Remark 4.2. We may denote $\mathrm{Cend}^D D^n$ and $\mathrm{gc}^D D^n$ by Cend_n^D and gc_n^D respectively. Then Cend_n^D is isomorphic to $H \otimes D \otimes \mathrm{End}\,\mathbf{k}^n$, with H acting by left multiplication on the first factor, endowed with the pseudoproduct

$$(h \otimes d \otimes A) * (h' \otimes d' \otimes A') = (h \otimes h'd_{(1)}) \otimes_H (1 \otimes d'd_{(2)} \otimes AA'),$$

where the pseudoaction of Cend_n^D on $D^n = D \otimes \mathbf{k}^n$ is given by

$$(h \otimes d \otimes A) * (d' \otimes v) = (h \otimes d'd) \otimes_D (1 \otimes Av),$$

for $h, h' \in H, d, d' \in D, A, A' \in \mathrm{End}\,\mathbf{k}^n, v \in \mathbf{k}^n$. The associative pseudoalgebra $\mathrm{Cend}_1^D = H \otimes D$ is similar to a pseudoalgebra structure considered by Kolesnikov [Ko].

The following statement is a straightforward consequence of definitions.

Proposition 4.2. *Let M be a finitely generated D-module, A an associative (resp. Lie) pseudoalgebra over H. Then giving on M a structure of A-module with coefficients in D is the same as giving an associative pseudoalgebra homomorphism $A \to \mathrm{Cend}^D M$ (resp. a Lie pseudoalgebra homomorphism $A \to \mathrm{gc}^D M$).*

4.2. **Equivalence of pseudoalgebra and annihilationalgebra representations.** The following statement generalizes the correspondence given in [BDK1, Proposition 9.1] to pseudoalgebra representation with coefficients. Indeed, every Lie pseudoalgebra representation of L with coefficients in D may be translated into a discrete representation of $\mathcal{L}$ possessing a compatible D-module structure.

Proposition 4.3. *Let D be a ring of coefficients over H, M a representation of the Lie H-pseudoalgebra L with coefficients in D, and $\mathcal{L} = X \otimes_H L$ the annihilation Lie algebra of L. Then M has a natural structure of discrete topological $\mathcal{L}$-module, given by*

$$(x \otimes_H a) \cdot m = \sum \langle xh^i, {d^i}_{(1)} \rangle\, {d^i}_{(2)} m_i\,, \quad \textit{if} \quad a * m = \sum (h^i \otimes d^i) \otimes_D m_i$$

for $a \in L$, $x \in X$, $m \in M$. Moreover, the D-module structure on M satisfies

$$d(g.m) = (d_{(1)}g).(d_{(2)}m), \tag{4.3}$$

for all $g \in \mathcal{L}, m \in M, d \in D$.

Conversely, any discrete $\mathcal{L}$-module M endowed with a D-module structure satisfying (4.3) *has a natural structure of representation of L with coefficients in D, given by*

$$a * m = \sum_{I \in \mathbb{N}^n} \left(S(\partial^{(I)}) \otimes 1\right) \otimes_D \left((t^I \otimes_H a) \cdot m\right),$$

where $\partial^{(I)} \in H = U(\mathfrak{d})$ are as in (2.1)*, and t^I are the corresponding dual basis elements.*

Proof. The proof that the pseudoalgebra action of L gives a continuous (discrete) representation of $\mathcal{L}$ and vice-versa is the same as in [BDK1, Proposition 9.1]. We are left with showing that $H \otimes D$-linearity of $a \otimes m \mapsto a * m$ is equivalent with (4.3).

Let us first show that $H \otimes D$-linearity implies (4.3), by computing $(d_{(1)}(x \otimes_H a)).(d_{(2)}m) = ((d_{(1)}x) \otimes_H a).(d_{(2)}m)$. By $H \otimes D$-linearity, if $a * m = \sum (h^i \otimes d^i) \otimes_D m_i$ then

$$a * (d_{(2)}m) = \sum_i (h^i \otimes d_{(2)}d^i) \otimes_D m_i,$$

so that

$$\begin{aligned} ((d_{(1)}x) \otimes_H a).(d_{(2)}m) &= \sum \langle d_{(1)}xh^i, d_{(2)}d^i_{(1)}\rangle \, d_{(3)}d^i_{(2)}m_i \\ &= \sum \langle xh^i, d^i_{(1)}\rangle \, \varepsilon(d_{(1)})d_{(2)} \, d^i_{(2)}m_i \\ &= d \sum \langle xh^i, d^i_{(1)}\rangle \, d^i_{(2)}m_i \\ &= d((x \otimes_H a).m). \end{aligned}$$

Conversely, assuming (4.3) holds, we immediately obtain

$$g.(dm) = d_{(2)}((S(d_{(1)})g).m),$$

for all choices of $g \in \mathcal{L}, d \in D, m \in M$, whence

$$\begin{aligned} a * dm &= \sum (S(\partial^{(I)}) \otimes 1) \otimes_D (t^I \otimes_H a).(dm) \\ &= \sum (S(\partial^{(I)}) \otimes 1) \otimes_D d_{(2)}((S(d_{(1)})t^I) \otimes_H a).m \\ &= \sum (S(d_{(1)}\partial^{(I)}) \otimes 1) \otimes_D d_{(2)}(t^I \otimes_H a).m \\ &= \sum (S(\partial^{(I)})S(d_{(1)}) \otimes 1) \otimes_D d_{(2)}(t^I \otimes_H a).m \\ &= \sum (S(\partial^{(I)})S(d_{(1)})d_{(2)} \otimes d_{(3)}) \otimes_D (t^I \otimes_H a).m \\ &= \sum (S(\partial^{(I)})\varepsilon(d_{(1)}) \otimes d_{(2)}) \otimes_D (t^I \otimes_H a).m \\ &= \sum (S(\partial^{(I)}) \otimes d) \otimes_D (t^I \otimes_H a).m \\ &= (1 \otimes d) \cdot (a * m). \end{aligned}$$

□

Remark 4.3. Notice that when $D = H$, Proposition 4.3 reduces to [BDK1, Proposition 9.1], and property (4.3) is the same as saying that the actions of elements in $\mathfrak{d} \subset H$ on both $\mathcal{L}$ and the $\mathcal{L}$-module M provides an action of the extended annihilation algebra $\widetilde{\mathcal{L}} = \mathfrak{d} \ltimes \mathcal{L}$.

Remark 4.4. The statement of Proposition 4.3 hints to the fact that the action of L on any representation with coefficients may be left-straightened, i.e., that $a * m \in (H \otimes D) \otimes_D M$ may be always be written as an expression in $(H \otimes \mathbf{k}) \otimes_D M$, see Lemma 2.1.

4.3. **Irreducible discrete representationsof P_{2N}.** Here we show that every irreducible discrete representation of P_{2N} admits a $D = D(\mathfrak{d}, \lambda\omega)$-module structure — where $D(\mathfrak{d}, \lambda\omega)$ is as in Example 2.2 — for some $\lambda \in \mathbf{k}$ which is compatible in the sense of (4.3) with the left H-module structure given on P_{2N} by the isomorphism $P_{2N} \simeq \mathcal{P} = \mathcal{A}(H(\mathfrak{d}, 0, \omega))$. In other words, the action of P_{2N} may be lifted to a pseudoalgebra representation of $H(\mathfrak{d}, 0, \omega)$ with coefficients in D.

Proposition 4.4. *Assume that* $\mathbf{k}$ *is uncountable. Let* M *be a continuous discrete irreducible representation of the linearly compact Lie algebra* $\mathcal{P} = P_{2N}$*. Then any central element in* $\mathcal{P}$ *acts via scalar multiplication by some element in* $\mathbf{k}$*.*

Proof. As the action of $\mathcal{P}$ on the discrete vector space M is continuous, then every element $m \in M$ is killed by an open subalgebra of $\mathcal{P}$, hence by $\mathcal{P}_k$ for a suitably large value of k. The subspace $U_k \subset M$ of all elements killed by $\mathcal{P}_k$ is then nonzero for sufficiently large $k \in \mathbb{N}$. Let us choose k so that $U_k \neq 0$: as $\mathcal{P}_0$ normalizes $\mathcal{P}_k$, then U_k is $\mathcal{P}_0$-stable. We immediately see that the action of $\mathcal{P}_0$ on U_k factors via the finite-dimensional Lie algebra $\mathcal{P}_0/\mathcal{P}_k$.

As $U(\mathcal{P}_0/\mathcal{P}_k)$ is countable-dimensional, we may find a countable-dimensional nonzero $\mathcal{P}_0$-submodule R of U_k. Then $U(\mathcal{P}) = U(\widehat{\mathfrak{d}})U(\mathcal{P}_0)$, hence $U = \operatorname{Ind}_{\mathcal{P}_0}^{\mathcal{P}} R = U(\mathcal{P}) \otimes_{U(\mathcal{P}_0)} R = U(\widehat{\mathfrak{d}}) \otimes R$ is still countable dimensional. By irreducibility of M, there is a surjective $\mathcal{P}$-module homomorphism $U \to M$, hence M is also countable-dimensional.

Using a countable Schur Lemma to M shows now that all central elements in $\mathcal{P}$ act by scalar multiplication on V. □

Remark 4.5. When $\mathbf{k}$ is countable, we may still assume that $-1 \otimes_H e = c \in Z(\mathcal{P})$ act by scalar multiplication up to replacing $\mathbf{k}$ with $\mathbf{k}(c)$.

Theorem 4.1. *Let M be a discrete irreducible representation of P_{2N} on which central elements act via scalar multiplication. Then there exists $\lambda \in \mathbf{k}$ such that M can be endowed with a $D = D(\mathfrak{d}, \lambda\omega)$-module structure, compatible in the sense of* (4.3) *with the $H = D(\mathfrak{d}, 0)$-module structure on P_{2N} obtained by identifying it with the annihilation algebra $\mathcal{P}$ of $H(\mathfrak{d}, 0, \omega)$. In other words, M may be lifted to a pseudoalgebra representation of $H(\mathfrak{d}, 0, \omega)$ with coefficients in D.*

Proof. We already know that $\widehat{\mathfrak{d}}$ embeds in the annihilation algebra $\mathcal{P}$ of $H(\mathfrak{d}, 0, \omega)$. Since $\mathcal{P} \simeq P_{2N}$, M may be given a left $U(\widehat{\mathfrak{d}})$-module structure via this embedding.

If $c = -1 \otimes_H e \in \mathcal{P}$ acts on M via multiplication by λ, then the left action of $U(\widehat{\mathfrak{d}})$ factors via the quotient $D = D(\mathfrak{d}, \lambda\omega)$. The compatibility between the H-module structure on $\mathcal{P}$ and the D-module structure on M is then stated in (3.7) for a set of algebra generators of D. □

Theorem 4.1 shows that all irreducible representations of P_{2N} may be read in the language of Lie pseudoalgebra representations with coefficients. This language has proved powerful in the study of representations of linearly compact Lie algebra $\mathcal{L}$ as it makes it easy to compute singular vectors in the so-called *tensor modules*, i.e., modules that are induced from a finite-dimensional representation of a maximal open primitive subalgebra of $\mathcal{L}$.

In order to make the use of pseudoalgebra language effective, one needs a *right-straightening formula* for the action of $\mathcal{L}$ on its tensor modules in order to give a bound for the degree of singular vectors, and then a *left-straightening formula* for the computation of singular vectors of a given degree, (see [BDK2, BDK3]) .

We have seen in Corollary 2.1 that existence of right- and left-straightening amounts to showing that $H \otimes D$ equals $(\mathbf{k} \otimes D)\Delta_D(D)$ and $(H \otimes \mathbf{k})\Delta_D(D)$ respectively: the latter always holds, whereas the former is false in general, but holds, by Lemma 2.2, in the ring of coefficients that is needed for expressing irreducible representations of P_{2N}. We will use this strategy towards studying irreducible representations of P_{2N} in a forthcoming paper.

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