THE KRUSKAL-KATONA THEOREM AND A CHARACTERIZATION OF SYSTEM SIGNATURES

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ABSTRACT. We show how to determine if a given vector can be the signature of a system on a finite number of components and, if so, exhibit such a system in terms of its structure function. The method employs combinatorial results from the theory of (finite) simplicial complexes, and provides a full characterization of signature vectors using a theorem of Kruskal and Katona. We also show how the same approach can provide new combinatorial proofs of further results, e.g., that the signature vector of a system cannot have isolated zeroes. Last, we prove that a signature with all nonzero entries must be the uniform distribution.

CONTENTS

1. INTRODUCTION

The concept of signature of a system is useful in providing knowledge of the lifetime distribution of the system in terms of its structure function and single components' lifetimes only; we refer to [2, 7] for a thorough introduction on the subject.

In this paper, we give a combinatorial characterization of signature vectors, which seems to be an open issue in the theory of system reliability (in [5] a complete study of systems with up to five components is provided). The approach provides a criterion to check whether a probability vector can be a signature. The method consists in simple tests involving entries of the candidate signature, and if the tests are positive it constructs explicitly the structure function of a system with the required signature. If the candidate signature vector does not fulfill a certain technical requirement, the procedure yields a family of sets of components which does not have the necessary algebraic properties, so that no system can have that vector as its signature.

The idea is to translate the problem in combinatorial terms, and then use a result of Kruskal and Katona, that offers a necessary and sufficient condition for a family of sets with a certain algebraic property to exist ([3, 4]). We show that the family of cut sets of any system enjoys such property, so that the characterization problem is eventually reduced to counting the number of cut sets of each possible cardinality.

The article is organized as follows. In Section 2 we recall the main definitions and notions in the theory of system reliability in terms of signatures, and in particular the relation between the signature and the number of cut sets of each cardinality. Section 3 is devoted to the definition

Date: May 20, 2014.

of simplicial complexes and its relation with cut sets and therefore with the signature as well. In Section 4 we recall the statement of the Kruskal-Katona theorem for simplicial complexes. Section 5 contains the main result as a summary of the observations of the previous sections: a criterion and a procedure testing whether a vector can be the signature of some system.

We then apply techniques from the theory of simplicial complexes to our context, which allows us to obtain more interesting results. We prove two more properties of system signatures in Section 6, namely that the signature cannot have isolated zeroes, and that the signature is uniform as soon as all of its entries are nonzero. The former results follows from [6, Theorem 2] as a consequence of the so-called IFRA property, yet here we provide a new proof of combinatorial nature. We conclude with some comments in Section 7.

2. Review of system signatures

In this section we recall some concepts and definitions in the theory of system reliability (see [7] for more details).

Let $\tau = \{\tau_1, \dots, \tau_n\}$ be a set of $n \in \mathbb{N}$ binary stochastic processes, interpreted as the state, as time evolves, of the components $\{X_1, \dots, X_n\} = X$ of a system. Each component X_l can be either down (or broken/off) or up (or working/on), e.g. $\tau_l = 0, 1$ respectively, $l = 1, \dots, n$. We assume that all components are initially up and when a component fails, it stays down forever, so each component X_l has a random lifetime T_l, whose distribution is assumed to be continuous, in order to avoid ties in failures. Lifetimes of components can be assumed to have the same distribution and to be independent, although exchangeability is enough. A system deploys its components according to some design architecture and is characterized by a *structure function* ϕ that indicates whether the whole system is up or down, for any given description of the states of individual components. In other words, the system may work even if some components are broken, and given a subset $G \subseteq X$, interpreted as the set of working components, the function

$$\phi : 2^X \to \{0, 1\}$$

tells us if the system is up ($\phi(G) = 1$) or down ($\phi(G) = 0$). Common sense requires ϕ to be non-decreasing, which means $A \subseteq B$ implies $\phi(A) \leq \phi(B)$, and to satisfy $\phi(\emptyset) = 0$, $\phi(X) = 1$.

At the beginning all components (hence the whole system) work, and then one at a time they fail (and stay broken), so that at some point the system stops working, say this occurs as the l-th failure of a components takes place. The order in which components fail is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the set $\{1, \dots, n\}$, and this means that $\phi(\{X_{\sigma(l)}, \dots, X_{\sigma(n)}\}) = 1$ but $\phi(\{X_{\sigma(l+1)}, \dots, X_{\sigma(n)}\}) = 0$. We may rephrase this by saying that (for a given system ϕ) one and only one breakdown index $l \in \{1, \dots, n\}$ is associated with any given ordering of the failures (permutation) σ. Let $N_l(\phi)$ be the number of permutations with breakdown index l, i.e., such that $\phi(\{X_{\sigma(l)}, \dots, X_{\sigma(n)}\}) = 1$ but $\phi(\{X_{\sigma(l+1)}, \dots, X_{\sigma(n)}\}) = 0$. Define $N(\phi) = (N_1(\phi), \dots, N_n(\phi)) \in \mathbb{N}^n$.

Definition 2.1. *The system signature is the probability vector $s(\phi) = N(\phi)/n!$, whose l-th entry $s_l(\phi)$ is the probability that the system stops working exactly as the l-th failure of a component takes place.*

An important question arises: given a vector, how can we determine whether it is the signature of some system? If so, what is a procedure to yield an explicit system inducing that singature? Further questions on the distributions of zero and nonzero entries have been raised by the observations of actual systems ([5]). We intend to address such questions in this article.

In the rest of this section we will recall the standard notations for the families of sets that determine the state of the system, revealing the combinatorial nature of the signature, that allows one to study the system reliability ([7]) combinatorially rather than as a stochastic process.

Definition 2.2. *A subset $B \subseteq X$ is called a* cut set *if the system cannot work when all its components are broken. A subset $G \subseteq X$ is called a* path set *if the system works whenever all its* *components work. A set of either type is said to be* minimal *if none of its proper subsets enjoys the same property.*

As the system evolves in time, choose from the set of all total orderings of $\{1, \dots, n\}$ the element σ that indicates the order in which the components failed. Say σ has breakdown index l, meaning that the system goes down as soon as the component $X_{\sigma(l)}$ breaks. Then [1]

$$S_l(\phi) = s_1(\phi) + \cdots + s_l(\phi) = \frac{1}{\binom{n}{l}} \sum_{B \subseteq X}^{|B|=l} (1 - \phi(B)) \,, \tag{2.1}$$

since $\sum_{B \subseteq X}^{|B|=l} (1 - \phi(B))$ counts the number of subsets B of X with cardinality l on which the structure function takes value zero, i.e., the cut sets of cardinality l. Elements of each cut set B and of those of its complement $X \setminus B$ can be (separately) freely permuted, so this term is multiplied by $l!(n-l)!$.

Definition 2.3. *We will call the complement of a cut set a co-cut set. By co-path, we will indicate the complement of a path set.*

Remark 2.1. *Equation* (2.1) *shows that the signature depends only on the number of co-cut sets of each cardinality. Equivalently, the signature depends only on the number of cut sets, or of path sets, or of co-path sets.*

Consider an ordering σ of components' failures with breakdown index l. This means that the set of the $l-1$ components that fail first does not contain a cut set, and that the remaining $n-l+1$ components include a path set (the system is still working at the time of the $l-1$-th failure) and therefore a minimal path set as well. We also know that the set of the l components that fail first does contain a cut set (and therefore a minimal cut set either), and the remaining $n-l$ components do not include any path set. The component $X_{\sigma(l)}$ giving place to the l-th failure belongs then to both a minimal cut set and a minimal path set. Since σ indicates the order in which the components fail, the component $X_{\sigma(l)}$ is the common element to the minimal cut and path sets that appear in the first l and last $n-l+1$ positions of the vector $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ respectively. This is a general fact.

Remark 2.2. *Each minimal cut set intersects all minimal path sets, and the intersection consists of exactly one element. Conversely, each minimal path set intersects all minimal cut sets, and the intersection has cardinality one.*

It is not difficult to see that the structure function is fully determined by the family of the minimal cut sets or equivalently by the family of the minimal path set. The system is thus completely defined by its structure function or by its family of minimal cut or path sets. This one-to-one correspondence that associates non decreasing functions $2^X \to \{0, 1\}$ with subsets of 2^X admitting no proper inclusions will be denoted by Ω, and justifies the following notation.

Definition 2.4. *Given a structure function ϕ, the corresponding family of minimal cut sets is $\Omega(\phi)$, and given a family $\bar{\Omega}$ of subsets of X without proper inclusions, the corresponding unique structure function will be denoted by $\phi_{\bar{\Omega}} = \Omega^{-1}(\bar{\Omega})$.*

3. Simplicial complexes and cut sets

In this section we recall some notions in the theory of simplicial complexes.

A *simplicial complex* K is a set of simplices such that any face of a simplex from K is also in K and so that the intersection of any two simplices $\Sigma_1, \Sigma_2 \in K$ is a face of both Σ_1 and Σ_2. A simplicial d-complex is a simplicial complex where the largest dimension of any of its simplices is d. The f-vector of a simplicial d-complex is the vector $(f_0, f_1, \dots, f_d)$ whose l-th component is the number of $(l-1)$-dimensional faces in the simplicial complex, and by convention $f_0 = 1$ unless the complex is empty.

It is important to notice that not all integral vectors can be f-vectors of a simplicial complex. In fact, there are constraints on the number of lower-dimensional simplices one must obtain when removing vertices from a given simplex in the complex. The Kruskal-Katona theorem provides a full characterization of f-vectors of simplicial complexes.

Now we want to relate the family of cut sets to the concept of simplicial complex. Let $\tilde{\Omega}$ be the family of cut sets of a system with structure function ϕ, and consider $\mathcal{C} \equiv 2^X \setminus \tilde{\Omega}$, i.e. the family of co-cut sets. Then $\mathcal{C} = \phi^{-1}(0)$ is a simplicial complex. In fact, since adding an element to a cut set yields a cut set of increased cardinality, removing an element from a co-cut sets yields a co-cut set of decreased cardinality: this is just the very essence of simplicial complexes. Similarly, using path sets instead of cut sets, the family of co-path sets $\mathcal{P} \equiv 2^X \setminus \phi^{-1}(1) = \phi^{*-1}(0)$ is also a simplicial complex (see Appedix 7 for the notation regarding ϕ^*).

We will focus on cut sets only, but our considerations stay unchanged if we consider path sets instead. In simpler terms, a superset of a cut or path set is still a cut or path set respectively. Denote by $\mathcal{C}_l$ and $\tilde{\Omega}_l$ the set of elements of $\mathcal{C}$ and $\tilde{\Omega}$ respectively of cardinality l, so that $\mathcal{C} = \cup_l \mathcal{C}_l$ and $\tilde{\Omega} = \cup_l \tilde{\Omega}_l$. Clearly $\mathcal{C}_l \cup \tilde{\Omega}_l = \binom{X}{l}$ and $|\mathcal{C}_l| + |\tilde{\Omega}_l| = \binom{n}{l}$. If $A \in \mathcal{C}_l$ and $x \in A$, then $A \setminus x \in \mathcal{C}_{l-1}$. Therefore the vector $(|\mathcal{C}_1|, \dots, |\mathcal{C}_n|)$ is the f-vector of the simplicial complex $\mathcal{C}$.

Knowledge of this vector is equivalent to knowledge of the family of cut sets, since $\tilde{\Omega}_l = \binom{X}{l} \setminus \mathcal{C}_l$. However, as noticed in Remark 2.1, the vector $(|\tilde{\Omega}_1|, \dots, |\tilde{\Omega}_n|)$ is the non-normalized cumulative signature whose l-th component coincides with $\binom{n}{l} S_l(\phi) = \binom{n}{l}(s_1(\phi) + \dots + s_l(\phi))$. Therefore the f-vector of $\mathcal{C}$, with components $f_l = \binom{n}{l}(1 - S_l)$, $l = 1, \dots, n$, is trivially related to the signature. Clearly, $1 - S_l$ equals $s_{l+1} + \dots + s_n$ by definition.

In the next section we will introduce the theorem of Kruskal-Katona, which provides a characterization of f-vectors, hence of signatures.

4. Kruskal-Katona Theorem

We recall here the Kruskal-Katona Theorem, which provides a characterization of f-vectors of simplicial complexes, which we will later apply towards understanding system signatures. Given two integers $k \geq 0$ and $l > 0$, it is known that there is a unique way to expand k as a sum of binomial coefficients as

$$k = \binom{n_l}{l} + \binom{n_{l-1}}{l-1} + \dots + \binom{n_j}{j},$$

with $n_l > n_{l-1} > \dots > n_j \geq j \geq 1$.

As an example, consider $n = 25$, $l = 3$. The largest integer of the form $\binom{n_3}{3}$ smaller than or equal to 25 is $20 = \binom{6}{3}$. The largest integer of the form $\binom{n_2}{2}$ smaller than or equal to $5 = 25 - 20$ is $3 = \binom{3}{2}$. The largest integer of the form $\binom{n_1}{1}$ smaller than or equal to $2 = 5 - 3$ is $2 = \binom{2}{1}$ and we are done since

$$\binom{6}{3} + \binom{3}{2} + \binom{2}{1} = 20 + 3 + 2 = 25.$$

Notice that $n_i \geq i$ for every i, so that all binomial summands are necessarily positive; as a consequence, when $k = 0$, the unique admissible expansion is the empty one.

Now for the given k and l define

$$k^+(l) = \binom{n_l}{l+1} + \binom{n_{l-1}}{l} + \dots + \binom{n_j}{j+1} \tag{4.1}$$

and

$$k^-(l) = \binom{n_l}{l-1} + \binom{n_{l-1}}{l-2} + \dots + \binom{n_j}{j-1} \tag{4.2}$$

from the previous expansion. When $k = 0$, this forces $k^+(l) = k^-(l) = 0$.

The next statement is a version of the Kruskal-Katona Theorem and offers a minimality constraint for simplicial complexes, with emphasis on the combinatorial aspects of the sets composing the complex. In fact, the term "complex" is not even used in the terminology.

Proposition 4.1 (Kruskal-Katona)**.** *Let $\underline{X}$ be a set of n elements, k and l be given integers such that*

$$1 \leq l \leq n\ ,\ \ 0 \leq k \leq \binom{n}{l}\ ,$$

and let

$$\mathcal{A} = \{A_1, \dots, A_k\}\ ,\ A_i \subseteq \underline{X}\ ,\ |A_i| = l\ ,\ i = 1, \dots, k\ .$$

If

$$\mathcal{A}^- = \{B : |B| = l-1\ ,\ \exists\, j : B \subset A_j\}\ , \tag{4.3}$$

then

$$\min_{\mathcal{A}} |\mathcal{A}^-| = k^-(l)\ ,$$

where the minimum runs over all collections $\mathcal{A}$ of k subsets of $\underline{X}$ of cardinality l, and $k^-(l)$ is defined as in (4.2).

Here and later, for the original proof and a more general analysis, see [3, 4]. Condition (4.3) states that families $\mathcal{A}$ of increasing or decreasing cardinality form a simplicial complex. The next statement is probably the most common version of the Kruskal-Katona Theorem, equivalent to the previous one, and provides a necessary and sufficient condition on the number of l-simplices in order for them to be induced from a complex. These numbers are the entries of the so called f-vector of the complex, whose definition has been recalled in Section 3.

Proposition 4.2 (Kruskal-Katona)**.** *A vector $(f_0, f_1, \dots, f_d)$ is the f-vector of a simplicial d-complex if and only if*

$$0 \leq f_l^-(l) \leq f_{l-1}\ ,\ 1 \leq l \leq d\ . \tag{4.4}$$

In the case of the application to system signature with n components, we will consider $d = n-1$, and any total ordering can be chosen for the system components. Choosing, at level l, initial segments (according to the reverse lexicographic order) of size f_l makes the number of implied elements al level $l-1$ minimal. There is a dual maximality condition which is equivalent to (4.4)

$$0 \leq f_{l+1} \leq f_l^+(l)\ ,\ 0 \leq l \leq d-1.$$

The reverse lexicographic order simply reads backwards the strings, then sorts lexicographically. The advantage of considering the reverse lexicographic order is that the list of the first (according to this order) $r \in \mathbb{N}$ elements does not depend on the size of the alphabet (the size n of the system, in our case).

5. Characterization of system signatures

In this section we sum up all observations made so far into the main result of this work. Let us start with a preliminary well known observation, basically equivalent to what we presented in Section 3. Consider the signature vector s of some system with n components $\{X_1, \dots, X_n\}$. The entry s_l is the probability that the system fails at the l-th failure of a component. In other words, the l-th entry of the cumulative signature $S_l = s_1 + \cdots + s_l$ is the probability that the first l components that broke form a cut set. This probability is in turn nothing but the fraction of cut sets of cardinality l among all subsets of $\{X_1, \dots, X_n\}$ with cardinality l.

Let us see how this applies in the context of system signatures, when translated in terms of simplicial sets. Recall that if we add a component to a cut set, we get again a cut set, i.e., all supersets of a cut set are cut sets. The algorithm that we are about to present is in fact the translation of the proof of the Kruskal-Katona Theorem where the role of f-vectors is played by the "complement" of the cumulative signature times the number of permutations of components, roughly speaking.

Theorem 5.1. *Let the probability vector $\bar{s} \in \mathbb{R}^n$ be the candidate signature. When $l = 1, \dots, n-1$, define $f_l = \binom{n}{l}(\bar{s}_{l+1} + \cdots + \bar{s}_n)$. Then $\bar{s}$ is the signature of a system if and only if all f_l are non-negative integers, and they satisfy*

$$0 \le f_l^-(l) \le f_{l-1}\ ,\ 1 \le l \le n-1. \tag{5.1}$$

Proof. From the candidate signature $\bar{s}$ we also know the non-normalized candidate cumulative signature $\bar{S}$. Clearly $1 - \bar{S}_l = \bar{s}_{l+1} + \cdots + \bar{s}_n$ by definition. Then the number $f_l = \binom{n}{l}(1 - \bar{S}_l)$ must be the number of co-cut sets of cardinality l, as we discussed in Section 3, if there is some system with signature $\bar{s}$, and is thus a non-negative integer; notice that, by the very definition, if $f_l = 0$, then $f_k = 0$ for all $k \ge l$, hence Condition (5.1) is satisfied for all such indices.

Moreover, the family of co-cut sets of our (hypothetical) system inducing $\bar{s}$ forms a simplicial complex, whose f-vector equals $(1, f_1, \dots, f_{n-1})$. The theorem of Kruskal-Katona provides a test to check whether this vector can actually be the f-vector of a simplicial complex. The test consists precisely of Condition (5.1), as explained in Proposition 4.2. □

Example 5.1. *Consider the probability vector*

$$\bar{s} = (s_1, s_2, s_3, s_4, s_5) = (0, 3/10, 2/5, 3/10, 0).$$

The corresponding vector is $f = (f_0, f_1, f_2, f_3, f_4) = (1, 5, 7, 3, 0)$. One has

$$f_1^-(1) = 1, \quad f_2^-(2) = 5, \quad f_3^-(3) = 6, \quad f_4^-(4) = 0.$$

For instance, $f_2 = 7$ can be written as $\binom{4}{2} + \binom{1}{1}$, hence $f_2^-(2)$ equals $\binom{4}{1} + \binom{1}{0} = 5$. We see that Condition (5.1) is satisfied, as $1, 5, 6, 0$ are not greater than, respectively, $1, 5, 7, 3$, and we conclude that $\bar{s}$ is a signature vector.

Sometimes an equivalent procedure might be handier, especially for small systems. Here it follows. Let $\bar{N} \equiv n!\bar{s} \in \mathbb{N}^n$.

(1) For each $l = 1, \dots, n$, sort in lexicographic order the subsets of $\{1, \dots, n\}$ of cardinality l.
(2) Take the family Ω^l of the first $(\bar{N}_1 + \cdots + \bar{N}_l)/(n-l)!l! = (\bar{s}_1 + \cdots + \bar{s}_l)\binom{n}{l}$ subsets, with respect to the lexicographic order.
(3) Take the union $\cup_{l=1}^n \Omega^l$ of all the Ω^l, $l = 1, \dots, n$, and extract the minimal family $\bar{\Omega}$.
(4) The function $\phi_{\bar{\Omega}}$ is the structure function of a system X with n components and signature $\bar{s}$.

Now considering Proposition 4.1, the arguments of Theorem 5.1 also prove the following test.

Criterion. *The family Ω^{l+1} should contain all the supersets (of cardinality $l+1$) of at least one element from Ω^l. If this is not the case, then the vector $\bar{s}$ cannot be the signature of a system since $\cup_h \Omega^l$ is not a simplicial complex.*

This criterion is equivalent to Theorem 5.1, and the algorithm we presented is simply the Kruskal-Katona algorithm adjusted to work directly with the candidate non-normalized cumulative signature as opposed to its "complementary" vector with components $\binom{n}{l}(1 - \bar{S}_l(\phi))$, $l = 1, \dots, n$. This is the reason why we sort strings lexicographically, because the collection $\mathcal{C}$ is a simplicial complex, as opposed to $\tilde{\Omega}$. So instead of taking, as in the original Kruskal-Katona algorithm, initial segments in each $\mathcal{C}_l$ according to reverse lexicographic order, we take final segments, i.e., initial segments according to the reverse ordering, which is the lexicographic order, in each $\tilde{\Omega}_l$.

We want to show that this second algorithm can be fairly fast in an explicit detailed example. The reader may want to compare this example with the systematic study of small systems in [5].

Example 5.2. *Consider the vector* $(0, 3/10, 2/5, 3/10, 0)$. *We pass easily to the non-normalized one* $(0, 36, 48, 36, 0)$ *by multiplying by* $5!$.

Start with $l = 1$. *We must take the first* $0/1!4! = 0$ *singletons.*

Take $l = 2$. *We must take the (lexicographically) first* $36/2!3! = 3$ *subsets with two elements. They are* $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$.

Take $l = 3$. *We must take the first* $(36 + 48)/3!2! = 84/12 = 7$ *subsets with three elements. These are* $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$.

Take $l = 4$. *We must take the first* $(36 + 48 + 36)/4!1! = 120/24 = 5$ *subsets with four elements. These are* $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$.

Take $l = 5$. *We must take the first* $120/120 = 1$ *subsets with five elements. This is* $\{1, 2, 3, 4, 5\}$.

From all these subsets we must extract a minimal family. It is not difficult to obtain $\bar{\Omega} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$. *In fact, all other listed sets are supersets of these four selected sets.*

Even finding $\bar{\Omega}$ can be automatized: in general, if f_l subsets are missing at level l, one should take, at level $l + 1$, all subsets that come after the first $\binom{n}{l+1} - f_l^+(l)$. Let us perform this computation explicitly in the above example: as

$$(f_0, f_1, f_2, f_3, f_4) = (1, 5, 7, 3, 0),$$

we obtain

$$f_1^+(1) = 10, \quad f_2^+(2) = 4, \quad f_3^+(3) = 0, \quad f_4^+(4) = 0.$$

Then we can recover $\bar{\Omega}$ by omitting, from the above lists, the (lexicographically) first $\binom{n}{l+1} - f_l^+(l)$ subsets of cardinality $l + 1$. This means we should omit the first $0 = \binom{5}{2} - 10$ subsets of cardinality 2, and take the remaining three; omit the first $6 = \binom{5}{3} - 4$ subsets of cardinality 3, and take just the last one; and omit all subsets of higher cardinality. This gives, indeed, $\bar{\Omega} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$.

This fully determines the system, and we can use the definition of minimal cut sets to determine the structure function $\phi_{\bar{\Omega}}$ and verify that $N(\phi_{\bar{\Omega}}) = (0, 36, 48, 36, 0)$.

6. Two more properties of signatures

In this section we provide two properties of system signatures.

6.1. **Signature vectors cannot have isolated zeroes.** In next theorem, we are going to provide a new combinatorial proof of a previously established property of signatures. Namely, system signature cannot have isolated zeroes, which follows from [6, Theorem 2].

We use the same notation as in the previous section: X is the set of system components, $\tilde{\Omega}$ is the family of cut sets, $\tilde{\Omega}_l$ is the family of all cut sets with cardinality l. Recall that $|\tilde{\Omega}_l| = \binom{n}{l} S_l(\phi)$, where $S_l(\phi) = s_1(\phi) + \cdots + s_l(\phi)$ is the cumulative signature, if $s(\phi)$ is the signature vector of a system with structure function ϕ. Should $s_l(\phi)$ be zero, we would have $S_l(\phi) = S_{l-1}(\phi)$. Therefore the condition

$$\frac{|\tilde{\Omega}_l|}{\binom{n}{l}} = \frac{|\tilde{\Omega}_{l-1}|}{\binom{n}{l-1}}$$

is equivalent to the vanishing of the l-th entry in the signature. We want to show that this cannot occur unless all subsequent (or preceding, by duality) entries are all zero.

Theorem 6.1. *Let* C_l *the number of cut sets of cardinality* l. *If*

$$C_{l-1} \neq 0, \binom{n}{l-1},$$

then

$$\frac{C_{l-1}}{\binom{n}{l-1}} < \frac{C_l}{\binom{n}{l}}.$$

Proof. We know from the previous section that $\mathcal{C}_l = \binom{X}{l} \setminus \tilde{\Omega}_l$ is a simplicial set, so that the Kruskal-Katona Theorem applies. Therefore we have

$$|\mathcal{C}_l| = \binom{c_l}{l} + \binom{c_{l-1}}{l-1} + \dots$$

for suitable integers $n \geq c_l > c_{l-1} > \dots$

We can assume $n > a_l$, since $|\mathcal{C}_l| \neq \binom{n}{l}$. Our aim is to show that

$$1 - \frac{|\mathcal{C}_l|}{\binom{n}{l}} > 1 - \frac{|\mathcal{C}_{l-1}|}{\binom{n}{l-1}}$$

or equivalently

$$\frac{|\mathcal{C}_l|}{\binom{n}{l}} < \frac{|\mathcal{C}_{l-1}|}{\binom{n}{l-1}}$$

In order to make the notation lighter, let us put $\mathbf{C}_l = |\mathcal{C}_l|$, so that, according to the notation of the previous section

$$\mathbf{C}_l^- = \binom{c_l}{l-1} + \binom{c_{l-1}}{l-2} + \cdots$$

and

$$\mathbf{C}_{l-1} \geq \mathbf{C}_l^- \, .$$

Hence we only have to prove that

$$\frac{\mathbf{C}_l}{\binom{n}{l}} < \frac{\mathbf{C}_l^-}{\binom{n}{l-1}}$$

which is the same as

$$\frac{\mathbf{C}_l^-}{\mathbf{C}_l} > \frac{\binom{n}{l-1}}{\binom{n}{l}} \, .$$

In other words, we want to show that the function $\mathbf{C}_l \mapsto \mathbf{C}_l^-/\mathbf{C}_l$, $\mathbf{C}_l > 0$, takes its only minimum at $\mathbf{C}_l = \binom{n}{l}$. We now use Lemma 6.1 below, the proof of which we postpone for the sake of clarity, with

$$a = \binom{c_l}{l-1}, \;\; a^* = \binom{c_l}{l-2}, \;\; a' = \binom{c_{l-1}}{l-2} + \cdots$$

$$b = \binom{c_l}{l}, \;\; b^* = \binom{c_l}{l-1}, \;\; b' = \binom{c_{l-1}}{l-1} + \cdots$$

By assumption, $a'/b' > a^*/b^*$. Moreover, $b^* > b'$ and $a^* \geq a'$. Since $a/b = l/(c_l - l + 1)$ and $a^*/b^* = (l-1)/(c_l - l + 2)$, then $a/b > a^*/b^*$. Lemma 6.1 yields now

$$\frac{\mathbf{C}_l^-}{\mathbf{C}_l} = \frac{a + a'}{b + b'} > \frac{a + a^*}{b + b^*} = \frac{\binom{c_l}{l-1} + \binom{c_l}{l-2}}{\binom{c_l}{l} + \binom{c_l}{l-1}} = \frac{\binom{c_l+1}{l-1}}{\binom{c_l+1}{l}} \geq \frac{\binom{n}{l-1}}{\binom{n}{l}}$$

which concludes the proof. □

We are only left with proving the following easy statement.

Lemma 6.1. *Let a, a', a^*, b, b', b^* be positive integers such that $a^* \geq a'$, $b^* > b'$. If a/b, $a'/b' > a^*/b^*$, then*

$$(a + a')/(b + b') > (a + a^*)/(b + b^*) \, .$$

Proof. It is well known that the mediant $(r + t)/(s + u)$ of any two given fractions r/s, t/u, lies in-between, provided that $r, s, t, u > 0$. Let us use this fact with a/b, $(a^* - a')/(b^* - b')$ to get

$$\frac{a^* - a'}{b^* - b'} < \frac{a^*}{b^*} < \frac{a'}{b'} \, .$$

Therefore

$$\frac{a}{b} > \frac{a^*}{b^*} > \frac{a^* - a'}{b^* - b'} .$$

Now both a/b and a'/b' are strictly larger than $(a^* - a')/(b^* - b')$, and hence so is $(a+a')/(b+b')$. Then

$$\frac{a + a'}{b + b'} > \frac{a + a' + a^* - a'}{b + b' + b^* - b'} = \frac{a + a^*}{b + b^*}$$

and the statement follows. □

6.2. **Singleton cut sets correspond to uniform signatures.** Let us conclude with a final observation, for which we use the notation presented in the Appendix. The signatures we computed so far, and almost all those appearing in the literature, have initial or final entry equal to zero. An exception is the signature of a system with structure function ϕ_1 associated with a minimal family of minimal cut sets consisting of the only subset $\Omega_1 = \{X_1\}$. This is no coincidence. As a matter of fact, the following result holds.

Theorem 6.2. *For any system, if both the first and last entry of the signature differ from zero, then all entries coincide (and equal the inverse of the size of the system).*

Proof. If both the first and last entry of the signature are different from zero, then if ϕ is the structure function both $\Omega(\phi)$ and $\Omega(\phi^*)$ contain a singleton, say $\{X_1\}$. Now all elements of $\Omega(\phi^*)$ must intersect all elements of $\Omega(\phi)$, and therefore they all contain X_1. For the same reason, all elements of $\Omega(\phi)$ contain X_1, so that the only minimal cut set is $\Omega(\phi) = \{\{X_1\}\}$. This means that $\phi = \phi_1$ and we know that $s(\phi_1)$ is the uniform distribution (over the components of the system). □

7. Conclusions and outlook

We have employed results obtained in the context of simplicial complexes to address questions in the theory of system reliability, with particular focus on system signatures. We have introduced a procedure, making use of the celebrated Kruskal-Katona theorem, that checks if a given probability vector can be a system signature, and in this case constructs a system with that signature. This completely characterizes the set of possible system signatures.

We have proved three properties of system signatures, that followed observation in many numerical studies carried out in the literature (see, e.g., [5]). Namely, we have showed in a new combinatorial way that no isolated zeroes may appear in the signature vector and that the only signature with first and last component both different from zero is the uniform one. Further applications can be given, for instance about the (partial) unimodal property of the signature, and we plan on reporting on some of them soon.

Appendix: Duality of systems

Let us recall a definition that is only need to introduce a notation that is used in Section 6.2.

Definition 7.1. *Given a system with structure function ϕ, the dual system has structure function ϕ^**

$$\phi^*(A) = 1 - \phi(X \setminus A)$$

for all $A \subseteq X$.

Let us study some elementary families of cut sets. Given a family $\bar{\Omega}$ of subsets of X without proper inclusions, and recalling Definition 2.4, we may define its dual family by

$$\bar{\Omega}^* = \Omega(\phi^*_{\bar{\Omega}})$$

independently of whether it is interpreted as family of cut or path set. The family of minimal cut sets of a system is also the family of the minimal path sets of the dual system (and vice versa, because duality is an involution). Duality is in essence the relation between minimal cut sets and minimal path sets, which ultimately consists of a time reversal (reading the signature in reverse order). Translated in terms of structure functions this means

$$\phi_{\bar{\Omega}^*} = \phi^*_{\bar{\Omega}} \, .$$

We can learn now how to find dual families of minimal cut or path sets. Let us introduce a basic example. If for some positive integer $h < n$ we choose $\Omega_h = \{\{X_1\}, \{X_2\}, \dots, \{X_h\}\}$, then $\Omega^*_h = \{\{X_1, X_2, \dots, X_h\}\}$. This is the special case of series-parallel duality. Now let $\phi_h = \phi_{\Omega_h}$. Using for simplicity the unnormalized signature $N(\phi) = n!s(\phi)$, it is immediate to recognize that

$$N_i(\phi_h) = (n-h)!h!\binom{n-i}{h-1} \, , \; i < n-h+1$$

and

$$N_i(\phi^*_h) = (n-h)!h!\binom{i-1}{h-1} \, , \; i > h \, .$$

ACKNOWLEDGMENTS

The authors would like to thank Mario Marietti for pointing out that a former characterization we had produced was actually equivalent to the one provided by Kruskal and Katona. They express their gratitude to Fabio Spizzichino for suggesting the problem of characterizing signatures. ADA was partially supported by "La Sapienza" Ateneo fundings. LDS was partially supported by research funds of C. De Concini. The authors thank the anonymous referee for pointing out reference [6].

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