A GRAPH-DYNAMICAL INTERPRETATION OF KISELMAN'S SEMIGROUPS

ELENA COLLINA AND ALESSANDRO D'ANDREA

ABSTRACT. A Sequential Dynamical System (SDS) is a quadruple (Γ, S_i, f_i, w) consisting of a (directed) graph $\Gamma = (V, E)$, each of whose vertices $i \in V$ is endowed with a finite set state S_i and an update function $f_i : \prod_{j, i \to j} S_j \to S_i$ — we call this structure an *update system* — and a word w in the free monoid over V, specifying the order in which update functions are to be performed. Each word induces an evolution of the system and in this paper we are interested in the dynamics monoid, whose elements are all possible evolutions.

When Γ is a directed acyclic graph, the dynamics monoid of every update system supported on Γ naturally arises as a quotient of the Hecke-Kiselman monoid associated with Γ. In the special case where $\Gamma = \Gamma_n$ is the complete oriented acyclic graph on n vertices, we exhibit an update system whose dynamics monoid coincides with Kiselman's semigroup K_n, thus showing that the defining Hecke-Kiselman relations are optimal in this situation. We then speculate on how these results may extend to the general acyclic case.

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1. INTRODUCTION

In this paper, we show how the recently defined notion of Hecke-Kiselman monoid finds a natural realization in the combinatorial-computational setting of Sequential Dynamical Systems.

Let Q be a mixed graph, i.e., a simple graph with at most one connection between each pair of distinct vertices; connections can be either oriented (arrows) or non-oriented (edges). In [GM11], Ganyushkin and Mazorchuk associated with Q a semigroup $\mathbf{HK}_Q$ generated by idempotents a_i indexed by vertices of Q, subject to the following relations

- $a_i a_j = a_j a_i$, if i and j are not connected;
- $a_i a_j a_i = a_j a_i a_j$, if $(\, i \text{---} j \,) \in Q$, i.e., i and j are connected by an edge;
- $a_i a_j = a_i a_j a_i = a_j a_i a_j$, if $(\, i \rightarrow j \,) \in Q$, i.e., i and j are connected by an arrow from i to j.

The semigroup $\mathbf{HK}_Q$ is known as the *Hecke-Kiselman monoid* attached to Q.

For the two extremal types of mixed graphs — graphs, where all sides are edges, and oriented graphs, in which all sides are arrows — Hecke-Kiselman monoids are well understood: when

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Date: May 20, 2014.

Q is an oriented graph with n vertices and no oriented cycles, then $\mathbf{HK}_Q$ is isomorphic to a quotient of Kiselman's semigroup K_n [Kis02, KM09], which is known to be finite [KM09].

On the other hand, when Q has only unoriented edges, $\mathbf{HK}_Q$ is finite if and only if Q is a (finite) disjoint union of finite simply laced Dynkin diagrams, and the corresponding semigroup is then variously known as *Springer-Richardson*, 0-*Hecke*, or *Coxeter monoid* attached to Q. The problem of characterizing mixed graphs inducing a finite Hecke-Kiselman monoid in the general case seems to be difficult and only very partial results are known [AD13]. The study of (certain quotients of) Hecke-Kiselman monoids and their representations has also attracted recent interest, see for instance [For12, Gre12, GM13].

The choice of a simple (i.e., without loops or multiple edges) graph Γ is also one of the essential ingredients in the definition of a Sequential Dynamical System (SDS). The notion of SDS has been introduced by Barrett, Mortveit and Reidys [BMR00, BMR01, BR99] in order to construct a mathematically sound framework for investigating computer simulations; this structure has found wide applicability in many concrete situations, cf. [MR08] and references therein. SDS on a directed acyclic graph Γ are related to Hecke-Kiselman monoids in that (see Proposition 2.4 below) the so-called Γ-local functions [MR08] satisfy the relations listed in the presentation of $\mathbf{HK}_\Gamma$; in other words, the evaluation morphism mapping each word (or update schedule) in the alphabet V to the corresponding composition of Γ-local functions factors through $\mathbf{HK}_\Gamma$.

One is naturally led to wonder whether $\mathbf{HK}_\Gamma$ is the smallest quotient through which all of the above evaluation morphisms must factor, or additional universal relations among Γ-local functions may be found. Henceforth, $\Gamma_n = (V, E)$ will denote the oriented graph where $V = \{1, \ldots, n\}$ and $(i, j) \in E$ if and only if $i < j$. In this paper, we show that $\mathbf{HK}_\Gamma$ is optimal in the special case $\Gamma = \Gamma_n$, by proving Theorem 6.3, which immediately implies the following statement.

Theorem. *There exists an update scheme $\mathcal{S}_n^\star = (\Gamma_n, S_i, f_i)$ such that the associated evaluation morphism factors through no nontrivial quotient of $\mathbf{HK}_{\Gamma_n}$.*

We believe that the same claim holds for every finite directed acyclic graph and mention in Section 8 some evidence in support of this conjecture.

The paper is structured as follows. In Section 2 we recall the definition of SDS, define the dynamics monoid of an update system supported on an oriented graph Γ, and show that it is a quotient of the Hecke-Kiselman monoid $\mathbf{HK}_\Gamma$ as soon as Γ has no oriented cycles. In Sections 3 and 4 we list the results on Kiselman's semigroup K_n that are contained in [KM09] and draw some useful consequences. Section 5 introduces the *join operation*, which is the key ingredient in the definition of the update system $\mathcal{S}_n^\star$, which is given in Section 6. The rest of the paper is devoted to the proof of Theorem 6.3.

2. SEQUENTIAL DYNAMICAL SYSTEMS

An *update system* is a triple $\mathcal{S} = (\Gamma, (S_i)_{i \in V}, (f_i)_{i \in V})$ consisting of

(1) a *base graph* $\Gamma = (V, E)$, which is a finite directed graph, with V as vertex set and $E \subseteq V \times V$ as edge set; we will write $i \to j$ for $(i, j) \in E$. The *vertex neighbourhood* of a given vertex $i \in V$ is the subset

$$x[i] = \{j \colon i \to j\}.$$

(2) a collection $S_i, i \in V$, of finite sets of *states*. We denote by $S = \prod_{i \in V} S_i$ the family of all the possible *system states*, i.e., n-tuples $\mathbf{s} = (s_i)_{i \in V}$, where s_i belongs to S_i for each vertex i. The *state neighbourhood* of $i \in V$ is $S[i] = \prod_{j \in x[i]} S_j$, and the restriction of $\mathbf{s} = (s_j)_{j \in V}$ to $x[i]$ is denoted by

$$\mathbf{s}[i] = (s_j)_{j \in x[i]} \in S[i].$$

(3) for every vertex i, a *vertex (update) function*

$$f_i \colon S[i] \quad \to \quad S_i,$$

computes the new state value on vertex i as a function of its state neighbourhood. In particular, if $x[i]$ is empty, then f_i is a constant $t \in S_i$ and we will write $f_i \equiv t$. Each vertex function f_i can be incorporated into a Γ-*local function* $\mathbf{F}_i \colon S \to S$ defined as

$$\mathbf{F}_i(\mathbf{s}) = \mathbf{t} = (t_j)_{j \in V}, \text{ where } t_j = \left\{ \begin{array}{ll} s_j, & \text{if } i \neq j \\ f_i(\mathbf{s}[i]), & \text{if } i = j \end{array} \right.$$

An SDS is an update system $\mathcal{S}$ endowed with

(4) an *update schedule*, i.e., a word $w = i_1 i_2 \ldots i_k$ in the free monoid $\mathrm{F}(V)$ over the alphabet V (from now on, we will often abuse the notation denoting both the alphabet and the vertex set with the same letter V, and both letters and vertices with the same symbols). The update schedule w induces a dynamical system map (SDS map), or an *evolution* of S, $\mathbf{F}_w \colon S \to S$, defined as

$$\mathbf{F}_w = \mathbf{F}_{i_1} \mathbf{F}_{i_2} \ldots \mathbf{F}_{i_k}.$$

Remark 2.1. As the graph Γ sets up a dependence relation between nodes under the action of update functions, it makes sense to allow Γ to possess self-loops, and arrows connecting the same vertices but going in opposite directions. However, we exclude the possibility of multiple edges between any two given vertices. Notice, however, that all SDS of interest in this paper will be supported on directed *acyclic* graphs, thus excluding in particular the possibility of self-loops.

Denote by $\mathrm{End}(S)$ the set of all maps $S \to S$, with the monoid structure given by composition. Then, the Γ-local functions $\mathbf{F}_i, i \in V$, generate a submonoid of S which we denote by $D(\mathcal{S})$. The monoid $D(\mathcal{S})$ is the image of the natural homomorphism

$$\begin{array}{rcl} \mathbf{F} \colon \mathrm{F}(V) & \to & \mathrm{End}(S) \\ w & \mapsto & \mathbf{F}_w. \end{array}$$

mapping each update schedule w to the corresponding evolution $\mathbf{F}_w$; in particular, we denote by $\mathbf{F}_\star$ the identity map, induced by the empty word $\star$.

Once an underlying update system has been chosen, our goal is to understand the monoid structure of $D(\mathcal{S})$.

Example 2.2. Let $\Gamma = (\{i\}, \emptyset)$ be a Dynkin graph of type A_1. It has only one vertex i, so there is only one vertex function f_i, which is constant as there are no arrows starting in i. The system dynamics monoid $D(\mathcal{S}) = \{\mathbf{F}_\star, \mathbf{F}_i\}$ contains exactly two elements, as soon as $|S_i| > 1$. If $|S_i| = 1$, then $D(\mathcal{S}) = \{\mathbf{F}_\star\}$.

Example 2.3. Let Γ be the graph

$$i \longrightarrow j$$

Let us consider $S_i = \{0, 1, 2\}$, $S_j = \{0, 1\}$. Set up an update system on Γ by requiring that $f_i \colon S[i] = S_j \to S_i$ acts as $f_i(s) = s + 1$ and that $f_j \equiv 1$. Evolutions induced by words $\star$ (the empty word), i, j, ij and ji on $S = S_i \times S_j$ all differ from each other, as they take different values on $(0, 0)$:

$$\begin{array}{rcl} \mathbf{F}_\star(0, 0) & = & (0, 0) \\ \mathbf{F}_i(0, 0) & = & (1, 0) \\ \mathbf{F}_j(0, 0) & = & (0, 1) \\ \mathbf{F}_{ij}(0, 0) = \mathbf{F}_i \mathbf{F}_j(0, 0) = \mathbf{F}_i(0, 1) & = & (2, 1) \\ \mathbf{F}_{ji}(0, 0) = \mathbf{F}_j \mathbf{F}_i(0, 0) = \mathbf{F}_j(1, 0) & = & (1, 1). \end{array}$$

Both $\mathbf{F}_i$ and $\mathbf{F}_j$ are idempotent. Moreover, it is easy to see that

$$\mathbf{F}_{iji} = \mathbf{F}_{jij} = \mathbf{F}_{ij},$$

as $\mathbf{F}_i\,\mathbf{F}_j\,\mathbf{F}_i = \mathbf{F}_j\,\mathbf{F}_i\,\mathbf{F}_j = \mathbf{F}_i\,\mathbf{F}_j$, so that $\mathbf{F}_i \mapsto a_1, \mathbf{F}_j \mapsto a_2$ extends to an isomorphism from $D(\mathcal{S})$ to Kiselman's semigroup K_2

$$\mathrm{K}_2 = \langle a_1, a_2 \,|\, a_1^2 = a_1,\ a_2^2 = a_2,\ a_1 a_2 a_1 = a_2 a_1 a_2 = a_1 a_2 \rangle.$$

These examples are instances of the following statement.

Proposition 2.4. *Let $\mathcal{S} = (\Gamma, S_i, f_i)$ be an update system defined on a directed acyclic graph Γ. Then the Γ-local functions $\mathbf{F}_i$ satisfy:*

(i) $\mathbf{F}_i^2 = \mathbf{F}_i$, *for every $i \in V$;*
(ii) $\mathbf{F}_i\,\mathbf{F}_j\,\mathbf{F}_i = \mathbf{F}_j\,\mathbf{F}_i\,\mathbf{F}_j = \mathbf{F}_i\,\mathbf{F}_j$, *if $i \to j$ (and hence, $j \nrightarrow i$);*
(iii) $\mathbf{F}_i\,\mathbf{F}_j = \mathbf{F}_j\,\mathbf{F}_i$, *if i and j are not connected.*

Proof. Every Γ-local function $\mathbf{F}_i$ only affects the vertex state s_i. As $\mathbf{F}_i(\mathbf{s})$ only depends on $\mathbf{s}[i]$, and $i \notin x[i]$, then each $\mathbf{F}_i$ is idempotent.

For the same reason, it is enough to check (ii) and (iii) on a graph with two vertices $i \neq j$. If they are not connected, then both f_i and f_j are constant, hence $\mathbf{F}_i$ and $\mathbf{F}_j$ trivially commute. If there is an arrow $i \to j$, then $\mathbf{F}_i(a_i, a_j) = (f_i(a_j), a_i)$, whereas $\mathbf{F}_j(a_i, a_j) = (a_i, t)$, since $f_j \equiv t$ is a constant. Then it is easy to check that the compositions $\mathbf{F}_i\,\mathbf{F}_j, \mathbf{F}_i\,\mathbf{F}_j\,\mathbf{F}_i, \mathbf{F}_j\,\mathbf{F}_i\,\mathbf{F}_j$ coincide, as they map every element to $(f_i(t), t)$. $\qquad\square$

These relations are remindful of those in the presentation of a Hecke-Kiselman monoid.

Definition 2.5. Let $\Gamma = (V, E)$ be a finite directed acyclic graph. The *Hecke-Kiselman monoid* associated with Γ is defined as follows

$$\mathbf{HK}_\Gamma = \langle a_i, i \in V \,|\quad a_i^2 = a_i,\ \text{for every } i \in V;$$

(2.1) $$a_i a_j a_i = a_j a_i a_j = a_i a_j,\ \text{for } i \to j;$$

(2.2) $$a_i a_j = a_j a_i,\ \text{for } i \nrightarrow j,\ \text{and } j \nrightarrow i \rangle$$

This structure has been first introduced in [GM11] for a finite mixed graph, i.e., a simple graph (without loops or multiple edges) in which edges can be either oriented or unoriented: there, an unoriented edge (i, j) is used to impose the customary braid relation $a_i a_j a_i = a_j a_i a_j$.

If $\mathcal{S} = (\Gamma, S_i, f_i)$ is an update system on a finite directed acyclic graph $\Gamma = (V, E)$, then Proposition 2.4 amounts to claiming that the evaluation homomorphism $\mathbf{F}\colon \mathrm{F}(V) \to \mathrm{End}(S)$ factors through the Hecke-Kiselman monoid $\mathbf{HK}_\Gamma$.

Our case of interest is when the graph $\Gamma = \Gamma_n$ is the complete graph on n vertices, where the orientation is set so that $i \to j$ if $i < j$. In this case, the semigroup $\mathbf{HK}_{\Gamma_n}$ coincides with Kiselman's semigroup K_n, as defined in [KM09]. The monoid K_n, however, only reflects immediate pairwise interactions between vertex functions. One may, in principle, wonder if K_n is indeed the smallest quotient of $\mathrm{F}(V)$ through which evaluation maps $\mathbf{F}$ factor, or additional identities may be imposed that reflect higher order interactions.

In this paper we will exhibit an update system $\mathcal{S}_n^\star$, defined on the graph Γ_n, whose dynamics monoid is isomorphic to K_n. In other words, we will show that $\mathrm{K}_n \to D(\mathcal{S}_n^\star)$ is indeed an isomorphism, once suitable vertex functions have been chosen.

3. COMBINATORIAL DEFINITIONS

Let $\mathrm{F}(A)$ be the free monoid over the alphabet A and denote by $\star$ the empty word. Recall that, for every subset $B \subseteq A$, the submonoid $\langle B \rangle \subseteq \mathrm{F}(A)$ is identified with the free monoid $\mathrm{F}(B)$.

Definition 3.1. Let $w \in \mathrm{F}(A)$. We define

- *subword of w* to be a substring of consecutive letters of w;
- *quasi-subword of w* to be an ordered substring u of not necessarily consecutive letters of w.

We will denote the relation of being a quasi-subword by $\leq$, so that

$$v \leq w$$

if and only if v is a quasi-subword of w.

Obviously, every subword is a quasi-subword. Also notice that $v \leq w$ and $w \leq v$ if and only if $v = w$.

Example 3.2. Set $w = acaab \in \mathrm{F}(\{a, b, c\})$; then

- aab is a subword (hence a quasi-subword) of w;
- aaa is a quasi-subword of w which is not a subword;
- abc is neither a subword nor a quasi-subword of w.

Trivial examples of subwords of w are the empty word $\star$ and w itself.

Definition 3.3. Let $w \in \mathrm{F}(A)$. Then

- if w is non-empty, the *head* of w, denoted $\mathrm{h}(w) \in A$, is the leftmost letter in w;
- if $a \in A$, then the *a-truncation* $\mathrm{T}_a\, w \in \mathrm{F}(A)$ of w is the longest (non-empty) suffix of w with head a, or the empty word in case a does not occur in w.

Similarly, if $I \subset A$, we denote by $\mathrm{T}_I\, w$ the longest (non-empty) suffix of w whose head lies in I, or the empty word in case no letter from I occurs in w.

The following observations all have trivial proofs.

Remark 3.4.

(i) If $w \in \mathrm{F}(A)$ does not contain any occurrence of $a \in A$, then

$$\mathrm{T}_a(ww') = \mathrm{T}_a(w'),$$

for every $w' \in \mathrm{F}(A)$.

(ii) For every $w \in \mathrm{F}(A)$ and $a \in A$, one may (uniquely) express w as

$$w = w'\, \mathrm{T}_a\, w$$

where $w' \in \langle A \setminus \{a\} \rangle$.

(iii) If $w \in \mathrm{F}(A)$ contains some occurrence of $a \in A$, then

$$\mathrm{T}_a(ww') = (\mathrm{T}_a\, w)w',$$

for every $w' \in \mathrm{F}(A)$.

Remark 3.5.

(i) If $w \in \mathrm{F}(A)$, and $a, b \in A$, then either $\mathrm{T}_a\, w$ is a suffix of $\mathrm{T}_b\, w$, or vice versa.

(ii) If $\mathrm{T}_b\, w$ is a suffix of $\mathrm{T}_a\, w$ for all $b \in A$, then $\mathrm{h}(w) = a$, hence $\mathrm{T}_a\, w = w$.

(iii) $\mathrm{T}_I\, w = \mathrm{T}_a\, w$ for some $a \in I$, and $\mathrm{T}_b\, w$ is a suffix of $\mathrm{T}_I\, w$ for every $b \in I$.

Definition 3.6. Given $I \subseteq A$, the *deletion morphism* is the unique semigroup homomorphism satisfying

$$\begin{aligned}
\partial_I \colon \mathrm{F}(A) &\rightarrow \mathrm{F}(A \setminus I) \subseteq \mathrm{F}(A) \\
a_i &\mapsto \star, \text{ for } i \in I \\
a_j &\mapsto a_j, \text{ for } j \notin I.
\end{aligned}$$

It associates with any $w \in \mathrm{F}(A)$ the longest quasi-subword of w containing no occurrence of letters from I.

Remark 3.7. For every $I, J \subseteq A$,

$$\partial_I\, \partial_J = \partial_{I \cup J}$$

4. KISELMAN'S SEMIGROUP AND CANONICAL WORDS OVER THE COMPLETE GRAPH

In this section, we recall results from [GM11] and draw some further consequences. Choose an alphabet $A = \{a_i, 1 \le i \le n\}$; Kiselman's semigroup K_n has the presentation

$$\mathrm{K}_n = \langle a_i \in A \,|\, a_i^2 = a_i \text{ for all } i; \ a_i a_j a_i = a_j a_i a_j = a_i a_j \text{ for } i < j \rangle.$$

In accordance with [KM09], let

$$\pi \colon \mathrm{F}(A) \to \mathrm{K}_n$$

denote the canonical evaluation epimorphism.

Definition 4.1 ([GM11]). Let $w \in \mathrm{F}(A)$. A subword of w of the form $a_i u a_i$, where $a_i \in A$ and $u \in \mathrm{F}(A)$, is *special* if u contains both some $a_j, j > i$, and some $a_k, k < i$.

Remark 4.2. Notice that a subword $a_i u a_i$ cannot be special if $i = 1$ or $i = n$.

Let us recall the following fact.

Theorem 4.3 ([KM09]). *Let $w \in \mathrm{F}(A)$. The set $\pi^{-1}\pi(w)$ contains a unique element whose only subwords of the form $a_i u a_i$ are special.*

Remark 4.4.
  (i) The unique element described in Theorem 4.3 contains at most one occurrence of a_1 and at most one occurrence of a_n.
  (ii) In order to prove Theorem 4.3, the authors of [KM09] define a binary relation $\to$ in $\mathrm{F}(A)$ as follows: $w \to v$ if and only if either
    ($\xrightarrow{1}$) $w = w_1 a_i a_i w_2$, $v = w_1 a_i w_2$, or
    ($\xrightarrow{2}$) $w = w_1 a_i u a_i w_2$, $v = w_1 a_i u w_2$ and $u \in \langle a_{i+1}, \ldots, a_n \rangle$, or
    ($\xrightarrow{3}$) $w = w_1 a_i u a_i w_2$, $v = w_1 u a_i w_2$ and $u \in \langle a_1, \ldots, a_{i-1} \rangle$.
      It is possible to iterate such simplifications and write (finite) sequences

(4.1) $w \to v_1 \to v_2 \cdots \to v_k,$

  so that each word contains exactly one letter less than the previous and they all belong to the same fiber with respect to π. Moreover, each v_i is a quasi-subword of w and of v_j, for all $j < i$.
      A sequence like (4.1) is called *simplifying sequence* if v_k is not simplifiable any further, and each $v_i \to v_{i+1}$ is called a *simplifying step*. It should be stressed that a simplifying step of type 1 can be seen as both of type 2 and of type 3.
  (iii) According to Theorem 4.3, every simplifying sequence for w ends on the same word.

We will refer to any word, whose only subwords of the form $a_i u a_i$ are special, as a *canonical word*, so that the above theorem claims existence of a unique canonical word v in each $\pi^{-1}\pi(w), w \in \mathrm{F}(A)$. Thus, the assignment $w \mapsto v$ is a well defined map $\mathrm{Can} : \mathrm{F}(A) \to \mathrm{F}(A)$ associating with each word its unique *canonical form*. Notice that w is canonical if and only if $w = \mathrm{Can}\, w$.

Remark 4.5.
  (i) If w is canonical then all of its subwords are canonical.
  (ii) A word $u \in \langle a_i, h \le i \le k \rangle$ is canonical if and only if, for every $j < h$ or $j > k$, the word $a_j u$ is canonical. Moreover, in such cases,

$$\mathrm{Can}(a_j u) = a_j \, \mathrm{Can}\, u.$$

Remark 4.6. More consequences of Theorem 4.3 are that
  (i) the canonical form $\mathrm{Can}\, w$ is the word of minimal length in $\pi^{-1}\pi(w)$;
  (ii) due to Remark 4.4(iii), $\mathrm{Can}\, w$ is the last element in every simplifying sequence starting from any of the words in $\pi^{-1}\pi(w)$;

    (iii) $\mathrm{Can}\, w$ is a quasi-subword of all the words in any simplifying sequence beginning from w;

    (iv) if w contains any given letter, so does $\mathrm{Can}\, w$.

Lemma 4.7. *For every* $u, v \in \mathrm{F}(A)$,

$$
\begin{aligned}
\mathrm{Can}(uv) &= \mathrm{Can}\left((\mathrm{Can}\, u)v\right) \\
&= \mathrm{Can}\left(u\, \mathrm{Can}\, v\right) \\
&= \mathrm{Can}\left(\mathrm{Can}\, u\, \mathrm{Can}\, v\right)
\end{aligned}
$$

Proof. Can is constant on fibres of π. $\qquad\square$

Definition 4.8. If $I \subseteq A$, then
$$
\mathrm{Can}_I\, w = \mathrm{Can}\, \partial_{A \setminus I} w,
$$
where $w \in \mathrm{F}(A)$. In particular $\mathrm{Can}_A\, w = \mathrm{Can}\, w$.

Before proceeding further, we need to make an important observation. Say we have a sequence of steps leading from a word y to a word x and that each step removes a single letter. The same procedure can be applied to every subword of y, and again at each step (at most) a single letter is removed, eventually yielding a subword of x. Every subword of x is obtained in this way from some (possibly non-unique) subword of y. Notice that in the cases we will deal with, some of the steps could be simplifications of type $\xrightarrow{1}$ for which there is an ambiguity on which of the two identical letters is to be removed.

We will say that a subword of x of the form $a_i u a_i$ *originates* from a subword w of y if w yields $a_i u a_i$ under the sequence of simplifications and w is of the form $a_i v a_i$. Let us clarify things with an example. In the following sequence of steps, we have highlighted the letter to remove at each step:

$$
bdbc\,\widehat{d}\,abcdc \to b\,\widehat{d}\,bcabcdc \to \widehat{bb}\,cabcdc \to bcabcdc.
$$

Notice that in the last step, there is an ambiguity on which letter is being removed. Then the subwords $bdbcdab, dbcdab, bcdab$ of $bdbcdabcdc$ all yield $bcab$; however, $bcab$ originates only from $bdbcdab$ and $bcdab$.

Henceforth, we will shorten the notation and denote by $\partial_i, \mathrm{T}_i, \dots$ the maps $\partial_{a_i}, \mathrm{T}_{a_i}, \dots$.

Lemma 4.9. *Assume* $y = \mathrm{Can}\, y$ *and let* $x = \partial_{\{1,\dots,k-1\}} y$, *where* $1 \le k \le n$. *Then any subword* $a_i u a_i \le x$ *is either special or satisfies* $u \in \langle a_i, \dots, a_n \rangle$.

Proof. Let $a_i u a_i$ be a subword of x. Then $i \ge k$, as $x = \partial_{\{1,\dots,k-1\}} y$ contains no $a_i, i < k$.

The above subword $a_i u a_i$ originates from a subword $a_i v a_i$ of y. If $a_i u a_i$ is not special, then either $u \in \langle a_j, j \le i \rangle$, or $u \in \langle a_j, j \ge i \rangle$. However, the former case does not occur, otherwise $a_i v a_i$ would fail to be special, as x is obtained from y by only removing letters $a_j, 1 \le j < k$, and $k \le i$. This is a contradiction, as y is canonical. $\qquad\square$

Lemma 4.10. *Assume* $y = \mathrm{Can}\, y$ *and let* $x = \partial_{\{1,\dots,k-1\}} y$, *where* $1 \le k \le n$. *Then any simplifying sequence*

$$
x = \partial_{\{1,\dots,k-1\}} y \to \cdots \to \mathrm{Can}\, x = \mathrm{Can}_{\{k,\dots,n\}}\, y
$$

is such that all simplifying steps are of type $\xrightarrow{2}$ *(possibly of type* $\xrightarrow{1}$*).*

Proof. By Lemma 4.9, any subword $a_i u a_i \le x$ is either special or satisfies $u \in \langle a_j, i \le j \le n \rangle$, hence the first simplifying step is of type $\xrightarrow{2}$ (and possibly of type $\xrightarrow{1}$).

We want to show that, starting from $x = \partial_{\{1,\dots,k-1\}} y$ and applying simplifications of type $\xrightarrow{2}$ or of type $\xrightarrow{1}$, one can never obtain a word admitting a subword $a_i u a_i, u \ne \star$, on which it is possible to apply a step of type $\xrightarrow{3}$ (which is not of type $\xrightarrow{1}$).

Assume therefore by contradiction that $a_i u a_i$, where $\star \neq u \in \langle a_j, k \leq j < i \rangle$, occurs as a subword after having performed some simplifying steps of type $\xrightarrow{2}$ on x. The subword $a_i u a_i$ originates from a subword $a_i v a_i$ of x, which is necessarily special, since v cannot lie in $\langle a_j, i \leq j \leq n \rangle$, as some $a_j, j < i$, occurs in $u \neq \star$, and u is obtained from v after removing some letters. This shows that letters $a_j, j > i$, must occur in v, whereas they do not occur in u.

However, a simplifying step of type $\xrightarrow{2}$ that removes a letter from between the two occurrences of a_i either occurs completely between them, or begins on the left of both. In the former case, it does not change the speciality of the subword, whereas in the latter case a simplification of type $\xrightarrow{2}$ can only remove a letter $a_j, j < i$, due to the presence of the left a_i; we thus obtain a contradiction. $\qquad \square$

Corollary 4.11. *If $u a_1 v$ is canonical, then $\mathrm{Can}(uv)$ admits u as a prefix.*

Proof. By Lemma 4.10, all simplifying sequences from $uv = \partial_1(u a_1 v)$ to $\mathrm{Can}(uv)$ only contain steps of type $\xrightarrow{2}$ and no such simplifying step does alter u. $\qquad \square$

Proposition 4.12. *Assume that $u a_j v$ is canonical, and that $u \in \langle a_i, k \leq i \leq n \rangle$, where $k > j$. Then $u a_j \mathrm{Can}_{\{k,\dots,n\}} v$ is also canonical.*

In particular, if $u a_1 v$ is canonical, then $u a_1 \mathrm{Can}_{\{k,\dots,n\}} v$ is also canonical.

Proof. We prove the latter claim, as the proof of the former more general statement is completely analogous.

We know that v is canonical. Then Lemma 4.10 shows that $u a_1 \partial_{\{1,\dots,k-1\}} v$ can be simplified into $u a_1 \mathrm{Can}_{\{k,\dots,n\}} v$ by only using steps of type $\xrightarrow{2}$ on the right of a_1.

If $u a_1 \mathrm{Can}_{\{k,\dots,n\}} v$ is not canonical, then we may find a subword $a_i x a_i$ that is not special. As both u and $\mathrm{Can}_{\{k,\dots,n\}} v$ are canonical, then a_1 must occur in x. Say that $a_i x a_i$ originates from the subword $a_i y a_i$ of $u a_1 v$. No simplification of type $\xrightarrow{2}$, when performed on the right of a_1, can change the set of letters that appear between the two a_i. This yields a contradiction, as $a_i y a_i$ is special, whereas $a_i x a_i$ is not. $\qquad \square$

Corollary 4.13. *For every choice of $u, u', v, v' \in \langle a_3, a_4, \dots, a_n \rangle$,*

    (i) if $u a_1 v a_2 v'$ is canonical, then $u a_1 \mathrm{Can}(vv')$ is canonical;

    (ii) if $u a_2 v a_1 v'$ is canonical, then $u a_2 \mathrm{Can}(vv')$ is canonical;

    (iii) if $u a_2 u' a_1 v a_2 v'$ is canonical, then both $u a_2 u' a_1 \mathrm{Can}(vv')$ and $u a_2 \mathrm{Can}(u'vv')$ are canonical.

Proof. (i) follows directly from Proposition 4.12, whereas (iii) follows by applying Proposition 4.12 and then (ii).

However, (ii) is equivalent to (i), as both $u a_1 v a_2 v'$ and $u a_2 v a_1 v'$ contain single occurrences of a_1 and a_2, and every simplifying sequence for the former can be turned into a simplifying sequence for the latter by switching a_1 with a_2. $\qquad \square$

Lemma 4.14. *Let $w \in \mathrm{F}(A)$, $1 \leq i \leq n$. Then*

$$\mathrm{T}_i \mathrm{Can}_{\{i,\dots,n\}} w = \mathrm{Can}_{\{i,\dots,n\}} \mathrm{T}_i w.$$

Proof. Once again, we may assume without loss of generality that $i = 1$. If there are no occurrences of a_1 in w, then both sides equal the empty word and we are done.

Otherwise, using Remark 3.4(ii), write $w = u \mathrm{T}_1 w$ and $\mathrm{Can} \mathrm{T}_1 w = a_1 v$. We are asked to show that

$$\mathrm{T}_1 \mathrm{Can}(u \mathrm{T}_1 w) = \mathrm{Can}(\mathrm{T}_1 w),$$

which is equivalent, using Lemma 4.7, to

$$\mathrm{T}_1 \mathrm{Can}(u a_1 v) = a_1 v.$$

Notice that v is canonical and $u \in \langle a_2, \ldots, a_n \rangle$. The simplifying steps that can occur on a word of type $u a_1 v$ may only affect u: indeed, v is canonical, there is an only occurrence of a_1 in $u a_1 v$, and the only special words that begin in u and end in v contain an occurrence of a_1, thus leading to a $\xrightarrow{3}$.

An easy induction now shows that $\mathrm{Can}(u a_1 v) = u' a_1 v$, where u' is a quasi-subword of u, hence $\mathrm{T}_1 \mathrm{Can}(u a_1 v) = a_1 v$. □

Lemma 4.15. *Let* $w \in \mathrm{F}(A)$, $1 \leq k \leq n$. *Then*

(i) $\mathrm{Can}_{\{k,\ldots,n\}} w = \mathrm{Can}_{\{k,\ldots,n\}} \mathrm{Can}\, w$;
(ii) $\mathrm{Can}_{\{k,\ldots,n\}} w$ *is a quasi-subword of* $\mathrm{Can}\, w$.

Proof. Let $k > 1$, the case $k = 1$ being trivial. It is easy to check that $\mathrm{Can} \circ \partial_{\{1,\ldots,k-1\}}$ is constant on fibres of π — it is invariant under all simplifying steps — which takes care of (i). We may therefore assume in (ii) that w is a canonical word; however $\mathrm{Can}_{\{k,\ldots,n\}} w$ is obviously a quasi-subword of w. □

5. THE JOIN OPERATION

Given an update system over the complete oriented graph Γ_n, Proposition 2.4 proves that its dynamics monoid is an epimorphic image of Kiselman's semigroup K_n. In next section, we will exhibit an update system $\mathcal{S}_n^\star$ over Γ_n whose dynamics monoid is isomorphic to K_n; from a dynamical point of view, $\mathcal{S}_n^\star$ serves as a *universal update system*.

We introduce the following operation in order to construct, later, a family of update functions.

Definition 5.1. Take $u, v \in \mathrm{F}(A)$. The *join* of u and v, denoted by $[u, v]$, is the shortest word admitting u as quasi-subword and v as suffix. Namely,

$$[u, v] = u^+ v$$

where $u = u^+ u^-$, so that u^- is the longest suffix of u which is a quasi-subword of v.

Notice that the decomposition $u = u^+ u^-$ strictly depends on the choice of v.

Example 5.2. For instance, consider $u = cbadc$ and $v = abdc$. Then,

$$[u, v] = cbabdc.$$

Remark 5.3.

(i) The empty word $\star$ is a subword of every $w \in \mathrm{F}(A)$, so that

$$[\star, u] = [u, \star] = u.$$

(ii) If u is a quasi-subword of v, then $[u, v] = v$.
(iii) If u is not a quasi-subword of v, then

$$[wu, v] = wu^+ v = w[u, v],$$

for every $w \in \mathrm{F}(A)$.
(iv) If $[u, v] = u^+ v$, then

$$[u, wv] = [u^+, w]v$$

Indeed, write $u = u^+ u^-$: u^- is the longest suffix of u which is a quasi-subword of v, but there could be a suffix of u^+ which is a quasi-subword of w.
(v) If $u, v \in \mathrm{F}(A)$ are canonical, $[u, v]$ may fail to be so. For instance, $[a_1 a_2, a_2 a_1] = a_1 a_2 a_1$, which is not canonical.

The following lemma will be used later in the proof of Proposition 6.2.

Lemma 5.4. *Let* $u, x, y \in \mathrm{F}(A)$, *if* $ux \leq uy$, *then* $x \leq y$.

Proof. As ux is a quasi-subword of uy, by Remark 5.3(ii),

$$[ux, uy] = uy.$$

Assume that x is not a quasi-subword of y and write it as $x = x^+x^-$, where x^- is its longest suffix which is a quasi-subword of y. Using Remark 5.3(iii),

$$[ux, y] = ux^+y.$$

Finally, using Remark 5.3(iv),

$$[ux, uy] = [ux^+, u]y$$

However, $[ux^+, u] = u$ can hold only if ux^+ is not longer than u, i.e., only if $x^+ = \star$, hence x is a quasi-subword of y. $\qquad\square$

6. AN UPDATE SYSTEM WITH UNIVERSAL DYNAMICS

If $U, V \subset F(A)$, denote by $[U, V] \subset F(A)$ the subset of all elements $[u, v], u \in U, v \in V$.

Definition 6.1. The update system $\mathcal{S}_n^\star$ is the triple (Γ_n, S_i, f_i), where

  (1) Γ_n is, as before, the complete oriented graph on n vertices, where $i \to j$ if and only if $i < j$.
  (2) On vertex i, the state set $S_i \subset F(A)$ is inductively defined as

$$S_i = \begin{cases} \{\star, a_n\} & \text{if } i = n, \\ \{\star, a_{n-1}, a_{n-1}a_n\} & \text{if } i = n-1, \\ \{\star\} \cup a_i[S_n, [\ldots, [S_{i+2}, S_{i+1}]\ldots]] & \text{if } 1 \le i \le n-2. \end{cases}$$

  (3) On vertex i, the vertex function is

$$\begin{aligned} f_i \colon S[i] &\to S_i \\ (s_{i+1}, \ldots, s_n) &\mapsto a_i[s_n, [\ldots, [s_{i+2}, s_{i+1}]\ldots]], \end{aligned}$$

   if $i \le n-2$, whereas $f_{n-1}(s_n) = a_{n-1}s_n$ and $f_n \equiv a_n$ is constant.

We will abuse the notation and denote by $\star$ also the system state $(\star, \ldots, \star) \in S$. The rest of the paper will be devoted to the proof of the following

Proposition 6.2. *Consider the evaluation morphism* $\mathbf{F} \colon F(A) \to D(\mathcal{S}_n^\star)$, *mapping each word* $w \in F(A)$ *to the corresponding evolution* $\mathbf{F}_w \in D(\mathcal{S}_n^\star)$. *If* $\mathbf{p} = (p_1, \ldots, p_n) = \mathbf{F}_w\star$, *then:*

  (i) *One has* $p_i = (\mathbf{F}_w\star)_i = \mathrm{Can}_{\{i,\ldots,n\}} \mathrm{T}_i\, w$.
  (ii) *For every choice of* $k, 1 \le k \le n$, *one may find* $j, 1 \le j \le k$, *so that*

$$[p_k, [\ldots, [p_2, p_1]\ldots]] = \mathrm{T}_{\{1,\ldots,k\}} \mathrm{Can}\, w.$$

Our central result then follows immediately.

Theorem 6.3. *With the same hypotheses and notation as in Proposition 6.2,*

  (i) $\mathrm{Can}\, w = [p_n, [\ldots, [p_2, p_1]\ldots]]$;
  (ii) *if* $u, v \in F(A)$, *then* $\mathbf{F}_u = \mathbf{F}_v$ *if and only if* $\mathrm{Can}\, u = \mathrm{Can}\, v$;
  (iii) K_n *is isomorphic to* $D(\mathcal{S}_n^\star)$.

Proof.

  (i) Use Proposition 6.2(ii) when $k = n$. Then

$$[p_n, [\ldots, [p_2, p_1]\ldots]] = \mathrm{T}_{\{1,\ldots,n\}} \mathrm{Can}\, w = \mathrm{Can}\, w.$$

  (ii) If $\mathbf{F}_u = \mathbf{F}_v$, then they certainly compute the same state on $\star$. However, by (i), one may recover both $\mathrm{Can}\, u$ and $\mathrm{Can}\, v$ from this state, hence $\mathrm{Can}\, u = \mathrm{Can}\, v$. The other implication follows trivially, as $\mathrm{Can}\, u = \mathrm{Can}\, v$ forces u and v to induce the same element in K_n, hence the same dynamics on $\mathcal{S}_n^*$.

(iii) This is just a restatement of (ii): distinct elements in K_n have distinct $\mathcal{S}_n^*$-actions, since they act in a different way on the system state $\star$.

$\square$

Remark 6.4. As an immediate consequence of Proposition 6.2, if $j = \mathrm{h}(\mathrm{Can}\, w)$, then $\mathrm{T}_j \, \mathrm{Can}\, w = \mathrm{Can}\, w$, hence

$$[p_j, [\ldots, [p_2, p_1] \ldots]] = [p_n, [\ldots, [p_2, p_1] \ldots]] = \mathrm{Can}\, w.$$

We will prove Proposition 6.2 by induction on the number n of vertices. The following technical fact is needed in the proof of the inductive step, and we assume in its proof that Proposition 6.2 and Theorem 6.3 hold on $\mathcal{S}_k^\star$, for $k < n$.

Proposition 6.5. *Let* $u, v \in \langle a_{j+1}, a_{j+2}, \ldots, a_n \rangle$ *be chosen so that* $u a_j \, \mathrm{Can}\, v$ *is a canonical word. Then* $[\mathrm{Can}(uv), v] = uv$. *In particular,* $[\mathrm{Can}(uv), a_j v] = u a_j v$.

Proof. We may assume, without loss of generality, that $j = 1$.

The statement is easily checked case by case when $n = 1$ or 2. Notice that u can have at most one occurrence of a_2, whereas v may have many. Let us therefore distinguish four cases:

(1) There is no occurrence of a_2 in either u or v.

  In this case, we can use inductive assumption, after removing the vertex indexed by 2.

(2) There is a single occurrence of a_2 in u.

  Write $u = u' a_2 u''$. As $u' a_2 u'' a_1 \, \mathrm{Can}\, v$ is canonical, then, using Corollary 4.13(ii) and Lemma 4.7,

$$\begin{aligned} \mathrm{Can}(u' a_2 u'' v) &= \mathrm{Can}(u' a_2 u'' \, \mathrm{Can}\, v) \\ &= u' a_2 \, \mathrm{Can}(u'' \, \mathrm{Can}\, v) \\ &= u' a_2 \, \mathrm{Can}(u'' v). \end{aligned}$$

  Thus, we need to compute $[u' a_2 \, \mathrm{Can}(u'' v), v]$. However, applying Case 1 gives

$$[\mathrm{Can}(u'' v), v] = u'' v,$$

  hence $[u' a_2 \, \mathrm{Can}(u'' v), v] = u' a_2 u'' v$ follows from Remark 5.3(iii).

(3) a_2 occurs in v, but not in u.

  Write $\mathrm{Can}\, v = v' a_2 v''$. As $u a_1 \, \mathrm{Can}\, v = u a_1 v' a_2 v''$ is canonical, then, using Corollary 4.13(i), also $u a_1 \, \mathrm{Can}(v' v'')$ is canonical. However Lemma 4.15 informs us that

$$\begin{aligned} \mathrm{Can}(v' v'') &= \mathrm{Can}\, \partial_2 \, \mathrm{Can}\, v \\ &= \mathrm{Can}_{\{3, \ldots, n\}} \, \mathrm{Can}\, v \\ &= \mathrm{Can}_{\{3, \ldots, n\}} \, v \\ &= \mathrm{Can}\, \partial_2 v, \end{aligned}$$

  so that $u a_1 \, \mathrm{Can}\, \partial_2 v$ is canonical. We may then use Case 1 to argue that

(6.1) $$[\mathrm{Can}(u \partial_2 v), \partial_2 v] = u \partial_2 v.$$

  Lemma 4.15 implies that $\mathrm{Can}(u \partial_2 v) = \mathrm{Can}\, \partial_2(uv)$ is a quasi-subword of $\mathrm{Can}(uv)$. By Corollary 4.11, both $\mathrm{Can}(uv)$ and $\mathrm{Can}(u \partial_2 v)$ admit u as a prefix, so, using Lemma 5.4, we can write

$$\mathrm{Can}(uv) = uy, \qquad \mathrm{Can}(u \partial_2 v) = uz, \qquad z \leq y.$$

  We need to show that $[\mathrm{Can}(uv), v] = uv$. Now, applying Corollary 4.13 (ii),

$$\mathrm{Can}(uv) = \mathrm{Can}(uv') a_2 v'' = uxa_2 v'',$$

  where x is some quasi-subword of v'. As

$$y = xa_2 v'' \leq v' a_2 v'' = \mathrm{Can}\, v \leq v,$$

then $[\mathrm{Can}(uv), v] = u^+v$ where $u = u^+u^-$ and $u^-y \le v$. However, this would force

$$u^-z \le u^-y \le v$$

hence also $u^-z \le \partial_2 v$; and this is only possible if $u^- = \star$, as Equation (6.1) shows.

(4) a_2 occurs both in u and in v.

This is similar to the previous case. Write $u = u'a_2u''$, $\mathrm{Can}\, v = v'a_2v''$. Applying Corollary 4.13(iii) to the canonical word $ua_1\,\mathrm{Can}\, v = u'a_2u''a_1v'a_2v''$, we obtain that both

$$u'a_2u''a_1\,\mathrm{Can}\,\partial_2 v = u'a_2u''a_1\,\mathrm{Can}(v'v'')$$

and

$$u'a_2\,\mathrm{Can}(u''\partial_2 v) = u'a_2\,\mathrm{Can}(u''\,\mathrm{Can}\,\partial_2 v) = u'a_2\,\mathrm{Can}(u''v'v'')$$

are canonical. Similarly,

$$
\begin{aligned}
\mathrm{Can}(uv) &= \mathrm{Can}(u'a_2u''v'a_2v'') \\
&= \mathrm{Can}(u'a_2u''v'v'') \\
&= \mathrm{Can}(u'a_2\,\mathrm{Can}(u''v'v'')) \\
&= u'a_2\,\mathrm{Can}(u''\partial_2 v).
\end{aligned}
$$

Now, as $u''a_1\,\mathrm{Can}\,\partial_2 v$, being a subword of $u'a_2u''a_1\,\mathrm{Can}\,\partial_2 v$, is canonical, we have

(6.2) $$[\mathrm{Can}(u''\partial_2 v), \partial_2 v] = u''\partial_2 v.$$

As a consequence,

$$
\begin{aligned}
[\mathrm{Can}(u'a_2u''v'a_2v''), \partial_2 v] &= [u'a_2\,\mathrm{Can}(u''\partial_2 v), \partial_2 v] \\
&= [u'a_2, u'']\partial_2 v \\
&= u'a_2u''\partial_2 v,
\end{aligned}
$$

and one may complete the proof as in Case 3.

$\square$

7. PROOF OF PROPOSITION 6.2

The basis of induction $n = 1$ being trivial, we assume that Proposition 6.2 and Theorem 6.3 hold for a complete graph on less than n vertices,

Proof of Proposition 6.2. Let us start by proving Part (i).

Let $w \in \mathrm{F}(A)$. By inductive hypothesis, for a complete graph on $n-1$ vertices $2, \ldots, n$, the SDS map $\mathbf{F}_{\partial_1 w}$ constructs on the ith vertex the state

$$p_i = \mathrm{Can}_{\{i,\ldots,n\}}\,\mathrm{T}_i\,\partial_1 w = \mathrm{Can}_{\{i,\ldots,n\}}\,\mathrm{T}_i\,w,$$

hence, the statement holds for all vertices $i > 1$.

As far as vertex 1 is concerned, we need to show that $p_1 = \mathrm{Can}\,\mathrm{T}_1\,w$. If w does not contain the letter a_1, then $p_1 = \star = \mathrm{T}_1\,w$ and there is nothing to prove; otherwise, $(\mathbf{F}_w)_1 = (\mathbf{F}_{\mathrm{T}_1\,w})_1$. We know that the state on vertex 1 depends only on the system state $(p'_2, \ldots, p'_n)$, which is computed by $\partial_1\,\mathrm{T}_1\,w$ on the subgraph indexed by $\{2, \ldots, n\}$, i.e.,

$$p_1 = a_1\,[p'_n, [\ldots, [p'_3, p'_2]\ldots]].$$

However, we may apply induction hypothesis and use Theorem 6.3(i), which yields

$$
\begin{aligned}
p_1 &= a_1\,[p'_n, [\ldots, [p'_3, p'_2]\ldots]] \\
&= a_1\,\mathrm{Can}\,\partial_1\,\mathrm{T}_1\,w \\
&= \mathrm{Can}\,\mathrm{T}_1\,w
\end{aligned}
$$

As for Part (ii) of Proposition 6.2, we proceed by induction on k. The basis of induction descends directly from Part (i) and Lemma 4.14, as

$$\begin{aligned} p_1 &= \operatorname{Can} T_1 w \\ &= T_1 \operatorname{Can} w. \end{aligned}$$

Assume now $k > 1$. By inductive hypothesis, there exists $j < k$ such that

$$[p_{k-1}, [\ldots, [p_2, p_1] \ldots]] = T_{\{1,\ldots,k-1\}} \operatorname{Can} w.$$

Recall that, by Remark 3.5, either $T_k \operatorname{Can} w$ is a suffix of $T_{\{1,\ldots,k-1\}} \operatorname{Can} w$ or vice versa. In the former case, we know by Part (i) and Lemma 4.14, that

(7.1) $$p_k = \operatorname{Can}_{\{k,\ldots,n\}} T_k w = T_k \operatorname{Can}_{\{k,\ldots,n\}} w.$$

By Lemma 4.15, $\operatorname{Can}_{\{k,\ldots,n\}} w$ is a quasi-subword of $\operatorname{Can} w$, hence $p_k = T_k \operatorname{Can}_{\{k,\ldots,n\}} w$ is a quasi-subword of $T_k \operatorname{Can} w$, which is a suffix of $T_{\{1,\ldots,k-1\}} \operatorname{Can} w$. Thus,

$$\begin{aligned} [p_k, [p_{k-1} [\ldots, [p_2, p_1] \ldots]]] &= [p_k, T_{\{1,\ldots,k-1\}} \operatorname{Can} w] \\ &= T_{\{1,\ldots,k-1\}} \operatorname{Can} w = T_{\{1,\ldots,k\}} \operatorname{Can} w. \end{aligned}$$

If, instead, $T_{\{1,\ldots,k-1\}} \operatorname{Can} w$ is a suffix of $T_k \operatorname{Can} w$, pick j so that $T_j \operatorname{Can} w = T_{\{1,\ldots,k-1\}} \operatorname{Can} w$ and argue as follows.

Choose u such that $T_k \operatorname{Can} w = u\, T_j \operatorname{Can} w$. As $u \in \langle a_i, k \le i \le n \rangle$, then by Part (i)

$$p_k = \operatorname{Can}_{\{k,\ldots,n\}}(u\, T_j \operatorname{Can} w) = \operatorname{Can}(u\partial_{\{1,\ldots,k-1\}} T_j \operatorname{Can} w).$$

By Proposition 4.12, as $T_k \operatorname{Can} w = u\, T_j \operatorname{Can} w$ is canonical, we may argue that the word $ua_j \operatorname{Can}_{\{k,\ldots,n\}} T_j \operatorname{Can} w$ is also canonical. Apply now Proposition 6.5 to $ua_j \partial_{\{1,\ldots,k-1\}} T_j \operatorname{Can} w$, to obtain

$$\begin{aligned} [p_k, a_j \partial_{\{1,\ldots k-1\}} T_j \operatorname{Can} w] &= [\operatorname{Can}(u\partial_{\{1,\ldots,k-1\}} T_j \operatorname{Can} w), a_j \partial_{\{1,\ldots,k-1\}} T_j \operatorname{Can} w] = \\ &= ua_j \partial_{\{1,\ldots,k-1\}} T_j \operatorname{Can} w, \end{aligned}$$

hence a fortiori $[p_k, T_j \operatorname{Can} w] = u\, T_j \operatorname{Can} w = T_k \operatorname{Can} w$, which equals $T_{\{1,\ldots,k\}} \operatorname{Can} w$. $\qquad\square$

8. Conclusions and further developments

If Γ is a finite directed acyclic graph, then any update system $\mathcal{S}$ supported on Γ induces a dynamics monoid $D(\mathcal{S})$ which naturally arises as a quotient of $\mathbf{HK}_\Gamma$. We refer to the smallest quotient of $\mathbf{HK}_\Gamma$ through which all evaluation maps $\mathbf{F} : F(V) \to D(\mathcal{S})$ factor as the *universal dynamics monoid* $D(\Gamma)$. In this paper, we have proved that $D(\Gamma_n) \simeq \mathbf{HK}_{\Gamma_n} = K_n$.

Conjecture 1. $D(\Gamma) \simeq \mathbf{HK}_\Gamma$ *for every finite directed acyclic graph.*

This has been computationally checked for all instances with ≤ 4 vertices and on most graphs on 5 vertices. A conceptual approach to the problem of determining the universal dynamics of a given graph Γ is by constructing an initial object, if there exists one, in the category of (pointed) update systems supported on Γ; here "pointed" means that a preferred system state has been chosen.

Conjecture 2. *The pair $(\mathcal{S}_n^\star, \star)$ is an initial object in the category of pointed update systems supported on Γ_n.*

The dynamics of an initial update system is clearly universal and its elements are told apart by their action on the marked system state $\star$: this has been the philosophy underneath the proof of Theorem 6.3.

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E-mail address: collina@dmmm.uniroma1.it, elena.collina@bancaditalia.it

DIPARTIMENTO DI SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA, UNIVERSITÀ DI ROMA "LA SAPIENZA", VIA ANTONIO SCARPA 14/16 – 00161 ROME, ITALY

SERVIZIO GESTIONE RISCHI FINANZIARI, BANCA D'ITALIA, VIA NAZIONALE 91 – 00184 ROME, ITALY

E-mail address: dandrea@mat.uniroma1.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA "LA SAPIENZA", P.LE ALDO MORO, 5 – 00185 ROME, ITALY