Semisimple Frobenius Manifolds and Conformal Algebras

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Introduction

As is well known, in recent years *string theory* has given particular stimulations and new input to algebraic geometry. The idea is to view a particle as a (possibly closed) vibrating string. Then the motion of a particle determines a surface in the ambient space. Such surfaces, under hypotheses that physically make sense, can be considered as complex algebraic curves. The *correlators* of the theory, called *Gromov-Witten invariants*, are numbers that (in some sense) count how many curves pass through a given set of algebraic cycles in the ambient space. There is a beautiful example of Kontsevich [Kon95] in which the ambient space is taken to be the complex projective plane $\mathbb{P}^2\mathbb{C}$. If we denote by $N(d)$ the number of rational curves of degree d passing through $3d - 1$ points in general position, then there is a recursive formula giving these numbers. This formula is a direct consequence of the fact that the generating series of the $N(d)$ determines, via its third derivatives, a product on $H^*(\mathbb{P}^2)$ that is commutative, unitary and associative. There is a general construction of a unitary and associative product on the cohomology of a projective manifold generalizing this case. This is *quantum cohomology*, which is given by taking as structure constants the third derivatives of the generating function of the genus-0 Gromov-Witten invariants (i.e. those correlators given by motions of strings that "sweep" genus-0 Riemann surfaces). Quantum cohomology is associative and unitary, but in general not commutative: it is *super-commutative*, which means that the $\mathbb{Z}_2$-grading (even cohomology plus odd cohomology) is taken into account in the commutation rules: two even classes commute, two odd classes anticommute and an even class and an odd class commute. This setting allows an important generalization, *Frobenius manifolds*, that are super-manifolds endowed with a super-commutative, unitary and associative product on the tangent sheaf. This concept is due to Dubrovin [Dub93] in the case of classical manifolds and its generalization to the $\mathbb{Z}_2$-graded case is due to Manin [Man99]. Frobenius manifolds can also be seen from a formal point of view, and from the point of view of deformation theory (see [DeV99]). We do not take the latter point of view here. Frobenius manifolds have many applications both in geometry and in mathematical physics. For example, mirror symmetry can be seen as a non-trivial isomorphism between two Frobenius manifolds associated in different ways with different projective manifolds (again, we do not treat this case here, but refer to [DP99] and [dB99]). One peculiarity of quantum cohomology is that it is often semisimple. This is unusual because cohomology is naturally nilpotent. For example, the cohomology of $\mathbb{P}^2$ is $H^*(\mathbb{P}^2) = \mathbb{C}[x]/(x^3)$, whereas the quantum cohomology is (essentially) $QH^*(\mathbb{P}^2) = \mathbb{C}[x]/(x^3-1)$. The semisemplicity of quantum cohomology, or, in general, of a Frobenius manifold is an important matter, since it ensures the integrability of a hierarchy of partial differential equations, as proved by Dubrovin and Zhang in [DZ]. This hierarchy is of central importance since, in many interesting cases, it is satisfied by the partition function of the theory at hand. Chapter 1 is dedicated to superstructures. Our main result here is the following.

1.1.2 THEOREM *Let A be a simple associative supercommutative superalgebra with unit. Then the odd part A_1 of A must be zero.*

This is specially important, because it implies (see remark 2.2.1) that a Frobenius super-manifold *must have no odd part* in order to be semisimple. In particular, *all projective manifolds with odd cohomology do not have semisimple quantum cohomology*.

In chapter 2 we deal with the problem of semisimplicity of Frobenius manifolds. We first give all the definitions and show the example of Grassmannians. We then turn to prove a criterion for semisimplicity, theorems 2.4.1 and 2.4.3.

2.4.1 THEOREM *Let M be a Frobenius manifold with Euler vector field E, coordinates $x_0, \dots, x_{n-1}$, corresponding vector fields $\partial_0, \dots, \partial_{n-1}$, element ∂_0 being the unit. The following are equivalent:*

1. *M is semisimple around a point $p \in M$;*
2. *The operator $\mathcal{E} : \mathcal{T}_M \to \mathcal{T}_M$ defined by $\mathcal{E}(X) = E * X$ is diagonalizable around p, its eigenvectors $\{e_i\}$ form a basis for $\mathcal{T}_M$, its eigenvalues $\{u_i\}$ are a system of functions on M around p such that $0 \neq u_i(0) \neq u_j(0)$ if $i \neq j$, $\overline{u}_i = u_i - u_i(0)$ is a system of coordinates on M around p and the functions $\eta_i = g(e_i, e_i)$ are invertible around p.*

2.4.3 THEOREM *Let H be a formal Frobenius manifold with metric g and potential Φ. Suppose that the operator $\mathcal{E}_0 : H \to H$ defined by $\mathcal{E}_0(h) = E(0) *_0 h$ has distinct non zero eigenvalues. Then H is semisimple.*

In the theory of quantum cohomology, this criterion is pontentially useful, because it reduces the problem of understanding whether a manifold has semisimple quantum cohomology to only three computations:

1. the Poincaré pairing of forms on the manifold;

2. the first Chern class of the manifold;

3. the 3-point Gromov-Witten invariants of the manifold.

The first two are quantities which can usually be calculated with little or no effort. The 3-point Gromov-Witten invariants are the simplest invariants, and often the only ones that can be calculated.

The hierarchy of partial differential equations we have cited above is of central importance. The simplest case is when the Frobenius manifold we deal with is the quantum cohomology of a point. Then the hierarchy produced is the *KdV* hierarchy. This is a family of integrable partial differential equations that plays an important role in many aspects of mathematics. Its origins lie in the study of solitonic waves, but it was soon seen also to play a central role in the study of the eigenvalues of the Schrödinger operator, in the study of Jacobians of algebraic curves and, more recently, in the so called *Virasoro conjecture* for gravitational descendants. A survey of the geometrical aspects of the theory of KdV equations can be found in [Arb], and all details of the Virasoro conjecture can be found in [Get99] and in the original papers of Eguchi, Hori and Xiong [EHX97, EX98]. This conjecture states that the generating function of the *gravitational descendants* (which are a generalization of Gromov-Witten invariants that play the role of Mumford-Morita classes on the moduli stacks of stable maps) is annihilated by a family of operators $\{L_k\}_{k\geq=-1}$ that satisfy the commutation relation

$$[L_h, L_k] = (h-k)L_{h+k}, \tag{1}$$

thus yielding the positive part of the Virasoro algebra (which is the infinite dimensional Lie algebra generated by $\{L_k\}_{k\in\mathbb{Z}}$ satisfying (1)). The conjecture, once proved, would produce an infinite amount of constraints the gravitational descendants must satisfy. The conjecture was proven by Kontsevich [Kon92] in the case of a point, which was the original form of the conjecture, as proposed by Witten in [Wit91], and partial results were obtained by various people, including Dubrovin, Zhang (see [DZ]) and Givental (see [Giv]). Here we concentrate on the approach of Dubrovin and Zhang, whose main ingredient is a Frobenius manifold (which is often semisimple) with which they associate a bihamiltonian structure on its loop space. The language they use is that of infinite-dimensional calculus of variations. Our proposal is that a tool which could be alternatively used is given by *conformal Poisson algebras*. These are introduced and studied in chapter 3. The idea comes from previous works of Kac, Bakalov, Beilinson, D'Andrea and Drinfeld [Kac98, BD, BDK01, DK98], in which algebraic counterparts of various aspect of physical theories are presented and brought to mathematical maturity. Our starting point is the theory of *Lie pseudo-algebras* in [BDK01] and *Lie*-algebras* in [BD]. These are Lie algebras over certain categories that have a tensorial structure in a weak sense. In particular, we are interested in *conformal algebras*, which can be seen as $\mathbb{C}[\partial]$-modules with a family of "Lie-brackets" that depend on a formal parameter λ. We have put quotation marks because the usual anticommutation and Jacobi identities are replaced by their conformal versions in which the parameter λ plays a vital role. The definition of conformal algebras is given in definition 3.1. Unfortunately, it does not fully suit our needs as we need to be able to express the bracket of a function by a *product* of functions. We therefore give the definition of a *conformal Poisson algebra*:

DEFINITION 3.2 *A* conformal Poisson algebra *is a* $\mathbb{C}[\partial]$*-module* L *endowed with a* $\mathbb{C}$*-bilinear map (called the* λ-bracket*)*

$$\begin{aligned} L \otimes L &\longrightarrow \mathbb{C}[\partial, \lambda] \otimes L \\ a \otimes b &\longmapsto \{a_\lambda b\} = \sum_k \lambda^k B_k(a, b) \end{aligned}$$

and a commutative and unitary product

$$\begin{aligned} L \times L &\longrightarrow L \\ (a, b) &\longmapsto ab \end{aligned}$$

such that ∂ *is a derivation with respect to this product and such that*

- $\{\partial a_\lambda b\} = -\lambda \{a_\lambda b\}$,
- $\{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\}$,
- $\{b_\lambda a\} = -\{a_{-\partial-\lambda} b\}$,
- $\{a_\lambda \{b_\mu c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\} + \{b_\mu \{a_\lambda c\}\}$,
- $\{ab_\lambda c\} = \{a_{\partial+\lambda} c\} b + \{b_{\partial+\lambda} c\} a$,
- $\{a_\lambda bc\} = b\{a_\lambda c\} + c\{a_\lambda b\}$,

- $\{\partial a_{\partial+\lambda} b\}c = \{a_{\partial+\lambda} b\}(-\partial - \lambda)c$,
- $\{a_{\partial+\lambda} \partial b\}c = (\partial + \lambda)(\{a_{\partial+\lambda} b\}c)$.

We point out some of the immediate properties of conformal Poisson algebras in lemmas 3.1.1, 3.1.2 and 3.1.3. Our first result, in this framework, is theorem 3.1.4, that gives a precise interpretation and meaning to the Lie structure induced by a conformal algebra L on its quotient $C = L/\partial L$. Denote by $\int : L \to C$ the natural projection.

3.1.4 THEOREM *The bracket on integrals can be expressed as*

$$[\textstyle\int F, \int G] = \int \{F_\lambda G\}|_{\lambda=0}.$$

We then give the definition of a *pencil* of conformal Poisson structures, i.e. two conformal Poisson brackets whose linear combinations are again conformal Poisson. This is very much in the style of the bihamiltonian formalism of [Mag78, DZ] and indeed inspired by these papers. This enables us to construct an infinite family of partial differential equations (3.24) for any pencil of conformal Poisson brackets. At this point we are able to study the Dubrovin-Zhang formalism of [DZ] with our new language. With a Frobenius manifold we associate a pencil of Poisson brackets and make this explicit for $M = QH^*(\mathbb{P}^1)$, writing down the equations of the *entire hierarchy* of PDEs in formulae (3.37), (3.38) and (3.39):

$$\begin{aligned}
\frac{dx}{dt_{n,1}} =& \sum_{j=1}^{\left[\frac{n+1}{2}\right]} 4^j \binom{n+1}{2j} \frac{(2j-1)!!^2}{j} x^{n+1-2j} \partial x e^{jy} + \\
&+ \sum_{j=1}^{\left[\frac{n+2}{2}\right]} 4^j \binom{n+1}{2j-1} (2j-1)!!^2 x^{n+2-2j} e^{jy} \partial y, \\
\frac{dy}{dt_{n,1}} =& (n+1)x^n \partial x + \sum_{j=1}^{\left[\frac{n}{2}\right]} 2^{2j+1} \binom{n+1}{2j+1} \frac{(2j+1)!!(2j-1)}{j} x^{n-2j} \partial x e^{jy} + \\
&+ \sum_{j=1}^{\left[\frac{n+1}{2}\right]} 2^{2j+1} \binom{n+1}{2j} (2j-1)!!^2 x^{n+1-2j} e^{jy} \partial y, \\
\frac{dx}{dt_{n,2}} =& \frac{dx}{dt_{n-1,1}}, \\
\frac{dy}{dt_{n,2}} =& \frac{dy}{dt_{n-1,1}}.
\end{aligned}$$

This is the explicit form of the genus-0 equations found by Eguchi and Yang in [EY94].

The Lie algebra structure on the integrals is explicitly studied in the rank 1 case ($L = \mathbb{C}[u, \partial u, \partial^2 u, \ldots]$) in section 3.6. This section contains a series of technical results of purely algebraic interest and is somehow apart from the rest of the dissertation. The first result we obtain is a theorem on the existence of an infinite-dimensional abelian sub-algebra.

3.6.5 THEOREM *The subalgebra $\mathcal{A} \subset C$ given by*

$$\mathcal{A} = \mathtt{Span}_{\mathbb{C}}\{\textstyle\int 1, \int u, \int u^2, \int u^3, \ldots\}$$

is abelian.

We then prove several formulae: equation (3.45)

$$[\int e^{\alpha\partial^i u}, \int e^{\beta\partial^j u}] = \sum_{\substack{\pi^1 \in P^j \\ \pi^2 \in P^{i+1}}} \alpha^{r(\pi^2)+1}\beta^{r(\pi^1)+1} N(\pi^1)N(\pi^2)\times$$
$$\times \int e^{\alpha\partial^i u + \beta\partial^j u} \prod_{h=1}^{r(\pi^1)} \partial^{\pi^1_h + j} u \prod_{k=1}^{r(\pi^2)} \partial^{\pi^2_k + i} u,$$

where $N(\pi)$ is an arithmetic function on partitions, and the following.

3.6.6 THEOREM *The following formulae hold true for all $m \geq 3, n \geq 2$*

$$[\int u^m, \int (\partial u)^n] = -6\binom{m}{3}(n-1)\int u^{m-3}(\partial u)^{n+1},$$
$$[\int (\partial u)^m, \int (\partial u)^n] = 4\binom{m}{2}\binom{n}{2}(m-n)\int (\partial u)^{m+n-5}(\partial^2 u)^3.$$

We are only interested in the rank 1 algebra because all the other conformal Poisson structures we deal with are (up to a basis change) a direct product of rank 1 ones.

Chapter 1

Superstructures

1.1 Superalgebras and semisimplicity

DEFINITION 1.1 *Let A be an algebra over an integral domain R with unit $1 \in R$ of characteristic 0. We say that A is a* superalgebra *if there exists an R-linear involution $\alpha : A \to A$. The eigenvalues of α can only be ± 1, so let us denote by A_i ($i \in \{0, 1\}$) the eigenspace of eigenvalue $(-1)^i$. We shall refer to the decomposition $A = A_0 \oplus A_1$ as the $\mathbb{Z}_2$-decomposition of A.*

We emphasize the fact that the superstructure endows A_0 with a structure of algebra and A_1 with a structure of A_0-module. A superalgebra is simple if it has no non-trivial homomorphic images:

DEFINITION 1.2 *Let A be a superalgebra. We say that A is* simple *if and only if every surjective morphism of superalgebras (i.e. morphism of algebras preserving $\mathbb{Z}_2$-degrees) $\varphi : A \to B \neq 0$ is an isomorphism. A superalgebra is* semisimple *if and only if it is direct sum of simple superalgebras.*

There is also a natural concept of *supercommutativity* for superalgebras which corresponds to commutativity: a superalgebra is supercommutative if, given $a \in A_i, b \in A_j$,

$$ab = (-1)^{ij} ba.$$

In particular, if A is supercommutative then A_0 is commutative.

Notice that the corresponding notion of superideal (i.e. the kernel of a morphism of superalgebras) is more refined than that of ideal of the underlying algebra structure.

EXAMPLE 1.1 Let $A = \mathbb{C} \oplus \mathbb{C}$. Define a $\mathbb{C}$-linear involution on A by

$$\alpha : \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C} \oplus \mathbb{C}$$
$$(x, y) \longrightarrow (y, x).$$

This yields a commutative superalgebra structure on A. Its $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$ is given by $A_0 = \{(x, x) | x \in \mathbb{C}\}$, $A_1 = \{(x, -x) | x \in \mathbb{C}\}$. Let $\varphi : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$ be the projection onto the first factor. The map φ is a morphism of algebras, but *not a morphism of superalgebras*. Therefore, $J = \mathtt{ker}\varphi$ is an ideal of A but not a superideal. In fact, A is a simple superalgebra but only a semisimple algebra.

Let J be an ideal of A. J is also a superideal if and only if $\alpha(J) \subset J$, and this occurs if and only if $J = (J \cap A_0) \oplus (J \cap A_1) = J_0 \oplus J_1$. However, things work out sufficiently well:

1.1.1 PROPOSITION *A superalgebra A is semisimple if and only if its underlying algebra structure is semisimple.*

Proof. Suppose that A is a semisimple algebra. Let I be a superideal of A. Then I is also an ideal, and there exists an ideal $J \subset A$ such that $A = I \oplus J$. Therefore, the exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow I \longrightarrow 0$$

endows J with the structure of a superideal. Conversly, suppose that A is a semisimple superalgebra. Let I be an ideal of A. Then I is defined by a short exact sequence of *algebras*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0.$$

Let $C = \varphi(A_0) \oplus \varphi(A_1)$. Notice that I is a superideal if and only if $C \simeq B = \varphi(A)_0 + \varphi(A_1)$ (i.e. if and only if the latter sum is direct). Then define a superideal $S(I) \subset I$ of A by the exact sequence of *superalgebras*

$$0 \longrightarrow S(I) \longrightarrow A \longrightarrow C \longrightarrow 0$$

(I is a superideal if and only if $S(I) = I$). B is naturally isomorphic to $C/C \cap I$. From the semisimplicity of A as a superalgebra, it follows that C is (isomorphic to) a superideal $J \subset A$; therefore, A is isomorphic to $I \oplus J/J \cap I$. $\square$

Throughout what follows we shall always assume that A is associative, unitary and supercommutative. Let $J = J_0 \oplus J_1$ be a subgroup of the additive group of A. Thus, the conditions for J to be a superideal are

$$\begin{aligned} A_0 J_0 &\subset J_0 \\ A_1 J_0 &\subset J_1 \\ A_0 J_1 &\subset J_1 \\ A_1 J_1 &\subset J_0. \end{aligned} \tag{1.1}$$

In particular J_0 is an ideal of A_0 and J_1 is a sub-A_0-module of A_1. Notice that $A_1^2 \oplus A_1$ and $A_0 \oplus A_0 A_1$ are *ideals* of A. If A is simple, we must have

$$A_1^2 = A_0 \qquad \text{and} \qquad A_0 A_1 = A_1.$$

1.1.2 THEOREM *Let A be a simple associative supercommutative superalgebra with unit. Then the odd part A_1 of A must be zero.*

Proof. Let us suppose that the odd part of A does not vanish. We begin by showing that if A is simple then A_0 is a simple algebra and A_1 is a simple A_0-module. Let J_0 be an ideal of A_0 and $J_1 = A_1 J_0$. Then $A_0 J_0 \subset J_0$ by hypothesis, whereas $A_0 J_1 = A_0 A_1 J_0$, which, by the simplicity of A is equal to $A_1 J_0 = J_1$. On the other hand, $A_1 J_0 = J_1$ and $A_1 J_1 = A_1 A_1 J_0 = A_0 J_0 \subset J_0$. Therefore, $J = J_0 \oplus J_1$ is an ideal of A; hence, it must be trivial, so that J_0 is either 0 or A_0; consequently, A_0 is simple. Similar calculations show that if I_1 is a sub-A_0-module of A_1, then $I = A_1 I_1 \oplus I_1$ is an ideal of A, and therefore I_1 must be trivial (i.e. either A_1 or 0), and thus A_1 is a simple A_0-module. Hence, there exists an isomorphism of A_0 modules $\psi : A_0 \to A_1$. Now, since A_0 is simple, it is generated by an idempotent element e. Call $f = \psi(e)$ the image of this element through the A_0-module isomorphism ψ. Then f generates A_1 as an A_0-module. However, the supercommutativity of A implies that $f^2 = \frac{1}{2}[f, f] = 0$, so that $A_1^2 = 0$, contradicting our hypotheses of simplicity of A and non-triviality of A_1. □

1.1.3 REMARK If we omit the hypothesis of supercommutativity in the theorem above, we are left with a dicotomy: A is a simple superalgebra if and only if it is either a simple algebra or the direct sum of a simple algebra A_0 and a simple A_0-module A_1.

1.2 Supermanifolds

In this section we follow the excellent book of Manin [Man99]. We begin with a very general definition that is intended to fit whatever category one prefers (real manifolds, complex manifolds, algebraic varieties, etc.).

DEFINITION 1.3 *A (real, complex, algebraic, etc.)* supermanifold M *is a ringed space* $(M, \mathcal{O}_M)$ *such that*

1. *the sheaf* $\mathcal{O}_M$ *decomposes* $\mathcal{O}_M = \mathcal{O}_{M,0} \oplus \mathcal{O}_{M,1}$ *(i.e. is a sheaf of superrings);*
2. $M_{\mathtt{red}} = (M, \mathcal{O}_{M,\mathtt{red}} = \mathcal{O}_M/(\mathcal{O}_{M,1}))$ *is a (real, complex, algebraic, etc.) manifold called the* reduced *manifold;*
3. *there exists a free* $\mathcal{O}_{M,\mathtt{red}}$*-sheaf* $\mathcal{E}$ *such that* $\mathcal{O}_M$ *is locally isomorphic to the exterior algebra* $\wedge\mathcal{E}$*.*

For a homogeneous element $x \in \mathcal{O}_M$ we shall denote by $\overline{x} \in \{0,1\}$ its degree with respect to the $\mathbb{Z}_2$-grading, i.e. $x \in \mathcal{O}_{M,\overline{x}}$. For a given set of coordinates $\{x_i\}$, $\partial_i = \partial/\partial x_i$ has the same degree as x_i (we are considering the tangent sheaf as a sheaf of modules over a superring, and hence giving it a natural superstructure). We give the opposite degree to forms, i.e. dx has degree $\overline{x} + 1 \bmod 2$. Whenever we have a superalgebra, the symmetric and exterior powers of the algebra "talk to each other" in a much more intricate way. This is because the symmetric second power of a superalgebra A is defined by $S^2 A = A \otimes A/I_S$, where I_S is the ideal generated by (homogeneous) elements of the form $a \otimes b - (-1)^{\overline{a}\overline{b}} b \otimes a$, whereas the exterior power is $\wedge^2 A = A \otimes A/I_\wedge$, where $I_\wedge$ is the ideal generated by (homogenous) elements of the form $a \otimes b + (-1)^{\overline{a}\overline{b}} b \otimes a$. This then implies certain isomorphisms between symmetric and exterior powers of suitable shifts of the algebra[1].

DEFINITION 1.4 *An* affine flat structure *on a supermanifold* M *is a pair of* $\mathcal{O}_M$*-sheaves* $\mathcal{T}_M^f \subset \mathcal{T}_M$ *such that*

1. $[\mathcal{T}_M^f, \mathcal{T}_M^f] = 0$*;*
2. $\mathcal{T}_M = \mathcal{T}_M^f \otimes_k \mathcal{O}_M$*, where* k *is the field we are working on.*

[1] It is not important here to be more specific, but the interested reader may find an accurate description of the more general $\mathbb{Z}$-graded case in [Gra99].

The sections of $\mathcal{T}_M^f$ are referred to as *flat vector fiels*. This can be thought of as the existence of a connection ∇ on the tangent sheaf, and $\mathcal{T}_M^f$ is the sheaf of vector fields that are flat with respect to ∇. We shall also need the concept of a Riemannian metric, i.e. a supersymmetric even pairing

$$g : S^2\mathcal{T}_M \longrightarrow \mathcal{O}_M \tag{1.2}$$

which is non-degenerate in the sense that it induces an isomorphism $g' : \mathcal{T}_M^* \to \mathcal{T}_M$. In the presence of an affine flat structure $\mathcal{T}_M^f$, a Riemannian metric is required to be *compatible* with the affine flat structure, meaning that, for all $X, Y \in \mathcal{T}_M^f$, we must have $g(X, Y) \in \boldsymbol{k}$.

Chapter 2

Frobenius manifolds

2.1 Definitions and first properties

Let $A : S^3\mathcal{T}_M \to \mathcal{O}_M$ be an even symmetric tensor. We shall denote by $A' : S^2\mathcal{T}_M \to \mathcal{T}^*_M$ the partial dualization of A.

Definition 2.1 *Let M be a supermanifold endowed with an affine flat structure $\mathcal{T}^f_M$ and a compatible Riemannian metric g structure on $\mathcal{T}_M$. A structure of* Frobenius manifold *on M is a pair consisting of a local function $\Phi \in \mathcal{O}_M$ called* the potential *and a vector field $E \in \mathcal{T}_M$ such that, letting*

$$\begin{aligned} A : S^3\mathcal{T}_M &\longrightarrow \mathcal{O}_M \\ X \otimes Y \otimes Z &\longmapsto XYZ\Phi, \end{aligned}$$

1. *the product $* : S^2\mathcal{T}_M \to \mathcal{T}_M$ defined by the composition*

   $$S^2\mathcal{T}_M \xrightarrow{A'} \mathcal{T}^*_M \xrightarrow{g'} \mathcal{T}_M, \qquad * = g' \circ A'$$

   is unitary and associative;

2. *E is an* Euler vector field*, i.e. satisfies*

   $$E(g(X,Y)) - g([E,X],Y) - g(X,[E,Y]) = d \cdot g(X,Y) \qquad (2.1)$$
   $$[E, X*Y] - [E,X]*Y - X*[E,Y] = X*Y \qquad (2.2)$$

   for all vector fields X, Y and for some constant d called the charge *of the Euler field.*

Notice that the definition of the product is equivalent to requiring the following diagram

$$
\begin{CD}
S^3\mathcal{T}_M @>{*\otimes 1}>> S^2\mathcal{T}_M \\
@V{1\otimes *}VV @VV{g}V \\
S^2\mathcal{T}_M @>>{g}> \mathcal{O}_M
\end{CD}
$$

commutes. Furthermore, the associativity of the product is equivalent to requiring Φ satisfies a set of partial differential equations first devised by Witten, Dijgraaf, E. Verlinde and H. Verlinde, and hence called WDVV equations:

$$\sum_{l,m} \Phi_{hil} g^{lm} \Phi_{mjk} = (-1)^{\overline{x}_h(\overline{x}_i+\overline{x}_j)} \sum_{l,m} \Phi_{ijl} g^{lm} \Phi_{mik}, \quad \forall h, i, j, k, \tag{2.3}$$

where all the x_a are flat coordinates on M, Φ_{abc} denote the third derivatives $\partial_a \partial_b \partial_c \Phi$ and g^{lm} is the l-th row and m-th column element of the inverse matrix to $(g_{ab}) = (g(\partial_a, \partial_b))$.

EXAMPLE 2.1 Let $M = \mathbb{C}^2$, with coordinates x and y. Denote by ∂_x and ∂_y the corresponding vector fields (i.e. $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$) and let

$$\Phi = \frac{x^2 y}{2} + e^y. \tag{2.4}$$

Moreover, set

$$E = x\partial_x + 2\partial_y. \tag{2.5}$$

The product on $\mathcal{T}_{\mathbb{C}^2}$ is then such that ∂_x is the identity and $\partial_y * \partial_y = e^y \partial_x$. This Frobenius manifold is in some sense the first (smallest) interesting one. It is the *quantum cohomology of* $\mathbb{P}^1$ and we shall refer to it throughout what follows as a nontrivial but simple example. Notice that if we take

$$\begin{aligned} e_1 &= \frac{1}{2}\partial_x + \frac{1}{2}e^{-\frac{y}{2}}\partial_y \\ e_2 &= \frac{1}{2}\partial_x - \frac{1}{2}e^{-\frac{y}{2}}\partial_y \end{aligned} \tag{2.6}$$

as generators of the tangent sheaf, we get a basis of orthogonal idempotents

$$e_i * e_j = \delta_{ij} e_j, \tag{2.7}$$

where δ_{ij} is the Kronecker symbol.

A problem which is often encountered is that in concrete examples there is little or no way to know whether the potential is an actual function or whether it is only a formal expression that comes from some idea (usually from physics). We therefore will need a formal version of definition 2.1: we take a free $\mathbb{Z}_2$-graded module H (that will play the role of M) over a super $\mathbb{Q}$-algebra k. We shall look at H as a formal neighbourhood of the $0 \in H$, identifying the functions on H with the formal power series $k[[H^\vee]]$. The affine flat structure will be the tangent space to H, that we shall identify with H itself, and the tangent sheaf will therefore be $H \otimes_k k[[H^\vee]]$. The compatible metric will be an even symmetric pairing $g : H \otimes H \to k$ (i.e. pairs of flat vectors map under g to constants). We shall also fix a basis $\{\Delta_i\}$ of H (whose relative coordinates we shall call $\{x_i\}$) and set $g_{ij} = g(\Delta_i, \Delta_j)$, $(g^{ij}) = (g_{ij})^{-1}$. The Frobenius structure will be given by:

1. a formal power series Φ such that the product defined on the basis of H as
$$\Delta_i * \Delta_j = \sum_{h,k} \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_h} g^{hk} \Delta_k$$
and extended linearly to $H \otimes_k k[[H^\vee]]$ is unitary (with identity Δ_0) and associative;

2. a formal power series $E = \sum_k E_k(x)\Delta_k$ that satisfies condition 2 of definition 2.1, i.e. the conditions are satisfied once we do the opposite identifications (for example considering E as $\sum_k E_k \partial/\partial x_k$).

An important aspect of Frobenius manifolds is that they can be viewed as manifolds with a *pencil of flat connections*. Namely, we define for all $z \in \mathbb{C}$ a connection $\nabla^{(z)} : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1(M)$ by letting the covariant derivative of a vector field Y with respect to another vector field X be

$$\nabla^{(z)}_X Y = \nabla_X Y + zX * Y. \tag{2.8}$$

A straightforward calculation shows that the flatness of the pencil of connections ($[\nabla_X, \nabla_Y]Z = \nabla_{[X,Y]}Z$) is equivalent to asking the associativity of $*$. Furthermore, we can consider $\widehat{M} = M \times \mathbb{C}$ (with z coordinate on $\mathbb{C}$) and naturally extend the family of connections to a single connection on $\widehat{M}$ by letting $\widehat{E} = E - z\partial_z$ and defining

$$\begin{aligned} \widehat{\nabla}_X Y &= \nabla^{(z)}_X Y \\ \widehat{\nabla}_{\widehat{E}} \partial_z &= 0 \\ \widehat{\nabla}_{\widehat{E}} Y &= [\widehat{E}, Y] + \kappa Y, \end{aligned} \tag{2.9}$$

where $\kappa = 2 - d/2$.

2.1.1 PROPOSITION *The connection $\widehat{\nabla}$ is flat.*

Proof. We only need to show that

$$(\widehat{\nabla}_{\widehat{E}}\widehat{\nabla}_{\partial_i} - \widehat{\nabla}_{\partial_i}\widehat{\nabla}_{\widehat{E}})\partial_j = \widehat{\nabla}_{[\widehat{E},\partial_i]}\partial_j$$

for any choice of $\partial_i = \partial/\partial x_i$ and $\partial_j = \partial/\partial x_j$, where x_i and x_j are elements of a local flat coordinate system. Let calculate the left hand side: the first term is

$$\begin{aligned}\widehat{\nabla}_{\widehat{E}}\widehat{\nabla}_{\partial_i}\partial_j =&\widehat{\nabla}_{\widehat{E}}(z\partial_i * \partial_j) = \widehat{E}(z)\partial_i * \partial_j + z\widehat{\nabla}_{\widehat{E}}(\partial_i * \partial_j) = \\ =&(E - z\partial_z)z\partial_i * \partial_j + z([\widehat{E}, \partial_i * \partial_j] + \kappa\partial_i * \partial_j) = \\ =&z(-\partial_i * \partial_j + [E, \partial_i * \partial_j] + \kappa\partial_i * \partial_j),\end{aligned}$$

whereas the second is

$$\begin{aligned}\widehat{\nabla}_{\partial_i}\widehat{\nabla}_{\widehat{E}}\partial_j =&\widehat{\nabla}_{\partial_i}([\widehat{E}, \partial_j] + \kappa\partial_j) = \widehat{\nabla}_{\partial_i}[E, \partial_j] + z\kappa\partial_i * \partial_j = \\ =&z(\partial_i * [E, \partial_j] + \kappa\partial_i * \partial_j).\end{aligned}$$

The difference of these two is then

$$\begin{aligned}\widehat{\nabla}_{\widehat{E}}\widehat{\nabla}_{\partial_i}\partial_j -& \widehat{\nabla}_{\partial_i}\widehat{\nabla}_{\widehat{E}}\partial_j = \\ =&z(-\partial_i * \partial_j + [E, \partial_i * \partial_j] + \kappa\partial_i * \partial_j - \partial_i * [E, \partial_j] + \kappa\partial_i * \partial_j) \\ =&z[E, \partial_i] * \partial_j.\end{aligned}$$

This is exactly what we wanted, because the term on the right hand side is

$$\widehat{\nabla}_{[\widehat{E},\partial_i]}\partial_j = \widehat{\nabla}_{[E,\partial_i]}\partial_j = z[E, \partial_i] * \partial_j.$$

□

2.2 Semisimple Frobenius manifolds

DEFINITION 2.2 *A* semisimple *Frobenius manifold is a Frobenius manifold such that $*$ endows $\mathcal{T}_M$ with the structure of a sheaf of semisimple $\mathcal{O}_M$-algebras.*

It is clear that situations may be found in which the semisimplicity of the algebra $\mathcal{T}_M$ is true only on some open subset of M. In fact semisimplicity is an open condition and we mentioned above we shall often be interested in formal manifolds rather than in geometric ones. It will suffice to verify, in such a case, the semisimplicity of the algebra given by the fiber over 0. This algebra we shall denote by H (see the identifications above) and its product by $*_0$:

$$\Delta_i *_0 \Delta_j = \sum_{h,k} \left. \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_h} \right|_{x=0} g^{hk} \Delta_k.$$

2.2.1 REMARK We only consider Frobenius *manifolds* because the "super" case does not make much sense. If the definition above is re-read using supermanifolds and superalgebras instead of manifolds and algebras, then theorem 1.1.2 guarantees that $\mathcal{T}_M$ must have *no odd part*, which means M must be a classical manifold.

Sunce we are dealing only with torsionless comutative objects, the following proposition is clear:

2.2.2 PROPOSITION *A Frobenius manifold M is semisimple if and only if there exists a local basis $\{e_i\}$ of $\mathcal{T}_M$ such that $e_i * e_j = \delta_{ij} e_j$.*

Notice that the metric is diagonal on this basis:

$$g(e_i, e_j) = g(e_i * e_i, e_j) = g(e_i, e_i * e_j) = g(e_i, \delta_{ij} e_j) = \delta_{ij} g(e_i, e_j).$$

The book of Manin [Man99] contains a complete review of the basic facts about semisimple Frobenius manifolds. In order to give a complete picture, we include, without proof, the main theorem.

2.2.3 THEOREM *Let M be a manifold. A structure of semisimple Frobenius manifold on M is given by the following data:*

1. *a product $*$ on $\mathcal{T}_M$;*
2. *a basis $\{e_i\}$ such that $e_i * e_j = \delta_{ij} e_j$;*
3. *a flat metric g such that $g(e_i, e_j) = \delta_{ij} g(e_i, e_i) = \delta_i^j \eta_j$ (i.e. $g = \sum_i \eta_i \nu_i^2$, where ν_i is dual to e_i);*
4. *a diagonal 3-tensor A with the same coefficients as g: $A = \sum_i \eta_i \nu_i^3$;*
5. *a set of coordinates $\{u_i\}$ on M (called* canonical coordinates*) such that $e_i = \partial/\partial u_i$ and $\nu_i = du_i$;*
6. *a local function η (called the metric potential) such that $\eta_i = e_i \eta$.*

The cubic tensor in the theorem is the tensor of third derivatives of the potential:

$$A(X,Y,Z) = XYZ\Phi.$$

It is clear that the metric potential η must satisfy certain constraints. These are called *Darboux-Egoroff equations*. We do not include them here.

2.3 Quantum cohomology of Grassmannians

Let X be a projective variety. Consider Kontsevich's *moduli stacks of stable maps* $\overline{\mathcal{M}}_{g,n}(X,\beta)$. These are suitable compactifications of $\mathcal{M}_{g,n}(X,\beta)$, which are the spaces whose points are maps $\mu : C \to X$ from an n-pointed genus g smooth algebraic curve such that the homology class $[\mu(C)]$ coincides with a given homology class $\beta \in H_2^+(X)$ (where the subscript $+$ denotes the positive cone generated by algebraic subvarieties od X of dimension 1). There is a great deal to say about $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Even in very simple cases they do not behave too well, thus forcing us to consider them as Deligne-Mumford stacks. One key fact is that their dimension does not coincide with their so-called *expected dimension*, which is the dimension obtained by writing down the Riemann-Roch theorem:

$$D = \mathrm{expdim}\overline{\mathcal{M}}_{g,n}(X,\beta) = 3g - 3 + n + \int_\beta c_1(X) + \mathrm{dim}X.$$

However, it can be proved that there exists a *virtual fundamental class*

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{v}} \in H_{2D}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

over which we can integrate and have geometrically useful and significant objects (see [BF97] for full details and proofs). Define the natural maps

$$\begin{aligned} \rho_i : \overline{\mathcal{M}}_{g,n}(X,\beta) &\longrightarrow X \\ [\mu, C, p_1, \ldots, p_n] &\longmapsto \mu(p_i), \end{aligned}$$

and consider their pullbacks

$$\rho_i^* : H^*(X) \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}(X,\beta)).$$

We are considering cohomology with coefficients in $\mathbb{C}$ in order to have algebraic closure. We shall need this later on to give concrete meaning to the orthogonal idempotents of semisimple quantum cohomology. Define now the *Gromov-Witten* invariants

$$I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)=\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{v}}}\rho_1^*(\gamma_1)\cup\ldots\cup\rho_n^*(\gamma_n). \tag{2.10}$$

There is a naïve way to look at these objects that corresponds to reality in only very few cases (such as convex varieties whose cohomology is purely algebraic): $I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)$ is the number of curves of genus g and "degree" β that pass through Poincaré duals of the γ_i taken in generic position, if this number is finite, and is 0 if this number is ∞. To define quantum cohomology we only need the genus zero invariants. Next, consider the generating series of these

$$\Phi(\gamma)=\sum_{\substack{n\geq 3\\ \beta\in H_2^+(X)}}\frac{1}{n!}I_{0,n,\beta}(\gamma,\ldots,\gamma). \tag{2.11}$$

Given a basis $\Delta_0=1,\Delta_1,\ldots,\Delta_N$ of $H^*(X)$ such that $\Delta_i\in H^{|\Delta_i|}(X)$ and the corresponding coordinates $x_0,x_1,\ldots,x_N$, we can view Φ as a formal power series in the x_i. It is well known that Φ satisfies the WDVV equations (2.3) and therefore defines a structure of Frobenius manifold on the tangent space to the domain in which it converges – or, in case we can say nothing about the convergence, it defines a formal Frobenius structure on $H^*(X)$. The Riemannian metric in this case is the Poincaré pairing $g(\Delta_i,\Delta_j)=\int_X\Delta_i\cup\Delta_j$. The Euler field has the form

$$E=\sum_i\left(1-\frac{|\Delta_i|}{2}\right)x_i\frac{\partial}{\partial x_i}+\sum_{j:|\Delta_j|=2}r_j\frac{\partial}{\partial x_j},$$

where the r_j are defined by

$$c_1(X)=\sum_{j:|\Delta_j|=2}r_j\Delta_j.$$

We shall denote the quantum cohomology of X by $QH^*(X)$. The central fiber will be denoted by $QH_0^*(X)$. Notice that this coincides with $H^*(X)$ as a vector space, but that its product is different since its structure constants are all the three-point Gromov-Wotten invariants, whereas the usual product structure constants are only those three-point Gromov-Witten invariants calculated for $\beta=0$.

There is a variant to this construction. Suppose $\Delta_1, \ldots, \Delta_t \in H^2(X)$ for some $t \leq N$. Then we can define

$$\Psi(\gamma) = \Phi(x_0, x_1, \ldots, x_t, 0, \ldots, 0).$$

This gives us another structure, called the *small* quantum cohomology. The basic properties of Gromov-Witten invariants (see [FP97] for full details) show that the product is completely defined by the whole set of three-point invariants. Therefore, the central fiber of the small quantum cohomology, which is defined in the obvious way, coincides with $QH_0^*(X)$.

By remark 2.2.1, the only projective varieties whose quantum cohomolgy can be semisimple are those whose odd cohomology is zero. Projective spaces, Grassmannians and certain Fano varieties have this property. We shall deal here only with the former two. There is a conjecture of Dubrovin [Dub98] on the nature of the projective varieties whose quantum cohomology is semisimple. A consequence of this conjecture is that the three types cited above are the only cases.

Next, let us see what this construction yields in the case of Grassmannians. The usual cohomology of the Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in n-space is given by

$$H^*(\mathrm{Gr}(k,n)) = \mathbb{C}[\sigma_1, \ldots, \sigma_k]/(S_{n-k+1}, \ldots, S_n),$$

where the S_j are defined by the formal power series

$$(1 + \sigma_1 t + \sigma_2 t^2 + \ldots + \sigma_k t^k)^{-1} = \sum_{j \geq 0} (-1)^j S_j t^j.$$

The small quantum cohomology of $\mathrm{Gr}(k,n)$ is (see [FP97] for a full account)

$$QH_s^*(\mathrm{Gr}(k,n)) = \mathbb{C}[\sigma_1, \ldots, \sigma_k][[q]]/(S_{n-k+1}, \ldots, S_n + (-1)^{n-k} q).$$

Its central fiber (i.e. the algebra in $q = 1$) is then given by

$$QH_0^*(\mathrm{Gr}(k,n)) = \mathbb{C}[\sigma_1, \ldots, \sigma_k]/(S_{n-k+1}, \ldots, S_n + (-1)^{n-k}).$$

Change variables:

$$\sigma_j = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq k} x_{i_1} x_{i_k} \ldots x_{i_j}.$$

2.3.1 LEMMA *In the above variables, $S_j(x_1, \ldots, x_k)$ is the elementary complete symmetric polynomial in the k variables $x_1, \ldots, x_k$.*

Proof. We use induction on j. When $j = 1$, $S_1 = \sigma_1 = \sum_i x_i$. So suppose the lemma is true for all $h < j$. Take $j_1, \dots j_k \geq 0$ such that $\sum_i j_i = j$. We want to write down the coefficient of $x_1^{j_1} \cdot \dots \cdot x_k^{j_k}$ in S_j. By definition, σ_h is the sum of $\binom{h}{k}$ monomials, therefore the contribution of $\sigma_h S_{j-h}$ to the coefficient of $x_1^{j_1} \cdot \dots \cdot x_k^{j_k}$ in S_j is $\binom{h}{k}$. Then, by the definition of the S_i, the coefficient of $x_1^{j_1} \cdot \dots \cdot x_k^{j_k}$ in S_j is

$$\sum_{h=1}^{k} (-1)^{h+1} \binom{h}{k} = (1-1)^k + 1 = 1.$$

□

We will be done once we show that the simultaneous equations

$$\begin{cases} S_{n-k+1}(x_1, \dots, x_k) = 0 \\ S_{n-k+2}(x_1, \dots, x_k) = 0 \\ \dots \\ S_{n-1}(x_1, \dots, x_k) = 0 \\ S_n(x_1, \dots, x_k) = \pm 1. \end{cases} \tag{2.12}$$

have solutions that are all distinct and non-zero. Since the S_j have the form described in the lemma above, we can write

$$S_{n-k+2}(x_1, \dots, x_k) = x_1 S_{n-k+1}(x_1, \dots, x_k) + S_{n-k+2}(x_2, \dots, x_k).$$

The first summand on the right hand side is zero, therefore we can write the second equation as $S_{n-k+2}(x_2, \dots, x_k) = 0$. The straightforward generalization of this allows us to write down (2.12) as

$$\begin{cases} S_{n-k+1}(x_1, \dots, x_k) = 0 \\ S_{n-k+2}(x_2, \dots, x_k) = 0 \\ \dots \\ S_{n-k+j}(x_j, \dots, x_k) = 0 \\ \dots \\ S_{n-1}(x_{k-1}, x_k) = 0 \\ S_n(x_k) = \pm 1. \end{cases} \tag{2.13}$$

It is easy to see that this implies that $x_1, \dots, x_k$ are distinct solutions of $S_n(x) = \pm 1$, i.e. of $x^n = \pm 1$. Therefore, we see that $QH_0^*(\mathrm{Gr}(k, n)) \cong$

$\mathbb{C}[x_1, \ldots, x_k]/(x_j^n)$, which is semisimple. In the case of the projective space $\mathbb{P}^n = \mathrm{Gr}(1, n+1)$ we can actually construct the basis of orthogonal idempotents. Let $\zeta = e^{\frac{2\pi\sqrt{-1}}{n+1}}$. Let Δ_i be the generator of $H^{2i}(\mathbb{P}^n)$, x_i be the corresponding coordinate and $\partial_i = \partial/\partial x_i$. Define $q \in QH^*(\mathbb{P}^n)$ to be the $(n+1)$-st root of $\partial_1^{*(n+1)}$ that satisfies $q \equiv e^{\frac{x_1}{r+1}} \bmod (x_2, \ldots, x_n)$. The orthogonal idempotents are then given by

$$e_i = \frac{1}{n+1} \sum_{j=0}^{n} \zeta^{-ij} (\partial_1 * q^{-1})^{*j}.$$

Notice that in the quantum setting the Plücker embedding implies an algebra isomorphism

$$QH_0^*(\mathrm{Gr}(k, n)) \cong QH_0^*(\mathbb{P}^{\binom{n}{k}-1})$$

which extends to a (formal) neighbourhood thus yielding

$$QH^*(\mathrm{Gr}(k, n)) \cong QH^*(\mathbb{P}^{\binom{n}{k}-1}).$$

2.4 A criterion for semisimplicity

2.4.1 THEOREM *Let M be a Frobenius manifold with Euler vector field E, coordinates $x_0, \ldots, x_{n-1}$, corresponding vector fields $\partial_0, \ldots, \partial_{n-1}$, element ∂_0 being the unit. The following are equivalent:*

1. *M is semisimple around a point $p \in M$;*
2. *The operator $\mathcal{E} : \mathcal{T}_M \to \mathcal{T}_M$ defined by $\mathcal{E}(X) = E * X$ is diagonalizable around p, its eigenvectors $\{e_i\}$ form a basis for $\mathcal{T}_M$, its eigenvalues $\{u_i\}$ form a system of functions on M around p such that $0 \neq u_i(0) \neq u_j(0)$ if $i \neq j$, $\overline{u}_i = u_i - u_i(0)$ is a system of coordinates on M around p and the functions $\eta_i = g(e_i, e_i)$ are invertible around p.*

Proof. The fact that 1 implies 2 is straightforward (see [Man99] or [Dub93]). The other implication is subtler. By the commutativity and associativity of the product, and since we have fixed notation $E * e_i = u_i e_i$, we see that if we take $i \neq j$ then $u_i e_i * e_j = E * e_i * e_j = e_i * E * e_j = u_j e_i * e_j$. Therefore the e_i are orthogonal. So, let $e_i * e_i = \sum_j \alpha_i^j e_j$. Again, using the associativity, we find that $E * e_i * e_i$ is, on the one hand equal to $u_i e_i * e_i = \sum_j u_i \alpha_i^j e_j$, and, on the other hand, to $\sum_j u_j \alpha_i^j e_j$. Therefore, for all i and j, $u_i \alpha_i^j = u_j \alpha_i^j$, which implies that $a_i^j = 0$ if $i \neq j$. Now, $u_i e_i = E * e_i = \sum_j E^j e_j * e_i$, where the last equality is given by expressing E in terms of the basis $\{e_h\}$. Therefore, $u_i e_i = E^i \alpha_i^i e_i$ which implies that each α_i^i is invertible, so that we can assume $\alpha_i^i = 1$. To sum up, we have found $e_i * e_j = \delta_{ij} e_j$. Notice that we have also found that $E = \sum_i u_i e_i$. Being the Euler field of the Frobenius manifold, E satisfies equation (2.2), which, choosing $X = Y = e_i$ reads

$$[E, e_i] - 2[E, e_i] * e_i = e_i,$$

which we will rewrite as

$$[E, e_i] * (\partial_0 - 2e_i) = e_i.$$

Notice however that $\partial_0 = \sum_j e_j$, so that $\partial_0 - 2e_i = -e_i + \sum_{j \neq i} e_j$, and this implies that $[E, e_i] = -e_i$. On the other hand, we may choose $X = E$ and $Y = e_i$ in equation (2.2), to get

$$[E, u_i e_i] - [E, E] * e_i - E * [E, e_i] = E * e_i$$

which implies that $[E, E * e_i] = 0$. Expanding the left hand side yields

$$\begin{aligned} 0 &= E(u_i)e_i + u_i E(e_i) - u_i e_i(E) \\ &= E(u_i)e_i + u_i[E, e_i] \\ &= E(u_i)e_i - u_i e_i, \end{aligned}$$

so that

$$E(u_i) = u_i.$$

Reading this equation on the lines $\{u_j = 0, \forall j \neq h\}$ yields

$$e_i(u_j) = \delta_{ij}.$$

If we now expand the right hand side of $-e_i = [E, e_i]$, we get

$$\begin{aligned} -e_i &= \sum_h (u_h e_h(e_i) - e_i(u_h)e_h - u_h e_i(e_h)) \\ &= -e_i + \sum_h u_h [e_h, e_i]. \end{aligned}$$

By calculating this last expression on the lines $\{u_j = 0, \forall j \neq h\}$, we get

$$[e_i, e_j] = 0,$$

therefore
$$e_i = \frac{\partial}{\partial u_i}.$$
To end the proof we need to show that there exists a function η on M around p such that $e_i\eta = \eta_i$. To do this we write down the multiplication in terms of the potential:
$$\delta_{ij}e_j = e_i * e_j = \sum_{h,k} e_i e_j e_h \Phi(u) g^{hk} e_k.$$
Notice that (g_{ij}) is the diagonal matrix with entries η_i, so that the last equation reduces to
$$\delta_{ij}e_j = \sum_h e_i e_j e_h \Phi(u) \eta_h^{-1} e_h,$$
which we rewrite as
$$e_i e_j e_h \Phi(u) = \delta_{ij}\delta_{jh}\eta_h.$$
In this way we have found
$$\eta_i = e_i^3 \Phi(u),$$
which implies that
$$e_i \eta_j = e_i e_j^3 \Phi(u) = e_j e_i e_j^2 \Phi(u) = 0$$
if $i \neq j$. In particular, $e_i\eta_j = e_j\eta_i$ whatever the choices of i and j, and, since we are around a point p, we have reached our goal. □

Unfortunately, as we already pointed out, it is often difficult to know the full structure of a Frobenius manifold. For example, in quantum cohomology, one might not know how to compute all the Gromov-Witten invariants except those calculated on three classes, $I_{0,3,\beta}$. Therefore, we look for a criterion which can be applied to such a situation. Furthermore, in quantum cohomology, we know from [Man99] that the explicit form of the Euler field is
$$E = \sum_i \left(1 - \frac{|\Delta_i|}{2}\right) x_i \frac{\partial}{\partial x_i} + \sum_{j:|\Delta_j|=2} r_j \frac{\partial}{\partial x_j},$$
where the Δ_i are a homogeneous basis for H^*, $\Delta_i \in H^{|\Delta_i|}$, the x_i are the coordinates corresponding to Δ_i and the r_j are the coefficients of the first Chern class of the manifold whose cohomology we are considering, i.e.
$$c_1(X) = \sum_{j:|\Delta_j|=2} r_j \Delta_j.$$

Therefore, $E|_{x_i=0} = c_1(X)$. We then have the problem of understanding what we can say about semisemplicity when we know so little. We shall first need a lemma which, for the sake of simplicity, we here prove for one variable only.

2.4.2 LEMMA *Let $P_x(t) = t^n + a_{n-1}(x)t^{n-1} + \ldots + a_0(x), a_i(x) \in \mathbb{C}[[x]]$. Suppose that $P_0(t) = t^n + a_{n-1}(0)t^{n-1} + \ldots + a_0(0)$ has only simple roots. Then for each simple root t_0 there exists a solution $t(x, t_0) \in \mathbb{C}[[x]]$ to $P_x(t) = 0$ such that $t(0, t_0) = t_0$.*

Proof. Suppose we have a solution $\bmod (x^N)$. This can be written as $t_{N-1} = t_0 + \alpha_1 x + \ldots + \alpha_{N-1} t^{N-1}$. We want to show that there exists a solution $\bmod (x^{N+1})$, which extends t_{N-1}. We denote $t_N = t_{N-1} + \alpha_N x^N$ and write down the equation $P_x(t_N) = 0$. We know that the equation is true $\bmod (x^N)$, so that we only need to show the existence of an α_N such that the coefficient of x^N in $P_x(t_N)$ be $0 \bmod (x^N)$. If we expand $P_x(t_N)$ in its Taylor series, we find that this coefficient is

$$P_0'(t_0)\alpha_N + A_N(t_0, \alpha_1, \ldots, \alpha_{N-1}),$$

where A_N is a polynomial which is easy to compute. We have reached our goal once we set

$$\alpha_N = \frac{A_N}{P_0'(t_0)},$$

which is possible because P_0 has only simple roots. Therefore, there is a unique way to extend the solution. Now, since $\bigcap_{r>0}(x^r) = 0$, we see that $\bar{t} = t_0 + \alpha_1 x + \alpha_2 x^2 + \ldots$ is in fact a solution to $P_x(t) = 0$ in $\mathbb{C}[[x]]$. □

2.4.3 THEOREM *Let H be a formal Frobenius manifold with metric g and potential Φ. Suppose that the operator $\mathcal{E}_0 : H \to H$ defined by $\mathcal{E}_0(h) = E(0) *_0 h$ has distinct non zero eigenvalues. Then H is semisimple.*

Proof. Denote the eigenvalues and the eigenvectors of $\mathcal{E}_0$ by v_i and ε_i, respectively. By the previous lemma we know that the v_i extend to eigenvalues u_i of $\mathcal{E}$ such that $u_i(0) = v_i$. By construction, the $u_i - v_i$ will be coordinates on the tangent sheaf, and the corresponding eigenvectors e_i of $\mathcal{E}$ will be a basis for the tangent sheaf because $e_i(0) = \varepsilon_i$. On the other hand, $\eta_i(0) = g(e_i, e_i)(0) = g(e_i(0), e_i(0)) = g(\varepsilon_i, \varepsilon_i)$ which is invertible since the ε_i are a basis for H and g is nondegenerate. Therefore we can apply the theorem above. □

Chapter 3

Conformal algebras and KdV

3.1 Basic definitions

DEFINITION 3.1 *A* conformal algebra *is a* $\mathbb{C}[\partial]$*-module* L *endowed with a* $\mathbb{C}$*-bilinear map (called the* λ-bracket*)*

$$\begin{aligned} L \otimes L &\longrightarrow \mathbb{C}[\partial, \lambda] \otimes L \\ a \otimes b &\longmapsto \{a_\lambda b\} = \sum_k \lambda^k B_k(a, b) \end{aligned}$$

satisfying

- $\{\partial a_\lambda b\} = -\lambda \{a_\lambda b\}$,
- $\{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\}$,
- $\{b_\lambda a\} = -\{a_{-\partial-\lambda} b\}$,
- $\{a_\lambda \{b_\mu c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\} + \{b_\mu \{a_\lambda c\}\}$.

Such an algebra can be seen as a Lie algebra over the category of $\mathbb{C}[\partial]$-modules, and a whole theory of Lie algebras over appropriate categories can be built in a very general setting (see [BD, BDK01, DK98]). Our goal is to define a *conformal Poisson* structure. For this we need a commutative and unitary[1] product on L

$$\begin{aligned} L \times L &\longrightarrow L \\ (a, b) &\longmapsto ab \end{aligned} \tag{3.1}$$

[1] The existence of a unit is not resctrictive.

that satisfies the Leibniz rule $\partial(ab) = (\partial a)b + a\partial b$. We also need to specify how the structure of $\mathbb{C}[\partial]$-module behaves when both products occur: if we have $\{a_\lambda b\} = \sum_k \lambda^k B_k(a, b)$, with $B_k(a, b) \in L$, we shall write

$$\{a_\lambda b\}c = \sum_k (\lambda^k c) B_k(a, b) \tag{3.2}$$

and

$$c\{a_\lambda b\} = (\{a_\lambda b\})c = \sum_k (\lambda^k B_k(a, b))c. \tag{3.3}$$

In this way we can read $\{a_\lambda b\}$ as $\{a_\lambda b\}1 = \{a_{\partial+\lambda} b\}1$.

DEFINITION 3.2 *A* conformal Poisson algebra *is a conformal algebra L with a commutative and unitary product (3.1) such that*

$$\begin{aligned} \{ab_\lambda c\} &= \{a_{\partial+\lambda} c\}b + \{b_{\partial+\lambda} c\}a \\ \{a_\lambda bc\} &= b\{a_\lambda c\} + c\{a_\lambda b\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \{\partial a_{\partial+\lambda} b\}c &= \{a_{\partial+\lambda} b\}(-\partial - \lambda)c, \\ \{a_{\partial+\lambda} \partial b\}c &= (\partial + \lambda)(\{a_{\partial+\lambda} b\}c). \end{aligned} \tag{3.5}$$

A typical case is when we have a conformal algebra L and consider the conformal Poisson structure on the free algebra generated by L induced by (3.4) and (3.5). For example, we shall often have a conformal structure on the free 1-dimensional $\mathbb{C}[\partial]$-module (call u its generator) $L = \bigoplus_{i\geq 0} \mathbb{C}\partial^i u$ and use (3.4) and (3.5) to give $\mathbb{C}[u, \partial u, \partial^2 u, \ldots]$ a conformal Poisson structure.

The motivation for this definition lies in all the mathematics involved in the definition of Poisson brackets over infinite dimensional manifolds, for example loop spaces. In such cases, the Poisson algebra is that of differential polynomials in functions $u_i(x)$, where $i \in \{1, \ldots, r\}$ and x varies on a compact manifold (S^1 in the case of loop spaces), and the bracket $\{u_i(x), u_j(y)\}$ lives only along the diagonal:

$$\{u_i(x), u_j(y)\} = \sum_k B_{ijk}(u(y)) \partial_y^k \delta(x - y). \tag{3.6}$$

Then, using the Leibniz rule, the bracket extends to all differential polynomials:

$$\{P(x), Q(y)\} = \sum_{h,k,i,j} \frac{\partial P(x)}{\partial u_i^{(h)}} \frac{\partial Q(y)}{\partial u_j^{(k)}} \partial_x^k \partial_y^h \{u_i(x), u_j(y)\}. \tag{3.7}$$

Therefore,

$$\{PQ, R\} = \sum_{h,k,i,j} \left(P(x) \frac{\partial Q(x)}{\partial u_i^{(h)}} + Q(x) \frac{\partial P(x)}{\partial u_i^{(h)}} \right) \frac{\partial R(y)}{\partial u_j^{(k)}} \partial_x^k \partial_y^h \{u_i(x), u_j(y)\} =$$
$$= \sum_{h,k,i,j,l} P(x) \frac{\partial Q(x)}{\partial u_i^{(h)}} \frac{\partial R(y)}{\partial u_j^{(k)}} \partial_x^k \partial_y^h (B_{ijl}(y) \delta^{(l)}(x-y)) + (P \leftrightarrow Q) =$$
$$= \sum_{h,k,i,j,l} \frac{\partial Q(x)}{\partial u_i^{(h)}} \frac{\partial R(y)}{\partial u_j^{(k)}} \partial_y^h (B_{ijl}(y) P(x) \delta^{(l+k)}(x-y)) + (P \leftrightarrow Q),$$

where $(P \leftrightarrow Q)$ denotes the same summand with P and Q exchanged. Recalling that

$$f(x)\delta^{(p)}(x-y) = \sum_{q=0}^{p} \binom{p}{q} f^{(q)}(y) \delta^{(p-q)}(x-y)$$

and applying the Fourier transform[2] $\delta^{(k)}(x-y) \to \lambda^k$ yields formula (3.4).

In particular we shall be interested in the case studied by Dubrovin and Zhang in [DZ, DZ98b, DZ98a, DZ99]. They consider Poisson brackets of the form (3.6) on the loop space of a manifold M: $\mathcal{L} = \mathcal{L}(M) = \{\varphi : S^1 \to M\}$ (M will later need to be a Frobenius manifold). In order to do this they identify the functions on $\mathcal{L}$ with elements of the algebra

$$\mathcal{A} = \mathbb{C}[u_i^{(j)}]_{i=1,\ldots,n; j\geq 1} \otimes \mathcal{D},$$

where $\mathcal{D}$ is the space of formal distributions on M. We shall write u_i' for $u_i^{(1)}$ and so on, whenever possible. Also, the (smooth) coefficients coefficients of the polynomials (that lie in $\mathcal{L}$ by definition) are written as $u(x) = (u_1(x), \ldots, u_n(x))$. The derivation with respect to the parameter of S^1 then acts in the natural way: for $f \in \mathcal{A}$ we set

$$\partial_x f = \frac{\partial f}{\partial x} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} u_i' + \sum_{i=1}^{n} \sum_{j \geq 1} \frac{\partial f}{\partial u_i^{(j)}} u_i^{(j+1)}.$$

In such a setting, given a Poisson structure (3.6), we can define a conformal Poisson structure on

$$L = \mathcal{O} \otimes \mathbb{C}[\partial^j u_i]_{i=1,\ldots,n; j\geq 1} \tag{3.8}$$

[2]See [BDK01] for full details.

by setting

$$\{u_{i\lambda}u_j\} = \sum_k \lambda^k B_{ijk}(u) \tag{3.9}$$

and then extending the λ-bracket to the whole algebra using (3.4) and (3.5). Here $\mathcal{O}$ denotes the ring of analytic functions of the variables $u_1, \ldots, u_n$, or, in case we are working on a *formal* Frobenius manifold, the ring of formal power series in the variables $u_1, \ldots, u_n$. In this procedure we *do not lose any information*. This is due to the fact there exists a construction that allows us to go back to the original Poisson structure. We briefly review it here, the main reference being [DK98]. The first thing to do is to look at the conformal algebra on the $\mathbb{C}[\partial]$-module generated by $u_1, \ldots, u_n$, R. For any $a, b \in R$ and any non-negative integer n, define the product $\cdot_{(n)}\cdot : R \otimes R \to R$ by

$$\{a_\lambda b\} = \sum_{n \geq 0} \lambda^n a_{(n)} b. \tag{3.10}$$

These products can obviously be easily expressed in terms of the $B_{ijk}(u)$ above, but it is not essential here. The axioms of conformal algebra translate into the following set of equations for these products

- $a_{(n)}b = 0$ for $n \gg 0$,
- $(\partial a)_{(n)}b = -na_{(n-1)}b$,
- $a_{(n)}(\partial b) = \partial(a_{(n)}b) + na_{(n-1)}b$,
- $a_{(n)}b = -\sum_{j \geq 0}(-1)^{n+j}\frac{1}{j!}\partial^j(b_{(n+j)}a)$,
- $a_{(m)}(b_{(n)}c) - b_{(n)}(a_{(m)}c) = \sum_{j=0}^{m}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}c$.

It is clear that a set of products satisfying them is equivalent to a structure of conformal algebra. We can now define

$$\mathfrak{Lie}(R) = R[t, t^{-1}]/(\partial + \partial_t)R[t, t^{-1}].$$

To give $\mathfrak{Lie}(R)$ a Lie structure, write a_n for at^n and define

$$[a_m, b_n] = \sum_{j \in \mathbb{Z}_+}\binom{m}{j}(a_{(j)}b)_{m+n-j}.$$

The family

$$\{a(z) = \sum_{n \in Z} a_n z^{-n-1}\}$$

spans $\mathfrak{Lie}(R)$. Now this Lie algebra can be seen to coincide with the Lie algebra $(\mathcal{A}, \{\cdot\,,\cdot\})$ defined above, thus showing that there is no information loss in the passage to the conformal setting.

Another few words must be spent on formula (3.2) to make its meaning clearer.

3.1.1 LEMMA *For all $k \geq 1$ and all $u \in L$,*

$$\{u^k{}_\lambda u\} = k\{u_{\partial+\lambda} u\} u^{k-1} \tag{3.11}$$

holds.

Proof. We use induction. The first step is clear: $\{u_\lambda u\} = \{u_{\partial+\lambda} u\}1$ and $\partial 1 = 0$. Next, suppose that $\{u^{k-1}{}_\lambda u\} = (k-1)\{u_{\partial+\lambda} u\} u^{k-2}$. We know from (3.2) that

$$\{u^k{}_\lambda u\} = \{u^{k-1}{}_{\partial+\lambda} u\} u + \{u_{\partial+l} u\} u^{k-1}.$$

We therefore only need to prove that

$$\{u^{k-1}{}_{\partial+\lambda} u\} u = (k-1)\{u_{\partial+\lambda} u\} u^{k-1}. \tag{3.12}$$

By setting $B_j = B_j(u, u)$, our inductive hypothesis is

$$\sum_{h \geq 0} B_h(u^{k-1}, u)\lambda^h = (k-1)\sum_{j \geq 0} B_j(\partial+\lambda)^j u^{k-2}.$$

The coefficient $B_h(u^{k-1}, u)$ of λ^h in this formula is equal to

$$B_h(u^{k-1}, u) = (k-1)\sum_{a \geq h} B_a \binom{a}{h} \partial^{a-h} u^{k-2}.$$

Formula (3.12) isequivalent to

$$\sum_{h \geq 0} B_h(u^{k-1}, u)(\partial+\lambda)^h u = (k-1)\sum_{j \geq 0} B_j(\partial+\lambda)^j u^{k-1}. \tag{3.13}$$

The left hand side is equal to

$$\sum_{h \geq 0}\sum_{a \geq h} B_a \binom{a}{h} \partial^{a-h} u^{k-2}(\partial+\lambda)^h u = \sum_{h \geq 0}\sum_{a \geq h}\sum_{b=0}^{h} B_a \binom{a}{h} \partial^{a-h} u^{k-2} \partial^b u \lambda^{h-b}.$$

Now, letting $c = a - h + b$, we can rewrite this as

$$\sum_{a\geq 0}\sum_{c=0}^{a}\sum_{b=0}^{c} B_a \binom{a}{c}\binom{c}{b}\partial^{c-b}u^{k-2}\partial^b u\lambda^{a-c} =$$
$$= \sum_{a\geq 0}\sum_{c=0}^{a} B_a \binom{a}{c}\partial^c u^{k-1}\lambda^{a-c}$$
$$= \sum_{a\geq 0} B_a(\partial+\lambda)^a u^{k-1}$$

thus proving (3.13). □

We omit the proof of the next two lemmas, since the technicalities involved are slightly more complicated, but of the same nature.

3.1.2 Lemma *For any given $a_1,\ldots,a_N,b_1,\ldots,b_M \in L$,*

$$\{\prod_{i=1}^{N} a_{i\lambda}\prod_{j=1}^{M} b\} = \sum_{i=1}^{N}\sum_{j=1}^{M}(\prod_{k\neq j} b_k)\{a_{i\partial+\lambda}b_j\}(\prod_{h\neq i} a_h) \tag{3.14}$$

holds.

Let $\{u_i\}$ be a basis of L as a $\mathbb{C}[\partial]$-module. Let $\partial_{i,k}$ be the derivation of L defined by

$$\partial_{i,k}(\partial^h u_j) = \begin{cases} 1 & \text{if } i=j \text{ and } h=k \\ 0 & \text{otherwise.}\end{cases}$$

Notice that $\partial_{i,k}\partial = \partial\partial_{i,k} + \partial_{i,k-1}$ for all i,k (where $\partial_{i,-1} = 0$).

3.1.3 Lemma *For all $F, G \in L$,*

$$\{F_\lambda G\} = \sum_{i,j,h,k}\partial_{j,k}G(\partial+\lambda)^k(\{u_{i\partial+\lambda}u_j\}(-\partial-\lambda)^h\partial_{i,h}F) \tag{3.15}$$

holds.

Example 3.1 Take $L = \mathbb{C}[\partial]$ with generator v. Define

$$\{u_\lambda u\} = \lambda \tag{3.16}$$

(the right way to write this would be $\{u_\lambda u\} = \lambda \cdot c$, where c is a non-zero multiple of the unit $1 \in \mathbb{C} \subset \mathbb{C}[\partial]$, but we shall omit the c throughout what follows as we can rescale u so as to have $c = 1$). If we consider the *Miura transformation*[3]

$$v = \frac{1}{4}u^2 + \partial u, \tag{3.17}$$

we can carry out the calculations to write down $\{v_\lambda v\}$:

$$\{v_\lambda v\} = \frac{1}{16}\{u^2{}_\lambda u^2\} + \frac{1}{4}\{u^2{}_\lambda \partial u\} + \frac{1}{4}\{\partial u_\lambda u^2\} + \{\partial u_\lambda \partial u\}.$$

The various terms are:

$$\begin{aligned}
\{u^2{}_\lambda u^2\} &= 4u\{u_{\partial+\lambda}u\}u = 4u(\partial+\lambda)u = 4u\partial u + 4\lambda u^2 \\
\{u^2{}_\lambda \partial u\} &= 2(\partial+\lambda)\{u_{\partial+\lambda}u\}u = 2(\partial+\lambda)^2 u, \\
\{\partial u_\lambda u^2\} &= 2u\{u_{\partial+\lambda}u\}(-\partial-\lambda)1 = -2u(\partial+\lambda)^2 1 = -2\lambda^2 u, \\
\{\partial u_\lambda \partial u\} &= -\lambda(\partial+\lambda)\lambda = -\lambda^3.
\end{aligned}$$

We then find

$$\{v_\lambda v\} = \frac{1}{4}u\partial u + \frac{1}{4}\lambda^2 u + \frac{1}{2}\partial^2 u + \lambda\partial u - \lambda^3 = \frac{1}{2}\partial v + \lambda v - \lambda^3. \tag{3.18}$$

We emphasize the fact that particular care must be taken in writing down the parentheses because of the Leibniz rule. This is the meaning of (3.2) and (3.3).

Let $C = L/\partial L$ and denote by

$$\textstyle\int : L \longrightarrow C$$

the natural projection[4]. C has a natural Lie algebra structure. To describe it, we must first introduce variational derivatives: define δ_i as the differential operator

$$\delta_i = \sum_{k\geq 0}(-\partial)^k \partial_{i,k}.$$

Then $\delta_i \partial F = 0$, because

$$\begin{aligned}
\delta_i \partial F = &\sum_{k\geq 0}(-\partial)^k \partial_{i,k}\partial F = \sum_{k\geq 0}\partial(-\partial)^k \partial_{i,k}F + \\
&+ \sum_{h\geq 0}(-\partial)^h \partial_{i,h-1}F = \partial\delta_i F - \partial\delta_i F = 0.
\end{aligned}$$

---

[3] In general a Miura transformation is a map $u_i \mapsto \sum_{k\geq 0} F^i_k(u)$, where F^i_k involves derivatives up to the k-th order.

[4] The notation is like this because in practice integrals of rapidly decreasing functions or integrals over the circle are often to be considered. We shall refer to elements in the quotient as *integrals*.

Let $B = (B_{ij}(u, \lambda))$ be the matrix with coefficients in $L \otimes \mathbb{C}[\lambda]$ given by the λ-bracket, i.e.

$$\{u_{i\lambda} u_j\} = B_{ij}(u, \lambda).$$

We can now write down the expression of the Lie bracket on integrals

$$\begin{aligned} [\cdot\,,\cdot] : C \otimes C &\longrightarrow C \\ \textstyle\int F \otimes \int G &\longmapsto [\textstyle\int F, \int G] = \int (B|_{\lambda=\partial} \delta F) \delta G, \end{aligned} \tag{3.19}$$

where δF denotes the vector whose components are the $\delta_i F$. In example 3.1, if we choose v as a generator, the Lie structure on itegrals is

$$[\textstyle\int F(v), \int G(v)] = \int (\partial \delta_v F(v)) \delta_v G(v).$$

If we take u as a generator (it is *not* a generator, since transormation (3.17) is not invertible, but we shall deal with this problem later), the Lie structure on itegrals is

$$[\textstyle\int F(u), \int G(u)] = \int (\frac{1}{2} \partial u \cdot \delta_u F(u) + \partial \delta_u F(u) - \partial^3 \delta_u F(u)) \delta_u G(u).$$

3.1.4 THEOREM *The bracket on integrals can be expressed as*

$$[\textstyle\int F, \int G] = \int \{F_\lambda G\}|_{\lambda=0}. \tag{3.20}$$

Proof. Let $\{u_i\}_{i=1,\dots,n}$ be a basis for L. By formula (3.15), we can write

$$\{F_\lambda G\}|_{\lambda=0} = \{F_0 G\} = \sum_{i,j=1}^{n} \sum_{h,k \geq 0} \partial_{j,k} G \partial^k (\{u_{i\partial} u_j\}(-\partial)^h \partial_{i,h} F).$$

In the quotient and using integration by parts (which, in our case tells us that $\int \partial F G = - \int F \partial G$), we get

$$\begin{aligned} \textstyle\int \{F_\lambda G\}|_{\lambda=0} &= \int \sum_{i,j=1}^{n} \sum_{k \geq 0} (-\partial)^k \partial_{j,k} G \{u_{i\partial} u_j\} \delta_i F \\ &= \int \sum_{i,j=1}^{n} \delta_j G B_{ij}(u, \partial) \delta_i F = \textstyle\int \delta G (B|_{\lambda=\partial} \delta F). \end{aligned}$$

□

3.1.5 PROPOSITION *Bracket (3.19) defines a Lie algebra structure on C.*

Proof. The bilinearity is obvious.

$$[\int G, \int F] = \int \{G_0 F\} = -\int \{F_{-\partial} G\} = -\int \{F_0 G\} = -[\int F, \int G]$$

because $\partial \equiv 0$ on C. The Jacobi identity is immediate as well:

$$\begin{aligned}[\int F, [\int G, \int H]] &= [\int F, \int \{G_0 H\}] = \int \{F_0 \{G_0 H\}\} \\ &= \int \{\{F_0 G\}_0 H\} + \int \{G_0 \{F_0 H\}\} \\ &= [\int \{F_0 G\}, \int H] + [\int G, \int \{F_0 H\}] \\ &= [[F, G], H] + [G, [F, H]].\end{aligned}$$

□

DEFINITION 3.3 *A pencil of conformal Poisson structures on a $\mathbb{C}[\partial]$-module L is a pair of conformal Poisson structures*

$$\{\cdot_\lambda \cdot\}_1, \{\cdot_\lambda \cdot\}_2 : L \otimes L \longrightarrow L \otimes \mathbb{C}[\lambda]$$

such that for any $t \in \mathbb{C}$ the map

$$\{\cdot_\lambda \cdot\}_{(t)} = \{\cdot_\lambda \cdot\}_1 - t\{\cdot_\lambda \cdot\}_2 \tag{3.21}$$

yields a conformal Poisson structure on L.

3.1.6 REMARK A pencil of conformal Poisson structures on L clearly descends to a pencil of Lie structures on C.

3.2 Drawing hierarchies with pencils

We give here the construction of an infinite family of partial differential equations. The procedure we carry out is due to Lenard and Magri (see [GGKM74] and [Mag78] for the full details). Suppose now that the $\mathbb{C}[\partial]$-module L is endowed with a pencil of conformal poisson structures (3.21). Also suppose that there exist a number $N \geq 1$ and N independent elements of L $c^1_{-1}, \dots, c^N_{-1}$ such that

$$[\int c^\alpha_{-1}, \cdot]_1 \equiv 0. \tag{3.22}$$

Such an element is called a *Casimir element* (or simply a *casimir*) of the first Poisson structure. We can then construct, for all $\alpha \in \{1, \dots, N\}$, a family of elements c^α_i by imposing

$$[\int c^\alpha_i, \cdot]_1 = [\int c^\alpha_{i-1}, \cdot]_2. \tag{3.23}$$

We then write down the *associated KdV-type hierarchy* by

$$\frac{du}{dt_{k,\alpha}} = \{c^\alpha_k {}_\lambda u\}_1|_{\lambda=0}. \tag{3.24}$$

Example 3.2 Let $L = \mathbb{C}[\partial]$ with generator u. Define the Poisson pencil by

$$\begin{aligned} \{u_\lambda u\}_1 &= \lambda \\ \{u_\lambda u\}_2 &= \frac{1}{2}\partial u + \lambda u - \lambda^3. \end{aligned} \tag{3.25}$$

There exists a unique casimir of the first Poisson structure, namely $c_{-1} = u$, since

$$[\textstyle\int u, \int G]_1 = \int (\lambda|_{\lambda=\partial}\delta u)\delta G = \int (\partial 1)\delta G = 0$$

for all $G \in L$. We then find

$$[\textstyle\int c_{-1}, \int G]_2 = \int (\frac{1}{2}\partial u \delta u + u\partial\delta u - \partial^3 \delta u)\delta g = \int \frac{1}{2} u \partial u \delta G,$$

so that, since

$$[\textstyle\int c_o, \int G]_1 = \int (\partial \delta c_0)\delta G,$$

we must have

$$\partial\delta c_0 = \frac{1}{2}\partial u.$$

Therefore,

$$\delta c_0 = \frac{1}{2} u$$

and

$$c_0 = \frac{u^2}{4}.$$

In a similar manner we can find

$$c_1 = \frac{1}{8}u^3 + \frac{1}{4}(\partial u)^2$$

and all the other c_i. So, the first two equations of the hierarchy are

$$\begin{aligned} \frac{du}{dt_0} &= \{c_{0\lambda} u\}_1|_{\lambda=0} = \frac{1}{2}\partial u, \\ \frac{du}{dt_1} &= \{c_{1\lambda} u\}_1|_{\lambda=0} = \frac{3u\partial u}{8} - \frac{\partial^3 u}{2}. \end{aligned} \tag{3.26}$$

The second of these equations is the original KdV equation (modulo some rescaling). In fact, all the hierarchy of this pencil coincides with the original KdV hierarchy.

The case we are interested in is when in a given basis $\{u_i\}$ of L, the first Poisson structure has the form

$$\{u_{i\lambda}u_j\}_1 = \eta^{ij}\lambda$$

for some invertible symmetric matrix (η^{ij}).

3.2.1 Remark There is a certain freedom in taking the c_i^α: we can choose the c_{-1}^α and replace them by a suitable number of linear combinations of the c_{-1}^α. Then, in the recursive procedure, we can add derivatives to the other c_i^α because $\delta\partial \equiv 0$.

3.3 Grading and dispersionless limits

Let $\mathcal{A}$ be a free $\mathbb{C}[\partial]$-module. Let $\{u_i\}_{i=1,\dots,n}$ be a basis for $\mathcal{A}$ (this is meant in a broad sense: if $\mathcal{A}$ is of finite rank then $\{u_i\}_{i=1,\dots,n}$ must naturally be a $\mathbb{C}[\partial]$-basis, but if $\mathcal{A}$ is *not* of finite rank, as in the case of (3.8), we require the u_i to generate $\mathcal{A}$ in a sense that varies from case to case). Assign degree 0 to each u_i and degree 1 to ∂. Introduce a formal parameter ε of degree -1 and define L to be formal Laurent series with coefficients in $\mathcal{A}$, i.e. $L = \mathcal{A}\otimes\mathbb{C}[[\varepsilon]][\varepsilon^{-1}]$. In order to define a λ-bracket on L which makes some sense from the graded point of view, we need to give λ degree 1 and define the bracket as a bilinear map

$$L \otimes L \longrightarrow (\mathbb{C}[\lambda] \otimes L)[-1],$$

satisfying the conditions of definition 3.1. Here we have denoted by $[-1]$ the shift on graded vector spaces: given a graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and an integer $a \in \mathbb{Z}$, denote by $V[a]$ the graded vector space whose degree i part is $V[a]_i = V_{i+a}$. This definition of grading corresponds to giving degree $k+1$ to the k-th derivative of the δ-function in expressions like (3.6) and (3.7).

Suppose that we have a Poisson structure on L. We shall write this as a formal sum

$$\{u_{i\lambda}u_j\} = \sum_{k\geq -1} \varepsilon^k \{u_{i\lambda}u_j\}^{[k]}, \tag{3.27}$$

with each $\{u_{i\lambda}u_j\}^{[k]}$ being a Poisson structure on $\mathcal{A}$ such that the polynomials $B_{ij}^{[k]}(u,\lambda)$ defining the brackets

$$\{u_{i\lambda}u_j\}^{[k]} = B_{ij}^{[k]}(u,\lambda)$$

are of degree $k + 2$.

The Miura transormations are now

$$u_i \longmapsto \sum_{k\geq 0} \varepsilon^k F^i_k(u)$$

with F^i_k of degree k. These transformations can now be inverted. This is one of the reasons for the introduction of the formal parameter ε. The other reason concerns the hierarchies of PDEs: if we have a pencil of conformal Poisson structures on L, as before we can write down the hierarchy. Example 3.2 would now read

$$\{u_\lambda u\}_1 = \lambda$$
$$\{u_\lambda u\}_2 = \frac{1}{2}\partial u + \lambda u - \varepsilon^2\lambda^3$$

and the second equation of the hierarchy would be

$$\frac{du}{dt_1} = \frac{3u\partial u}{8} - \varepsilon^2\frac{\partial^3 u}{2}$$

(we are actually rescaling $t_1 \mapsto \varepsilon t_1$). The *dispersionless limit* of the hierarchy is simply the hierarchy itself calculated in $\varepsilon = 0$. In this limit, the equation above becomes

$$\frac{du}{dt_1} = \frac{3u\partial u}{8}.$$

After rescaling we find the form in which this equation is usually written

$$\dot{u} = uu'.$$

3.4 From conformal algebras to Lie algebras

There exists another construction of Lie algebras using conformal algebras as ingredients, which is due to A. D'Andrea but was never published. It will be of particular interest to us and we therefore include it. Given a conformal algebra L, and given $a \in L$ we can define a_λ as the λ-adjoint of a, i.e.

$$a_\lambda(b) = \{a_\lambda b\}.$$

Then, after choosing an additive subgroup G of the additive group of the field we are working on, we have a natural Lie algebra structure on $\mathfrak{g}_G = \{a_\lambda | a \in L, \lambda \in G\}$ given by the Jacobi identity on the conformal algebra:

$$[a_\lambda, b_\mu] = \{a_\lambda b\}_{\lambda+\mu}.$$

Let us show that this actually yields a Lie algebra structure. The bilinearity is obvious.

$$[b_\mu, a_\lambda] = \{b_\mu\{a_\lambda\cdot\}\} - \{a_\lambda\{b_\mu\cdot\}\} = -[a_\lambda, b_\mu]$$

gives us skewcommutativity. To show that the Jacobi identity holds, we write down the three terms

$$\begin{aligned}
[a_\lambda,[b_\mu,c_\nu]] =& \{a_\lambda\{b_\mu\{c_\nu\cdot\}\}\} - \{b_\mu\{c_\nu\{a_\lambda\cdot\}\}\}+\\
&- \{a_\lambda\{c_\nu\{b_\mu\cdot\}\}\} + \{c_\nu\{b_\mu\{a_\lambda\cdot\}\}\}\\
[b_\mu,[a_\lambda,c_\nu]] =& \{b_\mu\{a_\lambda\{c_\nu\cdot\}\}\} - \{b_\mu\{c_\nu\{a_\lambda\cdot\}\}\}+\\
&- \{a_\lambda\{c_\nu\{b_\mu\cdot\}\}\} + \{c_\nu\{a_\lambda\{b_\mu\cdot\}\}\}\\
[[a_\lambda,b_\mu],c_\nu] =& \{a_\lambda\{b_\mu\{c_\nu\cdot\}\}\} - \{b_\mu\{a_\lambda\{c_\nu\cdot\}\}\}+\\
&- \{c_\nu\{a_\lambda\{b_\mu\cdot\}\}\} + \{c_\nu\{b_\mu\{a_\lambda\cdot\}\}\}.
\end{aligned}$$

By adding up the second and the third we get the first one, thus obtaining what we needed.

Let us look at what this construction does for the two conformal structures of example 3.2. Take the first structure $\{u_\lambda u\}_1 = \lambda c$, where c is the central element ($\{c_\lambda\cdot\} \equiv 0$ unless $\lambda = 0$, in which case it gives a constant that we call $\hbar$). Then we have

$$[u_\lambda, u_\mu]_1 = \{u_\lambda u\}_{\lambda+\mu} = \lambda c_{\lambda+\mu} = \lambda\hbar\delta_{\lambda,-\mu}.$$

By choosing $G = \mathbb{Z}$ as our group, we get the *Heisenberg algebra* with central charge $\hbar$. Taking the second structure, $\{u_\lambda u\}_2 = (\frac{1}{2}\partial + \lambda)u$, yields

$$[u_\lambda, u_\mu]_2 = (\frac{1}{2}\partial u + \lambda u)_{\lambda+\mu} = \frac{1}{2}(-\lambda-\mu)u_{\lambda+\mu} + \lambda u_{\lambda+\mu} = \frac{1}{2}(\lambda-\mu)u_{\lambda+\mu}$$

which, by taking $G = \mathbb{Z}$, is (isomorphic to) the *Virasoro algebra* with no central charge.

3.5 The Dubrovin-Zhang framework

We have introduced the algebra studied by Dubrovin and Zhang in (3.8) above. We now wish to present some of their results from our purely algebraic point of view. Let $M = \mathbb{C}^n$ be a (possibly be formal) Frobenius manifold, let Φ be its potential, g its metric, $\{x_1, \dots x_n\}$ a system of flat coordinates, $\partial_k = \partial/\partial x_k$ its corresponding flat vector fields and $E = \sum_{k=1}^n E^k\partial_k$ its Euler field. We fix notation $g_{ij} = g(\partial_i, \partial_j)$, $(g^{ij}) = (g_{ij})^{-1}$. We use, as in chapter 2, the matrices (g_{ij}) and (g^{ij}) to raise and lower indices. Define the matrix

$$h^{ij} = \sum_{h=1}^{n} E^h \Phi_h^{ij}.$$

Notice that the entries of this matrix are functions of the x_i. Further, define

$$\widehat{\Gamma}_k^{ij} = \sum_{h=1}^{n} \Phi_k^{ih} \left(\frac{d-1}{2} - \widehat{\nabla} E \right)_h^j,$$

where d is the charge of the Euler field and $\widehat{\nabla}$ is the extended structure connection of (2.9). Consider the differential algebra L of (3.8) (in which we replace the u_i and their derivatives with the x_i and the respective detivatives). We then have a pencil of conformal Poisson structures given by

$$\begin{aligned} \{x_{i\lambda} x_j\}_1 &= \lambda g^{ij}; \\ \{x_{i\lambda} x_j\}_2 &= \sum_{k=1}^{n} \widehat{\Gamma}_k^{ij} \partial x_k + \lambda h^{ij}(x). \end{aligned} \tag{3.28}$$

The fact that this is indeed a conformal Poisson structure is a mere calculation that we do not include here. Under some hypotheses the implication can be reversed, in the sense that it can be shown (see [DZ] for full details) that every pencil of conformal Poisson algebras of the form (3.28) satisfying some integrability conditions comes from a Frobenius manifold or from a degenerate version of a Frobenius manifold. In [DZ] it is also shown that the semisimplicity of the Frobenius manifold ensures the integrability of the hierarchy of PDEs given by (3.24). We shall not treat these topics here, but plan to consider them in the next future.

Let us now concentrate on a practical case. Let $M = QH^*(\mathbb{P}^1)$. Recall from example 2.1 that the multiplication table reads $\partial_x * \partial_x = \partial_x$, $\partial_x * \partial_y = \partial_y$, $\partial_y * \partial_y = e^y \partial_x$. The Euler field is $E = x\partial_x + 2\partial_y$. The Poincaré pairing is given by the matrix

$$(g_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The pencil of conformal Poisson structures is then given by

$$\begin{aligned} &\{x_\lambda x\}_1 = 0 && \{x_\lambda y\}_1 = \lambda \\ &\{y_\lambda x\}_1 = \lambda && \{y_\lambda y\}_1 = 0 \end{aligned}$$

$$\begin{aligned} &\{x_\lambda x\}_2 = \partial e^y + 2\lambda e^y && \{x_\lambda y\}_2 = \lambda x \\ &\{y_\lambda x\}_2 = \partial x + \lambda x && \{y_\lambda y\}_2 = 2\lambda. \end{aligned}$$

We wish to write the KdV-type kierarchy associated with this pencil. Let δ_x and δ_y denote variational derivatives with respect to x and y, respectively. The two Lie structures on the integrals are given by

$$\begin{aligned}[\int F, \int G]_1 &= \int \delta_x G \partial \delta_y F + \delta_y G \partial \delta_x F \\ [\int F, \int G]_2 &= \int \delta_x G((\partial e^y + 2e^y \partial)\delta_x F + \\ &\quad + x\partial \delta_y F) + \delta_y G((\partial x + x\partial)\delta_x F + 2\partial \delta_y F).\end{aligned}$$

The casimirs of the first bracket are then $c^1_{-1} = x$ and $c^2_{-1} = y$. Notice that $[\int c^2_{-1}, \cdot] \equiv 0$. We can therefore choose $c^2_0 = x$ and get $c^2_n = c^1_{n-1}$. All we have to calculate is then $c_n = c^1_n$ starting from $c_{-1} = x$. The recursive relation (3.23) now reads

$$\begin{aligned}\int \delta_x G \partial \delta_y c_{n+1} + \delta_y G \partial \delta_x c_{n+1} = \int \delta_x G((\partial e^y + 2e^y \partial)\delta_x c_n + x\partial \delta_y c_n) + \\ + \delta_y G((\partial x + x\partial)\delta_x c_n + 2\partial \delta_y c_n),\end{aligned}$$

so that we must have

$$\partial \delta_x c_{n+1} = (\partial x + x\partial)\delta_x c_n + 2\partial \delta_y c_n = \partial(x\delta_x c_n + 2\delta_y c_n)$$

and therefore

$$\delta_x c_{n+1} = x\delta_x c_n + 2\delta_y c_n. \tag{3.29}$$

From this it follows that $\deg_x c_n = n + 2$. We also have

$$\partial \delta_y c_{n+1} = (\partial e^y + 2e^y \partial)\delta_x c_n + x\partial \delta_y c_n). \tag{3.30}$$

The leading coefficient in x is given by $1/(n + 2)$. To show this we use induction; it is true for c_{-1}, so suppose it is true for c_n and use (3.29):

$$\delta_x c_{n+1} = x^{n+3} + \text{lower order terms in } x.$$

On the other hand it can be shown with no greater difficulty that c_n can be written in the form

$$c_n = \frac{x^{n+2}}{n+2} + \sum_{j=1}^{[\frac{n+2}{2}]} \alpha^j_n x^{n+2-2j} e^{jy}. \tag{3.31}$$

This allows us to rewrite the left hand side of (3.29) as

$$\delta_x c_{n+1} = x^{n+2} + \sum_{j=1}^{\left[\frac{n+2}{2}\right]} \alpha_{n+1}^j (n+3-2j) x^{n+2-2j} e^{jy}$$

and the right hand side as

$$x\delta_x c_n + 2\delta_y c_n = x^{n+2} + \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \alpha_n^j (n+2) x^{n+2-2j} e^{jy} + R,$$

where

$$R = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2\alpha_n^{\frac{n+2}{2}}(n+2) & \text{if } n \text{ is even.} \end{cases}$$

We therefore have

$$\sum_{j=1}^{\left[\frac{n+2}{2}\right]} \alpha_{n+1}^j (n+3-2j) x^{n+2-2j} e^{jy} = \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \alpha_n^j (n+2) x^{n+2-2j} e^{jy} + R, \quad (3.32)$$

which implies that

$$\alpha_{n+1}^j = \frac{n+2}{n+3-2j} \cdot \alpha_n^j. \quad (3.33)$$

This allows us to write down the actual values of the α_n^j recursively, but we need to determine the value of the α_n^j that has the lowest n with fixed j. Next, consider the formulae (3.30) and (3.31). The only case in which a term that does not depend on x appears in the right hand side of (3.29) is when n is odd, and this term is given by

$$(\partial e^y + 2e^y\partial)X,$$

where X is the term of $\delta_x c_n$ that does not depend on x, namely,

$$X = \alpha_n^{\frac{n+1}{2}} e^{\frac{n+1}{2}y}.$$

Therefore,

$$\begin{aligned}(\partial e^y + 2e^y\partial)X =& \alpha_n^{\frac{n+1}{2}} \left(e^{\frac{n+1}{2}y} e^y \partial y + 2e^y e^{\frac{n+1}{2}y} \frac{n+1}{2} \partial y \right) = \\ =& \alpha_n^{\frac{n+1}{2}} (n+2) e^{\left(\frac{n+1}{2}+1\right)y} \partial y = 2\alpha_n^{\frac{n+1}{2}} \frac{n+2}{n+3} \partial e^{\frac{n+3}{2}y}.\end{aligned}$$

On the left hand side of (3.30), the only term independent of x is

$$\partial\left(\alpha_{n+1}^{\frac{n+3}{2}}\delta_y e^{\frac{n+3}{2}y}\right),$$

so that we must have

$$\frac{n+3}{2}\alpha_{n+1}^{\frac{n+3}{2}} = 2\alpha_n^{\frac{n+1}{2}}\frac{n+2}{n+3},$$

that we rewrite as

$$\alpha_{n+1}^{\frac{n+3}{2}} = \frac{4(n+2)}{(n+3)^2}\alpha_n^{\frac{n+1}{2}}. \tag{3.34}$$

If we now apply (3.33) to this equation, we get

$$\alpha_{n+1}^{\frac{n+3}{2}} = \frac{4(n+2)(n+1)}{(n+3)^2}\alpha_{n-1}^{\frac{n+1}{2}}. \tag{3.35}$$

If we write down an infinite matrix $(a_{ij})_{i,j\geq 0}$ whose (i,j)-entry a_{ij} is the scalar part of the coefficient of x^j in the expression of c_i (i.e. $a_{ij} = \alpha_i^{\frac{i-j+2}{2}}$ if $i-j \equiv 0 \bmod 2$ and 0 otherwise), then (3.35) allows us to express any term in the 0-th coloumn in terms of the one above, whereas (3.33) allows us to write any entry in terms of the one that is up one place and left another place. The first row reads $1, 0, 1/2, 0, 0, \ldots$, so that the whole matrix is described by (3.33) and (3.35). Combining the two in order to have a closed expression yields

$$\alpha_n^j = 4^{j+1}\binom{n+1}{n+2-2j}\left(\frac{(2j-1)!!}{2j}\right)^2, \tag{3.36}$$

where $x!!$ is the product of the numbers not exceeding x that are congruent to $x \bmod 2$. Let us now write down the hierarchies of partial differential equations. We need to calculate

$$\{c_{n0}x\}_1$$

and

$$\{c_{n0}y\}_1.$$

First of all,

$$\{\frac{x^{n+2}}{n+2}{}_0x\}_1 = \{x_\partial x\}x_1^{n+1} \equiv 0.$$

Secondly,

$$\{x^a e^{by}{}_0 x\}_1 = \{x_\partial x\}_1 \partial_x(x^a e^{by}) + \{y_\partial x\}_1 \partial_y(x^a e^{by}) = b\partial(x^a e^{by})$$

for all a, b. In particular, when we set $a = n + 2 - 2j$ and $b = j$, we get

$$\{x^{n+2-2j} e^{jy}{}_0 x\}_1 = j(n+2-2j)x^{n+1-2j}\partial x e^{jy} + j^2 x^{n+2-2j} e^{jy}\partial y$$

so that

$$\{c_{n0} x\}_1 = \sum_{j=1}^{[\frac{n+2}{2}]} 4^{j+1}\binom{n+1}{n+2-2j}\frac{(2j-1)!!^2}{4j}\times \\ \times ((n+2-2j)x^{n+1-2j}\partial x + jx^{n+2-2j}\partial y)e^{jy}.$$

Therefore, the first part of the hierarchy is given by

$$\frac{dx}{dt_{n,1}} = \sum_{j=1}^{[\frac{n+1}{2}]} 4^j \binom{n+1}{2j}\frac{(2j-1)!!^2}{j} x^{n+1-2j}\partial x e^{jy} + \\ + \sum_{j=1}^{[\frac{n+2}{2}]} 4^j \binom{n+1}{2j-1}(2j-1)!!^2 x^{n+2-2j} e^{jy}\partial y. \tag{3.37}$$

On the other hand,

$$\{\frac{x^{n+2}}{n+2}{}_0 y\}_1 = \partial x^{n+1} = (n+1)x^n \partial x$$

and

$$\{x^{n+2-2j} e^{jy}{}_0 y\}_1 = (n+2-2j)((n+1-2j)x^{n-2j}\partial x e^{jy} + jx^{n+1-2j} e^{jy}\partial y)$$

so that

$$\frac{dy}{dt_{n,1}} = (n+1)x^n\partial x + \sum_{j=1}^{[\frac{n}{2}]} 2^{2j+1}\binom{n+1}{2j+1}\frac{(2j+1)!!(2j-1)}{j} x^{n-2j}\partial x e^{jy} + \\ + \sum_{j=1}^{[\frac{n+1}{2}]} 2^{2j+1}\binom{n+1}{2j}(2j-1)!!^2 x^{n+1-2j} e^{jy}\partial y. \tag{3.38}$$

Since, as we have noticed, $c_n^2 = c_{n-1}^1 = c_{n-1}$ we can write the other half of the hierarchy as

$$\frac{dx}{dt_{n,2}} = \frac{dx}{dt_{n-1,1}} \qquad \frac{dy}{dt_{n,2}} = \frac{dy}{dt_{n-1,1}}. \tag{3.39}$$

To summarize, the hierarchy of partial differential equations determined by the pencil of conformal Poisson structures induced by the quantum cohomology of $\mathbb{P}^1$ is given by (3.37), (3.38) and (3.39).

3.6 The Lie algebra structure on integrals

In this section we want to study the structure of the Lie algebra structure on integrals $C = L/\partial L$ in the rank 1 case $L = \mathbb{C}[u, \partial u, \partial_u^2, \ldots]$, with $\{u_\lambda u\} = \lambda$. We first need a few lemmas. Define, for all $m \geq 1$, $n \geq 0$,

$$A_m^n = \mathtt{Span}_{\mathbb{C}}\{(a_1, \ldots, a_m) | 0 \leq a_1 \leq \ldots \leq a_m, a_1 + \ldots + a_m = n\}.$$

We can also allow the existence of empty sequences by letting $A_0 = \mathbb{C}$. Then

$$A = A_0 \oplus \bigoplus_{\substack{m \geq 1 \\ n \geq 0}} A_m^n.$$

In what follows, our convention will be $A_0^n = \emptyset$ unless $n = 0$, in which case $A_0^0 = A_0$. There is a natural product on A

$$\begin{aligned} \cdot : A_{m_1}^{n_1} \times A_{m_2}^{n_2} &\longrightarrow A_{m_1+m_2}^{n_1+n_2} \\ ((a_1, \ldots, a_{m_1}), (b_1, \ldots, b_{m_2})) &\longmapsto (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}) \end{aligned}$$

where the indices on the right are to be reordered. Notice that this makes A isomorphic to L as an algebra:

$$(a_1, \ldots, a_m) \longmapsto \partial^{a_1} u \cdot \ldots \cdot \partial^{a_m} u$$

and elements in A_0 map to the constants. ∂ then acts on A by

$$\begin{aligned} \partial : A_m^n &\longrightarrow A_m^{n+1} \\ (a_1, \ldots, a_m) &\longmapsto \sum_{i=1}^m (a_1, \ldots, a_i + 1, \ldots, a_m), \end{aligned}$$

whereas ∂_h acts by

$$\begin{aligned}\partial_h : A_m^n &\longrightarrow A_{m-1}^{n-h}\\ (a_1, \ldots, a_m) &\longmapsto \sum_{i:a_i=h} (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m).\end{aligned}$$

These are both derivatives with respect to the product above. The variational derivative is

$$\delta = \sum_{h\geq 0} (-\partial)^h \partial_h : A_m^n \longrightarrow A_{m-1}^n.$$

∂ is clearly injective as long as we stay outside A_0, and $\partial^{-1} A_m^n = A_m^{n-1}$. Therefore,

$$A/\partial A = A_0 \oplus \bigoplus_{m\geq 1} A_m^0 \oplus \bigoplus_{\substack{m\geq 1\\ n\geq 1}} A_m^n/\partial A_m^{n-1}.$$

Denote as usual by $\int : A \to A/\partial A$ the projection. The Lie bracket

$$[\textstyle\int X, \int Y] = \int \delta Y \partial \delta X$$

then behaves in the following way:

$$[\cdot\,,\cdot] : A_m^h/\partial A_m^{h-1} \otimes A_n^k/\partial A_n^{k-1} \longrightarrow A_{m+n-2}^{h+k+1}/\partial A_{m+n-2}^{h+k}.$$

To write down the dimensions of the A_m^n, we need to make a simple remark. An m-tuple of non-decreasing numbers that add up to n can be seen as a pair consisting of its first number a_1 and of an $(m-1)$-tuple of numbers that add up to $n - a_1$. If we subtract a_1 from each term in the latter, we lose no information. And this allows us to think of the m-tuple as a_1 together with an $(m-1)$-tuple whose elements add up to $n - ma_1$. Therefore, letting a_1 vary as much as it may, we can write A_m^n as a *disjoint* union

$$A_m^n = A_{m-1}^n \cup A_{m-1}^{n-m} \cup \ldots \cup A_{m-1}^{n-[\frac{n}{m}]m}.$$

The dimensions $a_m^n = \dim A_m^n$ then add up:

$$a_m^n = \sum_{j=0}^{[\frac{n}{m}]} a_{m-1}^{n-jm}.$$

This enables us to calculate all the dimensions recursively starting from

$$a_1^n = 1.$$

Another way to calculate these numbers is the classical relation due to Euler (see [HW78])

$$\phi_m(x) = \sum_{n\geq 0} a_m^n x^n = \frac{1}{\prod_{j=1}^m (1-x^j)}. \tag{3.40}$$

This enables us to write down $a_m^n - a_m^{n-1}$ as the coefficient of x^n in

$$\psi_m(x) = \phi_m(x) - x\phi_m(x) = \frac{1}{\prod_{j=2}^m (1-x^j)}. \tag{3.41}$$

Take $(a_1, \dots, a_m) \in A$ and define the linear map $T : A \to A$ in the following way: if $a_1 = \dots = a_{h-1} = 0 \neq a_h$ then

$$T(0, \dots, 0, a_h, \dots, a_m) = (a_h - 1, \dots, a_m - 1).$$

Notice that T is injective on the subspace $\{F | \partial_0 F = 0\}$.

3.6.1 LEMMA *For every $F \in A$ such that F is not constant and $\partial_0 F = 0$, the following identities hold:*

1. *$\partial T F = T \partial F$;*
2. *$\partial_k T F = T \partial_{k+1} F$ for all $k \geq 0$.*

Proof. Suppose for simplicity that $F \in A_m^n$. Write

$$F = \sum_{i=1}^{a_m^n} \alpha_i F_i,$$

with $\alpha_i \in \mathbb{C}$ and $\{F_i\}$ the basis of monomials for A_m^n, $F_i = (b_1^i, \dots, b_m^i)$. Then

$$0 = \partial_0 F = \sum_{i: b_1^i = 0} \alpha_i \sharp\{b_j^i = 0\}(b_2^i, \dots, b_m^i).$$

But the $(b_2^i, \dots, a_m^i)$ form a basis for A_{m-1}^n. Therefore the corresponding α_i must be 0. Thus, we can write

$$F = \sum_{i: \partial_0 F_i = 0} \alpha_i F_i.$$

In this way it is clear that we can assume F to be a monomial

$$F = (a_1, \dots, a_m)$$

with $a_j \geq 1$ for all j. Then

$$\begin{aligned}\partial TF =& \partial(a_1 - 1, \ldots, a_m - 1)\\ =& \sum_{i=1}^{m} (a_1 - 1, \ldots, a_{i-1} - 1, a_i, a_{i+1} - 1, \ldots, a_m - 1),\end{aligned}$$

whereas

$$\begin{aligned}T\partial F &= T \sum_{i=1}^{m} (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_m)\\ &= \sum_{i=1}^{m} (a_1 - 1, \ldots, a_{i-1} - 1, a_i, a_{i+1} - 1, \ldots, a_m - 1),\end{aligned}$$

thus proving 1. On the other hand,

$$\begin{aligned}\partial_k TF = \partial_k(a_1 - 1, \ldots, a_m - 1) &= \sharp\{a_i - 1 = k\}(a_1 - 1, \ldots, \widehat{k}, \ldots, a_m - 1)\\ &= \sharp\{a_i = k + 1\} T(a_1, \ldots, \widehat{k+1}, \ldots, a_m) = \partial_{k+1} TF,\end{aligned}$$

where by $\widehat{x}$ we have denoted that x must be omitted. □

3.6.2 LEMMA *Suppose that $\partial_0 F = 0$. Then $\delta F = 0$ if and only if $\delta TF = 0$.*

Proof.

$$\delta TF = \sum_{k \geq 0} (-\partial)^k \partial_k TF = T \sum_{k \geq 0} (-\partial)^k \partial_{k+1} F$$

by lemma 3.6.1 above. Then

$$-\partial \delta TF = T \sum_{k \geq 0} T\delta F - T\partial_0 F.$$

The second term on the right hand side is zero by hypothesis. Therefore,

$$-\partial \delta TF = T\delta F.$$

Suppose that $T\delta F = 0$. Then $\partial \delta TF = 0$, and, since ∂ is injective (there are no constants in this expression), we get $\delta TF = 0$. Conversely, suppose that $\delta TF = 0$. Then $0 = -\partial \delta TF = T\delta F$. Notice that $\partial_0 \delta F = \delta \partial_0 F = 0$. Therefore, T annihilates only the zero, which yields $\delta F = 0$. □

3.6.3 PROPOSITION *Let $F \in A$. Then $\delta F = 0$ if and only if $F \in \partial A$.*

Proof. The δ annihilates derivatives, so that the implication to be proved is that the δ annihilates *only* derivatives. By lemma 3.6.2 above, we can assume that $\partial_0 F \neq 0$. Suppose that $F \in A_m^n$ and use induction on n. By hypothesis,

$$0 = \delta F = \partial_0 F - \partial \sum_{k \geq 0} (-\partial)^k \partial_{k+1} F.$$

If we let $G = \sum_{k \geq 0} (-\partial)^k \partial_{k+1} F$, this reads

$$\partial_0 F = \partial G. \tag{3.42}$$

Now define

$$\begin{aligned} \partial_0^{-1} : A_{m-1}^n &\longrightarrow A_m^n \\ (a_2, \ldots, a_m) &\longmapsto (\sharp\{a_i = 0\} + 1)^{-1}(0, a_2, \ldots, a_m). \end{aligned}$$

Notice that ∂_0^{-1} is right inverse to ∂_0 *but not* left inverse in general. Consider $H = F - \partial\partial_0^{-1} G$. This clearly satisfies $\delta H = 0$. Moreover,

$$\partial_0 H = \partial_0 F - \partial_0 \partial \partial_0^{-1} G = \partial_0 F - \partial \partial_0 \partial_0^{-1} G = \partial_0 F - \partial G = 0,$$

where the second equality holds because $[\partial_0, \partial] = 0$. If $n < m$ then we have found an element $H \in A_m^n$ that is a linear combination of monomials with no zeros, and this is impossible unless $H = 0$. If $n \geq m$ we use iduction: we can apply lemma 3.6.2 to get

$$\delta T^h H = 0,$$

where we choose $h = \min\{k | \partial_0 T^k H \neq 0\}$. Then $T^h H \in A_m^{n-hm}$ and induction tells us that $T^h H = \partial A$. Hence, $H = \partial T^{-h} A$, so that

$$F = H + \partial \partial_0^{-1} G = \partial (T^{-h} A + \partial_0^{-1} G).$$

□

This enables us to identify

$$C = \mathbb{C} \oplus L / \mathrm{ker}\delta.$$

Let P^a be the set of partitions of $a \in \mathbb{Z}_+$

$$P^a = \{\pi = (\pi_1, \ldots, \pi_m) | \pi_1 \geq \ldots \geq \pi_m \geq 1, \sum_{i=1}^m \pi_i = a\}.$$

Define two functions

$$\begin{aligned} r : P^a &\longrightarrow \mathbb{Z} \\ \pi = (\pi_1, \dots, \pi_m \geq 1) &\longmapsto m \end{aligned}$$

and

$$\begin{aligned} N : P^a &\longrightarrow \mathbb{Z} \\ \pi = (\pi_1, \dots, \pi_{r(\pi)}) &\longmapsto \frac{a!}{\prod_{j=1}^m \pi_j! \prod_{i=1}^{\pi_1} \varrho_i!} \end{aligned}$$

where $\varrho_i = \sharp\{\pi_h = i\}$. N has a natural combinatorial interpretation: it counts the number of ways in which a set A of a elements can be partitioned into $r(\pi)$ disjoint subsets $A = A_1 \cup \ldots \cup A_{r(\pi)}$ such that A_i has π_i elements. For all $v \in L$ and for an arbitrary parameter α, denote by $e^{\alpha v}$ the (formal) power series

$$e^{\alpha v} = \sum_{n \geq 0} \frac{\alpha^n v^n}{n!}$$

(we need exponentials to make faster calulations).

3.6.4 LEMMA *For all $a \in \mathbb{Z}_+$ the following identity holds*

$$\partial^a e^{\alpha v} = \sum_{\pi \in P^a} \alpha^{r(\pi)} N(\pi) e^{\alpha v} \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v. \tag{3.43}$$

Proof. We use induction on a. The first term is

$$\partial e^{\alpha v} = \alpha e^{\alpha v} \partial v,$$

which coincides with (3.43) since $P^1 = \{\pi = (1)\}$ and $r(\pi) = 1 = N(\pi) = \pi_1$. Next, suppose that (3.43) holds for a.

$$\begin{aligned} \partial^{a+1} e^{\alpha v} =& \partial(\partial^a e^{\alpha v}) = \partial \left(\sum_{\pi \in P^a} \alpha^{r(\pi)} N(\pi) e^{\alpha v} \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v \right) \\ =& \sum_{\pi \in P^a} \alpha^{r(\pi)+1} N(\pi) e^{\alpha v} \partial v \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v + \qquad (3.44) \\ & + \sum_{\pi \in P^a} \alpha^{r(\pi)} N(\pi) e^{\alpha v} \partial \left(\prod_{j=1}^{r(\pi)} \partial^{\pi_j} v \right). \end{aligned}$$

Denote by A and B the first and second term on the right hand side, respectively. To study A, introduce a map

$$\begin{aligned} \tilde{\cdot} : P^a &\longrightarrow P^{a+1} \\ \pi = (\pi_1, \ldots, \pi_m) &\longmapsto \tilde{\pi} = (\pi_1, \ldots, \pi_m, 1). \end{aligned}$$

We denote by $\tilde{\pi}_j$ the j-th element of $\tilde{\pi}$ and by

$$\tilde{\varrho}_i = \sharp\{\tilde{\pi}_j = i\} = \begin{cases} \varrho_i & \text{if } i \neq 1 \\ \varrho_i + 1 & \text{if } i = 1. \end{cases}$$

Then $r(\tilde{\pi}) = r(\pi) + 1$, $\tilde{\pi}_{r(\tilde{\pi})} = 1$ and

$$N(\tilde{\pi}) = \frac{(a+1)!}{\prod_{j=1}^{r(\tilde{\pi})} \pi_j! \prod_{i=1}^{\tilde{\pi}_1} \varrho_i!} = N(\pi) \frac{a+1}{\varrho_1 + 1}.$$

Therefore, we can write

$$A = \sum_{\pi \in P^a} \alpha^{r(\tilde{\pi})} N(\tilde{\pi}) \frac{\varrho_1 + 1}{a+1} e^{\alpha v} \prod_{j=1}^{r(\tilde{\pi})} \partial^{\tilde{\pi}_j} v.$$

To study B we need to introduce other maps

$$\begin{aligned} \bar{\cdot}^i : P^a &\longrightarrow P^{a+1} \\ \pi = (\pi_1, \ldots, \pi_m) &\longmapsto \bar{\pi}^i = (\pi_1, \ldots, \pi_{i-1}, \pi_i + 1, \pi_{i+1}, \ldots, \pi_m) \end{aligned}$$

with $i = 1, \ldots, m$. Notice that we can possibly look at $\tilde{\pi}$ as if it were $\bar{\pi}^{r(\pi)+1}$. With the natural notation, we have $\bar{\pi}^i_j = \pi_j + \delta_{ij}$, $\bar{\varrho}^i_{\pi_i} = \varrho_{\pi_i} - 1$, $\bar{\varrho}^i_{\pi_i+1} = \varrho_{\pi_{i+1}} + 1$ and $\bar{\varrho}^i_j = \varrho_j$ for all other j. Hence,

$$N(\bar{\pi}^i) = N(\pi) \frac{(a+1)\varrho_{\pi_i}}{(\pi_i + 1)(\varrho_{\pi_i+1} + 1)}.$$

Notice also that $r(\bar{\pi}) = r(\pi)$. We can then write

$$\begin{aligned} B &= \sum_{\pi \in P^a} \sum_{i=1}^{r(\pi)} a^{r(\pi)} N(\pi) e^{\alpha v} \partial^{\pi_i + 1} v \prod_{j \neq i} \partial^{\pi_j} v \\ &= \sum_{\pi \in P^a} \sum_{i=1}^{r(\pi)} \alpha^{r(\bar{\pi}^i)} N(\bar{\pi}^i) \frac{(\pi_i + 1)(\varrho_{\pi_i+1} + 1)}{\varrho_{\pi_i}(a+1)} e^{\alpha v} \prod_{j=1}^{r(\bar{\pi}^i)} \partial^{\bar{\pi}^i_j v}. \end{aligned}$$

Observe that $\{\tilde{\pi} | \pi \in P^a\} = \{\pi \in P^{a+1} | \pi_{r(\pi)} = 1\}$, whereas any $\pi \in P^{a+1}$ can be expressed as $\overline{(\pi^j)}^i$ for some $\pi^j \in P^a$, as long as $\pi^j \neq 1$. If we denote by (1^{a+1}) the partition of $a+1$ consisting of $a+1$ 1s, we can write

$$\begin{aligned}
A + B = &\sum_{\substack{\pi \in P^{a+1} \\ \pi_{r(\pi)}=1}} \alpha^{r(\pi)} N(\pi) \frac{\varrho_1}{a+1} \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v + \\
&+ \sum_{\substack{\pi \in P^{a+1} \\ \pi \neq (1^{a+1})}} \left(\sum_{\substack{1 \leq i \leq r(\pi) \\ \pi_i \neq 1}} \frac{\pi_i \varrho_{\pi_i}}{\varrho_{\pi_{i-1}} + 1} \right) \frac{1}{a+1} \alpha^{r(\pi)} N(\pi) \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v \\
= &\alpha^{a+1} e^{\alpha v} (\partial v)^{a+1} + \sum_{\substack{\pi \in P^{a+1} \setminus \{(1^{a+1})\} \\ \pi_{r(\pi)}=1}} C_\pi \alpha^{r(\pi)} N(\pi) \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v + \\
&+ \sum_{\substack{\pi \in P^{a+1} \\ \pi_{\varrho(\pi)} \neq 1}} D_\pi \alpha^{r(\pi)} N(\pi) \prod_{j=1}^{r(\pi)} \partial^{\pi_j} v,
\end{aligned}$$

where

$$C_\pi = \frac{1}{a+1} \left(\varrho_1 \sum_{\substack{1 \leq i \leq r \\ \pi_i \neq 1}} \frac{\pi_i \varrho_{\pi_i}}{\varrho_{\pi_i - 1} + 1} \right)$$

and

$$D_\pi = \frac{1}{a+1} \sum_{i=1}^{r(\pi)} \frac{\pi_i \varrho_{\pi_i}}{\varrho_{\pi_i - 1} + 1}.$$

It is easy to verify that both these constants are equal to 1 for all $\pi \in P^{a+1}$, which proves the statement. □

This enables us to write down the Lie bracket of the generating functions:

$$\begin{aligned}
[\int e^{\alpha \partial^i u}, \int e^{\beta \partial^j u}] = &\sum_{\substack{\pi^1 \in P^j \\ \pi^2 \in P^{i+1}}} \alpha^{r(\pi^2)+1} \beta^{r(\pi^1)+1} N(\pi^1) N(\pi^2) \times \\
&\times \int e^{\alpha \partial^i u + \beta \partial^j u} \prod_{h=1}^{r(\pi^1)} \partial^{\pi^1_h + j} u \prod_{k=1}^{r(\pi^2)} \partial^{\pi^2_k + i} u.
\end{aligned} \tag{3.45}$$

3.6.5 THEOREM *The subalgebra $\mathcal{A} \subset C$ given by*

$$\mathcal{A} = \mathtt{Span}_{\mathbb{C}}\{\textstyle\int 1, \int u, \int u^2, \int u^3, \dots\}$$

is abelian.

Proof. Apply formula (3.45) above to the case $i = j = 0$. We have then only one term, because P^0 is empty and P^1 has only one element. Thus,

$$[\int e^{\alpha u}, \int e^{\beta u}] = \alpha\beta^2 \int e^{(\alpha+\beta)u} \partial u = \frac{\alpha\beta^2}{\alpha+\beta} \int \partial e^{(\alpha+\beta)u} = 0.$$

Looking at the coefficients of $\alpha^m \beta^n$, for $m, n \geq 0$, yields

$$[\int \frac{u^m}{m!}, \int \frac{u^n}{n!}] = 0$$

□

3.6.6 THEOREM *The following formulae hold true for all $m \geq 3, n \geq 2$*

$$[\int u^m, \int (\partial u)^n] = -6\binom{m}{3}(n-1)\int u^{m-3}(\partial u)^{n+1},$$
$$[\int (\partial u)^m, \int (\partial u)^n] = 4\binom{m}{2}\binom{n}{2}(m-n)\int (\partial u)^{m+n-5}(\partial^2 u)^3.$$

Proof. To prove both formulae we use (3.45).

$$\begin{aligned}[\int e^{\alpha u}, \int e^{\beta \partial u}] =& \alpha^2\beta^2 \int e^{\alpha u + \beta \partial u} \partial^2 u \partial u \\ =& \sum_{k \geq 0} \sum_{h=0}^{k} \alpha^{h+2} \beta^{k-h+2} \binom{k}{h} \frac{1}{k!} \int u^h (\partial u)^{k-h+1} \partial^2 u.\end{aligned}$$

Therefore

$$\begin{aligned}[\int u^m, \int u^n] =& m!n!\binom{m+n-4}{m-2}\frac{1}{(m+n-4)!}\int u^{m-2}(\partial u)^{n-1}\partial^2 u \\ =& -m(m-1)(n-1)\int \partial u^{m-2}(\partial u)^n \\ =& -6\binom{m}{3}(n-1)\int u^{m-3}(\partial u)^n - 1,\end{aligned}$$

which is the first identity we wished to prove. To prove the second one, write

$$\begin{aligned}
[\int e^{\alpha\partial u}, \int e^{\beta\partial u}] =&\alpha^3\beta^2 \int e^{(\alpha+\beta)\partial u}(\partial^2 u)^3 + \alpha^2\beta^2 \int e^{(\alpha+\beta)\partial u}\partial^2 u \partial^3 u \\
=&\alpha^3\beta^2 \int e^{(\alpha+\beta)\partial u}(\partial^2 u)^3 - \frac{\alpha^2\beta^2}{2} \int \partial e^{(\alpha+\beta)\partial u}(\partial^2 u)^2) \\
=&\frac{\alpha^2\beta^2(\alpha-\beta)}{2} \int e^{(\alpha+\beta)\partial u}(\partial^2 u)^3 \\
=&\sum_{k\geq 0}\sum_{h=0}^{k} \frac{1}{2}\frac{1}{k!}\binom{k}{h}(\alpha^{h+3}\beta^{k-h+2} - \alpha^{h+2}\beta^{k-h+3})\times \\
&\times \int (\partial u)^k (\partial^2 u)^3.
\end{aligned}$$

Consequently,

$$\begin{aligned}
[\int (\partial u)^m, \int (\partial u)^n] =&m!n!\frac{1}{2}\left(\frac{1}{(m-3)!(n-2)!} - \frac{1}{(m-2)!(n-3)!}\right)\times \\
&\times \int (\partial u)^{m+n-5}(\partial^2 u)^3 = \\
=&4\binom{m}{2}\binom{n}{2}(m-n)\int (\partial u)^{m+n-5}(\partial^2 u)^3.
\end{aligned}$$

□

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