SAPIENZA
UNIVERSITÀ DI ROMA

ANNO ACCADEMICO 2008 - 2009

DIPARTIMENTO di MATEMATICA
"Guido Castelnuovo"

Tesi di Dottorato in Matematica

Some applications of Lie pseudoalgebra representations

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Contents

Introduction

The notion of Lie pseudoalgebra over a cocommutative Hopf algebra [BDK1, BeD] has recently emerged as a useful tool in algebra and representation theory. On the one hand, Lie pseudoalgebras can be viewed as a generalization of the concept of Lie conformal algebra, introduced by Kac [K] in connection with vertex algebras. On the other, Lie pseudoalgebras are intimately related to representation theory of linearly compact infinite dimensional Lie algebras [BDK2, BDK3, BDK4].

Vertex algebras were originally introduced by Borcherds [Bo] as an axiomatization of local families of quantum fields in the chiral sector of a Conformal Field Theory in dimension two. Algebraic properties of such families are described by the so-called Operator Product Expansion: its regular part is essentially captured by the normally ordered product of fields, whereas the singular part can be axiomatized into the notion of Lie conformal algebra.

Lie pseudoalgebras are a multivariable (and noncommutative) generalization of Lie conformal algebras. Algebraic properties of each element in a Lie pseudoalgebra may be recast in term of its Fourier coefficients, that are sometimes called creation and annihilation operators in the physical jargon: the space of all annihilation operators is then a (typically infinite dimensional) Lie algebra, and the Lie bracket is continuous with respect to a linearly compact topology. Cartan and Guillemin's study [Ca, Gu1, Gu2] of linearly compact infinite dimensional Lie algebras can then be usefully exploited in the study of such structures.

The present thesis applies representation theory of Lie pseudoalgebras in two distinct directions. In Section 2.6 we study a certain class of solvable Lie pseudoalgebras and show they enjoy a useful nilpotence property that may be applied towards characterizing finite vertex algebras. In later sections, we generalize the usual definition [BDK1] of Lie pseudoalgebra representation to what we call *representation with coefficients* [DM]. We show that this new concept can be used in order to study irreducible representations of the linearly compact Lie algebra P_N of formal functions in $N = 2n$ variables with respect to the standard Poisson bracket. We will now proceed to give a more detailed description of the ideas and techniques involved in my thesis.

A vertex algebra is a (complex) vector space V endowed with a linear *state-field correspondence* $Y : V \to (\operatorname{End} V)[[z, z^{-1}]]$, a *vacuum element* $\mathbf{1}$ and a linear endomorphism $\partial \in \operatorname{End} V$ satisfying:

- **Field axiom**: $Y(a, z)b \in V[[z]][z^{-1}]$ for all $a, b \in V$.

- **Locality axiom**: For every choice of $a, b \in V$

$$(z-w)^N [Y(a,z), Y(b,w)] = 0$$

for sufficiently large N.

- **Vacuum axiom**: The vacuum element 1 satisfies

$$\partial 1 = 0, \qquad Y(1,z) = \mathrm{id}_V, \qquad Y(a,z)1 \in a + zV[[z]],$$

for all $a \in V$.

- **Translation invariance**: ∂ satisfies

$$[\partial, Y(a,z)] = Y(\partial a, z) = \frac{d}{dz} Y(a,z),$$

for all $a \in V$.

Any vertex algebra V satisfies:

- **Skew-commutativity**:

$$Y(a,z)b = e^{z\partial} Y(b,-z)a,$$

for all $a, b \in V$.

Note that the vector space V carries a natural $\mathbb{C}[\partial]$-module structure. A vertex algebra is *finite* if it is a finitely generated $\mathbb{C}[\partial]$-module. Coefficients of vertex operators

$$Y(a,z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}$$

determine $\mathbb{C}$-bilinear products $a \otimes b \mapsto a_{(j)}b, j \in \mathbb{Z}$, on V. Locality can be rephrased by stating that commutators between coefficients of quantum fields satisfy:

$$[a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)}\, b)_{(m+n-j)}, \tag{0.1}$$

for all $a, b \in V$, $m, n \in \mathbb{Z}$. In other words, $[Y(a,z), Y(b,w)] = 0$ as soon as all positively labeled products $a_{(j)}b, j \geq 0$ vanish. This is always the case when V is finite dimensional [Bo].

If A and B are subsets of V, then we may define $A \cdot B$ as the $\mathbb{C}$-linear span of all products $a_{(j)}b$, where $a \in A, b \in B, j \in \mathbb{Z}$. A $\mathbb{C}[\partial]$-submodule $A \subset V$ (resp. $I \subset V$) is a subalgebra (resp. an ideal) of V if $A \cdot A \subset A$ (resp. $I \cdot V = V \cdot I = I$). Set now

$$I^1 = I, \qquad I^{k+1} = I \cdot I^k, \ k \geq 1.$$

Then an ideal I is a *nil-ideal* if $I^k = 0$ for sufficiently large values of k. Every finite vertex algebra has a unique maximal nil-ideal $\mathrm{Nil}(V)$ which is called the nilradical of V.

A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module R with a $\mathbb{C}$-bilinear product $(a,b) \mapsto [a\,{}_\lambda\, b] \in R[\lambda]$ satisfying the following axioms:

(C1) $[\partial a\,{}_\lambda\, b] = -\lambda[a\,{}_\lambda\, b]$, $[a\,{}_\lambda\, \partial b] = (\partial + \lambda)[a\,{}_\lambda\, b]$,

(C2) $[a\,{}_\lambda\, b] = -[b\,{}_{-\partial-\lambda}\, a]$,

(C3) $[a\,{}_\lambda\,[b\,{}_\mu\, c]] - [b\,{}_\mu\,[a\,{}_\lambda\, c]] = [[a\,{}_\lambda\, b]\,{}_{\lambda+\mu}\, c]$,

for every $a, b, c \in R$. Setting

$$[a\,{}_\lambda\, b] = \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} a_{(n)} b$$

endows every vertex algebra V with a Lie conformal algebra structure. Indeed, $[a\,{}_\lambda\, b] \in V[\lambda]$ follows from the field axiom, (C1) from translation invariance, (C2) from skew-commutativity, and (C3) from (0.1). If A and B are subspaces of R, then we set $[A, B]$ to be the $\mathbb{C}$-linear span of all λ-coefficients in the products $[a\,{}_\lambda b]$, where $a \in A, b \in B$. The Lie conformal algebra R is then *solvable* if, after defining

$$R^{(0)} = R, \qquad R^{(k+1)} = [R^{(k)}, R^{(k)}],\ k \geq 0,$$

we find that $R^{(K)} = 0$ for sufficiently large K. Similarly, R is *nilpotent* if, after defining

$$R^{[0]} = R, \qquad R^{[k+1]} = [R, R^{[k]}],\ k \geq 0, \tag{0.2}$$

we find that $R^{[K]} = 0$ for sufficiently large K. If R is finite, its central series $R^{[0]} \supset R^{[1]} \supset R^{[2]} \supset \dots$ always stabilizes on an ideal $R^{[\infty]}$.

In order to avoid confusion, we will denote by V^{Lie} the Lie conformal algebra structure underlying a vertex algebra V. Finiteness of V has strong algebraic consequences on the structure of V^{Lie}. More precisely, it is showed in [D1] that

- if V is a finite simple vertex algebra, then V^{Lie} is trivial, i.e., all quantum fields from V commute with each other;
- if V is a finite vertex algebra, then V^{Lie} is a solvable Lie conformal algebra;
- if V is a finite vertex algebra, then $(V/\operatorname{Nil}(V))^{Lie}$ is nilpotent. In particular, if $\operatorname{Nil}(V) = 0$ then V^{Lie} is nilpotent.

It is however not clear from [D1] whether finiteness of V implies nilpotence of V^{Lie}. The first result in this thesis is a refinement of the above statements. We prove the following

Theorem 3.1. *Let V be a finite vertex algebra and $N = V^{[\infty]} = (V^{Lie})^{[\infty]}$. Then $N \cdot N = 0$, and there exists a subalgebra $U \subset V$ such that U^{Lie} is nilpotent and $V = U \ltimes N$.*

In other words, if V^{Lie} is not nilpotent, then the central series $V^{[0]} \supset V^{[1]} \supset V^{[2]} \supset \dots$ stabilizes on an ideal $N = V^{[\infty]}$ contained in $\operatorname{Nil}(V)$. Moreover, the quotient V/N embeds as a subalgebra of V which is a complement of N, thus realizing V as a semidirect sum of V/N and N.

In order to prove the above statement, we study the adjoint action of V^{Lie}. This is a finite solvable Lie conformal algebra and by the conformal version of Lie's theorem, its action can be triangularized. It is well known that the action of a nilpotent Lie conformal algebra L on a finite $\mathbb{C}[\partial]$-module M decomposes it as a direct sum of generalized weight modules M^ϕ, $\phi \in L^*$. When V is a finite vertex algebra, and we consider the adjoint action on V of a nilpotent subalgebra $L \subset V^{Lie}$, [D1] shows that each V^ϕ is a nil-ideal, provided that its weight ϕ is nonzero, whereas V^0 is a subalgebra of V.

Recall now that one may locate all Cartan subalgebras of a finite dimensional Lie algebra by considering the largest subspace on which any given generic element acts nilpotently. Here the situation is almost identical: if we choose a generic element a in a finite vertex algebra V, the largest subspace of V on which the adjoint action of a is nilpotent is a vertex subalgebra $U \subset V$ such that U^{Lie} is nilpotent and self-normalizing in V^{Lie}, *provided a generates a nilpotent subalgebra $\langle a \rangle$ in V^{Lie}*. Moreover, the sum of the generalized weight spaces $V^\phi, \phi \neq 0$, for the action of U equals $V^{[\infty]}$ and $U = V^0$ is clearly a complement to it. The only technical problem is ensuring that we can choose a so that $\langle a \rangle$ is nilpotent. We do this by proving a nilpotence statement in the more general setting of Lie pseudoalgebras.

Let H be a cocommutative Hopf algebra over $\Bbbk$. A Lie pseudoalgebra L is a left H-module endowed with an $H \otimes H$-linear pseudobracket $L \otimes L \to (H \otimes H) \otimes_H L$ which satisfies axioms similar to those of a Lie algebra. Our main example of a cocommutative Hopf algebra is the universal enveloping algebra $H = U(\mathfrak{d})$ of a finite dimensional Lie algebra $\mathfrak{d}$. When $\mathfrak{d} = (0)$ and $H = \Bbbk$, a Lie pseudoalgebra is just a Lie algebra over $\Bbbk$ and when $\mathfrak{d} = \mathbb{C}\partial, H = \mathbb{C}[\partial]$, we obtain a notion equivalent to that of a Lie conformal algebra.

Each finite representation M of a Lie pseudoalgebra L is obtained by giving a Lie pseudoalgebra homomorphism $L \to \mathrm{gc}\, M$, where $\mathrm{gc}\, M$ plays a role similar to that of $\mathfrak{gl}\, V$ in the Lie algebra setting. However $\mathrm{gc}\, M$ is not finite, and its structure is highly noncommutative: for instance, in general an element of $\mathrm{gc}\, M$ does not generate an abelian, or even solvable subalgebra of $\mathrm{gc}\, M$. Nevertheless, there is a good control of solvable and nilpotent subalgebras, as one may prove pseudoalgebra analogues of Lie's and Engel's theorems. In Section 2.6, we prove the following statement on solvable subalgebras of $\mathrm{gc}\, M$ generated by a single element, here a *modification* of a is an element which differs from a by an element in the derived subalgebra of $\langle a \rangle$.

Theorem 2.40 *Let M be a finite H-module and S be a solvable Lie pseudoalgebra generated by $a \in \mathrm{gc}\, M$. Then some modification $\bar{a}$ of a generates a nilpotent Lie pseudoalgebra.*

We can exploit the above theorem to take care of our technical problem. After choosing a generic element a in a finite vertex algebra V, we may modify it so that it generates a nilpotent subalgebra of V^{Lie} but its weights are still generic. Then the techniques that work in the Lie algebra setting easily carry over to vertex algebras.

The description of finite vertex algebras given in Theorem 3.1 tells us how to construct an example V for which V^{Lie} is not nilpotent: this is accomplished in Corollary 3.3. We do so by choosing a commutative vertex algebra U and constructing a representation of U on a free $\mathbb{C}[\partial]$-module N of rank 1 in such a way that at least one quantum field $Y(u, z), u \in U$ acts on N with negative powers of z. Then the action of U on N has nonzero weights, and the central series of $(U \ltimes N)^{Lie}$ stabilizes on N. This shows that the statements in

Theorem 3.1 cannot be further improved. In this example, there are vertex algebra automorphisms fixing N but not U, thus showing that the semidirect sum decomposition provided by the theorem is not canonical. We may now step forward to the second part of the thesis, which uses the pseudoalgebra language in the study of representations of certain linearly compact infinite dimensional Lie algebras.

A *Lie H-pseudoalgebra* is an H-pseudoalgebra L endowed with a pseudoproduct $a \otimes b \mapsto [a * b]$, called *Lie pseudobracket*, satisfying the following skew-commutativity and Jacobi identity axioms:

$$[b * a] = -(\sigma \otimes_H \mathrm{id}_L)\,[a * b]\,,$$
$$[[a * b] * c] = [a * [b * c]] - ((\sigma \otimes \mathrm{id}_H) \otimes_H \mathrm{id}_L)\,[b * [a * c]]\,,$$

where $a, b, c \in L$, $\sigma : H \otimes H \to H \otimes H$ denotes the flip $\sigma(h \otimes k) = k \otimes h$, and expressions such as $[[a * b] * c]$ are given a suitable meaning as elements of $(H \otimes H \otimes H) \otimes_H L$, as described in Section 2.2.

The usefulness of the Lie pseudoalgebra language in the study of representations of infinite dimensional Lie algebras stems from the following observation. Let L be a Lie pseudoalgebra over a (Noetherian) cocommutative Hopf algebra H, and consider the tensor product $\mathcal{L} = H^* \otimes_H L$. When L is a finitely generated H-module, one may give $\mathcal{L}$ a natural linearly compact topology; moreover, setting

$$[x \otimes_H a, y \otimes_H b] = \sum (xh^i)(yk^i) \otimes_H c_i\,, \quad \text{if} \quad [a * b] = \sum (h^i \otimes k^i) \otimes_H c_i\,,$$

endows $\mathcal{L}$ with a Lie bracket which is compatible with the above topology. In other words, $\mathcal{L}$ is a linearly compact Lie algebra, which is called *annihilation algebra* of $\mathcal{L}$. Then, the concept of continuous discrete Lie algebra representation of $\mathcal{L}$ is equivalent to that of Lie pseudoalgebra representation of L, as soon as the representation space M satisfies the following technical condition,

$$h.(lm) = (h_{(1)}l).(h_{(2)}m), \tag{0.3}$$

for all $h \in H, l \in \mathcal{L},\ m \in M$. Thus, if a certain linearly compact Lie algebra is the annihilation algebra of some Lie pseudoalgebra, its representations may be also studied by means of pseudoalgebraic techniques, which has been done, for instance, in [BDK2, BDK3, BDK4].

Cartan and Guillemin's study of linearly compact Lie algebras shows that the Lie algebra W_N of all vector fields in N indeterminates, along with its subalgebras S_N, H_N, K_N of elements preserving a volume form, a symplectic form and a contact structure respectively, exhaust all examples of infinite dimensional simple linearly compact Lie algebras. All Lie algebras W_N, S_N, K_N are annihilation algebras of simple Lie pseudoalgebras, and their discrete irreducible representations all satisfy (0.3). As a consequence, the pseudoalgebraic techniques may be used, and this leads to a complete understanding of irreducibles for both the Lie algebras and the Lie pseudoalgebras. The description of irreducible representations is done, in both cases, in terms of quotients of some nice induced modules, called *tensor modules*, by maximal submodules. Tensor modules are always irreducible, but for a finite number of cases. Such exceptional modules may be grouped in exact complexes — which are remindful of the de Rham complex for W_N, S_N and of the Rumin complex [Ru] in type K_N — which easily provide all the irreducible quotients.

Treating the remaining case H_N leads to two extremely odd features. The first is that H_N is not the annihilation algebra of any Lie pseudoalgebra; there exist, however, Lie pseudoalgebras whose annihilation algebra is the unique irreducible one-dimensional central extension P_N of H_N. The Lie algebra P_N may be understood as the structure induced by the Poisson bracket on (formal) functions in $N = 2n$ indeterminates, its central ideal being spanned by constant functions. When considering irreducible discrete representations of P_N, one finds that condition (0.3) is only satisfied if the action of P_N factors via the simple quotient H_N, so that the pseudoalgebra language may only effectively handle irreducible representations of H_N. The other surprising aspect, when working with H_N and its pseudoalgebraic counterparts, is that reducible tensor modules group in an exact complex which is totally analogous to that used in the contact case.

In this thesis, we provide an explanation for both of the above phenomena. First of all, we generalize the concept of Lie pseudoalgebra representation to a wider setting. If H is a cocommutative Hopf algebra and L is a Lie pseudoalgebra over H, an H-module M is a *representation of* L if it is endowed with an $H \otimes H$-linear map

$$L \otimes M \to (H \otimes H) \otimes_H M\,, \quad a \otimes m \mapsto a * m$$

that satisfies ($a, b \in L$, $m \in M$)

$$[a * b] * m = a * (b * m) - ((\sigma \otimes \mathrm{id}_H) \otimes_H \mathrm{id}_M)(b * (a * m)).$$

We generalize this construction by asking that M be a D-module, rather than an H-module, where D is a comodule algebra over H, and providing an $H \otimes D$-linear map $L \otimes M \to (H \otimes D) \otimes_D M$, where the right D-module structure of $H \otimes D$ is described by the comodule map of D: this is called a representation of L with coefficients in D. Once again, discrete representations of $\mathcal{L}$ can be read in terms of representations of L with coefficients in D only when the condition (4.34) is satisfied. However, different choices of D impose different requirements on the representation space, and one is thus able to treat all irreducible representations of P_N, and not only those factoring via H_N, by choosing a suitable comodule algebra.

One of the ways of producing a new Lie pseudoalgebra from a given one is by extending scalars. This has been considered in [BDK1] in order to provide a complete description of simple Lie pseudoalgebras in terms of *primitive* ones. It is usually implemented by taking a (Lie) pseudoalgebra L over H, and taking its tensor product $H' \otimes_H L$, where H' is a larger Hopf algebra: H' is then an H-bimodule by means of the embedding homomorphism $\iota : H \to H'$, and $H' \otimes_H L = \mathrm{Cur}_H^{H'} L$ is called *current Lie pseudoalgebra*. Similarly, whenever a homomorphism $(H, D) \to (H', D')$ of Hopf algebra-comodule algebra pairs is given, one may extend scalars in order to obtain, from a pair (L, M), where L is a Lie pseudoalgebra over H and M is a representation of L with coefficients in D, an extended pair $(H' \otimes_H L, D' \otimes_D M)$ where the extended module is a representation of the extended Lie H'-pseudoalgebra with coefficients in D'.

However, scalars may also be extended by using non-injective maps, and the resulting objects still carry a (Lie) pseudoalgebra or representation structure.

When one applies this construction to a Lie pseudoalgebra of type K, using a suitable Hopf algebra homomorphism, it is possible to obtain a Lie pseudoalgebra of type H. Similarly, it is possible to obtain from the Rumin-like complex of type K the corresponding complex of reducible tensor modules of type H.

The most striking aspect of this construction is that from the Rumin-like complex of type K, which is composed of representation *with standard coefficients*, or ordinary representations — here we mean that $D = H$ for such tensor modules — one may extend scalars to obtain a complex of tensor modules with non-standard coefficients. These may be used in order to treat the representations of P_N that do not factor via H_N with the same strategy that proves successful for W_N, S_N, K_N, H_N. Techniques analogues to those used in [BDK2, BDK3, BDK4] should lead to a classification of all discrete irreducible representations of the linearly compact Lie algebra P_N.

Throughout this thesis all vector spaces, linear maps and tensor products will be considered over an algebraically closed field $\Bbbk$ of characteristic 0. In Chapters 1 and 3 we will assume that $\Bbbk = \mathbb{C}$.

Chapter 1

Vertex algebras and Lie conformal algebras

We denote by $\mathbb{N}$ the set of nonnegative integers. We recall definitions and basic results about Lie conformal algebras [DK] and vertex algebras [Bo, K]. Then we review results from [D1], where a characterization of the Lie conformal algebra structure underlying a vertex algebra V which satisfies a finiteness assumption is given.

1.1 Lie conformal algebras

In this section we recall the notion of Lie conformal algebra. Let $\mathbb{C}[\partial]$ be the ring of complex polynomials in the indeterminate ∂. A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module R endowed with a $\mathbb{C}$-bilinear product, called λ-bracket,

$$\begin{array}{rccc} [\ {}_\lambda\ ]: & R \otimes R & \longrightarrow & R[\lambda] \\ & a \otimes b & \mapsto & [a_\lambda b], \end{array}$$

such that for any $a, b, c \in R$ the following axioms are satisfied:

(C1) $[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$,

(C2) $[a_\lambda b] = -[b_{-\partial-\lambda} a]$,

(C3) $[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]$.

A *derivation* of a Lie conformal algebra R is a linear endomorphism D of R such that for every $a, b \in R$:

$$D[a_\lambda b] = [Da_\lambda b] + [a_\lambda Db].$$

Then axiom (C1) shows that ∂ is always a derivation of R.
A Lie conformal algebra is *finite* if it is finitely generated as a $\mathbb{C}[\partial]$-module.

Example 1.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$. Then setting*

$$[g_\lambda h] = [g, h], \quad g, h \in \mathfrak{g} \subset \mathbb{C}[\partial] \otimes \mathfrak{g},$$

extends uniquely by (C1) to a Lie conformal algebra structure on R, called current Lie conformal algebra.

Example 1.2. *Let $R = \mathbb{C}[\partial]L$ be a free $\mathbb{C}[\partial]$-module of rank one. Then*

$$[L_\lambda L] = (\partial + 2\lambda)L,$$

extends on R to a structure of a Lie conformal algebra, called the (centerless) Virasoro conformal (Lie) algebra *and denoted by* Vir*.*

Let R_1, R_2 be Lie conformal algebras. A *Lie conformal algebra homomorphism* from R_1 to R_2 is a $\mathbb{C}[\partial]$-linear map $\rho : R_1 \longrightarrow R_2$ such that

$$\rho([a_\lambda b]) = [\rho(a)_\lambda \rho(b)], \quad \forall\, a, b \in R_1.$$

Let R be a Lie conformal algebra, If $A, B \subset R$, then we denote by $[A, B]$ the linear span of all coefficient of polynomial $[a_\lambda b] = \sum_{i=0}^{n} c_i \lambda^i$, $a \in A$, $b \in B$.
Then if B is a $\mathbb{C}[\partial]$-submodule of R then $[A, B]$ is a $\mathbb{C}[\partial]$-submodule, and if A, B are both $\mathbb{C}[\partial]$-submodule of R then $[A, B] = [B, A]$. A *subalgebra* A of R is a $\mathbb{C}[\partial]$-submodule A such that $[A, A] \subset A$. An *ideal* I of R is a $\mathbb{C}[\partial]$-submodule such that $[R, I] = [I, R] \subset I$. An ideal I of R is said to be *central* if $[R, I] = 0$. The *center* $Z(R) = \{a \in R \mid [a_\lambda b] = 0, \forall\, b \in R\}$ of R is the maximal central ideal of R. A Lie conformal algebra R is *abelian* if $[R, R] = 0$, i.e., if $R = Z(R)$. R is a *simple* Lie conformal algebra if it is not abelian and its only ideals are trivial.
The main theorem in [DK] shows that, up to isomorphism, the only simple finite Lie conformal algebras are those described in Examples 1.1 and 1.2.
The *derived series* of a Lie conformal algebra R is defined inductively by

$$R^{(0)} = R,\ R^{(k+1)} = [R^{(k)}, R^{(k)}], \quad k \geq 0.$$

R is a *solvable* Lie conformal algebra if $R^{(K)} = 0$ for sufficiently large K.
The *central series* of a Lie conformal algebra R is similarly defined by

$$R^{[0]} = R,\ R^{[k+1]} = [R, R^{[k]}], \quad k \geq 0.$$

R is a *nilpotent* Lie conformal algebra if $R^{[K]} = 0$ for sufficiently large K.

1.1.1 Linear conformal maps

In this section we introduce one of the most important examples of a Lie conformal algebra, $\mathrm{gc}\, V$. It is the conformal analogue of the Lie algebra $\mathfrak{gl}\, V$ of all linear endomorphism of a vector space V.
Let U, V be $\mathbb{C}[\partial]$-modules. A *linear conformal map* from U to V is a $\mathbb{C}$-linear map $\phi_\lambda : U \longrightarrow V[\lambda]$ such that for any $u \in U$

$$\phi_\lambda(\partial u) = (\partial + \lambda)\phi_\lambda(u).$$

We denote by $\mathrm{Chom}(U, V)$ the vector space of all conformal linear maps from U to V. It is also a $\mathbb{C}[\partial]$-module via the action

$$\partial\phi_\lambda(u) = -\lambda\phi_\lambda(u).$$

Example 1.3. *Let R be a Lie conformal algebra, $a \in R$. The* adjoint action $(\mathrm{ad}\, a)_\lambda$ *is defined as*

$$(\mathrm{ad}\, a)_\lambda(b) = [a_\lambda b], \quad \forall\, b \in R. \tag{1.1}$$

It is a conformal linear map from R to itself as follows by axiom $(C1)$.

Lemma 1.4. *Let U, V be $\mathbb{C}[\partial]$-modules and $\phi_\lambda \in \mathrm{Chom}(U,V)$. Then $\phi_\lambda(u) = 0$ for any $u \in \mathrm{Tor}\, U$.*

Remark 1.1. *It follows by Lemma 1.4 that the torsion of a Lie conformal algebra is contained in its center.*

Let $\phi_\lambda \in \mathrm{Chom}(U,V)$ and suppose that:

$$\phi_\lambda(u) = \sum_{i=1}^{n} v_i \lambda^i.$$

We denote by $\phi \cdot U$ the $\mathbb{C}[\partial]$-submodule of V generated by the v_i. We will say that $\phi_\lambda \in \mathrm{Chom}(U,V)$ is *surjective* if $\phi \cdot U = V$. Let $\mathcal{F} \subset \mathrm{Chom}(U,V)$ be a family of conformal linear maps such that $\mathcal{F} \cdot U = V$. Then we say that $\mathcal{F}$ maps U *surjectively* on V.

Example 1.5. *The adjoint action* $\mathrm{ad}\, R$ *of R maps R surjectively on R'.*

For $U = V$ we denote $\mathrm{Chom}(V,V)$ by $\mathrm{Cend}\, V$. If V is a finite $\mathbb{C}[\partial]$-module then setting:

$$[\phi_\lambda \psi]_\mu v = \phi_\lambda(\psi_{\mu-\lambda} v) - \psi_{\mu-\lambda}(\phi_\lambda v), \quad \phi, \psi \in \mathrm{Cend}(V,V), \tag{1.2}$$

endows $\mathrm{Cend}\, V$ with a Lie conformal algebra structure, denoted by $\mathrm{gc}\, V$ (see [DK]).

Example 1.6. *Let V_0 be a finite dimensional vector space and $V = \mathbb{C}[\partial] \otimes V_0$ be the corresponding free $\mathbb{C}[\partial]$-module. A conformal linear map $\phi_\lambda \in \mathrm{Cend}\, V$ satisfies*

$$(\phi_\lambda(p(\partial) v_0) = p(\partial + \lambda)\phi_\lambda(v_0),$$

so that we can identify $\mathrm{Cend}\, V$ *with* $(\mathrm{End}\, V_0)[\partial, \lambda]$*, where the $\mathbb{C}[\partial]$-module structure on $(\mathrm{End}\, V_0)[\partial, \lambda]$ is given via multiplication by $-\lambda$. As a consequence,* $\mathrm{Cend}\, V$ *is a free $\mathbb{C}[\partial]$-module of infinite rank. When V is a free $\mathbb{C}[\partial]$-module of rank N then the corresponding Lie conformal algebra structure on* $\mathrm{Cend}\, V$ *is denoted by* gc_N.

1.1.2 Representation theory of solvable and nilpotent Lie conformal algebras

In this section we review the structure theory of finite representations of solvable and nilpotent Lie conformal algebras.
Let R be a Lie conformal algebra and V be a finite $\mathbb{C}[\partial]$-module. V is an *R-module*, or a *representation* of R, if for any $a \in R$ a conformal linear map $a_\lambda : V \longrightarrow V[\lambda]$ is defined, such that

$(R1)$ $(\partial a)_\lambda v = -\lambda\, a_\lambda v$,

$(R2)$ $[a_\lambda b]_{\lambda+\mu} v = a_\lambda(b_\mu v) - b_\mu(a_\lambda v)$.

Example 1.7. *Every finite Lie conformal algebra R is a module on itself via the adjoint action defined by (1.1). If* $\mathrm{ad} : R \ni a \longrightarrow (\mathrm{ad}\, a)_\lambda \in \mathrm{gc}\, R$ *then* ad *is a Lie conformal algebras homomorphism and* $\ker \mathrm{ad} = Z(R)$.

Lemma 1.8. *Let R be a Lie conformal algebra and V be an R-module. Then $a_\lambda v = 0$ for all $v \in \mathrm{Tor}\, V$ and $a_\lambda v = 0$ for all $a \in \mathrm{Tor}\, R$.*

Proposition 1.9. *Let R be Lie conformal algebra, V a finite $\mathbb{C}[\partial]$-module. Then there is a one-to-one correspondence between R-modules structures on V and Lie conformal algebra homomorphisms $R \longrightarrow \mathrm{gc}\, V$.*

Remark 1.2. *If R is a finite Lie conformal algebra then Proposition 1.9 implies that* $\mathrm{gc}\, R$ *contains a homomorphic image of R, as R is an R-module via its adjoint action. In particular, if R is centerless then we obtain an injective homomorphism of Lie conformal algebras from R to* $\mathrm{gc}\, R$.

Let R be a Lie conformal algebra and V be a finite R-module. An element $a \in R$ acts *nilpotently* on V if

$$a_{\lambda_1}(a_{\lambda_2}(\dots(a_{\lambda_n} v)\dots)) = 0,$$

for a sufficiently large n.

Theorem 1.10. *[Conformal version of Engel's theorem] Let R be a finite Lie conformal algebra. If any $a \in R$ has a nilpotent adjoint action then R is a nilpotent Lie conformal algebra.*

Let R be a Lie conformal algebra and V be a finite R-module.
Let $\phi : R \longrightarrow \mathbb{C}[\lambda]$ be a $\mathbb{C}[\partial]$-linear map, i.e., a $\mathbb{C}$-linear map such that for every $a, b \in R$:

$$\begin{aligned} \phi_{\partial a}(\lambda) &= -\lambda \phi_a(\lambda), \\ \phi_{a+b}(\lambda) &= \phi_a(\lambda) + \phi_b(\lambda), \end{aligned}$$

where the structure of a $\mathbb{C}[\partial]$-module on $\mathbb{C}[\lambda]$ is given by multiplication by $-\lambda$. We define

$$V_\phi = \{v \in V \mid a_\lambda v = \phi_a(\lambda) v, \ \forall\, a \in R\}.$$

If $V_\phi \neq 0$ then we call ϕ a *weight for the action of R on V* and nonzero elements $v \in V_\phi$ *weight vectors* of weight ϕ or *ϕ-weight vector.* V_ϕ is always a vector subspace of V and it is also a $\mathbb{C}[\partial]$-submodule for $\phi = 0$.

Theorem 1.11. *[Conformal version of Lie's theorem] Let R be a solvable Lie conformal algebra and V be a finite R-module. Then there exists a weight vector v and $\phi : R \ni a \mapsto \phi_a(\lambda) \in \mathbb{C}[\lambda]$ such that $a_\lambda v = \phi_a(\lambda) v$ for all $a \in R$.*

Now we define

$$V_0^\phi = 0, \ V_{i+1}^\phi = \{v \in V \mid a_\lambda v - \phi_a(\lambda) v \in V_i^\phi, \, a \in R\}, \ i \geq 0.$$

This is an increasing chain of subspaces of V. We call

$$V^\phi = \bigcup_i V_i^\phi$$

the *generalized weight subspace* of weight ϕ. Some of the properties of V^ϕ are described in the following

Proposition 1.12. *Let V be a finite R-module. Then:*

- V^ϕ *is a* $\mathbb{C}[\partial]$*-submodule of* V*,*
- V/V^ϕ *has no* ϕ*-weight vectors. In particular,* V/V^0 *is torsion free,*
- *if* $\phi \neq \psi$ *then* $V^\phi \cap V^\psi = 0$*.*

For nilpotent Lie conformal algebras we have

Theorem 1.13. *Let R be a nilpotent Lie conformal algebra and V be a finite R-module. Then V decomposes as a direct sum of generalized weight subspaces for the action of R.*

Remark 1.3. *Let R be a Lie conformal algebra, V be a finite R-module and $a \in R$. We denote by $\langle a \rangle$ the subalgebra of R generated by a. Saying that $a \in R$ acts nilpotently on V is equivalent to say that $V = V^0$ with respect to the action of $\langle a \rangle$ on V.*

The following results will be useful later.

Lemma 1.14. *Let R be a finite Lie conformal algebra and $\{R^{[k]}\}$ be its central series. If $\operatorname{rk} R^{[k]} = \operatorname{rk} R^{[k+1]}$ for some k then $R^{[k+1]} = R^{[k+2]}$.*

Proof. If $\operatorname{rk} R^{[k]} = \operatorname{rk} R^{[k+1]}$ then $R^{[k]}/R^{[k+1]}$ is a torsion $\mathbb{C}[\partial]$-module. As a consequence any conformal linear map maps $R^{[k]}/R^{[k+1]}$ to zero. By construction the adjoint action of R maps $R^{[k]}/R^{[k+1]}$ surjectively on $R^{[k+1]}/R^{[k+2]}$. Since all these maps are zero then we have $R^{[k+1]} = R^{[k+2]}$. □

Proposition 1.15. *Let R be a finite Lie conformal algebra. Then its central series $\{R^{[k]}\}$ stabilizes to an ideal of R which we denote by $R^{[\infty]}$.*

Proof. Since R is finite there exists an index k such that the hypothesis of Lemma 1.15 is satisfied. □

Corollary 1.16. *If R is a finite Lie conformal algebra then $R/R^{[\infty]}$ is nilpotent. In particular, R is nilpotent if and only if $R^{[\infty]} = 0$. If N is an ideal of R such that R/N is nilpotent then $N \supset R^{[\infty]}$.*

1.2 Vertex algebras

We will use the following definition of a vertex algebra given in [K].
Let V be a complex vector space and $(\operatorname{End} V)[[z, z^{-1}]]$ be the space of $\operatorname{End} V$-value formal distributions. An element $\phi(z) \in \operatorname{End}(V)[[z, z^{-1}]]$ is called a *field* if for any $v \in V$ it satisfies $\phi(z)(v) \in V[[z]][z^{-1}] = V((z))$. In other words, if we write

$$\phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j-1}$$

then $\phi_N(v)$ should vanish for sufficiently large $N = N(v) > 0$.
The data $(V, 1, \partial, Y)$, where $1 \in V$, $\partial \in \operatorname{End} V$ and $Y : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$ is a *vertex algebra* if the following properties are satisfied

(V1) **Field axiom:** $Y(a, z)$ is a *field* for every $a \in V$.

(V2) **Locality axiom:** for all choices of $a, b \in V$, there exists N such that

$$(z-w)^N [Y(a,z), Y(b,w)] = 0.$$

(V3) **Vacuum axiom:** the *vacuum element* 1 satisfies

$$\partial 1 = 0, \;\; Y(1,z) = \mathrm{id}_V, \;\; Y(a,z)1 \in a + zV[[z]].$$

(V4) **Translation invariance:**

$$[\partial, Y(a,z)] = Y(\partial a, z) = \frac{d}{dz} Y(a,z).$$

As a consequence of the axioms any vertex algebra V satisfies the following

(V5) **Skew-commutativity:**

$$Y(a,z)b = e^{z\partial} Y(b,-z)a.$$

Y is called *linear state-field correspondence*. Fields $Y(a,z)$ are called *vertex operators* or *quantum fields*. V has a natural structure of a $\mathbb{C}[\partial]$-module. A vertex algebra is *finite* if it is finitely generated as a $\mathbb{C}[\partial]$-module.

Remark 1.4. *Notice that* $1 \in \mathrm{Tor}\, V$*, so* V *is never a free* $\mathbb{C}[\partial]$*-module.*

Let V be a vertex algebra and

$$Y(a,z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}$$

be a vertex operator. For any $b \in V$, $j \in \mathbb{Z}$, we denote the elements $a_{(j)}(b) \in V$ by $a_{(j)} b \in V$. One may view the $a_{(j)} b$ as (infinitely many) bilinear products on V. Notice that by (V4), ∂ is a derivation of all such products.
Let

$$\delta(z-w) = \sum_{j \in \mathbb{Z}} w^j z^{-j-1}$$

be the *Dirac delta formal distribution.* We denote by $\delta^{(j)}$ the j-th derivative of $\delta(z-w)$ with respect to w. Locality $(V2)$ is equivalent to

$$[Y(a,z), Y(b,w)] = \sum_{j=0}^{N-1} c_j(w) \frac{\delta^{(j)}}{j!}. \tag{1.3}$$

If $Y(a,z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}$ then the (uniquely determined fields) $c_j(w)$ in (1.3) are vertex operators $Y(c_j, w)$ corresponding to elements $c_j = a_{(j)} b$.
Another way to define new vertex operators is given by the *normally ordered product* or *Wick product* that is

$$: Y(a,z) Y(b,w) := Y(a,z)_+ Y(b,w) + Y(b,w) Y(a,z)_-,$$

where

$$Y(a,z)_- = \sum_{j \geq 0} a_{(j)} z^{-j-1}, \quad Y(a,z)_+ = \sum_{j<0} a_{(j)} z^{-j-1}.$$

$: Y(a,z)Y(b,w):$ is a vertex operator and equals $Y(a_{(-1)}b,z)$.
Let $V_1,\ V_2$ be vertex algebras. A *vertex algebra homomorphism* ρ from V_1 to V_2 is a $\mathbb{C}[\partial]$-linear map $\rho: V_1 \longrightarrow V_2$ such that

$$\rho(a_{(j)}b) = \rho(a)_{(j)}\rho(b), \quad \forall\, a,b \in V_1,\ j \in \mathbb{Z}.$$

A *derivation* D of a vertex algebra V is a linear endomorphism of V such that

$$D(a_{(j)}b) = (Da)_{(j)}b + a_{(j)}(Db), \quad \forall\, a,b \in V,\ j \in \mathbb{Z}.$$

A derivation D of a vertex algebra V is *nilpotent* if there exists sufficiently large N such that $D^N = 0$. In this case the exponential e^D of D is a well defined automorphism of V. In particular, it is an automorphism of every $\mathbb{C}$-bilinear product $a_{(j)}b$.

Remark 1.5. *The same is true for locally nilpotent derivations D of V or when some suitable notion of convergence applies to e^D.*

Let V be a vertex algebra. Then the elements $a_{(j)} \in \operatorname{End} V$, $a \in V$, $j \in \mathbb{Z}$, span a Lie algebra with Lie bracket given by

$$[a_{(j)}, b_{(k)}] = \sum_{i \geq 0} \binom{m}{i} (a_{(i)}b)_{(j+k-i)}, \quad a,b \in V, m,n \in \mathbb{Z}. \tag{1.4}$$

Equation (1.4) shows that $a_{(0)} \in \operatorname{End} V$ is a derivation of V, for all $a \in V$.
Let A, B be subsets of V. We denote by $A \cdot B$ the $\mathbb{C}$-linear span of all $a_{(j)}b$ of any $Y(a,z)b$, with $a \in A$, $b \in B$, $j \in \mathbb{Z}$:

$$A \cdot B = \operatorname{span}\{a_{(j)}b \mid a \in A, b \in B, j \in \mathbb{Z}\}.$$

Then, if B is a $\mathbb{C}[\partial]$-submodule of V then $A \cdot B$ is a $\mathbb{C}[\partial]$-submodule too and if A, B are both $\mathbb{C}[\partial]$-submodules of V then $A \cdot B = B \cdot A$. Moreover, any subset A of V is always contained in the $\mathbb{C}[\partial]$-submodule $A \cdot V$. In particular, for any $a \in V$, $a \cdot V = \mathbb{C}a \cdot V$ is a $\mathbb{C}[\partial]$-submodule containing a. A *subalgebra* U of a vertex algebra V is a $\mathbb{C}[\partial]$-submodule containing 1 such that $U \cdot U = U$. An *ideal* I of a vertex algebra V is a $\mathbb{C}[\partial]$-submodule such that $I \cdot V = V$. A proper ideal I of a vertex algebra V cannot contain the vacuum element 1. A vertex algebra V is *simple* if its only ideals are trivial. A vertex algebra V is *commutative* if $[Y(a,z), Y(b,w)] = 0$ for any $a, b \in V$. Equivalently, V is a commutative vertex algebra if $a_{(j)}b = 0$, $\forall\, a,b \in V$, $j \in \mathbb{N}$. The *center* $Z(V)$ of a vertex algebra V is defined as the subspace

$$Z(V) = \{c \in V \mid a_{(j)}c = 0 = c_{(j)}a,\ \forall\, a \in V, j \in \mathbb{Z}_+\}.$$

1.2.1 The nilradical $\operatorname{Nil} V$

Let V be a vertex algebra and I be a vertex ideal of V. We set

$$I^1 = I,\ I^{k+1} = I \cdot I^k, \quad k \geq 1.$$

I is a *nil-ideal* of V if $I^k = 0$ for a sufficiently large value of k. The sum $I_1 + I_2$ of nil-ideals is a nil-ideal. If $I^2 = 0$ then I is an *abelian ideal* of V.
An element $a \in V$ is *nilpotent* if

$$Y(a,z_1)Y(a,z_2)\ldots Y(a,z_k)a = 0, \tag{1.5}$$

for a sufficiently large value of k.
It is shown in [D3] that (1.5) is equivalent to ask that the ideal generated by a, which equals $a \cdot V$, is a nil-ideal of V. If $a \in V$ is such that $Y(a,z)a = 0$ then $a \cdot V$ is an abelian ideal of V. If V is finite then there exists, by Noetherianity, a unique maximal nil-ideal of V. We call this ideal the *nil-radical* of V, and denote it by $\mathrm{Nil}\, V$. It contains all nilpotent elements of V and the quotient $V/\mathrm{Nil}\, V$ has no nonzero nilpotent elements.

1.2.2 The Lie conformal algebra V^{Lie}

Let V be a vertex algebra. We define for any $a, b \in V$ the following λ-bracket:

$$[a_\lambda b] = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} a_{(j)} b.$$

It endows V with a Lie conformal algebra structure that we denote by V^{Lie}.

Example 1.17. *Let V be a unital associative commutative algebra and d be a derivation of V. $Y(a,z)b = (e^{zd}a)b$, $a, b \in V$, endows V with a vertex algebra structure, in which the vacuum element is 1 and $\partial = d$.*

Example 1.17 describes the *trivial* case of a vertex algebra structure. It occurs if and only if all vertex operators of V are regular in z, i.e., if V is a commutative vertex algebra. This last assumption is equivalent to saying that the corresponding Lie conformal algebra structure V^{Lie} is abelian. It is shown in [Bo] that in this case setting $a \cdot b = Y(a,z)b \mid_{z=0}$ endows V with a differential commutative associative algebra structure, which completely determine the vertex algebra structure as in Example 1.17. This is the only possible vertex algebra structure on a finite dimensional vector space V.
Notice that while any ideal I of a vertex algebra V is also an ideal of V^{Lie} the converse is not true in general. For example, $\mathbb{C}1$ is always a central ideal of the Lie conformal algebra structure but it is never an ideal of V. We notice also that being an abelian ideal of V is a stronger requirement than being an abelian ideal of V^{Lie}.

Remark 1.6. *If A, B are subsets of a vertex algebra V then recall that the subspace $[A, B]$ of V is defined as*

$$[A, B] = \mathrm{span}\{a_{(j)}b \mid a \in A, b \in B, j \in \mathbb{N}\}.$$

In [D3] it is shown that if A, B, C are subspaces of vertex algebra V then $[A, B] \cdot C \subset [A, B \cdot C]$. As a consequence, if I is a vertex ideal of V then $[A, I]$ is an ideal of V for any subspace A of V. In particular, the elements of the derived series and of the central series of V^{Lie} are all ideals of V.

1.2.3 Finite vertex algebras

In this section we recall results from [D1, D2, D3], where the effects of a finiteness assumption for a vertex algebra V are investigated. It turns out that under this finiteness assumption the underlying Lie conformal algebra V^{Lie} has good algebraic properties. Precisely,

Theorem 1.18. *The following statements hold:*

- *if V is a finite simple vertex algebra, then V^{Lie} is trivial, i.e., all quantum fields from V commute with each other;*
- *if V is a finite vertex algebra, then V^{Lie} is a solvable Lie conformal algebra;*
- *if V is a finite vertex algebra, then $(V/\operatorname{Nil}(V))^{Lie}$ is nilpotent. In particular, if $\operatorname{Nil}(V) = 0$ then V^{Lie} is nilpotent.*

A natural question which comes from Theorem 1.18 is to investigate if V^{Lie} is always nilpotent whenever V is a finite vertex algebra. In the third chapter we solve negatively this question by giving a counterexample, whose construction naturally comes out from a refinement of Theorem 1.18 that we prove in Theorem 3.1.

1.2.4 The adjoint action of V^{Lie} on V

Let V be a finite vertex algebra. The adjoint action of V^{Lie} on V endows V with a structure of a V^{Lie}-module and gives rise to a homomorphism of conformal algebras from V^{Lie} to $\operatorname{gc} V$. By Theorem 1.18 we know that V^{Lie} and its Lie conformal subalgebras S are solvable. As V is a finite V^{Lie}-module then Theorem 1.11 holds. This follows by Lemma 1.8 that the vacuum element is a 0-weight vector for the action of any subalgebra $S \subset V^{Lie}$ on V. Properties of the generalized weight subspaces V^{ϕ} for the adjoint action of a Lie conformal subalgebra S on a finite vertex algebra V are studied in [D1].

Proposition 1.19. *Let V be a finite vertex algebra and S be a subalgebra of V^{Lie}. Denote by V_S^{ϕ} the generalized weight subspaces for the adjoint action of S on V. Then:*

- *V_S^{ϕ} is an ideal of V for every $\phi \neq 0$,*
- *if ϕ and ψ are distinct weights for the adjoint action of S on V then $V_S^{\phi} \cdot V_S^{\psi} \subset V_S^{\phi+\psi}$,*
- *as a consequence, if $\phi \neq 0$ then $V_S^{\phi} \cdot V_S^{\phi} = 0$, hence $V_S^{\phi} \subset \operatorname{Nil}(V)$.*

This follows by Proposition 1.19 that V_S^0 is a subalgebra of V and

$$V_S^{\neq 0} = \sum_{\phi \neq 0} V_S^{\phi}$$

is an abelian ideal of V.

Our goal is to continue the investigation of finite vertex algebras. In order to do so, we study the effects of a solvability assumption for the subalgebras of $\operatorname{gc} V$ generated by a single conformal linear map a. We approach this problem in the second chapter in the more general context of Lie pseudoalgebras and then apply Theorem 2.40 in the case of Lie conformal algebras to obtain the description of finite vertex algebras stated in Theorem 3.1.

Chapter 2

Lie pseudoalgebras

In this chapter we review Lie pseudoalgebras and their representations [BDK1], especially in the case of solvable and nilpotent structures. At the end of the chapter, in Section 2.6, we state and prove our result on 1-generated solvable subalgebras of $\mathrm{gc}\, M$, where M is a finite H-module: up to a "modification" of the generator they are essentially nilpotent.

2.1 Hopf algebras

In this section we review the definition of a Hopf algebra and we fix our notation, according to [Sw].
A *bialgebra* H is a unital associative algebra H endowed with a *coproduct* $\Delta : H \longrightarrow H \otimes H$ and a *counit* $\epsilon : H \longrightarrow \Bbbk$ such that Δ is a *homomorphism of associative algebras* and the following *coassociativity* and *counit axiom* are satisfied,

$$(2.1) \qquad (\Delta \otimes \mathrm{id}_H)\Delta(h) = (\mathrm{id}_H \otimes \Delta)\Delta(h),$$

$$(2.2) \qquad (\epsilon \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_H \otimes \epsilon)\Delta = \mathrm{id}_H,$$

where in (2.2) we use $\Bbbk \otimes H \simeq H \simeq H \otimes \Bbbk$.
By use of Sweedler's notation, $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $h \in H$, we can reformulate the fact that Δ is a homomorphism of associative algebras and the counit axiom as,

$$\begin{aligned} (fg)_{(1)} \otimes (fg)_{(2)} &= f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)} \\ \epsilon(h_{(1)})h_{(2)} &= h_{(1)}\epsilon(h_{(2)}) = h. \end{aligned}$$

An *antipode* of a bialgebra H is a map $S : H \longrightarrow H$ such that S is an *antihomomorphism* satisfying:

$$(2.3) \qquad S(h_{(1)})h_{(2)} = h_{(-1)}h_{(2)} = \epsilon(h) = h_{(1)}h_{(-2)} = h_{(1)}S(h_{(2)}),$$

where in (2.3) we use the notation $S(h_{(1)}) = h_{(-1)}$.
A bialgebra H endowed with an antipode S is called a *Hopf algebra*. A Hopf algebra H is said to be *cocommutative* if $h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}$ for any $h \in H$. Whenever H is a cocommutative Hopf algebra its antipode S is an *involution*, i.e., $S^2 = \mathrm{id}_H$.

Example 2.1. *Let G be a group and $\Bbbk G$ its group algebra. There exists a unique cocommutative Hopf algebra structure on $\Bbbk G$ satisfying*

$$\Delta(g) = g \otimes g, \;\; \epsilon(g) = 1, \;\; S(g) = g^{-1}, \quad \forall\, g \in G \subset \Bbbk G.$$

Example 2.2. *Let $\mathfrak{d}$ be a finite dimensional Lie algebra and $\mathcal{U}(\mathfrak{d})$ be its universal enveloping algebra. There exists a unique cocommutative Hopf algebra structure on $\mathcal{U}(\mathfrak{d})$ satisfying*

$$\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial, \;\; \epsilon(\partial) = 0, \;\; S(\partial) = -\partial, \quad \forall\, \partial \in \mathfrak{d} \subset \mathcal{U}(\mathfrak{d}).$$

Remark 2.1. *If $\mathfrak{d}$ is an abelian finite dimensional Lie algebra then the construction given in Example 2.2 endows the symmetric algebra $S(\mathfrak{d}) = \mathcal{U}(\mathfrak{d})$ with a structure of a cocommutative Hopf algebra.*

Let $(H_1, \Delta_1, S_1, \epsilon_1)$, $(H_2, \Delta_2, S_2, \epsilon_2)$ be Hopf algebras. An associative algebra homomorphism $\phi : H_1 \longrightarrow H_2$ is a *Hopf homomorphism* from H_1 to H_2 if for any $h \in H_1$ it satisfies:

$$\Delta_2(\phi(h)) = (\phi \otimes \phi)\Delta_1(h), \tag{2.4}$$

$$\epsilon_2(\phi(h)) = \epsilon_1(h). \tag{2.5}$$

Let (H, Δ, S, ϵ) be a Hopf algebra. We denote the n-fold tensor product $\underbrace{H \otimes \cdots \otimes H}_{n \text{ times}}$ of H simply by H^n. It leads to no confusion with the standard notation for the cartesian product since the last one is never involved in this work.
We set $\Delta_{1,1} = \Delta$ and

$$\Delta_{n,i} = (\mathrm{id}_H \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes \mathrm{id}_H) : \quad H^n \quad \longrightarrow \quad H^{n+1}, \qquad ,$$

for $n \geq 2$, $i = 1, \ldots, n$.
Then we define

$$\Delta^n = \Delta_{n,i_n} \circ \Delta_{n-1,i_{n-1}} \circ \cdots \circ \Delta_{2,i_2} \circ \Delta_{1,1},$$

for $1 \leq i_k \leq k \leq n$, so that $\Delta^1 = \Delta$ and $\Delta^n : H \longrightarrow H^{n+1}$. Coassociativity shows that Δ^n does not depend on the choice of the indices i_k.
We obtain the *generalized associativity property*, i.e., for every $h \in H$ we can write

$$\Delta^n(h) = (\mathrm{id}_H \otimes \Delta^{n-1})\Delta(h).$$

In the same way it is possible to obtain many different relations in H^n, like for example [Sw]:

$$(\mathrm{id}_H \otimes \cdots \otimes \epsilon \otimes \cdots \otimes \mathrm{id}_H)\Delta^n = \Delta^{n-1}(h). \tag{2.6}$$

The homomorphism $\Delta^{n-1} : H \longrightarrow H^n$ makes H^n into a left and right H-module. For any $n \geq 1$ the *generalized Fourier Transform* $\mathcal{F}_n$ [BDK1, Ko1] is the map $\mathcal{F}_n : H^{n+1} \longrightarrow H^{n+1}$ defined by:

$$\mathcal{F}_n(h_1 \otimes \cdots \otimes h_n \otimes f) = h_1 f_{(-n)} \otimes \cdots \otimes h_n f_{(-1)} \otimes f_{(n+1)},$$

where $\Delta^n(f) = f_{(1)} \otimes \cdots \otimes f_{(n)} \otimes f_{(n+1)}$.

Lemma 2.3. *The generalized Fourier Transform $\mathcal{F}_n : H^{n+1} \longrightarrow H^{n+1}$ is an isomorphism of vector spaces for any $n \geq 1$.*

Proof. The following map:

$$\begin{array}{rlcl} \mathcal{F}_n^{-1}: & H^{n+1} & \longrightarrow & H^{n+1} \\ & h_1 \otimes \cdots \otimes h_n \otimes f & \mapsto & h_1 f_{(1)} \otimes \cdots \otimes h_n f_{(n)} \otimes f_{(n+1)}, \end{array}$$

is both a left and right inverse of $\mathcal{F}_n$. This follows from a repeated application of (2.6). □

For $n = 1$ we will often denote $\mathcal{F}_1$ simply by $\mathcal{F}$. As a direct consequence of Lemma 2.3 we have

Proposition 2.4. *Let H be a cocommutative Hopf algebra and let $\{h_i \mid i \in I\}$ be a $\Bbbk$-basis of H. Then every element $f \in H^{n+1}$, $n \geq 1$, can be uniquely written as*

$$f = \sum_{i_1,\ldots,i_n} (h_{i_1} \otimes \cdots \otimes h_{i_n} \otimes 1).g_{i_1,\ldots,i_n}, \qquad g_{i_1,\ldots,i_n} \in H.$$

In other words, $H^{n+1} = (H^n \otimes \Bbbk)\Delta^n(H)$.

Proof. Observe that

$$\mathcal{F}_n^{-1}(h_1 \otimes \cdots \otimes h_n \otimes f) = (h_1 \otimes \cdots \otimes h_n \otimes 1).f, \quad f \in H,$$

and that, for every fixed $\Bbbk$-basis $\{h_i\}$ of H every f in H can be uniquely written as a $\Bbbk$-linear combination of h_i's. □

Corollary 2.5. *Let H be a cocommutative Hopf algebra and M be a finite H-module. Then every element of $(H \otimes H) \otimes_H M$ can be uniquely written in the form*

$$m = \sum_i (h^i \otimes 1) \otimes_H m_i, \tag{2.7}$$

for a suitable choice of $h^i \in H$, $m_i \in M$.

We will call (2.7) the *left-straightening* of $(H \otimes H) \otimes_H M$.

Remark 2.2. *If H is a cocommutative Hopf algebra there exist "right" analogues of Proposition 2.4 and Corollary 2.5. Precisely, equality $H^{n+1} = (\Bbbk \otimes H^n)\Delta^n(H)$ holds and for any H-module M there exist $k^i \in H$ and $m'_i \in M$ such that every element of $(H \otimes H) \otimes_H M$ can be uniquely written in the form*

$$m = \sum_i (1 \otimes k^i) \otimes_H m'_i. \tag{2.8}$$

Formula (2.8) is called the right-straightening *of $(H \otimes H) \otimes_H M$.*

The following claim follows from linear algebra.

Lemma 2.6. *Let M, N be vector space, $\alpha \in M \otimes N$. Then there exists a (non unique) choice of finitely many linear independent elements $\{m_i\} \subset M$, $\{n_i\} \subset N$ such that $\alpha = \sum_i m_i \otimes n_i$; moreover $M_\alpha = \operatorname{span}\langle m_i \rangle$ and $N_\alpha = \operatorname{span}\langle n_i \rangle$ only depend on α. In particular, if $M', M'' \subset M$, $N', N'' \subset N$ are linear subspaces, then $(M' \otimes N') \cap (M'' \otimes N'') = (M' \cap M'') \otimes (N' \cap N'')$.*

Lemma 2.7. *Let H be a cocommutative Hopf algebra and A be a finite H-module. For any $n \geq 1$ the map,*

$$\begin{array}{cccc} \pi_n : & H^{n+1} \otimes_H A & \longrightarrow & H^n \otimes A \\ & (h_1 \otimes \cdots \otimes h_n \otimes k) \otimes_H a & \mapsto & h_1 k_{(-n)} \otimes \cdots h_n k_{(-1)} \otimes k_{(n+1)} a, \end{array}$$

where $\Delta^n(k) = k_{(1)} \otimes \cdots \otimes k_{(n+1)}$, is an isomorphism of vector spaces.

Proof. First of all we have to check that π_n is a well defined, i.e., it must satisfies

$$\pi_n((h_1 \otimes \cdots \otimes h_n \otimes k) \otimes_H k'm) = \pi_n((h_1 k'_{(1)} \otimes \cdots \otimes h_n k'_{(n)} \otimes k k'_{(n+1)}) \otimes_H m).$$

By definition of π_n we have:

$$\pi_n((h_1 \otimes \cdots \otimes h_n \otimes k) \otimes_H k'm) = h_1 k_{(-n)} \otimes \cdots \otimes h_n k_{(-1)} \otimes k_{(n+1)} k'm,$$

and

$$\begin{aligned} &\pi_n((h_1 k'_{(1)} \otimes \cdots \otimes h_n k'_{(n)} \otimes k k'_{(n+1)}) \otimes_H m) \\ &= h_1 k'_{(1)} k'_{(-2n)} k_{(-n)} \otimes \cdots \otimes h_{n-1} k'_{(n-1)} k'_{(-n-2)} k_{(-2)} \\ &\otimes h_n k'_{(n)} k'_{(-n-1)} k_{(-1)} \otimes k_{(n+1)} k'_{(2n+1)} m \\ &= h_1 k'_{(1)} k'_{(-2n+1)} k_{(-n)} \otimes \cdots \otimes h_{n-1} k'_{(n-1)} k'_{(-n-1)} k_{(-2)} \\ &\otimes h_n \epsilon(k'_{(n)}) k_{(-1)} \otimes k_{(n+1)} k'_{(2n)} m \\ &= h_1 k'_{(1)} k'_{(-2n+1)} k_{(-n)} \otimes \cdots \otimes h_{n-1} k'_{(n-1)} \epsilon(k'_{(n)}) k'_{(-n-1)} k_{(-2)} \\ &\otimes h_n k_{(-1)} \otimes k_{(n+1)} k'_{(2n)} m = h_1 k'_{(1)} k'_{(-2n+2)} k_{(-n)} \otimes \cdots \\ &\otimes h_{n-1} k'_{(n-1)} k'_{(-n)} k_{(-2)} \otimes h_n k_{(-1)} \otimes k_{(n+1)} k'_{(2n-1)} m \\ &= \cdots = h_1 k_{(-n)} \otimes \cdots \otimes h_n k_{(-1)} \otimes k_{(n+1)} k'm. \end{aligned}$$

A straightforward computation proves that:

$$\begin{array}{cccc} (\pi_n)^{-1} : & H^n \otimes A & \longrightarrow & H^{n+1} \otimes_H A \\ & h_1 \otimes \cdots \otimes h_n \otimes a & \mapsto & (h_1 \otimes \cdots \otimes h_n \otimes 1) \otimes_H a, \end{array}$$

is both a left and right inverse of π_n. □

For $n = 1$ we will often denote the map π_1 simply by π.

Lemma 2.8. *Let H be a cocommutative Hopf algebra and M be a finite H-module. If $m \in M$ is not a torsion element then $\alpha \otimes_H m = 0$ if and only if $\alpha = 0$, where α is an element in $H \otimes H$.*

Proof. Let Hm be the cyclic module generated by m. Since m is not a torsion element, i.e., there not exists $h \in H$ such that $hm = 0$, the map ψ sending h to hm is an isomorphism of vector spaces from H to Hm. Let

$$\alpha = \sum_i \beta^i \otimes \gamma^i \in H \otimes H.$$

By Lemma 2.7 we have

$$\pi((\alpha \otimes_H m)) = \sum_i \beta^i \gamma^i{}_{(-1)} \otimes \gamma^i{}_{(2)} m \in H \otimes Hm. \tag{2.9}$$

Applying $(\mathrm{id}_H \otimes \psi^{-1})$ to (2.9) we obtain $\sum_i \beta^i \gamma^i{}_{(-1)} \otimes \gamma^i{}_{(2)} = \alpha'$. Since $\mathcal{F}^{-1}(\alpha') = \alpha$ and all these maps are isomorphisms of vector spaces we have the statement. □

Let $\sigma : H \otimes H \longrightarrow H \otimes H$ be the unique $\Bbbk$-linear map which satisfies

$$(h \otimes k)^\sigma = k \otimes h.$$

We will call σ the *flip map* of $H \otimes H$. If $h_1 \otimes \cdots \otimes h_n$ is an element of H^n then we denote by σ_{ij}, $i < j$, the unique $\Bbbk$-linear map from H^n to H^n, $n \geq 2$, such that

$$(h_1 \otimes \cdots \otimes h_i \otimes \cdots \otimes h_j \otimes \cdots \otimes h_n)^{\sigma_{ij}} = (h_1 \otimes \cdots \otimes h_j \otimes \cdots \otimes h_i \otimes \cdots \otimes h_n),$$

i.e., σ_{ij} is the flip map applied to the i-th and j-th component of any expression in H^n.

2.2 Pseudoalgebras over H

In this section we recall the notion of *pseudoalgebra over a Hopf algebra H*. All unproved statements are taken from [BDK1].
Let H be a Hopf algebra and A be a left H-module. An *H-pseudoproduct* on A is a $H \otimes H$-linear map

$$\begin{array}{rccc} * : & A \otimes A & \longrightarrow & (H \otimes H) \otimes_H A \\ & a \otimes b & \mapsto & a * b, \end{array}$$

where we consider $(H \otimes H) \otimes_H A$ with its structure of an $H \otimes H$-module.
A *pseudoalgebra over H* or an *H-pseudoalgebra* is a left H-module A endowed with an H-pseudoproduct. We say that a pseudoalgebra A is *finite* if it is finitely generated as an H-module.
Let A be an H-pseudoalgebra with an H-pseudoproduct $*$. The *expanded pseudoproduct* [BDK1, Ko1] is an H^{m+n}-linear map

$$* : \quad (H^m \otimes_H A) \otimes (H^n \otimes_H A) \quad \longrightarrow \quad H^{m+n} \otimes_H A,$$

defined as

$$(F \otimes_H a) * (G \otimes_H b) = (F \otimes G)(\Delta^{m-1} \otimes \Delta^{n-1} \otimes_H \mathrm{id}_A)(a * b), \tag{2.10}$$

where $F \in H^m, G \in H^n, a, b \in A$, $m, n \geq 1$.
We say that an H-pseudoalgebra A is *associative* if

$$a * (b * c) = (a * b) * c, \tag{2.11}$$

for any $a, b, c \in A$.
Both sides of equation (2.11) lie in $H^3 \otimes_H A$ and are defined by use of (2.10).
An H-pseudoproduct is said to be *commutative* (resp. *skew-commutative*) if for any $a, b \in A$ it satisfies:

$$a * b = (b * a)^\sigma \quad (\text{resp. } a * b = -(b * a)^\sigma), \tag{2.12}$$

where by σ we mean the map $\sigma \otimes_H \mathrm{id}_A$.
An H-pseudoproduct $[\ *\ ]$ is called a *Lie pseudobracket* if it is skew-commutative and it satisfies the *Jacobi identity*

$$[[a * b] * c] = [a * [b * c]] - [b * [a * c]]^{\sigma_{12}}, \tag{2.13}$$

where (2.13) is an equality in $H^3 \otimes_H A$ and σ_{12} denotes the map $(\sigma \otimes \mathrm{id}_H) \otimes_H \mathrm{id}_A$. An *associative pseudoalgebra* (resp. a *Lie pseudoalgebra*) over H is an H-pseudoalgebra endowed with an *associative pseudoproduct* (resp. a *Lie pseudobracket*).

Remark 2.3. *Notice that one may define associative pseudoalgebras over arbitrary Hopf algebras, whereas cocommutativity is necessary for handling Lie or commutative pseudoalgebras since otherwise (2.12) is not well defined.*

Remark 2.4. *If $(A, *)$ is an associative pseudoalgebra over a cocommutative Hopf algebra H then the following map*

$$[a * b] = (a * b) - (b * a)^{\sigma}, \quad a, b \in A, \tag{2.14}$$

is a Lie pseudobracket and endows A with a structure of a Lie pseudoalgebra over H.

Example 2.9. *If $H = \Bbbk$ then $H \otimes H \simeq H$ and $\Delta = \mathrm{id}_{\Bbbk}$. In this case the axioms of an associative H-pseudoalgebra (resp. of a Lie H-pseudoalgebra) are equivalent to those of an ordinary associative (resp. Lie) algebra over $\Bbbk$.*

Example 2.10. *Let $\mathfrak{d} = \Bbbk\partial$ be a one dimensional Lie algebra over $\Bbbk$. Then $H = \mathcal{U}(\mathfrak{d}) = \Bbbk[\partial]$ has a natural cocommutative Hopf algebra structure. The axioms of Lie pseudoalgebra over H are then equivalent to the axioms of a Lie conformal algebra given in Section 1.1.*
The equivalence between the λ-bracket and the pseudobracket $[\ * \ ]$ is given by

$$[a * b] = \sum_i P_i(\partial \otimes 1, 1 \otimes \partial) \otimes_H e_i \;\longleftrightarrow\; [a_\lambda b] = \sum_i P_i(-\lambda, \partial + \lambda) e_i.$$

Lie pseudoalgebras over a cocommutative Hopf algebra H can be viewed as a multivariable generalization of Lie conformal algebras.

Example 2.11. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$, H be a cocommutative Hopf algebra. Then setting*

$$[(1 \otimes a) * (1 \otimes b)] = (1 \otimes 1) \otimes_H (1 \otimes [a, b]), \quad a, b \in \mathfrak{g},$$

extends uniquely by $(H \otimes H)$-linearity to a Lie H-pseudoalgebra structure on $\mathrm{Cur}\,\mathfrak{g} = H \otimes \mathfrak{g}$, *with H acting on the first tensor factor, called* current Lie pseudoalgebra.

Example 2.12. *Let $W(\mathfrak{d}) = H \otimes \mathfrak{d}$, where $\mathfrak{d}$ is a finite dimensional Lie algebra over $\Bbbk$. Then setting*

$$[(1 \otimes a) * (1 \otimes b)] = (1 \otimes 1) \otimes_H (1 \otimes [a, b]) - (1 \otimes a) \otimes_H (1 \otimes b) + (b \otimes 1) \otimes_H (1 \otimes a),$$

extends uniquely on $W(\mathfrak{d})$ to a structure of a Lie pseudoalgebra over H, called Lie pseudoalgebra of type W.

Example 2.13. *Let $M = H \otimes M_0$ be a free H-module, where M_0 is a finite dimensional vector space over $\Bbbk$ with a trivial action of H. Let $\mathrm{Cend}\, M = H \otimes H \otimes \mathrm{End}\, M_0$. Then setting*

$$(1 \otimes a \otimes A) * (1 \otimes b \otimes B) = (1 \otimes a_{(1)}) \otimes_H (1 \otimes b a_{(2)} \otimes AB),$$

extends uniquely on Cend M *to a structure of associative pseudoalgebra over* H*, with* H *acting on the first tensor factor. We will see in the sequel that this is the associative pseudoalgebra over* H *of all* H-pseudolinear maps *from* M *to itself. The Lie* H*-pseudoalgebra obtained from* Cend M *by (2.14) is denoted by* gc M*.*

Let L_1, L_2 be Lie pseudoalgebras over H. A *Lie pseudoalgebra homomorphism* from L_1 to L_2 is an H-linear map $\rho : L_1 \longrightarrow L_2$ such that

$$((\mathrm{id}_H \otimes \mathrm{id}_H) \otimes_H \rho)([a * b]) = [\rho(a) * \rho(b)], \quad \forall\, a, b \in L_1. \tag{2.15}$$

Let L be a Lie pseudoalgebra over H, $a, b \in L$ and $\{h^i\}$ be a $\Bbbk$-basis of H. Then $[a * b]$ can be uniquely written in the form

$$[a * b] = \sum_i (h^i \otimes 1) \otimes_H e_i.$$

We will call the elements $e_i \in L$ the *coefficients of* $[a * b]$. If A, B are two subsets of a Lie pseudoalgebra L then we denote by $[A, B]$ the H-submodule generated by all coefficients of $[a * b]$, where $a \in A$, $b \in B$. A *subalgebra* of a Lie pseudoalgebra L is an H-submodule N such that $[N, N] \subset N$. An *ideal* of a Lie pseudoalgebra L is an H-submodule I such that $[L, I] \subset I$. A Lie pseudoalgebra L is *simple* if its only ideals are trivial and L is not abelian, i.e., if $[L, L] \neq 0$. A Lie pseudoalgebra is *semisimple* if it contains no nonzero abelian ideals.
The *derived series* of a Lie pseudoalgebra L is defined as usual as

$$L^{(0)} = L, \; L^{(k+1)} = [L^{(k)}, L^{(k)}], \quad k \geq 0.$$

L is a *solvable* Lie pseudoalgebra if there exists K such that $L^{(K)} = \{0\}$.
The *central series* of L is

$$L^{[0]} = L, \; L^{[k+1]} = [L, L^{[k]}], \quad k \geq 0.$$

L is a *nilpotent* Lie pseudoalgebra if $L^{[K]} = \{0\}$ for sufficiently large K.
As usual subalgebras and quotients of solvable (resp. nilpotent) Lie pseudoalgebras are solvable (resp. nilpotent).
Let $H^* = \mathrm{Hom}_\Bbbk(H, \Bbbk)$ and $\gamma \in H^*$. Then we have the following

Proposition 2.14. *Let* L *be a Lie pseudoalgebra over* H*. For any* $\gamma \in H^*$ *the map*

$$\begin{array}{rlcl} \tau: & (H \otimes H) \otimes_H L & \longrightarrow & L \\ & (h \otimes k) \otimes_H a & \mapsto & \gamma(h k_{(-1)}) k_{(2)} a. \end{array}$$

is well defined.

Proof. We have to check that for any $g \in H$ the condition

$$\tau((h \otimes k) \otimes_H ga) = \tau((h g_{(1)} \otimes k g_{(2)}) \otimes_H a)$$

is satisfied. Indeed

$$\begin{aligned} \tau((h g_{(1)} \otimes k g_{(2)}) \otimes_H a) &= \gamma(h g_{(1)} g_{(-2)} k_{(-1)}) k_{(2)} g_{(3)} a = \gamma(h \epsilon(g_{(1)}) k_{(-1)}) k_{(2)} g_{(2)} a \\ &= \gamma(h k_{(-1)}) k_{(2)} \epsilon(g_{(1)}) g_{(2)} a = \gamma(h k_{(-1)}) k_{(2)} g a \\ &= \tau((h \otimes k) \otimes_H ga). \end{aligned}$$

□

Remark 2.5. *The map τ defined in Proposition 2.14 equals $(\gamma \otimes \mathrm{id}_L) \circ \pi$.*

If L is a Lie pseudoalgebra over H and $[a * b] = \sum_i (h^i \otimes k^i) \otimes_H e_i$ then we call the elements $\tau([a * b]) \in L$ the *γ-coefficients* of $[a * b]$. Notice that if S is a subalgebra of L then all the γ-coefficients of products of elements in S still lie in S.

2.2.1 Extending scalars with pseudoalgebras

Let H, H' be Hopf algebras, A be an associative H-pseudoalgebra and $\phi : H \longrightarrow H'$ be a Hopf homomorphism. Then ϕ endows H' with a right H-module structure so that we may consider the tensor product $H' \otimes_H A$ for any left H-module A.

Proposition 2.15. *The left H'-module $A' = H' \otimes_H A$ has a structure of associative H'-pseudoalgebra satisfying:*

$$(h' \otimes_H a) * (k' \otimes_H b) = \sum_i (h'\phi(h^i) \otimes k'\phi(k^i)) \otimes_{H'} (1 \otimes_H e_i), \tag{2.16}$$

if $a * b = \sum_i (h^i \otimes k^i) \otimes_H e_i$, $a, b \in A$.

Proof. First of all we have to check that (2.16) gives a well defined map.
Let $h, k \in H$, then

$$\begin{aligned}
(h' \otimes_H ha) * (k' \otimes_H kb) &= \sum_i (h'\phi(hh^i) \otimes k'\phi(kk^i)) \otimes_{H'} (1 \otimes_H e_i) \\
&= \sum_i (h'\phi(h)\phi(h^i) \otimes k'\phi(k)\phi(k^i)) \otimes_{H'} (1 \otimes_H e_i) \\
&= (h'\phi(h) \otimes_H a) * (k'\phi(k) \otimes_H b).
\end{aligned}$$

We also have to check that for any $h \in H$ the elements $\sum_i (h^i \otimes k^i) \otimes_H he_i$ and $\sum_i (h^i h_{(1)} \otimes k^i h_{(2)}) \otimes_H e_i$ define the same element in $(H' \otimes H') \otimes_{H'} (H' \otimes_H L)$. Indeed,

$$\begin{aligned}
\sum_i (\phi(h^i) \otimes \phi(k^i)) \otimes_{H'} (1 \otimes_H he_i) &= \sum_i (\phi(h^i) \otimes \phi(k^i)) \otimes_{H'} (\phi(h) \otimes_H e_i) \\
&= \sum_i (\phi(h^i) \otimes \phi(k^i))(\Delta(\phi(h))) \otimes_{H'} (1 \otimes_H e_i) \\
&= \sum_i (\phi(h^i) \otimes \phi(k^i))((\phi \otimes \phi)\Delta(h)) \otimes_{H'} (1 \otimes_H e_i) \\
&= \sum_i (\phi(h^i)\phi(h_{(1)}) \otimes \phi(k^i)\phi(h_{(2)})) \otimes_{H'} (1 \otimes_H e_i) \\
&= \sum_i (\phi(h^i h_{(1)}) \otimes \phi(k^i h_{(2)})) \otimes_{H'} (1 \otimes_H e_i).
\end{aligned}$$

It is an $H' \otimes H'$-linear map by the very definition.
We are left with proving that (2.16) is an associative pseudoproduct. Let

$$a * b = \sum_i (h^i \otimes k^i) \otimes_H e_i, \quad e_i * c = \sum_j (h^{ij} \otimes k^{ij}) \otimes_H e_{ij},$$

so that by (2.10) we have

$$(a * b) * c = \sum_{i,j} (h^i h^{ij}_{(1)} \otimes k^i h^{ij}_{(2)} \otimes k^{ij}) \otimes_H e_{ij}.$$

Then

$$\begin{aligned}
&((h'\otimes_H a) * (k'\otimes_H b)) * (l'\otimes_H c)\\
&= (\sum_i (h'\phi(h^i) \otimes k'\phi(k^i)) \otimes_{H'} (1\otimes_H e_i)) * (l'\otimes_H c)\\
&= \sum_i (h'\phi(h^i) \otimes k'\phi(k^i) \otimes 1)(\Delta \otimes 1)(1\otimes_H e_i)) * (l'\otimes_H c)\\
&= \sum_{i,j} (h'\phi(h^i)\phi(h^{ij}_{(1)}) \otimes k'\phi(k^i)\phi(h^{ij}_{(2)}) \otimes l'\phi(k^{ij})) \otimes_{H'} (1\otimes_H e_{ij})\\
&= \sum_{i,j} (h'\phi(h^i h^{ij}_{(1)}) \otimes k'\phi(k^i h^{ij}_{(2)}) \otimes l'\phi(k^{ij})) \otimes_{H'} (1\otimes_H e_{ij}).
\end{aligned}$$

Similarly, if

$$a * (b * c) = \sum_{i,j} (f^{ij} \otimes f^i g^{ij}_{(1)} \otimes g^i g^{ij}_{(2)}) \otimes_H d_{ij},$$

then

$$\begin{aligned}
&(h'\otimes_H a) * ((k'\otimes_H b) * (l'\otimes_H c))\\
&= (h'\otimes_H a) * \sum_i (k'\phi(f^i) \otimes l'\phi(g^i)) \otimes_{H'} (1\otimes_H d_i)\\
&= (1 \otimes k'\phi(f^i) \otimes l'\phi(g^i))(1 \otimes \Delta)(h'\otimes_H a) * (1\otimes_H d_i)\\
&= \sum_{i,j} (h'\phi(f^{ij}) \otimes k'\phi(f^i)\phi(g^{ij})_{(1)} \otimes l'\phi(g^i)\phi(g^{ij})_{(2)})) \otimes_{H'} (1\otimes_H d_{ij})\\
&= \sum_{i,j} (h'\phi(f^{ij}) \otimes k'\phi(f^i g^{ij}_{(1)}) \otimes l'\phi(g^i g^{ij}_{(2)})) \otimes_{H'} (1\otimes_H d_{ij}).
\end{aligned}$$

Since $a * (b * c) = (a * b) * c$, we have

$$(h'\otimes_H a) * ((k'\otimes_H b) * (l'\otimes_H c)) = ((h'\otimes_H a) * (k'\otimes_H b)) * (l'\otimes_H c),$$

□

Remark 2.6. *A more conceptual proof of the above statement follows from the fact that if $\mu : A \otimes A \longrightarrow (H \otimes H)\otimes_H A$ is a H-pseudoproduct on the left H-module A then the corresponding H'-pseudoproduct on $H'\otimes_H A$ is given by the sequence of compositions*

$$\begin{aligned}
&(H'\otimes_H A) \otimes (H'\otimes_H A) \xrightarrow{\sim} (H' \otimes H') \otimes_{H\otimes H} (A \otimes A) \overset{(\mathrm{id}_{H'} \otimes \mathrm{id}_{H'})\otimes_{H\otimes H}\mu}{\longrightarrow}\\
&(H' \otimes H') \otimes_{H\otimes H} ((H \otimes H)\otimes_H A) \xrightarrow{\sim} (H' \otimes H')\otimes_H A \xrightarrow{\sim} (H' \otimes H') \otimes_{H'} (H'\otimes_H A).
\end{aligned}$$

Then associativity, commutativity, skew-commutativity, Jacobi identity, on A imply the corresponding properties on $\Bbbk\otimes_H A$, hence on all of $H'\otimes_H A$ by $(H' \otimes H')$-linearity.

As a consequence of Remark 2.6 we have that if L is a Lie H-pseudoalgebra then $L' = H'\otimes_H L$ has a structure of Lie H'-pseudoalgebra satisfying:

$$[(h'\otimes_H a) * (k'\otimes_H b)] = \sum_i (h'\phi(h^i) \otimes k'\phi(k^i)) \otimes_{H'} (1\otimes_H e_i), \tag{2.17}$$

if $[a * b] = \sum_i (h^i \otimes k^i)\otimes_H e_i$.

We will say that A' (resp. L') is obtained from A (resp. L) by *extension of scalars* or *base change*. Let $\phi : H \longrightarrow H'$ be a Hopf homomorphism, L be a Lie pseudoalgebra over H. Then we denote by $\mathrm{BC}_\phi(L) := H'\otimes_H L$ the Lie pseudoalgebra over H' obtained from L by extension of scalars. Similarly, if $\rho : L_1 \longrightarrow L_2$ is a Lie H-pseudoalgebra homomorphism, we set $\mathrm{BC}_\phi(\rho) = \mathrm{id}_{H'} \otimes_H \rho$. Let $\mathcal{P}s\mathcal{A}lg_H$ be the category of Lie pseudoalgebras over H.

Theorem 2.16. *Let H, H' be cocommutative Hopf algebras, $\phi : H \longrightarrow H'$ be a Hopf homomorphism. Then $\mathrm{BC}_\phi : \mathcal{P}s\mathcal{A}lg_H \longrightarrow \mathcal{P}s\mathcal{A}lg_{H'}$ is a (covariant) functor.*

Proof. As clearly, for any choice of Lie H-pseudoalgebras L_1, L_2, L_3, $\rho \in \mathcal{M}or(L_1, L_2)$, $\mu \in \mathcal{M}or(L_2, L_3)$ we have $\mathrm{BC}_\phi(\mu) \circ \mathrm{BC}_\phi(\rho) = \mathrm{BC}_\phi(\mu \circ \rho)$ and $\mathrm{BC}_\phi(\mathrm{id}_L) = \mathrm{id}_{\mathrm{BC}_\phi(L)}$ we are left with proving that if $\rho \in \mathcal{M}or(\mathcal{P}s\mathcal{A}lg_H)$ then $\mathrm{BC}_\phi(\rho) \in \mathcal{M}or(\mathcal{P}s\mathcal{A}lg_{H'})$. Let $\rho : L_1 \longrightarrow L_2$ be a Lie H-pseudoalgebra homomorphism. Notice that $\mathrm{BC}_\phi(\rho)$ is well defined since both $\mathrm{id}_{H'}$ and ρ are H-linear maps. Moreover,

$$
\begin{aligned}
&((\mathrm{id}_{H'} \otimes \mathrm{id}_{H'}) \otimes_{H'} \mathrm{BC}_\phi(\rho))(1 \otimes_H a) * (1 \otimes_H b) \\
&= ((\mathrm{id}_{H'} \otimes \mathrm{id}_{H'}) \otimes_{H'} (\mathrm{id}_{H'} \otimes_H \rho))(1 \otimes_H a) * (1 \otimes_H b) \\
&= (\mathrm{id}_{H'} \otimes \mathrm{id}_{H'}) \otimes_{H'} (1 \otimes_H \rho(a * b)) = (1 \otimes_H \rho(a)) * (1 \otimes_H \rho(b)) \\
&= (\mathrm{BC}_\phi(\rho)(1 \otimes_H a)) * (\mathrm{BC}_\phi(\rho)(1 \otimes_H b)),
\end{aligned}
$$

By $(H' \otimes H')$-linearity this proves that (2.15) is satisfied. □

Notice that if $\phi = \mathrm{id}_H$ then $\mathrm{BC}_{\mathrm{id}_H} = \mathrm{id}$ and if $i : H \longrightarrow H'$ is the inclusion of H in H' then BC_i coincides with $\mathrm{Cur}_H^{H'}$ (see [BDK1] for a definition).

2.3 The annihilation algebra of a Lie pseudoalgebra

In this section we present the construction given in [BDK1] which consents to associate to any Lie pseudoalgebra L an infinite dimensional Lie algebra $\mathcal{L}$.

2.3.1 Filtration and topology on H and X

Let $\mathfrak{d}$ be a finite dimensional Lie algebra and $\{\partial_1, \ldots, \partial_N\}$ be a $\Bbbk$ basis of $\mathfrak{d}$. Its universal enveloping algebra $\mathcal{U}(\mathfrak{d})$ has a Hopf algebra structure, that we explicitly described in Example 2.2.
We set:

$$\partial^{(I)} = \partial_1^{i_1} \cdots \partial_N^{i_N} / i_1! \cdots i_N!, \quad I = (i_1, \ldots, i_N) \in \mathbb{Z}_+^N. \tag{2.18}$$

Then $\{\partial^{(I)}\}$ is a basis of H such that

$$\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}.$$

$H = \mathcal{U}(\mathfrak{d})$ has a canonical increasing filtration, independent of the choice of a basis of $\mathfrak{d}$, given by

$$F^p H = \mathrm{span}_\Bbbk \{\partial^{(I)} \mid |I| = i_1 + \cdots + i_N \leq p\}, \quad p = 0, 1, 2, \ldots \tag{2.19}$$

This filtration satisfies

$$(F^p H)(F^q H) \subset F^{p+q} H, \tag{2.20}$$

$$\Delta(F^p H) \subset \sum_{i=0}^{p} F^i H \otimes F^{p-i} H, \tag{2.21}$$

$$S(F^p H) \subset F^p H. \tag{2.22}$$

Moreover $\bigcup_p F^p H = H$ and $\dim(F^p H) < \infty$ for any $p \in \mathbb{N}$. We will say that an element $h \in H$ is of *degree* p if $h \in F^p H \backslash F^{p-1} H$. The first terms of our filtration are the following,

$$F^{-1} H = \{0\}, \quad F^0 H = \Bbbk, \quad F^1 H = \Bbbk \oplus \mathfrak{d}.$$

We consider on $H \otimes H$ the filtration given by:

$$F^p(H \otimes H) = \sum_{l+m=p} F^l H \otimes F^m H.$$

Let $X = H^* = \mathrm{Hom}(H, \Bbbk)$ be the dual of $H = \mathcal{U}(\mathfrak{d})$. X is both a left and right H-module, with actions given respectively by:

$$\begin{aligned} \langle hx, f \rangle &= \langle x, S(h) f \rangle, \\ \langle xh, f \rangle &= \langle x, f S(h) \rangle, \end{aligned}$$

for any $f, h \in H$, $x \in X$.
The associativity of H implies that X is a H-*bimodule*, i.e.,

$$f(xg) = (fx)g, \quad f, g \in H, x \in X.$$

Cocommutativity of H implies commutativity of X. Moreover, X is an associative algebra which satisfies for any $f \in H$, $x, y \in X$:

$$\langle xy, f \rangle = \langle x, f_{(1)} \rangle \langle y, f_{(2)} \rangle. \tag{2.23}$$

We define an antipode $S : X \longrightarrow X$ as the dual of that of H:

$$\langle S(x), h \rangle = \langle x, S(h) \rangle.$$

Let $X \hat{\otimes} X = (H \otimes H)^*$ be the completed tensor product. We can define a map $\Delta : X \longrightarrow X \hat{\otimes} X$ as the dual of the multiplication of H. It satisfies:

$$\begin{aligned} \langle xy, f \rangle &= \langle x \otimes y, \Delta(f) \rangle = \langle x, f_{(1)} \rangle \langle y, f_{(2)} \rangle, \\ \langle x, fg \rangle &= \langle \Delta(x), f \otimes g \rangle = \langle x_{(1)}, f \rangle \langle x_{(2)}, g \rangle, \end{aligned}$$

for any $x, y \in X$, $f, g \in H$.
The increasing filtration $\{F^p H\}$ on H induces a decreasing filtration on X given by

$$F_p X = (F^p H)^{\perp} = \{x \in X \mid \langle x, f \rangle = 0, \ \forall f \in F^p H\}, \quad p \geq -1. \tag{2.24}$$

This two filtrations are compatible, i.e., the action of H on X satisfies

$$(F^q H).(F_p X) \subset F_{p-q} X \quad \text{for all } p, q.$$

It is easy to check that the filtration of X satisfies

$$(F_p X)(F_q X) \subset F_{p+q+1} X, \tag{2.25}$$

$$\mathfrak{d}(F_p X) \subset F_{p-1} X, \tag{2.26}$$

$$(F_p X)\mathfrak{d} \subset F_{p-1} X. \tag{2.27}$$

Moreover,

$$S(F_pX) \subset F_pX, \quad \Delta(F_pX) \subset \sum_{i=-1}^{p} F_iX \hat{\otimes} F_{p-i-1}X.$$

As a consequence of the properties of $\{F^pH\}$, the filtration $\{F_pX\}$ of X satisfies $\bigcap_p F_pX = 0$ and $\dim(X/F_pX) < \infty$, for any p. If we take $\{F_pX\}$ as a fundamental system of neighborhoods of 0 then X becomes a topological vector space.

2.3.2 Linearly compact algebras

In order to study the properties of X as a topological vector space we recall the following result (see [Gu1] for a proof).

Theorem 2.17. *Let $\mathcal{L}$ be a topological vector space over the discrete field $\Bbbk$. The following statements are equivalent:*

- *(a) $\mathcal{L}$ is the dual of a discrete vector space.*
- *(b) The topological dual $\mathcal{L}^*$ of $\mathcal{L}$ is a discrete topological space.*
- *(c) $\mathcal{L}$ is the topological product of finite dimensional discrete vector spaces.*
- *(d) $\mathcal{L}$ is the projective limit of finite dimensional discrete vector spaces.*
- *(e) $\mathcal{L}$ has a collection of finite codimensional open subspaces whose intersection is $\{0\}$, with respect to which it is complete.*

A vector space V satisfying one of the equivalent conditions of Theorem 2.17 is called a *linearly compact vector space*. Then X is a linearly compact vector space. We consider X with this topology and H, in particular $\mathfrak{d} \subset H$, with the discrete topology.
Let $\{x_I\}$ be the *dual basis* of X. It is a *topological basis of X which tends to* 0, i.e., such that for any p all but a finite number of x_I belong to F_pX.
For any $f \in H$, $y \in X$ we have:

$$f = \sum_I \langle x_I, f \rangle \partial^{(I)}, \quad y = \sum_I \langle y, \partial^{(I)} \rangle x_I.$$

An (associative or Lie) algebra is *linearly compact* if the underlying topological vector space is linearly compact.

Example 2.18. *Let $\mathcal{O}_N = \Bbbk[[t_1, \ldots, t_N]]$ be the associative algebra of formal power series in the N indeterminate $t_1, \ldots, t_N$. Let:*

$$F_p\mathcal{O}_N = (t_1, \ldots, t_N)^{p+1}, \quad p = -1, 0, 1, \ldots.$$

Then $\{F_p\mathcal{O}_N\}$ is a collection of finite-codimensional open ideals satisfying condition (e) of Theorem 2.17, i.e., $\mathcal{O}_N$ is a linearly compact associative algebra.

Example 2.19. *Let W_N be the Lie algebra of continuous derivations of $\mathcal{O}_N$. W_N has a canonical filtration induced by that of $\mathcal{O}_N$,*

$$F_qW_N = \{D \in W_N \mid D(F_p\mathcal{O}_N) \subset F_{p+q}\mathcal{O}_N,\ \forall\, i\}, \quad q = -1, 0, 1, \ldots$$

this follows by Theorem 2.17 that W_N is a simple linearly compact Lie algebra.

Proposition 2.20. *Let $H = \mathcal{U}(\mathfrak{d})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{d}$ of finite dimension N. Then $X = H^*$ can be identified with $\mathcal{O}_N = \Bbbk[[t_1, \ldots, t_N]]$ as a topological commutative algebra.*

Proof. Let $\{\partial^{(I)}\}$ be the basis of H. Any element $a \in X$ is uniquely determined by its values $a_I = \langle a, \partial^{(I)}\rangle$. Then we can write $a = \sum_I a_I x_I$ as the right-hand side becomes a finite sum when computed on any element of H. Then $a \mapsto \sum_I a_I t_1^{i_1} \cdots t_N^{i_N}$ is an isomorphism and is compatible with the topology. $\square$

Via the identification of Proposition 2.20 the elements of the filtration $\{F_pX\}$ of X become

$$F_pX \simeq F_p\mathcal{O}_N = (t_1, \ldots, t_N)^{p+1}\mathcal{O}_N, \quad p = -1, 0, 1, \ldots.$$

We now present two special instances of finite dimensional Lie algebras and of the corresponding universal enveloping algebra that we need later on.
Let $\{\partial_1, \ldots, \partial_{2n}\}$ be a basis of an abelian Lie algebra $\mathfrak{d}$. Its universal enveloping algebra $H = \mathcal{U}(\mathfrak{d})$ is then isomorphic to the symmetric algebra $S(\mathfrak{d}) = \Bbbk[\partial_1, \ldots, \partial_{2n}]$. In this case H is a *graded associative commutative algebra* with standard gradation given by:

$$G^pH = \{\partial^{(I)} \mid |I| = i_1 + \cdots + i_N = p\}, \quad p \geq 0, \tag{2.28}$$

so that

$$(G^pH)(G^qH) \subset G^{p+q}H.$$

It induces the canonical filtration

$$F^pH = \bigoplus_{q \leq p} G^qH = \{\partial^{(I)} \mid |I| \leq p\}$$

of H. Let $X = H^* \simeq \mathcal{O}_{2n} = \Bbbk[[t_1, \ldots, t_{2n}]]$. Since $\mathfrak{d}$ is abelian then the left and the right action of $\mathfrak{d}$ on X coincide and ∂_i, $i = 1, \ldots, 2n$, acts on X as $-\partial/\partial t_i$. Then

$$G^pX = \{\psi \in X \mid \psi|_{G^qH} \equiv 0, \quad p \neq q\}, \quad p \geq 0, \tag{2.29}$$

defines the standard grading on X. Namely, $X = \prod_{p \geq 0} G^pX$ and

$$(G^pX)(G^qX) \subset G^{p+q}X. \tag{2.30}$$

Notice that $\bigoplus_{p \geq 0} G^pX$ is a graded dense subalgebra of X.

Lemma 2.21. *The action of $H = S(\mathfrak{d})$ on $X = \Bbbk[[t_1, \dots, t_{2n}]]$ satisfies:*

$$(G^q H).(G^p X) \subset G^{p-q} X, \tag{2.31}$$

for all p, q. In particular,

$$\mathfrak{d}.(G^p X) \subset G^{p-1} X, \quad (G^p).\mathfrak{d} \subset G^{p-1} X.$$

Proof. It can be proved by induction on q. □

The above gradation induces the canonical filtration on X,

$$F_p X = \prod_{q \geq p+1} G^q X.$$

We have $F_p X / F_{p+1} X = G^{p+1} X$.
Let $\mathfrak{d}$ be a Heisenberg Lie algebra with a basis $\{\partial_0, \partial_1, \dots, \partial_{2n}\}$ such that $[\partial_i, \partial_{n+i}] = -\partial_0$, $i = 1, \dots, n$, are the only nonzero commutation relations. We denote by $\bar{\mathfrak{d}}$ the vector space linearly generated by ∂_i, $i = 1, \dots, 2n$. Let $H = \mathcal{U}(\mathfrak{d})$ and define:

$$G'^p H = \{\partial^{(I')} \mid |I'| = 2i_0 + i_1 + \cdots + i_{2n} = p\}, \quad p \geq 0. \tag{2.32}$$

With respect to $G'^p H$ the Lie bracket on $\mathfrak{d}$ is homogeneous, so that (2.32) makes H into a graded associative algebra. Namely, $H = \bigoplus_{p \geq 0} G'^p H$ and

$$(G'^p H)(G'^q H) \subset G'^{p+q} H.$$

The above *prime gradation* induces a *prime filtration* on H given by:

$$F'^p H = \bigoplus_{q \leq p} G'^q H = \{\partial^{(I')} \mid |I'| = 2i_0 + i_1 + \cdots + i_{2n} \leq p\}, \quad p \geq 0.$$

$\{F'^p H\}$ is compatible with the Hopf algebra structure of H and it satisfies $F'^0 H = \Bbbk$, $F'^1 H = \Bbbk \oplus \bar{\mathfrak{d}}$, $F'^2 H \supset \Bbbk \oplus \mathfrak{d} = F^1 H$. Moreover, it is equivalent to the canonical filtration $\{F^p H\}$ of H.
Let $X = \mathcal{O}_{2n+1} = \Bbbk[[t_0, t_1, \dots, t_{2n}]]$. Then

$$G'^p X = \{\psi \in X \mid \psi|_{G'^q H} \equiv 0, \quad p \neq q\}, p \geq 0, \tag{2.33}$$

defines a *prime gradation* on X. Namely, $X = \prod_{p \geq 0} G'^p X$ and

$$(G'^p X)(G'^q X) \subset G'^{p+q} X \tag{2.34}$$

Notice that $\bigoplus_{p \geq 0} G'^p X$ is a graded dense subalgebra of X.

Lemma 2.22. *The action of H on X satisfies*

$$(G'^q H).(G'^p X) \subset G'^{p-q} X, \tag{2.35}$$

for all p, q. In particular,

$$\bar{\mathfrak{d}}.(G'^p X) \subset G'^{p-1} X, \quad (G'^p X).\bar{\mathfrak{d}} \subset G'^{p-1} X, \tag{2.36}$$

$$\partial_0.(G'^p X) \subset G'^{p-2} X, \quad (G'^p X).\partial_0 \subset G'^{p-2} X. \tag{2.37}$$

As usual, we have an induced *prime filtration* on X given by:

$$F'_p X = \prod_{q \geq p+1} G'^q X. \tag{2.38}$$

It satisfies $F'_p X / F'_{p+1} X = G'^{p+1} X$. The prime filtration (2.38) is equivalent to the canonical filtration (2.24).

2.3.3 The linearly compact Lie algebras H_N, P_N and K_N

In this section we introduce other examples of linearly compact Lie algebras that will be the object of our interest in the rest of our thesis.
Let W_N be the linearly compact algebra described in Example 2.19 and suppose that $N = 2n$. Let ω be a symplectic form, i.e., a closed 2-form $\omega = \sum_{ij} \omega_{ij} dt_i \wedge dt_j$, where (ω_{ij}) is a skew-symmetric matrix such that $\det(\omega_{ij}) \neq 0$.
Then

$$H_N(\omega) = \{D \in W_N \mid D\omega = 0\}$$

is a simple Lie subalgebra of W_N.
By an automorphism of $\mathcal{O}_N$, any symplectic form ω can be transformed into the standard one, $\omega_0 = \sum_{i=1}^{n} dt_i \wedge dt_{n+i}$, so that we have an isomorphism of Lie algebras $H_N(\omega) \simeq H_N(\omega_0) = H_N$.
We consider on H_N the following filtration:

$$F_q H_N = \{D \in H_N \mid D(F_p \mathcal{O}_N) \subset F_{p+q} \mathcal{O}_N, \ \forall p\}, \quad q = -1, 0, 1, \ldots$$

If we denote by $H_{N,j} = \{D \in H_N \mid [E, D] = jD\}$ the j-th eigenspace for the adjoint action of the Euler vector field $E = \sum_{i=1}^{N} t_i(\partial/\partial t_i) \in W_N$ then

$$H_N = \prod_{j \geq -1} H_{N,j}, \quad [H_{N,i}, H_{N,j}] \subset H_{N,i+j}, \quad j \geq -1,$$

The following result is well known.

Lemma 2.23. *$H_{N,0}$ is isomorphic to symplectic algebra $\mathfrak{sp}_N(\Bbbk)$. The $\mathfrak{sp}_N(\Bbbk)$-modules $H_{N,j}$ are irreducibles and nontrivial.*

The *Poisson algebra* P_N is $\mathcal{O}_N = \Bbbk[[t_1, \ldots, t_N]]$ endowed with the Poisson bracket

$$\{\phi, \psi\} = \sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial t_i} \frac{\partial \psi}{\partial t_{n+i}} - \frac{\partial \phi}{\partial t_{n+i}} \frac{\partial \psi}{\partial t_i} \right).$$

It is a nontrivial central extension of the Lie algebra H_N:

$$0 \to \Bbbk \to P_N \xrightarrow{\pi} H_N \to 0,$$

where, for any $\phi \in P_N$,

$$\pi(\phi) = \sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial t_i} \frac{\partial}{\partial t_{n+i}} - \frac{\partial \phi}{\partial t_{n+i}} \frac{\partial}{\partial t_i} \right).$$

Let $N = 2n+1$, $n \geq 1$, and θ be a *contact form*, i.e., a 1-form such that $\theta \wedge (d\theta)^{(n)} \neq 0$. Then

$$K_N(\theta) = \{D \in W_N \mid D\theta = \phi\theta \text{ for some } \phi \in \mathcal{O}_N\}$$

is a simple Lie subalgebra of W_N.
By an automorphism of $\mathcal{O}_N$, any contact form θ can be transformed into the standard one $\theta_0 = t_0 + \sum_{i=1}^{n} t_i dt_{n+i}$. As a consequence $K_N(\theta) \simeq K_N(\theta_0) = K_N$.

2.3.4 The annihilation algebra

Let L be a finite Lie pseudoalgebra over H. We associate to L an infinite dimensional linearly compact Lie algebra as follows [BDK1]. Set $\mathcal{L} = X \otimes_H L$. Then

$$[x \otimes_H a, y \otimes_H b] = \sum_i (xh^i)(yk^i) \otimes_H e_i, \quad \text{if} \quad [a * b] = \sum_i (h^i \otimes k^i) \otimes_H e_i, \tag{2.39}$$

linearly extends to a Lie algebra structure on $\mathcal{L}$. $\mathcal{L}$ has a structure of a left H-module given by the action:

$$h.(x \otimes_H a) = hx \otimes_H a, \quad h \in H, x \in X, a \in L. \tag{2.40}$$

For any $h \in H, x, y \in X, a, b \in L$ the action defined in (2.40) satisfies
(2.41)

$$h.[x \otimes_H a, y \otimes_H b] = [h_{(1)}.(x \otimes_H a), h_{(2)}.(y \otimes_H b)] = [h_{(1)} x \otimes_H a, h_{(2)} y \otimes_H b],$$

The condition (2.41) is equivalent to say that $\mathfrak{d} \subset H$ acts by derivation on $\mathcal{L}$.
We define a topology on $\mathcal{L}$ as follows. Let L_0 be a finite dimensional subspace of L such that $L = HL_0$. Then we set

$$F_p\mathcal{L} = \{x \otimes_H a \in L \mid x \in F_pX,\ a \in L_0\}, \quad p \geq -1.$$

This filtration satisfies

$$[F_p\mathcal{L}, F_q\mathcal{L}] \subset F_{p+q-l}\mathcal{L}, \quad \mathfrak{d}(F_p\mathcal{L}) \subset F_{p-l}\mathcal{L},$$

where l is an integer depending only on the choice of L_0.
In [BDK1] it is shown that the topology that $\{F_p\mathcal{L}\}$ induces on $\mathcal{L}$ is independent of the choice of L_0. If we set $\mathcal{L}_p = F_{p+l}\mathcal{L}$ then $[\mathcal{L}_p, \mathcal{L}_q] \subset \mathcal{L}_{p+q}$.
With respect to this topology $\mathcal{L}$ is a linearly compact Lie algebra (i.e., the Lie bracket defined in (2.39) is continuous). It is called the *annihilation algebra* of L. Notice that $\mathcal{L}_0$ is a Lie subalgebra of $\mathcal{L}$ and that any $\mathcal{L}_p$, $p \geq 0$, is an ideal of $\mathcal{L}_0$. Let M be an $\mathcal{L}$-module. For every $p \geq -1$ we define

$$\ker_p M = \{m \in M \mid \mathcal{L}_p.m = 0\}.$$

An $\mathcal{L}$-module M is *conformal* if for evey $m \in M$ there exists p such that $\mathcal{L}_p.m = 0$. An $\mathcal{L}$-module M is $\mathcal{L}_0$*-locally finite* if any $m \in M$ is contained in a finite dimensional $\mathcal{L}_0$-submodule.

Example 2.24. *Let $W(\mathfrak{d})$ be the Lie pseudoalgebra described in Example 2.12. By definition its annihilation algebra is $\mathcal{W} = \mathcal{A}(W(\mathfrak{d})) = X \otimes_H (H \otimes \mathfrak{d}) \simeq X \otimes \mathfrak{d}$, endowed with the bracket:*

$$[x \otimes a, y \otimes b] = xy \otimes [a, b] - x(ya) \otimes b + (xb)y \otimes a, \quad x, y \in X, a, b \in \mathfrak{d}.$$

$\mathcal{W}$ has an H-module structure given by (2.40). If we take $L_0 = \Bbbk \otimes \mathfrak{d}$ then $W(\mathfrak{d}) = HL_0$. The corresponding filtration on $\mathcal{W}$ is given by

$$\mathcal{W}_p = F_p\mathcal{W} = F_pX \otimes_H L_0 \simeq F_pX \otimes \mathfrak{d}.$$

This decreasing filtration of $\mathcal{W}$ satisfies

$$\mathcal{W}_{-1} = \mathcal{W}, \quad \mathcal{W}/\mathcal{W}_0 \simeq \mathfrak{d}, \quad \mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{d} \otimes \mathfrak{d}^* \simeq \mathfrak{gl}\, \mathfrak{d}.$$

A proof of the last isomorphism can be found in [BDK2]. The annihilation algebra $\mathcal{W}$ acts on X by

$$(x \otimes a).y = -x(ya), \quad x, y \in X, a \in \mathfrak{d}.$$

Let $\mathfrak{d}$ be an abelian Lie algebra of dimension N and $\{\partial_1, \ldots, \partial_N\}$ be a basis of $\mathfrak{d}$. In this case the action of $\mathcal{W}$ on $X \simeq \mathcal{O}_N$ is given by

$$(x \otimes \partial_i).y = -x(y\partial_i) = x\frac{\partial y}{\partial t_i}, \quad x, y \in \mathcal{O}_N, \ i = 1, \ldots, N,$$

and the map

$$\begin{array}{rccc} \phi: & \mathcal{W} & \longrightarrow & W_N \\ & x \otimes \partial_i & \mapsto & x\,\partial/\partial t_i, \quad i = 1, \ldots, N, \end{array} \tag{2.42}$$

is an isomorphism of Lie algebras.

Remark 2.7. *It is shown in [BDK2] that $\mathcal{W}$ is isomorphic to W_N for any finite dimensional Lie algebra $\mathfrak{d}$.*

Remark 2.8. *In [BDK1] Cartan's classification of of linearly compact Lie algebras [Ca, Gu2] is used to obtain a classification of finite simple Lie pseudoalgebras over $H = \mathcal{U}(\mathfrak{d})$: $\mathrm{Cur}_{\Bbbk}^{H}\, \mathfrak{g}$, where $\mathfrak{g}$ is a simple finite dimensional Lie algebra over $\Bbbk$, along with all nonzero subalgebras of $W(\mathfrak{d})$ provide a complete list of non-isomorphic finite simple Lie pseudoalgebras over H.*

2.4 Finite modules over pseudoalgebras

In this section we introduce the notion of representation of a pseudoalgebra A as given in [BDK1]. In the fourth chapter we will introduce a new notion of representation of Lie pseudoalgebras [DM] in order to study a broader class of modules over Lie pseudoalgebras.
Let A be a pseudoalgebra over H and M be a left H-module. An *H-pseudoaction* $*$ on M is an $H \otimes H$-linear map from $A \otimes M$ to $(H \otimes H) \otimes_H M$, where we consider $(H \otimes H) \otimes_H M$ with its structure of a left $(H \otimes H)$-module. The *expanded pseudoaction* [BDK1, Ko1] is an H^{m+n}-linear map

$$* : \quad (H^m \otimes_H A) \otimes (H^n \otimes_H M) \quad \longrightarrow \quad H^{m+n} \otimes_H M,$$

defined as

$$(F\otimes_H a) * (G\otimes_H m) = (F\otimes G)(\Delta^{m-1}\otimes\Delta^{n-1}\otimes_H \mathrm{id}_M)(a*m), \tag{2.43}$$

where $F\in H^m, G\in H^n, a\in A, m\in M$, $m,n\geq 1$.
A *representation of an associative H-pseudoalgebra A*, or an *A-module*, is a left H-module M endowed with a pseudoaction $*$ satisfying:

$$(a*b)*m = a*(b*m), \tag{2.44}$$

for any $a,b\in A$, $m\in M$.
Similarly, a *representation of a Lie H-pseudoalgebra L* or an *L-module* is a left H-module M endowed with a pseudoaction $*$ satisfying:

$$[a*b]*m = a*(b*m) - (b*(a*m))^{\sigma_{12}}, \tag{2.45}$$

for any $a,b\in L$, $m\in M$.
Notice that both sides of (2.44) and (2.45) lie in $H^3\otimes_H M$ and are computed by means of (2.43). An A-module M, where A is an H-pseudoalgebra, is *finite* if it is finitely generated as an H-module.
Let A be an H-pseudoalgebra (associative or Lie) and M be a finite A-module. Let $a\in A, m\in M$ and assume that

$$a*m = \sum_i (h^i\otimes 1)\otimes_H m_i,$$

where $\{h^i\}$ is a system of linearly independent vectors of H.
We denote by $A\cdot M$ the H-submodule of M generated by the elements $m_i\in M$. An H-submodule N of M is an A-submodule of M if $A\cdot N\subset N$. If N is an A-submodule of M then $(H^m\otimes_H A)*(H^n\otimes_H N)\subset H^{m+n}\otimes_H N$, as follows by (2.43). An A-module M is *irreducible* if it does not contain nontrivial A-submodules and $A\cdot M\neq\{0\}$. Recall that if M is a left H-module then an element $m\in M$ is a torsion element if there exists a non zero-divisor $h\in H$ such that $hm=0$.

Proposition 2.25. *[BDK1] Let L be a Lie H-pseudoalgebra and M be an L-module. Then the torsion of L acts trivially on M, i.e., $(\mathrm{Tor}\,L)*M=0$ and the torsion of M is acted on trivially by L, i.e., $L*(\mathrm{Tor}\,M)=0$.*

Let M_1, M_2 be A-modules. An *A-module homomorphism* from M_1 to M_2 is an H-homomorphism $\rho: M_1\longrightarrow M_2$ such that for any $a\in A, m\in M_1$ we have

$$a*\rho(m) = ((\mathrm{id}_H\otimes\mathrm{id}_H)\otimes_H\rho)(a*m).$$

Let L be a Lie H-pseudoalgebra, M be a finite L-module. Let $a,b\in L, m\in M$ and assume that

$$[a*b] = \sum_i (h^i\otimes 1)\otimes_H e_i, \quad e_i*m = \sum_j (p^{ij}\otimes q^{ij})\otimes_H m_{ij}.$$

By (2.43) we have

$$[a*b]*m = \sum_{i,j}(h^i p^{ij}_{(1)}\otimes p^{ij}_{(2)}\otimes q^{ij})\otimes_H m_{ij}\in H^3\otimes_H L. \tag{2.46}$$

Applying $(\mathcal{F}\otimes \mathrm{id}_H)\otimes_H \mathrm{id}_M$ to (2.46) we obtain

$$((\mathcal{F}\otimes \mathrm{id}_H)\otimes_H \mathrm{id}_M)[a*b]*m = \sum_{i,j}(h^i\otimes p^{ij}\otimes q^{ij})\otimes_H m_{ij} \in H^3\otimes_H L. \tag{2.47}$$

Let $\gamma_j \in H^*$ be such that $\gamma_j(h^i)=\delta^i_j$. Then applying $(\gamma_j\otimes \mathrm{id}_H\otimes \mathrm{id}_H)\otimes_H \mathrm{id}_L$ to the right-hand side of (2.47) we obtain a description of the action on M of the coefficients e_i of $[a*b]$.

2.5 H-pseudolinear maps and $\mathrm{gc}\, M$

In this section we introduce the Lie pseudoalgebra $\mathrm{gc}\, M$ of all H-pseudolinear maps from an H-module M to itself.
Let M, N be H-modules. A *H-pseudolinear map* from M to N is a $\Bbbk$-linear map ϕ,

$$\begin{array}{cccc}\phi: & M & \longrightarrow & (H\otimes H)\otimes_H N\\ & m & \mapsto & \phi(m)\end{array}$$

such that for every $h\in H$, $m \in M$:

$$\phi(hm) = (1\otimes h)\phi(m).$$

We denote the set of all such applications by $\mathrm{Chom}(M,N)$. $\mathrm{Chom}(M,N)$ has a structure of a left H-module given by

$$(h\phi)(m) = (h\otimes 1)\phi(m); \quad h\in H.$$

Example 2.26. *Let L be a Lie pseudoalgebra over H. For any $a\in L$ the* adjoint action *of a defined as $(\mathrm{ad}\, a)b=[a*b]$ is an H-pseudolinear map from L to itself.*

Example 2.27. *Let A be an H-pseudoalgebra, M be an A-module. For any $a\in A$ the map:*

$$\begin{array}{ccccc}\lambda_a: & M & \longrightarrow & (H\otimes H)\otimes_H M & \\ & m & \mapsto & a*m, & m\in M,\end{array}$$

is an H-pseudolinear map, which satisfies $h\lambda_a = \lambda_{ha}$ for any $h\in H$.

If M is a *finite H-module* then [BDK1] $\mathrm{Chom}(M,M)$ has a unique structure of an associative pseudoalgebra over H denoted by $\mathrm{Cend}\, M$. Moreover, the pseudoaction given by $\phi * m = \phi(m)$, $\phi\in \mathrm{Cend}\, M, m\in M$ endows M with a structure of a $\mathrm{Cend}\, M$-module.
Let M_0 be a finite dimensional vector space, $M = H\otimes M_0$ be a left H-module by left multiplication on the first tensor factor and $\mathrm{Cend}\, M$ be the associative H-pseudoalgebra described in Example 2.13, i.e., the H-module $H\otimes H\otimes \mathrm{End}\, M_0$, with H acting on the first tensor factor, endowed with the associative pseudoproduct:

$$(f\otimes a\otimes A)*(g\otimes b\otimes B) = (f\otimes ga_{(1)})\otimes_H (1\otimes ba_{(2)}\otimes AB).$$

The pseudoaction of $\mathrm{Cend}\, M$ on M is then:

$$(f\otimes a\otimes A)*(h\otimes m) = (f\otimes ha)\otimes_H (1\otimes Am).$$

We denote by $\mathrm{gc}\, M$ the Lie H-pseudoalgebra obtained from $\mathrm{Cend}\, M$ by (2.14). Clearly, M is also a $\mathrm{gc}\, M$-module.

Proposition 2.28. *[BDK1] Let A be an associative H-pseudoalgebra, M be a finite A-module. Giving an A-module structure over M is equivalent to giving an associative H-pseudoalgebras homomorphism from A to* $\operatorname{Cend} M$.

A similar statement holds for finite L-modules, where L is a Lie H-pseudoalgebra.

2.6 Quasi-nilpotence of solvable subalgebras $\langle a \rangle$ of $\operatorname{gc} M$

In this section we present our result for 1-generated solvable subalgebras of $\operatorname{gc} M$. We will specialize this result to the Lie conformal algebra $\operatorname{gc} V$ to give a characterization of finite vertex algebras.
In the first part of this section we review main results from [BDK1] in representation theory of solvable and nilpotent Lie pseudoalgebras.

2.6.1 Generalized weight submodules

Let L be a Lie pseudoalgebra over $H = \mathcal{U}(\mathfrak{d})$ and M be a finite L-module.
For $\phi \in \operatorname{Hom}_H(L, H) = L^*$ we define,

$$M_\phi = \{m \in M \mid a * m = (\phi(a) \otimes 1) \otimes_H m, \forall\, a \in L\}.$$

M_ϕ is a $\Bbbk$-vector subspace of M and we call $\phi \in \operatorname{Hom}_H(L, H)$ a *weight for the action of L over M* if M_ϕ is nonzero. A nonzero vector $m \in M_\phi$ is called a *weight vector* of weight ϕ or *ϕ-weight vector*. If $\phi = 0$ then M_0 is an H-submodule of M. For $\phi \neq 0$, let HM_ϕ be the H-submodule generated by M_ϕ. It is shown in [BDK1] that if $\{m_1, \ldots, m_k\}$ is a $\Bbbk$-basis of M_ϕ then it is also an H-basis of HM_ϕ. In particular HM_ϕ is a free H-submodule of M.

Lemma 2.29. *Let M be a finite L-module. If $N \subset M_\phi$ is a vector subspace, then HN is an L-submodule of M.*

Proof. Let $n \in N$. Then, for any $h \in H$,

$$a * hn = (1 \otimes h)(a * n) = (\phi(a) \otimes h) \otimes_H n \in (H \otimes H) \otimes_H HN.$$

□

We set $M^\phi_{-1} = 0$ and inductively

$$M^\phi_{i+1} = \operatorname{span}_H\{m \in M \mid a * m - (\phi(a) \otimes 1) \otimes_H m \in (H \otimes H) \otimes_H M^\phi_i,\ \forall\, a \in L\}.$$

Then $M^\phi_0 = HM_\phi$ and $M^\phi_{i+1}/M^\phi_i = H(M/M^\phi_i)_\phi$. The M^ϕ_i form an increasing sequence of H-submodules of M. By Noetherianity of M this sequence stabilizes to an H-submodule $M^\phi = \bigcup_i M^\phi_i$ of M. We call M^ϕ the *generalized weight submodule* relative to the *weight* ϕ.

Lemma 2.30. *Let M be a finite L-module and $\phi \neq 0$ be a weight for the action of L on M. Then the generalized weight submodule M^ϕ is a free H-submodule of M.*

Proof. M^ϕ is an extension of free H-submodules so it is a free H-submodule. □

Proposition 2.31. *Let M be a finite L-module and $\phi \neq 0$ be a weight for the action of L on M. Then the generalized weight submodule M^ϕ is an L-submodule of M.*

Proof. By induction using Lemma 2.29. □

Now we present some results about generalized weight submodules that will be useful in the sequel.

Lemma 2.32. *Let M be a finite L-module and suppose that ϕ is a weight for the action of L on M. Then the generalized weight submodule M^ϕ has an increasing filtration of L-submodules*

$$(0) = M_{0,\phi} \subset M_{1,\phi} \subset \cdots \subset M_{n,\phi} = M^\phi,$$

such that each quotient $M_{k,\phi}/M_{k-1,\phi}$ is a cyclic H-module, i.e., it is 1-generated.

Proof. We may assume without loss of generality that $M = M^\phi$. Set $M_{0,\phi} = (0)$ and inductively

$$M_{k,\phi} = M_{k-1,\phi} + Hm_k,$$

where $0 \neq m_k$ is a lifting of a weight vector $\bar{m}_k \in (M^\phi/M_{k-1,\phi})$.
By construction $\{M_{k,\phi}\}$ is an increasing sequence of submodules which stabilizes to an H-submodule by Noetherianity of M, that coincides with all of M^ϕ. The fact that each $M_{k,\phi}$ is an L-submodule follows by Lemma 2.29 by induction on k. The second part of the statement is clear. □

Proposition 2.33. *Let L be a Lie pseudoalgebra and M be a finite torsion-free L-module. If $\phi \neq 0$ is a weight for the action of L on M then for every L-submodule $U \subset M^\phi$ the quotient M/U is torsion-free. In particular M/M^ϕ is torsion-free*

Proof. First of all we observe that $\phi \neq 0$ and $U \subset M^\phi$ imply that $U = U^\phi$, so that U is a free H-module. We proceed by induction on the rank of U.
As far as the basis of induction $\operatorname{rk} U = 1$ is concerned. Let $\bar{m} \in M/U$ be a nonzero torsion element, i.e., there exists $0 \neq h \in H$ such that $h\bar{m} = \bar{0}$. It is equivalent to say that there exists $0 \neq k \in H$ such that $hm = ku$, where m is any lifting of $\bar{m}$ and u is a free generator of U. Moreover, since $\bar{m}$ is a torsion element, we know by Proposition 2.25 that for any $a \in L$, $a * \bar{m} = \bar{0}$. Let $a \in L$ be such that $\phi(a) = \phi_a \neq 0$ and suppose that $a * m = \sum_i (\gamma_a^i \otimes \delta_a^i) \otimes_H u \in (H \otimes H) \otimes_H U$, where $\{\gamma_a^i\}$ and $\{\delta_a^i\}$ are two systems of linearly independent vectors and $\gamma_a^i = \phi_a$ for some i. Equality $hm = ku$ implies

$$a * hm = \sum_i (\gamma_a^i \otimes h\delta_a^i) \otimes_H u = (\phi_a \otimes k) \otimes_H u = a * ku \neq 0. \tag{2.48}$$

By applying Lemma 2.8 to (2.48) we obtain

$$\sum_i (\gamma_a^i \otimes h\delta_a^i) = \phi_a \otimes k \neq 0. \tag{2.49}$$

Notice that multiplication by h is an injective map, so that the $\{h\delta_a^i\}$ is still a system of linearly independent vectors. By our choice of γ_a^i's the left-hand side of (2.49) must contain just one term. We obtain,

$$(\gamma_a \otimes h\delta_a) = \phi_a \otimes k \neq 0. \tag{2.50}$$

Equation (2.50) has as a solution $\gamma_a = c\phi_a$, $h\delta_a = c^{-1}k$, $0 \neq c \in \Bbbk$. Replacing $k = ch\delta_a$ in $hm = ku$, we have

$$h(m - c\delta_a u) = 0, \quad h \neq 0.$$

Since M is torsion-free, then $m - c\delta_a u$ must be zero, i.e., $m = c\delta_a u$. It implies $\bar{m} = \bar{0}$, a contradiction. This proves the statement for the basis of our induction. Now we proceed with the inductive assumption. Let $\operatorname{rk} U = U^\phi = N + 1$ and consider the L-submodule Hu, generated by $0 \neq u \in U_\phi$. Since $U \subset M^\phi$ we have $U/Hu \subset (M/Hu)^\phi$, where U/Hu is an H-submodule of rank N. M/Hu is torsion-free by the basis of our induction and U/Hu is torsion-free by our inductive assumption. As

$$M/U = (M/Hu)/(U/Hu),$$

M/U is torsion-free. □

If M is a finite L-module then we call a *quasi-weight for the action of L on M* an element $\phi \in \operatorname{Hom}_H(L, H)$ such that ϕ is a weight for the action of L on some quotient $\bar{N}$ of M. It is clear that every weight for the action of L on M is also a quasi-weight.
Let M be a finite L-module and $N \subset M$ be an L-submodule. An element $m \in M$ is a *ϕ-weight vector* modulo N if $\pi_N(m) = \bar{m} \in (M/N)^\phi$.
For a finite L-module M a *good-set of generators of M* is a set of generators $\{m_1, \ldots, m_N\}$ of M such that every m_i is a ϕ_i-weight vector modulo HM^{i-1}, where $HM^0 = 0$ and HM^{i-1} is the H-submodule generated by $\{m_1, \ldots, m_{i-1}\}$.

2.6.2 Representation theory of solvable and nilpotent Lie pseudoalgebras

In this section we recall some results from [BDK1] about representation theory of solvable and nilpotent Lie pseudoalgebras.

Theorem 2.34. *[Pseudoalgebraic version of Lie's theorem] Let L be a solvable Lie pseudoalgebra over H and M be a finite L-module. Then there exists $\phi \in \operatorname{Hom}_H(L, H)$ such that $M_\phi \neq 0$.*

Corollary 2.35. *As a consequence of Theorem 2.34 if L is a solvable Lie pseudoalgebra and M is a finite L-module then M has a filtration by L-submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that for any i the L-module M_{i+1}/M_i is generated over H by weight vectors of some weight $\phi_i \in \operatorname{Hom}_H(L, H)$.*

Finite nilpotent Lie pseudoalgebras, as we are going to prove, have an important characterization in terms of their action on finite L-modules M.

Proposition 2.36. *Let M be a finite H-module and $L \subset \operatorname{gc} M$ be a Lie pseudoalgebra such that $M = M^0$ with respect to the action of L on M. Then L is a nilpotent Lie pseudoalgebra.*

Proof. Let $L^{[0]} = L$, $L^{[i]} = [L, L^{[i-1]}]$, $i \geq 1$, be the the central series of L and $\{M_{k,0}\}$ be an increasing sequence of L-submodules of M^0 defined as in Lemma 2.32. We will prove by induction on i that $L^{[i]} \cdot M_{k,0} \subset M^0_{k-i-1,0}$.

For $i=0$, we have $L^{[0]} \cdot M_{k,0} = L \cdot M_{k,0} \subset M_{k-1,0}$ by our choice of $M_{k,0}$. Suppose that $L^{[i]} \cdot M_{k,0} \subset M_{k-i-1,0}$. Then,

$$\begin{aligned} L^{[i+1]} \cdot M_{k,0} &= [L^{[0]}, L^{[i]}] \cdot M_{k,0} = L^{[0]} \cdot (L^{[i]} \cdot M_{k,0}) + L^{[i]} \cdot (L^{[0]} \cdot M_{k,0}) \\ &\subset L^{[0]} \cdot M_{k-i-1,0} + L^{[i]} \cdot M_{k-1,0} \subset M_{k-i-2,0}. \end{aligned}$$

To conclude the proof it is sufficient to observe that if $M^0 = M_{n+1,0}$ then, for $i = n$, we have $L^{[n]} \cdot M_{n+1,0} \subset M_{n+1-n-1,0} = M_{0,0} = (0)$. It implies $L^{[n]} = 0$, i.e., L is a nilpotent Lie pseudoalgebra. □

Proposition 2.37. *Let M be a finite H-module of rank n and $L \subset \mathrm{gc}\, M$ be finite Lie pseudoalgebra such that $M = M^{\phi}$, $\phi \neq 0$, with respect to the action of L on M. Then L is a nilpotent Lie pseudoalgebra.*

Proof. Let $\{L^{[i]}\}_{i \geq 0}$ be the the central series of L and $\{M_{k,\phi}\}$ be an increasing sequence of L-submodules of M^{ϕ} defined as in Lemma 2.32.
Since L is finite, for any $a \in L$, $m \in M$, $a * m \in (H \otimes H) \otimes_H M$ has an upper bound on the degree of the second tensor factor in H. We define $N_k^{-1} = 0$ and, for $j \geq 0$,

$$N_k^j = \{b \in L \mid b * m_i \in (H \otimes F^j H) \otimes_H m_{i-k} + M_{i-k-1,\phi},\ i = 1, \ldots, N\}.$$

Then we set $N_k = \bigcup_j N_k^j$. We want to prove by induction on p that $(\mathrm{ad}\, L)^p \cdot N_k^j \subset N_k^{j-p-1}$. The basis of our induction being $(\mathrm{ad}\, L)^0 \cdot N_k^j \subset N_k^{j-1}$. Let $a \in L$, $b \in N_k^j$. We can suppose, without loss of generality, that $a * m_i = (\phi \otimes 1) \otimes_H m_i$, for $i = 1, \ldots, n$. Assume that $b * m_i = \sum_i (h^i \otimes k^i) \otimes_H m_{i-k} + M_{i-k-1}$, where $\sum_i (h^i \otimes k^i) \in (H \otimes F^j H)$. Let us compute

$$\begin{aligned} [a * b] * m_i &= a * (b * m_i) - (b * (a * m_i))^{\sigma_{12}} \\ &= a * (\sum_i (h^i \otimes k^i) \otimes_H m_{i-k} + M^{\phi}_{i-k-1}) \\ &\quad - (b * ((\phi \otimes 1) \otimes_H m_i))^{\sigma_{12}} \\ &= \sum_i (1 \otimes h^i \otimes k^i)(1 \otimes \Delta) a * m_i + a * M^{\phi}_{i-k-1} \\ &\quad - \sum_i ((1 \otimes \phi \otimes 1)(1 \otimes \Delta)(b * m_i))^{\sigma_{12}} \\ &= \sum_i (\phi \otimes h^i \otimes k^i) \otimes_H m_{i-k} \\ &\quad - \sum_i (\phi k^i_{(1)} \otimes h^i \otimes k^i_{(2)}) \otimes_H m_{i-k} + M^{\phi}_{i-k-1} \\ &= \sum_i (\phi \otimes h^i \otimes k^i) \otimes_H m_{i-k} - \sum_i (\phi \otimes h^i \otimes k^i) \otimes_H m_{i-k} \\ &\quad + (H \otimes H \otimes F^{j-1} H) \otimes_H m_{i-k} + M^{\phi}_{i-k-1} \\ &\subset (H \otimes H \otimes F^{j-1} H) \otimes_H m_{i-k} + M^{\phi}_{i-k-1}. \end{aligned}$$

This proves that all the coefficients c_i of $[a * b]$ lie in N_k^{j-1}.
Suppose that $(\mathrm{ad}\, L)^p \cdot N_k^j \subset N_k^{j-p-1}$. Then we have

$$\begin{aligned} (\mathrm{ad}\, L)^{p+1} \cdot N_k^j &= (\mathrm{ad}\, L)^0 \cdot ((\mathrm{ad}\, L)^p \cdot N_k^j) + (\mathrm{ad}\, L)^p ((\mathrm{ad}\, L)^0 \cdot N_k^j) \\ &\subset (\mathrm{ad}\, L)^0 \cdot N_k^{j-p-1} + (\mathrm{ad}\, L)^p \cdot N_k^{j-1} \subset N_k^{j-p-2}. \end{aligned}$$

Since L is a finite Lie pseudoalgebra there exists an index D such that $N_k^{D+r} = N_k^D$, for any $r \geq 0$. For $p = D$ we obtain $(\mathrm{ad}\, L)^D \cdot N_k^D \subset N_k^{D-D-1} = 0$. Since

$N_n^j = 0$, where n is the rank of M, we can apply inductively the same argument to find a finite index P such that $(\operatorname{ad} L)^P = 0$, which proves that L is a nilpotent Lie pseudoalgebra. □

Theorem 2.38. *Let L be a finite nilpotent Lie pseudoalgebra over H and M be a (faithful) finite L-module. Then M decomposes as a direct sum of generalized weight submodules, $M = \bigoplus_{\phi \in L^*} M^\phi$. Conversely, if M is a faithful finite L-module of a finite Lie pseudoalgebra L such that $M = \bigoplus_{\phi \in L^*} M^\phi$ then L is a nilpotent Lie pseudoalgebra.*

Proof. A proof of the first statement can be found in [BDK1].
To prove the second statement we observe that under the assumption $M = \bigoplus_{\phi \in L^*} M^\phi$ we have $L \subset \bigoplus_{\phi \in L^*} \operatorname{gc}(M^\phi)$. By Proposition 2.36 and Proposition 2.37, for every weight ϕ, the image L_ϕ of L in $\operatorname{gc}(M^\phi)$ is nilpotent. As a consequence, $\bigoplus_{\phi \in L^*} L_\phi$ is nilpotent as it is a finite sum of nilpotent Lie pseudoalgebras. Finally, L is nilpotent Lie pseudoalgebra as it is a subalgebra of a nilpotent one. □

Remark 2.9. *Notice that the finiteness assumption on L in the statement of Theorem 2.38 on L is necessary as the following example proves.*

Example 2.39. *Let $M = Hm_1 + Hm_2$ be a free H-module of rank 2. Let $L \subset \operatorname{gc} M$ be the Lie pseudoalgebra generated by all the pseudolinear maps $A \in \operatorname{gc} M$ such that $A * m_1 = (\phi \otimes 1) \otimes_H m_1$, $A * m_2 = (H \otimes H) \otimes_H m_1 + (\phi \otimes 1) \otimes_H m_2$, for some $\phi \in H$. Since L is an H-module of infinite rank then $L^{[k]} \cdot Hm_2 = Hm_1$, for all $k \geq 1$, i.e., the central series $\{L^{[k]}\}$ of L stabilizes to the derived subalgebra $L' = L^{[1]}$. Notice that by Theorem 2.38 any finite subalgebra of L is nilpotent.*

2.6.3 A quasi-nilpotence result

Let M be a finite H-module and $a \in \operatorname{gc} M$ an element generating a solvable Lie pseudoalgebra $\langle a \rangle = S$.
A *modification* of $a \in S$ is an element $\bar{a} \in S$ such that $a \equiv \bar{a}$ modulo S'. It follows by the definition that the subalgebra generated by $\bar{a}$ is a subalgebra of S. The same inclusion holds for the corresponding derived subalgebras. As a consequence, a modification of a modification of a is still a modification of a.

Remark 2.10. *Let $\phi \in S^*$ be a weight for the action of S on M. If $\bar{a}$ is a modification of a then $\phi(a) = \phi(\bar{a})$, since the restriction of ϕ on S' is zero.*

Our main goal is to prove the following

Theorem 2.40. *Let M be a finite H-module and S be a solvable Lie pseudoalgebra generated by $a \in \operatorname{gc} M$. Then some modification $\bar{a}$ of a generates a nilpotent Lie pseudoalgebra.*

We divide the proof of Theorem 2.40 in several steps.

Lemma 2.41. *Let M be a finite H-module, S be a solvable Lie pseudoalgebra generated by $a \in \operatorname{gc} M$ and M^ϕ a generalized weight submodule for the action of S on M. Then $\bar{a} \cdot M^\phi \subset M^\phi$ for any modification $\bar{a}$ of a.*

Proof. Since $\langle \bar{a} \rangle = \bar{S} \subset S$ and $S \cdot M^\phi \subset M^\phi$ we have the statement. □

Proposition 2.42. *Let M be a finite H-module and $S = \langle a \rangle$ be a solvable Lie pseudoalgebra, $a \in \mathrm{gc}\, M$. Assume that $\phi \neq \psi \in S^*$, $M = Hu + Hv$, $0 \neq u \in M_\phi$ and v is a ψ-weight vector* modulo Hu. *Then there exists a modification of a generating a nilpotent subalgebra of* $\mathrm{gc}\, M$.

Proof. The strategy is as follows. We will locate a modification $\bar{a}$ of a such that M decomposes into a direct sum of the $\langle \bar{a} \rangle$-submodules Hu and Hv', where v' is a lifting of $\bar{v}$ of the form $v' = tu + v$, $t \in H$. Then the second part of Theorem 2.38 implies the statement. Assume that

$$\begin{aligned} a * u &= (\phi \otimes 1) \otimes_H u \\ a * v &= \sum_i (h^i \otimes k^i) \otimes_H u \quad + (\psi \otimes 1) \otimes_H v. \end{aligned}$$

We may assume both $\{h^i\}$ and $\{k^i\}$ to be systems of linearly independent vectors. We want to show that, up a modification of a, it is always possible to low the degree of the k^i's of maximum degree which appear in $a * v$. A reiteration of the same argument together with finiteness of the degree of k_i shows that, up to several modifications of a, there exists a modification $\bar{a}$ such that $\bar{a} * v' = (\psi \otimes 1) \otimes_H v'$.
Our first step is to compute the action on M of the coefficients c_i of $[a * a] = \sum_i (f^i \otimes g^i) \otimes_H c_i \in (H \otimes H) \otimes_H S'$. Clearly, it is sufficient to describe the action of $[a * a]$ on u and v. By a direct computation we obtain

$$\begin{aligned} a * (a * u) &= (\phi \otimes \phi \otimes 1) \otimes_H u, \\ a * (a * v) &= \sum_i (\phi \otimes h^i \otimes k^i) \otimes_H u + \sum_i (h^i \otimes \psi k^i_{(1)} \otimes k^i_{(2)}) \otimes_H u + (\psi \otimes \psi \otimes 1) \otimes_H v. \end{aligned}$$

Using (2.45) we obtain $[a * a] * u = 0$ and

$$\begin{aligned} [a * a] * v \quad &= \sum_i ((\phi \otimes h^i \otimes k^i) + (h^i \otimes \psi k^i_{(1)} \otimes k^i_{(2)})) \otimes_H u \\ &- \sum_i ((h^i \otimes \phi \otimes \otimes k^i) + (\psi k^i_{(1)} \otimes h^i \otimes k^i_{(2)})) \otimes_H u. \end{aligned}$$

Let K be the maximum degree between all the k^i. Up to elements in $(H \otimes H \otimes F^{K-1}) \otimes_H M$, we have

$$[a * a] * v = \sum_{i : \deg k^i = K} ((\phi - \psi) \otimes h^i \otimes k^i - h^i \otimes (\phi - \psi) \otimes k^i) \otimes_H u.$$

Let $\alpha = \phi - \psi$ and rewrite the above expression as

$$[a * a] * v = \sum_{i : \deg k^i = K} (\alpha_{(1)} \otimes h^i \alpha_{(-2)} \otimes k^i - h^i_{(1)} \otimes \alpha h_{(-2)} \otimes k^i) \otimes_H u,$$

modulo elements of lower degree. Suppose that for every i such that $\deg k^i = K$ we have $\deg \alpha \neq \deg h^i$. Then, up to elements of lower degree in the first tensor factor, we obtain $[a * a] * v = (\alpha \otimes h^i \otimes k^i) \otimes_H u$, if $\deg \alpha > \deg h^i$, or $[a * a] * v = -(h^i \otimes \alpha \otimes k^i) \otimes_H u$, if $\deg \alpha < \deg h^i$.
Let $\gamma \in X$ be such that γ is nonzero on the element in the second tensor factor. Then applying the map $(\mathrm{id}_H \otimes \gamma \otimes \mathrm{id}_H) \otimes_H \mathrm{id}\, L$ to $(\alpha \otimes h^i \otimes k^i) \otimes_H u$ in the first

case (or to $(h^i \otimes \alpha \otimes k^i) \otimes_H u$ in the second one) gives a coefficient $b \in S'$ of $[a * a]$ whose action on a k^i of maximum degree K is given by

$$\begin{aligned} b * u &= 0 \\ b * v &= (q \otimes k^i) \otimes_H u. \end{aligned} \tag{2.51}$$

Now we suppose that $\deg \alpha = \deg h^i$. Up to elements of lower degree in the first tensor factor, we have

$$[a * a] * v = (\alpha \otimes h^i \otimes k^i - h^i \otimes \alpha \otimes k^i) \otimes_H u.$$

If α and h^i are not proportional up to elements of lower degree then the above expression is nonzero. Let $\gamma' \in X$ be such that γ' is nonzero on the second tensor factor. Then applying $(\mathrm{id}_H \otimes \gamma' \otimes \mathrm{id}_H) \otimes_H \mathrm{id}\, L$ to the above expression gives a coefficient $b \in S'$ of $[a * a]$ whose action on M is as in (2.51).
Let α and h^i be proportional up to elements of lower degree. Let $v' = tu + v$, $t \in H$. Then

$$\begin{aligned} a * v' &= (\phi \otimes t) \otimes_H u + \sum_i (h^i \otimes k^i) \otimes_H u + (\psi \otimes 1) \otimes_H v \\ &= ((\alpha \otimes t) + \sum_i (h^i \otimes k^i)) \otimes_H u + (\psi \otimes 1) \otimes_H v'. \end{aligned}$$

Since $\alpha = ch^i$, then taking $t = -c^{-1} k^i$ gives rise to a vector v' for which $a * v'$ is an expression of lower degree in k_i with respect to $a * v$.
We now investigate the action on M of the coefficients of $[a * b]$, where b in an element in S' acting as in (2.51). We obtain

$$[a * b] * v = (\phi \otimes q \otimes k^i - \psi k^i_{(1)} \otimes q \otimes k^i_{(2)}) \otimes_H u.$$

Up to elements of lower degree in the second and in the third tensor factor, it equals:

$$[a * b] * v = (\alpha \otimes q \otimes k^i) \otimes_H u.$$

Let $\rho \in X$ a linear functional such that $\rho(\alpha) = 1$ and $\rho(g) = 0$, for any expression of the type $g \otimes q \otimes k^i$ of lower degree in the first tensor factor. This proves the existence of an element $b' \in S'$ whose action on M is

$$\begin{aligned} b' * u &= 0 \\ b' * v &= (1 \otimes k^i) \otimes_H u. \end{aligned}$$

Replacing a with $a + a'$, where $a' = -h^i b'$, and eventually v with v', we have that by construction the action of $a + a'$ on M is an expression of lower degree in k_i. An induction on the degree of k_i concludes the proof. □

Let V be a finite R-module, where R is a Lie conformal algebra. If V is a non-free $\mathbb{C}[\partial]$-module then necessarily $\mathrm{Tor}\, V \neq 0$ and as a consequence $\phi = 0$ is a weight for the action of R on V. In the following example we show that the same statement does not hold for finite representations M of a Lie H-pseudoalgebra, when H is any cocommutative Hopf algebra.

Example 2.43. *Let $L = \langle a \rangle$ be a 1-generated solvable Lie pseudoalgebra over H. Let $M = Hm \oplus In$, where $m \in M_\phi$, n is 0-weight vector* modulo Hm *and $I = \{h \in H \mid \epsilon(h) = 0\}$. Then we have*

$$\begin{aligned} a * m &= (\phi(a) \otimes 1) \otimes_H m, \\ a * in &= (1 \otimes i)(\alpha \otimes_H m), \end{aligned}$$

where $\phi(a) \neq 0$, $i \in I$, $\alpha \in H \otimes H$. For any $0 \neq k \in H$ we have

$$a * (in - km) = ((1 \otimes i)(\alpha) - (\phi \otimes k)) \otimes_H m. \tag{2.52}$$

By Lemma 2.8 Equation (2.52) is zero if and only if $(1 \otimes i)(\alpha) - (\phi \otimes k) = 0$. If we take $\alpha \in (H \backslash \Bbbk\phi) \otimes H$ then $(1 \otimes i)(\alpha) - (\phi \otimes k) \neq 0$, so that $a * (in - km) \neq 0$. Since M is a direct sum then we have $M/Hm \simeq In$ as an H-module and L acts trivially on In. By construction M is not free as an H-module, as I is not a free H-module, but $\phi = 0$ is not a weight for the action of L on M.

Proposition 2.44. *Let M be a finite H-module and $S = \langle a \rangle$ be a solvable Lie pseudoalgebra, $a \in \operatorname{gc} M$. Assume that $\phi \neq \psi \in S^* \setminus \{0\}$, $M^\phi \neq 0, M$ and $M/M^\phi = (M/M^\phi)^\psi$. Then there exists a modification of a generating a nilpotent Lie pseudoalgebra.*

Proof. Our strategy is to prove that there exists a modification $\bar{a}$ of a such that $M = M^\phi \oplus M^\psi$ with respect to the action of the Lie pseudoalgebra generated by $\bar{a}$. We proceed by induction on the rank of the free H-modules M^ϕ and $(M/M^\phi)^\psi$. Suppose that $\operatorname{rk} M^\phi = P$ and $\operatorname{rk}(M/M^\phi)^\psi = Q$. The basis of our induction $(P, Q) = (1, 1)$ is proved in Proposition 2.42.
The next step is to prove that if the statement holds for $(P, Q) = (P, 1)$ then it holds for $(P+1, 1)$ too. Since $\phi \neq 0$, M^ϕ is a free H-module it always contains an H-submodule N of corank 1. Moreover N satisfies $N = N^\phi$. This follows that $(M^\phi/N)^\phi = Hu$, where $u \neq 0$ is a free generator. By the basis of our induction up to a modification of a we have a decomposition $M/N = (M/N)^\phi \oplus (M/N)^\psi$. Denote by π_N be the canonical projection and observe that $\pi_N^{-1}((M/N)^\phi) = Hu + N$. Since Hu is an L-submodule we have $(M/Hu)^\phi = N = N^\phi$, where $\operatorname{rk} N^\phi = P$. By inductive assumption, up to a new modification, it is possible to decompose M/Hu as $(M/Hu)^\phi \oplus (M/Hu)^\psi$. By construction we have $M = M^\phi \oplus M^\psi$. Now it remains to prove that if the statement holds for $(P+1, Q)$ then it holds for $(P+1, Q+1)$ too. Since $\psi \neq 0$, $(M/M^\phi)^\psi$ is a free H-module which contains an H-submodule $\bar{M}$ of corank 1. By what we showed above, up to a modification of a, the following decomposition $M/\bar{M} = (M/\bar{M})^\phi \oplus (M/\bar{M})^\psi$ holds. Now we observe that $(M/\bar{M})^\psi = H\bar{v}$ and that $\bar{v}$ admits a lift v such that Hv is an L-submodule of M. Now we can apply our inductive assumption to M/Hv to prove that, up to a modification, $M/Hv = (M/Hv)^\phi \oplus (M/Hv)^\psi$. this proves that $M = M^\phi \oplus M^\psi$. □

Proposition 2.45. *Let M be a finite H-module and $S = \langle a \rangle$ be a solvable Lie pseudoalgebra, $a \in \operatorname{gc} M$. Let $\phi = 0$ be a weight for the action of S on M such that $M^0 \neq 0, M$ and $M/M^0 = (M/M^0)^\psi$, $\psi \neq 0$. Then there exists a modification of a generating a nilpotent Lie pseudoalgebra.*

Proof. Let $\{M_{k,0}\}$ be an increasing sequence of L-submodules of M^0 as described in Lemma 2.32. We can proceed by induction on the length of $\{M_{k,0}\}$ and the rank of $(M/M^0)^\psi$. The basis of our induction is Proposition 2.42, the rest of the proof is similar to Proposition 2.44. □

Proposition 2.46. *Let M be a finite H-module and $S = \langle a \rangle$ be a solvable Lie pseudoalgebra, $a \in \operatorname{gc} M$. Let $0 \neq \phi \in S^*$ such that $M^\phi \neq 0, M$ and $M/M^\phi = (M/M^\phi)^0$. Then there exists a modification of a generating a nilpotent Lie pseudoalgebra.*

Proof. Let $\operatorname{rk} M^{\phi} = P$ and $\{\bar{N}_{k,0}\}$ be an increasing sequence of L-submodules of M/M^{ϕ}. We proceed by induction on P and the length of $\{\bar{N}_{k,0}\}$.
We can assume, since 0 is not a weight for the action of S on M, that M is torsion free. By Proposition 2.33 we have that $M/M^{\phi} = (M/M^{\phi})^0$ is torsion free too. The basis of our induction is just Proposition 2.42. Let $\bar{N} \subset (M/M^{\phi})^0$ be a cotorsion free H-submodule. Recall that a Lie pseudoalgebra always acts trivially on torsion elements. Since both M^{ϕ} and $\bar{N}$ are free H-modules by the usual argument, up to a modification of a, there exists N in M such that $M^{\phi} \oplus N$ as an L-module.
The quotient $(M/N)^0$ is by construction a torsion H-module so that every Lie pseudoalgebra acts trivially on it. Let $\bar{v} \in (H/M^{\phi})/\bar{N}$, i.e., there exists $h \in H$, $h \neq 0$, such that $h\bar{v} \in \bar{N}$. Let $v \in M$ be a lifting of v. Then $hv = \alpha n + \beta u$, for some $0 \neq \alpha, \beta \in H$, $n \in N$. We have

$$a * hv = (1 \otimes\ h)a * v = (\phi \otimes \beta) \otimes_H u.$$

Let $a * v = \sum_i (\gamma_i \otimes \delta_i) \otimes_H u$, then

$$\sum_i (\gamma_i \otimes h\delta_i) = \phi \otimes \beta \implies i = 1,\, \gamma = c\phi,\, h\delta = c^{-1}\beta.$$

As a consequence,

$$h(v - c\delta u) = \alpha n.$$

If we set $v' = v - c\delta u$ then $hv' = \alpha n$ and $a * v' = 0$. Moreover v' is the unique lifting of $\bar{v}$ such that the action of a on there is trivial. In fact if $v'' = v' + h'u$ then $a * v'' = a * v' + (1 \otimes\ h')a * u = (1 \otimes\ h')a * u \neq 0$. This shows that, up to a modification of a and of the lifting of $\bar{v} \in (H/M^{\phi})/\bar{N}$, it is possible to decompose M as $M^{\phi} \oplus M^0$. □

Proof of Theorem 2.40.
Let $G = \{m_1, \ldots, m_n\}$ be a good-set of generators of M and $\phi_1, \ldots, \phi_n$ be the corresponding quasi-weights of M. We proceed by induction on the cardinality of a good-set of generators of M.
If $|G| = 1$ then $M = M^{\phi_1}$ and the statement holds with $\bar{a} = a$ as follows by the second part of Theorem 2.38.
If $|G| = 2$ then if $\phi_1 = \phi_2$ then the statement still follows by Theorem 2.38; if $\phi_1 \neq \phi_2$ then it follows by one of the above results.
Now we suppose that the statement holds for $|G| = n$ and we want to prove that it holds for $|G| = n + 1$. Let $\{m_1, \ldots, m_{n+1}\}$ be a good set of generators of M and let ψ the quasi-weight of m_{n+1}. Let HM^n be the L-submodule generated by $\{m_1, \ldots, m_n\}$. Then, up to a modification of a, that we still denote by a, we may suppose that HM^n decomposes as a direct sum of generalized weight submodules with respect to the action of a on M. Now we have to distinguish two cases, the case when $\psi \neq \phi_i$, for any $i = 1, \ldots, n$ and the case when $\psi = \phi_i$, for some i.
If $\psi \neq \phi_i$ for any $i = 1, \ldots, n$ then consider the L-submodule M/Hm_1.
$\{\bar{m}_2, \ldots, \bar{m}_{n+1}\}$ is a good set of generators of M/Hm_1. By inductive assumption, up to replacing a with one of its modifications, there exists an element m'_{n+1} such that $a * m'_{n+1} = (\psi \otimes 1) \otimes_H m'_{n+1} + Hm_1$. Then $\{m_1, \ldots, m'_{n+1}\}$ is still a good-set of generators of M. Now consider the L-submodule N generated

by $\{m_2, \ldots, m_n\}$ and the corresponding quotient M/N. By our assumption we have $\phi_1 \neq \psi$ and then we can apply one of the above proposition to conclude that there exists a modification of a, let ua say $\bar{a}$, with respect to which M/N decomposes as a direct sum of generalized weight submodules. By construction then we have $M = \bigoplus_{\phi \in L^*} M^\phi$, i.e., the subalgebra generated by $\tilde{a}$ is nilpotent.

If instead $\psi = \phi_i$ for some i, we may assume without loss of generality that $\psi \neq \phi_i$, $1 \leq i < k$ and $\psi = \phi_i$, $k \leq i \leq n$. Let N be the L-submodule generated by $m_k, \ldots, m_n$.

Then $\{m_1, \ldots, m_{k-1}, m_{n+1}\}$ is a good-set of $k \leq n$ generators of M/N. By our inductive assumption there exists a modification $\bar{a}$ of a with respect to M/N decomposes as a direct sum of generalized weight submodules. By construction the lifting of $\bar{m}_{n+1} \in M/N$ lies in $N + Hm_{n+1}$ and $M = \bigoplus_{\phi \in L^*} M^\phi$. □

Chapter 3

A characterization of finite vertex algebras

In this chapter we present a characterization of finite vertex algebras which improves the results stated in Theorem 1.18.
Let V be a finite vertex algebra, $a \in V$ and $\langle a \rangle$ be the subalgebra of V^{Lie} generated by a. The *zero-multiplicity* of a on V is the rank of the generalized weight submodule of weight 0 for the adjoint action of $\langle a \rangle$ on V.
It is shown in Proposition 1.15 that the central series $\{V^{[k]}\}$ of V^{Lie} stabilizes to $V^{[\infty]}$, which is a vertex ideal of V. We have the following,

Theorem 3.1. *Let V be a finite vertex algebra and $N = V^{[\infty]}$. Then $N \cdot N = 0$, and there exists a subalgebra $U \subset V$ such that U^{Lie} is nilpotent and $V = U \ltimes N$.*

Proof. Recall that under the finiteness assumption for V the Lie conformal algebra V^{Lie} is solvable. Let $a \in V^{Lie}$ be an element such that its zero-multiplicity is minimal and equal to k.
By Theorem 2.40 there exists a modification of $\operatorname{ad} a$ that generates a nilpotent conformal algebra. Since ad is a homomorphism of Lie conformal algebras this modification is the image of a suitable modification $\overline{a}$ of a. Moreover $\overline{a}$ is still an element with minimal zero-multiplicity. Then $\bar{a}$ generates a nilpotent Lie conformal algebra with respect to which V decomposes as a direct sum of generalized weight submodules,

$$V = \bigoplus_{\phi} V_{\bar{a}}^{\phi} = V_{\bar{a}}^{0} \oplus (\bigoplus_{\phi \neq 0} V_{\bar{a}}^{\phi}),$$

where $U = V_{\bar{a}}^{0}$ is a vertex subalgebra of V and $N = \bigoplus_{\phi \neq 0} V_{\bar{a}}^{\phi}$ is an abelian ideal of V as a consequence of Proposition 1.19.
We have to show that the Lie conformal algebra structure underlying U is nilpotent. We will prove that the adjoint action of any $b \in U$ on U is nilpotent. Since $\bar{a}$ is an element of minimal zero-multiplicity then the zero-multiplicity of b must be at least k. Suppose that $\phi \neq 0$ is a weight for the action of b on U. Then 0 must be a weight for the action of b on N. As a consequence, for a suitable choice of $c \in H$, 0 is not a quasi-weight for the action of $b + c\bar{a}$ on N. Then $b + c\bar{a}$ would be an element whose 0-multiplicity is lower than k, which

contradicts minimality of a. This proves that $U = U^0$ with respect to the adjoint action of any $b \in U$ and so by Engel's theorem U is nilpotent. It remains to prove that $N = V^{[\infty]}$: since N is an ideal such that V/N is nilpotent then $V^{[\infty]} \subset N$. We prove the other inclusion by showing that $N \subset V^{[k]}$ by induction on $k \in \mathbb{N}$, the basis of the induction being clear, as $N \subset V^0 = V$. Now we proceed with our inductive assumption. Notice that for any $a \in V$ we have $a \cdot N = N$, as N is a vertex ideal. As a consequence, $V^{[k+1]} = [V, V^{[k]}] \supset [a, N] = N$, for any k. □

Remark 3.1. *Notice that $V^{[\infty]}$ is the smallest nil-ideal of V having a complementary subalgebra U such that U^{Lie} is nilpotent.*

The statement of Theorem 3.1 suggests us how to construct a finite vertex algebra V such that the corresponding Lie conformal algebra V^{Lie} is not nilpotent. What we need is a nilpotent vertex algebra U with a suitable action on an $\mathbb{C}[\partial]$-module N. The simplest case is when U is a commutative vertex algebra, i.e., U^{Lie} is abelian, and N is a free $\mathbb{C}[\partial]$-module of rank 1.
Let $\mathcal{F} = \{a(t) \in \mathbb{C}[[t]][t^{-1}]\}$. $\mathcal{F}$ is a differential commutative associative algebra with 1, with derivation $\partial = \frac{\partial}{\partial t}$. Hence $\mathcal{F}$ has the vertex algebra structure described in Example 1.17. Explicitly, for any $a(t), b(t) \in \mathcal{F}$

$$Y(a(t), z)b(t) = (e^{z\partial}a(t))b(t) = i_{|z|<|t|}a(t+z)b(t), \tag{3.1}$$

where $i_{|z|<|t|}$ (see [K]) points to the fact that one should expand $a(t+z)$ in the domain $|z| < |t|$, i.e., using positive powers of $\frac{z}{t}$.
Let $N = \mathbb{C}[\partial]n$ be a free $\mathbb{C}[\partial]$-module of rank 1. We set:

$$Y(n, z)n = 0, \tag{3.2}$$

and define an action of $\mathcal{F}$ on N by setting

$$Y(a(t), z)n = a(z)n, \tag{3.3}$$

where $a(t) \in \mathcal{F}$.

Theorem 3.2. *There exists a unique vertex algebra structure on the $\mathbb{C}[\partial]$-module $V = \mathcal{F} \oplus N$ such that (3.1), (3.2), (3.3) are satisfied. Moreover, the central series $\{V^{[k]}\}$ of V^{Lie} stabilizes to N.*

Proof. We take as vacuum element on V the constant function $1 \in \mathcal{F}$. Then it is easy to verify that 1 satisfies the vacuum axiom. Also the field axiom easily follows by definition. Locality and translation invariance just requires some more computation. Notice that skew-commutativity forces:

$$Y(n, z)a(t) = e^{z\partial}Y(a(t), -z)n = a(-z)e^{z\partial}n. \tag{3.4}$$

Moreover, we set:

$$Y(a(t), z)\partial^K n = \sum_{i=1}^{K} \binom{K}{i} (-1)^i a^{(i)}(z)\partial^{K-i}n. \tag{3.5}$$

Then translation invariance follows by (3.5) and skew-commutativity (3.4).
We want to prove that:

$$(3.6)\qquad (z-w)^{N(a,b)} \quad [Y(a(t),z),Y(b(t),w)] \quad = 0,$$
$$(3.7)\qquad (z-w)^{N(a,n)} \quad [Y(a(t),z),Y(n,w)] \quad = 0,$$
$$(3.8)\qquad (z-w)^{N(n,n)} \quad [Y(n,z),Y(n,w)] \quad = 0,$$

for some choice of $N(a,b), N(a,n), N(n,n) \geq 0$. Equation (3.8) holds with $N(n,n) = 0$ as immediately follows by (3.2) and skew-commutativity.
Let $c \in V$. Applying ∂ to both sides of (3.6) we obtain

$$\begin{aligned} & (z-w)^{N(a,b)} \quad ([\partial,[Y(a(t),z),Y(b(t),w)]] + [Y(a(t),z),Y(b(t),w)]\partial c) \\ = \; & (z-w)^{N(a,b)} \quad (([Y(a'(t),z),Y(b(t),w)] + [Y(a(t),z),Y(b'(t),w)])c) \\ + \; & (z-w)^{N(a,b)} \quad ([Y(a(t),z),Y(b(t),w)]\partial c) = 0, \end{aligned}$$

which proves that it is sufficient to verify (3.6) on n. A similar computation shows that it is sufficient to verify (3.7) on $b(t)$. Since

$$Y(a(t),z)(Y(b(t),w)n)) = Y(a(t),z)b(w)n = b(w)Y(a(t),z)n = a(z)b(w)n,$$

and

$$Y(b(t),w)(Y(a(t),z)n)) = Y(b(t),w)a(z)n = a(z)Y(b(t),w)n = a(z)b(w)n,$$

we have

$$[Y(a(t),z),Y(b(t),w)] = 0,$$

so that (3.6) holds, and we can chooose $N(a,b) = 0$. Let us compute

$$\begin{aligned} Y(a(t),z)(Y(n,w)b(t)) & = Y(a(t),z)b(-w)e^{w\partial}n = b(-w)Y(a(t),z)e^{w\partial}n \\ & = b(-w)e^{w\partial}(e^{-w\partial}Y(a(t),z)e^{w\partial})n \\ & = e^{w\partial}b(-w)Y(a(t),z-w)n \\ & = i_{|w|<|z|}a(z-w)b(-w)e^{w\partial}n, \end{aligned}$$

and

$$\begin{aligned} Y(n,w)(Y(a(t),z)b(t)) & = Y(n,w)i_{|z|<|t|}a(t+z)b(t) \\ & = i_{|z|<|t|}Y(n,w)a(t+z)b(t) \\ & = i_{|z|<|w|}a(z-w)b(-w)e^{w\partial}n. \end{aligned}$$

Therefore,

$$(z-w)^{N(a,n)}[Y(a(t),z),Y(n,w)]b(t) = (i_{|w|<|z|} - i_{|z|<|w|})((z-w)^{N(a,n)}a(z-w)b(-w)e^{w\partial}n)$$

is zero for sufficiently large $N(a,n)$.
It remains to prove that $V^{[\infty]} = N$. Let $a = a(t) = \sum_{n\in\mathbb{Z}} a_n t^{-n-1} \in \mathcal{F}$ such that $a(t)$ contains some negative power of t. As a consequence, there exists $n \geq 0$ such that $a(t)_{(n)}n = a_n \cdot n \neq 0$. Therefore $a \cdot N = N$ and the central series of V^{Lie} stabilizes on N. □

The proof of Theorem 3.2 suggests us how to construct a finite vertex algebra whose underlying Lie conformal algebra structure is not nilpotent. It is sufficient to choose a finite subalgebra U of $\mathcal{F}$ whose conformal adjoint action on N has nonzero weights, i.e., a finite subalgebra $U \subset \mathcal{F}$ containing some element $a(t) \notin \mathbb{C}[[t]]$.

Corollary 3.3. *$M = \mathbb{C}[t^{-1}] \ltimes N \subset \mathcal{F} \ltimes N$ is a finite vertex algebra, generated by t^{-1}, 1 and n, as a $\mathbb{C}[\partial]$-module, such that M^{Lie} is not nilpotent.*

We conclude this chapter by observing that even trough the nil-ideal N in the decomposition stated in Theorem 3.1 is canonically determined, as it equals $V^{[\infty]}$, the subalgebra U is not. Indeed there are several possible choices of U as the following construction shows. Let $n \in N$. Then $n_{(0)}$ is a derivation of V, and as $N \cdot N = 0$, $n_{(0)}^2 = 0$. Recall that the exponential of a nilpotent derivation of a vertex algebra V gives an automorphism of V. If, in the vertex algebra described in Corollary 3.3, we choose n to be the free generator of N, then $\exp(kn_{(0)})(t^{-1}) = t^{-1} - kn$, $k \in \mathbb{C}$. Then, if we set $\psi = \exp(kn_{(0)})$, we obtain $\psi(N) = N$, $\psi(U) = \mathbb{C}[\partial](t^{-1} - kn) \oplus \mathbb{C}1$, and $\psi(U) \cap N = \psi(U \cap N) = 0$. Thus $\psi(U)$ is another subalgebra of V which complements N. In this example all such subalgebras can be showed be conjugated by an automorphism of V. It is not clear wether this holds in general.

Chapter 4

Lie pseudoalgebra representation with coefficients

In this chapter we generalize the notion of Lie pseudoalgebra representation given in Section 2.4. The proofs in this chapter are often verbatim translations of proofs given in [BDK1] to the new setting.

4.1 Rings of coefficients

Let $(H, \Delta_H, S, \epsilon)$ be a cocommutative Hopf algebra over $\Bbbk$.
A *left H-comodule* is a vector space A over $\Bbbk$ endowed with a $\Bbbk$-linear map

$$\begin{array}{rccc} \Delta_A : & A & \longrightarrow & H \otimes A \\ & a & \mapsto & a_{(1)} \otimes a_{(2)}, \end{array}$$

such that

$$(\Delta_H \otimes \mathrm{id}_A)\Delta_A = (\mathrm{id}_H \otimes \Delta_A)\Delta_A, \tag{4.1}$$

$$(\epsilon \otimes \mathrm{id}_A)\Delta_A = \mathrm{id}_A. \tag{4.2}$$

Δ_A is called the *comodule structure map* of A.
Let A be an H-comodule and Δ_A be its comodule structure map. Let

$$\begin{array}{rrcl} \Delta^A_{1,1} = \Delta_A : & A & \longrightarrow & H \otimes A \\ \Delta^A_{n,i} = (\mathrm{id}_H \otimes \cdots \otimes \Delta_H \otimes \cdots \otimes \mathrm{id}_D) : & H^{n-1} \otimes A & \longrightarrow & H^n \otimes A, \end{array}$$

for $n \geq 2, i = 1, \ldots, n-1$. For $n \geq 2$ we set

$$\Delta^A_{n,n} = (\mathrm{id}_H \otimes \cdots \otimes \mathrm{id}_H \otimes \Delta_A) : \quad H^{n-1} \otimes A \quad \longrightarrow \quad H^n \otimes A.$$

Then we set

$$\Delta^n_A = \Delta^A_{n,n} \circ \Delta^A_{n-1,n-1} \circ \cdots \circ \Delta^A_{2,2} \circ \Delta^A_{1,1},$$

Then $\Delta^1_A = \Delta_D$ and Δ^n_A is a map from A to $H^n \otimes A$ which by (4.1) does not depend on our choice of the indices i_k. From now on we often denote by Δ both the coproduct Δ_H on H that the comodule structure map Δ_A of A. This leads to no confusion because of (4.1). We can rewrite formulas (4.1) and (4.2) as

$$(\Delta \otimes \mathrm{id}_A)\Delta(a) = (\mathrm{id}_H \otimes \Delta)\Delta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}; \tag{4.3}$$

$$(\epsilon \otimes \mathrm{id}_A)\Delta(a) = \epsilon(a_{(1)})a_{(2)} = a. \tag{4.4}$$

Remark 4.1. *Let A be an H-comodule algebra, $a \in A$ and $\Delta^{n-1}(a) = a_{(1)} \otimes \cdots \otimes a_{(n-1)} \otimes a_{(n)}$. The notation in (4.3) is deceiving as $a_{(1)}, \dots, a_{(n-1)}$ lie in H whereas $d_{(n)}$ lies in A. As a consequence for any $a \in A$ we can apply the counit ϵ of H to all tensor factors of $\Delta^{n-1}(a)$ but for the last one.*

An *H-comodule algebra* is a unital associative algebra D endowed with an associative algebra homomorphism

$$\begin{array}{rccc} \Delta_D : & D & \longrightarrow & H \otimes D \\ & d & \mapsto & d_{(1)} \otimes d_{(2)}, \end{array}$$

making (D, Δ_D) into an H-comodule.
From now on we will call an H-comodule algebra D a *ring of coefficients over H*.

Lemma 4.1. *Let $(H_1, \Delta_1, S_1, \epsilon_1)$, $(H_2, \Delta_2, S_2, \epsilon_2)$ be Hopf algebras, $\phi : H_1 \longrightarrow H_2$ be a Hopf homomorphism and (D_1, Δ_{D_1}) be a ring of coefficients over H. Then $\Delta_{D_2} = (\phi \otimes 1)\Delta_{D_1}$ endows D_1 with a structure of a ring of coefficients over H_2.*

Proof. Clearly, Δ_{D_2} is an associative algebras homomorphism. We have to check that the axioms of a ring of coefficients are satisfied.

$$\begin{aligned} (\Delta_{H_2} \otimes 1)\Delta_{D_2}(d) &= (\Delta_{H_2} \otimes 1)(\phi(d_{(1)}) \otimes d_{(2)}) = \Delta_{H_2}(\phi(d_{(1)})) \otimes d_{(2)} \\ &= (\phi \otimes \phi \otimes 1)(\Delta_{H_1}(d_{(1)}) \otimes d_{(2)}) \\ &= (\phi \otimes \phi \otimes 1)(d_{(1)} \otimes d_{(2)} \otimes d_{(3)}) \\ &= \phi(d_{(1)}) \otimes \phi(d_{(2)}) \otimes d_{(3)}, \\ (1 \otimes \Delta_{D_2})\Delta_{D_2}(d) &= (1 \otimes \Delta_{D_2})(\phi(d_{(1)}) \otimes d_{(2)}) = \phi(d_{(1)}) \otimes \Delta_{D_2}(d_{(2)}) \\ &= \phi(d_{(1)}) \otimes \phi(d_{(2)}) \otimes d_{(3)}. \end{aligned}$$

This proves (4.3). Moreover,

$$(\epsilon_2 \otimes 1)\Delta_{D_2}(d) = (\epsilon_2 \otimes 1)(\phi(d_{(1)}) \otimes d_{(2)}) = \epsilon_1(d_{(1)})d_{(2)} = d.$$

□

Example 4.2. *Every Hopf algebra H is a ring of coefficients over itself with comodule structure map given by Δ_H.*

Example 4.3. *Let D be a Hopf subalgebra of a Hopf algebra H. Since D is a ring of coefficients over itself and the inclusion from D to H is a Hopf homomorphism from D to H this follows by Lemma 4.1 that D is a ring of coefficients over H. In particular $D = \Bbbk \subset H$ is a ring of coefficients over H.*

Let (D_1, Δ_{D_1}) and (D_2, Δ_{D_2}) be rings of coefficients over H_1 and H_2 respectively. Let ϕ be a Hopf homomorphism from H_1 to H_2 and ψ be an associative algebras homomorphism from D_1 to D_2. The pair $(\phi, \psi) : (H_1, D_1) \longrightarrow (H_2, D_2)$

is a *ring of coefficients homomorphism* from D_1 to D_2 if for any $d \in D_1$ it satisfies

$$\Delta_{D_2}(\psi(d)) = (\phi \otimes \psi)\Delta_{D_1}(d). \tag{4.5}$$

Let D be a ring of coefficients over H. For any $n \geq 1$ we define a map

$$\begin{array}{cccc} \mathcal{D}_n: & H^n \otimes D & \longrightarrow & H^n \otimes D \\ & h_1 \otimes \cdots \otimes h_n \otimes d & \mapsto & h_1 d_{(-n)} \otimes \cdots \otimes h_n d_{(-1)} \otimes d_{(n+1)}. \end{array} \tag{4.6}$$

We call the map (4.6) the *n-fold Fourier transform of D*. We often denote $\mathcal{D}_1$ simply by $\mathcal{D}$.

Proposition 4.4. *Let D be a ring of coefficients over H. For any $n \geq 1$ the n-fold Fourier transform of D is an isomorphism of vector spaces.*

Proof. The proof is the same as for Lemma 2.3, where the inverse of $\mathcal{D}_n$ is

$$\begin{array}{cccc} \mathcal{D}_n^{-1}: & H^n \otimes D & \longrightarrow & H^n \otimes D \\ & h_1 \otimes \cdots \otimes h_n \otimes d & \mapsto & h_1 d_{(1)} \otimes \cdots \otimes h_n d_{(n)} \otimes d_{(n+1)}, \end{array}$$

□

As a consequence of Proposition 4.4 we have the following analogous of Proposition 2.4.

Proposition 4.5. *Let H be a cocommutative Hopf algebra, $\{h_i \mid i \in I\}$ be a $\Bbbk$-basis of H and D be a a ring of coefficients over H. Every element $d \in H^n \otimes D$, $n \geq 1$, can be uniquely written as*

$$d = \sum_{i_1,\dots,i_n} (h_{i_1} \otimes \cdots \otimes h_{i_n} \otimes 1)\Delta^n(d').$$

In other words, $H^n \otimes D = (H^{n-1} \otimes \Bbbk)\Delta^n(D)$.

Corollary 4.6. *Let D be a ring of coefficients over H and M be a left D-module. Then every element in $(H \otimes D) \otimes_D M$ can be written in the form*

$$\sum_i (h^i \otimes 1) \otimes_D m_i, \tag{4.7}$$

for a suitable choice of $h^i \in H$, $m_i \in M$.

Proof. It is sufficient to use Proposition 4.5. Explicitly,

$$\sum_i (h^i \otimes d^i) \otimes_D m_i = \sum_i (h^i d^i_{(-1)} \otimes 1) \otimes_D d^i_{(2)} m_i.$$

□

We will refer to (4.7) as the *left-straightening* for $(H \otimes D) \otimes_D M$.

Remark 4.2. *Let D be a ring of coefficients over H. Conversely to what happens for a Hopf algebra H (see Remark 2.2) it is not true in general that $H \otimes D = (\Bbbk \otimes D)\Delta(D)$. For example if D is a proper Hopf subalgebra of a Hopf algebra H then $(\Bbbk \otimes D)\Delta(D) = D \otimes D$ is strictly contained in $H \otimes D$.*

Lemma 4.7. *Let D be a ring of coefficients over H and M be a finite left D-module.*
For every $n \geq 1$ the map
(4.8)

$$\begin{array}{rccc} \pi_n: & (H^n \otimes D)\otimes_D M & \longrightarrow & H^n \otimes M \\ & (h_1 \otimes \cdots \otimes h_n \otimes d)\otimes_D m & \mapsto & h_1 d_{(-n)} \otimes \cdots h_n d_{(-1)} \otimes d_{(n+1)} m, \end{array}$$

is an isomorphism of vector spaces.

Proof. We have to check that π_n is well defined, i.e., we have to verify that

$$\pi_n((h_1 \otimes \cdots \otimes h_n \otimes d)\otimes_D d'm) = \pi_n((h_1 {d'}_1 \otimes \cdots \otimes h_n {d'}_{(n)} \otimes d{d'}_{(n+1)})\otimes_D m).$$

By definition of π_n we have:

$$\begin{aligned} & \pi_n((h_1 \otimes \cdots \otimes h_n \otimes d)\otimes_D d'm) = h_1 d_{(-n)} \otimes \cdots \otimes h_n d_{(-1)} \otimes d_{(n+1)} d'm; \\ & \pi_n((h_1 {d'}_{(1)} \otimes \cdots \otimes h_n {d'}_{(n)} \otimes d{d'}_{(n+1)})\otimes_D m) = \\ & = h_1 {d'}_{(1)} {d'}_{(-2n)} d_{(-n)} \otimes \cdots \otimes h_{n-1} {d'}_{(n-1)} {d'}_{(-n-2)} d_{(-2)} \\ & \otimes h_n {d'}_{(n)} {d'}_{(-n-1)} d_{(-1)} \otimes d_{(n+1)} {d'}_{(2n+1)} m \\ & = h_1 {d'}_{(1)} {d'}_{(-2n+1)} d_{(-n)} \otimes \cdots \otimes h_{n-1} {d'}_{(n-1)} {d'}_{(-n-1)} d_{(-2)} \\ & \otimes h_n \epsilon({d'}_{(n)}) d_{(-1)} \otimes d_{(n+1)} {d'}_{(2n)} m \\ & = h_1 {d'}_{(1)} {d'}_{(-2n+1)} d_{(-n)} \otimes \cdots \otimes h_{n-1} {d'}_{(n-1)} \epsilon({d'}_{(n)}) {d'}_{(-n-1)} d_{(-2)} \\ & \otimes h_n d_{(-1)} \otimes d_{(n+1)} {d'}_{(2n)} m = h_1 {d'}_{(1)} {d'}_{(-2n+2)} d_{(-n)} \otimes \cdots \\ & \otimes h_{n-1} {d'}_{(n-1)} {d'}_{(-n)} d_{(-2)} \otimes h_n d_{(-1)} \otimes d_{(n+1)} {d'}_{(2n-1)} m \\ & = \cdots = h_1 d_{(-n)} \otimes \cdots \otimes h_n d_{(-1)} \otimes d_{(n+1)} d'm. \end{aligned}$$

By a straightforward computation, similar to that in Lemma 2.7, one can prove that:

$$\begin{array}{rccc} {\pi_n}^{-1}: & H^n \otimes M & \longrightarrow & (H^n \otimes D)\otimes_D M \\ & h_1 \otimes \cdots \otimes h_n \otimes m & \mapsto & (h_1 \otimes \cdots \otimes h_n \otimes 1)\otimes_D m, \end{array}$$

is both a left and right inverse of π_n. □

4.1.1 The ring of coefficients D_λ

In this section we introduce a family of ring of coefficients D_λ, $\lambda \in \Bbbk$, which will play an important role in the sequel. We also prove that for a such ring of coefficients equality $H \otimes D_\lambda = (\Bbbk \otimes D_\lambda)\Delta(D_\lambda)$ holds.
Let $\mathfrak{d}$ be a Heisenberg Lie algebra, that is a vector space of dimension $2n+1$ with a basis $\{p_i, q_i, c\}$ such that $[p_i, q_i] = -c$, $i = 1, \ldots, n$, are the only nonzero commutation relations. Its universal enveloping algebra $H = \mathcal{U}(\mathfrak{d})$ is graded associative algebra by the gradation defined in (2.32). H is a central extension by $\Bbbk(c-1)$ of the Weil algebra $A_{2n} = \langle t_1, \ldots, t_n, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_1} \rangle$.
Let $I_\lambda = (c-\lambda)H$, $\lambda \in \Bbbk$.

Lemma 4.8. *I_λ is a two-sided ideal of the associative algebra H such that*

$$\Delta(I_\lambda) \subset I_0 \otimes H + H \otimes I_\lambda.$$

In particular, I_0 is an Hopf ideal of H.

Proof. As $(c-\lambda)$ is central, the first claim is clear. The second claim follows by

$$\Delta(c-\lambda) = c\otimes 1 + 1\otimes (c-\lambda).$$

Since $S(I_0)\subset I_0$ and $\epsilon(I_0)=0$ then I_0 is an Hopf ideal of H. $\square$

For any $\lambda\in\mathbb{k}$ let

$$\pi_\lambda : H \longrightarrow H/I_\lambda \tag{4.9}$$

be the natural projection. We denote H/I_0 by $H_0 = D_0$ and H/I_λ by D_λ.

Corollary 4.9. *The Hopf algebra H induces a Hopf algebra structure on H_0 and a structure of a ring of coefficients over H_0 on each D_λ.*

The Hopf algebra H_0 is isomorphic to the graded commutative associative symmetric algebra $S(\bar{\mathfrak{d}})$, where $\bar{\mathfrak{d}}$ is the vector space linearly generated by p_i, q_i, $i=1,\dots,n$ and its gradation is given by (2.28). For any $\lambda\neq 0$ the ring of coefficients D_λ is isomorphic to the Weyl algebra A_{2n}.

Remark 4.3. *By construction π_0 is a Hopf homomorphism from H to H_0 and (π_0,π_λ) is a ring of coefficients homomorphism from H to D_λ.*

Proposition 4.10. *Let $\lambda\in\mathbb{k}$ and D_λ be the ring of coefficients over H_0 described in Corollary 4.9. Then $H\otimes D_\lambda = (\mathbb{k}\otimes D_\lambda)\Delta(D_\lambda)$.*

Proof. This follows by projecting $H\otimes H = (\mathbb{k}\otimes H)\Delta(H)$. $\square$

Corollary 4.11. *Let M be a left D_λ-module. Then every element in $(H_0\otimes D_\lambda)\otimes_{D_\lambda} M$ can be written in the form*

$$\sum_i (1\otimes d^i)\otimes_{D_\lambda} m'_i, \tag{4.10}$$

for a suitable choice of $d^i\in D_\lambda$, $m'_i\in M$.

Proof. It is sufficient to use Proposition 4.10. Explicitly,

$$(h^i\otimes d^i)\otimes_D m'_i = (1\otimes\, d^i h^i_{(-2)})\otimes_D h^i_{(1)} m'_i.$$

$\square$

We will refer to (4.10) as the *right-straightening* for $(H_0\otimes D_\lambda)\otimes_{D_\lambda} M$. From now on for the ring of coefficients D_λ we will use the following notation

$$\pi_0(p_i)=\partial_i,\quad \pi_0(q_i)=\partial_{n+i},\quad i=1,\dots,n, \tag{4.11}$$

$$\pi_\lambda(p_i)=\delta_i,\quad \pi_\lambda(q_i)=\delta_{n+i},\quad i=1,\dots,n, \tag{4.12}$$

so that, for example,

$$\begin{aligned}\Delta_\lambda(\delta_i) &= \partial_i\otimes 1 + 1\otimes\delta_i,\\ \Delta_\lambda(\delta_i\delta_j) &= \partial_i\partial_j\otimes 1 + \partial_i\otimes\delta_j + \partial_j\otimes\delta_i + 1\otimes\delta_i\delta_j.\end{aligned}$$

Remark 4.4. *Let us spell out explicitly some right-straightening that we will need later on. Let M be a D_λ-module. For any $m\in M$ we have equalities:*

$$(\partial_i\otimes 1)\otimes_{D_\lambda} m = (1\otimes\, 1)\otimes_{D_\lambda}\delta_i m - (1\otimes\,\delta_i)\otimes_{D_\lambda} m, \tag{4.13}$$

and

$$\begin{aligned}(\partial_i\partial_j\otimes 1)\otimes_{D_\lambda} m &= (1\otimes\, 1)\otimes_{D_\lambda}\delta_i\delta_j m - (1\otimes\,\delta_j)\otimes_{D_\lambda}\delta_i m\\ &\quad -(1\otimes\,\delta_i)\otimes_{D_\lambda}\delta_j m + (1\otimes\,\delta_i\delta_j)\otimes_{D_\lambda} m.\end{aligned} \tag{4.14}$$

4.1.2 A filtration on D_λ

Let D_λ be a ring of coefficients over H_0, we set

$$\delta^{(I)} = \frac{\delta_1^{i_1} \cdots \delta_{2n}^{i_{2n}}}{i_1! \cdots i_{2n}!},$$

similarly to the notation introduced in (2.18) for a Hopf algebra H.
We can define an increasing filtration on D_λ by

$$F^p D_\lambda = \mathrm{span}_{\Bbbk}\{\delta^{(I)} \mid |I| \leq p\}, \quad p = 0, 1, 2, \ldots, \tag{4.15}$$

This filtration satisfies

$$(F^p D_\lambda)(F^q D_\lambda) \subset F^{p+q} D_\lambda, \tag{4.16}$$

$$\Delta_\lambda(F^p D_\lambda) \subset \sum_{i=0}^{p} F^i H_0 \otimes F^{p-i} D_\lambda, \tag{4.17}$$

as one can prove immediately by projecting (2.20) and (2.21) to D_λ.
We will say that a nonzero element $d \in D_\lambda$ is of degree p if $d \in F^p D_\lambda \setminus F^{p-1} D_\lambda$.
In a similar way $H_0 \otimes D_\lambda$ is filtered by:

$$F^p(H_0 \otimes D_\lambda) = \sum_{l+m=p} F^l H_0 \otimes F^m D_\lambda. \tag{4.18}$$

Lemma 4.12. *Let M be a D_λ-module. With respect to the filtration of $H_0 \otimes D_\lambda$ defined in (4.18) we have equalities:*

$$(F^p H_0 \otimes \Bbbk) \otimes_{D_\lambda} M = F^p(H_0 \otimes D_\lambda) \otimes_{D_\lambda} M = (\Bbbk \otimes F^p D_\lambda) \otimes_{D_\lambda} M.$$

Proof. Recall that for D_λ we have $(H \otimes \Bbbk)\Delta(D_\lambda) = (\Bbbk \otimes D_\lambda)\Delta(D_\lambda)$. By use of left and right straightening formulas for $(H \otimes D_\lambda) \otimes_{D_\lambda} M$ and (4.17) we obtain the two opposite inclusions for $(F^p H_0 \otimes \Bbbk) \otimes_{D_\lambda} M$ and $(\Bbbk \otimes F^p D_\lambda) \otimes_{D_\lambda} M$, proving equality. Clearly, by (4.18) both $F^p H_0 \otimes \Bbbk$ and $\Bbbk \otimes F^p D_\lambda$ are contained in $F^p(H_0 \otimes D_\lambda)$. To prove the opposite inclusion we can apply an argument similar to the other one. □

Let

$$\mathrm{gr}\, D_\lambda = \bigoplus_{p \geq 0} F^p D_\lambda / F^{p-1} D_\lambda$$

be the associated graded space to D_λ. Then $\mathrm{gr}\, D_\lambda$ is isomorphic to $H_0 = S(\bar{\mathfrak{d}})$, for every $\lambda \in \Bbbk$.

4.2 Pseudoalgebra representations with coefficients

Let H be a cocommutative Hopf algebra, D be a ring of coefficients over H and A be an H-pseudoalgebra.
Let M be a left D-module. A *pseudoaction of A on M with coefficients in D* is an $(H \otimes D)$-linear map

$$\begin{array}{cccc} * : & A \otimes M & \longrightarrow & (H \otimes D) \otimes_D M \\ & a \otimes m & \mapsto & a * m. \end{array}$$

The *expanded pseudoaction with coefficients in* D is an $(H^{m+n-1}\otimes D)$-linear map

$$*: \quad (H^m \otimes_H A)\otimes((H^{n-1}\otimes D)\otimes_D M) \quad\longrightarrow\quad (H^{m+n}\otimes D)\otimes_D M,$$

defined as

$$(F\otimes_H b)*(G\otimes_D m) = (F\otimes G)(\Delta^{m-1}\otimes\Delta^{n-1})(a*m), \tag{4.19}$$

where $F\in H^m, G\in H^{n-1}\otimes D, a\in A, m\in M,\ m,n\geq 1$.
Let A be an associative H-pseudoalgebra, D be a ring of coefficients over H.
A *representation of* A *with coefficients in* D, or an A*-module with coefficients in* D, is a left D-module M endowed with a pseudoaction of A on M with coefficients in D satisfying:

$$(a*b)*m = a*(b*m), \tag{4.20}$$

for $a,b\in A$, $m\in M$.
Let L be a Lie H-pseudoalgebra, D be a ring of coefficients over H.
A *representation of* L *with coefficients in* D, or an L*-module with coefficients in* D, is a left D-module M endowed with a pseudoaction of L on M with coefficients in D satisfying:

$$[a*b]*m = a*(b*m)-(b*(a*m))^{\sigma_{12}}, \tag{4.21}$$

for $a,b\in L$, $m\in M$.
As usual, both sides of (4.20) and (4.21) lie in $(H^2\otimes D)\otimes_D M$ and are defined by use of (4.19).
Explicitly, if

$$b*m=\sum_i (k^i\otimes\, d^i)\otimes_D m_i, \quad a*m_i=\sum_j (h^{ij}\otimes\, d^{ij})\otimes_D m_{ij};$$

then,

$$a*(b*m)=\sum_{i,j}(h^{ij}\otimes k^i d^{ij}_{(1)}\otimes d^i d^{ij}_{(2)})\otimes_D m_{ij}. \tag{4.22}$$

If

$$[a*b]=\sum_i (h^i\otimes\, f^i)\otimes_H l_i, \quad l_i*m=\sum_j (h^{ij}\otimes\, e^{ij})\otimes_D n_{ij};$$

then,

$$[a*b]*m=\sum_{i,j}(h^i {h^{ij}}_{(1)}\otimes f^i h^{ij}_{(2)}\otimes e_{ij})\otimes_D n_{ij}. \tag{4.23}$$

An A-module M with coefficients in D, A an H-pseudoalgebra, is *finite* if it is finitely generated as a D-module. As a consequence of Lemma 4.7 we have the following,

Proposition 4.13. *Let* L *be a Lie pseudoalgebra over* H*,* D *be a ring of coefficients and* M *be a finite* L*-module with coefficients in* D*.*
Any element of $(H^{n+m}\otimes D)\otimes_D M$ *can be uniquely written as an element in* $H^{m+n}\otimes M$*.*

When $D = H$ the notion of representation with coefficients reduces to the usual notion of pseudoalgebra representation given in Section 2.4. From now on we call a pseudoalgebra representation with coefficients in H an *ordinary pseudoalgebra representation.*

Let M be an L-module with coefficients in D and let

$$a * m = \sum_i (h^i \otimes 1) \otimes_D m_i,$$

where $\{h^i\}$ is a $\Bbbk$-basis of H. For any subset N of M we denote by $L \cdot N$ the D-module generated by all the n_i of any $a * n$, $a \in L$, $n \in N$.

A D-submodule N of M is an *L-submodule with coefficients in D* if $L \cdot N \subset N$. An L-module with coefficients in D is *irreducible* if it contains no nontrivial submodules.

Let M_1, M_2 be L-modules with coefficients in D. An *L-module (with coefficients in D) homomorphism* from M_1 to M_2 is a D-linear map $\rho : M_1 \longrightarrow M_2$ such that for any $a \in L$, $m \in M_1$ we have

$$a * (\rho(m)) = ((\mathrm{id}_H \otimes \mathrm{id}_D) \otimes_D \rho)(a * m). \tag{4.24}$$

4.2.1 Extending scalars with pseudoalgebra representations

Here we show that under a compatibility condition it is possible to associate to any L-module M with coefficients in D an L'-module M' with coefficients in D', where L' is obtained from L by extension of scalars.

Let L be a Lie H-pseudoalgebra, D be a ring of coefficients over H, and M be an L-module with coefficients in D. Let D' be a ring of coefficients over H' and suppose that $\Psi = (\phi, \psi)$ is a ring of coefficients homomorphism from D to D'. Then ψ endows D' with a right D-module structure so that we may consider the tensor product $D' \otimes_D A$ for any left D-module A.

Proposition 4.14. *The left D'-module $M' = D' \otimes_D M$ has a structure of L'-module with coefficients in D', where $L' = \mathrm{BC}_\phi(L)$, satisfying*

$$(h' \otimes_H a) * (d' \otimes_D m) = \sum_i (h' \phi(h^i) \otimes d' \psi(d^i)) \otimes_{D'} (1 \otimes_D m_i), \tag{4.25}$$

if $a * m = \sum_i (h^i \otimes d^i) \otimes_D m_i \in (H \otimes D) \otimes_D M$.

Proof. First of all we have to verify that (4.25) gives a well defined map.

$$\begin{aligned}(h' \otimes_H ha) * (d' \otimes_H dm) &= \sum_i (h' \phi(hh^i) \otimes d' \psi(dd^i)) \otimes_{D'} (1 \otimes_D m_i) \\ &= \sum_i (h' \phi(h) \phi(h^i) \otimes d' \psi(d) \psi(d^i)) \otimes_{D'} (1 \otimes_D m_i) \\ &= (h' \phi(h) \otimes_H a) * (d' \psi(d) \otimes_D m).\end{aligned}$$

It is an $(H' \otimes D')$-linear map by the very definition. It remains to prove that it is a Lie pseudoaction. Let $a, b \in L$, $m \in M$ and suppose that:

$$[a * b] = \sum_i (h^i \otimes k^i) \otimes_H e_i, \quad e_i * m = \sum_j (h^{ij} \otimes k^{ij}) \otimes_D m,$$

then we have

$$[a*b]*m=\sum_{i,j}(h^i h^{ij}_{(1)}\otimes k^i h^{ij}_{(2)}\otimes k^{ij})\otimes_D m_{ij}.$$

Then

$$\begin{aligned}
&[(h'\otimes_H a)*(k'\otimes_H b)]*(d'\otimes_D m)\\
&=\sum_i (h'\phi(h^i)\otimes k'\phi(k^i))\otimes_{H'}(1\otimes_H e_i)*(d'\otimes_D m)\\
&=\sum_{i,j}(h'\phi(h^i)\phi(h^{ij})_{(1)}\otimes k'\phi(k^i)\phi(h^{ij})_{(2)}\otimes d'\psi(k^{ij}))\otimes_{D'}(1\otimes_D m_{ij})\\
&=(h'\phi(h^i h^{ij}_{(1)})\otimes k'\phi(k^i h^{ij}_{(2)})\otimes d'\psi(k^{ij})\otimes_{D'}(1\otimes_D m_{ij}).
\end{aligned}$$

Similarly, if

$$a*(b*m)=\sum_{i,j}(f^{ij}\otimes f^i g^{ij}_{(1)}\otimes g^i g^{ij}_{(2)})\otimes_D m_{ij}\in(H\otimes H\otimes D)\otimes_D M,$$

then

$$\begin{aligned}
&(h'\otimes_H a)*((k'\otimes_H b)*(d'\otimes_D m))\\
&=\sum_i (h'\otimes_H a)*(k'\phi(f^i)\otimes d'\psi(g^i))\otimes_{D'}(1\otimes_D m_i)\\
&=\sum_{i,j}(h'\phi(f^{ij})\otimes k'\phi(f^i)\psi(g^{ij})_{(1)}\otimes d'\psi(g^i)\psi(g^{ij})_{(2)})\otimes_{D'}(1\otimes_D m_{ij})\\
&=\sum_{i,j}(h'\phi(f^{ij})\otimes k'\phi(f^i)\phi(g^{ij}_{(1)})\otimes d'\psi(g^i)\psi(g^{ij}_{(2)}))\otimes_{D'}(1\otimes_D m_{ij})\\
&=\sum_{i,j}(h'\phi(f^{ij})\otimes k'\phi(f^i g^{ij}_{(1)})\otimes d'\psi(g^i g^{ij}_{(2)}))\otimes_{D'}(1\otimes_D m_{ij}).
\end{aligned}$$

In the same way we can compute $(k'\otimes_H b)*((h'\otimes_H a)*(d'\otimes_D m))$.
Then $[a*b]*m=a*(b*m)-(b*(a*m))^{\sigma_{12}}$ guarantees that:

$$\begin{aligned}
[h'\otimes_H a*k'\otimes_H b]*(d'\otimes_D m)= \quad &(h'\otimes_H a)*((k'\otimes_H b)*(d'\otimes_D m))\\
&-((k'\otimes_H b)*((h'\otimes_H a)*(d'\otimes_D m)))^{\sigma_{12}}.
\end{aligned}$$

□

Remark 4.5. *A more conceptual proof of the above statement can be given by an argument analogous to that in Remark 2.6.*

We will say that M' is obtained from M by *extension of scalars* or *base change*. Let $\Psi=(\phi,\psi):(H,D)\longrightarrow(H',D')$ be a ring of coefficients homomorphism, L a Lie pseudoalgebra over H, M a representation of L with coefficients in D. Then, by Proposition 4.14, $\mathrm{BC}_\Psi(M):=D'\otimes_D M$ is a representation of $\mathrm{BC}_\phi(L)=H'\otimes_H L$ with coefficients in D'. Similarly, if $\rho:M_1\longrightarrow M_2$ is an L-module with coefficients in D homomorphism, set $\mathrm{BC}_\Psi(\rho)=\mathrm{id}_{D'}\otimes_D\rho$. Let $\mathcal{M}od^D_L$ be the category of L-modules with coefficients in D.

Theorem 4.15. *Let D,D' be ring of coefficients over H and H' respectively, $\Psi=(\phi,\psi):(H,D)\longrightarrow(H',D')$ be a ring of coefficients homomorphism. Then $\mathrm{BC}_\Phi:\mathcal{M}od^D_L\longrightarrow\mathcal{M}od^{D'}_{\mathrm{BC}_\phi(L)}$ is a (covariant) functor.*

Proof. As clearly, for any choice of L-modules with coefficients in D, M_1, M_2, M_3, $\rho \in \mathcal{M}or(M_1, M_2)$, $\mu \in \mathcal{M}or(M_2, M_3)$ we have $\mathrm{BC}_\Psi(\mu) \circ \mathrm{BC}_\Psi(\rho) = \mathrm{BC}_\Psi(\mu \circ \rho)$ and $\mathrm{BC}_\Psi(\mathrm{id}_M) = \mathrm{id}_{\mathrm{BC}_\Psi(M)}$ we are left with proving that if $\rho \in \mathcal{M}or(\mathcal{M}od_L^D)$ then $\mathrm{BC}_\Psi(\rho) \in \mathcal{M}or(\mathcal{M}od^{D'}_{\mathrm{BC}_\phi(L)})$. Assume that $\rho : M_1 \longrightarrow M_2$ is a L-module with coefficients in D homomorphism. Notice that $\mathrm{BC}_\Psi(\rho)$ is well defined since both $\mathrm{id}_{D'}$ and ρ are D-linear maps. Moreover,

$$\begin{aligned}
&((\mathrm{id}_{H'} \otimes \mathrm{id}_{D'}) \otimes_{D'} \mathrm{BC}_\Psi(\rho))(1 \otimes_H a) * (1 \otimes_D m) \\
&= ((\mathrm{id}_{H'} \otimes \mathrm{id}_{D'}) \otimes_{D'} (\mathrm{id}_{D'} \otimes_D \rho))(1 \otimes_H a) * (1 \otimes_D m) \\
&= (\mathrm{id}_{H'} \otimes \mathrm{id}_{D'}) \otimes_{D'} (1 \otimes_H \rho(a * m)) = (1 \otimes_H a) * (1 \otimes_D \rho(m)) \\
&= (1 \otimes_H a) * (\mathrm{BC}_\Psi(\rho)(1 \otimes_H m)).
\end{aligned}$$

By $(H' \otimes D')$-linearity this proves that (4.24) is satisfied. □

Remark 4.6. *The same construction holds for associative H-pseudoalgebras.*

Remark 4.7. *When $D = H$, $D' = H'$ then BC_Ψ is a functor from the category of ordinary L-modules ${\mathcal{M}od_L}^H = \mathcal{M}od_L$ to $\mathcal{M}od_{\mathrm{BC}_\phi(L)}$. If $D = H$ and D' is a ring of coefficients over H' then BC_Ψ is a functor from $\mathcal{M}od_L$ to $\mathcal{M}od^{D'}_{\mathrm{BC}_\phi(L)}$.*

4.3 D-pseudolinear maps and $\mathrm{gc}^D M$

In this section we introduce the H-module $\mathrm{gc}^D M$ of all D-pseudolinear maps from a finite D-module M to itself, proving that $\mathrm{gc}^D M$ is a Lie H-pseudoalgebra. Let D be a ring of coefficients over H, M, N be D-modules. A D-pseudolinear map ϕ is a $\Bbbk$-linear map:

$$\begin{array}{rccc} \phi : & M & \longrightarrow & (H \otimes D) \otimes_D N \\ & m & \mapsto & \phi(m) \end{array}$$

such that

$$\phi(dm) = (1 \otimes d) \otimes_D \phi(m); \quad \forall\, d \in D. \tag{4.26}$$

We denote by $\mathrm{Chom}^D(M, N)$ the space of all D-pseudolinear maps from M to N. $\mathrm{Chom}^D(M, N)$ is a left H-module via the action

$$(h\phi)(m) = (h \otimes 1)\phi(m); \quad \forall\, h \in H. \tag{4.27}$$

As a consequence if M, N are D-modules and $\phi \in \mathrm{Chom}^D(M, N)$ then

$$\begin{array}{rccc} * : & \mathrm{Chom}^D(M, N) \otimes M & \longrightarrow & (H \otimes D) \otimes_D N \\ & \phi \otimes m & \mapsto & \phi * m = \phi(m), \end{array}$$

is an $(H \otimes D)$-linear map.

Example 4.16. *Let A be an associative pseudoalgebra over H and M be an A-module with coefficients in D. For any $a \in A$ the map*

$$\begin{array}{rccc} \lambda_a : & M & \longrightarrow & (H \otimes D) \otimes_D M \\ & m & \mapsto & \lambda_a(m) = a * m, \end{array} \tag{4.28}$$

is a D-pseudolinear map. Moreover for any $h \in H$ we have $h\lambda_a = \lambda_{ha}$.

Lemma 4.17. *Let $\phi \in \mathrm{Chom}^D(M,N)$ and $0 \neq h \in H$. Then $h\phi = 0$ implies $\phi = 0$.*

Proof. Let $m \in M$ and suppose that $\phi * m = \sum_i (h^i \otimes 1) \otimes_D m_i$, where we can choose the m_i to be linearly independent.
By (4.27) we have $(h\phi) * m = \sum_i (hh^i \otimes 1) \otimes_D m_i$. If $(h\phi) * m = 0$ then we must have $hh^i = 0$. Since $H = \mathcal{U}(\mathfrak{d})$ has no nonzero divisor this implies $h^i = 0$. □

Let $\phi \in \mathrm{Chom}^D(M,N)$ and suppose that $\phi * m = \sum_i (h^i \otimes d^i) \otimes_D n_i$.
For any $x \in X$ we define a map $\phi_x : M \longrightarrow N$ as

$$\phi_x(m) = \sum_i \langle S(x), h^i d^i_{(-1)} \rangle d^i_{(2)} n_i.$$

We will call ϕ_x the *x-coefficients* of ϕ. By Corollary 4.11 and properties of the filtration of X this follows that,

$$\mathrm{codim}\{x \in X \mid \phi_x m = 0\} < \infty, \quad \text{for any } m \in M. \tag{4.29}$$

Moreover, since ϕ satisfies (4.26) we have, for any $x \in X$:

$$\begin{aligned} \phi_x(dm) &= \sum_i \langle S(x), h^i d^i_{(-1) d_{(-1)}} \rangle d_{(2)} d^i_{(2)} n_i \\ &= \sum_i \langle S(x) S(d_{(-1)}), h^i d^i_{(-1)} \rangle d_{(2)} d^i_{(2)} n_i \\ &= d_{(2)} \sum_i \langle S(d_{(-1)} x), h^i d^i_{(-1)} \rangle d^i_{(2)} n_i \\ &= d_{(2)} \phi_{d_{(-1)} x}(m). \end{aligned} \tag{4.30}$$

On the other hand any collection of linear maps $\phi_x \in \mathrm{Hom}(M,N)$, $x \in X$ satisfying (4.29) and (4.30) comes from the D-pseudolinear map $\phi \in \mathrm{Chom}^D(M,N)$ defined by:

$$\phi * m = \sum_i (S(h^i) \otimes 1) \otimes_D \phi_{x_i} m,$$

where $\{h^i\}$ is a basis of H and $\{x_i\}$ is the corresponding dual basis of X.

Theorem 4.18. *Let M be a finite D-module and $\mathrm{Chom}^D(M,M)$ be the H-module of all D-pseudolinear maps from M to itself. Then*

$$(\phi * \psi) * m = \phi * (\psi * m),$$

where $\phi, \psi \in \mathrm{Chom}^D(M,M)$, $m \in M$, endows $\mathrm{Chom}^D(M,M)$ with a structure of an associative pseudoalgebra over H denoted by $\mathrm{Cend}\, M$.

Proof. The map $\phi * \psi$ is defined in term of its x-coefficients. The proof of the statement is the same as in [BDK1,Lemma 10.1]. □

If M is a free D-module of finite rank there exists an explicit description of $\mathrm{Cend}^D M$.

Proposition 4.19. *Let $M = D \otimes M_0$ where D acts trivially on M_0 and $\dim M_0 < \infty$. Then the associative pseudoalgebra $\mathrm{Cend}^D M$ is the H-module*

$H \otimes D \otimes \operatorname{End} M_0$ with H acting on the first tensor factor endowed with the following pseudoproduct

$$(f \otimes c \otimes A) * (g \otimes d \otimes B) = (f \otimes gc_{(1)}) \otimes_H (1 \otimes dc_{(2)} \otimes AB), \tag{4.31}$$

for $f, g \in H, c, d \in D, A, B \in \operatorname{End} M_0$.
Moreover, M is a $\operatorname{Cend}^D M$-module with coefficients in D via the pseudoaction
(4.32)

$$(f \otimes c \otimes A) * (e \otimes m) = (f \otimes ec) \otimes_D (1 \otimes Am), \quad f \in H, c, e \in D, A \in \operatorname{End} M_0, m \in M_0,$$

Proof. Since $M = D \otimes M_0$ a D-pseudolinear map is a map from M to $(H \otimes D) \otimes_D (D \otimes M_0) = H \otimes D \otimes M_0$. So, as a vector space, we can identify $\operatorname{Chom}^D(M, M)$ with $H \otimes D \otimes \operatorname{End} M_0$, with a structure of an H-module given by multiplication by h on the first tensor factor.
We have to verify that (4.31) is an $(H \otimes H)$-linear map and that it satisfies the associativity (2.11).
Let $f \otimes c \otimes A, g \otimes d \otimes B, h \otimes e \otimes C \in \operatorname{Cend}^D M$, $k, k' \in H$. Then

$$\begin{aligned}
&(kf \otimes c \otimes A) * (k'g \otimes d \otimes B) = (kf \otimes k'gc_{(1)}) \otimes_H (1 \otimes dc_{(2)} \otimes AB) \\
&= ((k \otimes k') \otimes_H 1)((f \otimes gc_{(1)}) \otimes_H (1 \otimes dc_{(2)} \otimes AB)) \\
&= ((k \otimes k') \otimes_H 1)((f \otimes c \otimes A) * (g \otimes d \otimes B)).
\end{aligned}$$

This proves that (4.31) is an $(H \otimes H)$-linear map.
By a direct computation we obtain

$$\begin{aligned}
&(f \otimes c \otimes A) * ((g \otimes d \otimes B) * (h \otimes e \otimes C)) \\
&= (f \otimes c \otimes A) * ((g \otimes hd_{(1)}) \otimes_H (1 \otimes ed_{(2)} \otimes BC)) \\
&= (1 \otimes g \otimes hd_{(1)})(1 \otimes \Delta)((f \otimes c \otimes A) * (1 \otimes ed_{(2)} \otimes BC)) \\
&= (1 \otimes g \otimes hd_{(1)})(1 \otimes \Delta)((f \otimes c_{(1)}) \otimes_H (1 \otimes ed_{(2)}c_{(2)} \otimes ABC)) \\
&= ((1 \otimes g \otimes hd_{(1)})(f \otimes c_{(1)} \otimes c_{(2)})) \otimes_H (1 \otimes ed_{(2)}c_{(3)} \otimes ABC) \\
&= (f \otimes gc_{(1)} \otimes hd_{(1)}c_{(2)}) \otimes_H (1 \otimes ed_{(2)}c_{(3)} \otimes ABC),
\end{aligned}$$

and

$$\begin{aligned}
&((f \otimes c \otimes A) * (g \otimes d \otimes B)) * (h \otimes e \otimes C) \\
&= ((f \otimes gc_{(1)}) \otimes_H (1 \otimes dc_{(2)} \otimes AB)) * (h \otimes e \otimes C) \\
&= (f \otimes gc_{(1)} \otimes 1)(\Delta \otimes 1)((1 \otimes dc_{(2)} \otimes AB) * (h \otimes e \otimes C)) \\
&= ((f \otimes gc_{(1)} \otimes 1)(\Delta \otimes 1)(1 \otimes hd_{(1)}c_{(2)} \otimes)) \otimes_H (1 \otimes ed_{(2)}c_{(3)} \otimes ABC) \\
&= ((f \otimes gc_{(1)} \otimes 1)(1 \otimes 1 \otimes hd_{(1)}c_{(2)})) \otimes_H (1 \otimes ed_{(2)}c_{(3)} \otimes ABC) \\
&= (f \otimes gc_{(1)} \otimes hd_{(1)}c_{(2)}) \otimes_H (1 \otimes ed_{(2)}c_{(3)} \otimes ABC),
\end{aligned}$$

We are left with proving that M is a $\operatorname{Cend}^D M$-module with coefficients in D. We have to verify that (4.32) is an $(H \otimes D)$-linear map which satisfies (4.20). Let $f \otimes c \otimes A, g \otimes d \otimes B \in \operatorname{Cend}^D M$, $e \otimes m \in M$, $h \in H$, $d \in D$. Then we have,

$$\begin{aligned}
(hf \otimes c \otimes A) * (de \otimes m) &= (hf \otimes edc) \otimes_D (1 \otimes Am) \\
&= ((h \otimes d) \otimes_H 1)((f \otimes ec) \otimes_D (1 \otimes Am)) \\
&= ((h \otimes d) \otimes_H 1)((f \otimes c \otimes A) * (e \otimes m)).
\end{aligned}$$

Let us compute:

$$
\begin{aligned}
&((f \otimes c \otimes A) * (g \otimes d \otimes B)) * (e \otimes m) \\
&= ((f \otimes gc_{(1)}) \otimes_H (1 \otimes dc_{(2)} \otimes AB)) * (e \otimes m) \\
&= (f \otimes gc_{(1)} \otimes 1)(\Delta \otimes 1)((1 \otimes dc_{(2)} \otimes AB) * (e \otimes m)) \\
&= (f \otimes gc_{(1)} \otimes 1)(\Delta \otimes 1)((1 \otimes edc_{(2)}) \otimes_D (1 \otimes ABm)) \\
&= (f \otimes gc_{(1)} \otimes 1)(1 \otimes 1 \otimes edc_{(2)}) \otimes_D (1 \otimes ABm) \\
&= (f \otimes gc_{(1)} \otimes edc_{(2)}) \otimes_D (1 \otimes ABm),
\end{aligned}
$$

and

$$
\begin{aligned}
&(f \otimes c \otimes A) * ((g \otimes d \otimes B) * (e \otimes m)) \\
&= (f \otimes c \otimes A) * ((g \otimes ed) \otimes_D (1 \otimes Bm)) \\
&= (1 \otimes g \otimes ed)(1 \otimes \Delta)((f \otimes c \otimes A) * (1 \otimes Bm)) \\
&= (1 \otimes g \otimes ed)(1 \otimes \Delta)((f \otimes c) \otimes_D (1 \otimes ABm)) \\
&= ((1 \otimes g \otimes ed)(f \otimes c_{(1)} \otimes c_{(2)})) \otimes_D (1 \otimes ABm) \\
&= (f \otimes gc_{(1)} \otimes edc_{(2)}) \otimes_D (1 \otimes ABm).
\end{aligned}
$$

□

We denote by $\mathrm{gc}^D M$ the Lie pseudoalgebra structure on H obtained from $\mathrm{Cend}^D M$ by use of (2.14). Clearly, M is a $\mathrm{gc}^D M$-module too.
If M is a free D-module of rank N then we denote $\mathrm{Cend}^D M$ and $\mathrm{gc}^D M$ respectively by Cend^D_N and gc^D_N.

Example 4.20. *Let M be a free D-module of rank 1. The associative H-pseudoalgebra Cend^D_1 is the H-module $H \otimes D$, with H acting on the first tensor factor, endowed with the associative pseudoproduct:*

$$(f \otimes c) * (g \otimes d)) = (f \otimes gc_{(1)}) \otimes_H (1 \otimes dc_{(2)}),$$

for $f, g \in H$, $c, d \in D$.
As a consequence of (2.14), gc^D_1 is $H \otimes D$ endowed with the pseudobracket:

$$[(f \otimes c) * (g \otimes d)] = (f \otimes gc_{(1)}) \otimes_H (1 \otimes dc_{(2)}) - (fd_{(1)} \otimes g) \otimes_H (1 \otimes cd_{(2)}),$$

for $f, g \in H$, $c, d \in D$.

Remark 4.8. *The algebraic structures in Example 4.20 are remindful to those introduced by Kolesnikov in [Ko2] and Retakh in [Re].*

Proposition 4.21. *Let D be a ring of coefficients over H, M be a finite D-module, A be an associative H-pseudoalgebra. Giving an A-module structure with coefficients in D over M is equivalent to giving an associative pseudoalgebras homomorphism from A to $\mathrm{Cend}^D M$. A similar statement holds for Lie H-pseudoalgebras L.*

Proof. The equivalence between A-modules with coefficients in D and associative pseudoalgebras homomorphism from A to $\mathrm{Cend}^D M$ is obtained via the map (4.28). □

Remark 4.9. *For $D = H$ we reobtain the results stated in Proposition 2.28.*

Corollary 4.22. *Let M be an L-module with coefficients in D. Then any torsion element of L acts trivially on M.*

Proof. It follows by Lemma 4.17. □

In the rest of this section we will assume that D is a ring of coefficients over H such that $H \otimes D = (\Bbbk \otimes D)\Delta(D)$.
Let $\phi \in \mathrm{Chom}^D(M, N)$, we set:

$$\ker_p \phi = \{m \in M \mid \phi_x m = 0, \ \forall\, x \in F^p X\}.$$

Lemma 4.23. *The space* $\ker_{-1} \phi = \ker \phi = \{m \in M \mid \phi_x m = 0, \ \forall\, x \in X\}$ *is a* D*-submodule of* M.

Proof. By (4.30), for any $d \in D, m \in \ker M$, we have:

$$\phi_x(dm) = d_{(2)}\phi_{d_{(-1)}x} m = 0 \implies dm \in \ker \phi.$$

□

Proposition 4.24. *Let* $\phi \in \mathrm{Chom}^D(M, N)$*. Then the vector space* $\ker_p \phi / \ker \phi$ *is finite dimensional for any* $p \geq -1$*.*

Proof. By Lemma 4.23 after replacing M by $M/\ker\phi$ we can assume without loss of generality that $\ker\phi = 0$. By definition we have $\ker_p \phi = \phi^{-1}((F^pH \otimes \Bbbk) \otimes_D N)$. By Lemma 4.12 we have $(F^pH \otimes \Bbbk) \otimes_D N = (\Bbbk \otimes F^pD) \otimes_D N$, so that $\phi(\ker_p \phi) \subset (\Bbbk \otimes F^pD) \otimes_D N \simeq F^pD \otimes N$ via the isomorphism π given in (4.8). On the other hand M, since M is finite and ϕ satisfies (4.26) there exists a finite dimensional subspace N' of N such that $\phi(\ker_p \phi) \subset (\Bbbk \otimes D) \otimes_D N' \simeq D \otimes N'$. Using Lemma 2.6 we obtain $\phi(\ker_p \phi) \subset (F^pD \otimes N')$. Injectivity of $\pi \circ \phi$ implies that $\ker_p \phi$ is finite-dimensional. □

Lemma 4.25. *Let* $\phi \in \mathrm{Chom}^D(M, N)$*. If* $d \in D$ *is not a divisor of zero then* $dm \in \ker\phi = \{m \in M \mid \phi_x m = 0, \forall\, x \in X\}$ *implies* $m \in \ker\phi$*.*

Proof. Let $m \in M$ and suppose that $\phi * m = \sum_i (1 \otimes d^i) \otimes_D n_i$, where $\{n_i\}$ is a system of linearly independent vectors. By (4.26),

$$\phi * (dm) = \sum_i (1 \otimes dd^i) \otimes_D n_i.$$

If $\phi * (dm) = 0$ then $dd^i = 0$. By our assumption this forces d^i to be zero. □

Corollary 4.26. *Let* M *be an* L*-module with coefficients in* D*, where* D *is such that* $H \otimes D = (\Bbbk \otimes D)\Delta(D)$*. Then any torsion element from* M *is acted on trivially by* L*.*

Proof. It follows by Lemma 4.25. □

4.4 The correspondence between L-modules with coefficients and conformal $\mathcal{L}$-modules

In this section we prove that there exists a one-to-one correspondence between L-modules with coefficients and conformal modules satisfying a suitable technical condition on the annihilation algebra $\mathcal{L}$ of L. This result is a generalization

of the same correspondence given in [$BDK1$,Proposition 9.1] for ordinary L-modules.
We will apply this result in the next chapter in order to study irreducible modules of the Poisson algebra P_N.

Theorem 4.27. *Let D be a ring of coefficients over H, L be a Lie pseudoalgebra and $\mathcal{L}$ be its annihilation algebra. Any L-module M with coefficients in D has a structure of conformal $\mathcal{L}$-module given by the following action*

$$(x\otimes_H a).m = \sum_i \langle x, S(h^i)\rangle m_i, \quad \textit{if} \quad a * m = \sum_i (h^i \otimes 1)\otimes_D m_i. \tag{4.33}$$

Moreover the action defined in (4.33) is D-compatible, i.e., it satisfies

$$d.(lm) = (d_{(1)}l).(d_{(2)}m), \tag{4.34}$$

for any $d \in D$, $l \in \mathcal{L}$, $m \in M$.
Conversely, if M is a conformal $\mathcal{L}$-module whose Lie action satisfies (4.34) then M has a structure of L-module with coefficients in D given by

$$a * m = \sum_i (S(h^i) \otimes 1)\otimes_D ((x_i\otimes_H a)\cdot m), \tag{4.35}$$

where $\{h^i\}$ is a basis of H and $\{x_i\}$ the corresponding dual basis of X.

Proof. We have to prove that (4.33) is a Lie action, i.e.,

$$[(x\otimes_H a),(y\otimes_H b)].m = (x\otimes_H a).((y\otimes_H b).m) - (y\otimes_H b).((x\otimes_H a).m), \tag{4.36}$$

for $x, y \in X, a, b \in L, m \in M$. Let

$$b * m = \sum_i (k^i \otimes 1)\otimes_D m_i, \qquad a * m_i = \sum_j (g^{ij} \otimes 1)\otimes_D m_{ij},$$
$$a * m = \sum_p (h^p \otimes 1)\otimes_D m_p, \qquad b * m_p = \sum_q (f^{pq} \otimes 1)\otimes_D m_{pq}.$$

By (4.19) we have:

$$a * (b * m) = \sum_{i,j} (g^{ij} \otimes k^i \otimes 1)\otimes_D m_{ij}, \tag{4.37}$$

$$(b * (a * m))^{\sigma_{12}} = \sum_{p,q} (h^p \otimes f^{pq} \otimes 1)\otimes_D m_{pq}. \tag{4.38}$$

By definition we have:

$$\begin{aligned}(x\otimes_H a).((y\otimes_H b).m) &= (x\otimes_H a).(\sum_i \langle y, S(k^i)\rangle m_i) \\ &= \sum_i \langle y, S(k^i)\rangle (x\otimes_H a).m_i \\ &= \sum_{i,j} \langle x, S(g^{ij})\rangle \langle y, S(k^i)\rangle m_{ij}.\end{aligned}$$

A similar computation shows that:

$$(y\otimes_H b).((x\otimes_H a).m) = \sum_{p,q} \langle x, S(h^p)\rangle \langle y, S(f^{pq})\rangle m_{pq}.$$

The Lie action (4.33) is such that in the right-hand side of (4.36) $x \in X$ is acted on by the first tensor factor in H in the right-hand side of (4.37) and (4.38) and $y \in X$ is acted on by the second tensor factor in H of the same equations. We are left with proving that the same holds for the left-hand side of (4.36) . Let

$$[a * b] = \sum_r (l^r \otimes 1) \otimes_H c_r, \qquad c_r * m = \sum_s (t^{rs} \otimes 1) \otimes_D m_{rs}.$$

By (4.19) we obtain:

$$[a * b] * m = \sum_{r,s} (l^r t^{rs}_{(1)} \otimes t^{rs}_{(2)} \otimes 1) \otimes_D m_{rs}.$$

Recall that by (2.39) we have:

$$[x \otimes_H a, y \otimes_H b] = \sum_r (x l^r)(y) \otimes_H c_r,$$

so that,

$$\begin{aligned} [(x \otimes_H a), (y \otimes_H b)].m &= \sum_{r,s} \langle (x l^r)(y), S(t^{rs}) \rangle m_{rs} \\ &= \sum_{r,s} \langle x l^r, t^{rs}_{(-1)} \rangle \langle y, t^{rs}_{(-2)} \rangle m_{rs} \\ &= \sum_{r,s} \langle x, S(l^r) S(t^{rs}_{(1)}) \rangle \langle y, S(t^{rs}_{(2)}) \rangle m_{rs} \\ &= \sum_{r,s} \langle x, S(l^r t^{rs}_{(1)}) \rangle \langle y, S(t^{rs}_{(2)}) \rangle m_{rs}. \end{aligned}$$

Then (4.36) follows from (4.21). It remains to check that (4.34) is satisfied.

$$(d_{(1)}(x \otimes_H a)).(d_{(2)}.m) = (d_{(1)} x \otimes_H a) d_{(2)} m = \sum_i \langle d_{(1)} x, S(h^i d_{(-2)}) \rangle d_{(3)} m_i,$$

since

$$a * (d_{(2)} m) = (1 \otimes d_{(2)}) a * m = \sum_i (h^i d_{(-2)} \otimes 1) \otimes_D d_{(3)} m_i.$$

Then,

$$\begin{aligned} \sum_i \langle d_{(1)} x, S(h^i d_{(-2)}) \rangle d_{(3)} m_i &= \sum_i \langle d_{(1)} x, d_{(2)} S(h^i) \rangle d_{(3)} m_i \\ &= \sum_i \langle x, d_{(-1)} d_{(2)} S(h^i) \rangle d_{(3)} m_i \\ &= \sum_i \langle x, S(h^i) \epsilon(d_{(1)}) \rangle d_{(2)} m_i \\ &= \sum_i \langle x, S(h^i) \rangle d m_i = d \sum_i \langle x, S(h^i) \rangle m_i \\ &= d.((x \otimes_H a) m). \end{aligned}$$

Now, let M be a conformal $\mathcal{L}$-module which satisfies the D-compatibility condition (4.34). We want to prove that (4.35) endows M with a structure of an L-module with coefficients in D. First of all we have to check that it is an $(H \otimes D)$-linear map.

$$\begin{aligned} ha * m &= \sum_i (S(h^i) \otimes 1) \otimes_D ((x_i \otimes ha).m) = \sum_i (S(h^i) \otimes 1) \otimes_D ((x_i h \otimes_H a).m) \\ &= \sum_i (h S(h^i)) \otimes 1) \otimes_D ((x_i \otimes_H a).m) \\ &= ((h \otimes 1) \otimes_D 1) \sum_i ((S(h^i) \otimes 1) \otimes_D ((x_i \otimes_H a).m) \\ &= ((h \otimes 1) \otimes_D 1) (a * m). \end{aligned}$$

And

$$\begin{aligned}
(1\otimes\, d)\otimes_D (a*m) &= \sum_i (S(h^i)\otimes d)\otimes_D ((x_i\otimes_H a).m)\\
&= \sum_i (S(h^i)d_{(-1)}\otimes 1)\otimes_D d_{(2)}[(x_i\otimes_H a).m]\\
&= \sum_i (S(d_{(1)}h^i)\otimes 1)\otimes_D d_{(2)}[(x_i\otimes_H a).m]\\
&= \sum_i (S(h^i)\otimes 1)\otimes_D d_{(2)}[d_{(-1)}x_i\otimes_H a).(d_{(3)}m)]\\
&= \sum_i (S(h^i)\otimes 1)\otimes_D ((d_{(2)}d_{(-1)}x_i\otimes_H a).d_{(3)}m)\\
&= \sum_i (S(h^i)\otimes 1)\otimes_D ((\epsilon(d_{(1)})x_i\otimes_H a).d_{(2)}m)\\
&= \sum_i (S(h^i)\otimes 1)\otimes_D ((x_i\otimes_H a).\epsilon(d_{(1)})d_{(2)}m)\\
&= \sum_i (S(h^i)\otimes 1)\otimes_D ((x_i\otimes_H a).dm) = a*dm.
\end{aligned}$$

Now let

$$[a*b] = \sum_r (l^r\otimes\, 1)\otimes_H c_r,$$

so that

$$[x_i\otimes_H a, x_j\otimes_H b] = \sum_r (x_i l^r)x_j\otimes_H c_r.$$

Then

$$[a*b]*m = \sum_{r,s} (l^r S(h^{rs}_{(1)})\otimes S(h^{rs}_{(2)})\otimes 1)\otimes_D ((x_{rs}\otimes c_r).m).$$

On the other hand

$$\begin{aligned}
a*(b*m) - (b*(a*m))^{\sigma_{12}} = \; &\sum_{p,q} (S(h^{pq})\otimes S(h^p)\otimes 1)\otimes_D ((x_{pq}\otimes_H a).((x_p\otimes_H b).m))\\
&- \sum_{i,j} (S(h^i)\otimes S(h^{ij})\otimes 1)\otimes_D ((x_{ij}\otimes_H b).((x_i\otimes_H a).m)).
\end{aligned}$$

Then (4.21) follows from (4.36). □

Chapter 5

Representations of P_N

We now apply the pseudoalgebraic language towards the study of discrete modules of the Lie algebra P_N. Throughout this chapter, $\mathfrak{d}$ denotes an abelian Lie algebra of dimension $N = 2n$, of which we fix a basis $\{\partial_1, \dots, \partial_N\}$. We will first recall the definition of $H(\mathfrak{d}, 0, \omega)$ and describe its annihilation algebra according to [BDK1]. We will then show that discrete irreducible modules of P_N can be understood as representations of $H(\mathfrak{d}, 0, \omega)$ with coefficients in D_λ via the correspondence given in Chapter 4. Here we follow [DM]. We will then study tensor modules and singular vectors using the same strategy as in [BDK2, BDK3, BDK4]: it extends to representation with coefficients in D_λ without special effort.

5.1 The Lie pseudoalgebra $H(\mathfrak{d}, 0, \omega)$

The universal enveloping algebra $H = \mathcal{U}(\mathfrak{d}) = S(\mathfrak{d})$ is isomorphic to $\Bbbk[\partial_1, \dots, \partial_N]$ and, as we have seen in Section 2.3, it is a graded associative commutative algebra, with $\deg \partial_i = 1$, $i = 1, \dots, N$. Its dual $X = O_N = \Bbbk[[t_1, \dots, t_N]]$, where $\{t_i\}$ is a basis of $\mathfrak{d}^*$ dual to $\{\partial_i\}$, also posses a gradation.
Section 8.5 in [BDK1] shows that all Lie pseudoalgebra structures on a free H-module $L = He$ of rank 1 satisfy

$$[e * e] = (r + s \otimes 1 - 1 \otimes s) \otimes_H e,$$

for some $r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d}$. Moreover, if r is of maximal rank, and $N > 2$, then necessarily $s = 0$. In what follows, we will focus on such structures, that are denoted by $H(\mathfrak{d}, 0, \omega)$, where $\omega \in \wedge^2 \mathfrak{d}^*$ is the symplectic 2-form corresponding to the inverse of r. $H(\mathfrak{d}, 0, \omega)$ is a simple Lie pseudoalgebra. If $r = \sum_{i,j=1}^{N} r^{ij} \partial_i \otimes \partial_j$ and $\omega_{ij} = \omega(\partial_i, \partial_j)$, then the matrices (r^{ij}) and (ω_{ij}) are inverse to each other.

Lemma 5.1. *[BDK1] The following*

$$\begin{array}{rccc} \iota: & H(\mathfrak{d}, 0, \omega) & \longrightarrow & W(\mathfrak{d}) \\ & e & \mapsto & -r, \end{array} \tag{5.1}$$

is the only nonzero Lie pseudoalgebra homomorphism from $H(\mathfrak{d}, 0, \omega)$ to $W(\mathfrak{d})$.

We set:

$$\partial^i = \sum_{j=1}^{N} r^{ij} \partial_j. \tag{5.2}$$

Then,

$$\partial_i = \sum_{j=1}^{N} \omega_{ij} \partial^j, \qquad r = \sum_{i=1}^{N} \partial_i \otimes \partial^i = -\sum_{i=1}^{N} \partial^i \otimes \partial_i.$$

By a direct computation we obtain the following relations

$$\omega(\partial^i, \partial_j) = \delta^i_j = -\omega(\partial_i, \partial^j), \qquad \omega(\partial^i, \partial^j) = -r^{ij} = \langle t_i, \partial^j \rangle. \tag{5.3}$$

Remark 5.1. *Since r and ω are nondegenerate it is always possible to choose a basis $\{\partial_1, \ldots, \partial_N\}$ of $\mathfrak{d}$ such that*

$$\omega(\partial_i \wedge \partial_{n+j}) = -\delta^i_j, \quad i, j = 1, \ldots, n.$$

In this case

$$\partial^i = \partial_{n+i}, \quad \partial^{n+i} = -\partial_i, \quad i = 1, \ldots, n,$$

and

$$r = \sum_{i=1}^{n} (\partial_i \otimes \partial_{n+i} - \partial_{n+i} \otimes \partial_i). \tag{5.4}$$

5.1.1 The symplectic algebra $\mathfrak{sp}(\mathfrak{d}, \omega)$

Let $\{t_i\}$ be the basis of $\mathfrak{d}^*$ dual to $\{\partial_i\}$. We consider the identification of $\operatorname{End} \mathfrak{d} = \operatorname{Hom}(\mathfrak{d}, \mathfrak{d})$ with $\mathfrak{d} \otimes \mathfrak{d}^*$ given by $e^j_i \mapsto \partial_i \otimes t_j$. Let $*$ be the antihomomorphism of associative algebras which uniquely extends

$$(\partial^i \otimes t_j)^* = -\partial^j \otimes t_i,$$

where ∂^i is defined in (5.2).
Then the *symplectic algebra* is

$$\mathfrak{sp}(\mathfrak{d}, \omega) = \{A \in \operatorname{End} \mathfrak{d} \mid A^* = -A\}.$$

It is a Lie subalgebra of $\mathfrak{gl}\,\mathfrak{d}$. Since ω is nondegenerate $\mathfrak{sp}(\mathfrak{d}, \omega)$ is isomorphic to the *symplectic Lie algebra* $\mathfrak{sp}_N$. In particular, it is a simple Lie algebra.
The elements

$$f^{ij} = -\frac{1}{2}(\partial^i \otimes t_j + \partial^j \otimes t_i)$$

linearly generate $\mathfrak{s}p(\mathfrak{d}, \omega)$ and $\{f^{ij}\}_{i \leq j}$ is a basis of $\mathfrak{s}p(\mathfrak{d}, \omega)$. Notice that $f^{ij} = f^{ji}$.

5.2 The annihilation algebra of $H(\mathfrak{d},0,\omega)$

Let $\mathcal{P} = X\otimes_H He \simeq X\otimes_H e$ be the annihilation algebra of $H(\mathfrak{d},0,\omega)$. Due to (2.39), its Lie bracket is given by

$$[\phi\otimes_H e, \psi\otimes_H e] = \sum_{ij} r^{ij}(\phi\partial_i)(\psi\partial_j)\otimes_H e = \sum_i (\phi\partial_i)(\psi\partial^i)\otimes_H e. \tag{5.5}$$

Recall that $\mathcal{P}$ is an H-module via the action defined in (2.40) and that ∂_i act as $-\partial/\partial t_i$.

Lemma 5.2. *The annihilation algebra $\mathcal{P}$ has a one-dimensional center linearly generated by $1\otimes_H e$.*

Proof. By formula (5.5) we have

$$[1\otimes_H e, \psi\otimes_H e] = \sum_{ij} r^{ij}(1\partial_i)(\psi\partial_j)\otimes_H e = 0,$$

for any $\psi\otimes_H e \in \mathcal{P}$.
Conversely, let $\phi\otimes_H e \in Z(\mathcal{P})$. Then by (5.5)

$$0 = [t_k\otimes_H e, \phi\otimes_H e] = \sum_{ij} r^{ij}(t_k\partial_i)(\phi\partial_j)\otimes_H e = -\delta^i_k r^{ij}(\phi\partial_j)\otimes_H e.$$

This follows that $(\phi\partial_j) = 0$ for $j = 1,\dots,N$ and so $\phi \in \Bbbk$. $\square$

The canonical injection ι of the subalgebra $H(\mathfrak{d},0,\omega)$ in $W(\mathfrak{d})$ defined in (5.1) induces a Lie algebra homomorphism $\iota_* : \mathcal{P} \longrightarrow \mathcal{W}$. Its kernel coincides with the center of $\mathcal{P}$. In particular, $\mathcal{P}$ is a central extension of $\iota_*(\mathcal{P}) = \mathcal{H} \subset \mathcal{W}$ by the one-dimensional ideal $\Bbbk\otimes_H e$. If $\{\partial_1,\dots,\partial_N\}$ is such that (5.4) holds then the Lie bracket on $\mathcal{P}$ reads as,

$$[\phi\otimes_H e, \psi\otimes_H e] = \sum_{i=1}^{n}\left(\frac{\partial\phi}{\partial t_i}\frac{\partial\psi}{\partial t_{n+i}} - \frac{\partial\phi}{\partial t_{n+i}}\frac{\partial\psi}{\partial t_i}\right)\otimes_H e = \{\phi,\psi\}\otimes_H e.$$

As an immediate consequence,

Proposition 5.3. *The Poisson algebra P_N is isomorphic to the annihilation algebra $\mathcal{P}$ of $H(\mathfrak{d},0,\omega)$. The isomorphism being given by:*

$$\begin{array}{rccc} \rho: & P_N & \longrightarrow & X\otimes_H e \\ & \phi & \mapsto & \phi\otimes_H e. \end{array}$$

Remark 5.2. *Let $\{\partial_1,\dots,\partial_{2n}\}$ be a basis of $\mathfrak{d}$ such that (5.4) is satisfied. Then the isomorphism between $\mathcal{W}$ and W_{2n} given in (2.42) identifies $\mathcal{H}$ with the subalgebra $H_{2n} \subset W_{2n}$ of all formal vector fields preserving the standard symplectic form $\sum_{i=1}^{n} dt_i \wedge dt_{n+i}$.*

Let $\{G^pX\}$ be the gradation of X defined in (2.29). If we set

$$\mathcal{P}^p = G^{p+2}X\otimes_H e, \quad p \geq -2, \tag{5.6}$$

then

Lemma 5.4. $\{\mathcal{P}^p\}_{p\geq -2}$ *defines a gradation on* $\mathcal{P}$ *which satisfies:*

$$\mathfrak{d}.(\mathcal{P}^p) \subset \mathcal{P}^{p-1}, \tag{5.7}$$

$$[\mathcal{P}^p, \mathcal{P}^q] \subset \mathcal{P}^{p+q}, \tag{5.8}$$

$$(G^q H).(\mathcal{P}^p) \subset \mathcal{P}^{p-q}. \tag{5.9}$$

Proof. The first statement is clear. Let $\phi \in \mathcal{P}^p$, $\psi \in \mathcal{P}^q$. Then by (5.5), (5.7) and (2.30) it follows that $[\phi \otimes_H e, \psi \otimes_H e] \in G^{p+q+2} X \otimes_H e = \mathcal{P}^{p+q}$. The last statement follows from (2.31). □

We consider on $\mathcal{P}$ the filtration induced by (5.6) given by:

$$\mathcal{P}_p = F_p \mathcal{P} = \prod_{q \geq p} \mathcal{P}^q = F^{p+1} X \otimes_H e, \quad p \geq -2. \tag{5.10}$$

Lemma 5.5. *The filtration* $\{\mathcal{P}_p\}_{p\geq -2}$ *of* $\mathcal{P}$ *defined in (5.10) satisfies*

(*a*) $\mathfrak{d}(\mathcal{P}_p) \subset \mathcal{P}_{p-1}$,

(*b*) $[\mathcal{P}_p, \mathcal{P}_q] \subset \mathcal{P}_{p+q}$.

Notice that $\mathcal{P}_0$ is a Lie subalgebra of $\mathcal{P}$ and all $\mathcal{P}_p$, $p \geq 0$, are ideals of $\mathcal{P}_0$. The filtration $\{\mathcal{P}_p\}$ satisfies

$$\mathcal{P}_{-2}/\mathcal{P}_{-1} \simeq \Bbbk(1 \otimes_H e), \;\; \mathcal{P}_{-1}/\mathcal{P}_0 \simeq \mathfrak{d}^* \otimes_H e. \tag{5.11}$$

The filtrations of $\mathcal{P}$ and $\mathcal{W}$ are compatible, i.e., $\iota_*(\mathcal{P}_p) = \iota_*(\mathcal{P}) \cap \mathcal{W}_p$.

Lemma 5.6. *The homomorphism of Lie algebras* $\iota_* : \mathcal{P} \longrightarrow \mathcal{W}$ *is such that:*

$$\begin{array}{lll} \iota_*(t_i \otimes_H e) & = 1 \otimes \partial^i & \text{modulo}\, \mathcal{W}_1, \\ \iota_*(t_i t_j \otimes_H e) & = t_j \otimes \partial^i + t_i \otimes \partial^j & \text{modulo}\, \mathcal{W}_1. \end{array}$$

Proof. It follows by $\iota_*(x \otimes_H e) = \sum_k (\partial x / \partial t_k) \otimes \partial^k$ by a direct computation. □

Proposition 5.7. *Let* $\{\mathcal{P}_p\}_{p\geq -2}$ *be the filtration of* $\mathcal{P}$ *defined in (5.10). There exists an isomorphism of Lie algebras from* $\mathcal{P}_0/\mathcal{P}_1$ *to* $\mathfrak{sp}(\mathfrak{d}, \omega)$.

Proof. It is a consequence of injectivity of $\pi : \mathcal{H}_0 \longrightarrow \mathcal{W}_0/\mathcal{W}_1$, compatibility between the filtrations of $\mathcal{P}$ and $\mathcal{W}$ and Lemma 5.6, using the identification between $\mathfrak{gl}(\mathfrak{d})$ and $\mathcal{W}_0/\mathcal{W}_1$ given in [BDK2]. □

Proposition 5.8. *The normalizer* $\mathcal{N}_{\mathcal{P}}$ *of* $\mathcal{P}_p$, $p \geq 0$, *in* $\mathcal{P}$ *is independent of* p *and it equals* $\Bbbk(1 \otimes_H e) + \mathcal{P}_0$.

Proof. This follows by Lemma 5.2 and $[\mathcal{P}_0, \mathcal{P}_p] \subset \mathcal{P}_p$ that $\Bbbk(1 \otimes_H e) + \mathcal{P}_0 \subset \mathcal{N}_{\mathcal{P}}$. By (5.11) we have $\mathcal{P} = \Bbbk(1 \otimes_H e) + \mathfrak{d}^* \otimes_H e + \mathcal{P}_0$. It remains to prove that $t_i \otimes_H e \notin \mathcal{N}_{\mathcal{P}}$. This follows by point (*a*) of Lemma 5.5. □

5.2.1 A central extension of $\mathfrak{d}$

Let $\{\partial_i\}$ be a basis of $\mathfrak{d}$, $\{t_i\}$ be the corresponding dual basis of $\mathfrak{d}^*$.
We define a map $\mathfrak{d} \ni \partial^i \mapsto \widehat{\partial^i} = -t_i \otimes_H e \in \mathfrak{d}^* \otimes_H e$ and we extend it by linearity to a map from $\mathfrak{d}$ to $\mathcal{P}$.

Lemma 5.9. *For any $\partial \in \mathfrak{d}, \phi \otimes_H e \in \mathcal{P}$ the map from $\mathfrak{d}$ to $\mathcal{P}$ just defined satisfies*

$$[\widehat{\partial}, \phi \otimes_H e] = \partial.(\phi \otimes_H e). \tag{5.12}$$

Proof. It is sufficient to prove (5.12) for $\widehat{\partial^i}$.

$$\begin{aligned}
[\widehat{\partial^i}, \phi \otimes_H e] &= [-t_i \otimes_H e, \phi \otimes_H e] = -\sum_{kj} r^{kj}(t_i \partial_k)(\phi \partial_j) \otimes_H e \\
&= \sum_j r^{ij}(\phi \partial_j) \otimes_H e = (\sum_j r^{ij} \partial_j \phi) \otimes_H e \\
&= (\partial^i \phi) \otimes_H e = \partial^i.(\phi \otimes_H e).
\end{aligned}$$

□

Let $\widehat{\mathfrak{d}} = \text{span}_{\Bbbk}\{c = -1 \otimes_H e, \widehat{\partial^i}, i = 1, \dots, N\}$. Then we have the following

Proposition 5.10. *$\widehat{\mathfrak{d}}$ is a Lie subalgebra of $\mathcal{P}$. It is a central extension of $\mathfrak{d}$ of 2-cocycle ω.*

Proof. The fact that c is a central element follows by Lemma 5.2. The second part of the statement follows by Lemma 5.9 and (5.3). □

Remark 5.3. *The Lie algebra $\mathcal{P}$ decomposes as a direct sum of vector spaces $\mathcal{P} = \widehat{\mathfrak{d}} \oplus \mathcal{P}_0$.*

Let $\iota : \widehat{\mathfrak{d}} \longrightarrow \mathcal{P}$ be the canonical embedding of $\widehat{\mathfrak{d}}$ in $\mathcal{P}$ and π be the canonical projection from $\widehat{\mathfrak{d}}$ to $\mathfrak{d}$. Let $\widehat{\partial} \in \widehat{\mathfrak{d}}$ and $\phi \otimes_H e \in \mathcal{P}$. We can reformulate (5.12) as

$$[\iota(\widehat{\partial}), \phi \otimes_H e] = \pi(\widehat{\partial}).(\phi \otimes_H e).$$

Corollary 5.11. *Let M be a $\mathcal{P}$-module. For any $\partial \in \mathfrak{d}$, $\phi \otimes_H e \in \mathcal{P}$, $m \in M$ we have*

$$\widehat{\partial}.((\phi \otimes_H e).m) = (\partial.(\phi \otimes_H e)).m + (\phi \otimes_H e).(\widehat{\partial}.m). \tag{5.13}$$

Proof. Since M is a $\mathcal{P}$-module we have

$$[\widehat{\partial}, \phi \otimes_H e].m = \widehat{\partial}.(\phi \otimes_H e.m) - (\phi \otimes_H e).(\widehat{\partial}.m).$$

Then the statement immediately follows by Lemma 5.9. □

5.3 Irreducible conformal modules over P_N

In this section we show that the action of the Poisson algebra on an irreducible conformal module M can be lifted to a pseudoaction with coefficients in D_λ of $H(\mathfrak{d}, 0, \omega)$ on M.
Let $\mathcal{P}$ be the annihilation algebra of $H(\mathfrak{d}, 0, \omega)$, $\{\mathcal{P}_p\}_{p \geq -2}$ be the decreasing

filtration of $\mathcal{P}$ defined in (5.10) and M be a $\mathcal{P}$-module.
We set

$$\ker_p M = \{m \in M \mid \mathcal{P}_p.m = 0\}. \tag{5.14}$$

Recall that a $\mathcal{P}$-module M is *conformal* if $M = \bigcup_p \ker_p M$ and it is $\mathcal{P}_0$-*locally finite* if any $m \in M$ is contained in a finite-dimensional $\mathcal{P}_0$-module.
Let M be a conformal $\mathcal{P}$-module. Then $\ker_p M$ is stable for the action of the normalizer $\mathcal{N}_\mathcal{P}$ of $\mathcal{P}_p$ in $\mathcal{P}$. Every $\ker_p M$, $p \geq 0$, is therefore a $\mathcal{N}_\mathcal{P}/\mathcal{P}_p$-module. In particular, since $\mathcal{P}_0 \subset \mathcal{N}_\mathcal{P}$ is a Lie subalgebra of $\mathcal{P}$ then any $\ker_p M$ is a $\mathcal{P}_0/\mathcal{P}_p$-module.

Lemma 5.12. *The subalgebra $\mathcal{P}_1 \subset \mathcal{P}_0$ acts trivially on any irreducible finite-dimensional conformal $\mathcal{P}_0$-module R. Irreducible finite-dimensional conformal $\mathcal{P}_0$-modules are in one-to-one correspondence with irreducible finite-dimensional modules over the Lie algebra $\mathcal{P}_0/\mathcal{P}_1 \simeq \mathfrak{sp}(\mathfrak{d}, \omega)$.*

Proof. A finite-dimensional vector space R is a conformal $\mathcal{P}_0$-module if and only if it is a $\mathcal{P}_0$-module on which $\mathcal{P}_p$ acts trivially for some $p \geq 0$. It means that R is an irreducible $\mathcal{P}_0/\mathcal{P}_p$-module, where $\mathfrak{g} = \mathcal{P}_0/\mathcal{P}_p$ is a finite-dimensional Lie algebra. Let $\mathfrak{g}_0 = \mathcal{P}_0/\mathcal{P}_1 \simeq \mathfrak{s}p(\mathfrak{d}, \omega)$ and $\mathfrak{r} = \mathcal{P}_1/\mathcal{P}_p$. Note that $\mathfrak{r}$ is a solvable ideal of $\mathfrak{g}$ so it is contained in $\operatorname{Rad}\mathfrak{g}$. By Cartan-Jacobson's Theorem $\mathfrak{r}$ acts by scalar multiplication on R. On the other hand since all $\mathcal{P}_i/\mathcal{P}_{i+1}$ are irreducible $\mathfrak{g}_0$-modules we can apply Lemma 2.23 to conclude that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{r}$ is surjective. Then $\mathfrak{r}$ acts trivially on R. □

Proposition 5.13. *Let $\Bbbk$ be an uncountable field of characteristic zero. Let M be an irreducible conformal $\mathcal{P}$-module. Then M is at most countable dimensional and any central element in $\mathcal{P}$ acts via scalar multiplication by $\lambda \in \Bbbk$.*

Proof. Since M is a conformal $\mathcal{P}$-module there exists k such that $\ker_p M = M_p \neq 0$. We know that M_p is a $\mathcal{P}_0/\mathcal{P}_p$-module, where $\mathcal{P}_0/\mathcal{P}_p$ is a finite-dimensional Lie algebra. Its universal enveloping algebra $\mathcal{U}(\mathcal{P}_0/\mathcal{P}_p)$ is therefore countable dimensional so that we can find a countable dimensional nonzero $\mathcal{P}_0$-submodule R of M_p. Let us consider the induced representation $U = \operatorname{Ind}_{\mathcal{P}_0}^{\mathcal{P}} R = \mathcal{U}(\mathcal{P}) \otimes_{\mathcal{U}(\mathcal{P}_0)} R$ and recall that $\mathcal{P} = \widehat{\mathfrak{d}} \oplus \mathcal{P}_0$. This follows by PBW-theorem that $\mathcal{U}(\mathcal{P})$ is a free $\mathcal{U}(\mathcal{P}_0)$-module. Then we have

$$\mathcal{U}(\mathcal{P}) = \mathcal{U}(\widehat{\mathfrak{d}})\mathcal{U}(\mathcal{P}_0).$$

The map

$$\begin{array}{ccc} \mathcal{U}(\widehat{\mathfrak{d}}) \otimes \mathcal{U}(\mathcal{P}_0) & \longrightarrow & \mathcal{U}(\widehat{\mathfrak{d}})\mathcal{U}(\mathcal{P}_0) \\ d^I \otimes p^J & \mapsto & d^I p^J \end{array}$$

is an isomorphism of vector spaces. Therefore

$$\begin{aligned} U &= \mathcal{U}(\mathcal{P}) \otimes_{\mathcal{U}(\mathcal{P}_0)} R \simeq \mathcal{U}(\widehat{\mathfrak{d}})\mathcal{U}(\mathcal{P}_0) \otimes_{\mathcal{U}(\mathcal{P}_0)} R \\ &\simeq \mathcal{U}(\widehat{\mathfrak{d}}) \otimes \mathcal{U}(\mathcal{P}_0) \otimes_{\mathcal{U}(\mathcal{P}_0)} R \simeq \mathcal{U}(\widehat{\mathfrak{d}}) \otimes R. \end{aligned}$$

this proves that U is still countable dimensional. By irreducibility of M there exists a surjective homomorphism from U to M, proving that M is countable dimensional. The last part of the statement follows from a countable Schur Lemma. □

Theorem 5.14. *Let M be an irreducible conformal P_N-module on which central elements act via scalar multiplication. There exists $\lambda \in \Bbbk$ such that M can be lifted to a $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ.*

Proof. We may use the embedding of $\widehat{\mathfrak{d}}$ in $\mathcal{P}$ to endow M with a $\mathcal{U}(\widehat{\mathfrak{d}})$-module structure. We have seen in the proof of Proposition 5.13 that M is a quotient of $\mathcal{U}(\widehat{\mathfrak{d}}) \otimes R$, where R is a countable dimensional $\mathcal{P}_0$-submodule. If $c = -1 \otimes_H e \in \mathcal{P}$ acts on M via multiplication by $\lambda \in \Bbbk$ then the left action of $\mathcal{U}(\widehat{\mathfrak{d}})$ factors via the quotient $D_\lambda = \mathcal{U}(\widehat{\mathfrak{d}})/(c-\lambda)\mathcal{U}(\widehat{\mathfrak{d}})$. This proves that M has a structure of a D_λ-module. The compatibility condition (4.34) of Theorem 4.27 between the H-module structure of $\mathcal{P}$ and the D_λ-structure of M has been proved in Corollary 5.11 for a set of algebra generators of D_λ. □

Notice that if the central element $c \in \mathcal{P}$ acts trivially on M then we obtain a correspondence between irreducible conformal $\mathcal{P}$-modules and ordinary $H(\mathfrak{d}, 0, \omega)$-modules [BDK4].

Remark 5.4. *In the statement of Theorem 5.14 we use the isomorphism between the Poisson algebra P_N and the annihilation algebra $\mathcal{P}$ of $H(\mathfrak{d}, 0, \omega)$.*

5.4 Singular vectors of finite $H(\mathfrak{d}, 0, \omega)$-modules with coefficients

In this section we show how the language of Lie pseudoalgebras with coefficients in D simplifies the computation of singular vectors. To make this strategy effective we need equalities $H \otimes D = (H \otimes \Bbbk)\Delta_D(D)$ and $H \otimes D = (\Bbbk \otimes D)\Delta_D(D)$. We showed in Proposition 4.10 that the latter equality holds for the ring of coefficients D_λ.

Lemma 5.15. *Let M be a finite $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ. If $p \geq -1$, then $\ker_p M / \ker M$ is finite dimensional.*

Proof. Let $\phi_e \in \mathrm{Chom}^{D_\lambda}(M, M)$ the D_λ-pseudolinear map associated to the action of the free generator e of $H(\mathfrak{d}, 0, \omega)$. Recall that $\mathcal{P}_p = F_{p+1}X \otimes_H e \simeq F_{p+1}X$ as a vector space. Then the statement follows by Proposition 4.24 as

$$\ker_{p+1} \phi_e = \{m \in M \mid {\phi_e}_x m = 0, \ \forall\, x \in F^{p+1}X\} \simeq \ker_p M.$$

□

It follows by Lemma 5.15 that if M is irreducible then all $\ker_p M$ are finite dimensional $\mathcal{P}_0$-modules. In other way, M is a $\mathcal{P}_0$-locally finite $\mathcal{P}$-module.

Proposition 5.16. *Let M be a $\mathcal{P}_0$-locally finite irreducible conformal $\mathcal{P}$-module on which the central element acts via scalar multiplication. There exists $\lambda \in \Bbbk$ such that M can be endowed with a structure of a finite $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ.*

Proof. We know by Theorem 5.14 that M can be endowed with a structure of $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ. To prove that M is finite as a D_λ-module it is sufficient to observe that under the locally finite assumption it is possible to find a finite-dimensional irreducible $\mathcal{P}_0$-submodule $R \neq 0$. Then the same argument used in the proof of Theorem 5.14 shows that M is a quotient of the finite D_λ-module $D_\lambda \otimes R$. □

Let M be an $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ. We will call each nonzero element in $\operatorname{sing} M = \ker_1 M$ a *singular vector* of M. It follows by Theorem 4.27 that $m \in M$ is a singular vector if and only if

$$e * m \in (F^2 H \otimes \Bbbk) \otimes_{D_\lambda} M.$$

By Lemma 4.12 this condition is equivalent to

$$e * m \in (\Bbbk \otimes F^2 D_\lambda) \otimes_{D_\lambda} M.$$

Notice that $\ker M \subset \operatorname{sing} M$ and $\operatorname{sing} M / \ker M$ is finite-dimensional if M is finite. If M is irreducible then $\ker M = 0$. Recall that $\operatorname{sing} M$ is an $\mathfrak{sp}(\mathfrak{d},\omega)$-module, via the isomorphism $\mathcal{P}_0/\mathcal{P}_1 \simeq \mathfrak{sp}(\mathfrak{d},\omega)$ of Proposition 5.7. Let $\rho_{\mathrm{sing}} : \mathfrak{sp}(\mathfrak{d},\omega) \longrightarrow \mathfrak{gl} \operatorname{sing} M$ be the corresponding representation. Then

$$\rho_{\mathrm{sing}}(f^{ij})m = \frac{1}{2}(t_i t_j \otimes_H e).m, \quad m \in \operatorname{sing} M.$$

Proposition 5.17. *Let M be an $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ. Let $m \in \operatorname{sing} M$. Then*

$$\begin{aligned} e * m \quad = \quad & \sum_{i,j=1}^{N} (\partial_i \partial_j \otimes 1) \otimes_{D_\lambda} \rho_{\mathrm{sing}}(f^{ij})m \\ & + \sum_{i=1}^{N} (\partial_i \otimes 1) \otimes_{D_\lambda} (\widehat{\partial}^i.m) \\ & - (1 \otimes 1) \otimes_{D_\lambda} (\lambda m). \end{aligned}$$

Proof. Since $m \in \operatorname{sing} M$ we have $\mathcal{P}_1.m = 0$. By Theorem 4.27 the action of $H(\mathfrak{d},0,\omega)$ on m is

$$\begin{aligned} e * m \quad = \quad & \sum_{i<j}^{N} (S(\partial_i \partial_j) \otimes 1) \otimes_{D_\lambda} ((t_i t_j \otimes_H e).m) \\ & + \sum_{i=j}^{N} (S((\partial_i)^2/2) \otimes 1) \otimes_{D_\lambda} (((t_i)^2 \otimes_H e).m) \\ & + \sum_{i=1}^{N} (S(\partial_i) \otimes 1) \otimes_{D_\lambda} ((t_i \otimes_H e).m) \\ & + (S(1) \otimes 1) \otimes_{D_\lambda} ((1 \otimes_H e).m) \\ = \quad & \sum_{i<j}^{N} (\partial_i \partial_j \otimes 1) \otimes_{D_\lambda} (2\rho_{\mathrm{sing}}(f^{ij})m) \\ & + \tfrac{1}{2}((\partial_i)^2 \otimes 1) \otimes_{D_\lambda} (2\rho_{\mathrm{sing}}(f^{ii})m) \\ & + \sum_{i=1}^{N} (\partial_i \otimes 1) \otimes_{D_\lambda} (\widehat{\partial}^i.m) - (1 \otimes 1) \otimes_{D_\lambda} (\lambda m). \end{aligned}$$

Since $\mathfrak{d}$ is an abelian Lie algebra then we have $\partial_i \partial_j = \frac{1}{2}\partial_i \partial_j + \frac{1}{2}\partial_j \partial_i$. □

Corollary 5.18. *Let M be an $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ. Let R be a nontrivial $\mathfrak{sp}(\mathfrak{d},\omega)$-submodule of $\operatorname{sing} M$. Denote by $D_\lambda R$ the D_λ-submodule of M generated by R. Then $D_\lambda R$ is an $H(\mathfrak{d},0,\omega)$-submodule of M.*

Proof. By (5.17) we have $H(\mathfrak{d},0,\omega) * R \subset (H \otimes D_\lambda) \otimes_{D_\lambda} D_\lambda R$. Since $*$ is an (H, D_λ)-linear map we have $H(\mathfrak{d},0,\omega) * D_\lambda R \subset (H \otimes D_\lambda) \otimes_{D_\lambda} D_\lambda R$. □

Proposition 5.19. *Let M be a nontrivial finite $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ. Then* $\operatorname{sing} M \neq 0$ *and* $\operatorname{sing} M/\ker M$ *is finite dimensional.*

Proof. The second part of the statement is a special case of Lemma 5.14. We can assume without loss of generality that $\ker M = 0$.
M is a conformal $\mathcal{P}$-module so that there exists p such that $U = \ker_p M \neq 0$. By Lemma 5.15 U is a finite-dimensional $\mathcal{P}_0$-module. Let R be a minimal $\mathcal{P}_0$-submodule of U. Then R is an irreducible $\mathcal{P}_0$-module. By Lemma 5.12 $\mathcal{P}_1$ acts trivially on R. This proves that $0 \neq R \subset \operatorname{sing} M$. □

5.4.1 Tensor modules for $H(\mathfrak{d},0,\omega)$

In this section in analogy with [BDK2, BDK3, BDK4] we introduce a special class of $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ, that we call *tensor modules*. We show that every irreducible finite $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ is an homomorphic image of a tensor module. We also explain how the study of singular vectors of $D_\lambda \otimes R$ can be used in order to obtain a classification of an important class of irreducible modules over P_N.
Let M be a $\mathcal{P}_0$-locally finite conformal $\mathcal{P}$-module on which central elements act via scalar multiplication. Therefore M has a structure of a finite $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ, $\lambda \in \Bbbk$. Let $R \subset \operatorname{sing} M$ be an irreducible $\mathfrak{sp}(\mathfrak{d},\omega)$-module with an action ρ_R. Then, since the central element c acts via scalar multiplication by λ, we obtain a ($\mathcal{P}_0$-locally finite) conformal $\mathcal{P}$-module $D_\lambda \otimes R$. By Theorem 4.27 $D_\lambda \otimes R$ can be endowed with a structure of a finite $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ. We call $D_\lambda \otimes R$ a tensor module for $H(\mathfrak{d},0,\omega)$. We now explicitly describe this structure.
Let $D_\lambda \otimes R$ be the free D_λ-module generated by R, where D_λ acts by a left multiplication on the first tensor factor. Then for $r \in R \subset \operatorname{sing} M$ we define a pseudaction with coefficients in D_λ on $D_\lambda \otimes R$ by setting:

$$
\begin{aligned}
e*(1\otimes r) \;=\; & \sum_{i,j=1}^{N} (\partial_i\partial_j \otimes 1)\otimes_{D_\lambda}(1\otimes \rho_R(f^{ij})r) \\
& + \sum_{i=1}^{N} (\partial_i \otimes 1)\otimes_{D_\lambda}(\delta^i \otimes r) \\
& - (1\otimes 1)\otimes_{D_\lambda}(1\otimes \lambda r).
\end{aligned} \tag{5.15}
$$

Extending (5.15) to all $D_\lambda \otimes R$ by $(H\otimes D_\lambda)$-linearity endows $D_\lambda \otimes R$ with a structure of a $H(\mathfrak{d},0,\omega)$-module with coefficients in D_λ.

Remark 5.5. *Using formulas (4.13) and (4.14) we can rewrite (5.15) as*

$$
\begin{aligned}
e*(1\otimes r) \;=\; & \sum_{ij}(1\otimes \delta_i\delta_j)\otimes_{D_\lambda}(1\otimes \rho_R(f^{ij})r) \\
& + \sum_{i}((1\otimes \delta_i)\otimes_{D_\lambda}((\delta^i\otimes r)+\sum_{j}(2\delta_j\otimes \rho_R(f^{ij})r)) \\
& + (1\otimes 1)\otimes_{D_\lambda}((n-1)\lambda\otimes r).
\end{aligned} \tag{5.16}
$$

Let

$$
\begin{aligned}
\phi:\; & D_\lambda\otimes R \longrightarrow M \\
& d\otimes r \mapsto d.\rho_R(r),
\end{aligned}
$$

and denote by $D_\lambda R$ the image of $D_\lambda \otimes R$ in M.

Proposition 5.20. *Let M be an irreducible $\mathcal{P}_0$-locally finite conformal $\mathcal{P}$-module. Then $M = D_\lambda R$, where R is a finite-dimensional irreducible $\mathfrak{sp}(\mathfrak{d}, \omega)$-module.*

Proof. By construction ϕ is a homomorphism of $H(\mathfrak{d}, 0, \omega)$-modules with coefficients in D_λ. Then $D_\lambda R$ is a $\mathcal{P}$-submodule of M. Irreducibility M of implies the statement. □

Corollary 5.21. *Any $\mathcal{P}_0$-locally finite conformal $\mathcal{P}$-module M is a homomorphic image of a tensor module $D_\lambda \otimes R$.*

5.4.2 Irreducibility of tensor modules for H

As a consequence of Corollary 5.21 we have that any irreducible $H(\mathfrak{d}, 0, \omega)$-modules with coefficients in D_λ is a quotient of $D_\lambda \otimes R$.
For this reason it is important to study the irreducibility of tensor modules $D_\lambda \otimes R$. In this section we establish an irreducibility criterion for tensor modules. We also determine explicit conditions which must be satisfied by nonconstant singular vectors of tensor modules for tensor modules which not satisfy the irreducibility criterion.

Lemma 5.22. *Let $D_\lambda \otimes R$ be the $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ defined by (5.15). Then $\Bbbk \otimes R \subset \operatorname{sing}(D_\lambda \otimes R)$.*

Proof. By construction we have $e * (1 \otimes r) \in (F^2 H \otimes \Bbbk) \otimes_{D_\lambda} (D_\lambda \otimes R)$ for any $r \in R$. □

Remark 5.6. *We will call singular vectors $u \in \Bbbk \otimes R$* constant singular vectors.

Proposition 5.23. *Let N be a proper $H(\mathfrak{d}, 0, \omega)$-submodule with coefficients in D_λ of $D_\lambda \otimes R$. Then $N \cap (\Bbbk \otimes R) = 0$.*

Proof. N is a $\mathcal{P}$-module and $(\Bbbk \otimes R)$ is an $\mathfrak{sp}(\mathfrak{d}, \omega)$-module. Their intersection $N \cap (\Bbbk \otimes R)$ is an $\mathfrak{sp}(\mathfrak{d}, \omega)$-submodule. Since $\Bbbk \otimes R \simeq R$ is an irreducible $\mathfrak{sp}(\mathfrak{d}, \omega)$-module we have the statement. □

Remark 5.7. *This follows by Proposition 5.23 that if $D_\lambda \otimes R$ is reducible it must contain nonconstant singular vectors.*

Corollary 5.24. *Suppose that $\operatorname{sing}(D_\lambda \otimes R) = \Bbbk \otimes R$. Then $D_\lambda \otimes R$ is an irreducible $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ.*

Proof. Assume that N is a proper $H(\mathfrak{d}, 0, \omega)$-submodule with coefficients in D_λ. N is finite so that $\operatorname{sing} N \neq 0$. On the other hand by Proposition 5.23 we have $N \cap \operatorname{sing}(D_\lambda \otimes R) = N \cap (\Bbbk \otimes R) = 0$. □

Let $u \in D_\lambda \otimes R$. Then u can expressed as

$$u = \sum_K \delta^{(K)} \otimes u_K, \quad u_K \in R. \tag{5.17}$$

We will call each nonzero elements u_I in (5.17) a *coefficient* of $u \in D_\lambda \otimes R$.
If U is a submodule of $D_\lambda \otimes R$ we denote by $\operatorname{coeff} U$ the subspace of R linearly

spanned by all coefficients of elements in U.
For any $r \in R$ we set

$$\psi(r) = \sum_{kl=1}^{N} \delta_k \delta_l \otimes \rho_R(f^{kl})r.$$

Remark 5.8. *Notice that $\psi(r)$ is an element of degree two in $D_\lambda \otimes R$. Moreover since $f^{kl} = f^{lk}$ it can be written also as*

$$\psi(r) = 2 \sum_{k \leq l} \delta^{(\epsilon_k + \epsilon_l)} \otimes \rho_R(f^{kl})r, \tag{5.18}$$

where $\epsilon_k = (0, \dots, \underbrace{1}_{k}, \dots, 0)$ and $\delta^{\epsilon_k} = \delta_k$, $k = 1, \dots, N$.

Lemma 5.25. *Let $u = \sum_K \delta^{(K)} \otimes u_K \in D_\lambda \otimes R$. Then*

$$\begin{aligned} e * u = \; & \sum_K (1 \otimes \delta^{(K)}) \otimes_{D_\lambda} (\psi(u_K)) \\ & + \text{terms in } (\Bbbk \otimes \delta^{(K)}) \otimes_{D_\lambda} (F^1 D_\lambda \otimes (\Bbbk + \rho_R(\mathfrak{sp}(\mathfrak{d}, \omega))).u_K). \end{aligned}$$

In particular the coefficients multiplying $1 \otimes \delta^{(K)}$ equals $\psi(u_K)$ modulo $F^1 D_\lambda \otimes R$.

Proof. This follows by (5.16) and $(H \otimes D_\lambda)$-linearity. ☐

Proposition 5.26. *Let U be a nontrivial proper submodule of $D_\lambda \otimes R$. Then* $\operatorname{coeff} U = R$.

Proof. Since U is a submodule of $D_\lambda \otimes R$ then we have $H(\mathfrak{d}, 0, \omega) \subset (H \otimes D_\lambda) \otimes_{D_\lambda} U$. Let $0 \neq u = \sum_K \delta^{(K)} \otimes u_K \in U$. Then by Lemma 5.25 we have $\psi(u_K) \in U$. This proves that $\operatorname{coeff} U$ is a nontrivial $\mathfrak{sp}(\mathfrak{d}, \omega)$-submodule of R. Since R is irreducible we have the statement. ☐

Corollary 5.27. *Let U be a nontrivial proper submodule of $D_\lambda \otimes R$. Then for every $r \in R$ there exists an element $u \in U$ such that $u = \psi(r)$ modulo $F^1 D_\lambda \otimes R$.*

Proof. By Proposition 5.26 we have $\operatorname{coeff} U = R$. It is sufficient to prove the statement for the coefficients of elements $u \in U$. By Lemma 5.25 the coefficient multiplying $1 \otimes \delta^{(K)}$ coincides with $\psi(u_K)$ modulo $F^1 D_\lambda \otimes R$ and it lies in U since U is a submodule. ☐

Proposition 5.28. *Assume that $D_\lambda \otimes R$ is a reducible $H(\mathfrak{d}, 0, \omega)$-module with coefficients in D_λ. Then the action of $\mathfrak{sp}(\mathfrak{d}, \omega)$ on R satisfies*

$$\sum f^{ij} f^{kl}(r) = 0, \quad r \in R, \tag{5.19}$$

for all $1 \leq i, j, k, l \leq N$, where the sum is over all permutations of i, j, k, l.

Proof. Let U be a nontrivial proper submodule of $D_\lambda \otimes R$. By Corollary 5.27 there exists $u \in U$ such that $u = \psi(r)$ modulo $F^1 D_\lambda \otimes R$. If

$$u = \sum_{ij} \delta_k \delta_l \otimes \rho_R(f^{kl})r + \sum_k \delta_k \otimes r_k + 1 \otimes r',$$

then by (5.16) and $(H \otimes D_\lambda)$-linearity we have

$$e * u = \sum_{ijkl} (1 \otimes \delta_i\delta_j\delta_k\delta_l) \otimes_{D_\lambda} (1 \otimes \rho_R(f^{ij})\rho_R(f^{kl})r)$$
$$+\text{terms in } (\Bbbk \otimes F^3 D_\lambda) \otimes_{D_\lambda} (D_\lambda \otimes R).$$

Since the sum over all permutations of $1 \otimes \rho_R(f^{ij})\rho_R(f^{kl})$ lies in $\Bbbk \otimes R$ by Proposition 5.23 these coefficients must cancel each other, giving (5.19). □

Now we can provide a sufficient condition for irreducibility of $D_\lambda \otimes R$.

Theorem 5.29. *Let U be a nontrivial proper submodule of $D_\lambda \otimes R$. Then either R is the trivial $\mathfrak{sp}(\mathfrak{d}, \omega)$-module or it is isomorphic to one of the fundamental representation $R(\pi_k)$, for some $k = 1, \ldots, n$, of $\mathfrak{sp}(\mathfrak{d}, \omega)$.*

Proof. R satisfies (5.19), then the claim follows from [BDK3, Theorem 7.1]. □

Corollary 5.30. *If R is a nontrivial representation of $\mathfrak{sp}(\mathfrak{d}, \omega)$ which is not isomorphic to $R(\pi_k)$, for any $k = 1, \ldots, n$, then $D_\lambda \otimes R$ is irreducible for all $\lambda \in \Bbbk$.*

5.4.3 Singular vectors of tensor modules

We are left with studying the irreducibility of $D_\lambda \otimes R$ when R is either the trivial $\mathfrak{sp}(\mathfrak{d}, \omega)$-representation or it is isomorphic to some of the irreducible $\mathfrak{sp}(\mathfrak{d}, \omega)$-modules $R(\pi_k)$. We have already seen that the presence of nontrivial submodule of a tensor module $D_\lambda \otimes R$ implies the existence of nonconstant singular vectors. A possible strategy is then an explicit computation of singular vectors of $D_\lambda \otimes R$ whenever R is either the trivial or a fundamental representation of $\mathfrak{sp}(\mathfrak{d}, \omega)$. In this section we list a few computations about singular vectors of tensor modules $D_\lambda \otimes R$ and we omit the proof, which are obtained by computation totally analogous to those in [BDK3].

Lemma 5.31. *Let R be an irreducible nontrivial $\mathfrak{sp}(\mathfrak{d}, \omega)$-module. If $u \in \operatorname{sing}(D_\lambda \otimes R)$ then u is of degree at most two, i.e., it is of the form*

$$u = \sum_{kl} \delta_k\delta_l \otimes u_{kl} + \sum_k \delta_k \otimes u_k + 1 \otimes u'. \tag{5.20}$$

Lemma 5.32. *Let $u = 1 \otimes u' + \sum_k \delta_k \otimes u_k \in D_\lambda \otimes R$ be a singular vector. Then $\sum_k \delta_k \otimes u_k$ is a singular vector.*

Proposition 5.33. *Let $u = \sum_k \delta_k \otimes u_k \in D_\lambda \otimes R$. Then $u \in \operatorname{sing}(D_\lambda \otimes R)$ if and only if:*

$$\sum_{ijk} (1 \otimes \rho_R(f^{ik})u_k) = 0,$$

where the sum is over all permutations of i, j, k.

Proposition 5.34. *Let $u = 1 \otimes u' + \sum_k \delta_k \otimes u_k + \sum_{kl} \delta_k\delta_l \otimes u_{kl} \in \operatorname{sing}(D_\lambda \otimes R)$. Then $\sum_{kl} \delta_k\delta_l \otimes u_{kl} \in \operatorname{sing}(D_\lambda \otimes R)$.*

Corollary 5.35. *Let* $u = \sum_{kl} \delta_k \delta_l \otimes u_{kl}$*. Then* $u \in \operatorname{sing}(D_\lambda \otimes R)$ *if and only if:*

$$\begin{aligned} \sum_{ijkl} (1 \otimes \rho_R(f^{ij}) u_{kl}) &= 0, \\ \sum_{ijkl} (\delta_j \otimes \rho_R(f^{ij}) u_{kl} + \delta_i \otimes \rho_R(f^{ij}) u_{kl}) + \sum_{ikl} \delta_i \otimes u_{kl} &= 0, \end{aligned}$$

where the sum is over all permutations of i, j, k, l.

An investigation of the relations described in Proposition 5.33 and Corollary 5.35 can give an explicit description of singular vectors of degree one and of degree two respectively. Notice that the same identities are obtained when studying ordinary tensor modules of $H(\mathfrak{d}, 0, \omega)$ and $K(\mathfrak{d}, \theta)$ [BDK3, BDK4]. Finally we study the case when R is the trivial $\mathfrak{sp}(\mathfrak{d}, \omega)$-module. In this case formula (5.15) just becomes

$$\begin{aligned} e * (1 \otimes r) = \; & \sum_i (\partial_i \otimes 1)) \otimes_{D_\lambda} (\delta^i \otimes r) \\ & -(1 \otimes 1) \otimes_{D_\lambda} (1 \otimes \lambda r). \end{aligned}$$

Proposition 5.36. *If* R *is the trivial* $\mathfrak{sp}(\mathfrak{d}, \omega)$*-module then* $\operatorname{sing}(D_\lambda \otimes R) = F^1 D_\lambda \otimes R$.

Chapter 6

Extending scalars with ordinary $K(\mathfrak{d}, \theta)$-modules

We begin this chapter by recalling some results on ordinary representations of the Lie pseudoalgebra $K(\mathfrak{d}, \theta)$. Later on we show that extending scalars in a suitable way, one obtains from the exact contact pseudo de Rham complex introduced in [BDK3] the exact symplectic pseudo de Rham complex described in [BDK4]. Other ways of extending scalars produce, instead, new complexes of representations of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$ with coefficients in D_λ. We show that all such complexes are exact and we use this information in order to study the irreducibility of tensor modules $D_\lambda \otimes R$ for $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$, when R is the trivial $\mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega})$-module or it is isomorphic to $R(\pi_k)$, $k = 1, \ldots, n$.

6.1 The Lie pseudoalgebra $K(\mathfrak{d}, \theta)$

Let $\mathfrak{d}$ be a Lie algebra of dimension $2n+1$, $\{\partial_0, \partial_1, \ldots, \partial_{2n}\}$ be a basis of $\mathfrak{d}$ and $H = \mathcal{U}(\mathfrak{d})$ be the corresponding universal enveloping algebra. Let $L = He$ be a free H-module of rank 1.
It is shown in [BDK1] that if $r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d}$ satisfy the following system,

$$(6.1) \qquad \begin{cases} [r, \Delta(s)] = 0, \\ ([r_{12}, r_{13}] + r_{12}s_3) + \text{cyclic} = 0, \end{cases}$$

then

$$[e * e] = (r + s \otimes 1 - 1 \otimes s) \otimes_H e,$$

endows L with a Lie pseudoalgebra structure.

Example 6.1. *Let $\mathfrak{d}$ be an Heisenberg Lie algebra with basis $\{\partial_0, \partial_1, \ldots, \partial_{2n}\}$ and only non zero commutation relations $[\partial_i, \partial_{n+i}] = -\partial_0$. Then*

$$(6.2) \qquad r = \sum_{i=1}^{n} (\partial_i \otimes \partial_{n+i} - \partial_{n+i} \otimes \partial_i), \quad s = \partial_0,$$

is a solution of (6.1).

Henceforth, $\mathfrak{d}$ will be a Heisenberg Lie algebra with a basis $\{\partial_0, \partial_1, \ldots, \partial_{2n}\}$ as in Example 6.1. Let $\theta \in \mathfrak{d}^*$ be the *contact form* defined by

$$\theta(\partial_0) = -1, \quad \theta(\partial_i) = 0, \quad \text{for } i = 1, \ldots, 2n. \tag{6.3}$$

Then $\bar{\mathfrak{d}} = \ker\theta = \{\partial \in \mathfrak{d} \mid \theta(\partial) = 0\}$ is a vector subspace of dimension $2n$ of $\mathfrak{d}$ linearly generated by ∂_i, $i = 1, \ldots, 2n$.
The *Chevalley-Eilenberg differential* d_0 of the cohomology complex of $\mathfrak{d}$ from $\mathfrak{d}^*$ to $\mathfrak{d}^* \wedge \mathfrak{d}^*$ is explicitly given by

$$(d_0\theta)(\partial_l \wedge \partial_k) = \sum_{l<k} (-1)^{l+k}(\theta[\partial_l, \partial_k]). \tag{6.4}$$

We set $\omega = d_0\theta$. We have:

$$\begin{array}{lll} \omega(\partial_0 \wedge \partial_k) & = 0, & \forall\, k = 0, \ldots, 2n, \\ \omega(\partial_i \wedge \partial_j) & = 0, & \text{if } j \neq n+i, \\ \omega(\partial_i \wedge \partial_{n+i}) & = -1, & \text{for } i = 1, \ldots, n. \end{array}$$

For any $\partial_l \in \mathfrak{d}$ we define:

$$(\iota_{\partial_l}\omega)(\partial_k) = \omega(\partial_l \wedge \partial_k),$$

so that $(\iota_{\partial_l}\omega) \in \mathfrak{d}^*$.
By a direct computation we obtain:

$$\begin{array}{lll} (\iota_{\partial_l}\omega)(\partial_0) & = 0, & \text{for any } l = 0, \ldots, 2n, \\ (\iota_{\partial_i}\omega)(\partial_j) & = 0, & \text{if } j \neq n+i, \\ (\iota_{\partial_i}\omega)(\partial_{n+i}) & = -1 & \text{for } i = 1, \ldots, n. \end{array}$$

If we set $\ker\omega = \{\partial_l \in \mathfrak{d} \mid \iota_{\partial_l}\omega = 0\}$ then $\ker\omega$ is a 1-dimensional subspace generated by ∂_0, $\ker\omega = \Bbbk\partial_0$.
The following proposition is clear.

Proposition 6.2. *Let $\mathfrak{d}$ be the Heisenberg Lie algebra and $\{\partial_0, \partial_1, \ldots, \partial_{2n}\}$ be a basis of $\mathfrak{d}$ such that the only nonzero commutation relations are $[\partial_i, \partial_{n+i}] = -\partial_0$. Let $\theta \in \mathfrak{d}^*$ be such that $\theta(\partial_0) = -1$. $\theta(\partial_i) = 0$, for $i = 1, \ldots, 2n$ and $\omega = d_0\theta \in \mathfrak{d}^* \wedge \mathfrak{d}^*$ is defined by (6.4). Then $\mathfrak{d} = \bar{\mathfrak{d}} \oplus \Bbbk\partial_0$, where $\bar{\mathfrak{d}} = \ker\theta = \langle\partial_1, \ldots, \partial_{2n}\rangle$. Moreover, the restriction $\bar{\omega}$ to $\bar{\mathfrak{d}} \wedge \bar{\mathfrak{d}}$ of ω is non degenerate, $\iota_{\partial_0}\omega = 0$ and $[\partial_0, \bar{\mathfrak{d}}] \subset \bar{\mathfrak{d}}$.*

Proposition 6.2 shows that to the contact form θ defined in (6.3) corresponds a solution (r, s) of (6.1), obtained choosing by r the unique element in $\bar{\mathfrak{d}} \wedge \bar{\mathfrak{d}}$ determined by the symplectic form $\bar{\omega}$ and by s the unique element in $\mathfrak{d}$ such that $\theta(s) = -1$. Conversely, if (r, s) is a solution of (6.3) like in (6.2) we can define a contact form θ on $\mathfrak{d}$ as follows. Let $\bar{\mathfrak{d}}$ be the vector space linearly generated by ∂_i, $i = 2n$. Then $r \in \bar{\mathfrak{d}} \wedge \bar{\mathfrak{d}}$ is an element of maximal rank which defines a symplectic form $\bar{\omega}$ on $\bar{\mathfrak{d}}$ given by $\bar{\omega}(\partial_i \wedge \partial_{n+i}) = -1$, $i = 1, \ldots, n$. We extend $\bar{\omega}$ to a 2-form on $\mathfrak{d}$ setting $\iota_{\partial_0}\bar{\omega} = 0$, which implies $d_0\omega = 0$. Then we can define a contact form θ on $\mathfrak{d}$ setting $\theta(\partial_0) = -1$ and $\theta(\partial_i) = 0$ for $i = 1, \ldots, n$. Let $\mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega})$ be the symplectic Lie algebra linearly generated by the elements

$$f^{ij} = -\frac{1}{2}(\partial^i \otimes t_j + \partial^j \otimes t_i) \in \bar{\mathfrak{d}} \otimes \bar{\mathfrak{d}}^* \subset \mathfrak{gl}\bar{\mathfrak{d}}.$$

Let

$$I' = 2\partial_0 \otimes t_0 + \sum_{i=1}^{2n} \partial_i \otimes t_i \in \mathfrak{gl}\mathfrak{d}.$$

Then $\mathfrak{csp}(\bar{\mathfrak{d}}, \bar{\omega}) = \mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega}) + \Bbbk I'$ is a Lie subalgebra of $\mathfrak{gl}\mathfrak{d}$. It is a (trivial) central extension of $\mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega})$ by the ideal $\Bbbk I'$.
According to [BDK1] we denote the Lie pseudoalgebra over H corresponding to the contact form θ on $\mathfrak{d}$ in (6.3) by $K(\mathfrak{d}, \theta)$.

6.2 The annihilation algebra of $K(\mathfrak{d}, \theta)$

Let $K(\mathfrak{d}, \theta)$ be the Lie pseudoalgebra over $H = \mathcal{U}(\mathfrak{d})$ defined in the previous section. Recall that $\mathcal{U}(\mathfrak{d})$ is a graded associative algebra via the gradation $\{G'^p H\}$ defined in (2.32). Notice that with respect to (2.32) we have $\deg(\partial_i) = 1$, for all $i = 1, \ldots, 2n$, and $\deg(\partial_0) = 2$. We consider on $X \simeq \mathcal{O}_{2n+1}$ the corresponding structure of a graded associative algebra given by (2.33).
Let $\mathcal{K} = \mathcal{A}(K(\mathfrak{d}, \theta)) = X \otimes_H (He) \simeq X \otimes_H e$ be the annihilation algebra of $K(\mathfrak{d}, \theta)$. Due to (2.39), its Lie bracket is given by

$$[\phi \otimes_H e, \psi \otimes_H e] = \sum_i (\phi\partial_i)(\psi\partial^i) \otimes_H e + (\phi\partial_0)\psi \otimes_H e - \phi(\psi\partial_0) \otimes_H e \tag{6.5}$$

We set:

$$\mathcal{K}'^p = G'^{p+2} X \otimes_H e, \quad p \geq -2. \tag{6.6}$$

Lemma 6.3. $\{\mathcal{K}'^p\}_{p \geq -2}$ *defines a gradation on* $\mathcal{K}$ *which satisfies:*

$$\bar{\mathfrak{d}}.(\mathcal{K}'^p) \subset \mathcal{K}'^{p-1}, \tag{6.7}$$

$$\partial_0.(\mathcal{K}'^p) \subset \mathcal{K}'^{p-2} \tag{6.8}$$

$$[\mathcal{K}'^p, \mathcal{K}'^q] \subset \mathcal{K}'^{p+q}, \tag{6.9}$$

$$(G'^q H).(\mathcal{K}'^p) \subset \mathcal{K}'^{p-q}. \tag{6.10}$$

Proof. The first two statements follow from (2.36) and (2.37).
Let $\phi \in \mathcal{K}'^p$, $\psi \in \mathcal{K}'^q$. Then by (6.5), (6.7), (6.8) and (2.34) it follows that $[\phi \otimes_H e, \psi \otimes_H e] \in G'^{p+q+2} X \otimes_H e = \mathcal{K}'^{p+q}$. The last statement follows from (2.35). □

We consider on $\mathcal{K}$ the filtration induced by (6.6) given by:

$$\mathcal{K}'_p = F_p \mathcal{K} = \prod_{q \geq p} \mathcal{K}'^q = F'^{p+1} X \otimes_H e, \quad p \geq -2. \tag{6.11}$$

There exists an embedding of $K(\mathfrak{d}, \theta)$ in $W(\mathfrak{d})$ which induces a map from $\mathcal{K}$ to $\mathcal{W}$ which we explicitly describe in the following

Lemma 6.4. *The map*

$$\begin{array}{rccc} \iota_* : & \mathcal{K} = X \otimes_H e & \longrightarrow & \mathcal{W} = X \otimes \mathfrak{d} \\ & \phi \otimes_H e & \mapsto & \phi \otimes \partial_0 - \sum_{i=1}^{2n} \phi\partial_i \otimes \partial^i \end{array}$$

is an injective homomorphism of Lie algebras.

Remark 6.1. *If $\mathfrak{d}$ is as in Example 6.1 then the isomorphism between $\mathcal{W}$ and W_{2n} (see Remark 2.7) identifies $\mathcal{K}$ with the subalgebra $K_{2n+1} \subset W_{2n+1}$ of all formal vector fields preserving the standard contact form $dt_0 + \sum_{i=1}^{n} t_i dt_{n+i}$.*

We introduce a prime filtration on $\mathcal{W}$ too by setting

$$\mathcal{W}'_p = F'_p\mathcal{W} = (F'_pX \otimes \bar{\mathfrak{d}}) \oplus (F'_{p+1}X \otimes \Bbbk\partial_0).$$

A proof of the following lemma can be found in [BDK3].

Lemma 6.5. *The prime filtrations of $\mathcal{K}$ and $\mathcal{W}$ are compatible, i.e., one has $\mathcal{K}'_p = \mathcal{K} \cap \mathcal{W}'_p$. In particular, $[\mathcal{K}'_p, \mathcal{K}'_q] \subset \mathcal{K}'_{p+q}$.*

Remark 6.2. *As usual, $\mathcal{K}'_0$ is a Lie subalgebra of $\mathcal{K}$ and each $\mathcal{K}'_p$, $p \geq 0$, is an ideal of $\mathcal{K}'_0$.*

It is shown in [BDK3] that the filtration $\{\mathcal{K}'_p\}_{p\geq -2}$ of $\mathcal{K}$ defined in (6.11) is such that $\mathcal{K}'_0/\mathcal{K}'_1 \simeq \mathfrak{csp}(\bar{\mathfrak{d}}, \bar{\omega}) = \mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega}) + \Bbbk I'$. By an argument analogous to that in [BDK2] the representation theory of a special class of $\mathcal{K}$-modules is proved to be equivalent to representation theory of ordinary $K(\mathfrak{d}, \theta)$-modules. Let M be an ordinary $K(\mathfrak{d}, \theta)$-module. An element $m \in M$ is a *singular vector* if $\mathcal{K}'_1.m = 0$. As usual, the space of all singular vectors of M is denoted by $\operatorname{sing} M$. As a consequence of Theorem 4.27 a vector $m \in M$ is singular if and only if

$$e * m \in (F'^2H \otimes 1) \otimes_H M.$$

6.3 Tensor modules for $K(\mathfrak{d}, \theta)$

Let R be an irreducible $\mathfrak{csp}(\bar{\mathfrak{d}}, \bar{\omega})$-module with an action ρ_R. $H \otimes R$ is a free H-module, where H acts by left multiplication on the first tensor factor. It is possible to endow $H \otimes R$ with a structure of ordinary $K(\mathfrak{d}, \theta)$-module as follows [BDK3]. For $r \in R$ we set

$$\begin{aligned} e * (1 \otimes r) \;=\; & \sum_{i,j=1}^{2n} (\partial_i\partial_j \otimes 1) \otimes_H (1 \otimes \rho_R(f^{ij})r) \\ & + \sum_{i=1}^{2n} (\partial_i \otimes 1) \otimes_H (\partial^i \otimes r) \\ & + \tfrac{1}{2}(\partial_0 \otimes 1) \otimes_H (1 \otimes \rho_R(I')r) \\ & - (1 \otimes 1) \otimes_H (\partial_0 \otimes r), \end{aligned} \tag{6.12}$$

and we extend it to all of $H \otimes R$ by $(H \otimes H)$-linearity.
We will call the $K(\mathfrak{d}, \theta)$-module $H \otimes R$ a *tensor module* for $K(\mathfrak{d}, \theta)$ and we denote it by $\mathcal{V}(R)$. If $I' \in \mathfrak{csp}(\bar{\mathfrak{d}}, \bar{\omega})$ acts on R as multiplication by a scalar $k \in \Bbbk$ we denote the corresponding tensor module by $\mathcal{V}(R, k)$.

Remark 6.3. *The tensor module $\mathcal{V}(R, k)$ described above corresponds to $\mathcal{V}(\Bbbk, R, k)$, according to notations in [BDK3].*

6.3.1 Contact Pseudo de Rham complex

In order to study the reducibility of tensor modules $\mathcal{V}(R)$, where R is either the trivial $\mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega})$-module or one of the fundamental representations $R(\pi_k)$, a pseudoalgebraic version of the *Rumin complex* [Ru], called *contact pseudo de Rham complex* is introduced in [BDK3].
Let

$$\begin{aligned} \mathcal{V}_k &= \mathcal{V}(R(\pi_k), k), \\ \mathcal{V}_{2n+2-k} &= \mathcal{V}(R(\pi_k), 2n+2-k), \quad k = 0, 1, \ldots, n. \end{aligned}$$

The main result from [BDK3] (see Section 6 of this reference for the details) is the following,

Theorem 6.6. *There exists an exact complex of $K(\mathfrak{d}, \theta)$-modules with all nonzero homomorphisms*
(6.13)

$$0 \longrightarrow \mathcal{V}_{2n+2} \xrightarrow{d} \mathcal{V}_{2n+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{V}_{n+2} \xrightarrow{d^R} \mathcal{V}_n \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{V}_1 \xrightarrow{d} \mathcal{V}_0.$$

For an explicit description of maps d, d^R in (6.13) see [BDK3]. We just recall that d is homogeneous of degree 1, whereas the *Rumin map* d^R is homogeneous of degree 2, with respect to the grading of $\mathcal{K} \simeq K_N$ defined in (6.6). We denote the exact complex in (6.13) by $\mathcal{V}^\bullet$.

6.4 Extending scalars with ordinary $K(\mathfrak{d}, \theta)$-modules

In this section we show that any ordinary $K(\mathfrak{d}, \theta)$-module gives rise by an extension of scalars to an $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$-module with coefficients in D_λ, for every $\lambda \in \Bbbk$. Recall that $L = K(\mathfrak{d}, \theta)$ is the free H-module He endowed with the following Lie pseudobracket,

$$[e * e] = \left(\sum_{i=1}^{2n} (\partial_i \otimes \partial_{n+i} - \partial_{n+i} \otimes \partial_i) + \partial_0 \otimes 1 - 1 \otimes \partial_0 \right) \otimes_H e.$$

Let H_0 be the Hopf algebra structure described in Corollary 4.9, $\pi_0 : H \longrightarrow H_0$ be the Hopf homomorphism in (4.9). Extending scalars from H to H_0 using π_0 we obtain a Lie H_0-pseudoalgebra $\mathrm{BC}_{\pi_0}(L) = L_0$. L_0 isomorphic as an H_0-module to $H_0 \otimes_H e$, with H_0 acting on the first tensor factor, and by (2.17) its pseudobracket satisfies:

$$[(1 \otimes_H e) * (1 \otimes_H e)] \quad = \quad \sum_{i=1}^{2n} (\partial_i \otimes \partial_{n+i} - \partial_{n+i} \otimes \partial_i) \otimes_{H_0} (1 \otimes_H e), \tag{6.14}$$

where in (6.14) we use notations introduced in (4.11).
Recall that $H_0 = \mathcal{U}(\mathfrak{d}_0)$, where $\pi_0(\mathfrak{d}) = \mathfrak{d}_0$ is an abelian Lie algebra of dimension $2n$. In other words, $\mathrm{BC}_{\pi_0}(L)$ is isomorphic to the Lie pseudoalgebra $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$, where $\bar{\mathfrak{d}} = \ker \theta$ and $\bar{\omega}$ is the restriction to $\bar{\mathfrak{d}}$ of ω.
Let M be an ordinary $K(\mathfrak{d}, \theta)$-module. Since $\Psi_\lambda = (\pi_0, \pi_\lambda) \in \mathcal{M}or(H, D_\lambda)$, we know that $\mathrm{BC}_{\Psi_\lambda}(M)$ is a $\mathrm{BC}_{\pi_0}(L) = H(\bar{\mathfrak{d}}, 0, \bar{\omega})$-module with coefficients in D_λ. Explicitly, $\mathrm{BC}_{\Psi_\lambda}(M)$ is the D_λ-module $D_\lambda \otimes_H M$ with $\mathrm{BC}_{\pi_0}(L)$-action with coefficients given by

$$[(1 \otimes_H e) * (1 \otimes_H m)] = \sum_i (\pi_0(h^i) \otimes \pi_\lambda(k^i)) \otimes_{D_\lambda} (1 \otimes_H m_i), \tag{6.15}$$

if $e * m = \sum_i (h^i \otimes k^i) \otimes_H m_i$.
Notice that for $\lambda = 0$ for any ordinary $K(\mathfrak{d}, \theta)$-module M we obtain an ordinary $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$-module $\mathrm{BC}_{\Psi_0}(M)$.
Extending scalars by Ψ_λ with M is particular useful when $M = \mathcal{V}(R)$ is a tensor modules of $K(\mathfrak{d}, \theta)$.

Proposition 6.7. *Let $\mathcal{V}(R)$ be a tensor module of $K(\mathfrak{d}, \theta)$. Then $\mathrm{BC}_{\Psi_\lambda}(\mathcal{V}(R))$ is isomorphic as an $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$-module with coefficients to the tensor module $D_\lambda \otimes R$ of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$.*

Proof. First of all notice that R is an irreducible $\mathfrak{sp}(\bar{\mathfrak{d}}, 0, \bar{\omega})$-module too. Moreover, since $\mathcal{V}(R) = H \otimes R$ we have $\mathrm{BC}_{\Psi_\lambda}(\mathcal{V}(R)) = D_\lambda \otimes_H (H \otimes R) \simeq D_\lambda \otimes R$. Now, applying (6.15) to (6.12) we obtain

$$
\begin{aligned}
(1 \otimes_H e) * (1 \otimes r) \quad = \quad & \sum_{i,j=1}^{2n} (\partial_i \partial_j \otimes 1) \otimes_{D_\lambda} (1 \otimes \rho_R(f^{ij}) r) \\
& + \sum_{i=1}^{2n} (\partial_i \otimes 1) \otimes_{D_\lambda} (\delta^i \otimes r) \\
& - (1 \otimes 1) \otimes_{D_\lambda} (1 \otimes \lambda r),
\end{aligned}
$$

which coincides with (5.15). □

As a consequence of Proposition 6.7, applying $\mathrm{BC}_{\Psi_\lambda}$ to the exact complex $\mathcal{V}^\bullet$ of $K(\mathfrak{d}, \theta)$ we obtain a complex of tensor modules of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$. If we set $\mathrm{BC}_{\Psi_\lambda}(\mathcal{V}_k) = \mathcal{V}_k^\lambda$, $\mathrm{BC}_{\Psi_\lambda}(\mathcal{V}_{2n+2-k}) = \mathcal{V}_{2n+2-k}^\lambda$, $\mathrm{BC}_{\Psi_\lambda(d)} = d_\lambda$ and $\mathrm{BC}_{\Psi_\lambda}(d^R) = d_\lambda^R$ then:
(6.16)

$$
0 \longrightarrow \mathcal{V}_{2n+2}^\lambda \xrightarrow{d_\lambda} \mathcal{V}_{2n+1}^\lambda \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} \mathcal{V}_{n+2}^\lambda \xrightarrow{d_\lambda^R} \mathcal{V}_n^\lambda \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} \mathcal{V}_1^\lambda \xrightarrow{d_\lambda} \mathcal{V}_0^\lambda .
$$

We denote the complex in (6.16) by $\mathcal{V}_\lambda^\bullet$. This complex is studied in [BDK4] in the special case $\lambda = 0$.

Theorem 6.8. *[BDK4] The complex $\mathcal{V}_0^\bullet$ is an exact complex of tensor modules of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$.*

Our main goal is to prove that the complex $\mathcal{V}_\lambda^\bullet$ is exact for any $\lambda \in \Bbbk$. Let $D_\lambda \otimes R$ be a tensor module for $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$. By use of the filtration on D_λ defined in (4.15) we can introduce an increasing filtration on $D_\lambda \otimes R$ by setting

$$
F^p(D_\lambda \otimes R) = F^p D_\lambda \otimes R = \mathcal{D}_\lambda^p R, \quad p = -1, 0, \ldots,
$$

so that $\mathcal{D}_\lambda^{-1} R = 0$ and $\mathcal{D}_\lambda^0 R = \Bbbk \otimes R$.
Let

$$
\mathrm{gr}(D_\lambda \otimes R) = \bigoplus_{p \geq 0} \mathcal{D}_\lambda^p R / \mathcal{D}_\lambda^{p-1} R, \tag{6.17}
$$

be the associated graded space to the tensor module $D_\lambda \otimes R$.
Since $\mathrm{gr}(D_\lambda \otimes R) = \mathrm{gr}\, D_\lambda \otimes R \simeq H_0 \otimes R$, the associated graded space has a structure of a left H_0-module given by left multiplication on the first tensor factor.

Theorem 6.9. $\mathrm{gr}(D_\lambda \otimes R)$ *has a structure of an ordinary tensor module of* $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$.

We will prove Theorem 6.9 showing that it is possible to endow the H_0-module $\mathrm{gr}(D_\lambda \otimes R)$ with a structure of a conformal module over the annihilation algebra $\mathcal{P} = P_{2n}$ of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$, with a trivial action of central elements of $\mathcal{P}$ and such that (4.34) is satisfied.
Let $\mathcal{P}$ the annihilation algebra of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$ and $\{\mathcal{P}^p\}_{p\geq -2}$ be the gradation of $\mathcal{P}$ defined in (5.6). Notice that $\mathcal{P}^{-1} \oplus \mathcal{P}^{-2} = \hat{\mathfrak{d}}$ and that the structure of a D_λ-module on $D_\lambda \otimes R$ is induced by the action of $\hat{\mathfrak{d}}$ on R.

Lemma 6.10. *For every* $p \geq -2$ *the action of* $\mathcal{P}^p$ *on* $\mathcal{D}^0_\lambda R = \Bbbk \otimes R$ *is such that* $\mathcal{P}^p . \mathcal{D}^0_\lambda R \subset \mathcal{D}^{-p}_\lambda R$.

Proof. It can be proved by a direct computation. Since $\Bbbk \otimes R \subset \mathrm{sing}(D_\lambda \otimes R)$ we have $\mathcal{P}^p . \mathcal{D}^0_\lambda R = 0$ for every $p \geq 1$. It remains to prove that $\mathcal{P}^{-2} . \mathcal{D}^0_\lambda R \subset \mathcal{D}^2_\lambda R$, $\mathcal{P}^{-1} . \mathcal{D}^0_\lambda R \subset \mathcal{D}^1_\lambda R$ and $\mathcal{P}^0 . \mathcal{D}^0_\lambda R \subset \mathcal{D}^0_\lambda R$. They follow by (5.15). □

Proposition 6.11. *The action of* $\mathcal{P}^p$, $p \geq -2$, *on* $\mathcal{D}^q_\lambda R$ *is such that* $\mathcal{P}^p . \mathcal{D}^q_\lambda R \subset \mathcal{D}^{q-p}_\lambda R$ *for every* q.

Proof. We proceed by induction on p. Since the center of $\mathcal{P}$ acts via scalar multiplication by λ this follows that $\mathcal{P}^{-2} . \mathcal{D}^q_\lambda R \subset \mathcal{D}^q_\lambda \subset \mathcal{D}^{q+2}_\lambda R$. Then the basis of our induction holds. Now we proceed with our inductive assumption.
Suppose that $\mathcal{P}^p . \mathcal{D}^q_\lambda R \subset \mathcal{D}^{q-p}_\lambda R$ for every q. We want to prove that $\mathcal{P}^{p+1} . \mathcal{D}^q_\lambda R \subset \mathcal{D}^{q-p-1}_\lambda R$ for every q. We proceed by induction on q. The basis of our induction is proved in Lemma 6.10. Now we proceed with our inductive assumption. We want to prove that $\mathcal{P}^{p+1} . \mathcal{D}^{q+1}_\lambda R \subset \mathcal{D}^{q-p}_\lambda R$. Since $\mathcal{D}^{q+1}_\lambda R$ is generated by elements of the form $\delta_s \delta^{(Q)}$, with $|Q| = q$, it is sufficient to prove the statement for a such element.

$$\begin{aligned}\mathcal{P}^{p+1} . (\delta_s \delta^{(Q)}) &= \mathcal{P}^{p+1} . (\hat{\partial}^s . \delta^{(Q)}) = \hat{\partial}^s . (\mathcal{P}^{p+1} . \delta^{(Q)}) + [\mathcal{P}^{p+1}, \hat{\partial}^s] . \delta^{(Q)} \\ &\subset \hat{\partial}^s . \mathcal{D}^{q-p-1}_\lambda R + \mathcal{P}^p . \delta^{(Q)} \subset \mathcal{D}^{q-p}_\lambda R.\end{aligned}$$

□

As a consequence of Proposition 6.11 the action of $\mathcal{P} = \prod_{p \geq -2} \mathcal{P}^p$ on $\mathrm{gr}(D_\lambda \otimes R)$ is graded.

Proposition 6.12. *The center* $\mathcal{P}^{-2}$ *of* $\mathcal{P}$ *acts trivially on the* $\mathcal{P}$*-module* $\mathrm{gr}(D_\lambda \otimes R)$.

Proof. We know that $\mathcal{P}^{-2} . (\mathcal{D}^q_\lambda R / \mathcal{D}^{q-1}_\lambda R) \subset (\mathcal{D}^{q+2}_\lambda R / \mathcal{D}^{q+1}_\lambda R)$ by Proposition 6.11. On the other hand, since the action of $\mathcal{P}^{-2}$ is by scalar multiplication by $\lambda \in \Bbbk$, we have $\mathcal{P}^{-2} . \mathcal{D}^q_\lambda R \subset \mathcal{D}^q_\lambda R \subset \mathcal{D}^{q+1}_\lambda R$. □

Lemma 6.13. *For every* $\bar{d} \in H_0$, $x \in \mathcal{P}$, $m \in \mathrm{gr}(D_\lambda \otimes R)$ *the* $\mathcal{P}$*-module* $\mathrm{gr}(D_\lambda \otimes R)$ *satisfies*

$$\bar{d} . (x . m) = (\bar{d}_{(1)} x) . (\bar{d}_{(2)} m). \tag{6.18}$$

Proof. We know that the tensor module $D_\lambda \otimes R$ satisfies (4.34). Let $d \in F^p D_\lambda$ and recall that $\Delta(d) \subset \sum_{i=0}^{p} F^i H_0 \otimes F^{p-i} D_\lambda$. This proves (6.18). □

By Theorem 4.27 we can conclude that it is possible to endow the H_0-module $\mathrm{gr}(D_\lambda \otimes R)$ with a structure of an ordinary $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$-module.

Proposition 6.14. *The action of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$ on $\mathrm{gr}(D_\lambda \otimes R) \simeq H_0 \otimes R$ is given by*

$$(6.19)\qquad \begin{aligned}(1 \otimes_H e) * (1 \otimes r) \;&= \sum_{i,j=1}^{2n} (\partial_i \partial_j \otimes 1) \otimes_{H_0} (1 \otimes \rho_R(f^{ij}) r) \\ &+ \sum_{i=1}^{2n} (\partial_i \otimes 1) \otimes_{H_0} (\partial^i \otimes r).\end{aligned}$$

Proof. This follows by (5.15) and Proposition 6.12. □

It is shown in [BDK4] that (6.19) describes the action of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$ on an ordinary tensor module.

Theorem 6.15. *For any $\lambda \in \Bbbk$ the graded complex $\mathrm{gr}(\mathcal{V}_\lambda^\bullet)$ associated to $(\mathcal{V}_\lambda^\bullet)$ is exact.*

Proof. $\mathrm{gr}(\mathcal{V}_\lambda^\bullet)$ is isomorphic to the complex $(\mathcal{V}_0^\bullet)$ which is exact by Theorem 6.8. □

Corollary 6.16. *For any $\lambda \in \Bbbk$ the complex $(\mathcal{V}_\lambda^\bullet)$ is exact and any tensor module in (6.16) is reducible.*

Proof. The statement follows by the isomorphism between $\mathrm{gr}(\mathcal{V}_\lambda^\bullet)$ and $(\mathcal{V}_0^\bullet)$ and results in [BDK4], where it is shown that all the maps of the complex $\mathcal{V}_0^\bullet$ are nonzero. As a consequence, the same holds for $\mathrm{gr}(\mathcal{V}_\lambda^\bullet)$ and then for $(\mathcal{V}_\lambda^\bullet)$. This proves that the image of the maps $\mathrm{BC}_{\Psi_\lambda}(d)$ and $\mathrm{BC}_{\Psi_\lambda}(d^R)$ is always a nontrivial proper submodule in the next tensor module. □

Notice that we have not determined irreducible quotients of reducible tensor modules of $H(\bar{\mathfrak{d}}, 0, \bar{\omega})$. This is done in [BDK4].

Bibliography

[BDK1] B. Bakalov, A. D'Andrea, V. G. Kac, *Theory of finite pseudoalgebras*, Adv. Math. 162 (2001), 1-140.

[BDK2] B. Bakalov, A. D'Andrea, V. G. Kac, *Irreducible representations of primitive simple Lie pseudoalgebras I. Primitive pseudoalgebras of type W and S*, Adv. Math. 204 (2006), 278-346.

[BDK3] B. Bakalov, A. D'Andrea, V. G. Kac, *Irreducible representations of primitive simple Lie pseudoalgebras II. Primitive pseudoalgebras of type K*, preprint, arXiv:1003.6055v1 [math. QA].

[BDK4] B. Bakalov, A. D'Andrea, V. G. Kac, *Irreducible representations of primitive simple Lie pseudoalgebras III. Primitive pseudoalgebras of type H*, work in progress.

[BeD] A. Beilinson, V. Drinfeld, *Chiral algebras*, AMS Colloquium Publications, vol. 51, American Mathematical Society, Providence, RI, 2004.

[Bo] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras and the monster*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), 3068-3071.

[Ca] E. Cartan, *Les groupes de transformation continus, infinis, simples*, Ann. Sci. ENS 26 (1909), 93-161.

[D1] A. D'Andrea, *Finite vertex algebras and nilpotence*, J. of Pure and Applied Algebra, 212, no. 4 (2008), 669-688.

[D2] A. D'Andrea, *Simplicity of vertex and Lie conformal algebras*, J. of Algebra, 319, no. 5 (2008), 2106-2112.

[D3] A. D'Andrea, *Associativity and commutativity in vertex algebras*, Bulg. J. of Phys. 35, no. 1 (2008), 43-50.

[DK] A. D'Andrea, V. G. Kac, *Structure theory of finite conformal algebras*, Selecta Mathematica, 4 (1998), 377-418.

[DM] A. D'Andrea, G. Marchei, *Representations of Lie pseudoalgebras with coefficients*, J. of Algebra (2010), doi:101016/j.jalgebra.2010.04.007.

[Gu1] V. Guillemin, *A Jordan-Hölder decomposition for a certain class of infinite dimensional Lie algebras*, J. Diff. Geom. 2 (1968), 313-345.

[Gu2] V. Guillemin, *Infinite-dimensional primitive Lie algebras*, J. Diff. Geom. 4 (1970), 257-282.

[K] V. G. Kac, *Vertex Algebras for Beginners*, Mathematical Lecture Series, AMS, Vol.10, 1996 (2nd Ed. AMS, 1998).

[Ko1] P. Kolesnikov, *Identities of conformal algebras and pseudoalgebras*, Communications in Algebra, 34 (2006), 1965-1979.

[Ko2] P. Kolesnikov, *Simple finite Jordan pseudoalgebras*, SIGMA, 14 (2009), 1-17.

[Re] A. Retakh, *Unital associative pseudoalgebras and their representations*, math.QA/0109110.

[Ru] M. Rumin, *Formes différentielles sur les variétés de contact*, J. Diff. Geom. 39 (1994), 281-330.

[Sw] M. Sweedler, *Hopf algebras*, University Lecture Note Series, W. A. Benjamin, Inc. New York, 1969.