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RESEARCH/REVIEW ARTICLE

On projective wonderful models for toric arrangements and their cohomology

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In reverent and grateful memory of Ştefan Papadima

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Abstract

This paper is divided into two parts. The first part is a brief survey, accompanied by concrete examples, on the main results of the papers (De Concini and Gaiffi in Adv Math 327:390–09, 2018; Algebr Geom Topol 19(1):503–532, 2019): the construction of projective models of toric arrangements and the presentation of their cohomology rings by generators and relations. In the second part we focus on the notion of *well-connected building set* that appears in the cohomological computations mentioned above: we explore some of its properties in the more general context of arrangements of subvarieties of a variety X.

Keywords Toric arrangements · Compact models · Configuration spaces

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1 Introduction

Let $T \simeq (\mathbb{C}^*)^n$ be an n-dimensional torus and let $X^*(T)$ be its group of characters; a *layer* in T is a subvariety of T of the form

$$\mathcal{K}(\Gamma, \phi) := \left\{ t \in T \mid \chi(t) = \phi(\chi) \text{ for all } \chi \in \Gamma \right\}$$

where $\Gamma < X^*(T)$ is a split direct summand and $\phi \colon \Gamma \to \mathbb{C}^*$ is a homomorphism. A *toric arrangement* $\mathcal{A}$ is a (finite) set of layers $\{\mathcal{K}_1, \ldots, \mathcal{K}_r\}$ in T. We remark that in literature the usual definition of "toric arrangement" requires the layers to be 1-codimensional; instead we allow layers of any codimension and use the term *divisorial arrangement* for the case where all layers are 1-codimensional.

Toric arrangements have been studied since the early 1990s, and over the last two decades several aspects have been investigated: in particular, as far as the topology of the complement is concerned, De Concini and Procesi [12] determined the generators of the cohomology modules over $\mathbb{C}$ in the divisorial case, as well as the ring structure in the case of totally unimodular arrangements; d'Antonio and Delucchi, generalizing an algebraic complex first introduced by Moci and Settepanella [25], provided a presentation of the fundamental group for the complement of a divisorial complexified arrangement [5,6]; Callegaro, Delucchi and Pagaria computed the graded cohomology ring with integer coefficients (see [3,4,26]); the cohomology ring itself was computed by Callegaro et al. [2].

The problem of studying wonderful models for toric arrangement was first addressed by Moci [24], where he described a construction of a non-projective model. Wonderful models for subspace arrangements were introduced by De Concini and Procesi [10,11], where they provided both a projective and a non-projective version of their construction. A *wonderful model* for the complement of an arrangement $\mathcal{M}(\mathcal{A})$ is a smooth variety $Y_{\mathcal{A}}$ containing $\mathcal{M}(\mathcal{A})$ as an open set and such that $Y_{\mathcal{A}} \setminus \mathcal{M}(\mathcal{A})$ is a divisor with normal crossings and smooth irreducible components. In [10] the integral cohomology ring of these wonderful models was presented by generators and relations. Subsequently, many authors studied the cohomology rings of the models of subspace arrangements: among others Yuzvinsky [31] and Gaiffi [17], where some integer bases were provided; and Etingof et al. [13,30] in the real case.

The connections between the geometry of these models and the Chow rings of matroids were pointed out first by Feichtner and Yuzvinsky [16] and then by Adiprasito et al. [1], where they also played a crucial role in the study of some log-concavity problems—see also [19].

The first part (Sects. 2–5) of this paper is a short survey, enriched with examples, on the main results of the papers [8] and [9]: the construction of a *projective wonderful model* $Y_{\mathcal{A}}$ for a toric arrangement $\mathcal{A}$ and the presentation of its cohomology ring by generators and relations.

The key ingredient in the construction of $Y_{\mathcal{A}}$ is a toric variety $X_{\mathcal{A}}$ with some good properties. In [8] this variety is obtained by subdividing a given fan in a suitable way. De Concini and Gaiffi provided an algorithm to do so, and Papini implemented it in his Ph.D. thesis [27] in order to produce some meaningful examples.

Inspired by the computations in [10] and using [8], in [9] the same authors describe a presentation of the cohomology ring of the wonderful model $Y_{\mathcal{A}}$ with integer coefficients; more precisely, they show that $H^*(Y_{\mathcal{A}}; \mathbb{Z})$ is isomorphic to a quotient of a polynomial ring with coefficients in $H^*(X_{\mathcal{A}}; \mathbb{Z})$. Papini in his Ph.D. thesis provided a code that computes the ideal of the relations involved in the presentation of $H^*(Y_{\mathcal{A}}; \mathbb{Z})$ as the quotient of a polynomial ring with $\mathbb{Z}$ coefficients, thus being able to give first examples of computed presentations. Actually, the wonderful model $Y_{\mathcal{A}}$ depends on the choice of a so-called *building set* and the algorithm allows the user to choose one, even computing the minimal one if requested.

This part of the paper is structured as follows. Section 2 describes the properties of a "good" toric variety in which we have to embed the toric arrangement in order to construct the model. Section 3 outlines the algorithms that subdivides the fan to compute the toric variety, and illustrates a couple of examples. Section 4 recalls the theoretic construction of a wonderful model for an arrangement of subvarieties. Section 5 shows how to construct a projective wonderful model for a toric arrangement, using the results from the previous sections, and exhibits generators and relations of a presentation of the cohomology ring, together with some examples.

The second part (Sect. 6) of this paper explores the notion of *well-connected building set* of subvarieties (see Definition 5.1), which was introduced in [9] and turned out to play a crucial role in the computation of the cohomology rings of projective models for toric arrangements. In more detail, the construction of a wonderful model of a variety X is obtained by a sequence of blowups, whose centres are the (transforms of the) elements of a building set of subvarieties. The order in which the blowups are performed respects the inclusion relation of the subvarieties (i.e., the minimal subvarieties with respect to inclusion are blown up first, and so on).

In the case of projective models of toric arrangements, if the building set is well-connected, it is possible to compute the cohomology ring by induction on the steps of the blowup process. The key points in this toric case are the following:

(a) the property of being well-connected turns out to be stable under the blowup process;
(b) at each step of the blowup process, the restriction map in cohomology from the blown up variety to the transform of an element of the building set is surjective; this allows for the application of a Keel's lemma (see Theorem 6.6) to compute the cohomology ring;
(c) at every step of the blowup process, some polynomials are chosen in such a way that their restrictions to the transforms of the elements of the building set coincide with the Chern polynomials of the normal bundles of the strata; then one proves that the ideal generated by these polynomials does not depend on their choice.

In Sect. 6 we consider an arrangement of subvarieties in a variety X. We consider a well-connected building set of subvarieties and we prove that the properties (a) and (b) mentioned above are still valid in this general setting (see Theorems 6.5 and 6.8). The proof of property (c) instead uses some specific features of toric arrangements. In our opinion this remark points out the interest of the definition of well-connected building set: if for a specific family of wonderful models one can prove property (c), the same methods used in [9] can be applied to compute their cohomology rings. This is the case, for instance, of the wonderful models of subspace arrangements, where the methods of [9] can be used to give a different proof of the presentation of the cohomology ring provided in [10].

2 A good toric variety for the arrangement

Let $\mathcal{A} = \{\mathcal{K}_1, \dots, \mathcal{K}_r\}$ be a toric arrangement in the n-dimensional torus T, where $\mathcal{K}_i = \mathcal{K}(\Gamma_i, \phi_i)$ with Γ_i split direct summands of $X^*(T)$ and $\phi_i : \Gamma_i \to \mathbb{C}^*$ homomorphisms, and let $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Notice that a layer $\mathcal{K}(\Gamma, \phi)$ is a coset with respect to the torus

$$H = \bigcap_{\chi \in \Gamma} \ker(\chi) \subseteq T. \tag{1}$$

In [8] it is shown how to build projective wonderful models for the complement $\mathcal{M}(\mathcal{A})$. The goal is to apply Li's machinery [22], thus we have to identify a toric arrangement $\mathcal{A}$ as an *arrangement of subvarieties* in the sense of Li's paper.

Definition 2.1 Let X be a non-singular variety. A *simple arrangement of subvarieties* of X is a finite set Λ of non-singular closed connected subvarieties properly contained in X such that

- for every two $\Lambda_i, \Lambda_j \in \Lambda$, either $\Lambda_i \cap \Lambda_j \in \Lambda$ or $\Lambda_i \cap \Lambda_j = \varnothing$;
- if $\Lambda_i \cap \Lambda_j \neq \varnothing$, the intersection is *clean*, i.e., it is non-singular and for every $y \in \Lambda_i \cap \Lambda_j$ we have the following conditions on the tangent spaces:

$$\mathrm{T}_y(\Lambda_i \cap \Lambda_j) = \mathrm{T}_y(\Lambda_i) \cap \mathrm{T}_y(\Lambda_j).$$

Definition 2.2 Let X be a non-singular variety. An *arrangement of subvarieties* of X is a finite set Λ of non-singular closed connected subvarieties properly contained in X such that

- for every two $\Lambda_i, \Lambda_j \in \Lambda$, either $\Lambda_i \cap \Lambda_j$ is a disjoint union of elements of Λ or $\Lambda_i \cap \Lambda_j = \varnothing$;
- if $\Lambda_i \cap \Lambda_j \neq \varnothing$, the intersection is clean.

By choosing a basis of $X^*(T)$, we get an isomorphism $T \simeq (\mathbb{C}^*)^n$. Therefore we can embed T into $(\mathbb{P}^1)^n$ which, as a toric variety, is associated with the fan Ω induced by the decomposition of V into orthants. Then we consider the closures of the layers $\overline{\mathcal{K}_i}$ in $(\mathbb{P}^1)^n$. These do *not* give an arrangement of subvarieties in the sense of Li, since these closures of the layers, or their intersections, may be singular.

To see a simple example, let us consider the torus $(\mathbb{C}^*)^n$ as the open set of $\mathbb{C}^n$ consisting of points with non-zero coordinates. If we take $n = 2$, the closure in $\mathbb{C}^2$ of the layer of equation $\mathcal{K}(\mathbb{Z}(2, -3), 1)$ is the cubic curve of equation $z_1^2 = z_2^3$ which has a cusp in the origin.

On the other hand, it can be easily shown (see for instance Theorem 2.5 below, which recalls [8, Theorem 3.1]) that, given a layer $\mathcal{K}(\Gamma, \phi) \subset (\mathbb{C}^*)^n$, if Γ has a basis $(\chi_1, \dots, \chi_s) \subset \mathbb{Z}^n$ consisting of vectors of positive coordinates, then the closure of $\mathcal{K}(\Gamma, \phi)$ in $\mathbb{C}^n$ is non-singular.

Definition 2.3 Let Δ be a fan in V. A character $\chi \in X^*(T)$ has the *equal sign property* with respect to Δ if, for every cone $C \in \Delta$, either $\langle \chi, c\rangle \geqslant 0$ for all $c \in C$ or $\langle \chi, c\rangle \leqslant 0$ for all $c \in C$.

Definition 2.4 Let Δ be a fan in V and let $\mathcal{K}(\Gamma, \phi)$ be a layer. A $\mathbb{Z}$-basis $(\chi_1, \dots, \chi_m)$ for Γ is an *equal sign basis* with respect to Δ if χ_i has the equal sign property for all $i = 1, \dots, m$.

This suggests the following idea (see [8], in particular Proposition 6.1). Let $\mathcal{A} = \{\mathcal{K}_1, \dots, \mathcal{K}_r\}$ be a toric arrangement. Let us start with a fixed projective toric variety (for example $(\mathbb{P}^1)^n$) and suppose that we can subdivide its fan to obtain a smooth projective fan $\Delta(\mathcal{A})$ such that for every layer $\mathcal{K}_i = \mathcal{K}(\Gamma_i, \phi_i)$ in $\mathcal{A}$ there is an equal sign basis $(\chi_{i,1}, \dots, \chi_{i,s_i})$ of Γ_i with respect to $\Delta(\mathcal{A})$. Then all the intersections of the closures $\overline{\mathcal{K}(\Gamma_i, \phi_i)}$ of the layers of $\mathcal{A}$ in the toric variety $X_{\mathcal{A}} := X_{\Delta(\mathcal{A})}$ defined by $\Delta(\mathcal{A})$ are smooth. We will say that $X_{\mathcal{A}}$ is a *good toric variety* for $\mathcal{A}$.

The behaviour of the layers in this variety $X_{\mathcal{A}}$ has been described in [8]. In fact, consider the closure $\overline{\mathcal{K}(\Gamma, \phi)}$ of a layer in $X_{\mathcal{A}}$. It turns out that this closure is a toric variety itself, whose explicit description is provided by the following result.

Theorem 2.5 ([8, Proposition 3.1 and Theorem 3.1]) *For every layer $\mathcal{K}(\Gamma, \phi)$ let H be the corresponding homogeneous subtorus as defined in (1) and let $V_\Gamma := \{\mathbf{v} \in V \mid \langle \chi, \mathbf{v}\rangle = 0$ for all $\chi \in \Gamma\}$.*

(i) *For every cone $C \in \Delta(\mathcal{A})$, its relative interior is either entirely contained in V_Γ or disjoint from V_Γ.*
(ii) *The collection of cones $C \in \Delta(\mathcal{A})$ which are contained in V_Γ is a smooth fan $\Delta(\mathcal{A})_H$.*
(iii) *$\overline{\mathcal{K}(\Gamma, \phi)}$ is a smooth H-variety whose fan is $\Delta(\mathcal{A})_H$.*
(iv) *Let $\mathcal{O}$ be an orbit of T in $X_{\mathcal{A}}$ and let $C_{\mathcal{O}} \in \Delta(\mathcal{A})$ be the corresponding cone. Then*
   (a) *if $C_{\mathcal{O}}$ is not contained in V_Γ, $\overline{\mathcal{O}} \cap \overline{\mathcal{K}(\Gamma, \phi)} = \varnothing$;*
   (b) *if $C_{\mathcal{O}} \subset V_\Gamma$, $\mathcal{O} \cap \overline{\mathcal{K}(\Gamma, \phi)}$ is the orbit of H in $\overline{\mathcal{K}(\Gamma, \phi)}$ corresponding to $C_{\mathcal{O}} \in \Delta(\mathcal{A})_H$.*

Once we have the toric variety $X_{\mathcal{A}}$, the next step is to build the wonderful model. Let $\mathcal{Q}'$ be the set

$$\mathcal{Q}' := \{\overline{\mathcal{K}} \mid \mathcal{K} \in \mathcal{A}\}$$

and let

$$\mathcal{Q} := \mathcal{Q}' \cup \left\{ D \mid D \text{ is an irreducible component of } X_{\mathcal{A}} \setminus T \right\}.$$

As a consequence of Theorem 2.5, the family $\mathcal{L}$ of all the connected components of intersections of elements of $\mathcal{Q}$ gives an arrangement of subvarieties in the sense of Definition 2.2.

Notice that also the family $\mathcal{L}'$ of all the connected components of intersections of elements of $\mathcal{Q}'$ is an arrangement of subvarieties, because it is contained in $\mathcal{L}$ and it is closed under intersection. This allows, by a series of blowups, to build a projective wonderful model associated with $\mathcal{A}$: the construction is outlined in Sect. 4.

3 Subdividing the fan

3.1 The first algorithm

The core of the process outlined in Sect. 2 is the construction of the T-variety $X_{\mathcal{A}}$. Section 4 of [8] is devoted to a combinatorial algorithm that, starting from a finite set $\Xi \subset X^*(T)$ of characters (here identified with vectors of $\mathbb{Z}^n$) and a fan Δ in $V \simeq \mathbb{R}^n$, produces a new fan Δ' with the same support as Δ (in fact, a proper subdivision of Δ) and such that each vector of Ξ has the equal sign property with respect to Δ'. We describe it here briefly.

First of all, we suppose that $\Xi = \{\chi\}$, i.e., we subdivide a fan Δ in such a way that a single fixed character χ has the equal sign property with respect to the new fan Δ'; in general, if $\Xi = \{\chi_1, \dots, \chi_s\}$ we set $\Delta_{(0)} := \Delta$ and repeat the previous step s times, where we obtain $\Delta_{(i)}$ by subdividing $\Delta_{(i-1)}$ using the character χ_i, for $i = 1, \dots, s$. The output is $\Delta' := \Delta_{(s)}$.

So, suppose that Δ is a fan in V and $\chi \in X^*(T)$. In accordance with [8], we have to consider only the 2-dimensional cones of Δ: in fact, let $C \in \Delta$ be a k-dimensional cone generated by $(r_1, \dots, r_k)$ and suppose that for each 2-*dimensional* cone $C' \in \Delta$ we have either $\langle \chi, c\rangle \geqslant 0$ or $\langle \chi, c\rangle \leqslant 0$ for all $c \in C'$. For each r_i, r_j let $C(r_i, r_j) \in \Delta$ be the 2-dimensional cone generated by r_i and r_j. Now, without loss of generality we may assume that $\langle \chi, r_1\rangle \geqslant 0$ and $\langle \chi, r_2\rangle \geqslant 0$, since we have the property for $C(r_1, r_2)$. But now also $\langle \chi, r_3\rangle \geqslant 0$, due to the property applied to the cone $C(r_2, r_3)$. By induction then we have $\langle \chi, r_i\rangle \geqslant 0$ for all $i = 1, \dots, k$.

Notice that for 2-dimensional cones of the form $C(v_1, v_2)$ the property translates as

$$\langle \chi, v_1\rangle \langle \chi, v_2\rangle \geqslant 0,$$

and the algorithm checks it for each 2-dimensional cone of Δ, building a list of "bad" cones for which the property is not satisfied. If there are no bad cones, the algorithm terminates returning the fan; otherwise, it chooses a bad cone $C = C(v_1, v_2)$ and defines a new fan $\Delta(C)$ obtained from Δ by substituting each cone $C(v_1, v_2, w_1, \dots, w_k)$ containing C with two new cones generated by $(v_1, v_1 + v_2, w_1, \dots, w_k)$ and $(v_1 + v_2, v_2, w_1, \dots, w_k)$ respectively. After that it restarts with $\Delta(C)$ as the new input fan.

Proposition 3.1 ([8, Proposition 4.1]) *The new fan $\Delta(C)$ is smooth, and a projective subdivision of Δ. Moreover, if X_Δ and $X_{\Delta(C)}$ are the two toric varieties associated with the fans Δ and $\Delta(C)$ respectively, then $X_{\Delta(C)}$ is obtained from X_Δ by blowing up the closure of the 2-codimensional orbit in X_Δ associated with C.*

The only thing to do is to find a way to choose wisely the bad cone that has to be replaced. We follow the choice of [8]: if $\Delta_{(N)}$ is the set of the bad 2-dimensional cones, define

$$\begin{array}{rcl} P_\Delta : \quad \Delta_{(N)} & \to & \mathbb{N} \times \{0, 1\} \\ C(v_1, v_2) & \mapsto & (M_C, \epsilon_C) \end{array}$$

where $M_C = \max\{|\langle \chi, v_1\rangle|, |\langle \chi, v_2\rangle|\}$ and

$$\epsilon_C = \begin{cases} 1 & \text{if } |\langle \chi, v_1\rangle| = |\langle \chi, v_2\rangle|, \\ 0 & \text{otherwise.} \end{cases}$$

Fix the lexicographic order on $\mathbb{N} \times \{0, 1\}$, i.e.,

$$(0, 0) < (0, 1) < (1, 0) < (1, 1) < (2, 0) < \cdots$$

Lemma 3.2 ([8, Lemma 4.2]) *Assume $\Delta_{(N)} \neq \varnothing$ and choose $C \in \Delta_{(N)}$ so that $P_\Delta(C) = (M_C, \epsilon_C)$ is maximum in* $\mathrm{Im}(P_\Delta)$.

(i) *If $\epsilon_C = 1$, then $\Delta(C)_{(N)} = \Delta_{(N)} \setminus \{C\}$.*
(ii) *If $\epsilon_C = 0$, then* $\max(\mathrm{Im}(P_{\Delta(C)})) \leqslant (M_C, \epsilon_C)$*, and*

$$\left|P^{-1}_{\Delta(C)}((M_C, \epsilon_C))\right| < \left|P^{-1}_{\Delta}((M_C, \epsilon_C))\right|.$$

The previous lemma proves that, if we choose $C \in \Delta_{(N)}$ such that $P_\Delta(C) = (M_C, \epsilon_C)$ is maximum in $\mathrm{Im}(P_\Delta)$, we are guaranteed that the number of bad cones eventually decreases and the algorithm stops.

We are now ready to describe (the fan associated with) a good toric variety $X_{\mathcal{A}}$ for a toric arrangement $\mathcal{A}$. Let

$$\mathcal{A} = \{\mathcal{K}(\Gamma_1, \phi_1), \ldots, \mathcal{K}(\Gamma_r, \phi_r)\}$$

be a toric arrangement in the n-dimensional torus T; for $i = 1, \ldots, r$ let $\mathcal{B}_i$ be a $\mathbb{Z}$-basis for the lattice Γ_i and define

$$\Xi = \bigcup_{i=1}^{r} \mathcal{B}_i.$$

A good toric variety $X_{\mathcal{A}}$ is obtained by using this set Ξ in the algorithm described above, starting with any fan that gives a projective smooth variety: in [8] the authors choose the one generated by the orthants of $\mathbb{R}^n$ (the associated variety is $(\mathbb{P}^1)^n$).

3.2 The second algorithm

The algorithm described in the previous section begins with a smooth fan and subdivides it in such a way that:

- each intermediate subdivision of the fan remains smooth;
- in the end we have a fan such that each vector of the set Ξ has the equal sign property with respect to that fan.

Let us show that in the 2-dimensional case we can use a different strategy. Given a set of integral vectors Ξ that span $V \simeq \mathbb{R}^2$ (which is the relevant case—see [8, Remark 6.2]),[1] we compute the fan Δ whose rays are the vectors orthogonal to the ones in Ξ: this fan is projective and by construction the vectors of Ξ have the equal sign property with respect to it, but it can be non-smooth. Hence we proceed as follows: at first for each 2-dimensional cone $C \in \Delta$ we check if C is smooth. This is easily done because, if the two rays delimiting a cone C are $v_1 = (x, y)$ and $v_2 = (z, w)$, smoothness is guaranteed as long as

$$p := \left| \det \begin{pmatrix} x & z \\ y & w \end{pmatrix} \right| = 1.$$

Therefore we compute p for each cone; if $p = 1$, we leave the cone untouched and proceed with the next one; otherwise we subdivide it with the following method.

Let c_1, c_2 be integer numbers such that $c_1 x + c_2 y = 1$ (they exist because v_1 is supposed to be primitive) and notice that the value $c_1 z + c_2 w \pmod{p}$ does not depend on the choice of c_1 and c_2. In fact, let c_1', c_2' be another such choice; therefore

$$0 = 1 - 1 = (c_1 x + c_2 y) - (c_1' x + c_2' y) = (c_1 - c_1')x + (c_2 - c_2')y$$

and on the other hand there exists k such that

$$\begin{pmatrix} z \bmod p \\ w \bmod p \end{pmatrix} = k \begin{pmatrix} x \bmod p \\ y \bmod p \end{pmatrix}$$

(this is because $\det \equiv 0 \pmod{p}$, but none of the vectors can be the zero vector modulo p since they are primitive). It follows that

$$\begin{aligned} (c_1 z + c_2 w) - (c_1' z + c_2' w) &= (c_1 - c_1')z + (c_2 - c_2')w \\ &\equiv k\big((c_1 - c_1')x + (c_2 - c_2')y\big) \equiv 0 \pmod{p}. \end{aligned}$$

Now let $q = c_1 z + c_2 w$ and let q_0 be the remainder of the division of q by p. Notice that $0 \leqslant q_0 < p$ and that $\mathrm{GCD}(q_0, p) = 1$ (by absurd: let $\mathrm{GCD}(q_0, p) = h > 1$; since $h \mid q_0$ and $h \mid p$, by definition $h \mid q$, hence the vector (q, p) is not primitive; but

$$\begin{pmatrix} c_1 & c_2 \\ -y & x \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}$$

[1] For technical reason we suppose also that all vectors of Ξ are primitive and that there are no parallel vectors.

and the determinant of the matrix is $c_1 x + c_2 y = 1$, so it sends primitive vectors to primitive vectors). Therefore we define the vector

$$\overline{v} := \frac{1}{p}\left((p-q_0)v_1 + v_2\right) = \frac{1}{p}\begin{pmatrix}(p+q-q_0)x - c_2 p \\ (p+q-q_0)y + c_1 p\end{pmatrix},$$

where it is an easy check that the division is exact, i.e., $\overline{v} \in \mathbb{Z}^2$. Now notice that $\overline{v}$ belongs to the cone generated by v_1 and v_2, because it is a linear combination of them with positive coefficients (remember that $q_0 < p$); moreover,

$$\det\left(v_1 \mid \overline{v}\right) = \frac{p-q_0}{p}\det\left(v_1 \mid v_1\right) + \frac{1}{p}\det\left(v_1 \mid v_2\right) = \pm\frac{p}{p} = \pm 1$$

so the cone $C(v_1, \overline{v})$ is smooth. On the other hand,

$$\det\left(\overline{v} \mid v_2\right) = \frac{p-q_0}{p}\det\left(v_1 \mid v_2\right) + \frac{1}{p}\det\left(v_2 \mid v_2\right) = \pm\frac{p(p-q_0)}{p} = \pm(p-q_0)$$

and $|p - q_0| < p$, so we can reapply the algorithm to the cone $C(\overline{v}, v_2)$. Since the absolute value of the new determinant strictly decreases, we can prove by induction that this procedure terminates with $p = 1$.

3.3 Examples

An implementation of the aforementioned procedures was done by one of the authors in his Ph.D. thesis [27], using the `SageMath` language.

Example 3.3 Consider in $(\mathbb{C}^*)^2$ with coordinates x, y the following sets:

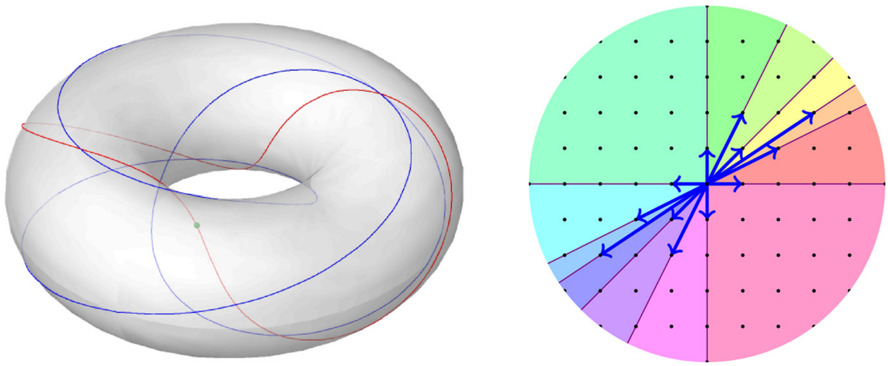
$$\begin{aligned} L_1 &= \{(x, y) \in (\mathbb{C}^*)^2 \mid x^2 = y^3\}, \\ L_2 &= \{(x, y) \in (\mathbb{C}^*)^2 \mid y = -x^2\}, \\ L_3 &= \{(1, -1)\}. \end{aligned}$$

They are layers of a toric arrangement $\mathcal{A} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ with

$$\begin{aligned} \Gamma_1 &= \langle(2, -3)\rangle,\ \ \phi_1\colon (2, -3) \mapsto 1; \\ \Gamma_2 &= \langle(-2, 1)\rangle,\ \ \phi_2\colon (-2, 1) \mapsto -1; \\ \Gamma_3 = \langle(1, 0), (0, 1)\rangle,&\ \ \phi_3\colon \begin{cases}(1, 0) \mapsto 1, \\ (0, 1) \mapsto -1.\end{cases} \end{aligned}$$

Therefore we have to give the set

$$\Xi = \{(2, -3), (-2, 1), (1, 0), (0, 1)\}$$



(a) (b)

Fig. 1 Arrangement of Example 3.3. (a) Picture of the corresponding arrangement in $(S^1)^2$: L_1 is the blue subtorus, L_2 is the red one, and L_3 is the green dot. (b) Fan associated to the arrangement (Color figure online)

to the algorithm. The resulting fan is pictured in Fig. 1. Actually, both algorithms return the same fan, and this behaviour takes place in many other examples that we computed, even if it is possible to construct examples in which this does not happen.

Example 3.4 By running the code, we notice that even relatively small arrangements can give rise to quite large objects. Consider the arrangement in $(\mathbb{C}^*)^4$ with two layers

$$L_1 = \{(x, y, z, w) \in (\mathbb{C}^*)^4 \mid xy = zw\},$$
$$L_2 = \{(x, y, z, w) \in (\mathbb{C}^*)^4 \mid y = x^2, z = -1\}.$$

In this case

$$\Xi = \{(1, 1, -1, -1), (2, -1, 0, 0), (0, 0, 1, 0)\}$$

and if we apply the algorithm of Sect. 3.1 we get a fan with 52 rays, which are the following:

$$\begin{array}{cccc}
(1, 2, 2, 0) & (0, 0, 1, -1) & (1, 2, 3, 0) & (0, 0, -1, 0) \\
(1, 2, 0, 0) & (-1, -2, 0, -3) & (1, 2, 1, 0) & (-1, -2, -2, 0) \\
(-1, -1, -1, 0) & (-1, -2, -2, -1) & (-1, 1, 0, 0) & (-1, -2, -1, 0) \\
(-1, -1, -1, -1) & (-1, -2, 0, -2) & (-1, -2, -3, 0) & (-1, -2, -1, -2) \\
(1, 2, 2, 1) & (1, -1, 0, 0) & (0, -1, -1, 0) & (1, 2, 0, 1) \\
(-1, 0, -1, 0) & (1, 2, 0, 2) & (-1, -2, 0, -1) & (1, 1, 0, 1) \\
(1, 2, 1, 2) & (1, 0, 0, 0) & (-1, -1, -2, 0) & (1, 0, 1, 0) \\
(0, 0, 0, 1) & (0, 1, 0, 1) & (0, -1, 0, 0) & (0, 1, 0, 0) \\
(1, 1, 1, 1) & (0, 1, 1, 0) & (0, 0, 0, -1) & (0, 0, -1, 1) \\
(-1, -1, 0, 0) & (-1, 0, 0, 0) & (1, 1, 0, 2) & (1, 1, 1, 0) \\
(1, 2, 0, 3) & (-1, 0, 0, -1) & (1, 1, 0, 0) & (-1, -1, 0, -1) \\
(1, 2, 1, 1) & (-1, -2, -1, -1) & (1, 0, 0, 1) & (0, 0, 1, 0) \\
(1, 1, 2, 0) & (-1, -2, 0, 0) & (-1, -1, 0, -2) & (0, -1, 0, -1)
\end{array}$$

4 General construction of wonderful models for arrangements of subvarieties

In this section we recall the construction of a wonderful model associated with a variety X using Li's techniques [22], which were inspired by the original work of De Concini and Procesi [10] and by the work of MacPherson and Procesi [23].

We start by giving the definitions of building sets and nested sets of subvarieties; we refer the reader to the papers [14,15,18,28,29] where building and nested sets are studied from a more combinatorial point of view (in particular in [28, Section 2] one can find a short comparison among various definitions and notations in the literature).

Definition 4.1 Let Λ be a simple arrangement of subvarieties of X (see Definition 2.1). A subset $\mathcal{G} \subseteq \Lambda$ is a *building set* for Λ if for each subvariety $\Lambda_i \in \Lambda \setminus \mathcal{G}$ the minimal[2] elements of the set $\{G \in \mathcal{G} \mid G \supset \Lambda_i\}$ intersect transversally and their intersection is Λ_i. These minimal elements are called the $\mathcal{G}$-*factors* of Λ_i.

Definition 4.2 Let $\mathcal{G}$ be a building set for a simple arrangement Λ. A subset $\mathcal{T} \subseteq \mathcal{G}$ is called $\mathcal{G}$-*nested* if for any subset $\{A_1, \ldots, A_k\} \subseteq \mathcal{T}$ of pairwise non-comparable[3] elements there is an element in Λ of which $A_1, \ldots, A_k$ are the $\mathcal{G}$-factors.

Let $U \subseteq X$ be an open set. The *restriction* of an arrangement of subvarieties Λ to U is the set

$$\Lambda|_U := \{\Lambda_i \cap U \mid \Lambda_i \in \Lambda,\ \Lambda_i \cap U \neq \varnothing\}.$$

Definition 4.3 Let Λ be an arrangement of subvarieties of X. A subset $\mathcal{G} \subseteq \Lambda$ is a *building set* for Λ if there is an open cover $\mathcal{U}$ of X such that

- for every $U \in \mathcal{U}$, the restriction $\Lambda|_U$ is simple;
- for every $U \in \mathcal{U}$, $\mathcal{G}|_U$ is a building set for $\Lambda|_U$.

Definition 4.4 Let $\mathcal{G}$ be a building set for an arrangement Λ. A subset $\mathcal{T} \subseteq \mathcal{G}$ is called $\mathcal{G}$-*nested* if there is an open cover $\mathcal{U}$ of X such that, for every $U \in \mathcal{U}$, $\mathcal{G}|_U$ is simple and $\mathcal{T}|_U$ is $\mathcal{G}|_U$-nested.

We have first introduced the notion of arrangement of subvarieties and then defined a building set for the arrangement. However it is often convenient to go in the opposite direction.

Definition 4.5 A finite set $\mathcal{G}$ of connected subvarieties of X is called a *building set* if the set of the connected components of all the possible intersections of collections of subvarieties from $\mathcal{G}$ is an arrangement of subvarieties Λ (the arrangement *induced* by $\mathcal{G}$) and $\mathcal{G}$ is a building set for Λ according to Definition 4.3.

[2] With respect to the inclusion.

[3] This means that for any i, j we have $A_i \not\subseteq A_j$ and $A_i \not\supseteq A_j$.

Given an arrangement Λ of a non-singular variety X and a building set $\mathcal{G}$ for Λ, we can construct a wonderful model $Y(X;\mathcal{G})$ by considering (by analogy with [10]) the closure of the image of the locally closed embedding

$$\left(X \setminus \bigcup_{\Lambda_i \in \Lambda} \Lambda_i\right) \to \prod_{G \in \mathcal{G}} \mathrm{Bl}_G X$$

where $\mathrm{Bl}_G X$ is the blowup of X along G. This can be done one step at a time, as the following results show.

Proposition 4.6 (see [22, Proposition 2.8]) *Let $\mathcal{G}$ be a building set in the variety X. Let $F \in \mathcal{G}$ be a minimal element in $\mathcal{G}$ under inclusion and, for a subvariety $V \subseteq X$, denote by $t(V)$ its proper transform. Then the set $t(\mathcal{G}) := \{t(G) \mid G \in \mathcal{G}\}$ consisting of the proper transforms of the elements in $\mathcal{G}$ is a building set in $\mathrm{Bl}_F X$.*

Proof In fact, Li showed this for a building set of a simple arrangement. But since the definition of building set is local, one can easily adapt his proof. ◻

Theorem 4.7 (see [22, Theorem 1.3]) *Let $\mathcal{G}$ be a building set in a non-singular variety X. Let us order the elements $G_1, \dots, G_m$ of $\mathcal{G}$ in such a way that for every $1 \leqslant k \leqslant m$ the set $\mathcal{G}_k := \{G_1, \dots, G_k\}$ is building. Then if we set $X_0 := X$ and $X_k := Y(X;\mathcal{G}_k)$ for $1 \leqslant k \leqslant m$, we have*

$$X_k = \mathrm{Bl}_{t(G_k)} X_{k-1},$$

where $t(G_k)$ denotes the dominant transform[4] of G_k in X_{k-1}.

Remark 4.8 1. We notice that any total ordering of the elements of a building set $\mathcal{G} = \{G_1, \dots, G_m\}$ which refines the ordering by inclusion, that is $i < j$ if $G_i \subset G_j$, satisfies the condition of Theorem 4.7.

2. In particular, using the above ordering we deduce that $Y(X;\mathcal{G})$ is obtained from X by a sequence of blowups, each with centre a minimal element in a suitable building set.

To finish this section let us mention further result of Li describing the boundary of $Y(X;\mathcal{G})$ in terms of $\mathcal{G}$-nested sets.

Theorem 4.9 (see [22, Theorem 1.2]) *The complement in $Y(X;\mathcal{G})$ of $X \setminus \bigcup \Lambda_i$ is the union of the divisors $t(G)$, where G ranges among the elements of $\mathcal{G}$. An intersection of the form $t(T_1) \cap \cdots \cap t(T_k)$ is non-empty if and only if $\{t_1, \dots, T_k\}$ is $\mathcal{G}$-nested; moreover, if the intersection is non-empty then it is transversal.*

[4] In the blowup of a variety M along a centre F the dominant transform of a subvariety Z coincides with the proper transform if $Z \nsubseteq F$ (and therefore it is isomorphic to the blowup of Z along $Z \cap F$), and with $\pi^{-1}(Z)$ if $Z \subseteq F$, where $\pi : \mathrm{Bl}_F M \to M$ is the projection. We will use the same notation $t(Z)$ for both the proper and the dominant transform of Z, if no confusion arises.

5 Projective wonderful model in the toric case: the cohomology

Let $\mathcal{A} = \{\mathcal{K}_1, \dots, \mathcal{K}_r\}$ be a toric arrangement in the n-dimensional torus T, where $\mathcal{K}_i = \mathcal{K}(\Gamma_i, \phi_i)$. Recall that $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $V_\Gamma := \{v \in V \mid \langle \chi, v \rangle = 0 \text{ for all } \chi \in \Gamma\}$. Let $X_{\mathcal{A}}$ be a good toric variety for $\mathcal{A}$, as we have seen in Sect. 2, and consider the arrangement of subvarieties $\mathcal{L}'$ as defined in the same section. We choose a building set $\mathcal{G}$ for $\mathcal{L}'$ and construct a projective wonderful model $Y_{\mathcal{A}}(\mathcal{G}) := Y(X_{\mathcal{A}}; \mathcal{G})$ according to the strategy of Sect. 4.

5.1 Computing the cohomology

In order to compute the cohomology $H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ using the methods of [9], the building set is required to have an additional property.

Definition 5.1 A building set $\mathcal{G}$ is *well-connected* if for any subset $\{G_1, \dots, G_k\} \subseteq \mathcal{G}$, if the intersection $G_1 \cap \dots \cap G_k$ has two or more connected components, then each of these components belongs to $\mathcal{G}$.

Example 5.2 If Λ is a simple arrangement, then each building set $\mathcal{G}$ for Λ is well-connected. In fact every intersection $G_1 \cap \dots \cap G_k$ is either empty or connected, therefore the condition of Definition 5.1 is vacuously true.

Remark 5.3 One may choose $\mathcal{G}$ to be the whole arrangement Λ, which is always a well-connected building set. However bigger building sets imply more complicated wonderful models (and cohomologies).

Remark 5.4 If $\mathcal{G}$ is well-connected and $F \in \mathcal{G}$ is minimal, we have that for every $G \in \mathcal{G}$ the intersection $G \cap F$ is either empty or connected.

In [9] the cohomology ring $H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ is presented as a quotient of a polynomial ring with coefficients in $H^*(X_{\mathcal{A}}; \mathbb{Z})$. We recall here a well-known presentation of this ring, which was proven by Danilov.

Theorem 5.5 ([7, Theorem 10.8]) *Let $X = X_\Delta$ be a smooth complete T-variety. Let $\mathcal{R}$ be the set of primitive generators of the rays of Δ and define a polynomial indeterminate C_r for each $r \in \mathcal{R}$. Then*

$$H^*(X_\Delta; \mathbb{Z}) \simeq \mathbb{Z}[C_r \mid r \in \mathcal{R}]/(I_{\mathrm{SR}} + I_{\mathrm{L}})$$

where

- I_{SR} *is the* Stanley–Reisner ideal

$$I_{\mathrm{SR}} := (C_{r_1} \cdots C_{r_k} \mid r_1, \dots, r_k \textit{ do not belong to a cone of } \Delta);$$

- I_{L} *is the* linear equivalence ideal

$$I_{\mathrm{L}} := \left(\sum_{r \in \mathcal{R}} \langle \beta, r \rangle C_r \;\middle|\; \beta \in X^*(T) \right).$$

Notice that for I_{SR} it is sufficient to take only the square-free monomials, and for I_{L} it is sufficient to take only the β's belonging to a basis of $X^*(T)$.

Let $\mathcal{L}'$ be as before; let $X_{\mathcal{A}}$ be a good toric variety for $\mathcal{A}$ and let $B \coloneqq H^*(X_{\mathcal{A}};\mathbb{Z})$ be its cohomology ring. For each $G \in \mathcal{G}$ let T_G be a polynomial indeterminate. We are going to produce an ideal I_{W} of the polynomial ring $B[T_G \mid G \in \mathcal{G}]$ such that the cohomology ring of $Y_{\mathcal{A}}(\mathcal{G})$ is isomorphic to the quotient $B[T_G \mid G \in \mathcal{G}]/I_{\mathrm{W}}$. To do so, we need some auxiliary polynomials.

Let Z be an indeterminate and, for every $G \in \mathcal{L}'$ denote by Γ_G the lattice such that $G = \mathcal{K}(\Gamma_G, \phi)$. For every pair $(M, G) \in \mathcal{L}' \times \mathcal{L}'$ with $G \subseteq M$ choose a basis $(\chi_1, \ldots, \chi_s)$ for Γ_G such that $(\chi_1, \ldots, \chi_k)$, with $k \leqslant s$, is a basis for Γ_M and such that it is equal sign with respect to the fan Δ associated with the variety $X_{\mathcal{A}}$.[5] If M is the whole torus T, then choose any (equal sign) basis of Γ_G and let $k = 0$. Define the polynomials $P_G^M \in B[Z]$ as

$$P_G^M \coloneqq \prod_{j=k+1}^{s} \left(Z - \sum_{r \in \mathcal{R}} \min(0, \langle \chi_j, r \rangle) C_r \right). \tag{2}$$

Notice that we allow $G = M$: in that case $P_G^G \coloneqq 1$ since it is an empty product.

Now consider the following set:[6]

$$\mathcal{W} \coloneqq \left\{ (G, A) \in \mathcal{G} \times \mathcal{P}(\mathcal{G}) \mid G \subsetneq \mathcal{K} \text{ for all } \mathcal{K} \in A \right\}.$$

For each $G \in \mathcal{G}$ define

$$B_G \coloneqq \{H \in \mathcal{G} \mid H \subseteq G\}$$

and for each $(G, A) \in \mathcal{W}$ with $A = \{G_1, \ldots, G_k\}$ let M be the unique connected component of $G_1 \cap \cdots \cap G_k$ that contains G (if $A = \varnothing$, let $M = T$). Define the polynomial in $B[T_G \mid G \in \mathcal{G}]$

$$F(G, A) \coloneqq P_G^M \left(\sum_{H \in B_G} - T_H \right) \prod_{K \in A} T_K.$$

Finally let $\mathcal{W}_0 \coloneqq \{A = \{G_1, \ldots, G_k\} \in \mathcal{P}(\mathcal{G}) \mid G_1 \cap \cdots \cap G_k = \varnothing\}$. For each $A \in \mathcal{W}_0$ define the polynomial in $B[T_G \mid G \in \mathcal{G}]$

$$F(A) \coloneqq \prod_{K \in A} T_K.$$

Theorem 5.6 ([9, Theorem 7.1]) *The cohomology ring $H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ is isomorphic to the quotient of $B[T_G \mid G \in \mathcal{G}]$ by the ideal I_{W} generated by*

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5 This last property was not explicitly stated in [9], although the authors assumed it; we would like to thank Roberto Pagaria for pointing this out. This remark should be considered as an erratum to [9], first lines of Section 8 at p. 528.

6 For a set S, we denote by $\mathcal{P}(S)$ the power set of S.

- *the products* $C_r T_G$*, with* $G \in \mathcal{G}$ *and* $r \in \mathcal{R}$ *such that* r *does* not *belong to* V_{Γ_G}*;*
- *the polynomials* $F(G, A)$ *for every pair* $(G, A) \in \mathcal{W}$*;*
- *the polynomials* $F(A)$ *for every* $A \in \mathcal{W}_0$.

Putting all together, we have

$$H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z}) \simeq B[T_G \mid G \in \mathcal{G}]/I_{\mathrm{W}} \simeq \mathbb{Z}[C_r, T_G \mid r \in \mathcal{R}, G \in \mathcal{G}]/(I_{\mathrm{SR}} + I_{\mathrm{L}} + I_{\mathrm{W}}).$$

Remark 5.7 It is known (see [8, Theorem 9.1]) that the cohomology of the projective wonderful model $Y_{\mathcal{A}}(\mathcal{G})$ satisfies:

- $H^i(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z}) = 0$ for i odd;
- $H^i(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ is torsion-free for i even.

Notice that the ideal I_{W} is homogeneous and that the image of an indeterminate T_G under the isomorphism stated in Theorem 5.6 belongs to $H^2(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ ([9, Theorem 7.1] describes this isomorphism explicitly). This means that $H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ and $B[T_G \mid G \in \mathcal{G}]/I_{\mathrm{W}}$ are isomorphic as *graded* rings and that

$$\big(B[T_G \mid G \in \mathcal{G}]/I_{\mathrm{W}}\big)_i \simeq H^{2i}(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z}).$$

In particular, if $\mathcal{B}$ is a monomial basis of $B[T_G \mid G \in \mathcal{G}]/I_{\mathrm{W}}$ as $\mathbb{Z}$-module, we have

$$\mathrm{rk}(H^{2i}(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})) = |\{m \in \mathcal{B} \mid \deg(m) = i\}|\,.$$

5.2 Examples

To compute these examples we use a quite straightforward implementation of the algorithm that computes the cohomology $H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$ written in the `SageMath` language, which can be found in [27], with a minor correction: in order to make sure that all the bases required for the P_G^M polynomials (see formula (2)) have the equal sign property, we compute them *before* computing the fan with the procedure of Sect. 3.1, and we include all the vectors of those bases in the set Ξ. The result is a fan Δ with respect to which all the bases are automatically equal sign.

Example 5.8 Let us begin with a small arrangement $\mathcal{A}$ of three 1-dimensional layers in $(\mathbb{C}^*)^2$, namely

$$\begin{aligned} L_1 &= \{(x, y) \in (\mathbb{C}^*)^2 \mid x^2 = y^3\}, \\ L_2 &= \{(x, y) \in (\mathbb{C}^*)^2 \mid y = i\}, \\ L_3 &= \{(x, y) \in (\mathbb{C}^*)^2 \mid x = \omega\}, \end{aligned}$$

where i is the imaginary unit and $\omega = e^{2\pi i/3}$ is a primitive third root of unity. We want to compute a presentation of the cohomology ring $H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z})$, where $\mathcal{G}$ is the *minimal* (i.e., with least elements) well-connected building set.

We begin computing the well-connected building set $\mathcal{G}$: in this case it is not difficult, even by hand (see Fig. 2). Then we compute the fan Δ with one of the algorithms

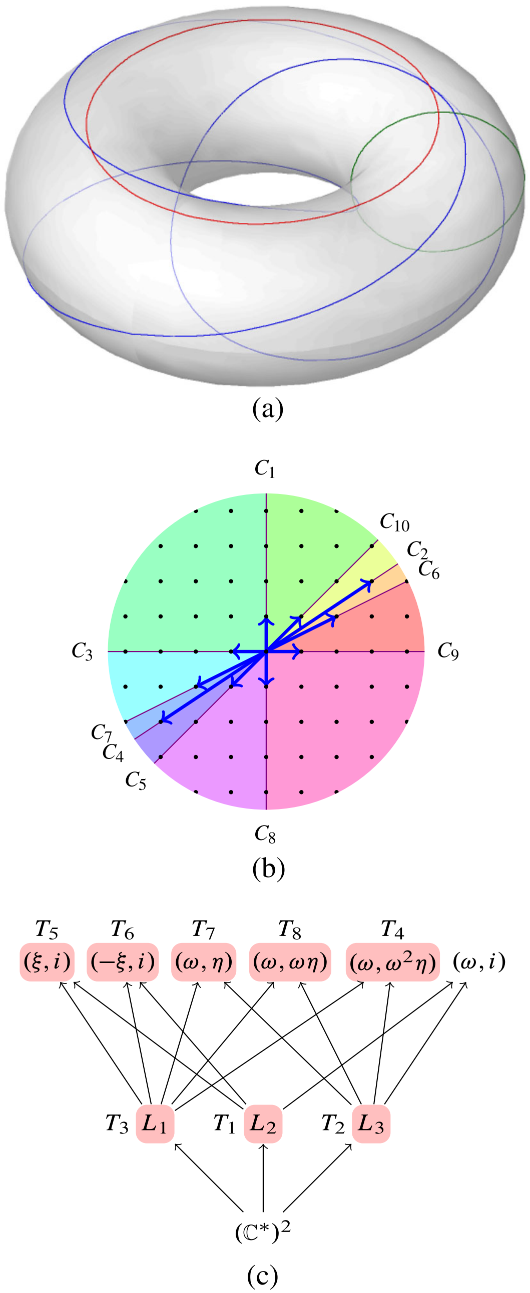

Fig. 2 Arrangement of Example 5.8. (a) Picture of the corresponding arrangement in $(S^1)^2$: L_1 is the blue subtorus, L_2 is the red one, and L_3 is the green one. (b) Fan associated to the arrangement, with polynomial indeterminates C_r corresponding to the generators of the rays (see Theorem 5.5). (c) Poset of the connected components of the intersections of the layers in $\mathcal{A}$, with elements of the building set highlighted, together with the associated polynomial indeterminates T_G; note that it contains the whole torus $(\mathbb{C}^*)^2$ as the intersection of zero layers. The values are $\xi = e^{3\pi i/4}$, $\eta = e^{2\pi i/9}$ (Color figure online)

described in Sect. 3 (in this case they actually return the same fan). The result is pictured in Fig. 2.

Once we have all the ingredients, we compute all the relations as stated in Theorem 5.6:

- the relations of the form C_rT_G are: C_1T_1, C_2T_1, C_4T_1, C_5T_1, C_6T_1, C_7T_1, C_8T_1, $C_{10}T_1$, C_2T_2, C_3T_2, C_4T_2, C_5T_2, C_6T_2, C_7T_2, C_9T_2, $C_{10}T_2$, C_1T_3, C_3T_3, C_5T_3, C_6T_3, C_7T_3, C_8T_3, C_9T_3, $C_{10}T_3$, C_1T_4, C_2T_4, C_3T_4, C_4T_4, C_5T_4, C_6T_4, C_7T_4, C_8T_4, C_9T_4, $C_{10}T_4$, C_1T_5, C_2T_5, C_3T_5, C_4T_5, C_5T_5, C_6T_5, C_7T_5, C_8T_5, C_9T_5, $C_{10}T_5$, C_1T_6, C_2T_6, C_3T_6, C_4T_6, C_5T_6, C_6T_6, C_7T_6, C_8T_6, C_9T_6, $C_{10}T_6$, C_1T_7, C_2T_7, C_3T_7, C_4T_7, C_5T_7, C_6T_7, C_7T_7, C_8T_7, C_9T_7, $C_{10}T_7$, C_1T_8, C_2T_8, C_3T_8, C_4T_8, C_5T_8, C_6T_8, C_7T_8, C_8T_8, C_9T_8, $C_{10}T_8$;
- the relations of the form $F(G, A)$ are: $2C_4+C_5+C_7+C_8-T_1-T_5-T_6$, $C_3+3C_4+C_5+2C_7-T_2-T_4-T_7-T_8$, $3C_1+2C_3+C_7+C_{10}-T_3-T_4-T_5-T_6-T_7-T_8$, $2C_3C_4+6C_4^2+C_3C_5+5C_4C_5+C_5^2+C_3C_7+7C_4C_7+3C_5C_7+2C_7^2+C_3C_8+3C_4C_8+C_5C_8+2C_7C_8-C_3T_4-5C_4T_4-2C_5T_4-3C_7T_4-C_8T_4+T_4^2$, $2C_4T_2+C_5T_2+C_7T_2+C_8T_2-T_2T_4$, $C_1T_3+C_3T_3+C_4T_3+C_7T_3-T_3T_4$, T_2T_3, $2C_3C_4+6C_4^2+C_3C_5+5C_4C_5+C_5^2+C_3C_7+7C_4C_7+3C_5C_7+2C_7^2+C_3C_8+3C_4C_8+C_5C_8+2C_7C_8-C_3T_5-5C_4T_5-2C_5T_5-3C_7T_5-C_8T_5+T_5^2$, $C_3T_1+3C_4T_1+C_5T_1+2C_7T_1-T_1T_5$, $C_1T_3+C_3T_3+C_4T_3+C_7T_3-T_3T_5$, T_1T_3, $2C_3C_4+6C_4^2+C_3C_5+5C_4C_5+C_5^2+C_3C_7+7C_4C_7+3C_5C_7+2C_7^2+C_3C_8+3C_4C_8+C_5C_8+2C_7C_8-C_3T_6-5C_4T_6-2C_5T_6-3C_7T_6-C_8T_6+T_6^2$, $C_3T_1+3C_4T_1+C_5T_1+2C_7T_1-T_1T_6$, $C_1T_3+C_3T_3+C_4T_3+C_7T_3-T_3T_6$, $2C_3C_4+6C_4^2+C_3C_5+5C_4C_5+C_5^2+C_3C_7+7C_4C_7+3C_5C_7+2C_7^2+C_3C_8+3C_4C_8+C_5C_8+2C_7C_8-C_3T_7-5C_4T_7-2C_5T_7-3C_7T_7-C_8T_7+T_7^2$, $2C_4T_2+C_5T_2+C_7T_2+C_8T_2-T_2T_7$, $C_1T_3+C_3T_3+C_4T_3+C_7T_3-T_3T_7$, $2C_3C_4+6C_4^2+C_3C_5+5C_4C_5+C_5^2+C_3C_7+7C_4C_7+3C_5C_7+2C_7^2+C_3C_8+3C_4C_8+C_5C_8+2C_7C_8-C_3T_8-5C_4T_8-2C_5T_8-3C_7T_8-C_8T_8+T_8^2$, $2C_4T_2+C_5T_2+C_7T_2+C_8T_2-T_2T_8$, $C_1T_3+C_3T_3+C_4T_3+C_7T_3-T_3T_8$;
- the relations of the form $F(A)$ are: T_1T_8, T_2T_6, T_6T_8, $T_1T_2T_6T_8$, T_4T_8, T_1T_4, T_4T_6, $T_1T_2T_3T_8$, $T_2T_3T_6T_8$, $T_1T_2T_3T_6$, $T_1T_3T_6T_8$, $T_2T_3T_4T_8$, $T_1T_2T_3T_4$, $T_1T_3T_4T_8$, $T_2T_3T_4T_6$, $T_3T_4T_6T_8$, $T_1T_3T_4T_6$, T_7T_8, T_1T_7, T_6T_7, T_4T_7, $T_2T_3T_7T_8$, $T_1T_2T_3T_7$, $T_1T_3T_7T_8$, $T_2T_3T_6T_7$, $T_3T_6T_7T_8$, $T_1T_3T_6T_7$, $T_2T_3T_4T_7$, $T_3T_4T_7T_8$, $T_1T_3T_4T_7$, $T_3T_4T_6T_7$, T_2T_5, T_5T_8, $T_1T_2T_5T_8$, T_5T_6, T_4T_5, $T_2T_3T_5T_8$, $T_1T_2T_3T_5$, $T_1T_3T_5T_8$, $T_2T_3T_5T_6$, $T_3T_5T_6T_8$, $T_1T_3T_5T_6$, $T_2T_3T_4T_5$, $T_3T_4T_5T_8$, $T_1T_3T_4T_5$, $T_3T_4T_5T_6$, T_5T_7.

Therefore, adding the relations that come from the cohomology of $X_{\mathcal{A}}$ (see Theorem 5.5), we conclude that

$$H^*(Y_{\mathcal{A}}(\mathcal{G}); \mathbb{Z}) \simeq \mathbb{Z}[C_1, \dots, C_{10}, T_1, \dots, T_8]/I$$

where I is the ideal generated by T_8^3, $C_6^2-2T_8^2$, C_6C_7, $C_7^2-2T_8^2$, C_6C_8, C_7C_8, $C_8^2-T_8^2$, $C_6C_9+T_8^2$, C_7C_9, $C_8C_9+T_8^2$, $C_9^2-2T_8^2$, C_6C_{10}, C_7C_{10}, C_8C_{10}, C_9C_{10}, $C_{10}^2-3T_8^2$, C_6T_1, C_7T_1, C_8T_1, $C_9T_1+T_8^2$, $C_{10}T_1$, $T_1^2-2T_8^2$, C_6T_2, C_7T_2, $C_8T_2+T_8^2$, C_9T_2, $C_{10}T_2$, $T_1T_2+T_8^2$, $T_2^2-3T_8^2$, C_6T_3, C_7T_3, C_8T_3, C_9T_3, $C_{10}T_3$, T_1T_3, T_2T_3, $T_3^2-5T_8^2$, C_6T_4, C_7T_4, C_8T_4, C_9T_4, $C_{10}T_4$, T_1T_4, $T_2T_4+T_8^2$, $T_3T_4+T_8^2$, $T_4^2-T_8^2$, C_6T_5, C_7T_5, C_8T_5, C_9T_5, $C_{10}T_5$, $T_1T_5+T_8^2$, T_2T_5, $T_3T_5+T_8^2$, T_4T_5, $T_5^2-T_8^2$, C_6T_6,

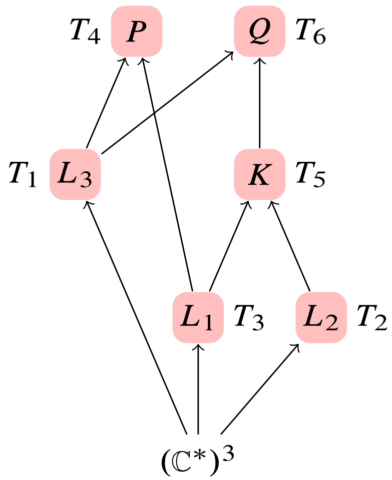

Fig. 3 Poset of the connected components of the intersections of layers for the arrangement in Example 5.9, with elements of the well-connected building set $\mathcal{G}$ highlighted, together with the associated polynomial variable T_G

$C_7T_6, C_8T_6, C_9T_6, C_{10}T_6, T_1T_6+T_8^2, T_2T_6, T_3T_6+T_8^2, T_4T_6, T_5T_6, T_6^2-T_8^2, C_6T_7, C_7T_7, C_8T_7, C_9T_7, C_{10}T_7, T_1T_7, T_2T_7+T_8^2, T_3T_7+T_8^2, T_4T_7, T_5T_7, T_6T_7, T_7^2-T_8^2, C_6T_8, C_7T_8, C_8T_8, C_9T_8, C_{10}T_8, T_1T_8, T_2T_8+T_8^2, T_3T_8+T_8^2, T_4T_8, T_5T_8, T_6T_8, T_7T_8, 3C_1-C_6-2C_9+C_{10}-3T_1+2T_2+2T_4-3T_5-3T_6+2T_7+2T_8, 3C_2+2C_6+C_9+C_{10}-T_2-T_4-T_7-T_8, 2C_3+C_6+C_7+2C_9+3T_1-2T_2-T_3-3T_4+2T_5+2T_6-3T_7-3T_8, 2C_4-C_6+C_7-2C_8-2C_9-T_1+T_3+T_4+T_7+T_8, C_5+C_6+3C_8+2C_9-T_3-T_4-T_5-T_6-T_7-T_8$. In particular, the computation of a monomial normal basis for $\mathbb{Z}[C_1,\ldots,C_{10},T_1,\ldots,T_8]/I$ gives us the Betti numbers: such a basis is

$$\left(T_8^2,\ T_8,\ T_7,\ T_6,\ T_5,\ T_4,\ T_3,\ T_2,\ T_1,\ C_{10},\ C_9,\ C_8,\ C_7,\ C_6,\ 1\right),$$

so we have $H^0(Y_{\mathcal{A}}(\mathcal{G});\mathbb{Z})\simeq\mathbb{Z}$, $H^2(Y_{\mathcal{A}}(\mathcal{G});\mathbb{Z})\simeq\mathbb{Z}^{13}$ and $H^4(Y_{\mathcal{A}}(\mathcal{G});\mathbb{Z})\simeq\mathbb{Z}$, as one could also check by hand, since in this case we are blowing up points in a 2-dimensional toric variety.

Example 5.9 Consider the (non-divisorial) arrangement in $(\mathbb{C}^*)^3$ with three layers

$$\begin{aligned}
L_1 &= \{(x,y,z)\in(\mathbb{C}^*)^3 \mid x=1\},\\
L_2 &= \{(x,y,z)\in(\mathbb{C}^*)^3 \mid y=1\},\\
L_3 &= \{(x,y,z)\in(\mathbb{C}^*)^3 \mid z=1,\ xy^2=1\}.
\end{aligned}$$

The poset of the connected components of the intersections of the layers, as well as the well-connected building set $\mathcal{G}$ that we use to build $Y_{\mathcal{A}}(\mathcal{G})$ (in this case the minimal one coincides with the set of all the elements of the poset except the whole torus), is represented in Fig. 3.

The fan Δ as computed with the algorithm of Sect. 3 has 10 rays:[7]

$$
\begin{aligned}
&C_1 \to (-1,1,0), && C_2 \to (-2,1,0),\\
&C_3 \to (1,0,0), && C_4 \to (2,-1,0),\\
&C_5 \to (0,0,1), && C_6 \to (1,-1,0),\\
&C_7 \to (0,0,-1), && C_8 \to (0,-1,0),\\
&C_9 \to (-1,0,0), && C_{10} \to (0,1,0).
\end{aligned}
$$

The ideal I of $\mathbb{Z}[C_1,\dots,C_{10},T_1,\dots,T_6]$ of the relations for $H^*(Y_{\mathcal{A}}(\mathcal{G});\mathbb{Z})$ has a Gröbner basis DEGREVLEX with 68 polynomials, which are the following: T_6^4, $T_1^3-4T_6^3$, $T_4^3-T_6^3$, $T_5^3-2T_6^3$, $T_1^2T_6+T_6^3$, $T_5^2T_6+T_6^3$, $T_1T_6^2$, $T_5T_6^2$, $C_6^2-2T_5^2-4T_5T_6-2T_6^2$, $C_6C_7+2C_7C_8+C_7C_9+T_1^2+T_4^2+4T_1T_6+T_6^2$, C_7^2, $C_6C_8+T_5^2+2T_5T_6+T_6^2$, $C_8^2-T_5^2-2T_5T_6-T_6^2$, C_6C_9, $C_8C_9+T_5^2+2T_5T_6+T_6^2$, $C_9^2-2T_5^2-4T_5T_6-2T_6^2$, C_6C_{10}, C_8C_{10}, C_9C_{10}, $C_{10}^2-T_5^2-2T_5T_6-T_6^2$, C_6T_1, C_7T_1, C_8T_1, C_9T_1, $C_{10}T_1$, C_6T_2, C_8T_2, $C_9T_2+T_5^2+2T_5T_6+T_6^2$, $C_{10}T_2$, T_1T_2, $T_2^2-T_5^2-2T_5T_6-T_6^2$, C_6T_3, $C_8T_3+T_5^2+2T_5T_6+T_6^2$, C_9T_3, $C_{10}T_3+T_5^2+2T_5T_6+T_6^2$, T_1T_3, T_2T_3, $T_3^2-T_4^2-T_5^2-2T_5T_6-T_6^2$, C_6T_4, C_7T_4, C_8T_4, C_9T_4, $C_{10}T_4$, $T_1T_4-T_1T_6$, T_2T_4, $T_3T_4+T_4^2$, C_6T_5, $C_7T_5-T_5T_6$, C_8T_5, C_9T_5, $C_{10}T_5$, T_1T_5, $T_2T_5+T_5^2+T_5T_6$, $T_3T_5+T_5^2+T_5T_6$, T_4T_5, C_6T_6, C_7T_6, C_8T_6, C_9T_6, $C_{10}T_6$, $T_2T_6+T_5T_6+T_6^2$, $T_3T_6+T_5T_6+T_6^2$, T_4T_6, $C_1-C_9+2C_{10}-2T_2+T_3+T_4-T_5-T_6$, $C_2+C_9-C_{10}+T_2-T_3-T_4$, $C_3-C_6-2C_8+2T_2-T_3-T_4+T_5+T_6$, $C_4+C_6+C_8-T_2-T_5-T_6$, C_5-C_7.

Finally the computation of a monomial basis gives us the following Betti numbers:

i	0	2	4	6
$\mathrm{rk}(H^i(Y_{\mathcal{A}}(\mathcal{G});\mathbb{Z}))$	1	11	11	1

6 More on well-connected building sets

In Sect. 5 we outlined how the cohomology of the wonderful model has been computed in [9]. In this section we are going to show under which circumstances a similar strategy can be, at least in theory, used in a more general situation.

Let us start from a well-connected building set $\mathcal{G}$ in a variety X and blow up a minimal element $F \in \mathcal{G}$. Consider the set $t(\mathcal{G}) := \{t(G) \mid G \in \mathcal{G}\}$ in $\mathrm{Bl}_F X$. Our two main observations are

- The set $t(\mathcal{G}) := \{t(G) \mid G \in \mathcal{G}\}$ in $\mathrm{Bl}_F X$ is a building set which is still a well-connected building set (Theorem 6.5).

[7] On the left of each ray there is the corresponding polynomial variable C_r.

- If for any subset $\{G_{j_1}, \dots, G_{j_s}\}$ of $\mathcal{G}$ for which $G_{j_1} \cap \dots \cap G_{j_s}$ is connected, the map

$$H^*(X) \to H^*(G_{j_1} \cap \dots \cap G_{j_s}),$$

induced by the inclusion, is surjective, then for every subset $\{t(G_{i_1}), \dots, t(G_{i_s})\}$ of $t(\mathcal{G})$ for which $t(G_{i_1}) \cap \dots \cap t(G_{i_s})$ is connected, the map

$$H^*(\mathrm{Bl}_F X) \to H^*(t(G_{i_1}) \cap \dots \cap t(G_{i_s})),$$

induced by the inclusion, is surjective (see Theorem 6.8).

We remark that the concrete application to the computation of the cohomology ring of compact models for toric arrangements as it is described above, also depends on some further specific properties of toric arrangements and toric varieties (see [9, Section 6]). Anyway, we think that it is useful to point out these general properties of a well-connected building set in a variety X and of the model constructed starting from it.

Remark 6.1 A description of the cohomology of a wonderful model of subvarieties as a module was already found by Li [21].

We begin by recalling some useful lemmas for building sets that are not necessarily well-connected. In the following, let $\mathcal{A}$ be an arrangement of subvarieties in a connected, non-singular variety X and let $\mathcal{G}$ be a building set for $\mathcal{A}$. Moreover, let F be a minimal element of $\mathcal{G}$ with respect to inclusion; for any subvariety D of X, we denote by $t(D)$ the dominant transform of D in the blowup $\mathrm{Bl}_F X$.

Lemma 6.2 ([9, Lemma 3.1]; see [22, Lemma 2.9]) *Let A, B, A_1, A_2, B_1, B_2 be non-singular subvarieties of X.*

(i) *Suppose that $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$, and suppose that $A_1 \cap A_2 = F$ cleanly. Then $t(A_1) \cap t(A_2) = \varnothing$.*
(ii) *Suppose that A_1 and A_2 intersect cleanly and that $F \subsetneq A_1 \cap A_2$. Then $t(A_1) \cap t(A_2) = t(A_1 \cap A_2)$.*
(iii) *Suppose that B_1 and B_2 intersect cleanly and that F is transversal to B_1, B_2 and $B_1 \cap B_2$. Then $t(B_1) \cap t(B_2) = t(B_1 \cap B_2)$.*
(iv) *Suppose that A is transversal to B, F is transversal to B and $F \subset A$. Then $t(A) \cap t(B) = t(A \cap B)$.*

Lemma 6.3 ([9, Lemma 3.2]) *Let $\mathcal{G}$ be a building set for $\mathcal{A}$ and let U be an open set belonging to an open cover $\mathcal{U}$ as in Definition 4.3. Consider two subsets $\{H_1, \dots, H_k\}$ and $\{G_1, \dots, G_s\}$ of $\mathcal{G}$, and let $H' := H_1 \cap \dots \cap H_k$ and $G' := G_1 \cap \dots \cap G_s$. Finally let $H^0 := U \cap H'$ and $G^0 := U \cap G'$. If $H^0 \neq \varnothing$ and $H^0 \subset G^0$, then the connected component of H' containing H^0 is contained in the connected component of G' containing G^0.*

Lemma 6.4 ([9, Corollary 3.4]; see [22, Lemma 2.6]) *Let $\mathcal{G}$ be a building set for $\mathcal{A}$ and let U be an open set belonging to an open cover $\mathcal{U}$ as in Definition 4.3. In order* *to keep the notation simple, in the following every object is considered restricted to U but we avoid repeating the symbol. Let F be a minimal element in $\mathcal{G}$ with respect to inclusion.*

(i) *For any $G \in \mathcal{G}$, either G contains F, or $F \cap G = \varnothing$, or $F \cap G$ is transversal.*
(ii) *Let $G_1, \ldots, G_k$ be elements of $\mathcal{G}$ such that $F \not\subseteq G_i$ for every $i = 1, \ldots, k$. Suppose that $G_1 \cap \cdots \cap G_k \neq \varnothing$. Then the intersection between $G_1 \cap \cdots \cap G_k$ and F is either empty or transversal.*

Theorem 6.5 *Let $\mathcal{G}$ be a* well-connected *building set of X, and let F be a minimal element in $\mathcal{G}$. Then $t(\mathcal{G}) := \{t(G) \mid G \in \mathcal{G}\}$ is a well-connected building set of subvarieties in $Y := \mathrm{Bl}_F X$.*

Proof We know from Proposition 4.6 that $t(\mathcal{G})$ is still a building set in Y; we still need to check that it is well-connected, i.e., that for any subset $\{t(G_1), \ldots, t(G_k)\}$ of $t(\mathcal{G})$, whenever the intersection $t(G_1) \cap \cdots \cap t(G_k)$ is non-empty, it is either connected or it is the union of connected components each belonging to $t(\mathcal{G})$.

We notice that the statement is trivial for $k = 1$, so from now on we assume $k \geqslant 2$. We also assume that $t(G_1) \cap \cdots \cap t(G_k)$ is non-empty and that the elements $G_1, \ldots, G_k$ are pairwise non-comparable. We analyze all the possible relations of inclusion between F and $G_1, \ldots, G_k$:

1. F is properly contained in all the G_i's;
2. F is not contained in any G_i;
3. F is properly contained in some of the G_i's, say in the first s, and not contained in the others;
4. F belongs to $\{G_1, \ldots, G_k\}$, say $F = G_1$.

We proceed by induction on $k \geqslant 2$ in each case. It is useful to introduce notation: for $I \subseteq \mathcal{G}$ let G be the intersection of the elements of I; we denote by $t^*(G)$ the union of the transforms of the connected components of G which are not equal to F.

Case 1: $F \subsetneq G_i$ for every $i = 1, \ldots, k$. We start with the case $k = 2$. If $G_1 \cap G_2 = F$, by Lemma 6.2 (i) it would follow that $t(G_1) \cap t(G_2) = \varnothing$ but this is excluded by assumption; therefore we have $F \subsetneq G_1 \cap G_2$.

Now, since $\mathcal{G}$ is well-connected, we know that $G_1 \cap G_2$ is either connected or it is the union of connected components $A_1, \ldots, A_r$ belonging to $\mathcal{G}$. In the first case, $t(G_1) \cap t(G_2) = t(G_1 \cap G_2)$ by Lemma 6.2 (ii), so it is connected. In the second case, one of the connected components $A_1, \ldots, A_r$, say A_1, contains F, while the others have empty intersection with F. We have again two cases:

- if $F \subsetneq A_1$, we may apply again Lemma 6.2 (ii) and deduce that $t(G_1) \cap t(G_2) = t(G_1 \cap G_2)$: we conclude that $t(G_1) \cap t(G_2)$ is the disjoint union of the connected components $t(A_1), \ldots, t(A_r)$, which belong to $t(\mathcal{G})$;
- if $F = A_1$, then $t(G_1) \cap t(G_2)$ is the disjoint union of the connected components $t(A_2), \ldots, t(A_r)$, which belong to $t(\mathcal{G})$.

Therefore, in all the previous cases we have

$$t(G_1) \cap t(G_2) = t^*(G_1 \cap G_2)$$

using the notation introduced before.

The case $k \geqslant 3$ follows by a straightforward induction, whose essential ingredient is the $k = 2$ step illustrated above.

Case 2: $F \nsubseteq G_i$ for every $i = 1, \dots, k$. In this case $F \cap G_i$ is either empty or transversal by Lemma 6.4 (i), and it is connected by well-connectedness (see Remark 5.4). Suppose $k = 2$. If $(G_1 \cap G_2) \cap F = \varnothing$, it follows immediately that $t(G_1) \cap t(G_2) = t(G_1 \cap G_2)$. If $(G_1 \cap G_2) \cap F$ is not empty, then it is connected by well-connectedness (otherwise each connected component of $G_1 \cap G_2 \cap F$ would belong to $\mathcal{G}$, against the minimality of F) and it is transversal by Lemma 6.4 (ii). Then, using Lemma 6.2 (iii), we conclude again that $t(G_1) \cap t(G_2) = t(G_1 \cap G_2)$.

In either case, since $\mathcal{G}$ is well-connected we know that $G_1 \cap G_2$ is either connected or the union of connected components $A_1, \dots, A_r$ each belonging to $\mathcal{G}$. This implies that $t(G_1) \cap t(G_2) = t(G_1 \cap G_2)$ is either connected[8] or it is the disjoint union of the connected components $t(A_1), \dots, t(A_r)$ each belonging to $t(\mathcal{G})$.

Suppose now $k \geqslant 3$. We will prove by induction that

$$t(G_1) \cap \cdots \cap t(G_k) = t(G_1 \cap \cdots \cap G_k),$$

from which we can conclude immediately by well-connectedness. The base step is $k = 2$, which we did before. Let us denote $G \coloneqq G_1 \cap \cdots \cap G_{k-1}$. By inductive hypothesis

$$t(G_1) \cap \cdots \cap t(G_{k-1}) = t(G_1 \cap \cdots \cap G_{k-1}) = t(G),$$

so we need to prove that $t(G) \cap t(G_k) = t(G \cap G_k)$. The situation is similar to the case $k = 2$:

- if $G \cap G_k \cap F$ is empty, we immediately obtain $t(G) \cap t(G_k) = t(G \cap G_k)$;
- if $G \cap G_k \cap F$ is not empty, then we notice that the intersections $G_k \cap F$, $G \cap F$ and $(G \cap G_k) \cap F$ are transversal and connected, again by Corollary 6.4 and well-connectedness, and we conclude $t(G) \cap t(G_k) = t(G \cap G_k)$ by Lemma 6.2 (iii).

Case 3: there exists $1 \leqslant s < k$ such that $F \subsetneq G_i$ for every $i = 1, \dots, s$ and $F \nsubseteq G_i$ for every $i = s+1, \dots, k$. Let

$$U_1 \coloneqq \bigcap_{i=1}^{s} G_i \quad \text{and} \quad U_2 \coloneqq \bigcap_{i=s+1}^{k} G_i.$$

By the previous cases we know that

$$t(G_1) \cap \cdots \cap t(G_k) = t^*(U_1) \cap t(U_2).$$

From the analysis made before we know that one of the following cases occurs:

(a) U_1 is connected (and properly contains F, otherwise $t^*(U_1)$ would be empty);

[8] It is the blowup of the connected subvariety $G_1 \cap G_2$ along $G_1 \cap G_2 \cap F$.

(b) U_1 is the disjoint union of some connected components $A_1, \dots, A_r$ belonging to $\mathcal{G}$, with $F \subsetneq A_1$, while the other components have empty intersection with F;
(c) U_1 is the disjoint union of some connected components $A_1, \dots, A_r$ belonging to $\mathcal{G}$, with $F = A_1$.

For case (a), we have $t^*(U_1) = t(U_1)$ and $U_1 \cap U_2 \cap F = U_2 \cap F$. If $U_2 \cap F = \varnothing$ we deduce $t(U_1) \cap t(U_2) = t(U_1 \cap U_2)$ and conclude by well-connectedness of $\mathcal{G}$. Suppose now that $U_2 \cap F \neq \varnothing$: then, using again the same reasoning than the previous cases, it is connected by well-connectedness. Now notice that, if C is the unique connected component of $U_1 \cap U_2$ that intersects F, then the intersection of U_1 and U_2 in C is transversal. This follows again from Lemma 6.4 (ii): we know that U_2 is tranversal to F, but $F \subset U_1$, therefore for every point $y \in F$ the set of linear equations that describe the tangent space $\mathrm{T}_y(F)$ includes the equations that describe $\mathrm{T}_y(U_1)$. Since $U_2 \cap F \subseteq C$ and C is smooth, this implies our claim.

All the properties listed above allow us to apply Lemma 6.2 (iv) and deduce that $t(U_1) \cap t(U_2) = t(U_1 \cap U_2)$.

Case (b) can be reduced to case (a): also in this case $t^*(U_1) = t(U_1)$, therefore $t(U_1) \cap t(U_2)$ is the disjoint union of the $t(A_i) \cap t(U_2)$. For $i = 1$, the same argument of case (i) allows us to conclude $t(A_1) \cap t(U_2) = t(A_1 \cap U_2)$; for $i \geqslant 2$, we already know that $t(A_i) \cap t(U_2) = t(A_i \cap U_2)$. In conclusion, we have $t(U_1) \cap t(U_2) = t(U_1 \cap U_2)$ and the claim follows again from well-connectedness.

In case (c) we have that

$$t(G_1) \cap \cdots \cap t(G_k) = t^*(U_1) \cap t(U_2) = \bigcup_{i=2}^{r} (t(A_i) \cap t(U_2)) = \bigcup_{i=2}^{r} t(A_i \cap U_2)$$

and everything follows from well-connectedness.

Case 4: $F = G_1$. Notice that in this case $F \nsubseteq G_i$ for all $i = 2, \dots, k$ because we suppose that the G_i's are pairwise non-comparable. We put $U := G_2 \cap \cdots \cap G_k$; from the preceding points we have that

$$t(F) \cap t(G_2) \cap \cdots \cap t(G_k) = t(F) \cap t(U).$$

Now, if $F \cap U$ is empty, then it immediately follows that $t(F) \cap t(U) = \varnothing$, which cannot happen under our assumptions. If $F \cap U$ is not empty, then it is connected, by the well-connectedness of $\mathcal{G}$ and the minimality of F. Only one connected component of U contains $F \cap U$ (the others, if any, have empty intersection with F), and $t(F) \cap t(U)$ coincides with the exceptional divisor of the blowup of this component along $F \cap U$, therefore it is connected. ◻

We recall here Keel's Theorem (Theorem 1 of the Appendix of [20]). Let Y be a smooth variety and suppose that X is a regularly embedded subvariety of codimension d (we denote by $i : X \to Y$ the injection). Let $\mathrm{Bl}_X Y$ be the blowup of Y along X, with the usual map $\pi : \mathrm{Bl}_X Y \to Y$, and let E be the exceptional divisor.

Theorem 6.6 (Keel) *Suppose that the map $i^*\colon H^*(Y) \to H^*(X)$ is surjective with kernel J and let T be an indeterminate. Then $H^*(\mathrm{Bl}_X Y)$ is isomorphic to*

$$H^*(Y)[T]/(J\cdot T, P(t))$$

where $P(t) \in H^*(Y)[T]$ is any polynomial whose constant term is $[X]$ and whose restriction to $H^*(X)$ is the Chern polynomial of the normal bundle $N = \mathrm{N}_X Y$, that is to say

$$i^*(P(t)) = T^d + c_1(N)T^{d-1} + \cdots + c_d(N).$$

This isomorphism is induced by $\pi^*\colon H^*(Y) \to H^*(\mathrm{Bl}_X Y)$ and by sending $-T$ to $[E]$.

In the sequel we will denote by $J(Y, X)$ and by $P(Y, X)(t)$ respectively the kernel J and the polynomial $P(t)$ that appear in the statement of Theorem 6.6.

Let X be a smooth variety and let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be a well-connected building set of subvarieties of X whose elements are ordered in a way that refines inclusion, i.e., if $G_i \subsetneq G_j$ then $i < j$. For every $k = 1, \ldots, m$, let us denote by $\mathcal{G}_k$ the set $\mathcal{G}_k := \{G_1, \ldots, G_k\}$. We extend this notation by putting $\mathcal{G}_0 := \varnothing$ and $Y(X; \mathcal{G}_0) := X$.

Lemma 6.7 ([9, Proposition 4.3]) *For every $k = 1, \ldots, m$, the set $\mathcal{G}_k$ is a building set in the sense of Definition 4.5 and it is well-connected.*

Notice that by Proposition 4.6 the set

$$\{t^1(G_1), \ldots, t^1(G_m)\}$$

is a building set in $Y(X; \mathcal{G}_1)$, where $t^1(G_i)$ denotes the transform of G_i in this blowup. Moreover, since $t^1(G_1)$ is a divisor, hence maximal with respect to inclusion, we have that also

$$\mathcal{G}^1 := \{t^1(G_2), \ldots, t^1(G_m)\}$$

is a building set in $Y(X; \mathcal{G}_1)$, well-connected by Theorem 6.5, and $t^1(G_2)$ is a minimal element.

By a simple induction, we obtain that

$$\mathcal{G}^s := \{t^s(G_{s+1}), \ldots, t^s(G_m)\}$$

is a well-connected building set in $Y(X; \mathcal{G}_s)$ (again, $t^s(G_i)$ denotes the transform of G_i in this blowup) and $t^s(G_{s+1})$ is a minimal element. We also observe that, by Theorem 4.7, for every $s = 1, \ldots, m$ we have

$$Y(X; \mathcal{G}) = Y(Y(X; \mathcal{G}_s); \mathcal{G}^s).$$

Moreover we extend this notation to $\mathcal{G}^0 := \mathcal{G}$.

The following theorem, which is about the surjectivity of some restriction maps in cohomology, shows that in this general setting one can start an inductive process which involves the repeated application of Keel's Theorem 6.6; by dealing with this process in the particular case of compact models of toric arrangements we managed to compute their cohomology rings (Theorem 5.6 illustrated above); this motivates our interest in the following more general statement.

Theorem 6.8 *Assume that, for any subset* $\{G_{j_1}, \ldots, G_{j_s}\}$ *of* $\mathcal{G}$ *for which* $G_{j_1} \cap \cdots \cap G_{j_s}$ *is connected, the map*

$$H^*(X) \to H^*(G_{j_1} \cap \cdots \cap G_{j_s}),$$

induced by the inclusion, is surjective. Then for all $k = 0, \ldots, m-1$ *and every subset* $\{t^k(G_{i_1}), \ldots, t^k(G_{i_s})\}$ *of* $\mathcal{G}^k$ *for which* $t^k(G_{i_1}) \cap \cdots \cap t^k(G_{i_s})$ *is connected, the map*

$$H^*(Y(X; \mathcal{G}_k)) \to H^*\big(t^k(G_{i_1}) \cap \cdots \cap t^k(G_{i_s})\big),$$

induced by the inclusion, is surjective.

Proof We prove this by induction on k. For $k = 0$ the statement is true by assumption. Now suppose that $k \geqslant 1$ and that the theorem is true for $k-1$.

Let the indices $i_1, \ldots, i_s$ be such that

$$k < i_1 < i_2 < \cdots < i_s \leqslant m$$

and $t^k(G_{i_1}) \cap \cdots \cap t^k(G_{i_s})$ is connected. We have to prove that the map

$$\phi\colon H^*(Y(X; \mathcal{G}_k)) \to H^*\big(t^k(G_{i_1}) \cap \cdots \cap t^k(G_{i_s})\big)$$

is surjective. Let us put for brevity

$$\begin{aligned} G &:= t^{k-1}(G_{i_1}) \cap \cdots \cap t^{k-1}(G_{i_s}), \\ G' &:= t^k(G_{i_1}) \cap \cdots \cap t^k(G_{i_s}). \end{aligned}$$

Recall that $Y(X; \mathcal{G}_k)$ is obtained by blowing up $Y(X; \mathcal{G}_{k-1})$ along $t^{k-1}(G_k)$. We have two cases.

Case 1: $G \cap t^{k-1}(G_k)$ *is empty.* In this case we have $G' = t(G)$ which is isomorphic to G. In this case the surjectivity follows by the inductive hypothesis, since G' is connected and so is G.

Case 2: $G \cap t^{k-1}(G_k)$ *is not empty.* By inductive hypothesis we know that

$$H^*(Y(X; \mathcal{G}_{k-1})) \to H^*(t^{k-1}(G_k))$$

is surjective, so we can apply Keel's Theorem 6.6 and deduce that

$$H^*(Y(X; \mathcal{G}_k)) \simeq \frac{H^*(Y(X; \mathcal{G}_{k-1}))[T]}{J(Y(X; \mathcal{G}_{k-1}), t^{k-1}(G_k)) \cdot T,\ P(Y(X, \mathcal{G}_{k-1}), t^{k-1}(G_k))(t)} \tag{3}$$

where the indeterminate T is evaluated at $-[t^k(G_k)] \in H^*(Y(X; \mathcal{G}_k))$.

Now notice that $G \cap t^{k-1}(G_k)$ is connected (by Remark 5.4: $t^{k-1}(G_k)$ is minimal in the well-connected building set $\mathcal{G}^{k-1}$), therefore the map

$$H^*(Y(X; \mathcal{G}_{k-1})) \to H^*(G \cap t^{k-1}(G_k))$$

is surjective again by inductive hypothesis. Moreover, notice that in this case $G' = t(G)$: checking the proof of Theorem 6.5, one realizes that $G' \neq t(G)$ can happen only if $t^{k-1}(G_k)$ is contained properly in all the $t^{k-1}(G_{i_j})$, for $j = 1, \ldots, s$, and it is equal to one of the connected components of G; but

- G is connected, because G' and $G \cap t^{k-1}(G_k)$ are;
- $t^{k-1}(G_k) \neq G$, otherwise G' would be empty by Lemma 6.2 (i).

The surjectivity of the map $H^*(G) \to H^*(G \cap t^{k-1}(G_k))$ induced by the inclusion allows us to apply again Keel's Theorem 6.6 to $t(G) = G'$, which is the blowup of the variety G along $G \cap t^{k-1}(G_k)$. We obtain

$$H^*(t(G)) \simeq \frac{H^*(G)[T]}{J(G, t^{k-1}(G_k) \cap G) \cdot T,\ P(G, t^{k-1}(G_k) \cap G)(t)} \tag{4}$$

where the indeterminate T is evaluated at $-[t^k(G_k) \cap t(G)] \in H^*(t(G))$.

We can now compare the isomorphisms (3) and (4): looking at the right members of these equations, we conclude that the surjectivity of the map $\phi\colon H^*(Y(X; \mathcal{G}_k)) \to H^*(t(G))$ follows from the surjectivity of $H^*(Y(X; \mathcal{G}_{k-1})) \to H^*(G)$, which is provided by the inductive hypothesis, and from $\phi([t^k(G_k)]) = [t^k(G_k) \cap t(G)]$. $\square$

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