Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Corrado de Concini

# A necessary condition for the existence of vector bundles on the infinite dimensional quaternionic projective space

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **59** (1975), n.1-2, p. 1–7.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_1-2_1_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

# RENDICONTI

DELLE SEDUTE

# DELLA ACCADEMIA NAZIONALE DEI LINCEI

**Classe di Scienze fisiche, matematiche e naturali**

*Ferie 1975 (Luglio–Agosto)*

(Ogni Nota porta a pie' di pagina la data di arrivo o di presentazione)


## SEZIONE I

**(Matematica, meccanica, astronomia, geodesia e geofisica)**

**Matematica.** — *A necessary condition for the existence of vector bundles on the infinite dimensional quaternionic projective space.* Nota (*) di CORRADO DE CONCINI, presentata dal Corrisp. E. MARTINELLI.


RIASSUNTO. — Si stabilisce che le radici del polinomio totale di Chern dei fibrati vettoriali complessi sullo spazio proiettivo quaternionale di dimensione infinita sono quadrati di interi. Si trattano inoltre i casi di fibrati reali e quaternionali.


### INTRODUCTION

The purpose of this paper is to give a restriction on the Chern classes of the complex vector bundles on the quaternionic projective space of infinite dimension, $HP^{\infty}$.

In order to obtain such a result we use the integrality theorems of Atiyah-Hirzebruch [1] plus some elementary number theory.

In § 1 we recall the results of Atiyah–Hirzebruch.

In § 2 we prove the main theorem, and as a consequence we also give conditions for the quaternionic and real case.

I finally wish to express my thanks to G. Lusztig for helpful conversations.

### § 1. PRELIMINARIES

Let M be a compact differentiable manifold of dimension $m$, without boundary, with Pontrjagin classes $p_i \in H^{4i}(M, Z)$. Let $\{\hat{A}_k(p_1, \cdots, p_k)\}$ be the multiplicative sequence with characteristic power series $Q(z) = \dfrac{\frac{1}{2}\sqrt{z}}{\sinh \frac{1}{2}\sqrt{z}}$ (for the definition and the properties of the multiplicative sequences, see [3, ch. 1]).

(*) Pervenuta all'Accademia l'11 agosto 1975.

We have the following [1]:

THEOREM 1. *Let* d *be an element of* $H^2(M, Z)$, *whose reduction mod 2 is the Whitney class* $w_2(M)$, *and* $\eta$ *a continous complex vector bundle over* M. *Then, if* $ch\ \eta \in H^*(M, Q)$ *denotes the Chern character of* $\eta$,

$$\hat{A}(M, \tfrac{1}{2}d, \eta) = k^m \left[ e^{\frac{1}{2}d} \cdot \mathrm{ch}\,\eta \sum_{j=0}^{\infty} \hat{A}_j(p_1, \cdots, p_j) \right]$$

*where* $k^m$ *denotes evaluation on the fundamental class of* M, *is an integer.*

*This theorem has the following:*

COROLLARY. *Mantaining the notation of Theorem 2*, if $\dim X \equiv 0 \mod 8$ *and if* $\eta$ *is a quaternionic vector bundle, then* $\hat{A}(M, \frac{1}{2}d, \eta)$ *is an even integer.*

*If* $\dim X \equiv 4 \mod 8$ *and if* $\eta$ *is a real vector bundle, then* $\hat{A}(M, \frac{1}{2}d, \eta)$ *is an even integer.*

*Proof.* The second case is proved in [1] using periodicity of real K-theory. The first case can be proved in exactly the same fashion by substituting quaternionic to real K-theory.

q.e.d.

## § 2. BUNDLES OVER $HP^n$

Let us consider now $HP^n$, the quaternionic projective space of (quaternionic) dimension $n$, and let $HP^\infty = \lim_{n\to\infty} HP^n$.

It is well known that $H^*(HP^\infty, Z) \cong Z[u]$ where $u$ can be taken equal to $-c_2$, $c_2$ being the second Chern class of the universal quaternionic line bundle over $HP^\infty$. It is also well known that $H^*(HP^n, Z) \cong Z[u]/(u^{n+1})$.

For the Pontrjagin classes we have by Hirzebruch's computation [2], that, if $P(HP^n)$ denotes the total Pontrjagin class of $HP^n$.

$$P(HP^n) = (1+u)^{2n+2}(1+4u)^{-1}.$$

Now consider a continuous complex vector bundle over $HP^\infty$ and let

$$c(\eta) = 1 + \gamma_1 u + \cdots + \gamma_{[s/2]} u^{[s/2]},$$

where $\gamma_1, \cdots, \gamma_{[s/2]}$ are integers, be its total Chern class.

By the definition of $\mathrm{ch}(\eta)$ [3], and by the fact that the odd Chern classes are clearly zero, it follows easily that

$$\mathrm{ch}(\eta) = \sum_{i=1}^{[s/2]} e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}} \qquad \text{with}$$

$$\gamma_i u^i = \sigma_i(-d_1, \cdots, -d_{[s/2]})\, u^i, \qquad \text{where} \quad \sigma_i(-d_1, \cdots, -d_{[s/2]})$$

denotes the *i-th* elementary symmetric function in $d_1, \cdots, d_{[s/2]}$. It is clear that $d_1, \cdots, d_{[s/2]}$, being roots of the polynomial with integral coefficients $x^{[s/2]} + \gamma_1 x^{[s/2]-1} + \cdots + \gamma_{[s/2]}$, are complex numbers.

We are now ready to prove the following

THEOREM 2. *Let $\eta$ be a complex vector bundle of dimension $s$ on $\mathrm{HP}^\infty$, then if $d_1, \cdots, d_{[s/2]}$ are the complex numbers associated to $\eta$ in the above manner, $d_1, \cdots, d_{[s/2]}$ must be square integers or zero.*

*Proof.* We shall divide the proof in two sections. In the first section we shall obtain a combinatorial formula for the $d_i$'s using Theorem 1, and in the second one we shall derive the result from the formula using elementary number theory.

## SECTION 1

Let $\eta$ be a complex vector bundle of dimension $s$ over $\mathrm{HP}^n$, and let $\mathrm{ch}(\eta) = \sum_{i=1}^{[s/2]} e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}$ be its Chern character. Now by the expression for $\mathrm{P}(\mathrm{HP}^n)$ recalled above together with the Cauchy integral formula and the fact that $\mathrm{H}^2(\mathrm{HP}^n, \mathrm{Z}) = 0$, we get:

$$\hat{\mathrm{A}}(\mathrm{HP}^n, 0, \eta) = \mathscr{H}^{4n}\left[\left(\sum_{i=1}^{[s/2]} e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}\right)\left(\frac{\frac{1}{2}\sqrt{u}}{\sinh\frac{1}{2}\sqrt{u}}\right)^{2n+2}\left(\frac{\sinh\sqrt{u}}{\sqrt{u}}\right)\right] =$$

$$= \frac{1}{2\pi(-1)^{\frac{1}{2}}}\int\left(\sum_{i=1}^{[s/2]} e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}\right)\left(\frac{\frac{1}{2}\sqrt{u}}{\sinh\frac{1}{2}\sqrt{u}}\right)^{2n+2}\left(\frac{\sinh\sqrt{u}}{\sqrt{u}}\right)\frac{1}{\sqrt{u}^{2n+1}}\, d\sqrt{u}$$

where the integral is taken over a small circle centered at the origin in the $\sqrt{u}$ plane; and where we can integrate with respect to the variable $\sqrt{u}$ since the series expands in the same way with respect to the variables $\sqrt{u}$ and $u$.

Now by applying some easy transformations we get:

$$\frac{1}{2\pi(-1)^{\frac{1}{2}}}\int\left(\sum_{i=1}^{[s/2]} e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}\right)\left(\frac{\frac{1}{2}\sqrt{u}}{\sinh\frac{1}{2}\sqrt{u}}\right)^{2n+2}\left(\frac{\sinh\sqrt{u}}{\sqrt{u}\,(\sqrt{u})^{2n+1}}\right) d\sqrt{u} =$$

$$= \frac{1}{2\pi(-1)^{\frac{1}{2}}}\sum_{i=1}^{[s/2]}\int\left(e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}\right)\left(\frac{\frac{1}{2}\sqrt{u}}{\sinh\frac{1}{2}\sqrt{u}}\right)\frac{\sinh\sqrt{u}}{\sqrt{u}\,(\sqrt{u})^{2n+1}}\, d\sqrt{u} =$$

$$= \frac{1}{2\pi(-1)^{\frac{1}{2}}}\sum_{i=1}^{[s/2]}\int\frac{1}{2^{2n+2}}\left(e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}\right)\frac{\sinh\sqrt{u}}{\sinh\frac{1}{2}\sqrt{u}^{\,2n+2}}\, d\sqrt{u} =$$

$$= \frac{1}{2}\,\frac{1}{2\pi(-1)^{\frac{1}{2}}}\sum_{i=1}^{[s/2]}\int\left(e^{\sqrt{d_i u}} + e^{-\sqrt{d_i u}}\right)\frac{e^{-(n+1)\sqrt{u}}}{(1-e^{-\sqrt{u}})^{2n+2}}\left(e^{\sqrt{u}} - e^{-\sqrt{u}}\right) d\sqrt{u} =$$

$$= \frac{1}{2}\,\frac{1}{2x(-1)^{\frac{1}{2}}}\sum_{i=1}^{[s/2]}\int\left[\frac{e^{(\sqrt{d_i}+1-(n+1))\sqrt{u}} + e^{(-\sqrt{d_i}+1-(n+1))\sqrt{u}}}{(1-e^{-\sqrt{u}})^{2n+2}}\, d\sqrt{u} + \right.$$

$$\left. + \frac{e^{(\sqrt{d_i}-1-(n+1))\sqrt{u}} + e^{(-\sqrt{d_i}-1-(n+1))\sqrt{u}}}{(1-e^{-\sqrt{u}})^{2n+2}}\, d\sqrt{u}\right].$$

Substituting $1 - e^{-1} = t$ we get:

$$\hat{A}(\mathrm{HP}^n, 0, \eta) = \tfrac{1}{2} \sum_{i=1}^{[s/2]} \left[ \binom{2n+1+\sqrt{d_i}+1-(n+1)}{2n+1} - \binom{2n+1+\sqrt{d_i}-1-(n+1)}{2n+1} + \right.$$

$$\left. + \binom{2n+1-\sqrt{d_i}+1-(n+1)}{2n+1} - \binom{2n+1-\sqrt{d_i}-1-(n+1)}{2n+1} \right] =$$

$$= \tfrac{1}{2} \sum_{i=1}^{[s/2]} \left[ \frac{(1+\sqrt{d_i}+1-(n+1)) \cdots (2n+1+\sqrt{d_i}+1-(n+1))}{(2n+1)!} - \right.$$

$$- \frac{(1+\sqrt{d_i}-1-(n+1)) \cdots (2n+1+\sqrt{d_i}-1-(n+1))}{(2n+1)!} +$$

$$+ \frac{(1-\sqrt{d_i}+1-(n+1)) \cdots (2n+1-\sqrt{d_i}+1-(n+1))}{(2n+1)!} -$$

$$\left. - \frac{(1-\sqrt{d_i}-1-(n+1) \cdots (2n+1-\sqrt{d_i}-1-(n+1))}{(2n+1)!} \right] =$$

$$= \sum_{i=1}^{[s/2]} \frac{\sqrt{d_i}\,(d_i-1)(d_i-4) \cdots (d_i-(n-1)^2\,(d_i+(n+1)\sqrt{d_i}+}{(2n+1)!}$$

$$\frac{+ n\sqrt{d_i} + n(n+1) - d_i + (n+1)\sqrt{d_i} + n\sqrt{d_i} - n(n+1))}{} =$$

$$= \sum_{i=1}^{[s/2]} \frac{2\,d_i\,(d_i-1)(d_i-4) \cdots (d_i-(n-1)^2)}{(2n)!} = \mathrm{H}_n(d_1 \cdots d_{[s/2]}).$$

Now by Theorem 1 we have that $\mathrm{H}_n(d_1 \cdots d_{[s/2]})$ must be an integer.

When $\eta$ is a complex vector bundle over $\mathrm{HP}^\infty$, by restricting to $\mathrm{HP}^n$ for each $n$ we clearly get the condition that $\mathrm{H}_n(d_1 \cdots d_{[s/2]})$ is equal to an integer for each $n$, $d_1 \cdots d_{[s/2]}$ being the $d_i$'s relative to $\eta$.

## Section 2

In this section we shall deduce the Theorem from the conditions obtained in section 1.

First of all let us consider the sequence $\{\mathrm{H}_n(d)\}_{n=1}^{\infty}$ where $d$ is any complex number; we have $\lim_{n\to\infty} \mathrm{H}_n(d) = 0$.

In fact we have

$$|\mathrm{H}_n(d)| = \left| \frac{2d(d-1)(d-4)\cdots(d-(n-1)^2)}{(2n)!} \right| \le \frac{2|d|(|d|+1)\cdots(|d|+(n-1)^2)}{(2n)!};$$

now if we take an integer $t$ such that $t^2 \ge d$ we get

$$|\mathrm{H}_n(d)| \le \frac{2t^2(t+1)^2\cdots(t+n-1)^2}{(2n)!} \le \frac{2((t+n-1)!)^2}{(2n)!}$$

and such a sequence is easily seen to converge to zero. Now since $H_n(d_1, \cdots, d_\nu) = \sum_{i=1}^{\nu} H_n(d_i)$ we also clearly have that $\lim_{n\to\infty} H_n(d_1, \cdots, d_\nu) = 0$ for arbitrary complex numbers, $d_1, \cdots, d_\nu$; and if we require $H_n(d_1, \cdots, d_\nu)$ to be an integer for each $n$ this clearly implies that there exists $n_0$ such that $H_n(d_1, \cdots, d_\nu) = 0$ for $n \geq n_0$.

Now if we put $(d_i - )(d_i - 4) \cdots (d_i - n_0^2) = \alpha_i$ we get

LEMMA. *If* $H_n(d_1, \cdots, d) = 0$ *for* $n \geq n_0$ *then we have* $\sum_{i=1}^{\nu} d_i^s \alpha_i = 0$ *for each* $s \geq 0$.

*Proof*. Since $\sum_{i=1}^{\nu} \alpha_i = \frac{(2n)!}{2} H_{n_0}(d_1, \cdots, d_\nu) = 0$ we get that the lemma is true for $s = 0$. Suppose it is true for each $t \leq s - 1$. Then

$$0 = \frac{(2(n_0 + s))!}{2} H_{n_0+s}(d_1, \cdots, d_\nu) =$$

$$= \sum_{i=1}^{\nu} \alpha_i (d_i - n_0^2)(d_1 - (n_0 + 1)^2) \cdots (d_i - (n_0 + s - 1)^2) = \sum_{i=1}^{\nu} d_i^s \alpha_i + \alpha_i \cdot f(d_i)$$

where $f(d_i)$ is a polynomial of degree $s - 1$ in $d_i$. Now the inductive hypothesis implies that $\sum_{i=1}^{\nu} \alpha_i f(d_i) = 0$ so we get $\sum_{i=1}^{\nu} d_i^s \alpha_i = 0$.

q.e.d.

We come back now to the proof of our theorem.

Suppose that $q \leq \gamma$ is the number of different $d_i$'s; then, by identifying the $d_i$'s which are equal we get, by the above lemma, that $\sum_{j=1}^{q} \lambda_j d_j^s \alpha_j = 0$ for each $s$ where $(\lambda_1, \cdots, \lambda_q)$ is a partition of $\nu$ and where $d_j \neq d_{j'}$ if $j \neq j'$. Then if we consider the conditions $\left\{\sum_{j=1}^{q} \lambda_j d_j^s \alpha_j = 0\right\}_{s=0}^{q-1}$ as a linear system in the variables $\{\alpha_1, \cdots, \alpha_q\}$, we get that the determinant associated to this system is just $\lambda_1, \cdots, \lambda_q$ times the Vandermonde determinant

$$\begin{vmatrix} 1 & \cdots & 1 \\ d_1 & \cdots & d_q \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ d_1^{q-1} & \cdots & d_q^{q-1} \end{vmatrix}$$

which is different from zero since we suppose $d_j \neq d_j$, if $j \neq j'$. This implies that the system $\sum_{j=1}^{q} \lambda_j d_j^s \alpha_j = 0$ has solutions if and only if each of the $\alpha_i$'s is equal to zero but, since $\alpha_i = d_i (d_i - 1)(d_i - 4) \cdots (d_i - (n_0 - 1)^2)$ this is verified if and only if $d_1 = 0$ or $d_i$ is equal to a square integer.

Even if what we have done so far is sufficient, we want to show that using our methods our condition is the best we can find. In fact, let $t$ be an integer and let us consider $H_n(t^2)$. We have

$$H_n(t^2) = \frac{2t^2(t^2-1)(t^2-4)\cdots(t^2-(n-1)^2)}{(2n)!} =$$
$$= \frac{2t^2(t-1)(t+1)\cdots(t-(n-1))(t+(n-1))}{(2n)!}$$

which is zero if $0 \le t \le n-1$ and which is equal to $2\binom{t+n}{2n} - \binom{t+n-1}{2n-1}$ if $t \ge n$. This clearly implies that $H_n(t^2)$ is an integer and since $H_n(d_1, \cdots, d_\nu) = \sum_{i=1}^{\nu} H_n(d_i)$ we have that if each of the $d_i$'s is a square integer then $H_n(d_1, \cdots, d_\nu)$ is an integer for each $n$.

q.e.d.

Using the corollary to Theorem 1 we get:

THEOREM 3. *Given a quaternionic vector bundle $\xi$ of dimension $m$ on* $HP^\infty$, *let $\eta$ be the complex vector bundle of dimension $2m$ associated to $\xi$. We have, using the same notations as above, that if one of $d_i$'s of $\eta$, say $d_h$, is even, then there must exist an even number of $k$'s including $h$ such that $d_h = d_k$.*

*Proof.* It follows from the corollary to Theorem 1 that $H_n(d_1, \cdots, d_m)$ must be an even integer whenever $n$ is even since $HP^{2n}$ has real dimension $\equiv 0 \pmod 8$.

We have already seen that $d_1, \cdots, d_m$ must be square integers.

Now let us note that $H_n(n^2) = 2\binom{2n}{2n} - \binom{2n-1}{2n-1} = 1$, and $H_n(t^2) = 0$ if $t < n$. Now suppose $d_h = e_h^2$ to be the largest even square such that there is an odd number of $k$'s, including $h$, such that $d_h = d_k$. Then since $H_n(d_1 \cdots d_n) = \sum_{i=1}^{m} H_n(d_i)$ we have that $H_{e_h}(d_1 \cdots d_m) = a + g$, where $a = \sum_{j=1}^{\Upsilon} H_{e_h}(d_{ij})$ with $d_{ij}$ running over the $d_i$'s which are larger than that $d_h$, and where $t$ is the (odd) number of $d_k$'s which are equat to $d_h$. Now, since we suppose that if $d_e > d_h$ and $d_e$ is even there is an even number of $d_i$'s including $d_e$ which are equal to $d_e$, we have that the contribution given by those $d_i$'s to a is even. For an odd $d > e_h$ we have that

$$H_{e_h}(d^2) = 2\binom{d+e_h}{2e_h} - \binom{d+e_h-1}{2e_h-1} = 2\binom{d+e_h}{2e_h} - \frac{d+e_h-1}{2e_h-1}\binom{d+e_h-2}{2e_h-2}.$$

Since $2e_h - 1$ is odd and $d + e_h - 1$ is even it follows that $H_{e_h}(d^2)$ is even, and so a must be even. Since $t$ is odd this implies that $H_{e_h}(d_1 \cdots d_\nu)$ is odd thus contradicting the Corollary to Theorem 1 and proving the Theorem.

q.e.d.

*Remark.* 1) As an immediate consequence we get that if a self map of $S^4$ can be extended to a self map of $HP^\infty$, such a map must have odd square degree. In fact a self map of $HP^\infty$ corresponds to a quaternionic projective line bundle and Theorem 2 can be applied.

2) Since $Sp(1) \cong SU(2)$ and $H^2(HP^\infty, Z) = 0$ it follows that any complex plane bundle on $HP^\infty$ lifts to a quaternionic line bundle so we must have that the second Chern class of a complex plane bundle on $HP^\infty$ must be equal to $-d\,u$ where $d$ is an odd square.

Finally we note that for real bundles on $HP^\infty$ we get:

THEOREM 4. *Given a real vector bundle $\xi$ of dimension $m$ on* $HP^\infty$, *let $\eta$ be the complex vector bundle of dimension $m$ associated to $\xi$. We have, using the same notations as above, that if one of the $d_i$'s* of $\eta$, *say $d_h$, is odd, then there must exist an even number of $k$'s, including $h$, such that $d_k = d_h$.*

*Proof.* The proof is identical to that of Theorem 3, using, instead of the quaternionic projective spaces of even (quaternionic) dimension, those of odd (quaternionic) dimension.

q.e.d.

## REFERENCES

[1] M. F. ATIYAH and F. HIRZEBRUCH (1959) – *Riemann-Roch theorems for differentiable manifolds*, « Bull. Am. Math. Soc. », *65*, 276–281.

[2] F. HIRZEBRUCH (1954) – *Uber die quaternionalen projektiven Raume*, « S.-B. Math., Nat. Kl. Bayer. Akad. Wiss. ».

[3] F. HIRZEBRUCH (1956) – *Neue topologische Methoden in der algebraischen Geometrie*, Springer, Berlin.