HOPF ALGEBRAS WITH TRACE AND CLEBSCH-GORDAN COEFFICIENTS

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Abstract. In this lecture I shall report on some joint work with Procesi, Reshetikhin and Rosso [1].

1. Recollections and the problem

Let $G$ be a simply connected semisimple algebraic group over $\mathbb{C}$. $\mathfrak{g} = \text{Lie } G$.

$T \subset G$ a maximal torus, $\mathfrak{t} = \text{Lie } T$, $R$ the root system, $R_+$ positive roots, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the simple roots, $A = (a_{i,j})$, the Cartan matrix, $D = \text{diag}(d_1, \ldots, d_n)$ the diagonal matrix with $DA$ symmetric. $B \supset T \subset B^-$ the corresponding Borel subgroup and its opposite.

Fix $\ell$ odd (and prime with $3$ if there are $G_2$ components). Let $\varepsilon$ be a primitive $\ell$ root of $1$.

The quantized enveloping algebra is the $\mathbb{C}$-algebra $U_\varepsilon(\mathfrak{g})$ generated by elements

$\{E_1, \ldots E_n\}, \{F_1, \ldots F_n\}, K_\lambda, \lambda \in X^*(T)$ the character group of $T$, with relations:

$K_\lambda E_i K_\mu^{-1} = \varepsilon^{\langle \alpha_i, \lambda \rangle} E_i$

$K_\lambda F_i K_\mu^{-1} = \varepsilon^{-\langle \alpha_i, \lambda \rangle} F_i$

$E_i F_j = F_j E_i \quad i \neq j, \quad [E_i, F_i] = \frac{K_\alpha_i - K^{-1}_\alpha_i}{\varepsilon_i - \varepsilon_i^{-1}} \varepsilon_i = \varepsilon^{d_i}$

$\sum \left[ \begin{array}{c} 1 - a_{ij} \kappa \\ k \\ \varepsilon_i \end{array} \right] E_i^{1-k-a_{ij}} F_j E_i^k = 0 , \quad i \neq j$

$\sum \left[ \begin{array}{c} 1 - a_{ij} \kappa \\ k \\ \varepsilon_i \end{array} \right] F_i^{1-k-a_{ij}} F_j F_i^k = 0 , \quad i \neq j$

$\left[ \begin{array}{c} m \\ h \varepsilon \end{array} \right] = \frac{[m]_{\varepsilon}^!}{[m-h]_{\varepsilon} [h]_{\varepsilon}^!}, \quad [h]_{\varepsilon}^! = [h]_{\varepsilon} \cdots [2]_{\varepsilon} [1]_{\varepsilon}$ and $[h]_{\varepsilon} = \frac{\varepsilon^h - \varepsilon^{-h}}{\varepsilon - \varepsilon^{-1}}$. $\Delta$ acts on generators by

$\Delta K_\mu = K_\mu \otimes K_\mu, \quad \Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i,$

$\Delta F_i = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i$

The main peculiarity of being at roots of unity is that $U_\varepsilon(\mathfrak{g})$ has a very big center $Z$.

Indeed, $U_\varepsilon(\mathfrak{g})$ is a finite $Z$-module. $Z$ is the coordinate ring of an algebraic variety $X$ of dimension equal to $\dim \mathfrak{g}$. So:

1) Every irreducible $U_\varepsilon(\mathfrak{g})$-module is finite dimensional.
2) If $\hat{U}_\gamma(g)$ denotes the set of irreducible $U_\gamma(g)$-modules, taking central characters we get a surjective map

$$\gamma : \hat{U}_\gamma(g) \to X.$$ 

Problem 1. Given two irreducible $U_\gamma(g)$-modules, $V, W$, describe the composition factors of $V \otimes W$ and their multiplicities.

Consider the sub Hopf algebra $U_\gamma(b) \subset U_\gamma(g)$ generated by the $K_\lambda$ and $E_1, \ldots, E_n$.

Problem 2. Given an irreducible $U_\gamma(g)$-module, $V$ “decompose” its restriction to $U_\gamma(b)$.

Notice that as Problem 2, also Problem 1 is a problem of branching. In fact it consists of decomposing the restriction of the irreducible $U_\gamma(g) \otimes U_\gamma(g)$-module $V \otimes W$ to the subalgebra $\Delta(U_\gamma(g))$. To explain our results recall that $Z$ contains a sub Hopf algebra $Z_0$ such that $U_\gamma(g)$ is a free $Z_0$ module of rank $\ell \dim g$. $Z_0$ is the coordinate ring of an algebraic group $H$ (the Poisson dual of $G$) which is the kernel of the homomorphism $p : B \times B \to T$ defined by $p((b, b')) = p_+(b)p_-(b')^{-1}$ where $p_{\pm}$ are the quotients modulo the unipotent radicals. The inclusion $Z_0 \subset Z$ gives a map $\pi : X \to H$.

**Theorem 1.** There is a non empty Zariski open set $\mathcal{V} \subset H \times H$ such that if $V$ and $W$ are two irreducible representations of $U_\gamma(g)$, such that if $(h_1, h_2) = (\pi\gamma(V), \pi\gamma(W)) \in \mathcal{V}$, as a $U_\gamma(g)$ module,

$$V \otimes W = \bigoplus_{(\pi, h_1, h_2) \in \mathcal{V}} U^m$$

with $m = \ell |R_+| - \tau g$.

As for problem 2, $Z_0^+ = Z_0 \cap U_\gamma(b)$ is a sub Hopf algebra of $U_\gamma(b)$. $Z_0^+$ is the coordinate ring of $B^-$ and the inclusion $Z_0^+ \subset Z_0$ induces the homomorphism $\mu : H \to B^-$ given by the projection on the second factor. Again, taking central character we get a map

$$\gamma' : \hat{U}_\gamma(b) \to B^-.$$ 

**Theorem 2.** There is a non empty Zariski open set $\mathcal{V} \subset H$ such that if $V$ is an irreducible representations of $U_\gamma(g)$, such that if $h = \pi\gamma(V) \in \mathcal{V}$, as a $U_\gamma(b)$ module,

$$V = \bigoplus_{(\gamma', h) \in \mathcal{V}} U^m$$

with $m = \ell |R_+| - s/2$ where $s$ is the number of orbits of $-w_0$ on the set of simple roots.

The main content of these theorems is that in both cases, at least generically, the multiplicities are uniformly distributed among the various irreducible components.

### 2. Cayley-Hamilton algebras.

The results we have just stated follow from a careful study of the centers of the algebras involved, by applying the theory of Cayley-Hamilton algebras which we now briefly recall.

A trace on a algebra $R$ over a field $k$ (here $\kappa = \mathbb{C}$) is a 1-ary operation $t : R \to R$ such that

1. $t$ is $k$-linear. (This implies that $t(R)$ is an $k$-subspace)
2. $t(ab) = b t(a), \ \forall a, b \in R$. (This implies that $t(R)$ is in the center of $R$),
Theorem 4. \( F_n \) by traces of monomials modulo the ideal generated by evaluating the \( F_s \) sider the algebra \( C \) Hamilton identity.

Definition 3. An algebra with trace \( r \) coefficients in the trace algebra and define briefly an \( n \)-CH algebra) if

\( (M_n, tr) \) have the following extra properties:

1. \( tr(1) = n \)
2. Given \( A \in M_n \), set \( p^{(n)}_a(x) = det(xI - A) \), the characteristic polynomial, then we get the Cayley-Hamilton identity

\[
p^{(n)}_a(a) = 0
\]

Remark that \( p^{(n)}_a(x) \) has coefficients which are universal polynomials in \( tr(a), \ldots, tr(a^n) \).

For example for \( n = 2 \),

\[
p^{(2)}_a(x) = x^2 - tr(a)x + \frac{1}{2}(tr(a)^2 - tr(a^2)).
\]

Thus given \( (R, t) \) we can consider \( p^{(n)}_a(x) \) for every \( a \in R \) as a polynomial with coefficients in the trace algebra and define

**Definition 3.** An algebra with trace \( (A, t) \) is a \( n \)-Cayley Hamilton algebra (or briefly an \( n \)-CH algebra) if

1. \( t(ab) = t(ba), \forall a, b \in R \). (This implies that \( t \) is 0 on the space of commutators \([R, R]\).)
2. \( t(t(a)b) = t(ab) \), \( \forall a, b \in R \). (This implies that \( t(R) \) is an subalgebra, called the trace algebra, and that \( t \) is also \( t(R) \)-linear).

The most famous example is that of the algebra \( M_n \) of \( n \times n \) matrices with the usual trace \( tr \).

Algebras with trace form a category. Given \( (R_1, t_1) \), \( (R_2, t_2) \) a morphism \( \phi \) from \( (R_1, t_1) \) to \( (R_2, t_2) \) is an algebra homomorphism such that \( \phi t_1(a) = t_2 \phi(a) \) for each \( x \in R_1 \).

A morphism \( \phi : (R, t) \rightarrow (M_n, tr) \) is called a representation.

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The following results describes the free \( n \)-CH algebra \( F_n(x_1, \ldots, x_m) \). Consider the algebra \( C_n(\xi_1, \ldots, \xi_m) \) of \( GL(n) \)-equivariant polynomial maps of matrices \( M_n(k)^m \rightarrow M_n(k), \xi(A_1, \ldots, A_m) = A_i \), with trace \( t(\psi) = tr \cdot \psi \). We have a universal map \( F_n(x_1, \ldots, x_m) \rightarrow C_n(\xi_1, \ldots, \xi_m) \)

**Theorem 4.** (Procesi)

1. The universal map

\[
F_n(x_1, \ldots, x_m) \rightarrow C_n(\xi_1, \ldots, \xi_m)
\]
is an isomorphism.

(2) The trace algebra $T_n(\xi_1, \ldots, \xi_m)$ of the algebra $C_n(\xi_1, \ldots, \xi_m)$ is the algebra of invariants of $m$-tuples of matrices and, if $m > 1$, it is the center of $C_n(\xi_1, \ldots, \xi_m)$.

(3) $C_n(\xi_1, \ldots, \xi_m)$ is a finite $T_n(\xi_1, \ldots, \xi_m)$ module.

The above theorem tells us in particular that if $(R, t)$ is a $n$-CH algebra which is finitely generated also its trace algebra $A$ is finitely generated and $R$ is a finite $A$-module.

It follows that every irreducible representation of $R$ (as an algebra) is finite dimensional.

$A$ is a ring of functions on an affine algebraic variety $X$, so if $\hat{R}$ denotes the set of irreducible representations of $R$, we get, by taking central characters, a surjective map

$$\gamma : \hat{R} \to X.$$ 

The points of $X$ itself are a "moduli space" for a class of representations. Indeed

**Theorem 5.** Given a point $x \in X$, there is a unique semisimple, trace compatible, representation $W_x$ of the form

$$W_x = \oplus_{V \in \gamma^{-1}(x)} V^\oplus m_V,$$

and each semisimple, trace compatible, representation of $R$ is of this form.

In fact there is a Zariski open set (possibly empty), called the unramified locus, in $X$ such that if $m_x \subset A$ is the maximal ideal in $A$, $R/m_x R$ is semisimple and so $W_x$ is the unique representation "lying over $x$" which is compatible with the trace.

If $A$ is the center of $R$ and $R$ satisfies suitable conditions, for example it is a domain, there is a non empty Zariski open set for which $W_x$ is irreducible.

In general however, the determination of $\gamma^{-1}(x)$ and of the multiplicities $m_V$ is a very hard problem.

If we consider $(R, h t)$, with $h$ a positive integer, then the semisimple, trace compatible, representation corresponding to $x \in X$ with respect to the new trace is just $W_x^\oplus h$.

4. Reduced trace

In our example of quantized enveloping algebra we have no trace, so if we want to use the results about the CH algebras we need to find ways to canonically introduce a trace.

Here is a way of doing this under some assumptions.

A prime algebra $R$ is an algebra in which the product of two non-zero ideals is non-zero. Let $R$ be a prime algebra, assume that $A$ is a subalgebra of the center and $R$ is an $A$–module of finite type. Then:

1) $A$ is an integral domain.

2) $R$ is a torsion free module.

So if $F$ is the field of fractions of $A$, then $R \subset R \otimes_A F$ and $S := R \otimes_A F$ is a finite dimensional simple algebra isomorphic to $M_k(D)$ with $D$ a finite dimensional division ring.

Denote by $Z$ the center of $S$ and of $D$. Then $\dim_Z D = h^2.$
If \( Z \) is an algebraic closure of \( \mathbb{Z} \), \( M_k(D) \otimes Z Z = M_{hk}(Z) \). Setting \( p := [Z : F] = \dim_F Z \) we also have
\[
S \otimes_F Z = M_k(D) \otimes_F Z = M_{hk}(Z)^{\otimes p}
\]
The number \( n := hkp \) is called degree of \( S \) over \( F \). Consider the \( F \)-linear operator \( a^L : S \rightarrow S, a^L(b) := ab \). We define the reduced trace
\[
t_{S/F}(a) = \frac{1}{h^k} tr(a^L).
\]
This theorem tells us that, provided \( A \) is integrally closed, we have constructed a nice trace:

**Theorem 6.** If \( S = R \otimes_A F \) as before and \( A \) is integrally closed we have that the reduced trace \( t_{S/F} \) maps \( R \) into \( A \), so we will denote by \( t_{R/A} \) the induced trace. The algebras \( R \) and \( S \) with their reduced trace are \( n \)-Cayley Hamilton algebras.

Even more important for us is the fact that, in the situation above, the nice trace is essentially unique. Indeed,

**Theorem 7.** Under the same circumstances of Theorem 6, if \( \tau : R \rightarrow A \) is any trace for which \( R \) is an \( m \)-Cayley Hamilton algebra then there is a positive integer \( r \) for which:
\[
m = rn, \quad \tau = rt_{R/A}
\]
Notice that in particular, this implies that if \( x \) is a point in \( X \) and \( W_x \) is the unique semisimple representation above \( x \), compatible with \( t_{R/A} \) then \( W_x^{\otimes r} \) is the unique semisimple representation above \( x \) compatible with \( \tau \).

5. **Compatibility**

Since our final goal is to analyze the restriction of certain representation of an algebra \( R \) to a subalgebra, what I am going to explain now is how the reduced trace of \( R \) relative to a subring \( A \) restricts to a subalgebra.

Assume that we have two prime algebras \( R_1 \subset R_2 \) with two central subrings \( A_1 \subset R_1, A_2 \subset R_2 \) such that \( R_1 \) is a finite \( A_1 \) module, \( R_2 \) is a finite \( A_2 \) module and \( A_1, A_2 \) are integrally closed. We have the reduced traces \( t_{R_1/A_1} \) and \( t_{R_2/A_2} \) for which both algebras are Cayley-Hamilton. We need a criterion ensuring that the restriction of \( t_{R_2/A_2} \) to \( R_1 \) is a multiple of \( t_{R_1/A_1} \) (in this case we say that they are compatible). The criterion is the following,

**Theorem 8.** Denote by \( Z_1 \) the center of \( R_1 \). If the algebra \( Z_1 \otimes_{A_1} A_2 \) is a domain then there is a positive integer \( r \) such that
\[
r t_{R_1/A_1} = t_{R_2/A_2} \quad \text{on} \quad R_1.
\]

6. **Back to quantum algebras**

In this final section I am going to explain how the theory of Cayley Hamilton algebras can be applied to the quantum group situation described in Section 1.

As a first step we need to show that quantized enveloping algebras have reduced traces. This is not hard. One knows that both \( U_\varepsilon(g) \) and \( U_\varepsilon(b) \) are domains and
hence prime and that their centers $Z$, $Z'$ are integrally closed. $Z_0$ and $Z_0^+$ are clearly integrally closed, so we have reduced traces

$$t_{U,(g)}/Z_0 \quad t_{U,(g)}/Z' \quad t_{U,(g)}/Z \cdot Z^2 \quad t_{U,(b)}/Z_0^+.$$

Next we have to compare them.

We start with Problem 1 and the inclusion $\Delta(U,(g)) \subset U,(g) \otimes U,(g)$. Unfortunately we have that $\Delta(Z)$ is not contained in $Z \otimes Z$. On the other hand the fact that $Z_0$ is a sub Hopf algebra means that $\Delta(Z_0) \subset Z_0 \otimes Z_0 \subset Z \otimes Z$. So, we may try to compare $t_{U,(g)}/Z^2 \otimes t_{U,(g))/Z_0}$ with $t_{U,(g)}/Z_0$.

In order to use Theorem 8 we need to show that the variety $\tilde{X}$ defined as the fiber product

$$\begin{array}{ccc}
\tilde{X} & \overset{\phi}{\longrightarrow} & X \\
q \downarrow & & \pi \downarrow \\
X \times X & \overset{\pi \times \pi}{\longrightarrow} & H \times H \overset{m}{\longrightarrow} H.
\end{array}$$

with $m : H \times H \to H$ the group multiplication, is irreducible.

Let us recall that $X$ is obtained as the fiber product

$$\begin{array}{ccc}
X & \overset{\rho}{\longrightarrow} & T/W \\
\pi \downarrow & & \ell \downarrow \\
H & \overset{\rho}{\longrightarrow} & G \overset{\sigma}{\longrightarrow} G//G \cong T/W.
\end{array}$$

Where for $(h,k) \in H$, $\rho((h,k)) = h k^{-1}$, $\sigma$ is the quotient modulo the adjoint action, $W$ is the Weyl group and the map $\ell$ is the map induced by the homomorphism $t \to t^\ell$ of the torus $T$ on $W$-invariants.

Putting the two diagrams together we obtain $\tilde{X}$ as the fiber product

$$\begin{array}{ccc}
\tilde{X} & \overset{\rho \circ}{\longrightarrow} & T/W \\
q \downarrow & & \ell \downarrow \\
X \times X & \overset{\sigma \circ \rho \circ \pi \times \pi}{\longrightarrow} & T/W.
\end{array}$$

From this one proves:

Step 1. $\tilde{X}$ is a complete intersection over $X \times X$ which is Cohen-Macaulay so it is Cohen-Macaulay.

Step 2. $\tilde{X}$ is non singular in codimension 1 so in particular it is reduced and normal.

Step 3. $\tilde{X}$ is connected. This is the hardest step and one needs to use a variation of Steinberg section for the quotient $G \to G//G$ ($G//G$ denoted the categorical quotient defined using invariant polynomial functions).

Using the irreducibility of $\tilde{X}$ we deduce that the restriction to $\Delta(U,(g))$ of $t_{U,(g)}/Z^2 \otimes Z^2$ is $m t_{U,(g)}/Z_0$ with $m = \ell | H_{a-1} - k q$ and Theorem 1 follows.

The open set $V \subset H \times H$ consists of the set of pairs $((h_1, k_1), (h_2, k_2))$ with $h_1 k_1^{-1}$, $h_2 k_2^{-1}$, $h_1 h_2 k_2^{-1} k_1^{-1}$ all regular semisimple.

The solution to Problem 2 is quite similar with the extra difficulty that one has to compute the center of $U,(b)$ first since this has not been determined before. We have seen that $Z_0^+$ is the coordinate ring of the group $B^-$. For a dominant weight $\lambda$, consider the irreducible $G$-module $V_\lambda$ of highest weight $\lambda$. Take a highest weight vector $v \in V_\lambda$, a highest weight vector $\phi$ and a lowest vector $\psi$ in $V_\lambda^*$. Define the
function \( f_\lambda(b) = \phi(bv)\psi(bv) \) for \( b \in B^\ast \). These functions generate a subring \( Z \) in \( Z_0^+ \), isomorphic to the polynomial ring \( \mathbb{C}[f_\omega_1, \ldots, f_\omega_r] \), \( \omega_i \) the fundamental weights.

Embed \( Z \) into the polynomial ring

\[
Z' = \mathbb{C}[f_{\omega_1}^{1/\ell}, \ldots, f_{\omega_r}^{1/\ell}].
\]

\( Z \) is the ring of invariants in \( Z' \) under an obvious action of the group \( \mathbb{Z}/\ell\mathbb{Z} \) which we can consider as the group \( T_\ell \) of \( \ell \)-torsion points in \( T \). Set \( \Gamma = \{ t \in T_\ell \mid t^{v_0} = t^{-1} \} \) and \( Z_1^+ = (Z')^{\Gamma} \).

**Theorem 9.** The center of the algebra \( U_\varepsilon(b) \) is isomorphic to \( Z^+ = Z_0^+ \otimes_\mathbb{Z} Z_1^+ \).

Once we know the center of \( U_\varepsilon(b) \), in order to solve Problem 2 we need to compare \( t_{U_\varepsilon(g)/Z} \) and \( t_{U_\varepsilon(b)/Z_0^+} \). In order to use Theorem 8 we need to show that the fiber product \( \tilde{Y} \)

\[
\begin{array}{ccc}
\tilde{Y} & \overset{\pi}{\longrightarrow} & Y \\
q \downarrow & & \downarrow \ell \\
X & \overset{p}{\longrightarrow} & B^-
\end{array}
\]

with \( Y = \text{Spec} \ Z^+ \), is a irreducible variety. This is done as in the previous case. After these facts have been established the proof of Theorem 2 follows rather easily.

### 7. Final remarks and problems

The technique explained can be used in other cases. One example is the so called quantized function algebra \( \mathbb{F}_\varepsilon[G] \). In this case the role played by \( H \) is played by \( G \) and we have a projection \( \pi : F_\varepsilon[G] \to G \).

**Theorem 10.** There is a non empty Zariski open set \( V \subset G \times G \) such that if \( V \) and \( W \) are two irreducible representations of \( \mathbb{F}_\varepsilon[G] \), such that if \( (g_1, g_2) = (\pi(V), \pi(W)) \in V \), as a \( \mathbb{F}_\varepsilon[G] \) module,

\[
V \otimes W = \bigoplus_{U \in (\pi)^{-1}(g_1,g_2)} U^m
\]

with \( m = \ell^{|R_+| - \text{rk}_G} \).

### References