EXERCISES FOR THE MINI–COURSE ON LDS

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1. Notation

We write $LDP(\mu_n, r_n, I)$ if the sequence of probability measures (μ_n) on some Polish space $\mathcal{X}$ satisfies a LDP with speed r_n and rate function I (note that we do not assume a priori I to be *good*).

2. Varadhan's Lemma

We have seen Varadhan's lemma in its simplest form [2, Thm. III.13]: suppose $LDP(\mu_n, r_n, I)$ holds and let $f : \mathcal{X} \to \mathbb{R}$ be a continuous function bounded from above, then

$$\lim_{n\to\infty} \frac{1}{r_n} \log \int e^{r_n f} d\mu_n = \sup_{x\in\mathcal{X}} \{f(x) - I(x)\} . \tag{1}$$

In [3, Chp. 3] the following extention is proved:

Varadhan's lemma. *Suppose $LDP(\mu_n, r_n, I)$ holds and let $f : \mathcal{X} \to [-\infty, \infty]$ be a continuous function such that*

$$\lim_{b\to+\infty} \overline{\lim_{n\to\infty}} \frac{1}{r_n} \log \int_{f\geq b} e^{r_n f} d\mu_n = -\infty . \tag{2}$$

Then it holds

$$\lim_{n\to\infty} \frac{1}{r_n} \log \int e^{r_n f} d\mu_n = \sup \{f(x) - I(x) \, : \, f(x) \wedge I(x) < \infty\} . \tag{3}$$

Trivially, if f is real, continuous and bounded from above the above extended version implies Varadhan's lemma as in [2, Thm. III.13].

Exercise 1. *Prove that* (2) *is satisfied if there exists $\alpha > 1$ such that*

$$\sup_n \left(\int e^{\alpha r_n f} d\mu_n \right)^{1/r_n} < \infty . \tag{4}$$

Exercise 2. *Consider the following game. Let S_n be the number of tails you get when you flip n times a fair coin. Determine the asymptotics of* $\mathbb{E}[3^{S_n}]$.

Comments: One could try as follows: by the law of large numbers with high probability $S_n/n \approx \frac{1}{2}$, and therefore one would expect that $\mathbb{E}[3^{S_n}] = 3^{(n/2)(1+o(1))}$, i.e. $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[3^{S_n}] = (\log 3)/2$. But solve the exercise!
Suggestion: By Cramer's theorem we know the LDP for the law μ_n of S_n/n. Write $\mathbb{E}[3^{S_n}]$ as $\mu_n(e^{nf})$ for some f. Use the extended Varadhan's lemma and Exercise 1.

Exercise 3. *(taken from* [3]*). Let Z_k be i.i.d. with $\mathbb{P}(Z_k = 1) = \mathbb{P}(Z_k = 2) = 1/2$. Let $W_n := Z_1 \cdot Z_2 \cdots Z_n$.*

(a) *Show that $\mathbb{E}[W_n] = (3/2)^n$*
(b) *Show that, given $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left((\sqrt{2}-\varepsilon)^n < W_n < (\sqrt{2}+\varepsilon)^n\right) = 1$.*

Comments: comparing the two Items we get that $\mathbb{E}[W_n] = (3/2)^n \gg \sqrt{2}^n$, note that the r.h.s. is the value one would guess for $\mathbb{E}[W_n]$ assuming that rare events do not contribute much to $\mathbb{E}[W_n]$. The aim of the exercise is to show that rare events can even give the main contribution to some expectations.
Suggestion: do not use LDP's

3. Relation between Sanov's theorem and Cramer's theorem

The aim of this section is to derive the multidimensional Cramer's theorem from Sanov's theorem. In particular, we consider in some detail Exercise 5.18 in [3], restricting to ϕ bounded there. The unbounded case can be treated following the Hint after Exercise 5.18 in [3].

Exercise 4. *Let $\mathcal{S}$ be a Polish space. Let $\{X_n\}$ be a sequence of i.i.d. $\mathcal{S}$–valued random variables with common distribution λ and let $\phi : \mathcal{S} \to \mathbb{R}^d$ be a continuous function. Assume that*

$$\mathbb{E}[e^{a|\phi(X_1)|}] < \infty \qquad \forall a > 0\,. \tag{5}$$

Prove by contraction from Sanov's theorem that

$$\frac{S_n}{n} := \frac{1}{n}\sum_{k=1}^{n} \phi(X_k)$$

satisfies a LDP with speed n and good and convex rate function $I : \mathbb{R}^d \to [0,\infty]$ given by

$$I(z) := \inf\{H(\nu|\lambda) \,:\, \nu(\phi) = z\} = \sup_{\theta\in\mathbb{R}^d}\left\{z\cdot\theta - \log\lambda(e^{\theta\cdot\phi})\right\}\,. \tag{6}$$

For simplicity suppose that ϕ is bounded.

We give some comments:

If we take $\mathcal{S} = \mathbb{R}^d$ and $\phi(x) = x$ then we get the multidimensional Cramer's theorem for iid random vectors with value in $\mathbb{R}^d$. In this case S_n/n is simply the arithmetic average of $X_1, X_2, \ldots, X_n$. Note that for $d = 1$ the above condition (5) is equivalent to $\mathbb{E}[e^{tX_1}] < \infty$ for any $t \in \mathbb{R}$ which is the condition of the 1d Cramer's theorem in [2].

We give some suggestions to solve the exercise:
1) Apply the contraction principle and recall that $\mathcal{M}_1(\mathcal{S})$ is endowed of the weak topology (which can be metrized to make $\mathcal{M}_1(\mathcal{S})$ itself to be a Polish space). You get between others the first identity in (6).
2) Prove that $I(z)$ given by the first identity in (6) is convex.
3) Compute the Legendre transform $I^*(\theta) := \sup_{z\in\mathbb{R}^d}\{z\cdot\theta - I(z)\}$ of I using the first identity of (6) and show that it equals $\sup_{\nu\in\mathcal{M}_1(\mathcal{S})}\{\nu(\theta\phi) - H(\nu|\lambda)\}$.
4) Use that $H(\cdot|\lambda)$ and $p(g) := \log\lambda(e^g)$ are convex conjugate as in Theorem 5.6 of [3] and deduce that $I^*(\theta) = p(\theta\phi)$.
5) To finally get the second identity apply the Fenchel–Moreau theorem (see below)

Theorem 3.1 (Fenchel–Moreau theorem). *Suppose $f : \mathcal{X} \to (-\infty,\infty]$ is not identically ∞. Then, $f = f^{**}$ if and only if f is convex and lower semicontinuous.*

If interested to a proof, see e.g. [3, Chp.4].

4. Esercizi misti

Exercise 5. *Solve Exercise 2.16 in* [3]

Exercise 6. *Solve Exercise 5.20 in* [3]

Exercise 7. *Solve Exercise 6.5 in* [3]

Exercise 8. *Solve Exercise 6.10 in* [3]

Exercise 9. *Consider μ,ν in $\mathcal{M}_1(\mathcal{X})$ with $\nu\neq\nu$. As a consequence, there exits $f\in\mathcal{C}_b(\mathcal{X})$ such that $\mu(f)\neq\nu(f)$. Use this observation and Birkhoff's ergodic theorem to deduce that $\mu^{\otimes\mathbb{Z}^d}$ and $\nu^{\otimes\mathbb{Z}^d}$ are mutually singular (in agreement with the fact that different probability measures in $\mathcal{M}_\theta$ are mutually singular).*

Exercise 10. *Solve Exercise 8.5 in* [3]

References

[1] A. Dembo, O. Zeitouni; *Large deviations techniques and applications.* Second edition. Springer Verlag.
[2] F. den Hollander; *Large deviations.* Fields Institute Monographs, AMS.
[3] F. Rassoul-Agha, T. Seppäläinen; *A Course on Large Deviations with an Introduction to Gibbs Measures.* Graduate Studies in Mathematics Volume 162, AMS.