Groupe de travail Viscosity Solutions

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Wentzell-Freidlin estimates of non-smooth reflected diffusion processes
a PDE method.

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- $\mathcal{O}$ be a smooth ($W^{3,\infty}$) open bounded subset of $I\!R^d$, d the distance to the boundary, $n = -Dd$ the outword normal.
- γ be Lipschitz continuous and s.t. there exists $c_0 > 0$ such that for all $x \in \partial\mathcal{O}$, $n(x) \cdot \gamma(x) \geq c_0 > 0$,
- $b : I\!R^+ \times I\!R^d \to I\!R^d$ and $\sigma : I\!R^+ \times I\!R^d \to I\!R^{d\times m}$ be Lipschitz continuous with respect to $x \in I\!R^d$ and continuous with respect to $t \in I\!R^+$.
- $W = (W_t)$ an R^m-valued Wiener process defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathcal{F}_t$ the associated standard filtration.

We consider the solution $(X^{t,x,\varepsilon}, k^{t,x,\varepsilon})$ (X continuous and k BV) of the reflected stochastic differential equation in $\overline{\mathcal{O}}$

$$dX^\varepsilon_s = b_\varepsilon(s, X^\varepsilon_s)ds + \varepsilon\sigma_\varepsilon(s, X^\varepsilon_s)dW_s - dk^\varepsilon_s, \qquad dk^\varepsilon_s = \mathbf{1}_{\partial\mathcal{O}}(X^\varepsilon_\tau)\gamma(X^\varepsilon_\tau)d|k^\varepsilon|_s, \quad X^\varepsilon_t = x. \tag{1}$$

(We refer to [7] for the existence and to [3] for the uniqueness of such processes.) As ε goes to 0 it converges in probability to the solution $(X^{t,x}, k^{t,x})$ of the deterministic reflected differential equation

$$dX_s = b(s, X_s)ds - dk_s, \qquad dk_s = \mathbf{1}_{\partial\mathcal{O}}(X_\tau)\gamma(X_\tau)d|k|_s, \quad X_t = x, \tag{2}$$

Wentzell-Freidlin estimates give some precision on the rate of this convergence. Indeed, for all Borel subset $\mathcal{G}$ of $C([0,T] \times \overline{\mathcal{O}})$ one has

$$\Lambda_{t,x}(\overline{\mathcal{G}}) \leq \liminf_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln \mathbf{P}[X^{t,x,\varepsilon} \in \mathcal{G}_t]\right\} \leq \limsup_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln \mathbf{P}[X^{t,x,\varepsilon} \in \mathcal{G}_t]\right\} \leq \Lambda_{t,x}(\mathring{\mathcal{G}}), \tag{3}$$

where, $\Lambda_{t,x}(\mathcal{G}) = \inf_{g\in\mathcal{G}} \lambda_{t,x}(g)$ and for $g \in C([t,T]; \overline{\mathcal{O}})$, $\lambda_{t,x}(g)$ is defined by

$$\lambda_{t,x}(g) = \inf\{\frac{1}{2}\int_t^T |\alpha(s)|^2 ds;\ \alpha \in L^2(0,T) \text{ s.t. } Y^{t,x,\alpha} = g \text{ on } [t,T]\}. \tag{4}$$

and $(Y^{t,x,\alpha}, \zeta^{t,x,\alpha})$ is the solution of the following deterministic reflected dynamic in $\overline{\mathcal{O}}$

$$dY_s = (b(s, Y_s) - \sigma(s, Y_s)\alpha(s))ds - d\zeta^{t,x}_s, \quad d\zeta_s = \mathbf{1}_{\partial\mathcal{O}}(Y_s)\gamma(Y_s)d|\zeta|_s, \quad Y_t = x. \tag{5}$$

(thus we generalize the results of Doss Priouret [5] who use probabilistic methods.)

A PDE method (starting from Fleming [6], [1] ,[2]) consists in

1. estimate “small probabilities” by the solution u^ε of some partial differential equation,
2. make the logarithmic change $v^\varepsilon = -\varepsilon^2 \ln(u^\varepsilon)$, and consider the PDE$_\varepsilon$ satisfied by v_ε,
3. let ε to 0 and have $v^\varepsilon \to v$ the solution of PDE,
4. interpret v in terms of control in order to obtain the action fonctional $\Lambda_{t,x}$.

In order to prove (3) we proceed in three steps. (We note $\mathcal{B}=\mathcal{B}(f,r)$ a ball in $C([0,T]\times\overline{\mathcal{O}})$)

$$\begin{array}{ll} \text{(I)} & \liminf_{\varepsilon\to 0}[-\varepsilon^2\ln \mathbf{P}[X^{t,x,\varepsilon}\in\mathcal{B}]]\le \Lambda_{t,x}(\mathcal{B}) \\ \text{(II)} & \limsup_{\varepsilon\to 0}[-\varepsilon^2\ln \mathbf{P}[X^{t,x,\varepsilon}\in\bigcap_{i=1}^N \mathcal{B}_i^c]]\le \Lambda_{t,x}(\bigcap_{i=1}^N \mathcal{B}_i^c) \\ \text{(III)} & \text{it is enough.} \end{array}$$

Proof of (I)

1. We set $\mathcal{T}=\{(t,x) s.t. |f(t)-x|<r\}$ and we remark that

$$u^\varepsilon(t,x)=\mathbf{P}\{X_s^{t,x,\varepsilon}\in\mathcal{B}\}=\inf_{\theta\in\Theta_T}\mathbb{E}\mathbf{1}_{\mathcal{T}}(\theta,X_\theta^{t,x,\varepsilon}).$$

   In view of the exponential change of variable we approximate $\mathbf{1}_{\mathcal{T}}+e^{-A/\varepsilon^2}$ from upon in the following way : Let $\chi_{\delta,A}$ be an approximation of $A\mathbf{1}_{\mathcal{T}^c}$ from above then $\phi^\varepsilon_{\delta,A}=e^{(-\chi_{\delta,A}/\varepsilon^2)}$ converges to $\mathbf{1}_{\mathcal{T}}+e^{-A/\varepsilon^2}$ from upon as $\varepsilon,\delta\to 0$.
   We set $u^\varepsilon_{\delta,A}=\inf_{\theta\in\Theta_T}\mathbb{E}\phi^\varepsilon_{\delta,A}(\theta,X_\theta^{t,x,\varepsilon})$. This function is the solution of the following obstacle problem

$$\max(-\frac{\partial u}{\partial t}-\frac{\varepsilon^2}{2}\mathrm{Tr}[\sigma_\varepsilon\sigma_\varepsilon^T D^2u]-b_\varepsilon\cdot Du, u-\psi_\varepsilon^{\delta,A})=0,\quad \frac{\partial u}{\partial\gamma}=0,\quad u(T)=\psi_\varepsilon^{\delta,A}(T).$$

2. we set $v^\varepsilon_{\delta,A}=-\varepsilon^2\ln u^\varepsilon_{\delta,A}$. It is the solution of

$$\min(-\frac{\partial v}{\partial t}-\frac{\varepsilon^2}{2}\mathrm{Tr}[\sigma_\varepsilon\sigma_\varepsilon^T D^2u]+\frac{1}{2}|\sigma^\varepsilon Dv|^2-b_\varepsilon\cdot Dv, v-\chi^{\delta,A})=0,\quad \frac{\partial v}{\partial\gamma}=0,\quad v(T)=\chi^{\delta,A}(T).$$

3. we proove (**Theorem 1**) that there is a strong comparison result for viscosity solutions of equation

$$\min(-\frac{\partial v}{\partial t}+\frac{1}{2}|\sigma Dv|^2-b\cdot Dv, v-\chi^{\delta,A})=0,\quad \frac{\partial v}{\partial\gamma}=0,\quad v(T)=\chi^{\delta,A}(T).$$

   and therefore $(v^\varepsilon_{\delta,A})_\varepsilon$ converges to $v_{\delta,A}$ the solution of the above equation.

4. we proove (**Theorem 2**) the existence and continuity of the flow $(Y^{t,x,\alpha},z^{t,x,\alpha})$ defined by equation (5) as well as the convergence of $(Y^{t,x,\alpha^\varepsilon})_\varepsilon$ to $Y^{t,x,\alpha}$ if $(\alpha^\varepsilon)_\varepsilon$ to α weakly in L^2. Thus we have the following representation

$$v_{\delta,A}(t,x)=\inf_{\alpha\in L^2(0,T)}\sup_{t\le\theta\le T}\int_t^\theta|\alpha_s|^2ds+\chi_{\delta,A}(Y_\theta^{t,x,\alpha})$$

   and we show that the limit when δ goes to 0 of the right-hand side of the above equation is equal to $\Lambda_{t,x}(\mathcal{B})\wedge A$. Therefore

$$\liminf_{\varepsilon\to 0}[-\varepsilon^2u^\varepsilon(t,x)\wedge A\le\liminf_{\varepsilon\to 0}[-\varepsilon^2(u^\varepsilon(t,x)+e^{-A/\varepsilon^2})\le\Lambda_{t,x}(\mathcal{B})\wedge A.$$

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