Workshop on

Numerical Methods for Viscosity Solutions and Applications

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ABSTRACTS

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Approximation of degenerate Hamilton-Jacobi equations

Fabio Camilli

Dipartimento di Energetica, Università dell'Aquila,
67040 Roio Poggio (l'Aquila), Italy, e-mail: camilli@caspur.it.

The Shape–from–Shading model in image analysis leads to the study of the following type of problems

$$H(x, Du) = f(x) \qquad x \in \Omega, \tag{1}$$

$$u(x) = g(x) \qquad x \in \partial\Omega, \tag{2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, H and f are continuous nonnegative functions.

Since Lions [?], it is well known that, if $K = \{x \in \Omega : \; f(x) = 0\}$ is not empty, then the uniqueness of viscosity solution for problem (1)-(2) fails. The lack of uniqueness is obviously a drawback in proving convergence of approximation schemes by standard methods.

We present a characterization of the maximal viscosity solution of problem (1)-(2) recently obtained in [1], [2]. Without any assumption on K, we show that the maximal solution of (1)-(2) can be characterized as the unique viscosity subsolution of the problem which is also a supersolution, in an appropriate sense, of the singular equation

$$\rho(x, Du) = 1 \qquad x \in \Omega, \tag{3}$$

where $\rho(x,p) = \inf\{\lambda : H(x, p/\lambda) \leq f(x)\}$.

We also give stability properties for the previous notion of solution which allow us to prove convergence to the maximal solution of (1)-(2) both of discretization schemes and of the vanishing viscosity approximation.

References

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3. P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Pitman, London, 1982.

Non convex problems with wells in phase transitions

M. Chipot
Angewandte Mathematik
Üniversität Zürich
Winterthurerstr. 190, CH-8057 Zürich
e-mail: chipot@amath.unizh.ch

We will make a summary of the state of the art regarding these problems from a theoretical point of view and also regarding the numerical analysis.

Error analysis for a difference scheme approximating mean curvature flow

Klaus Deckelnick

School of Mathematical Sciences,
University of Sussex
Falmer, Brighton BN1 9QH, England
e-mail: K.P.Deckelnick@sussex.ac.uk

We analyze a finite difference scheme for the approximation of level set solutions to mean curvature flow. The scheme, which was proposed by Crandall and Lions is a monotone and consistent discretization of a regularized version of the underlying problem. We derive a uniform bound between the numerical solution and the viscosity solution to the level set equation provided that the space and time step sizes are appropriately related to the regularization parameter.

A large time step, non-oscillatory Godunov-type scheme for Hamilton-Jacobi equations

Maurizio Falcone

Dipartimento di Matematica, Università di Roma "La Sapienza"
P. Aldo Moro 2, 00185, Roma, e-mail: `falcone@caspur.it`.

Roberto Ferretti

Dipartimento di Matematica, Università di Roma "Tor Vergata"
via della Ricerca Scientifica, 00133 Roma, Italy,
e-mail: `ferretti@mat.uniroma2.it`

We present a class of large time step numerical schemes for the evolutive Hamilton–Jacobi equation

$$u_t + H(x, \nabla u) = 0$$

complemented with an initial condition and appropriate boundary conditions.
The schemes are of Godunov type [1], and their construction is based on a suitable generalization of the representation formula of the solution via the method of characteristics. These methods are interesting since they allow large time steps, so that they have low computational complexity. Moreover, their numerical viscosity is quite limited and this results in a good resolution around singularities and discontinuities.
We will discuss a multidimensional implementation where a non–oscillatory n–dimensional reconstruction is obtained by the direct sum of one–dimensional ENO interpolations. We will also show numerical tests involving the generation of singularities in the solution.

References

1. M. Falcone and R. Ferretti, *Fully discrete high–order schemes for viscosity solutions of Hamilton–Jacobi–Bellman equations*, in preparation.

Error estimates and numerical simulations of mean curvature flow through a modified reaction-diffusion equation

F. Fierro

Dipartimento di Matematica, Università di Milano,
20133 Milano, Italy, e-mail: `fierroares.mat.unimi.it`

We study the approximation of forced mean curvature flow $V = \kappa + g$, trough the following singularly perturbed reaction-diffusion equation $\varepsilon\, \partial u_\varepsilon t - \varepsilon\, \Delta u_\varepsilon + \frac{1}{2\varepsilon}\Psi'(u_\varepsilon) \ni \varepsilon|\nabla u_\varepsilon| g$ with a small parameter $\epsilon > 0$ and double obstacle potential Ψ. The zero level set of $u_\varepsilon(\cdot, t)$ approximates the evolution of a surface $\Sigma(t) \subset \mathbb{R}^n$ This approach is insensitive to the choice of the potential Ψ and thus differs from the usual Allen-Cahn approximation. We investigate smooth flows and evolutions past singularities. Under suitable assumptions we prove that the zero level sets of the solution to the double obstacle problem converge to the generalized motion by mean curvature with forcing term. Further interface error estimate are established. Finally we present some numerical results. This work was partially done in collaboration with Roberta Goglione (University of Milano).

Adaptive numerical methods for Hamilton-Jacobi equations

Lars Grüne

FB Mathematik, University "J.W. Goethe"
60054 Frankfurt, Germany, e-mail: gruene@math.uni-frankfurt.de

In the numerical solution of Hamilton-Jacobi equations the spatial discretization is one of the most expensive parts of the whole discretization procedure, especially for equations in three or even more space dimensions. Here a fine equidistant discretization results in a huge number of nodes requiring both long computation times and enormous memory. Often, however, the structure of the solution requires fine discretization only in a small part of the whole computational domain. During the last years suitable adaptive procedures, usually based on a-posteriori error estimates have been developed for a huge number of partial differential equations.
In this talk we will present such techniques for the computation of visco= sity solutions of first order Hamilton-Jacobi equations, discussing the underlying mathematical theory as well as implementational aspects and a number of examples.

Processing of 3D image sequences by degenerate diffusion equations

Karol Mikula

Department of Mathematics, Slovak University of Technology,
Radlinského 11, 813 68 Bratislava, Slovakia
e-mail: mikula@ops.svf.stuba.sk

We discuss models and numerical methods for multiscale analysis of space-time images. Nonlinear partial differential equations are used for processing of the time sequence considering spatial as well as temporal coherence. We describe adaptive finite element and finite volume methods for solving regularized Perona-Malik anisotropic diffusion equation and geometrical equations of mean curvature flow type used for spatial diffusion process. The idea of movie multiscale analysis of Alvarez, Guichard, Lions and Morel combined with geometry driven spatial diffusions is then used for processing of the image sequences. The computational results with artificial and real (echocardiographic) images are discussed and video simulations of image multiscale analysis are presented.

Robust a posteriori Estimators for Non-uniformly Elliptic Obstacle Problems Approximating Minimization of Functionals with Curvature

Andreas Veeser

FB Mathematik, University of Freiburg
Freiburg, Germany, e-mail: `veeser@mat.unimi.it`

A surface of prescribed mean curvature and contact angle may be found by a minimization over the set of the characteristic functions $\mathrm{BV}(\Omega, \{-1, 1\})$ of a functional $\mathcal{F}$ involving the total variation $\int_\Omega |\nabla\chi|$. Besides of its own interest, this minimization is used in an implicit time discretization of the mean curvature flow [1, 3, 4] .

The functional $\mathcal{F}$ can be equivalently minimized on the larger, convex set $\mathrm{BV}(\Omega, [-1, 1])$ and, on that space, $\mathcal{F}$ may be approximated by "semi-strictly convex" functionals $\mathcal{F}_\epsilon$, in which the total variation $\int_\Omega |\nabla\chi|$ is replaced by $\int_\Omega \sqrt{\epsilon^2 + |\nabla\chi|^2}$; see [2].

Supposing that the minimum points of $\mathcal{F}_\epsilon$, $\epsilon > 0$, are regular, we base our a posteriori analysis on the corresponding non-uniformly elliptic obstacle problems. We derive residual type estimators, which yield global upper and (partial) local lower bounds for a relevant geometric quantity, provided that a certain a posteriori condition holds. The estimators are robust in the sense that the constants of these bounds do not depend on the relaxation parameter ϵ. Finally, we illustrate the performance of the estimator with numerical simulations.
Joint work with F. Fierro (Università di Milano).

References

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