EQUICONTINUITY OF SOME LARGE TIME–STEP APPROXIMATIONS TO CONVEX HAMILTON–JACOBI EQUATIONS

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Abstract. We consider a class of semi–lagrangian high–order approximation schemes for convex Hamilton–Jacobi equations. In this framework, we prove that under certain restrictions on the relationship between Δx and Δt, the sequence of approximate solutions is uniformly Lipschitz continuous (and hence, by standard arguments, that it admits a subsequence converging to the exact solution). The argument is suitable for most reconstructions of interest, including high–order polinomials and Essentially Non–Oscillatory (ENO) reconstructions.

Key words. Convex Hamilton–Jacobi equations, high–order schemes, semi–lagrangian schemes, convergence.

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1. Introduction.

Although the numerical solution of first order Hamilton–Jacobi equations is a widely studied topic in many applications, it is well known that, when high–order schemes are considered (see [OS], [JP], [LT1]), very few convergence results are available (one such result is given in [LS]). A key tool for proving convergence of numerical approximations would be to prove some stability property for discrete solutions; when working with continuous solutions it is common to require boundedness of the family in $W^{1,\infty}$, so that compactness

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would be proved by the classical Ascoli–Arzela theorem. $W^{1,\infty}$ boundedness can generally be proved with reasonable effort in the case of low–order (usually monotone) schemes, but can be really troublesome to prove in high–order schemes, in which oscillations may occur. We must also mention that a recent paper (see [LT2]) has proposed a different tool for proving convergence, that is, uniform semiconcavity of discrete solutions.

This paper is devoted to the study of the HJ equation

$$\begin{cases} v_t(x,t) + H(\nabla v(x,t)) = 0 & \text{in } \Omega \times [0,T] \\ v(x,0) = v_0(x) \end{cases} \tag{1.1}$$

(with Ω a bounded open subset of $\mathbb{R}^N$, $H(\cdot)$ smooth and convex). The purpose of the paper is precisely to obtain a result of uniform Lipschitz continuity for a semi–lagrangian, high–order scheme. The scheme is described in several versions in [FF1], [FF2], [FFM], [FG] (see also [SC] for a general review on semi–lagrangian approach in Computational Fluid Dynamics), but for reader's convenience, we briefly sketch its construction for the Hamilton–Jacobi equation (1.1). Since the solution (sse [L]) may be represented as

$$v(x,t) = \inf_{\alpha \in \mathbb{R}^N} \{tH^*(\alpha) + v_0(x + \alpha t)\} \tag{1.2}$$

(where $H^*(\alpha)$ is the Legendre transform of $H(p)$), using the representation formula (1.2) at each node and over each single time step, we get

$$v(x_j, t_{n-1} + \Delta t) = \inf_{\alpha} \{\Delta t H^*(\alpha) + v(x_j + \alpha \Delta t, t_{n-1})\} \tag{1.3}$$

and further replacing $v(x_j + \alpha\Delta t, t_{n-1})$ with a numerical reconstruction $I[V^{n-1}](x_j + \alpha\Delta t)$, we obtain at last the general form of the scheme for (1.1), namely

$$\begin{cases} v_j^n = \min_{\alpha} \{\Delta t H^*(\alpha) + I[V^{n-1}](x_j + \alpha\Delta t)\} \\ v_j^0 = v_0(x_j). \end{cases} \tag{1.4}$$

In (1.4), we have omitted for simplicity to include the treatment of boundary conditions, for which an explicit form can be found in [FFM].

In the sequel of the paper, we will consider for clarity of exposition a simplified version of (1.1), that is

$$\begin{cases} v_t(x,t) + \frac{1}{2}|v_x(x,t)|^2 = 0 & \text{in } \mathbb{R} \times [0,T] \\ v(x,0) = v_0(x), \end{cases} \tag{1.5}$$

posed on the whole of $\mathbb{R}$ so that it is possible to avoid treating boundary conditions in (1.4). Most arguments used in the proof are not specifically suited for this simplified case, which seems to be a good model for any HJ equation with smooth and strictly convex hamiltonian.

The outline of the paper is the following. We will prove in Section 2 a bound on the values α appearing in (1.4), and apply in Section 3 this estimate to the main proof of boundedness and Lipschitz continuity for numerical solutions.

2. A preliminary estimate.

We assume that equation (1.5) is discretized on an infinite uniform grid with nodes $x_j = j\Delta x$ ($j = 0, \pm 1, \pm 2, \ldots$), and that given a Lipschitz continuous function $v(x)$ and the sequence $V = \{v(x_j)\}_j$, the reconstruction (interpolation) operator $I[V]$ which extends the sequence V on the whole of $\mathbb{R}$ satisfies

$$I[V](x_j) = v(x_j) \ , \quad |I[V](x) - I_1[V](x)| \le C\Delta x. \tag{2.1}$$

Here and in the sequel, $I_1[V]$ will denote the P_1 (piecewise linear) interpolation on the sequence V. For example, if $I[V]$ is a quadratic reconstruction, we would have

$$|I[V](x) - I_1[V](x)| \le \frac{1}{8} \max |I''[V]| \Delta x^2 \le \frac{1}{8}\frac{2L}{\Delta x}\Delta x^2 = \frac{L}{4}\Delta x$$

with L the Lipschitz constant of v. A similar argument (as the reader can easily check) could be applied to any piecewise polynomial interpolation, including ENO–type reconstructions.

We also recall that for the special case of (1.5), the Legendre transform of the Hamiltonian function is $H^*(\alpha) = |\alpha|^2/2$.

We can now prove the following

Lemma 2.1. *Consider the scheme* (1.4) *applied to equation* (1.5). *Assume that* (2.1) *holds, that $\Delta x/\Delta t \to 0$ and that the sequence V^{n-1} in* (1.4) *satisfies the discrete Lipschitz estimate*

$$\left| \frac{v_{j+1}^{n-1} - v_j^{n-1}}{\Delta x} \right| \le L_{n-1}. \tag{2.2}$$

Then, the value $\bar\alpha$ achieving the minimum in (1.4) *satisfies the bound*

$$|\bar\alpha| \le L_{n-1}\left(1 + C_1 \frac{\Delta x^{1/2}}{\Delta t^{1/2}}\right). \tag{2.3}$$

for some constants C_1, C_2 independent of Δx, Δt.

Proof. Let us assume for example that $\bar\alpha > 0$. Define the two functions

$$F(\alpha) := \frac{\Delta t|\alpha|^2}{2} + I[V^{n-1}](x_j + \alpha\Delta t)$$

$$F_1(\alpha) := \frac{\Delta t|\alpha|^2}{2} + I_1[V^{n-1}](x_j + \alpha\Delta t)$$

(note that we have dropped for simplicity the dependence on n and j). We look for a value A such that $F(\alpha)$ cannot be a minimum of F for any $\alpha > A$.

Consider first the case of P_1 reconstructions. In this situation, the maximum value of $\bar{\alpha}$ occurs when $v_{j+k}^{n-1} = v_j^{n-1} - L_{n-1}k\Delta x$ for v_{j+k}^{n-1} in some suitable right neighbourhood of v_j^{n-1}, that is, when V^{n-1} decreases at its maximum rate. Since in this case

$$I_1[V^{n-1}](x_j + \alpha\Delta t) = v_j^{n-1} - L_{n-1}\alpha\Delta t,$$

an easy computation shows that the point at which the minimum of F_1 is achieved is $\bar{\alpha}_1 = L_{n-1}$, and accordingly

$$F_1(\bar{\alpha}_1) = v_j^{n-1} - \frac{\Delta t L_{n-1}^2}{2}. \tag{2.4}$$

To treat the general, high–order case we first take into account the values attained when the point $x_j + \alpha\Delta t$ coincides with a grid node; for such choices of α, F_1 and F have the same value. It is easy to show that the minimum of such values is attained in one of the nodes closest to the point $x_j + \bar{\alpha}_1\Delta t$, and that the corresponding value of F is less or equal to

$$F_1\left(L_{n-1} + \frac{\Delta x}{2\Delta t}\right) = v_j^{n-1} - \frac{\Delta t L_{n-1}^2}{2} + \frac{\Delta x^2}{8\Delta t}. \tag{2.5}$$

Now, from (2.5) we also have

$$\min_\alpha F(\alpha) \le v_j^{n-1} - \frac{\Delta t L_{n-1}^2}{2} + \frac{\Delta x^2}{8\Delta t} \tag{2.6}$$

and since, by (2.1),

$$F(\alpha) \ge F_1(\alpha) - C\Delta x = \frac{\Delta t|\alpha|^2}{2} + v_j^{n-1} - L_{n-1}\alpha\Delta t - C\Delta x, \tag{2.7}$$

combining (2.6) and (2.7) we obtain the following condition, necessary for a given α to be a minimum point for F:

$$v_j^{n-1} - \frac{\Delta t L_{n-1}^2}{2} + \frac{\Delta x^2}{8\Delta t} \ge \frac{\Delta t|\alpha|^2}{2} + v_j - L_{n-1}\alpha\Delta t - C\Delta x$$

and solving for positive α we get at last

$$\bar{\alpha} \le L_{n-1} + \sqrt{2C\frac{\Delta x}{\Delta t} + 4\frac{\Delta x^2}{\Delta t^2}}$$

which yelds (2.2), after considering the reverse case $\bar{\alpha} < 0$ and taking into account that $\Delta x/\Delta t \to 0$.

■

Remark. The condition $\Delta x/\Delta t \to 0$ is not unnatural in this class of schemes. As it has been remarked elsewhere, the numerical domain of dependence enlarges (as it is easy to see from (1.4)) as Δt increases, so that very large Courant numbers are allowed without loss of stability.

3. The main result.

In this section we state and prove the main stability results. In order to do this, it will be necessary to make further restrictions on the $\Delta x/\Delta t$ relationship. We point out that such restrictions seem necessary for the proof, but not in computational practice.

We first prove Lipschitz continuity:

Theorem 3.1. *Consider the scheme* (1.4) *applied to equation* (1.5). *Assume that* (2.1) *holds, that* $\Delta x = O(\Delta t^3)$ *and that* v_0 *is Lipschitz continuous. Then, the numerical solutions* V^n *satisfy the discrete Lipschitz estimate*

$$\left| \frac{v^n_{j+1} - v^n_j}{\Delta x} \right| \le L \tag{3.1}$$

for a constant L *independent of* Δx *and* Δt, *and for* $0 \le n \le T/\Delta t$.

Proof. First, we introduce the notation $\bar\alpha_j$ to denote the argmin for α in (1.4):

$$v^n_j = \min_\alpha \left\{ \frac{\Delta t |\alpha|^2}{2} + I[V^{n-1}](x_j + \alpha\Delta t) \right\} = \frac{\Delta t |\bar\alpha_j|^2}{2} + I[V^{n-1}](x_j + \bar\alpha_j \Delta t). \tag{3.2}$$

In order to estimate the discrete incremental ratio on V^n we give on v_{j+1} the bound

$$\begin{aligned} v^n_{j+1} &= \frac{\Delta t |\bar\alpha_{j+1}|^2}{2} + I[V^{n-1}](x_{j+1} + \bar\alpha_{j+1}\Delta t) \le \\ &\le \frac{\Delta t \left|\bar\alpha_j - \frac{\Delta x}{\Delta t}\right|^2}{2} + I[V^{n-1}]\left(x_{j+1} + \left(\bar\alpha_j - \frac{\Delta x}{\Delta t}\right)\Delta t \right) = \\ &= \frac{\Delta t \left|\bar\alpha_j - \frac{\Delta x}{\Delta t}\right|^2}{2} + I[V^{n-1}](x_j + \bar\alpha_j \Delta t). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we obtain the unilateral estimate

$$\frac{v^n_{j+1} - v^n_j}{\Delta x} \le \frac{\Delta t}{\Delta x} \left[\frac{\left|\bar\alpha_j - \frac{\Delta x}{\Delta t}\right|^2}{2} - \frac{|\bar\alpha_j|^2}{2} \right]. \tag{3.4}$$

Interchanging the role of $\bar\alpha_j$ and $\bar\alpha_{j+1}$ we get the reverse estimate

$$\frac{v^n_j - v^n_{j+1}}{\Delta x} \le \frac{\Delta t}{\Delta x} \left[\frac{\left|\bar\alpha_{j+1} + \frac{\Delta x}{\Delta t}\right|^2}{2} - \frac{|\bar\alpha_{j+1}|^2}{2} \right] \tag{3.5}$$

so that combining (3.4) with (3.5), and using the local Lipschitz continuity of H^* we have

$$\left|\frac{v_{j+1}^n - v_j^n}{\Delta x}\right| \leq \frac{\Delta t}{\Delta x} L_{H^*} \frac{\Delta x}{\Delta t} = L_{H^*} \tag{3.6}$$

where L_{H^*} is the Lipschitz constant of $H^*(\alpha) = \alpha^2/2$ on the smallest interval containing $\bar{\alpha}_j$, $\bar{\alpha}_{j+1}$, $\bar{\alpha}_j - \Delta x/\Delta t$, $\bar{\alpha}_{j+1} + \Delta x/\Delta t$. Recalling that $(\alpha^2/2)' = \alpha$, by lemma 2.1 we obtain

$$\left|\frac{v_{j+1}^n - v_j^n}{\Delta x}\right| \leq L_{H^*} = L_{n-1}\left(1 + C_1 \frac{\Delta x^{1/2}}{\Delta t^{1/2}} + C_2 \frac{\Delta x}{\Delta t}\right)$$

and hence

$$L_n \leq L_{n-1}\left(1 + C_1 \frac{\Delta x^{1/2}}{\Delta t^{1/2}} + o\left(\frac{\Delta x^{1/2}}{\Delta t^{1/2}}\right)\right). \tag{3.7}$$

Iterating back the estimate, (3.7) gives

$$L_n \leq L_0\left(1 + C_1 \frac{\Delta x^{1/2}}{\Delta t^{1/2}} + o\left(\frac{\Delta x^{1/2}}{\Delta t^{1/2}}\right)\right)^{T/\Delta t}. \tag{3.8}$$

In order to get a finite limit in (3.8) it is necessary to have

$$\frac{\Delta x^{1/2}}{\Delta t^{1/2}} = O(\Delta t)$$

which holds true if $\Delta x = O(\Delta t^3)$.

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Remark. It is necessary to note that the constants C_1 and C_2 do not depend on n in a critical way. The constant C (see the remark in section 2) behaves like L_{n-1}, thus C_1 behaves like a constant. C_2 decreases at the increase of L_{n-1}.

Once proved Lipschitz continuity, we can easily obtain L^∞ boundedness as follows:

Theorem 3.2. *Consider the scheme* (1.4) *applied to equation* (1.5). *Assume that* (2.1) *holds, that* $\Delta x = O(\Delta t^3)$ *and that* v_0 *is Lipschitz continuous. Then, for* $n = T/\Delta t$, *the numerical solutions* V^n *satisfies the bounds*

$$v_0(x_j) - \frac{L^2 T}{2} + O(\Delta t^2) \leq v_j^n \leq v_0(x^j) \tag{3.9}$$

L being defined as in theorem 3.1.

Proof. Working as in Lemma 2.1 (see (2.4), (2.7)) we can give a lower bound on v_j^n as

$$v_j^n \geq v_j^{n-1} - \frac{\Delta t L^2}{2} - C\Delta x$$

so that iterating back $n = T/\Delta t$ times and recalling that $\Delta x = O(\Delta t^3)$ we obtain the first inequality in (3.9). To get the second inequality, we set $\alpha = 0$ in (1.4), thus obtaining

$$v_j^n \leq I[V^{n-1}](x_j) = v_j^{n-1} \leq \cdots \leq v_j^0 = v_0(x^j).$$

■

We finally remark that, by standard arguments, theorems 3.1–3.2 imply that if v_0 is bounded and Lipschitz continuous, then the family $I_1[V_\Delta]$ (V_Δ being a numerical approximation satisfying the basic assumptions) satisfies the same $W^{1,\infty}$ bound, and therefore admits locally uniformly convergent subsequences. Using (2.1) one proves that the same also holds for the corresponding subsequences of the family $I[V_\Delta]$ which, by consistency (compare (1.3) and (1.4)) necessarily converge to the exact solution.

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